Title: Why Does Nature Like the Square Root of Negative One?

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Abstract: Is there a theory yet to be discovered that underlies quantum theory and explains its structure? If there is such a theory, one of the features it will have to explain is the central role of complex numbers as probability amplitudes. In this talk I explore the physical meaning of the statement â€œprobability amplitudes are complexâ€• by comparing ordinary complex-vector- space quantum theory with the real-vector-space theory having the same basic structure. Specifically, I discuss three questions that bring out qualitative differences between the two theories: (i) Is information about a preparation expressed optimally in the outcomes of a measurement? (ii) Are multipartite states locally accessible? (iii) Is entanglement â€œmonogamousâ€•?

Why does nature like the square root of negative one?

William K. Wootters

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and

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The Question

Motivating question: Is there a theory yet to be discovered that will explain the structure of quantum theory, particularly the role of complex numbers?

Not my question: How do empirical observations lead us to a theory with complex probability amplitudes?

Actual question: Are there qualitative differences between real-vector-space quantum theory and complex-vector-space quantum theory that might gives us clues to the origin of the complex-vector-space structure?

What I will actually do: Discuss three specific questions to which the real and complex theories give very different answers.

The Two Theories I'm Comparing

	The real case	The complex case
ure states	rays in R^N	rays in C^N
omplete orthogonal neasurements	orthogonal bases for R^N	orthogonal bases for C^N
eversible evolution	orthogonal (det=1)	unitary
nixed states	positive unit-trace operators (real)	positive unit-trace operators (complex)
omposition rule	tensor product	tensor product

Note: These two theories can *simulate* each other.
(Stueckelberg, 1960)

They can even simulate each other locally.
(McKague, Mosca, Gisin, 2009)

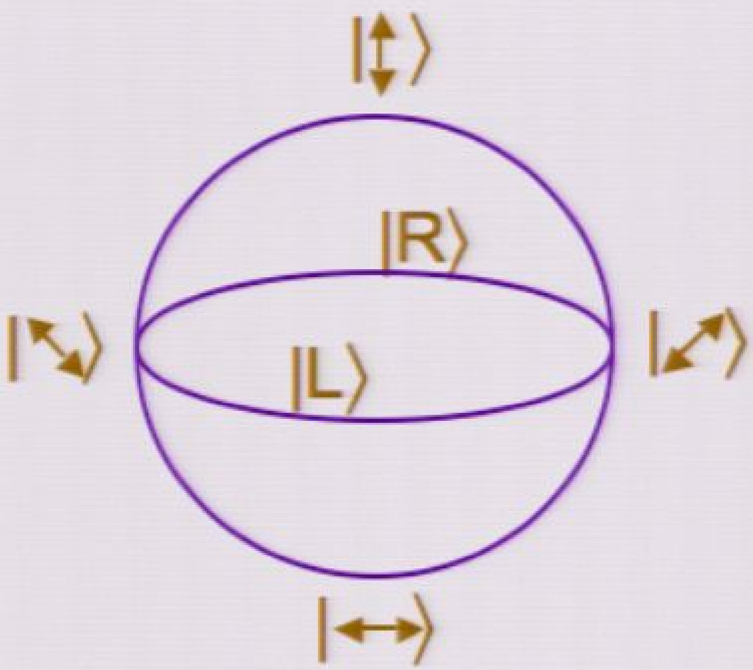
But in the simulation, one has to *restrict* the simulating theory in order not to get too many possibilities.

So the two theories can be distinguished by what they allow.

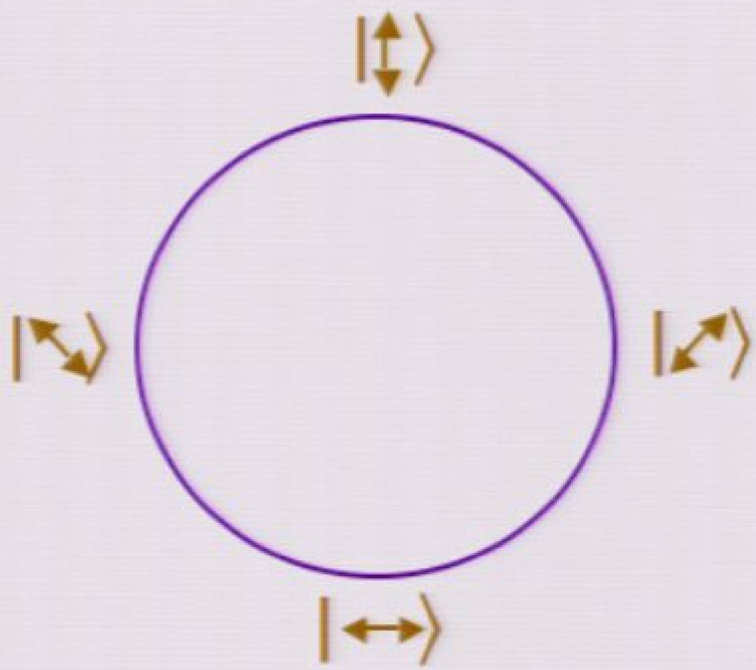
A few of the other people who have addressed the general problem:
Gudder and Piron (1971), Maczynski (1973), Maczynski/Lahti (1987), Bohm (1951), Myrheim (1999), Hardy (2001), Goyal (2008).

States of a single binary object in the two theories (Example: photon polarization)

The complex case



The real case



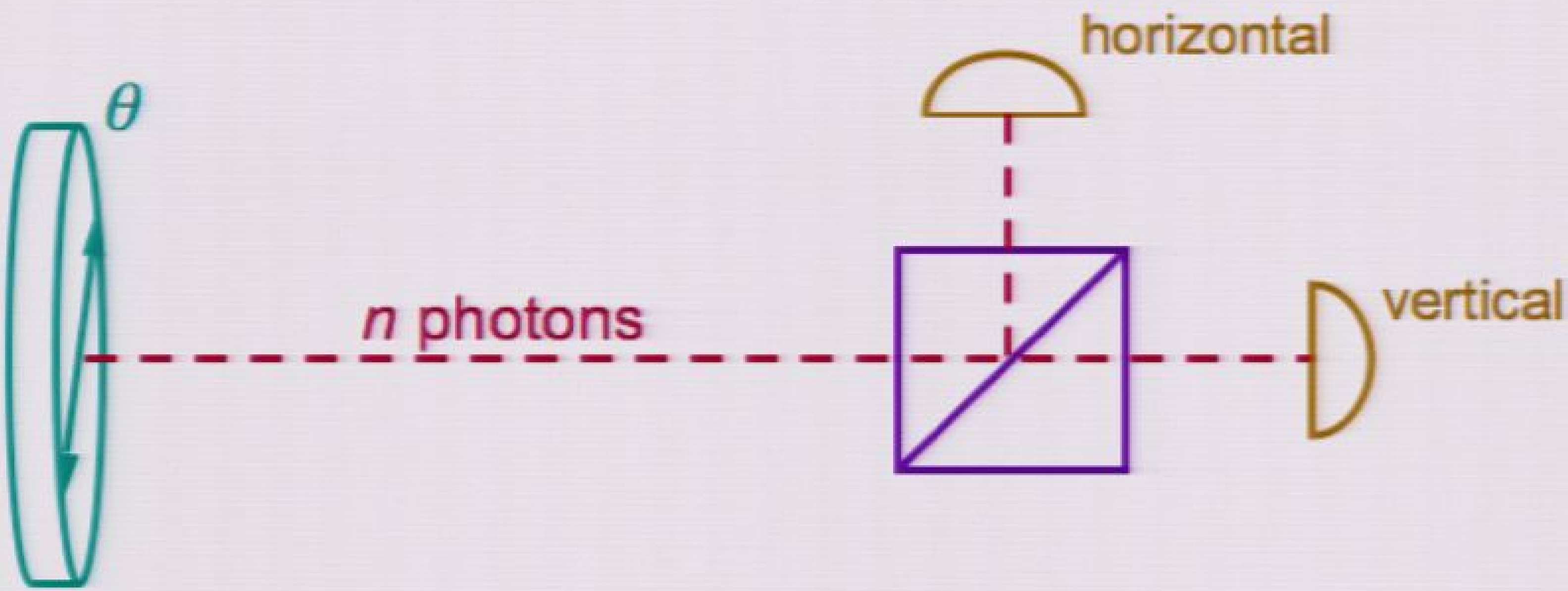
The three questions

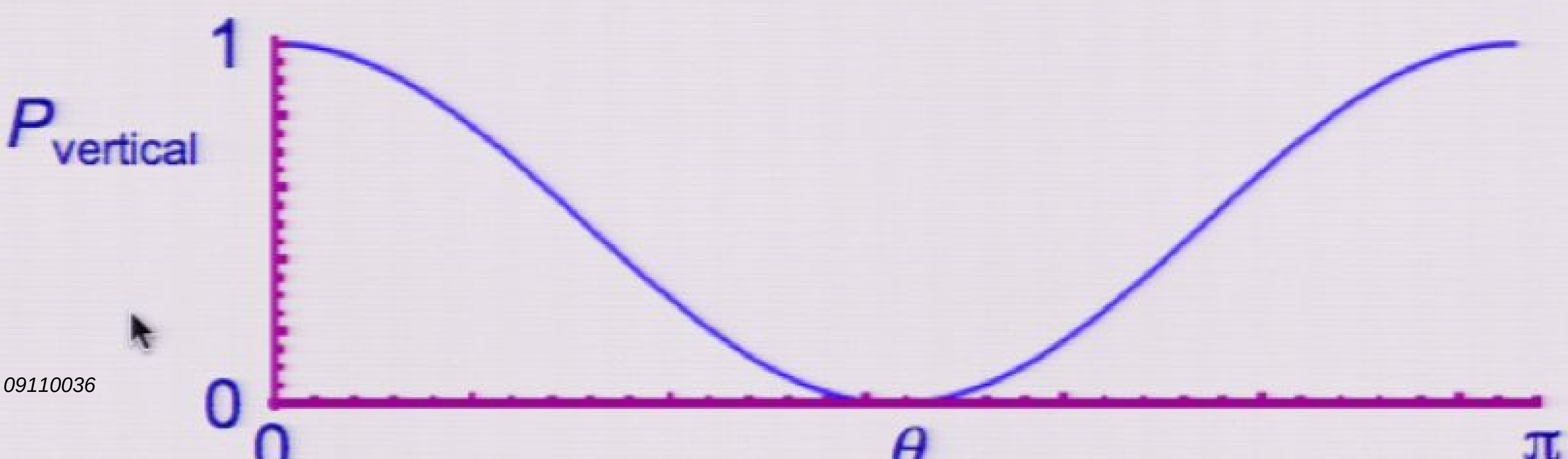
I. How well is information about a pure state expressed in the outcomes of an orthogonal measurement?

