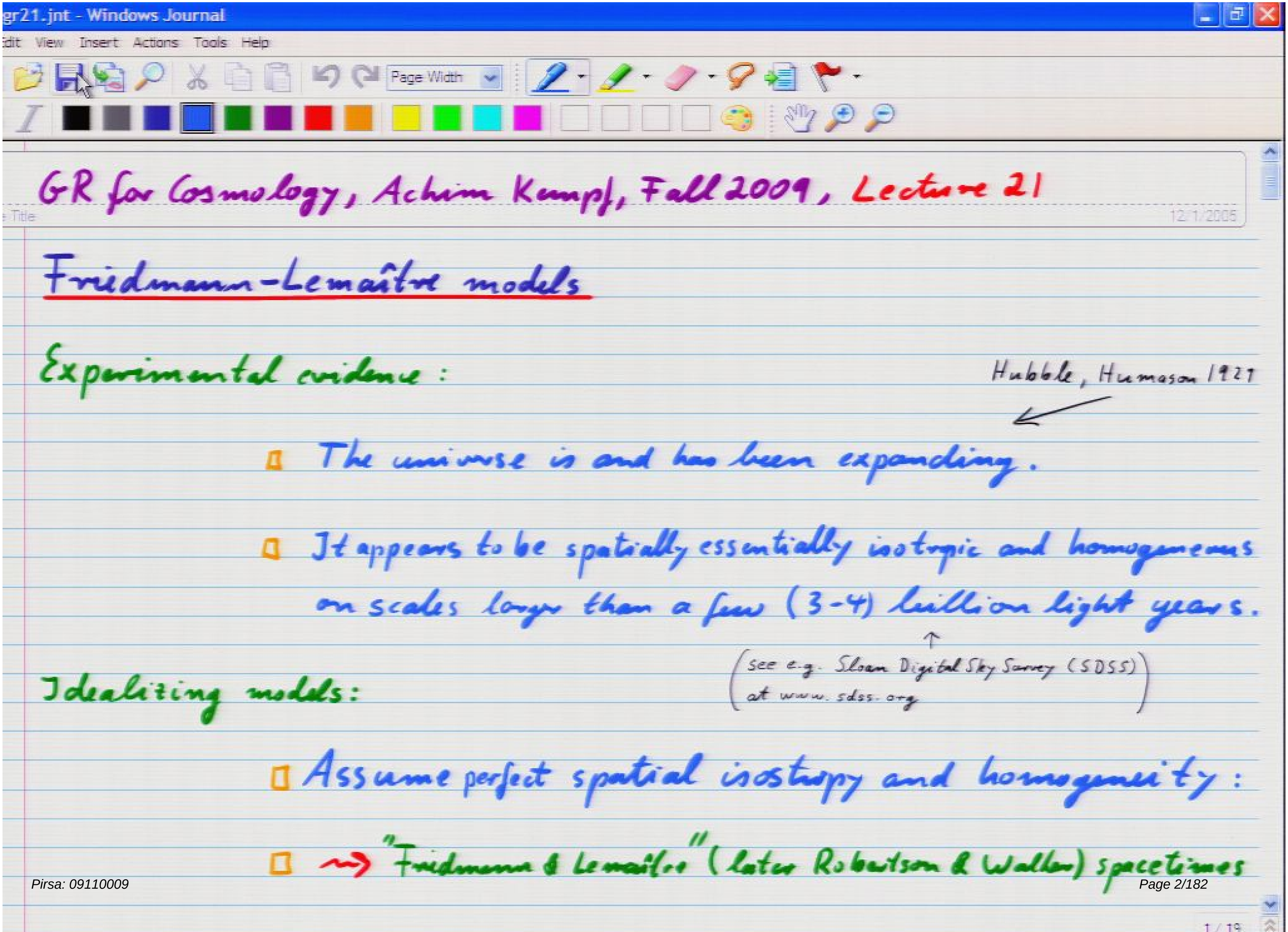
Title: General Relativity for Cosmology - Lecture 18A

Date: Nov 19, 2009 04:00 PM

URL: http://pirsa.org/09110009

Abstract:



GR for Cosmology, Achim Kempf, Fall 2009, Lecture 21

Friedmann-Lemaître models

Experimental evidence:

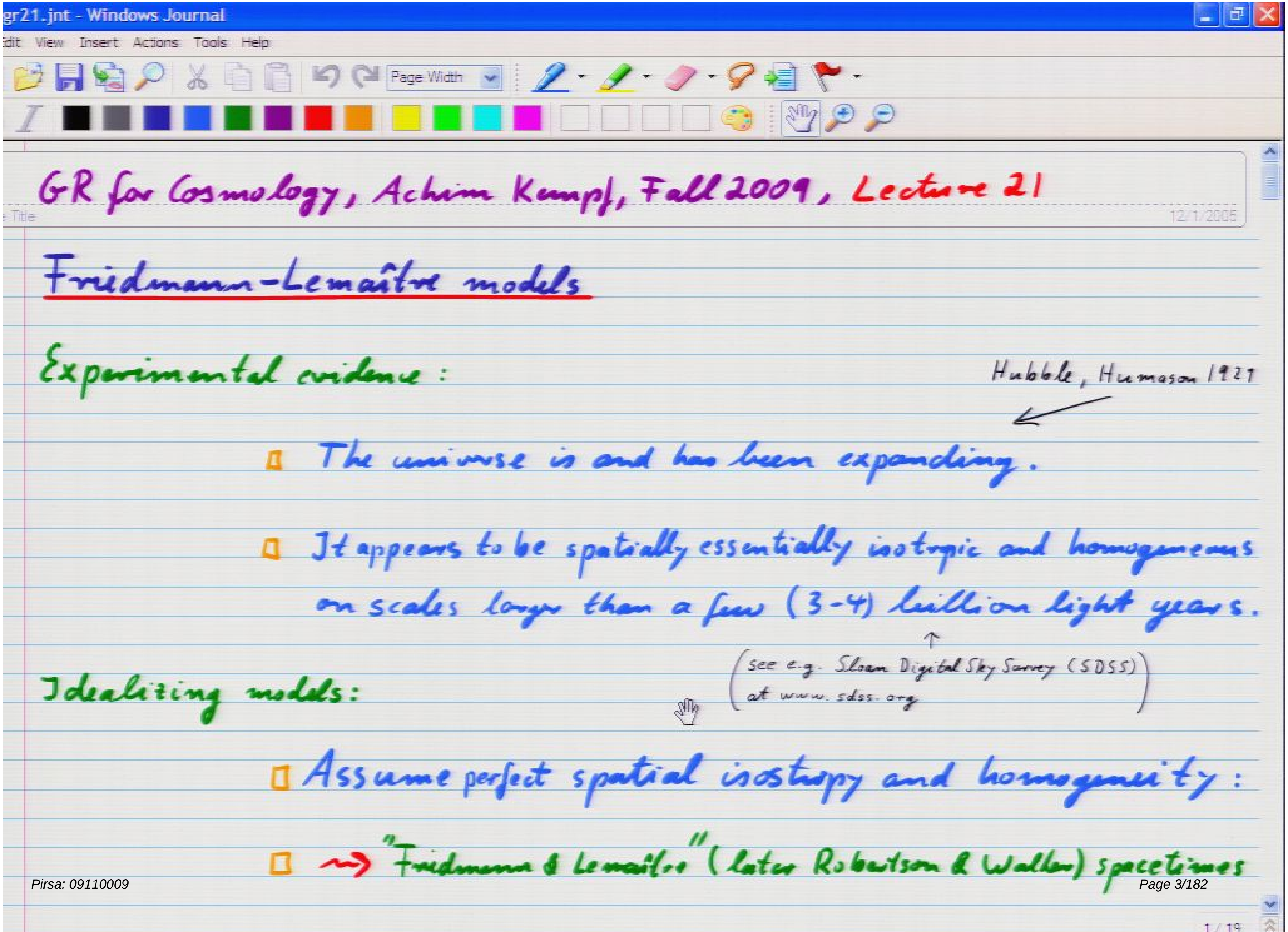
Hubble, Humason 1929

- ☐ The universe is and has been expanding.
- ☐ It appears to be spatially essentially isotropic and homogeneous on scales larger than a few (3-4) billion light years.

(see e.g. Sloan Digital Sky Survey (SDSS) at www.sdss.org)

Idealizing models:

- ☐ Assume perfect spatial isotropy and homogeneity:
- ☐ $\rightsquigarrow$ "Friedmann & Lemaître" (later Robertson & Walker) spacetimes



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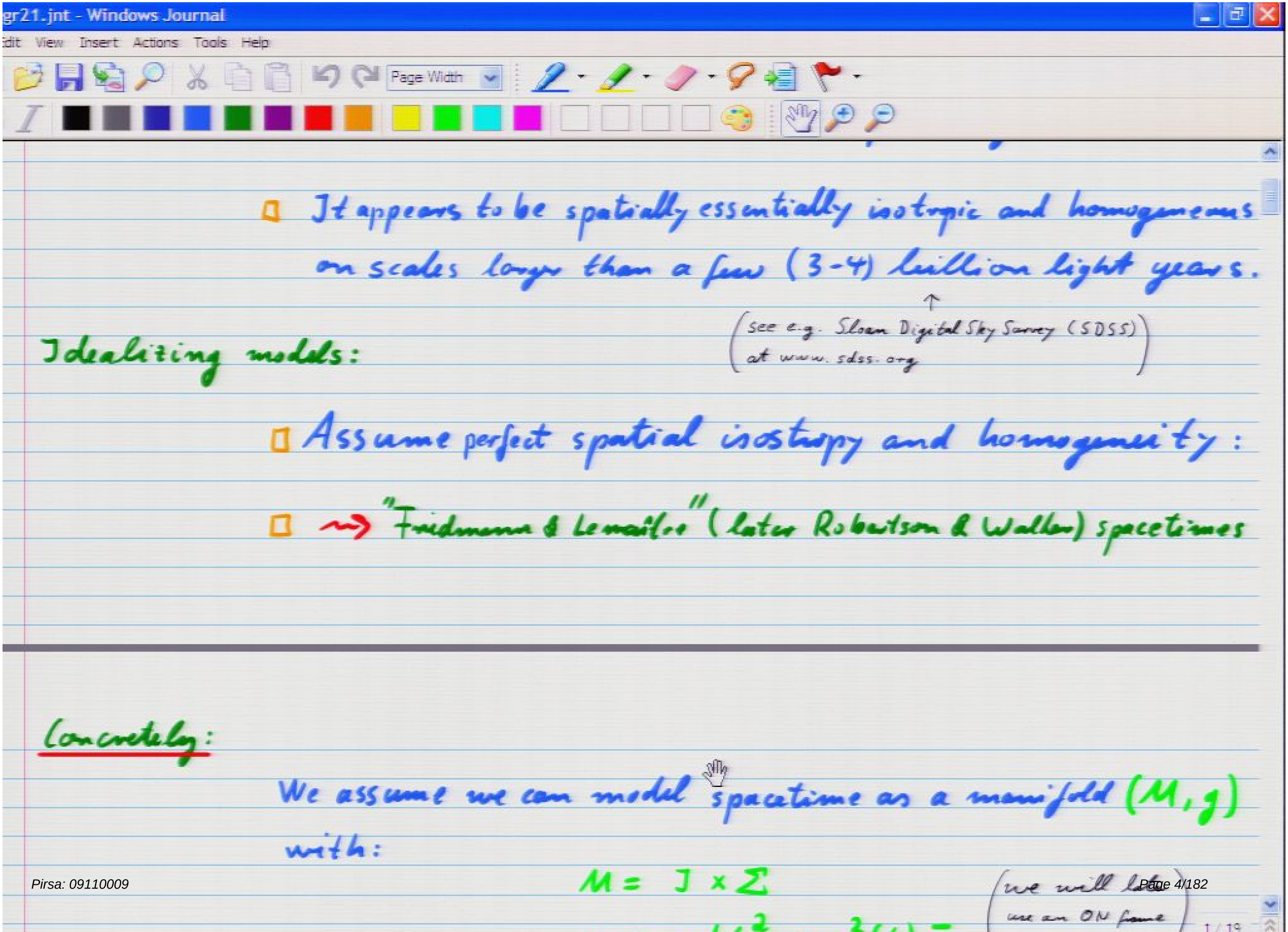

☐ It appears to be spatially essentially isotropic and homogeneous on scales longer than a few (3-4) billion light years.

