Title: Asymptotic safety: a review

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Abstract: I shall review on field theory examples, the meaning of the concept of asymptotic safety in the context of low energy effective field theories.

ASYMPTOTIC SAFETY: A SHORT REVIEW

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The notion of asymptotic safety initially has been directly connected to the search for UV (large momentum) fixed points of the renormalization group (RG) of quantum field theory (QFT). However, the meaning and importance of UV fixed points have themselves evolved with a deepening understanding of QFT and its relation with the theory of critical phenomena. In a modern interpretation of QFT, the problem of asymptotic safety has to be discussed within the general framework of effective quantum field theories. Moreover, the notion of asymptotic safety may also apply to the existence of IR (small momentum) fixed points. To illustrate the problem, we first review a few classical examples and then try to draw some general conclusions.

As a general reference see

J. ZINN-JUSTIN, *Quantum Field Theory and Critical Phenomena* (Oxford University Press, 1989) International Series of Monographs on Physics 113, 1054 pp. (2002), Fourth Edition

Early considerations

A first discussion of the relevance of UV stable RG fixed can be found in the article of Gell-Mann–Low (1954) in the context of QED. The hope there was to give some meaning to the bare Lagrangian. If QED would have exhibited a UV fixed point, the bare coupling constant could have had a finite limit for infinite cut-off. But, as it was found, in QED the origin is an IR fixed point. Thus, a possible UV fixed point could only be non-trivial. Such a fixed point has never been found in pure QED. This negative result comforted another scheme, later called the BHPZ scheme, which claimed that only renormalized perturbation theory is meaningful, and the bare theory is just a book-keeping device. Moreover, as an apparent consequence, QFT could only describe weakly-coupled theories. For example, the strong nuclear force could not be described by QFT and this led in the 1960ties to the alternative development of the S-matrix theory (but alternatively series summation methods like Padé approximants were also eventually proposed).

The non-Abelian gauge theory revolution: quarks and asymptotic freedom

The end of the sixties witnessed a series of spectacular new results. Deep inelastic experiments at SLAC confirmed the quark idea and indicated that Strong Interactions were in fact weak at short distance. The possible relevance of renormalization group was emphasized (Callan–Symanzik). Non-Abelian gauge theories were quantized and allowed completing the construction of a consistent model describing Weak and Electromagnetic Interactions. Finally, the discovery that a class of non-Abelian gauge theories is asymptotically free, that is, that zero coupling is an UV fixed point, yielded an explanation for the weak interaction of quarks at short distance. Quantum Chromodynamics (QCD) became part of the Standard Model of all interactions, but gravity. This achievement was a triumph of the concept of renormalizable quantum field theories.

However, in QCD the problem of quark confinement, that is, that conversely interactions become very large at large distance, forced to envisage the global existence of quantum field theory, which no longer could be entirely reduced to renormalized perturbative expansion.

Other theories, like the non-linear σ-model, were discovered that are non-renormalizable by power counting but, nevertheless, behave like renormalizable theories thanks to the presence of non-trivial UV fixed points. The concept of asymptotic safety (Weinberg 1976) eventually emerged from such examples, the existence of UV fixed points seeming a sufficient condition for the existence of renormalized field theories consistent on all scales. This idea was later applied to quantum gravity.

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Wilson's theory of critical phenomena

Another development (on which I started working in 1972 with Brézin and Le Guillou) strongly influenced our view of QFT: Wilson's theory of critical phenomena (1971). It showed how QFT emerges in a completely different context: the large distance properties of a large class of continuous phase transitions with short range interactions can be described by statistical field theories (QFT in Euclidean or imaginary time).

There, physics is initially described by some microscopic model, like a lattice model, which is not of field theory type and has no infinities.

However, to describe its large distance properties, the initial model can be replaced by an effective local QFT. The effective theory contains all possible local interactions consistent with symmetries, but the renormalizable or super-renormalizable terms are distinguished by the property that they dominate long distance physics (at least in some neighbourhood of the Gaussian fixed point).