II. Are multipartite states locally accessible?

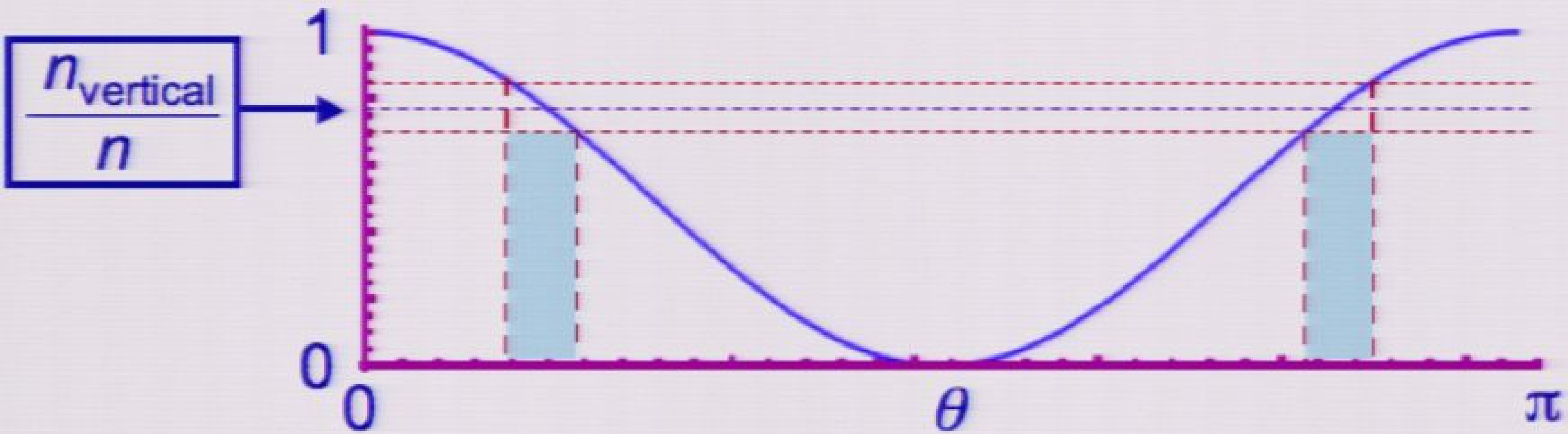
III. Is entanglement "monogamous"?

I. How well is information about a pure state expressed in the outcomes of an orthogonal measurement?





Information gained about θ on average



A reasonable measure is

$$\lim_{n\to\infty}\left[I(\theta : n_{\text{vertical}}) - \frac{1}{2}\log\left(\frac{n}{2\pi e}\right)\right]$$

where I is the mutual information between θ and n_{vertical}, assuming a uniform a priori distribution over θ.

Definition of mutual information:

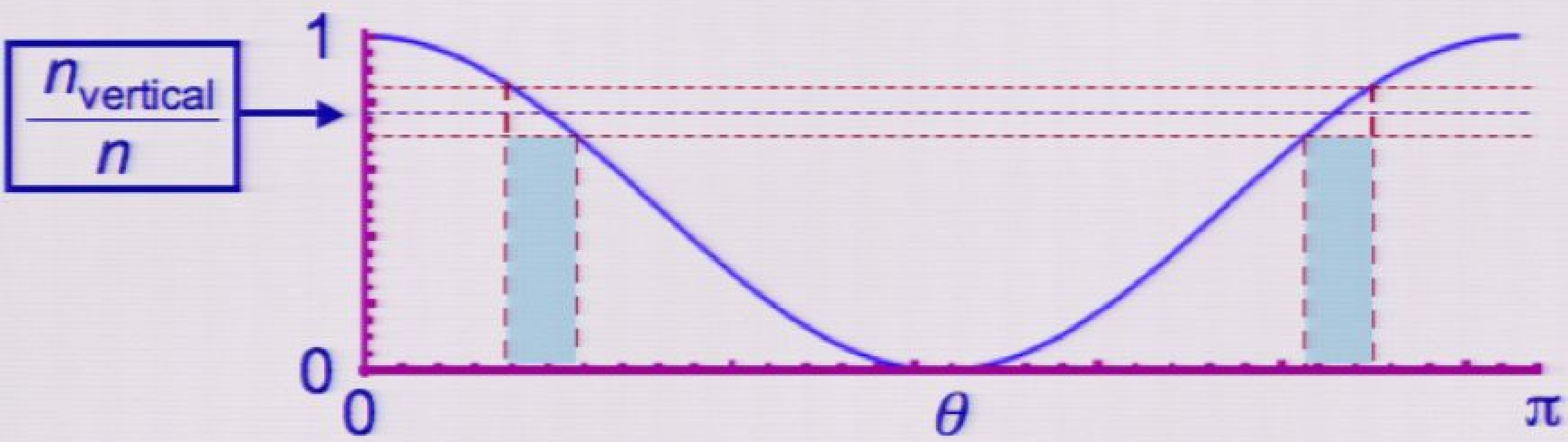
$$I(\theta : n_v) = \left\langle \sum_{n_v} p(n_v|\theta) \log p(n_v|\theta) \right\rangle_\theta - \sum_{n_v} \left\langle p(n_v|\theta) \right\rangle_\theta \log \left\langle p(n_v|\theta) \right\rangle_\theta$$

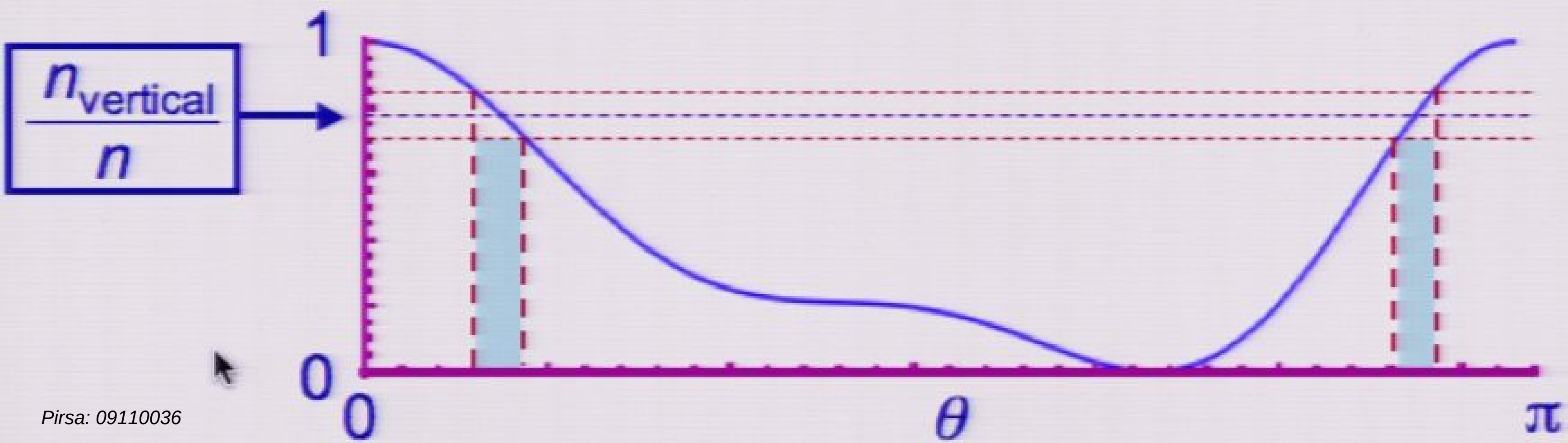
A reasonable measure is

$$\lim_{n \to \infty} \left[I(\theta : n_{\text{vertical}}) - \frac{1}{2} \log \left(\frac{n}{2\pi e} \right) \right]$$

where I is the mutual information between q and n_{vertical}, assuming a uniform a priori distribution over q.

Comparison with other possible worlds





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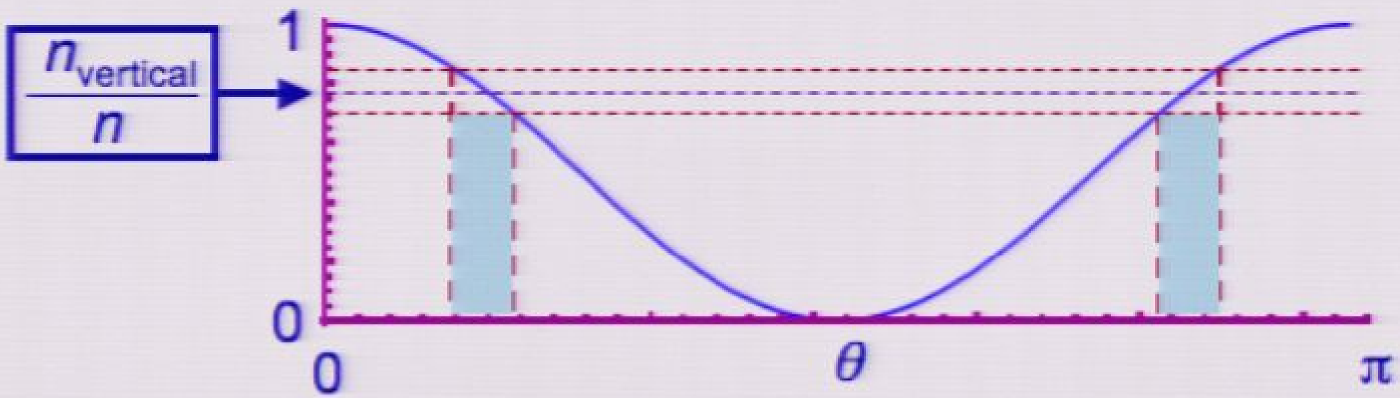
Our world's probability law is optimal (for linear polarization)

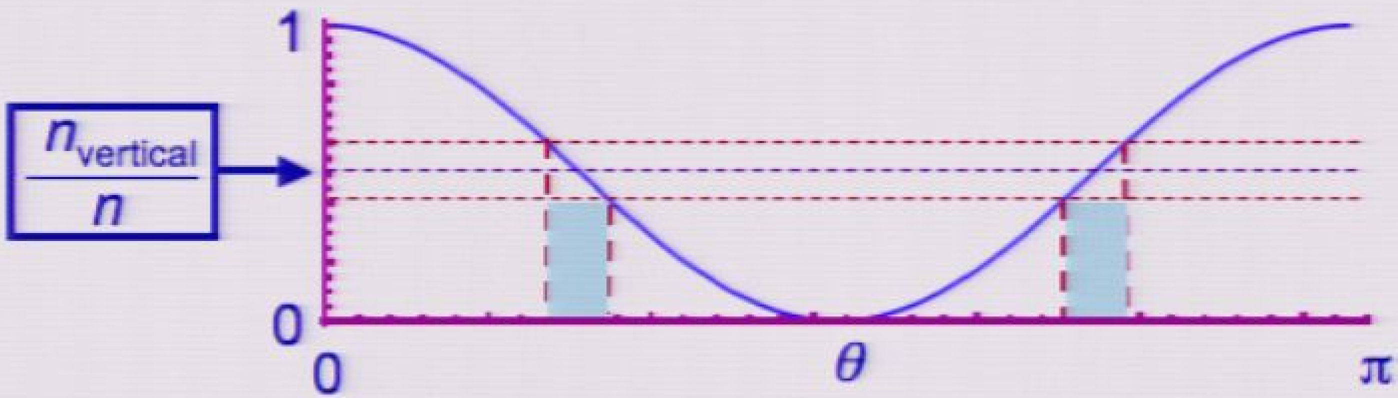
For any probability law, one can show that

$$\lim_{n\to\infty}\left[I(\theta : n_{\text{vertical}}) - \frac{1}{2}\log\left(\frac{n}{2\pi e}\right)\right] \leq \log \pi$$

The upper bound is achieved for $p(\theta) = \cos^2\theta$.