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Concretely:

We assume we can model spacetime as a manifold (M, g) with:

$$M = I \times \Sigma$$

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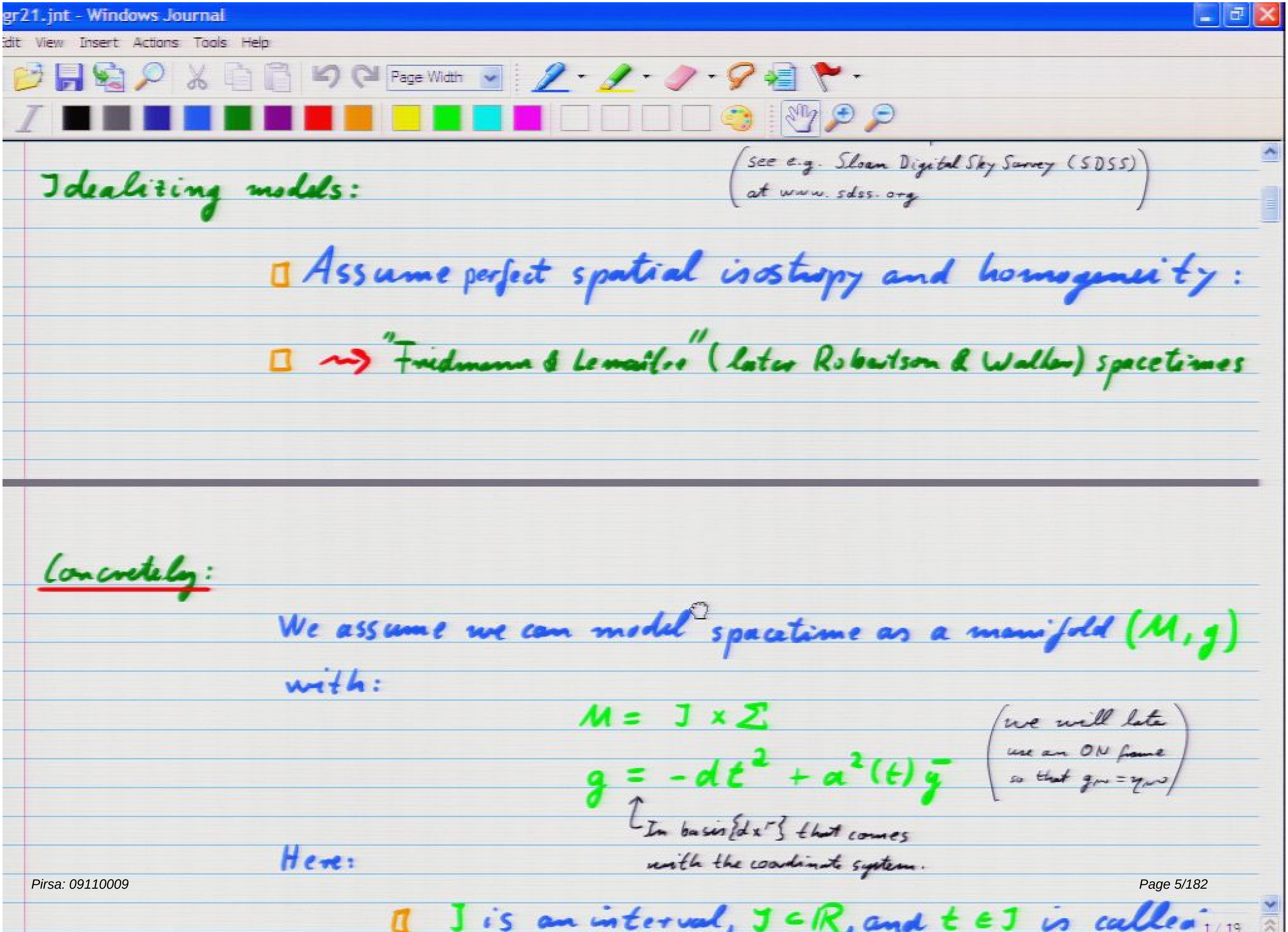

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$$g = -dt^2 + a^2(t)\,\bar{g}$$

In basis $\{dx^\mu\}$ that comes with the coordinate system.

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Here:

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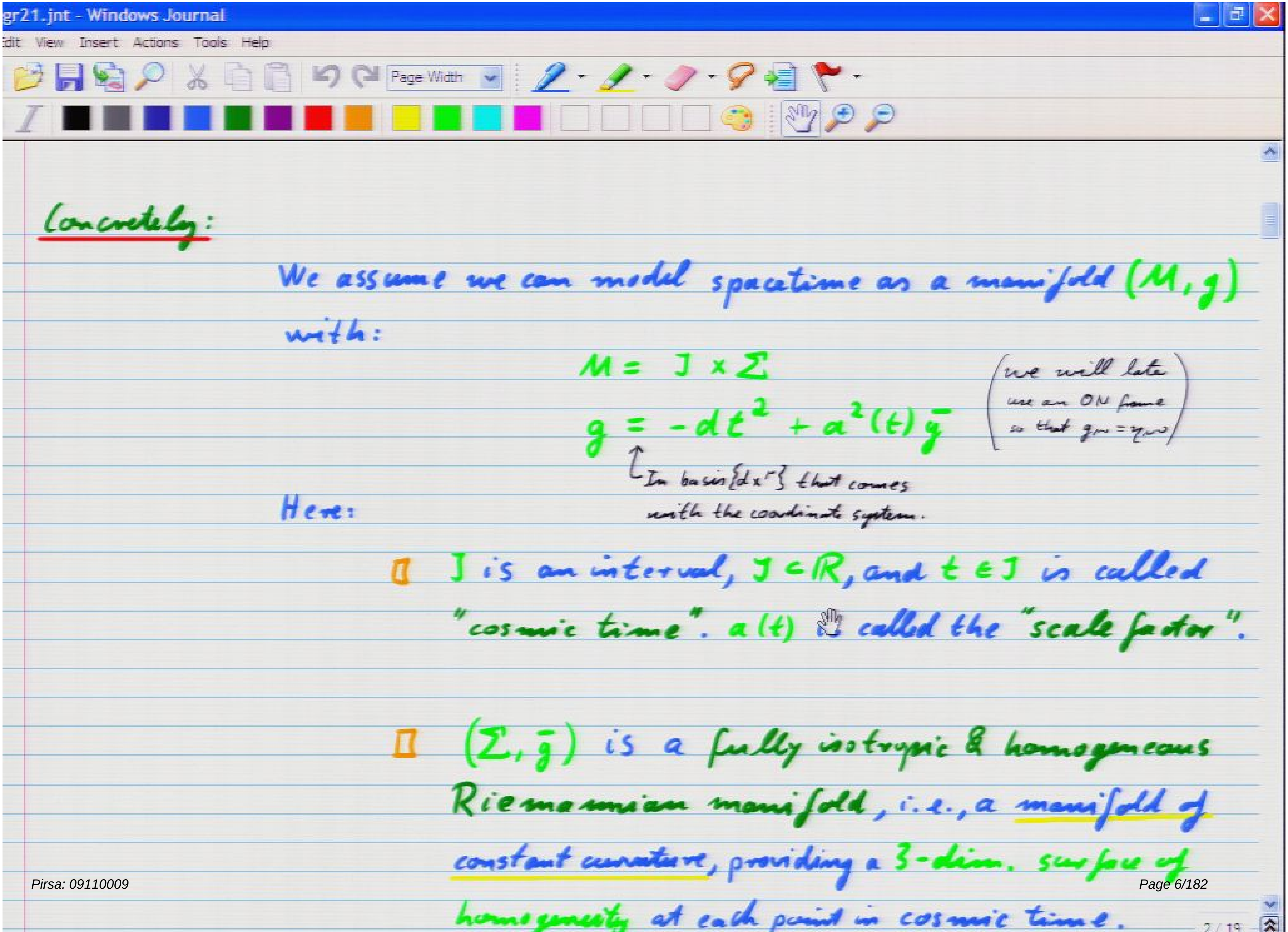

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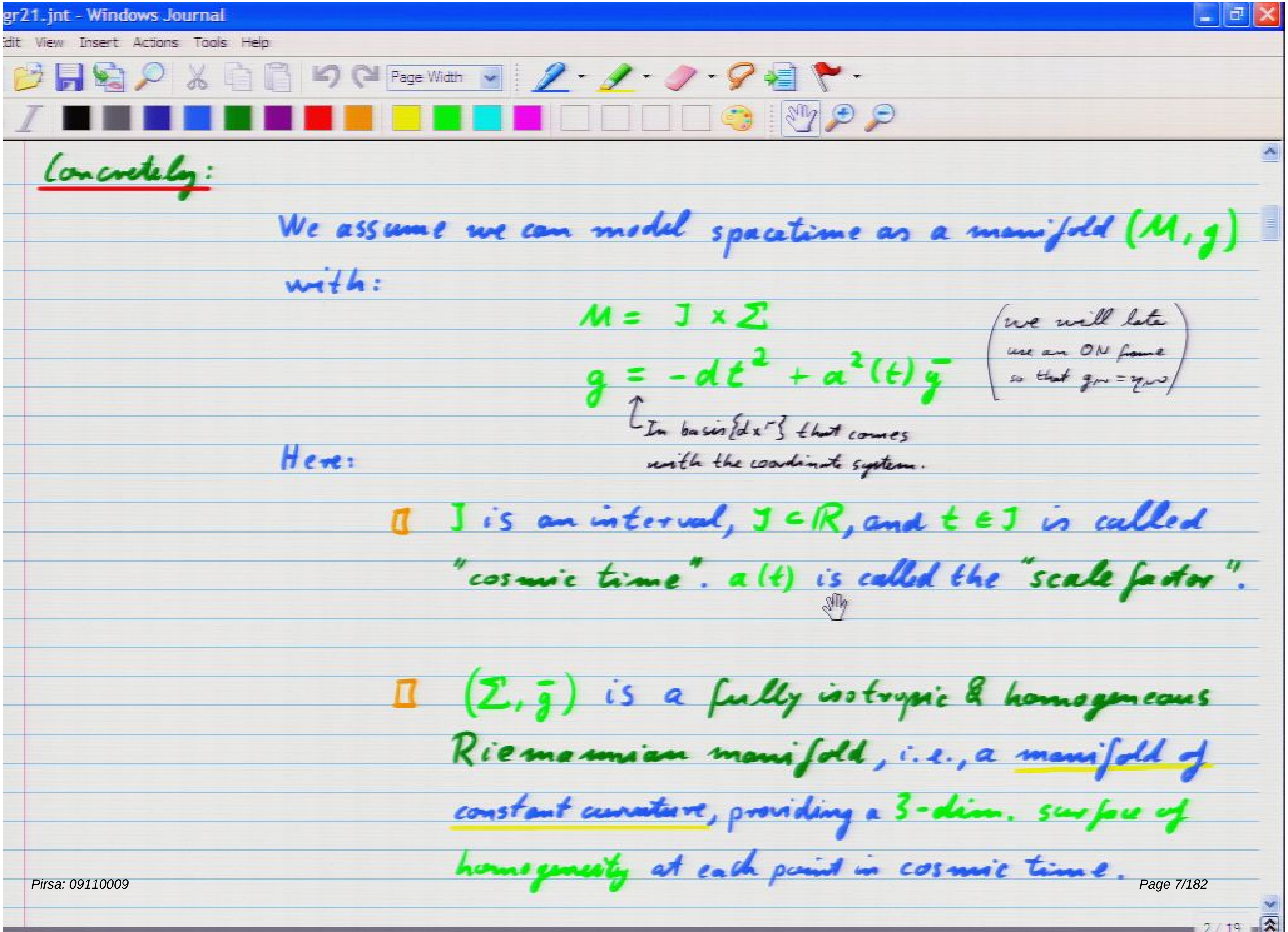
↑ In basis $\{dx^\mu\}$ that comes with the coordinate system.