Such a QFT is equipped with a natural cut-off, reflection of the initial microscopic scale (like the range of interactions).

In this framework, bare field theory and cut-off have some physical reality: the usual bare theory is the field theory reduced to the renormalizable and super-renormalizable interactions.

Then, the large distance properties of the bare theory can be derived directly from QFT bare RG equations (Zinn-Justin 1973).

In particular, the existence of the Wilson–Fisher IR fixed point (1972) implies insensitivity to the cut-off regularization, that is, the *ad hoc* explicit cut-off structure plays no role, and universal behaviour.

Finally, one can also introduce a QFT renormalized at a suitable large distance scale (like the correlation length scale) and employ, for example, Callan–Symanzik RG equations (Brézin, Le Guillou, ZJ 1972). The renormalized QFT appears as a convenient calculation tool if one is interested only in determining the asymptotic, universal, large distance properties.

Super-renormalizable effective field theories: the $(\phi^2)^2$ example

In dimensions $d < 4$, the large distance properties of a large class of continuous phase transitions can be described by the $O(N)$ symmetric $(\phi^2)^2$ scalar field theory. The Euclidean action can be written as

$$\mathcal{S}(\phi) = \int^{\Lambda} \left\{ \frac{1}{2} \left[\nabla_x \phi(x)\right]^2 + \frac{1}{2}\Lambda^2 r \phi^2(x) + \frac{1}{4!} g \Lambda^{4-d} \left[\phi^2(x)\right]^2 \right\} \mathrm{d}^d x \,,$$

where Λ is a cut-off and the constants g, r are dimensionless.

For $d < 4$, the theory is super-renormalizable. It requires only a fine tuning of the ϕ^2 coefficient $\Lambda^2 r$ to a vicinity of a value $\Lambda^2 r_c$, which from the RG viewpoint is a UV fixed point. This critical value exists for any N at $d = 3$ and for $N \le 2$ at $d = 2$. Then, at $g\Lambda^{4-d}$ fixed, correlation functions have a $\Lambda \to \infty$ limit and the existence of a renormalized field theory can be proved beyond perturbation theory.

From the RG viewpoint, the origin $g = 0$ is a UV fixed point and the theory is clearly UV asymptotically safe.

However, from the viewpoint of effective field theories, the $\Lambda \to \infty$ limit has to be taken at g fixed and, in a generic situation, the coupling $g\Lambda^{4-d}$ is very large in the relevant physical, large distance scale. The existence of such a limit requires the existence of an IR fixed point, which in this example is known to exist (Wilson–Fisher 1972). This can be called a condition of IR asymptotic safety.

By contrast, to approach the UV fixed point $g = 0$ requires a specific fine tuning and the absence of other less IR relevant interactions like ϕ^6 and higher dimension monomials, a situation that, remarkably enough, is physically realized in the weakly interacting dilute Bose gas.

The meaning of this fine tuning can be better understood by solving RG equations.

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The meaning of this fine tuning can be better understood by solving RG equations.

The inverse two-point function $\tilde{\Gamma}^{(2)}(p)$ at the transition point $r = r_c$ (thus $\tilde{\Gamma}^{(2)}(0) = 0$), satisfies the RG equation (Zinn-Justin 1973)

$$\left(\Lambda\frac{\partial}{\partial\Lambda} + \beta(g)\frac{\partial}{\partial g} - \eta(g)\right)\tilde{\Gamma}^{(2)}(p,\Lambda,g) = 0\,,$$

where $\eta(g) = O(g^2)$ and

$$\beta(g) = -(4-d)g + \beta_2(d)g^2 + O(g^3), \quad \beta_2(d) > 0\,.$$

The β-function has a trivial zero $g = 0$, which is the UV fixed point, and we know that it has also a non-trivial IR stable zero. For $0 < \varepsilon = 4 - d \ll 1$,

$$g = g^*_{\rm IR} = \frac{48\pi^2}{(N+8)}\varepsilon + O(\varepsilon^2).$$

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$$\tilde{\Gamma}^{(2)}(p, \Lambda, g) = p^2 Z(g) F\big(p/\Lambda(g)\big)$$

with

$$\beta(g)\frac{\partial \ln Z(g)}{\partial g} = \eta(g), \quad \beta(g)\frac{\partial \ln \Lambda(g)}{\partial g} = -1\,.$$