Why this works: Wider deviation matches greater slope.

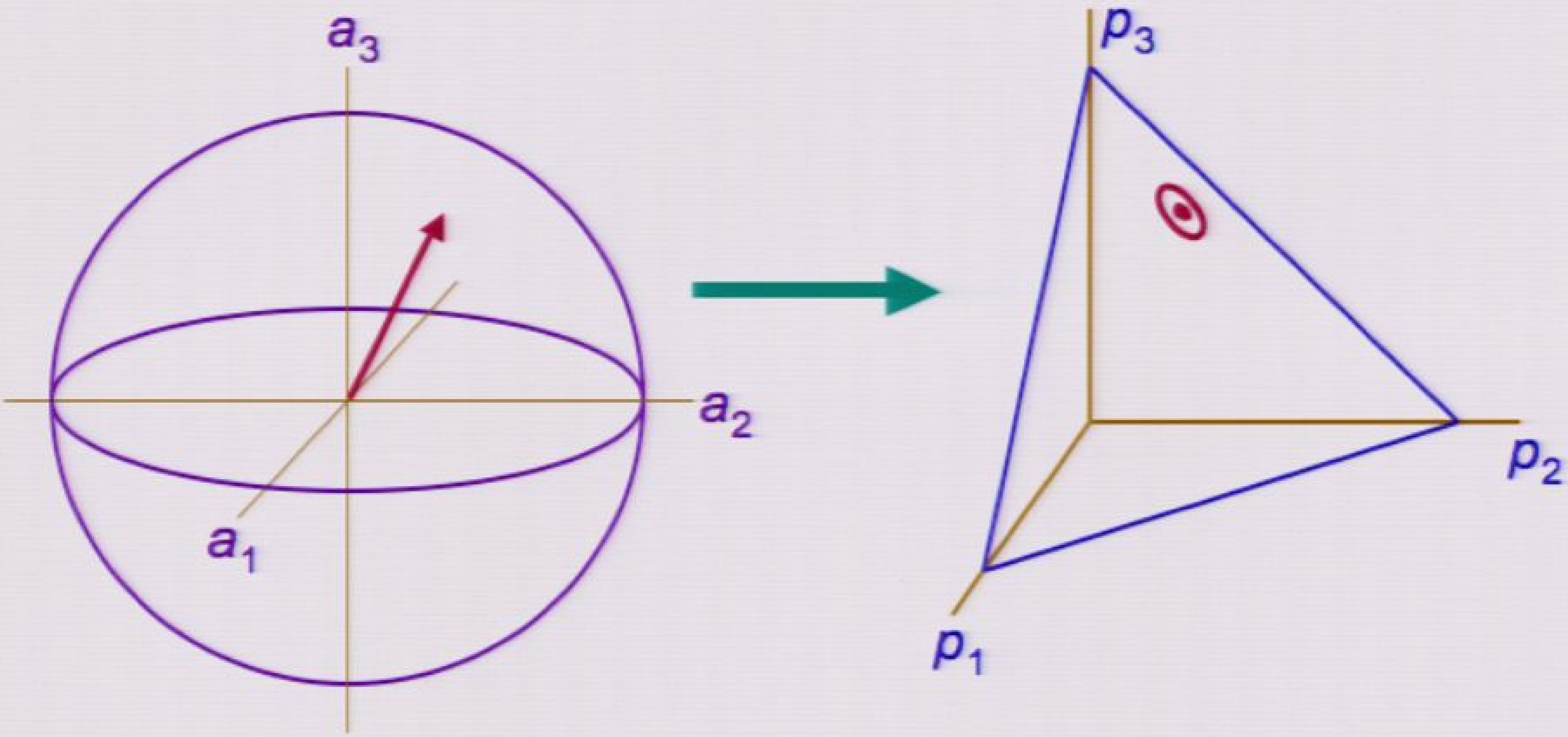




$$\Delta\left(\frac{n_{\text{vertical}}}{n}\right) = \sqrt{\frac{p(1-p)}{n}}$$

$$\left|\frac{dp}{d\theta}\right| = 2\sqrt{p(1-p)}$$

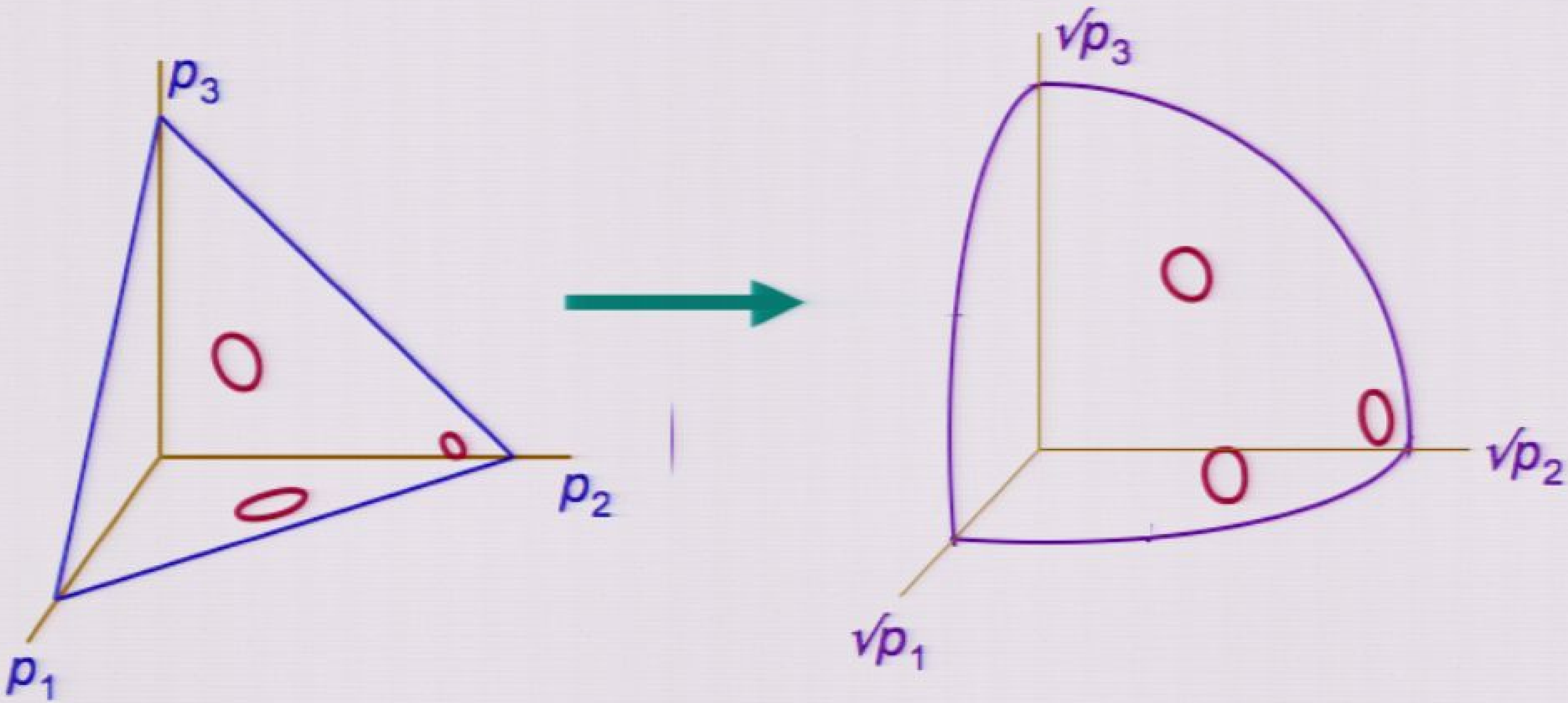
Real-vector-space quantum mechanics in N dimensions



The rule $p_k = a_k^2$ again maximizes the information gained about $\boldsymbol{a}$, compared with other conceivable probability rules.

Why this works: Uncertainty in $\boldsymbol{a}$ is uniform and isotropic.

Making statistical fluctuations uniform and isotropic



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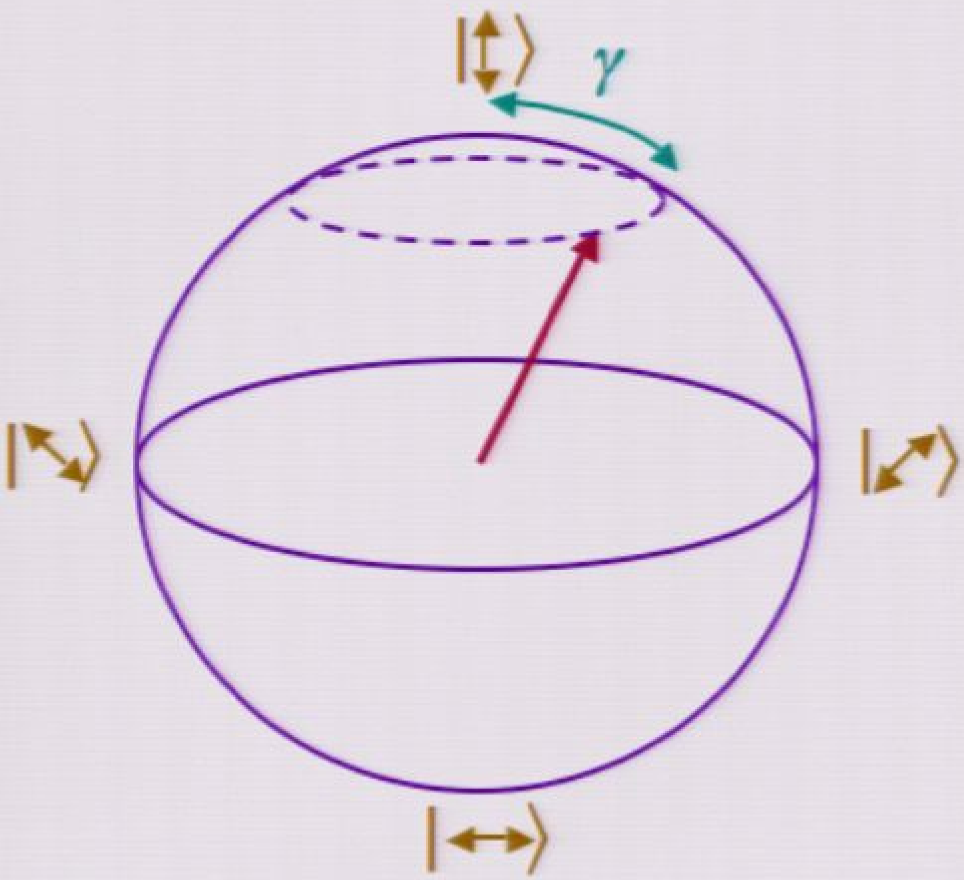
No information maximization for the *complex* theory.