Here:

- I J is an interval, $J \subset \mathbb{R}$, and $t \in J$ is called "cosmic time". $a(t)$ is called the "scale factor".

- II $(\Sigma, \bar{g})$ is a fully isotropic & homogeneous Riemannian manifold, i.e., a manifold of constant curvature, providing a 3-dim. surface of homogeneity at each point in cosmic time.



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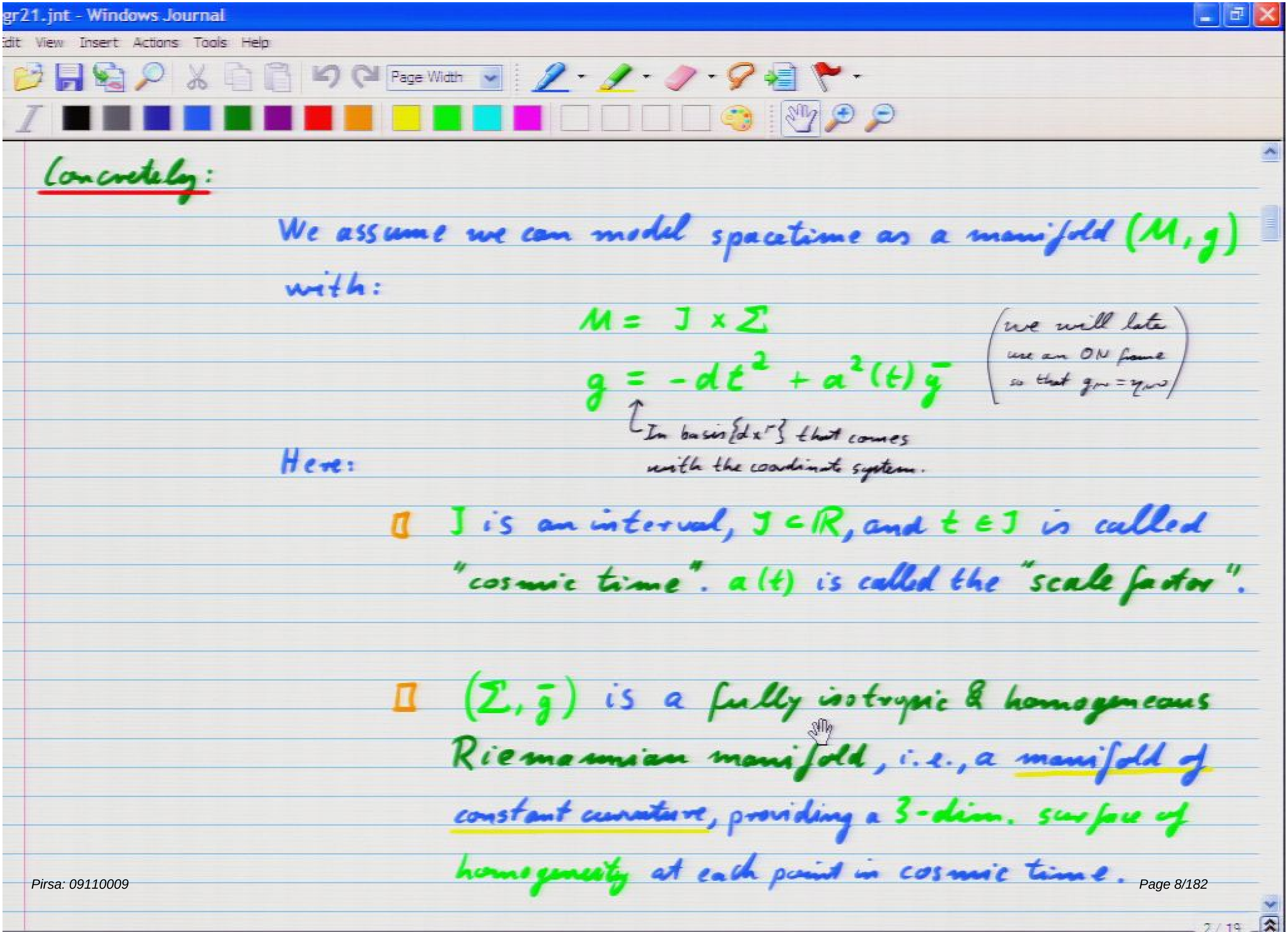

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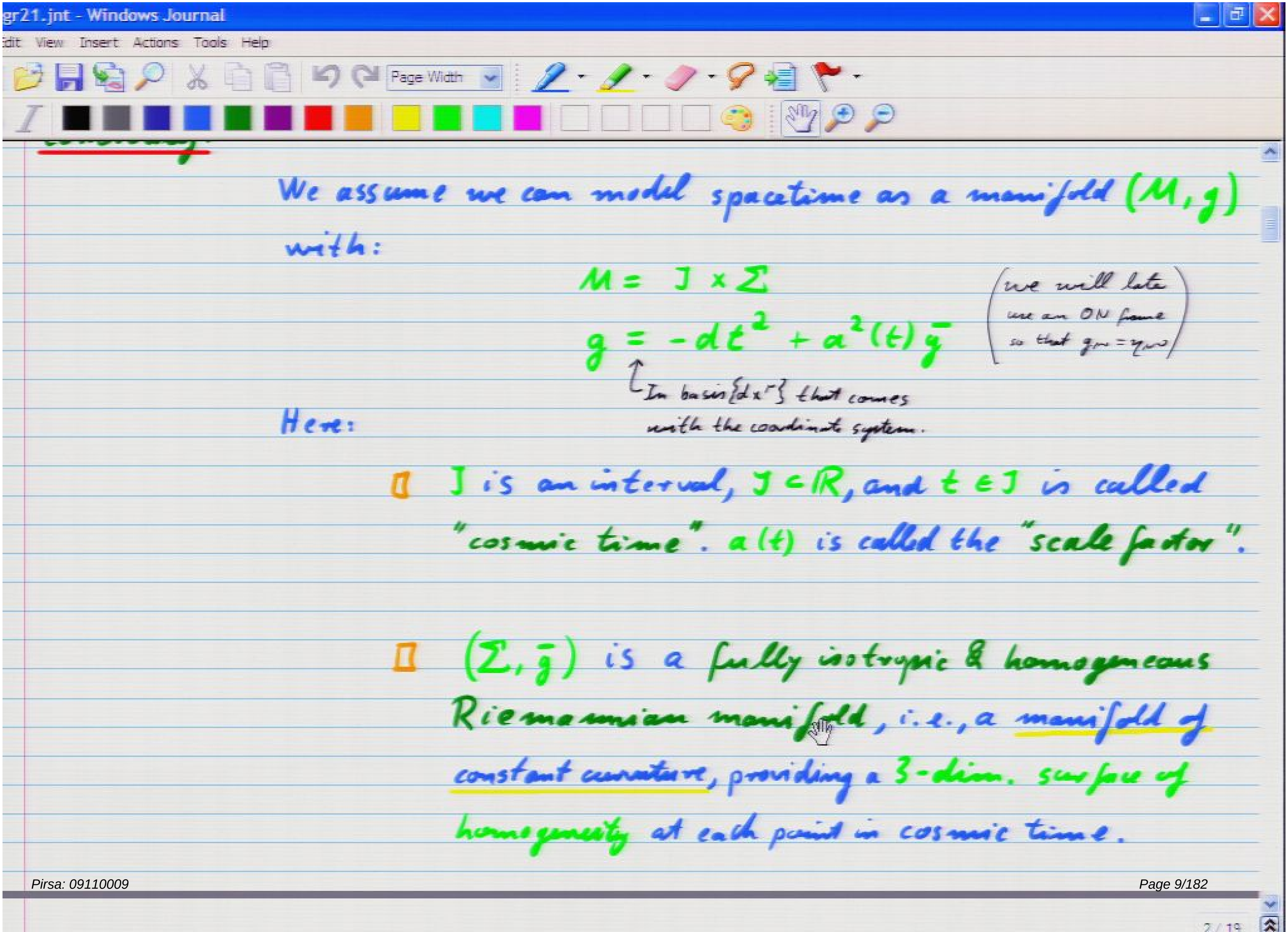

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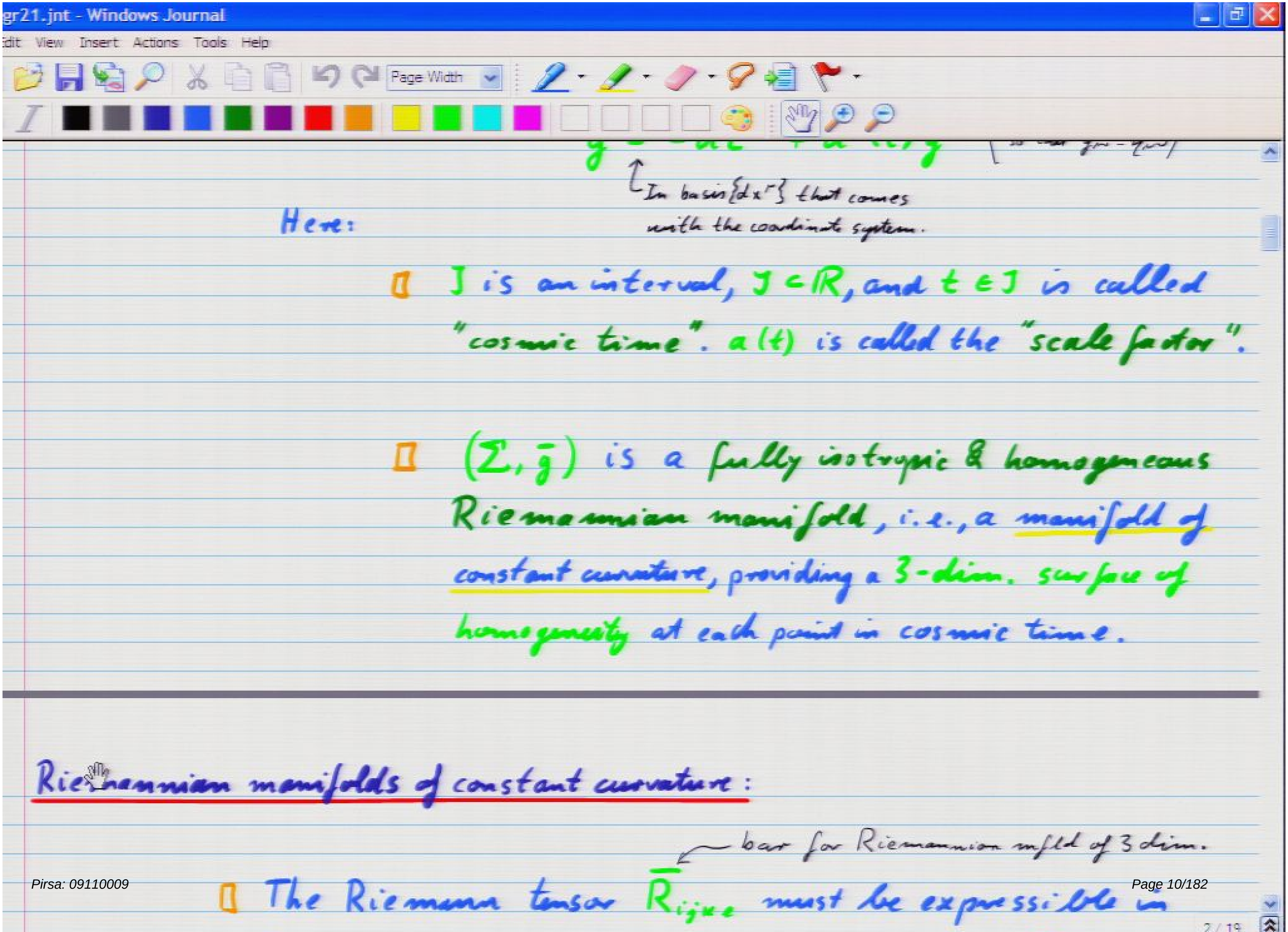