On dimensional grounds $\Lambda(g)$ is proportional to Λ. The function $\Lambda(g)$ is then obtained by integration:

$$\Lambda(g) = \Lambda g^{1/(4-d)} \exp\left[-\int_0^g \mathrm{d}g' \left(\frac{1}{\beta(g')} + \frac{1}{(4-d)g'}\right)\right].$$

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In a generic situation g is of order unity and, thus, $\Lambda(g)$ is of order Λ: only the universal IR behaviour of correlation functions can be observed.

By contrast, $g \ll 1$ implies $\Lambda(g) \sim g^{1/(4-d)}\Lambda \ll \Lambda$.

Then, the intermediate scale $\Lambda(g)$ becomes a mass crossover scale separating a universal long-distance regime governed by the non-trivial zero $g^*_{\mathrm{IR}} > 0$ of the β-function, from a universal short distance regime governed by the Gaussian fixed point, $g = 0$.

One finds:

$$\tilde{\Gamma}^{(2)}(p) \propto p^{2-\eta} \text{ for } p \ll \Lambda(g),$$

where $\eta > 0$ is a critical exponent, and

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A renormalizable field theory: the $(\phi^2)^2$ theory in dimension 4

The $(\phi^2)^2$ is renormalizable in dimension 4 and IR free, like QED. To the best of our knowledge, the RG β-function has no other zero. The field theory is not asymptotically safe. This situation has different interpretations, depending whether one follows the conventional renormalized QFT or the statistical physics inspired viewpoints.

In the former, one insists on taking the infinite cut-off limit at fixed renormalized parameters. In the 4D $(\phi^2)^2$ theory, this problem has no solution except the trivial one $g_{\text{ren.}} = 0$. This is the triviality problem.

In the latter, one simply concludes that the $(\phi^2)^2$ QFT has a limited distance or momentum range validity and that this range decreases when the renormalized coupling increases:

$$\ln(\Lambda/\mu) \le \frac{1}{\beta_2(4) g_{\text{ren.}}(\mu)},$$

where μ is the renormalization or physical scale.

Moreover, in the absence of a UV fixed point, correlation functions have no universal large momentum behaviour.

In QED, the small value of α implies a validity even beyond the Planck scale and, thus, sufficient for any physical purpose. Indeed, taking into account all leptons and quarks, neglecting weak interactions (which strongly modify the RG flow), one finds

$$\beta(\alpha) = \frac{16}{3\pi}\alpha^2 + O(\alpha^3)$$

and, thus,

$$\ln(\Lambda/\mu) \le \frac{3\pi}{16\alpha(\mu)} .$$

If Λ is the Planck mass and μ the top mass, the inequality implies

$$\alpha(\mu) \le \frac{3\pi}{16\ln(\Lambda/\mu)} \approx 0.016 ,$$

which is satisfied by the physical value, but nevertheless, yields the right order of magnitude.

The non-linear σ-model

With the non-linear σ-model, one comes closer to the QCD situation. The model is renormalizable, asymptotically free in $d = 2$ dimensions and non-renormalizable in higher dimensions. We limit ourselves here to the $O(N)$ symmetric vector model, which has been more extensively studied.

The action can then be written, for example, as

$$\mathcal{S}(\phi) = \frac{\Lambda^{d-2}}{2g} \int^{\Lambda} \mathrm{d}^d x \, [\partial_\mu \phi(x)]^2$$

with the constraint

$$\phi^2(x) = 1 \, .$$

Here, Λ is the cut-off and g is dimensionless (with the possible interpretation of a temperature).

The perturbative phase corresponds to spontaneous symmetry breaking of the $O(N)$ symmetry, with $(N-1)$ massless Goldstone modes.