$P_{\text{vertical}} = \cos^2(\gamma/2)$,

but γ is not uniformly distributed.

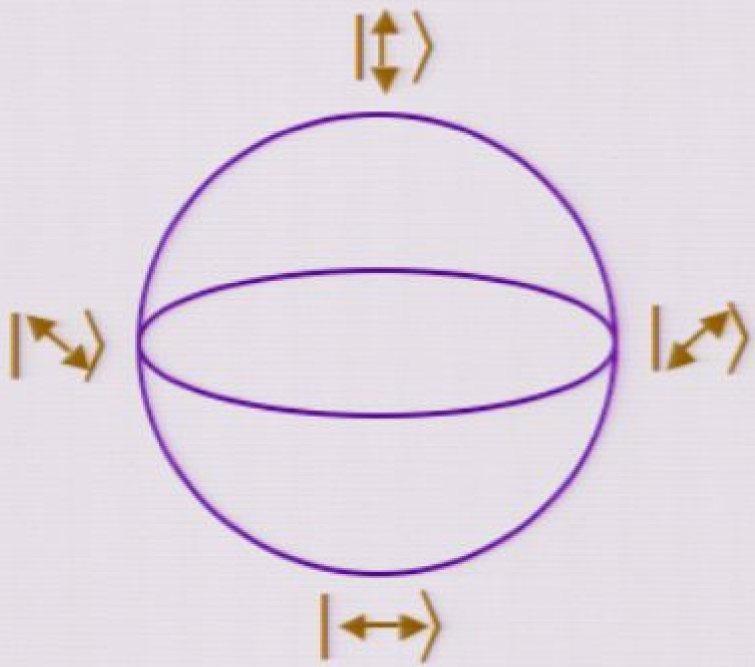
In N dimensions, a pure state holds **2(*N*–1)** real parameters, but there are ***N*–1** independent probabilities.

Is there some simple underlying explanation of this doubling?

(cf Spekkens, 2004; Goyal, 2008)

II. Are (Mixed) States Locally Accessible?

State estimation for a single qubit (complex case):



C

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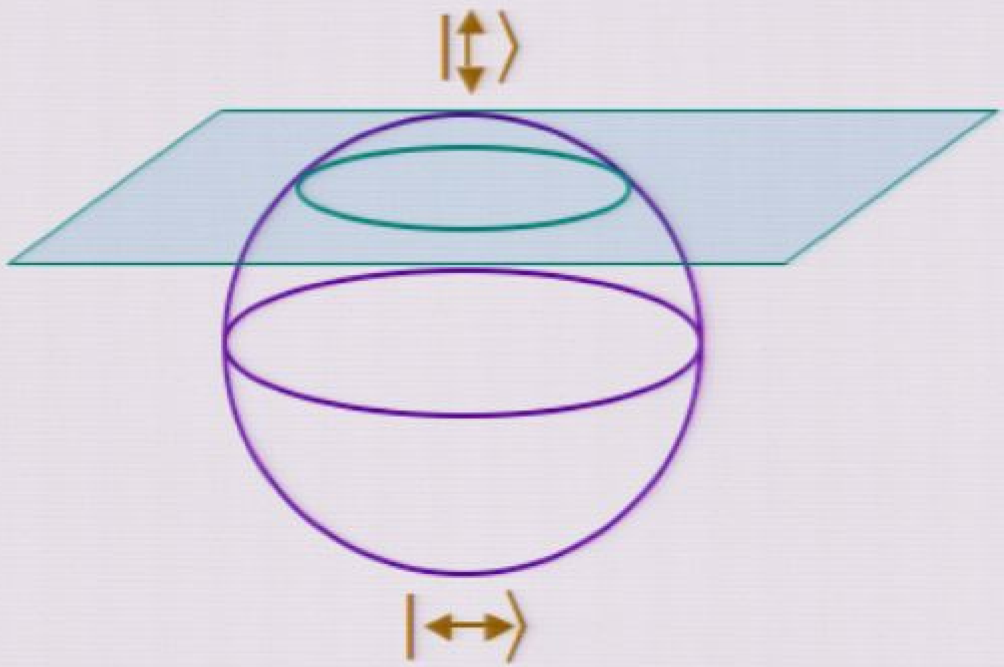

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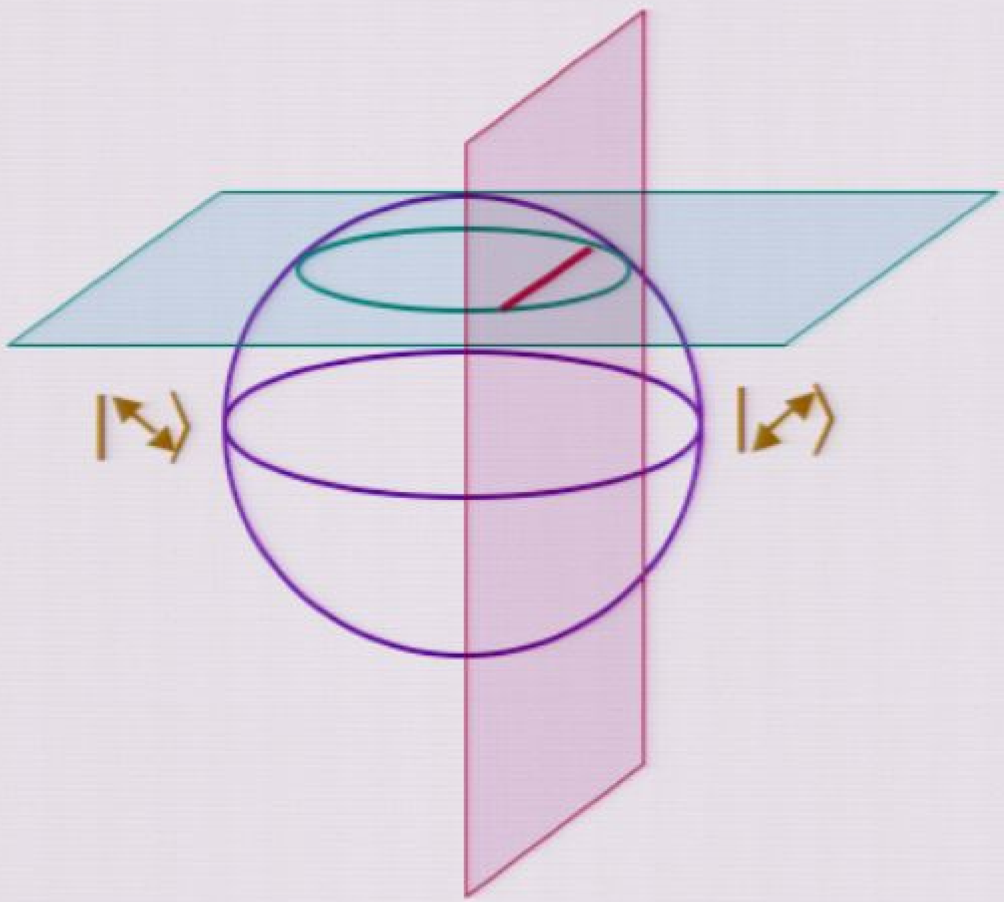

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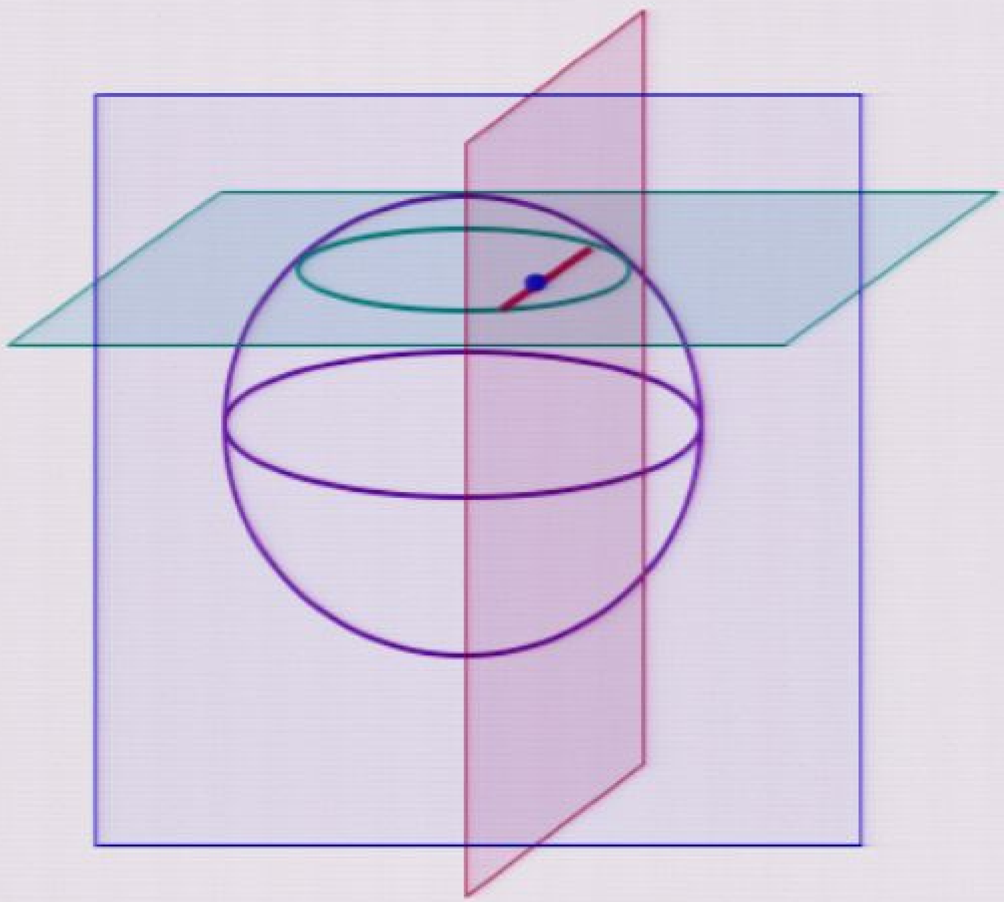
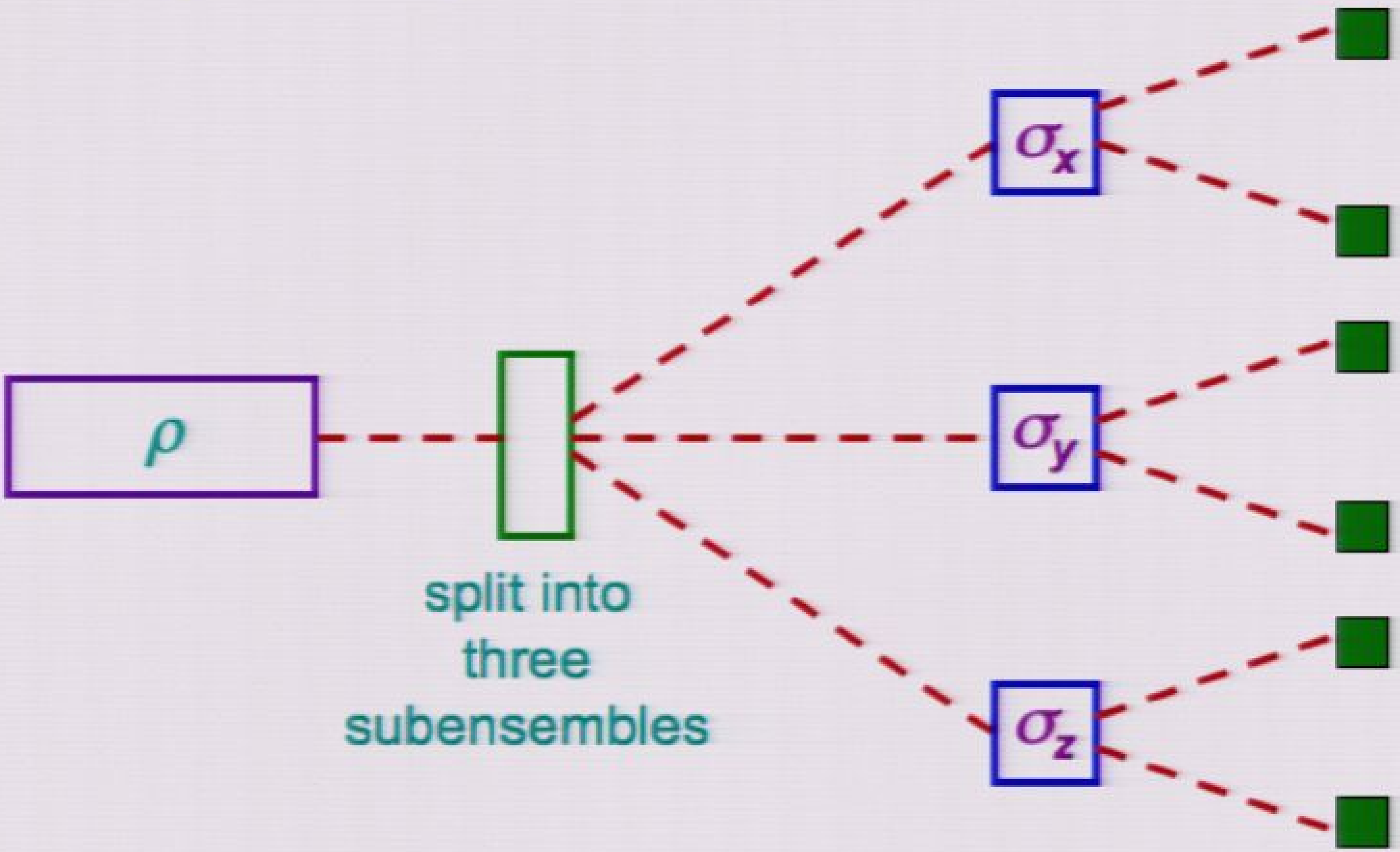