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bar for Riemannian mfld of 3 dim.

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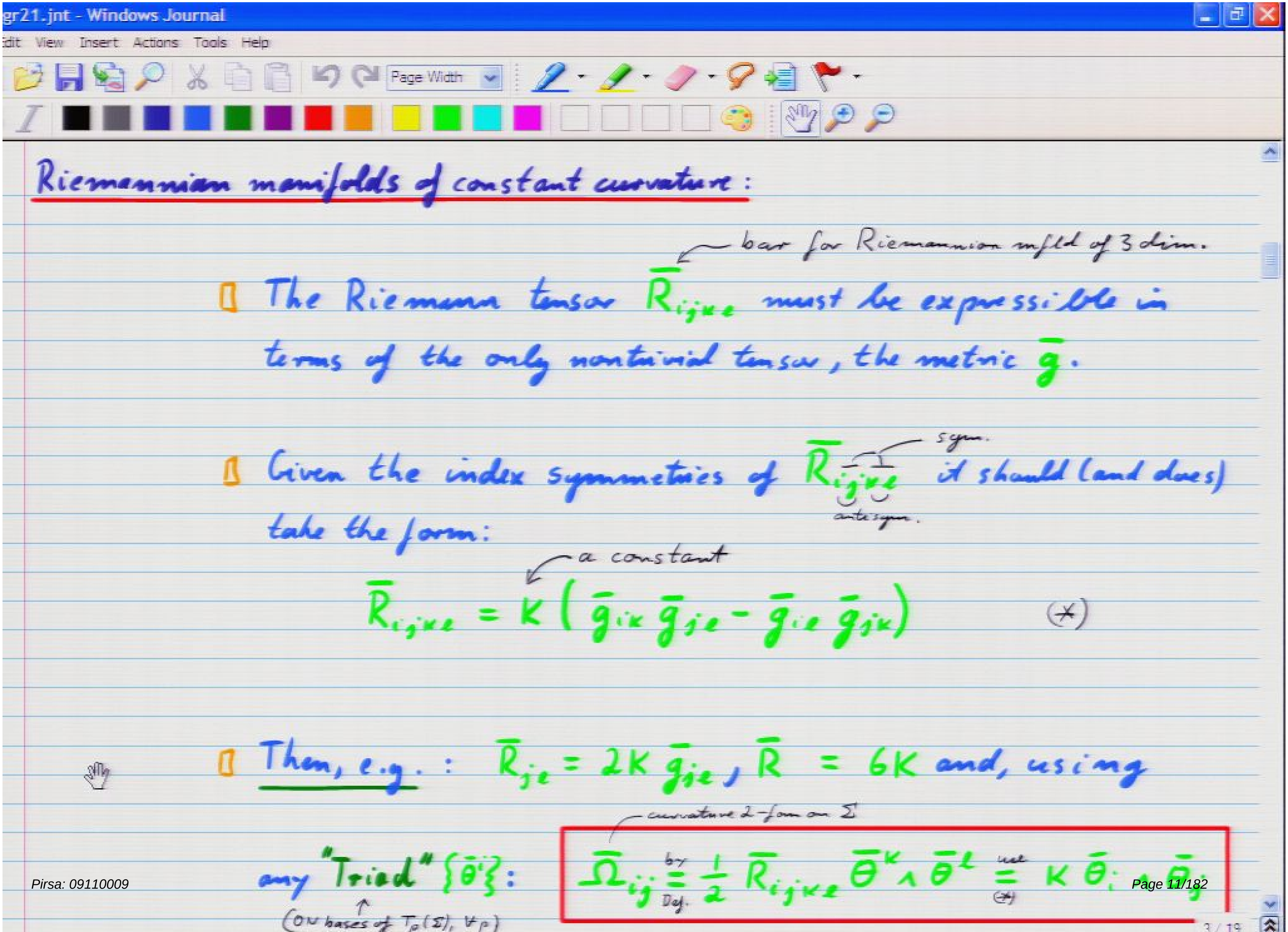

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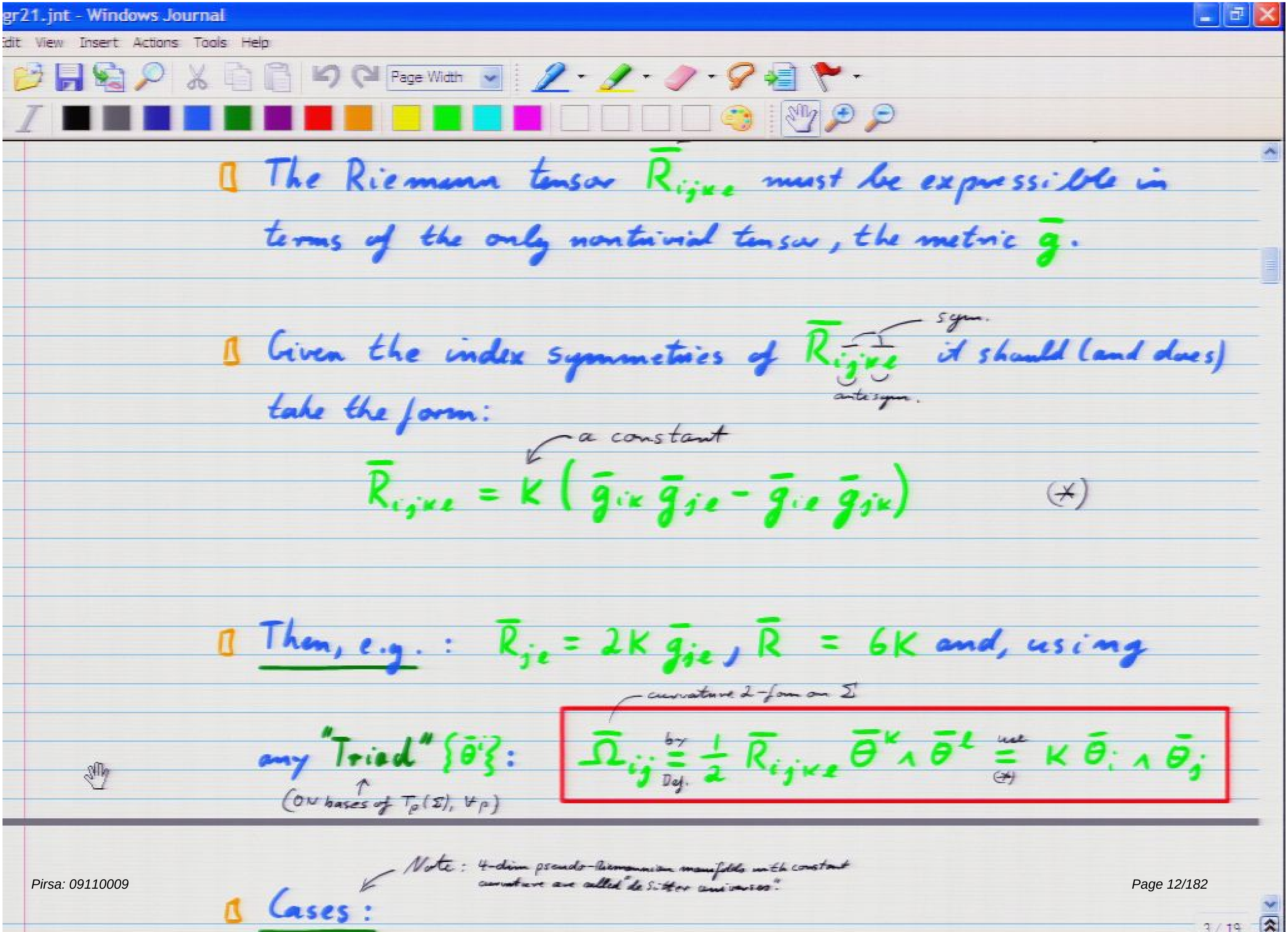
- Given the index symmetries of $\underbrace{\bar{R}_{\overset{\text{sym.}}{\overbrace{ij\,kl}}}}_{\text{antisym.}}$ it should (and does) take the form:

$$\bar{R}_{ijkl} = \overset{\text{a constant}}{K}\left(\bar{g}_{ik}\bar{g}_{jl} - \bar{g}_{il}\bar{g}_{jk}\right) \qquad (*)$$

- Then, e.g.: $\bar{R}_{jl} = 2K\bar{g}_{jl}$, $\bar{R} = 6K$ and, using any $\underset{\uparrow \text{ (ON bases of } T_p(\Sigma),\ \forall p)}{\text{"Triad"}}$ $\{\bar{\theta}^i\}$:

$$\boxed{\overset{\text{curvature 2-form on }\Sigma}{\bar{\Omega}_{ij}} \overset{\text{by}}{\underset{\text{Def.}}{=}} \frac{1}{2}\bar{R}_{ijkl}\,\bar{\theta}^k \wedge \bar{\theta}^l \overset{\text{use}}{\underset{(*)}{=}} K\,\bar{\theta}_i \wedge \bar{\theta}_j}$$