Dimension two

In dimension two, the massless modes are responsible for IR divergent corrections and an IR cut-off is required, for example, one can add to the action an explicit $O(N)$ breaking term of the form

$$\mathbf{c} \cdot \int \mathrm{d}^d x \, \phi(x).$$

A few first terms of the RG β-function have been calculated. At leading order, one finds

$$\beta(g) = -(N-2)\frac{g^2}{2\pi} + O(g^3).$$

For $N > 2$ (the non-Abelian situation), $g = 0$ is an UV fixed point. Like QCD, the theory is asymptotically free and thus asymptotically safe. One can define a renormalized field theory consistent on all scales. Unlike what perturbation theory suggests, in the symmetric limit $\mathbf{c} = 0$, the $O(N)$ symmetry is unbroken and the spectrum consists in N massive states.

In the symmetric $\mathbf{c} = 0$ limit, the mass scale is given by

$$m(g) = \Lambda \exp\left[-\int^{g} \frac{\mathrm{d}g'}{\beta(g')}\right] \underset{g\to 0}{\propto} \Lambda\, \mathrm{e}^{-2\pi/(N-2)g}\,.$$

Note that the physical condition $m(g) \ll \Lambda$ here again requires some mild fine tuning of the coupling constant since

$$g(m) \sim \frac{2\pi}{(N-2)\ln(\Lambda/m)} \ll 1\,.$$

Higher dimensions

In higher dimensions, the non-linear σ-model is not renormalizable by power counting. However, with a suitable $O(N)$ symmetric cut-off, the perturbation series is defined and, at least for small coupling, one finds a broken phase with Goldstone particles. As the action

$$\mathcal{S}(\phi) = \frac{\Lambda^{d-2}}{2g} \int \mathrm{d}^d x \, [\partial_\mu \phi(x)]^2 ,$$

shows, for $d > 2$, the coupling constant $g\Lambda^{2-d}$ is generically small in the physical domain. Therefore, generically, one finds a model with $(N-1)$ weakly interacting Goldstone particles and the physics is completely perturbative.

However, in the case of the $O(N)$ non-linear σ-model, various consistent arguments (large N expansion, $(d-2)$ expansion, lattice regularization) indicate that the non-linear σ-model behaves like a renormalizable field theory at least in dimension 3 (Polyakov 1975, Brézin–ZJ 1976).

For example, to all orders in a double g and $\varepsilon = d-2$ expansions, one proves that the vertex functions satisfy the RG equation

$$\left[\Lambda\frac{\partial}{\partial\Lambda} + \beta(g)\frac{\partial}{\partial g} - \frac{n}{2}\zeta(g)\right]\tilde{\Gamma}^{(n)}(p_i; g, \Lambda) = 0\,,$$

where

$$\beta(g) = \varepsilon g - \frac{(N-2)}{2\pi}g^2 + O\left(g^3, g^2\varepsilon\right),$$

$$\zeta(g) = \frac{(N-1)}{2\pi}g + O\left(g^2, g\varepsilon\right).$$

The RG β-function has two zeros, $g = 0$, which is an IR fixed point and a non-trivial value

$$g = g^*_{\mathrm{UV}} = \frac{2\pi\varepsilon}{(N-2)} + O(\varepsilon^2),$$

which is a UV fixed point and, from the point of view of phase transitions, a critical temperature separating a broken phase for $g < g^*_{\mathrm{UV}}$ from a symmetric phase for $g > g^*_{\mathrm{UV}}$.

The two-point vertex function, solution of the RG equation, can then be written as

$$\tilde{\Gamma}^{(2)}(p; g, \Lambda) = \Lambda^d(g) Z(g) F\big(p/\Lambda(g)\big)$$

with

$$\Lambda(g) = \Lambda \exp\left[-\int^g \frac{\mathrm{d}g'}{\beta(g')}\right]$$
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$\Lambda(g)$ is thus the physical mass scale. For generic values of g, the integral is finite and $\Lambda(g)/\Lambda$ is thus finite. For $g < g^*_{\mathrm{UV}}$, this means that one finds only almost free Goldstone bosons and for $g > g^*_{\mathrm{UV}}$ no particle propagates.