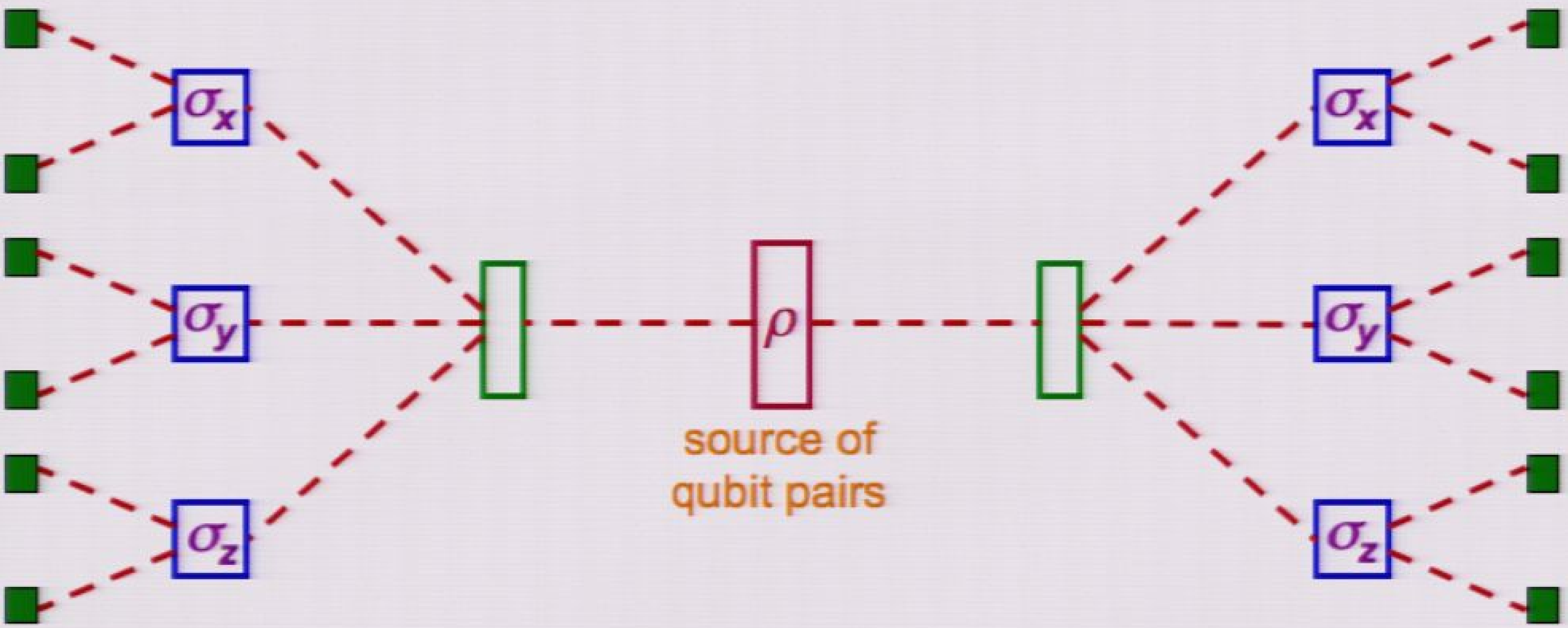
Summary of this procedure



Need 3 real parameters.
Each measurement supplies 1 parameter.

C

State estimation for a pair of qubits



Need $4^2 - 1 = 15$ real parameters.

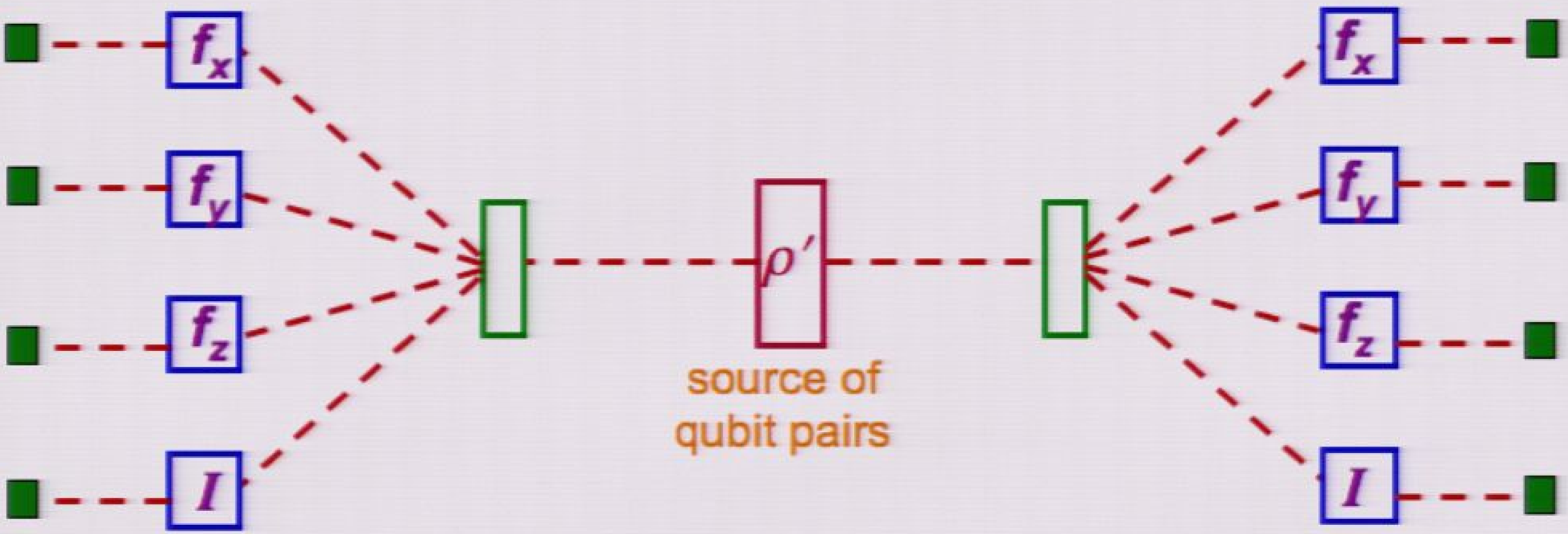
Get: coefficients in ρ of

$$I \otimes \sigma_j\,,\ \ \sigma_j \otimes I\,,\ \ \sigma_j \otimes \sigma_k$$

$$3 \quad + \quad 3 \quad + \quad 9 \quad = \quad 15$$

C

A simpler way of counting: unnormalized state (Hardy, 2001)



Need $4^2 = 16$ real parameters.

Get: coefficients in ρ' of

$\sigma_j \otimes \sigma_k$ (where σ_0 is the identity)

$4 \times 4 = 16$

C

A composite system with $N_1 \times N_2$ dimensions (complex case).

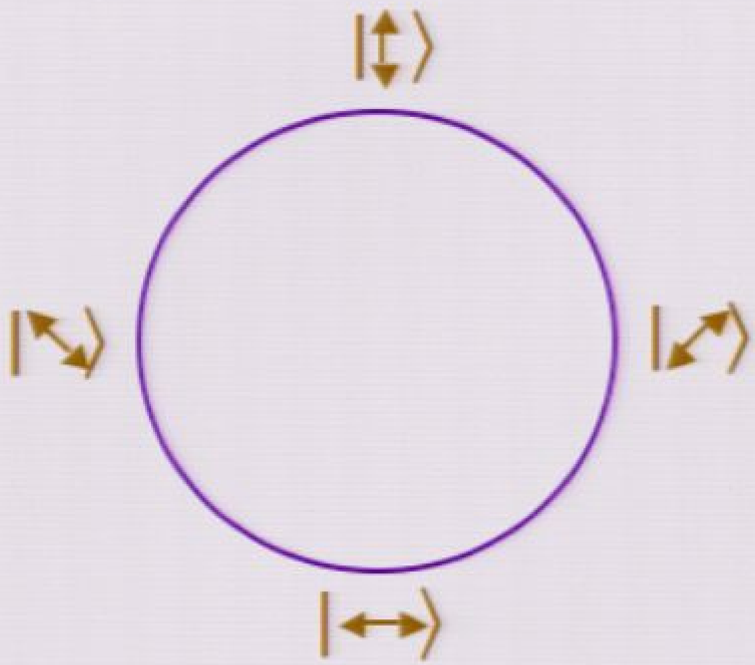
Need $(N_1 N_2)^2$ real parameters (unnormalized state).