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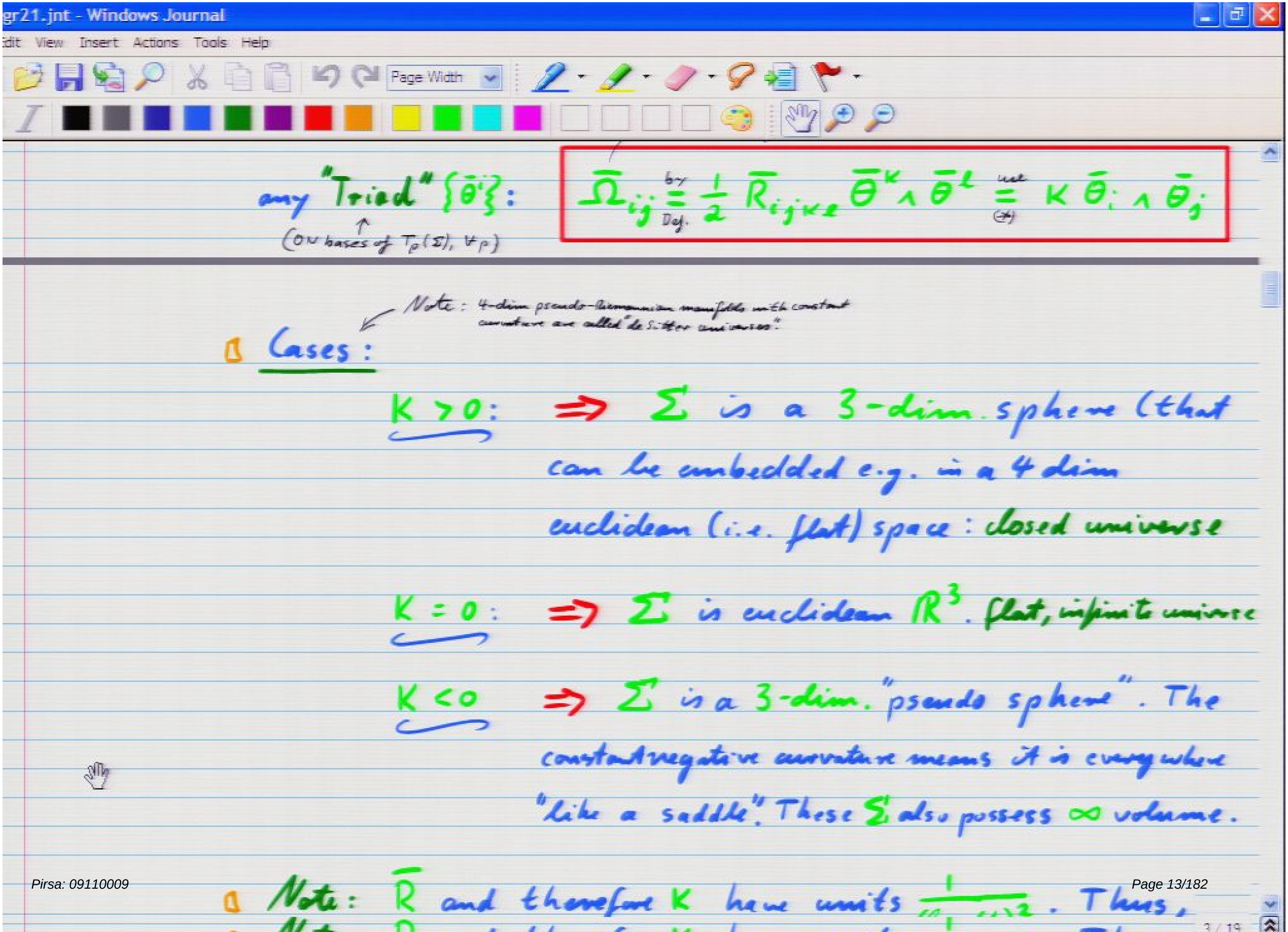
Note: 4-dim pseudo-Riemannian manifolds with constant curvature are called "de Sitter universes".

- Cases:



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Cases:

$K > 0$: $\Rightarrow$ Σ is a 3-dim. sphere (that can be embedded e.g. in a 4 dim euclidean (i.e. flat) space: closed universe

$K = 0$: $\Rightarrow$ Σ is euclidean $\mathbb{R}^3$. flat, infinite universe

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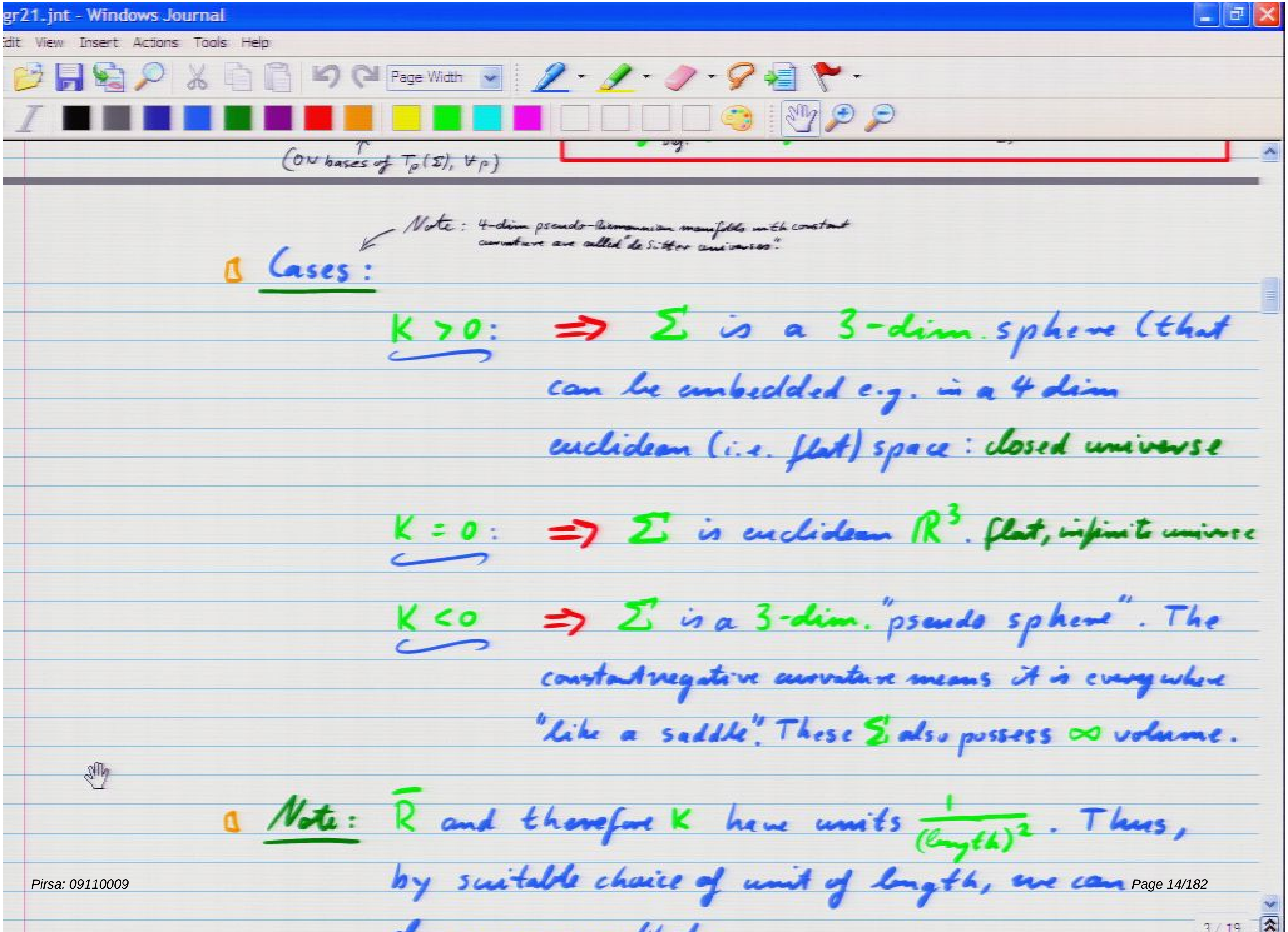

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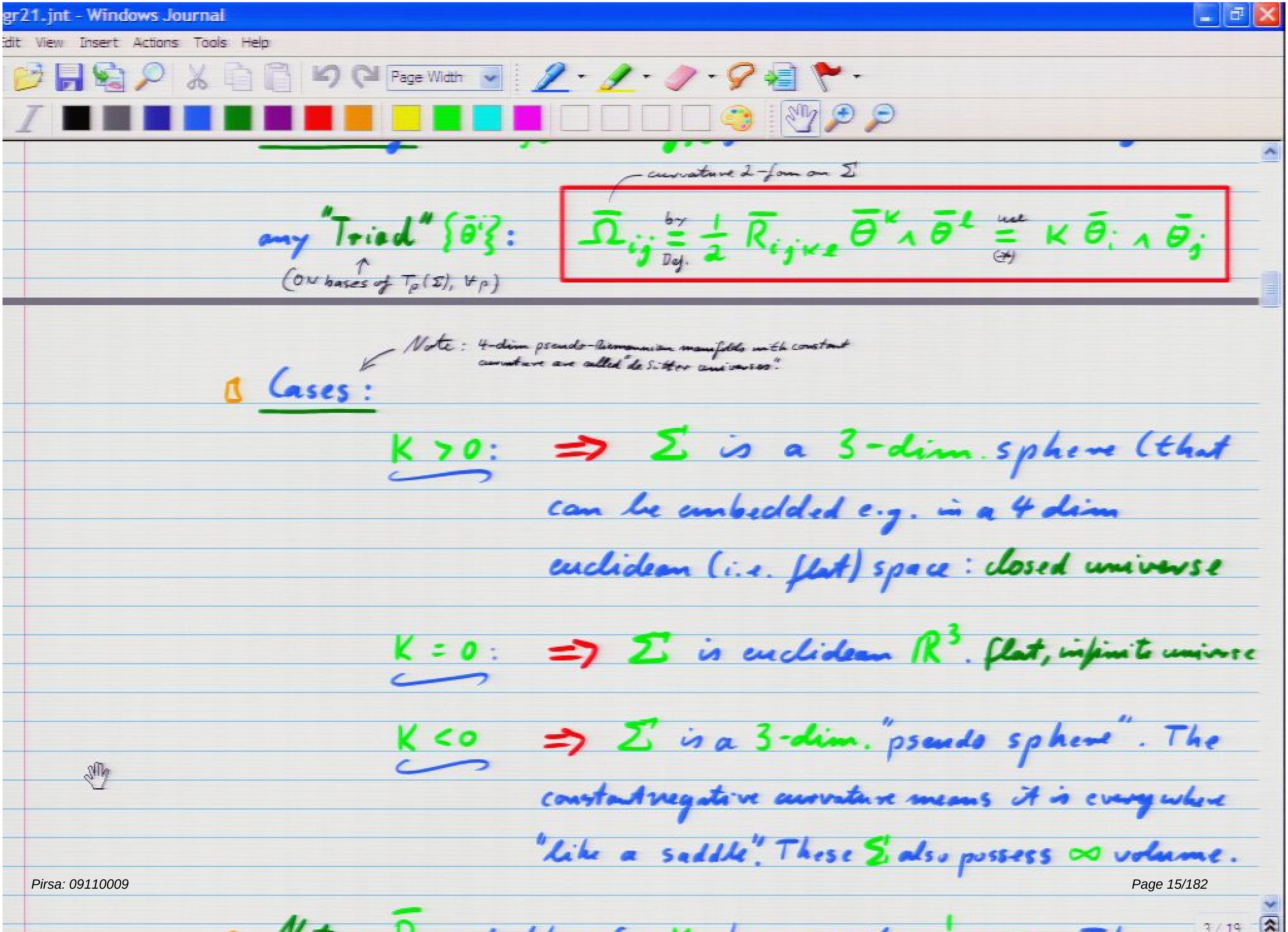