Only for $|g - g^*_{\mathrm{UV}}| \ll 1$, and this again implies fine tuning, does the integral diverge. From the statistical viewpoint, this is the critical domain.

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Then, $\Lambda(g) \ll \Lambda$ and a new mass scale is generated:

$$\Lambda(g) \propto \Lambda|g - g^*_{\mathrm{UV}}|^\nu \ll \Lambda, \quad \nu = -\frac{1}{\beta'(g^*_{\mathrm{UV}})}.$$

The physics is then the same as for the generic $((\phi)^2)^2$ field theory.

For $g > g^*_{\mathrm{UV}}$, $\Lambda(g)$ provides a physical mass scale, which is now much smaller than the cut-off.

For $g < g^*_{\mathrm{UV}}$, $\Lambda(g)$ is a crossover scale between a universal small momentum free behaviour and a large momentum critical behaviour.

Finally, note that a similar analysis can be made, for example, for the Gross–Neveu model with $U(N)$ symmetry in $d \geq 2$ dimensions.

Quantum ChromoDynamics (QCD)

QCD is asymptotically free and, thus, asymptotically safe as the 2D non-linear σ-model. A mathematical question remains open: is the renormalized, fine tuned field theory consistent on all scales?

I insist that this is mainly a mathematical question.

First, due its initial infinities, QCD has to be embedded in another theory that provides the cut-off that renders the theory finite.

Then, consistency on all scales is not a physical requirement. One only needs consistency up to some energy scale where physics will be modified anyway, at latest the Planck scale.

General interactions and summary

From the viewpoint of critical phenomena, which we now consider also as the relevant viewpoint for the theory of microscopic interactions, all field theories are effective large distance theories. In a generic effective field theory, all local interactions consistent with field content, symmetries... are present. All monomials are affected by powers of the cut-off (which is the scale of some new physics) dictated by power counting.

Leading effects of terms with positive powers of the cut-off either are removed by fine tuning (this necessarily includes scalar mass terms) or generate infinite couplings, like in the example of the 3D $(\phi^2)^2$ field theory. IR asymptotic safety then requires the existence of IR fixed points.

The coefficients with renormalizable interactions have by definition no cut-off dependence and, under RG, have a logarithmic flow.

In IR free theories, the effective or renormalized interactions go to zero for large cut-off and these theories have a limited energy range of validity.

Only the existence of non-trivial UV fixed points (UV asymptotic safety) and some fine tuning could produce a non-trivial theory.

UV asymptotically free theory like QCD or the 2D non-linear σ-model require some fine tuning to generate a mass scale much smaller than the cut-off. They are then asymptotically safe and mathematical candidates to be fully consistent renormalized theories.

Of course, in the case of several independent coupling constants, this does not exhaust all possible situations. In a general situation, a more detailed analysis is required. For example, situations with weak order phase transitions can appear.

Non-renormalizable interactions are multiplied by negative powers of the cut-off. The theory of renormalization of composite operators tell us that a given monomial renormalizes terms of lower dimension and, in addition, simply generates corrections that vanish at large cut-off.

At least this is the generic situation.

However, the examples of the non-linear σ-model or the Gross–Neveu model in three dimensions, show that by some fine tuning the effect of these operators can be enhanced if an UV fixed point can be found. The theory is then UV asymptotically safe. In this situation, non-renormalizable interactions can generate non-trivial interacting field theories.

Finally, the application of these ideas to quantum gravity, obviously suggests that Einstein–Hilbert's action is just the first, most important at large distance, term of a general local expansion. Whether such a theory possesses a non-trivial UV fixed point is a very interesting question. A UV fixed point would imply a crossover scale, much larger than Planck length, between long distance weak gravity and shorter distance strong gravity. It would also raise the question of fine tuning.

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