Local measurements give $N_1^2 \times N_2^2 = (N_1 N_2)^2$

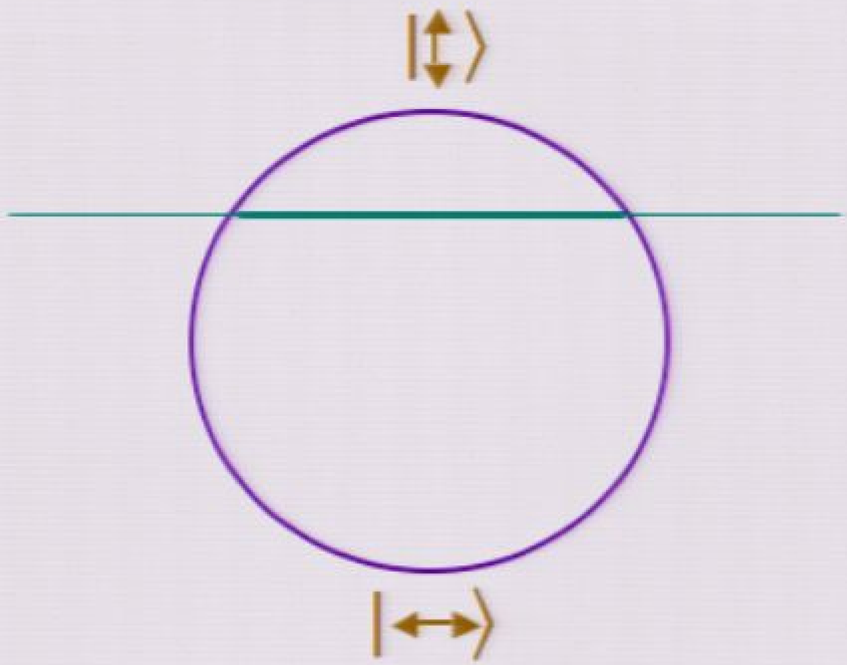
independent parameters, exactly as many as needed.

So in the complex case, measurements on the parts (with attention paid to correlations) provide exactly the information needed about the whole.

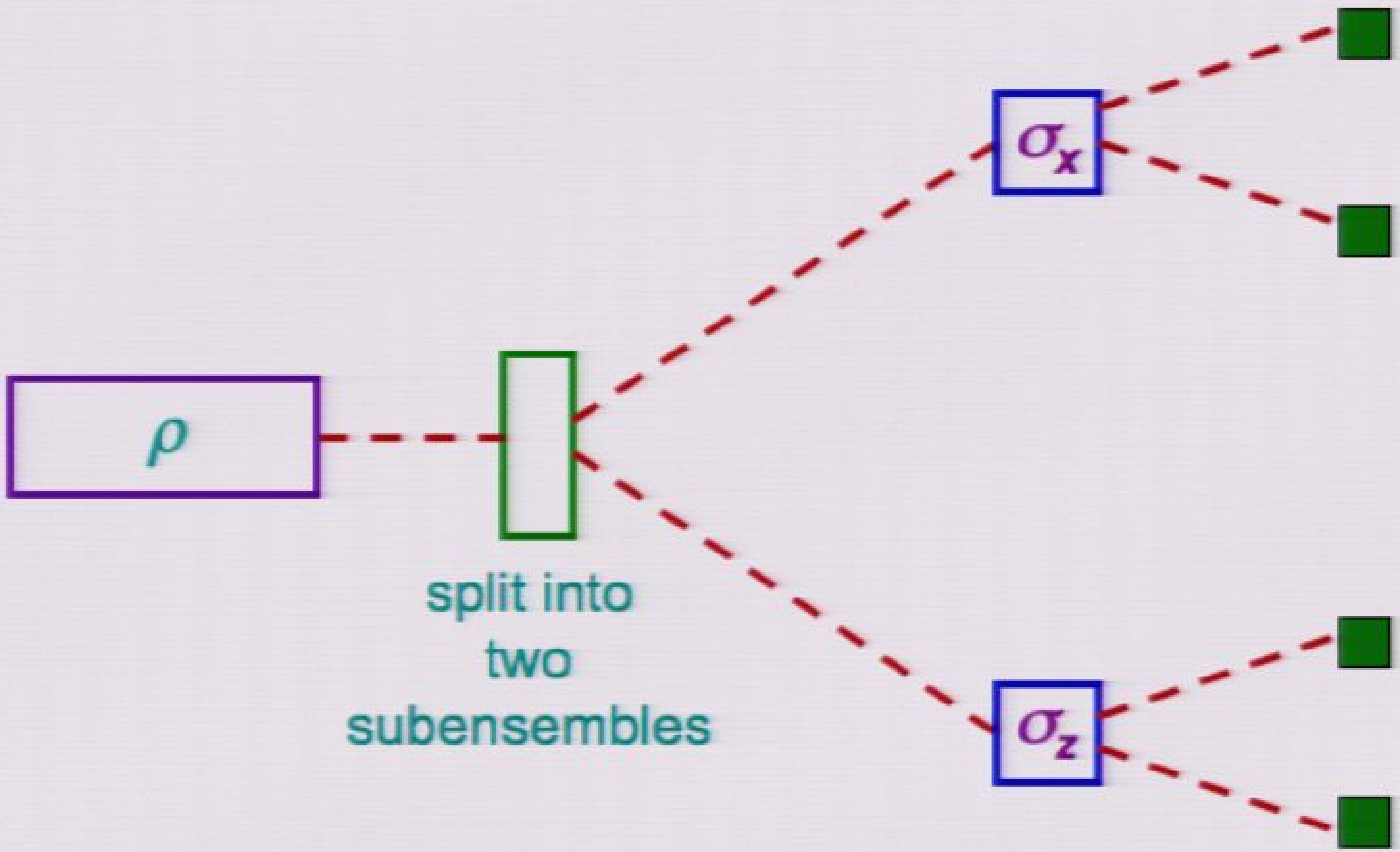
State estimation for a single "rebit" (real quantum bit)



State estimation for a single "rebit" (real quantum bit)

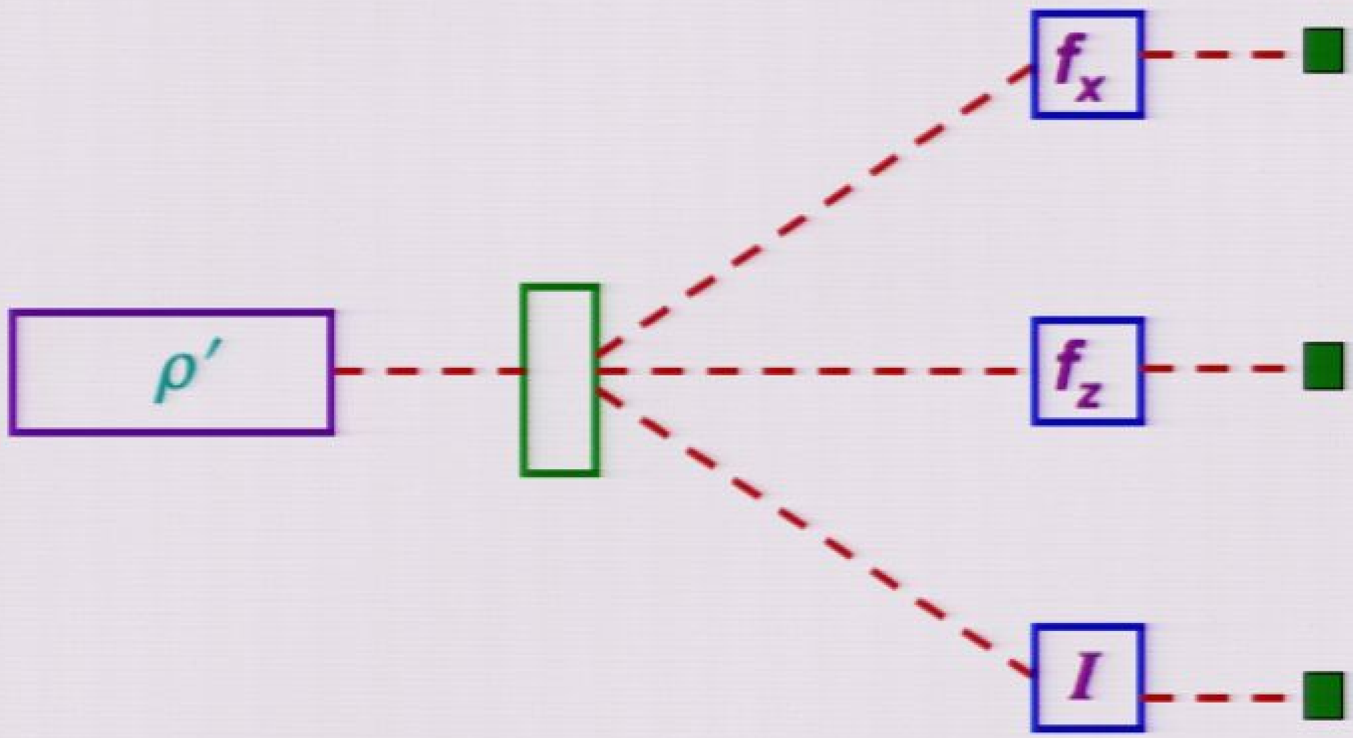


Summary of this process



Need 2 real parameters.
Each measurement supplies 1 parameter.

How it looks with an unnormalized state



Need 3 real parameters.
Each measurement supplies 1 parameter.

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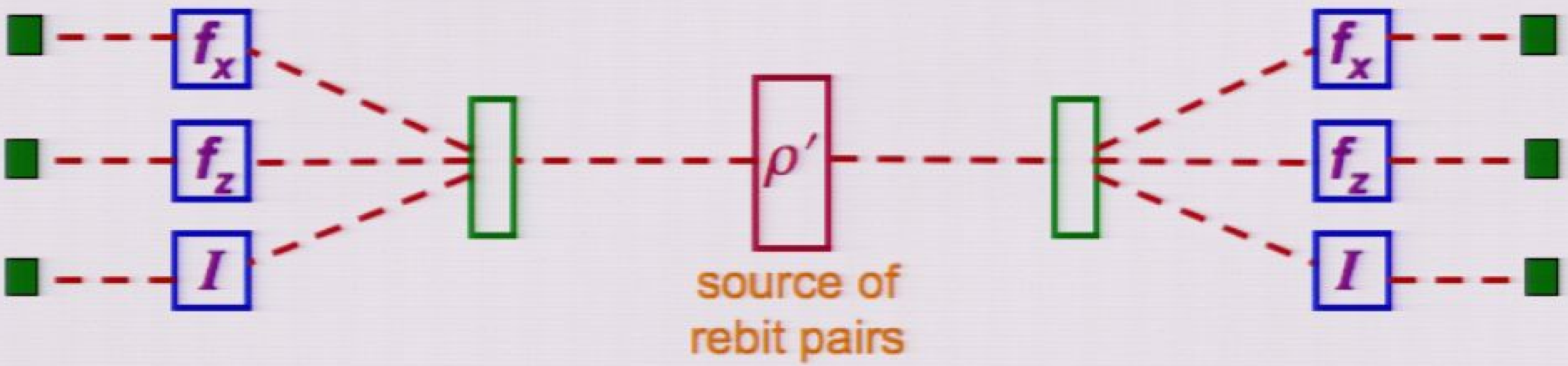
State estimation for a pair of rebits (unnormalized)



Need 4(4+1)/2 = 10 real parameters.