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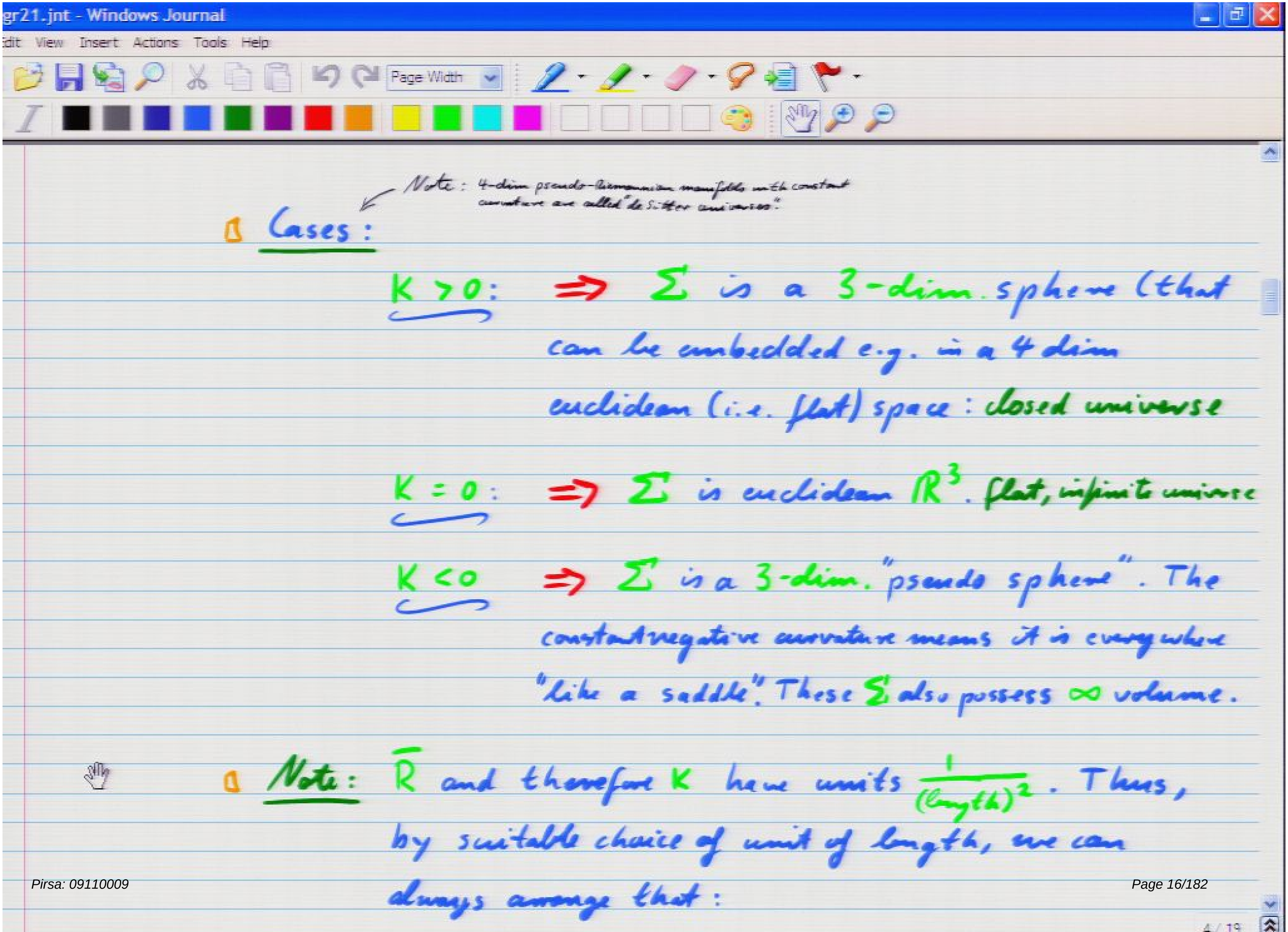

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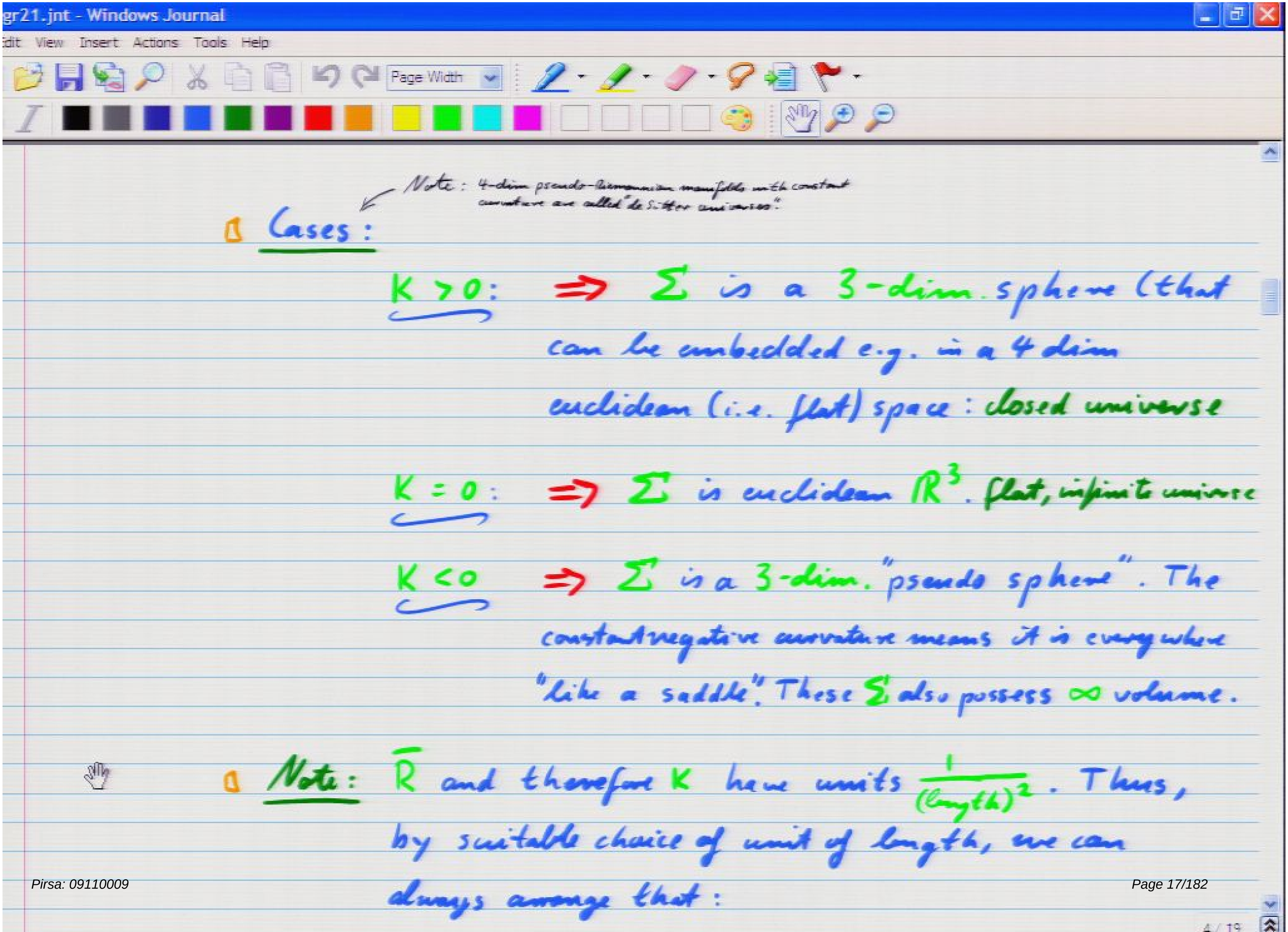

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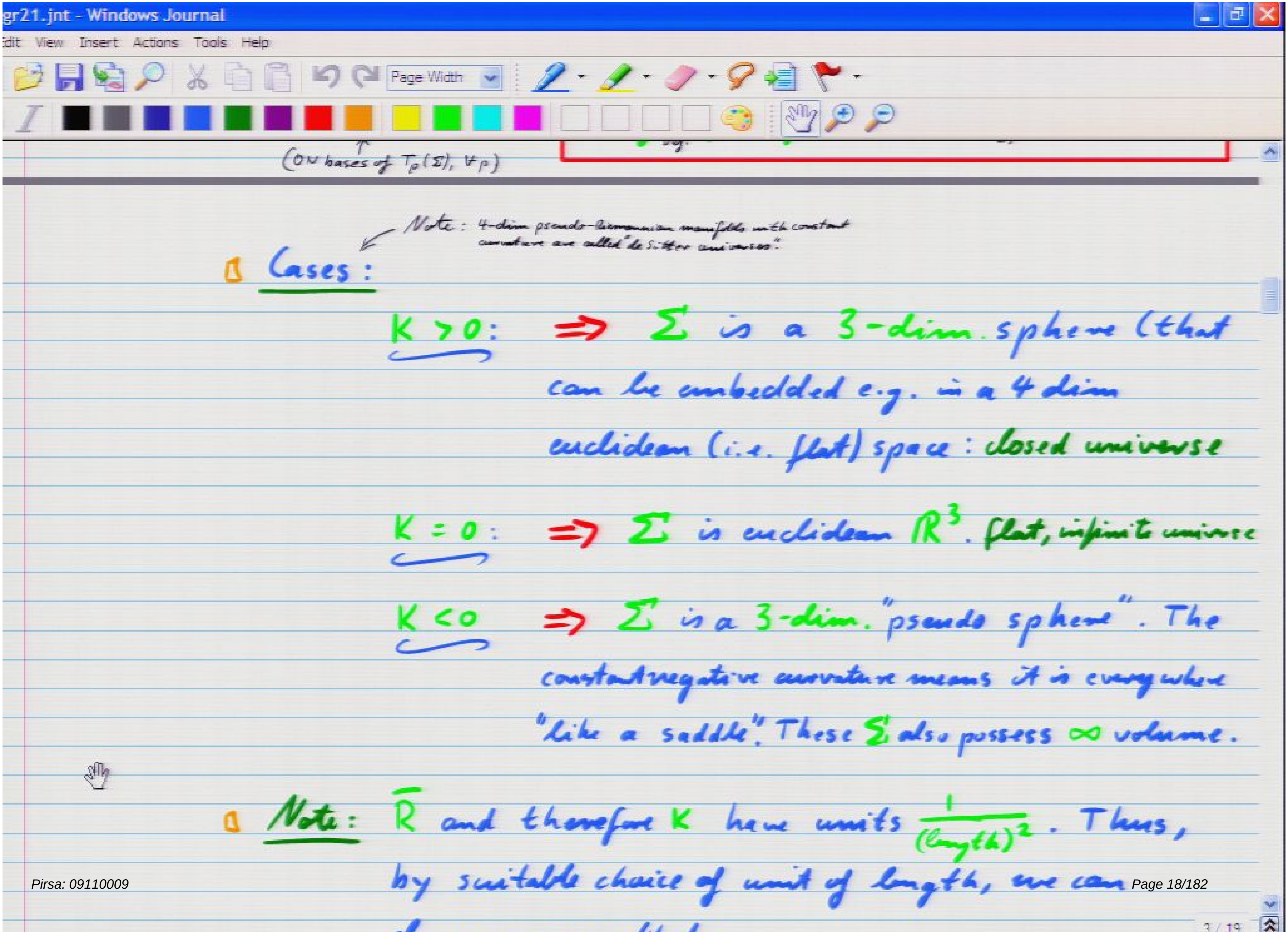

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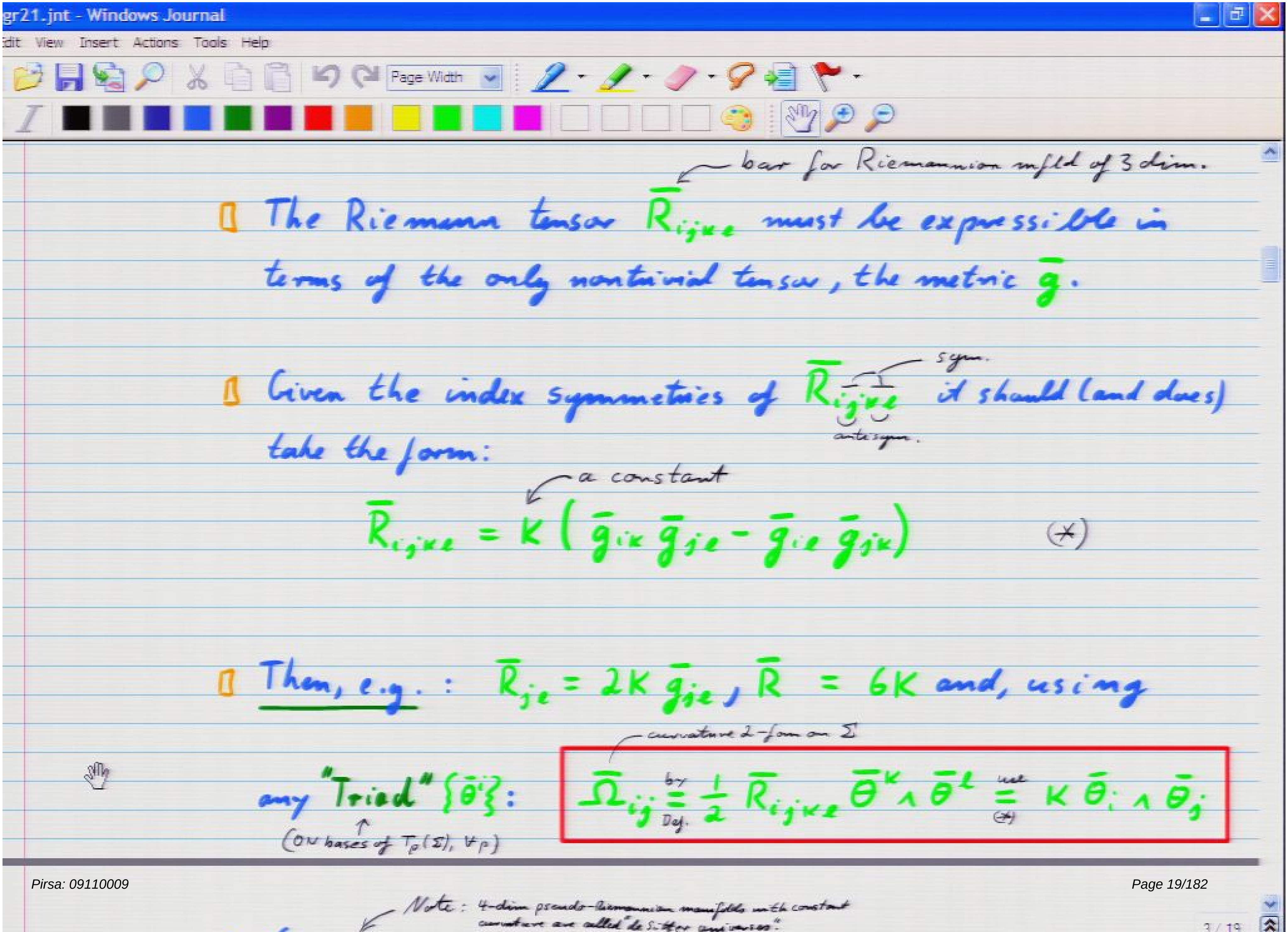

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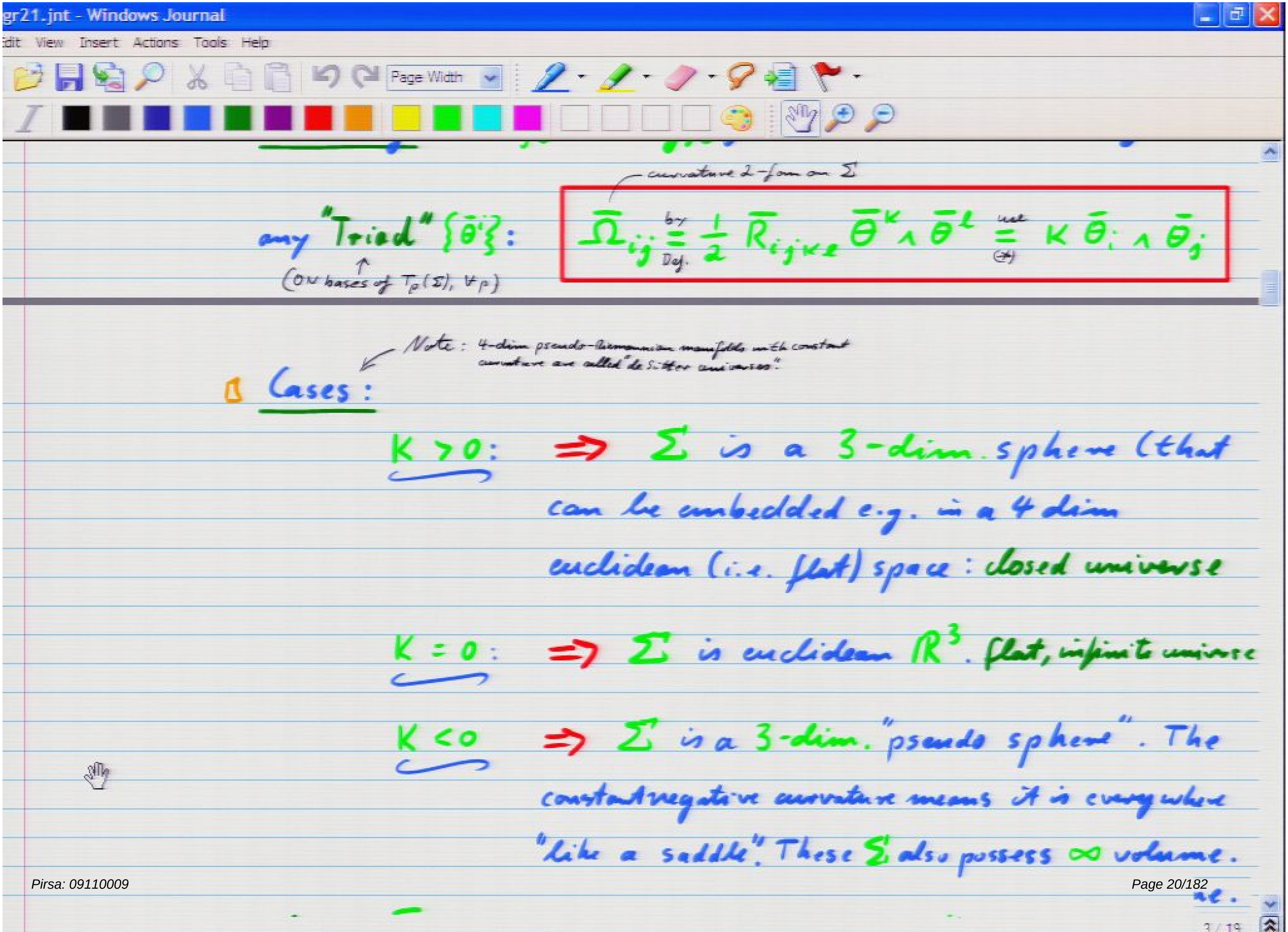

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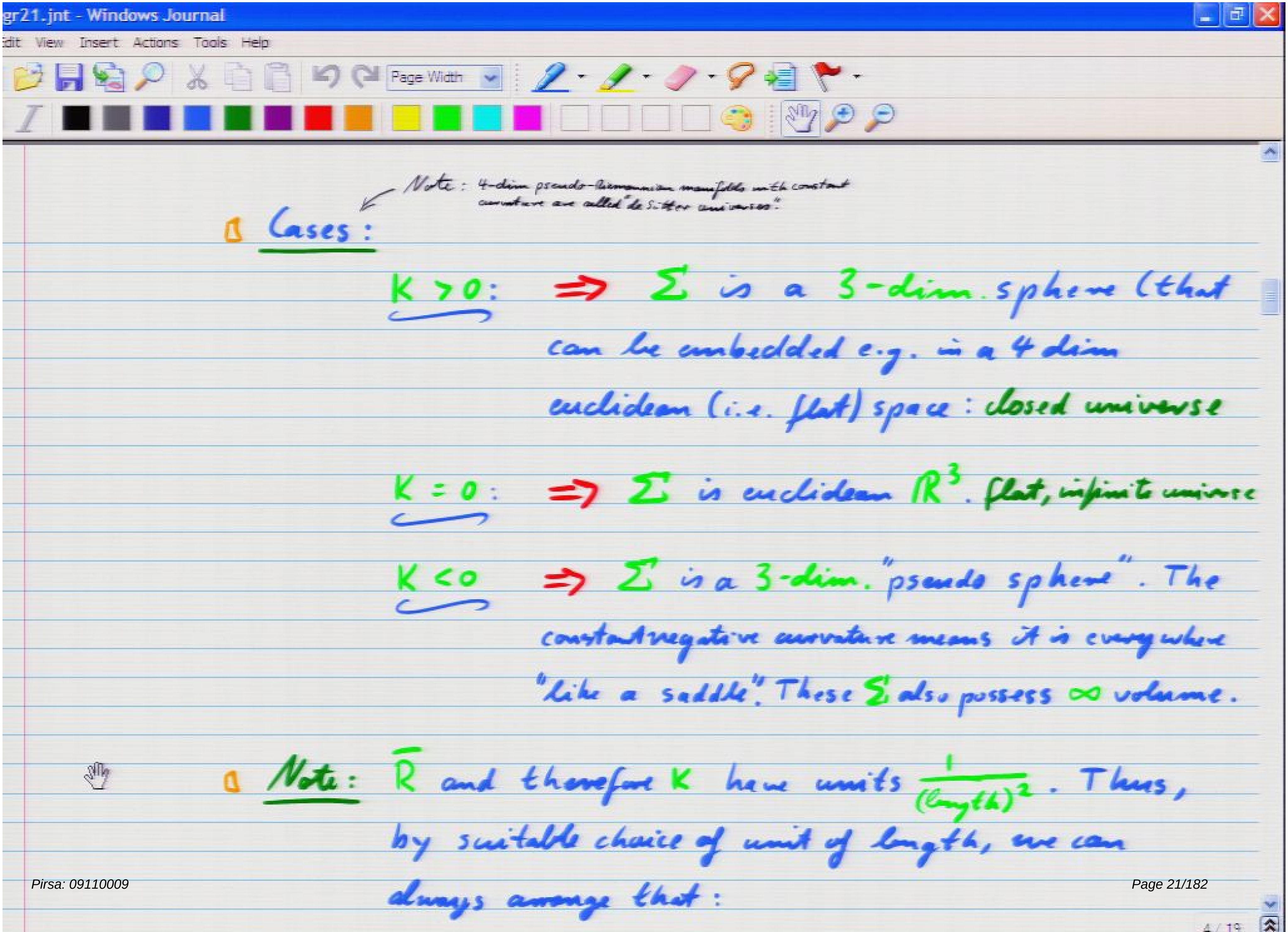