Get: coefficients in ρ' of

$\sigma_j \otimes \sigma_k$ (where σ_0 is the identity)

3 x 3 = 9

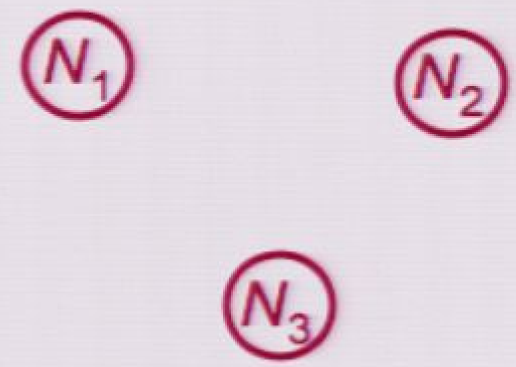
The missing contribution, from $\sigma_y \otimes \sigma_y$, must be accessed globally.

R

In the real theory, states are “bilocally accessible”.

Three objects with dimensions N_1, N_2, and N_3:

Let $K(N)$ be the number of parameters in an unnormalized state in N dimensions.



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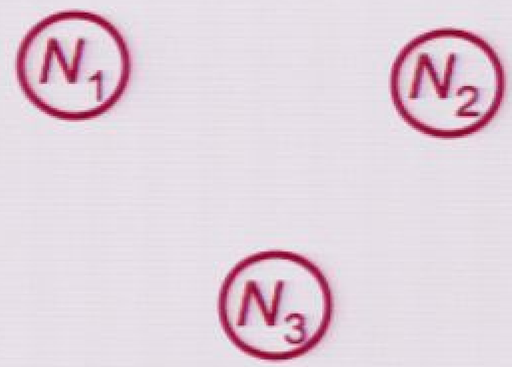
Number of parameters accessible bilocally:



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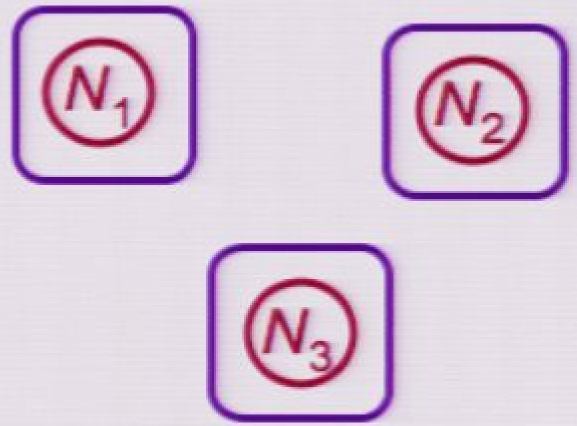

Number of parameters accessible bilocally:

$K(N_1)K(N_2)K(N_3)$

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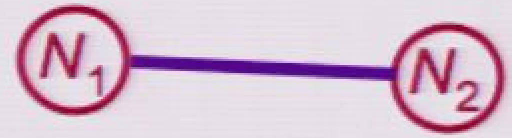

Let $K(N)$ be the number of parameters in an unnormalized state in N dimensions.

Number of parameters accessible bilocally:

$$K(N_1)K(N_2)K(N_3) + [K(N_1N_2) - K(N_1)K(N_2)]\, K(N_3)$$

R

In the real theory, states are “bilocally accessible”.

Three objects with dimensions N_1, N_2, and N_3:

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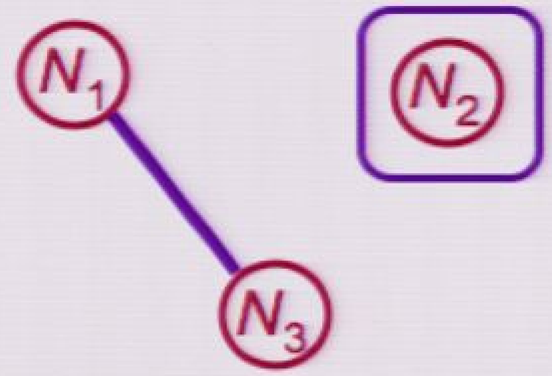

Number of parameters accessible bilocally:

$$K(N_1)K(N_2)K(N_3) + [K(N_1N_2) - K(N_1)K(N_2)]\,K(N_3) + [K(N_1N_3) - K(N_1)K(N_3)]\,K(N_2)$$

R

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Three objects with dimensions N_1, N_2, and N_3:

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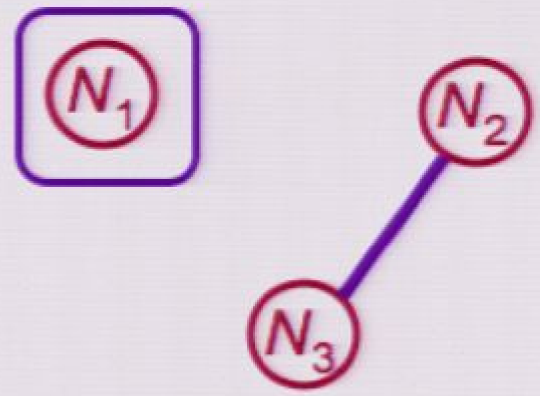


Number of parameters accessible bilocally:

$$\begin{aligned} K(N_1)K(N_2)K(N_3) &+ [K(N_1N_2) - K(N_1)K(N_2)]\ K(N_3) \\ &+ [K(N_1N_3) - K(N_1)K(N_3)]\ K(N_2) \\ &+ [K(N_2N_3) - K(N_2)K(N_3)]\ K(N_1) \end{aligned}$$

R

In the real theory, states are “bilocally accessible”.

Three objects with dimensions N_1, N_2, and N_3:

Let $K(N)$ be the number of parameters in an unnormalized state in N dimensions.

Number of parameters accessible bilocally:

$$\begin{aligned} K(N_1)K(N_2)K(N_3) &+ [K(N_1N_2) - K(N_1)K(N_2)]\ K(N_3) \\ &+ [K(N_1N_3) - K(N_1)K(N_3)]\ K(N_2) \\ &+ [K(N_2N_3) - K(N_2)K(N_3)]\ K(N_1) \end{aligned}$$

For the real theory, this exactly equals $K(N_1\ N_2\ N_3)$.

Functions $K(N)$ corresponding to exact bilocal accessibility:

$K(N) = N$ (ordinary probability theory)

$K(N) = N(N+1)/2$ (real-vector-space quantum)

$K(N) = N^2$ (quantum theory)

$\vdots$

$K(N) = (N^r + N^s)/2, \quad r \neq 0, s \neq 0$

(Hardy and Wootters, in preparation)

Functions $K(N)$ corresponding to exact ***local*** accessibility:

$K(N) = N$ (ordinary probability theory)

~~$K(N) = N(N+1)/2$ (real-vector-space quantum)~~

$K(N) = N^2$ (quantum theory)

⋮

$K(N) = N^r$

In this sense the real-vector-space theory is more nonlocal, or more holistic, than actual quantum theory.

III. Is Entanglement “Monogamous”?

Entanglement in the complex case:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

This state is maximally entangled: to create it,
one needs to transmit one qubit between the two sites.

C

Mixed-state entanglement in the complex case

Let ρ be an equal mixture of

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \quad \text{and} \quad |\Phi\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

One might think this mixture is also maximally entangled.

But no. The same ρ is also an equal mixture of

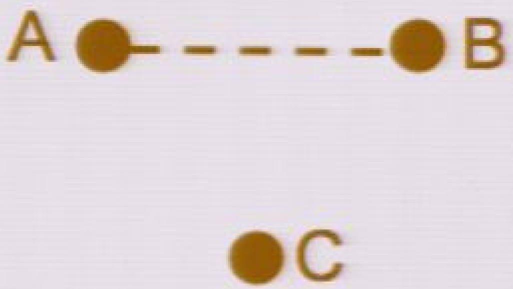
$$\frac{1}{2}(|0\rangle + i|1\rangle) \otimes (|0\rangle - i|1\rangle) \quad \text{and} \quad \frac{1}{2}(|0\rangle - i|1\rangle) \otimes (|0\rangle + i|1\rangle)$$

which can be created locally.

So the state ρ is unentangled.

C

Entanglement monogamy in the complex case



If A and B are maximally entangled, then neither can be at all entangled with C.

The reason: If either A or B were entangled with C, then AB would be in a mixed state, but every maximally entangled state is pure.

R

Mixed-state entanglement in the real case

Let ρ be an equal mixture of

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \quad \text{and} \quad |\Phi\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

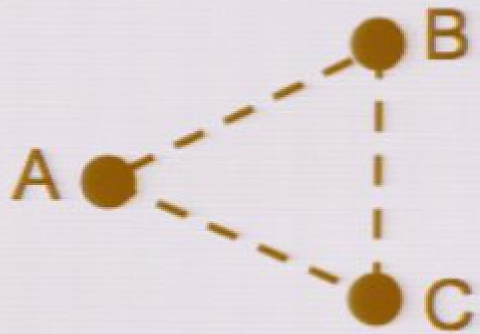
This mixture is maximally entangled!
Every decomposition of ρ into real pure states consists of maximally entangled states. The decomposition

$$\frac{1}{2}(|0\rangle + i|1\rangle) \otimes (|0\rangle - i|1\rangle) \quad \text{and} \quad \frac{1}{2}(|0\rangle - i|1\rangle) \otimes (|0\rangle + i|1\rangle)$$

is not allowed. So a mixed state can be maximally entangled.

R

No entanglement monogamy in the real case



Three rebits can be pairwise maximally entangled.

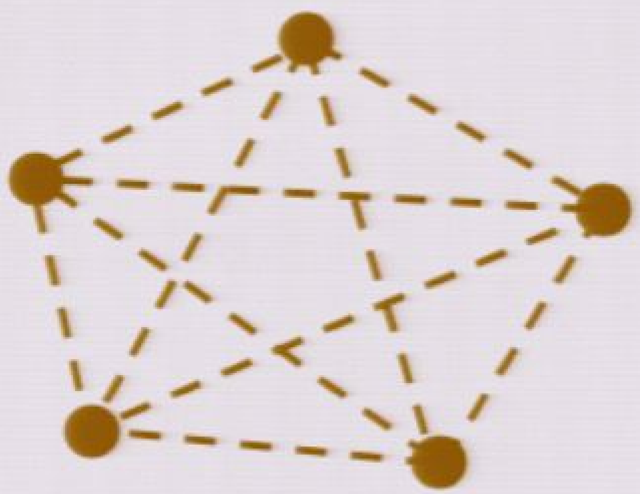
$$\rho_{ABC} = \frac{1}{8}(I \otimes I \otimes I + \sigma_y \otimes \sigma_y \otimes I + \sigma_y \otimes I \otimes \sigma_y + I \otimes \sigma_y \otimes \sigma_y)$$

The reduced two-rebit state is

$\rho_{AB} = \frac{1}{4}\left(I \otimes I + \sigma_y \otimes \sigma_y\right)$, which is maximally entangled.

R

No entanglement monogamy in the real case



For n rebits, there exist $\mathbf{2}^{(n-1)}$ mutually orthogonal states, each of which has maximal entanglement between any two rebits.

$$\rho_{s_1,\cdots,s_{n-1}} = \frac{1}{2^n}\mathrm{Re}\sum_{k_1,\cdots,k_n=0}^{1}(s_1\sigma_y)^{k_1}\otimes\cdots\otimes(s_{n-1}\sigma_y)^{k_{n-1}}\otimes\sigma_y^{k_n}$$

$$s_j = \pm 1$$

So one can hide ***n*–1** classical bits in ***n*** rebits. The local observers cannot access any of these bits, even with unlimited classical communication.

Summary

Complex:	An orthogonal measurement accesses only half the parameters of a pure state.
Real:	Information about a pure preparation is *optimally* expressed in the outcomes of a measurement.
Complex:	Multipartite states are locally accessible.
Real:	Multipartite states are *bilocally* accessible.
Complex:	Entanglement is monogamous.
Real:	Arbitrarily many objects can be pairwise maximally entangled.

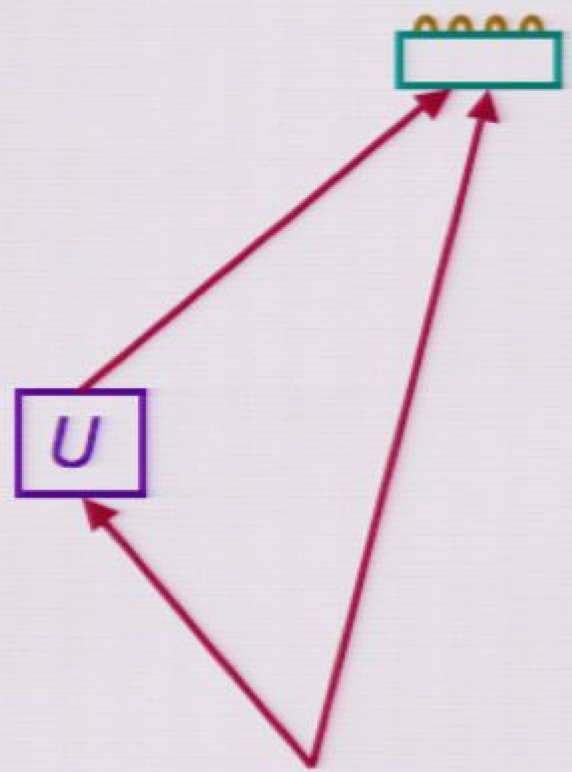
Tentative conclusion, and a nagging question

"Complex" may be telling us to what extent nature is limited in its nonlocality, or its holism. The real case would also be limited, but less so (bilocal accessibility).

But "limited holism" does not give us a direct answer to the question: Why does a complete orthogonal measurement access only *half* the parameters of a pure state?

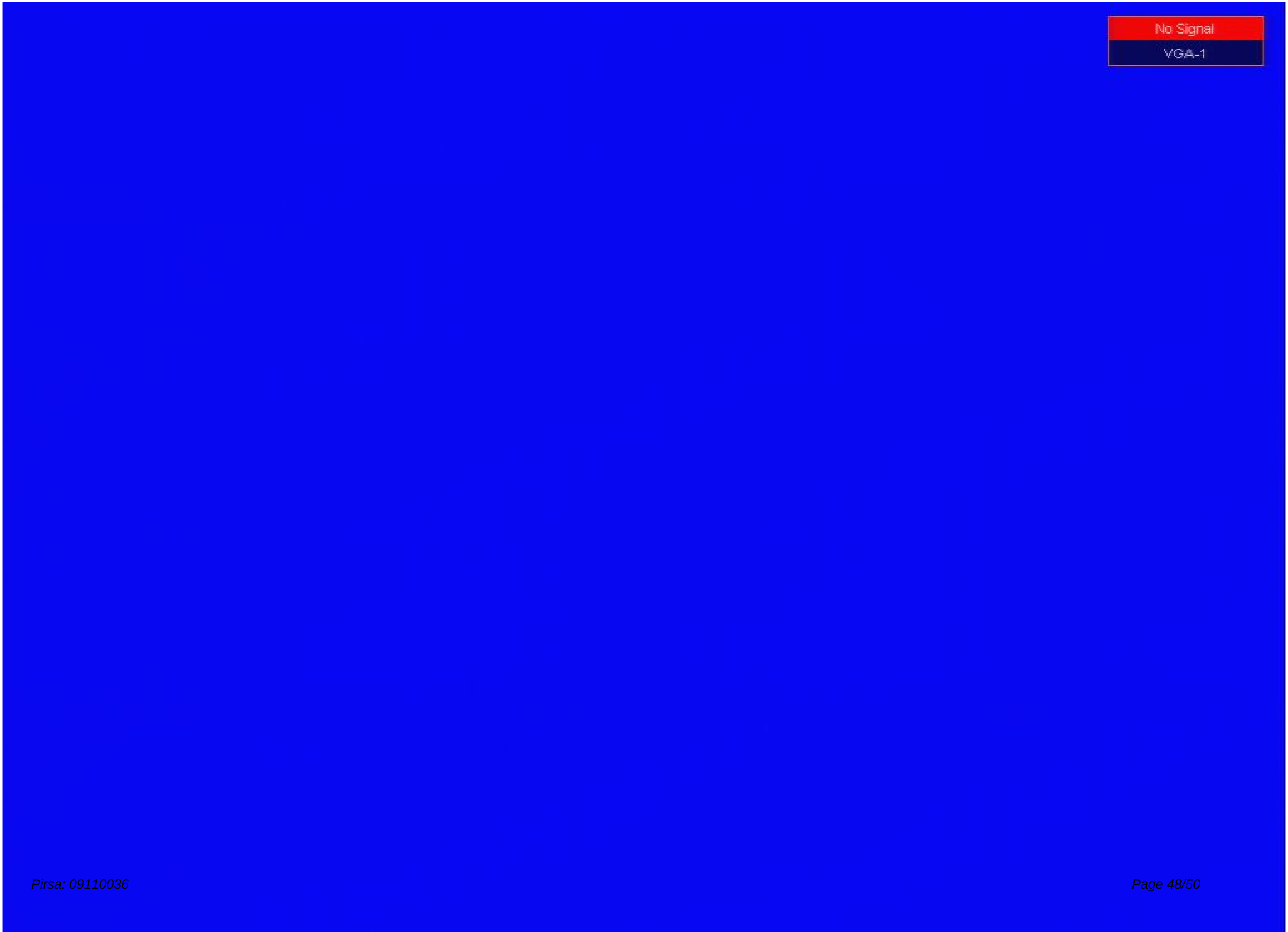
One more try: Information about a *transformation*

$$\begin{cases} |00\rangle + |11\rangle \\ |00\rangle - |11\rangle \\ |01\rangle + |10\rangle \\ |01\rangle - |10\rangle \end{cases}$$



$|00\rangle + |11\rangle$

Information about the special unitary transformation *U* is expressed optimally in the outcomes.


No Signal
VGA-1

No Signal

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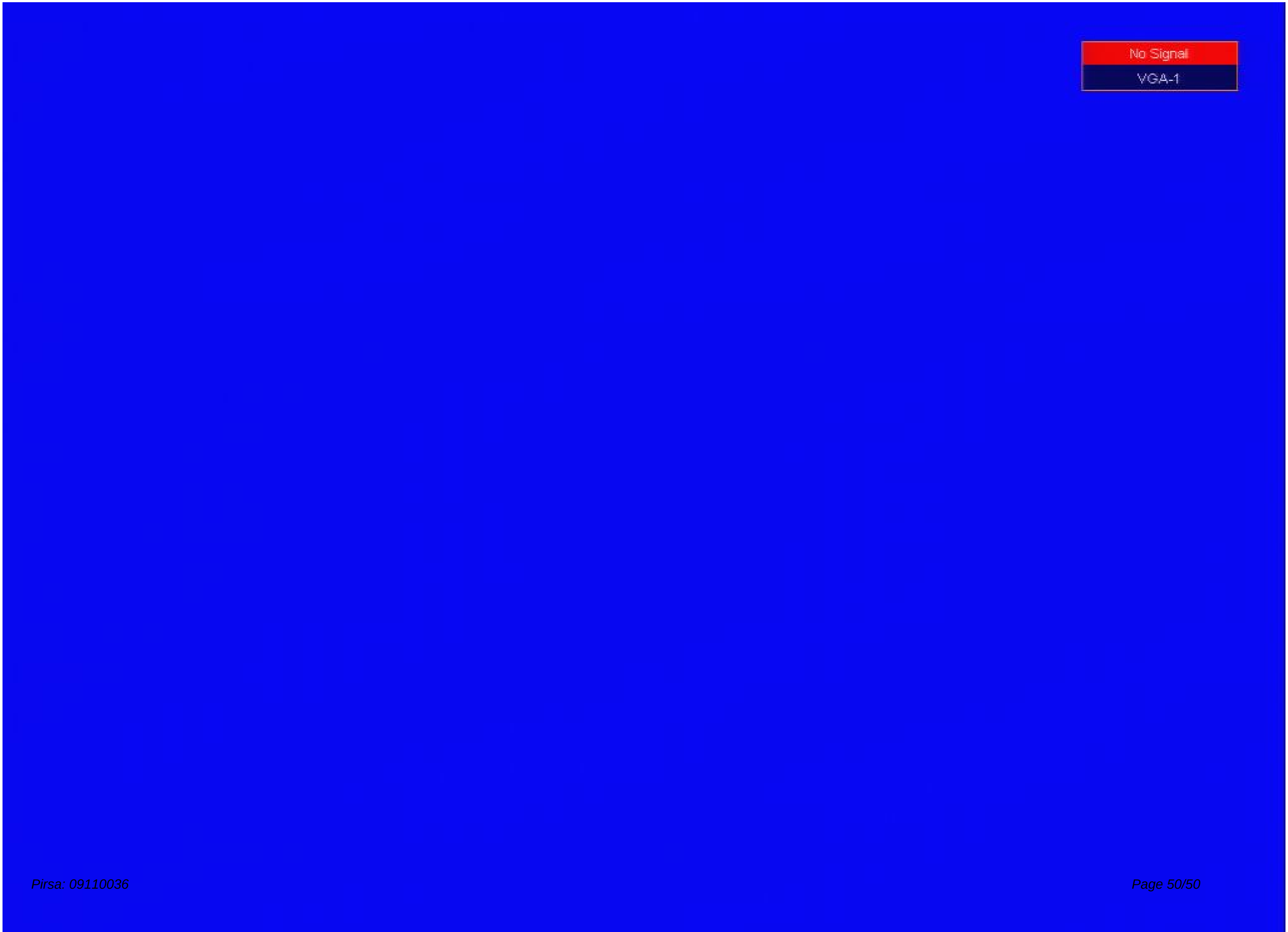
No Signal
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