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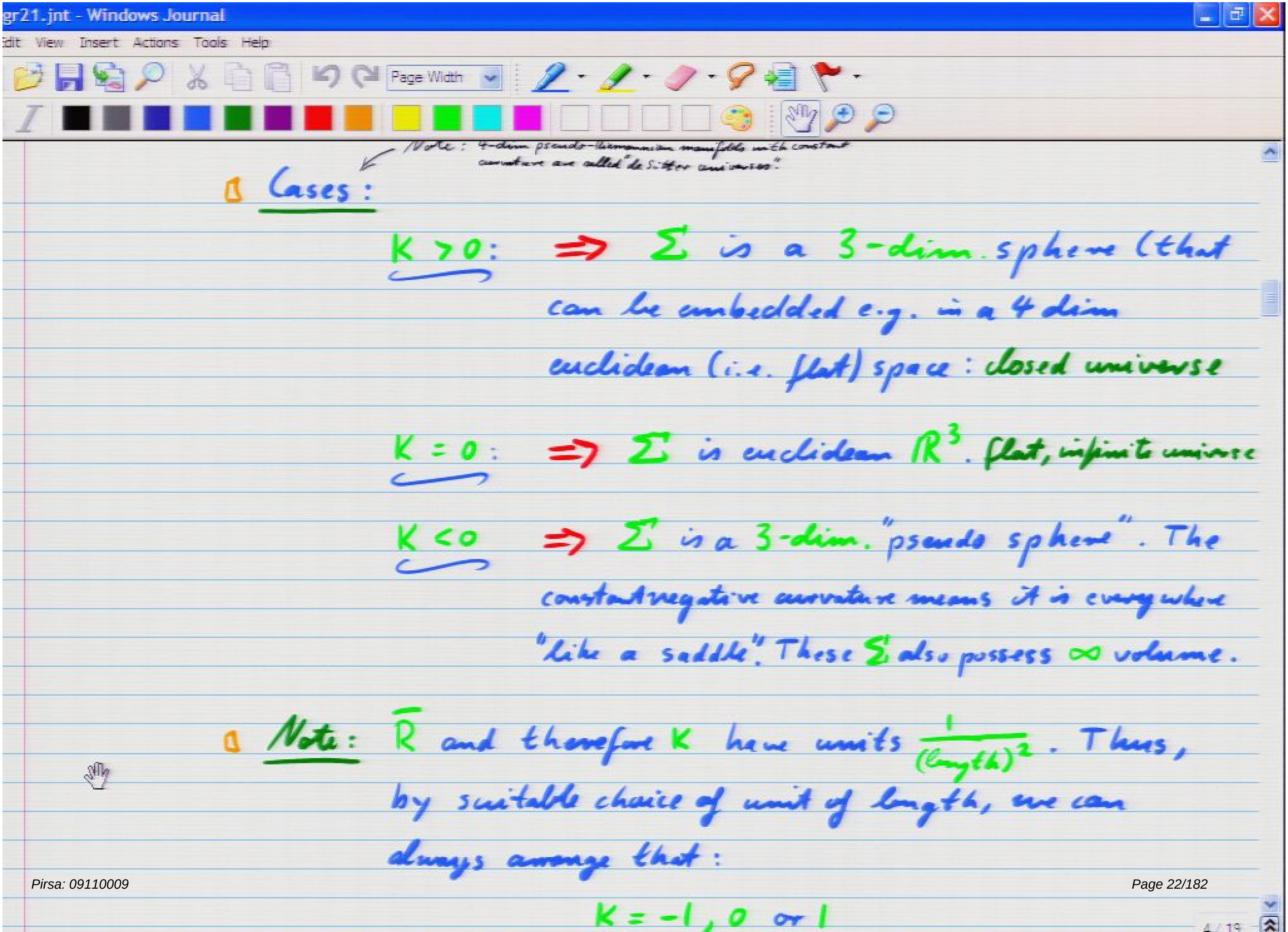

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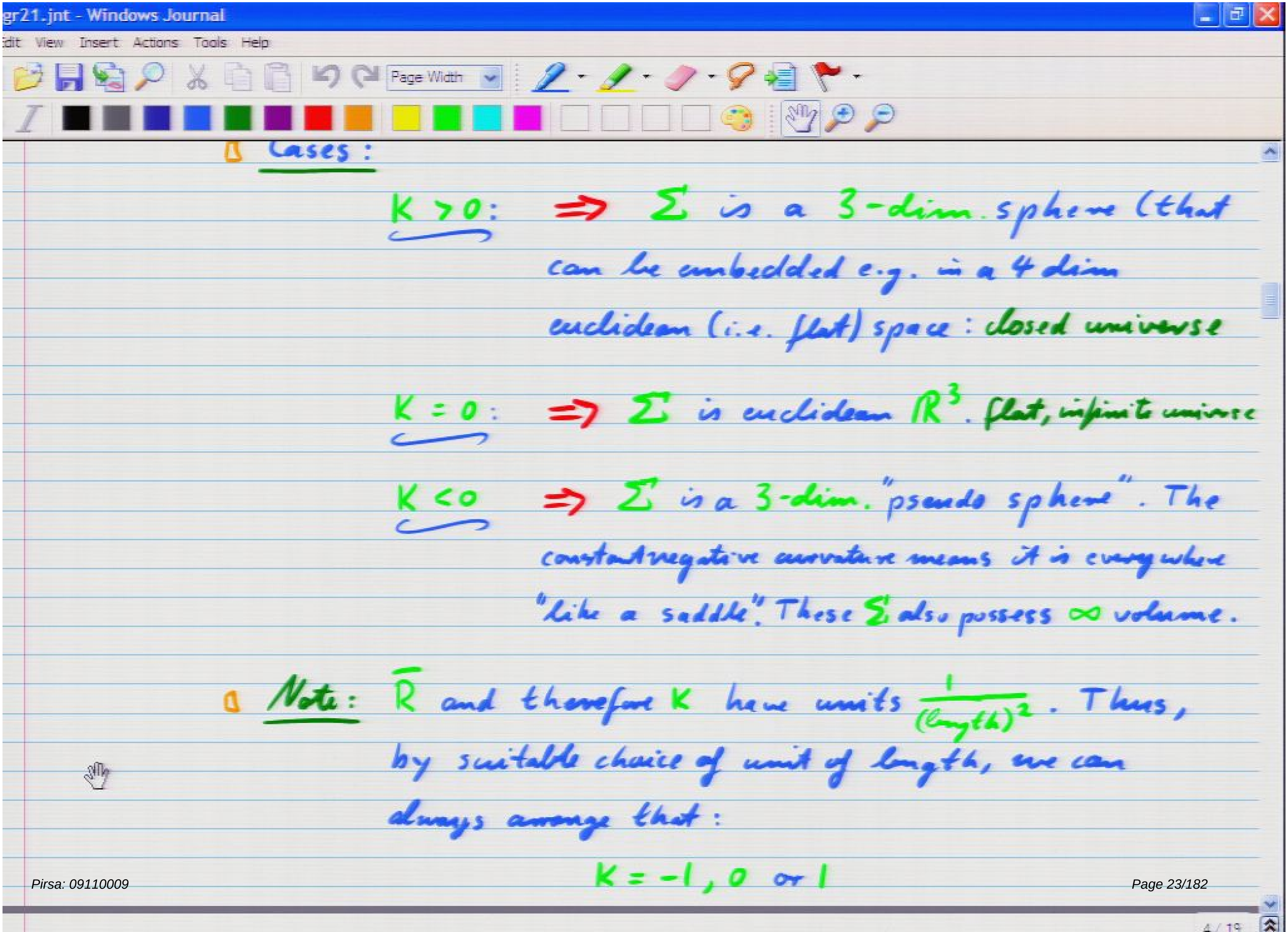

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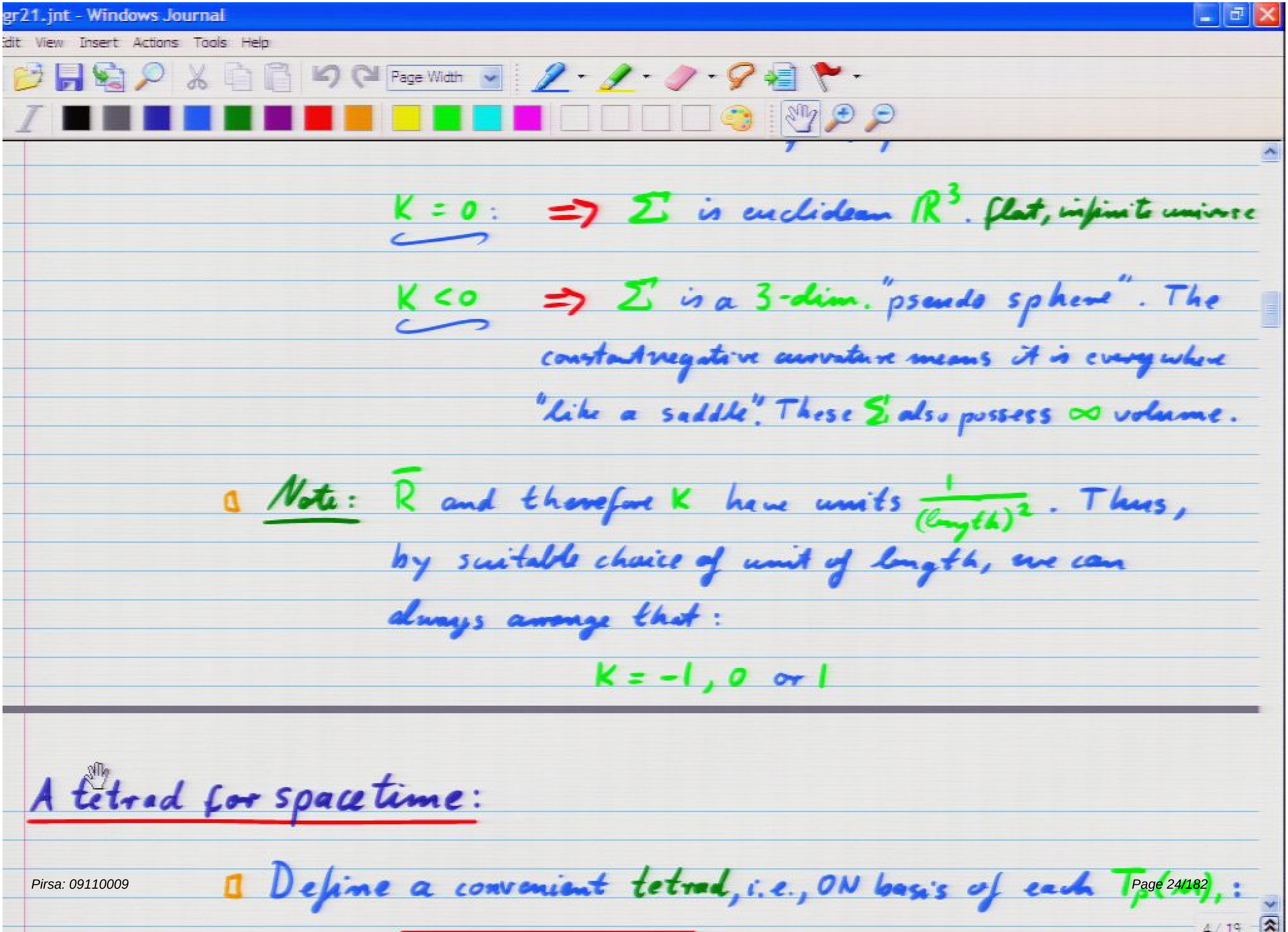

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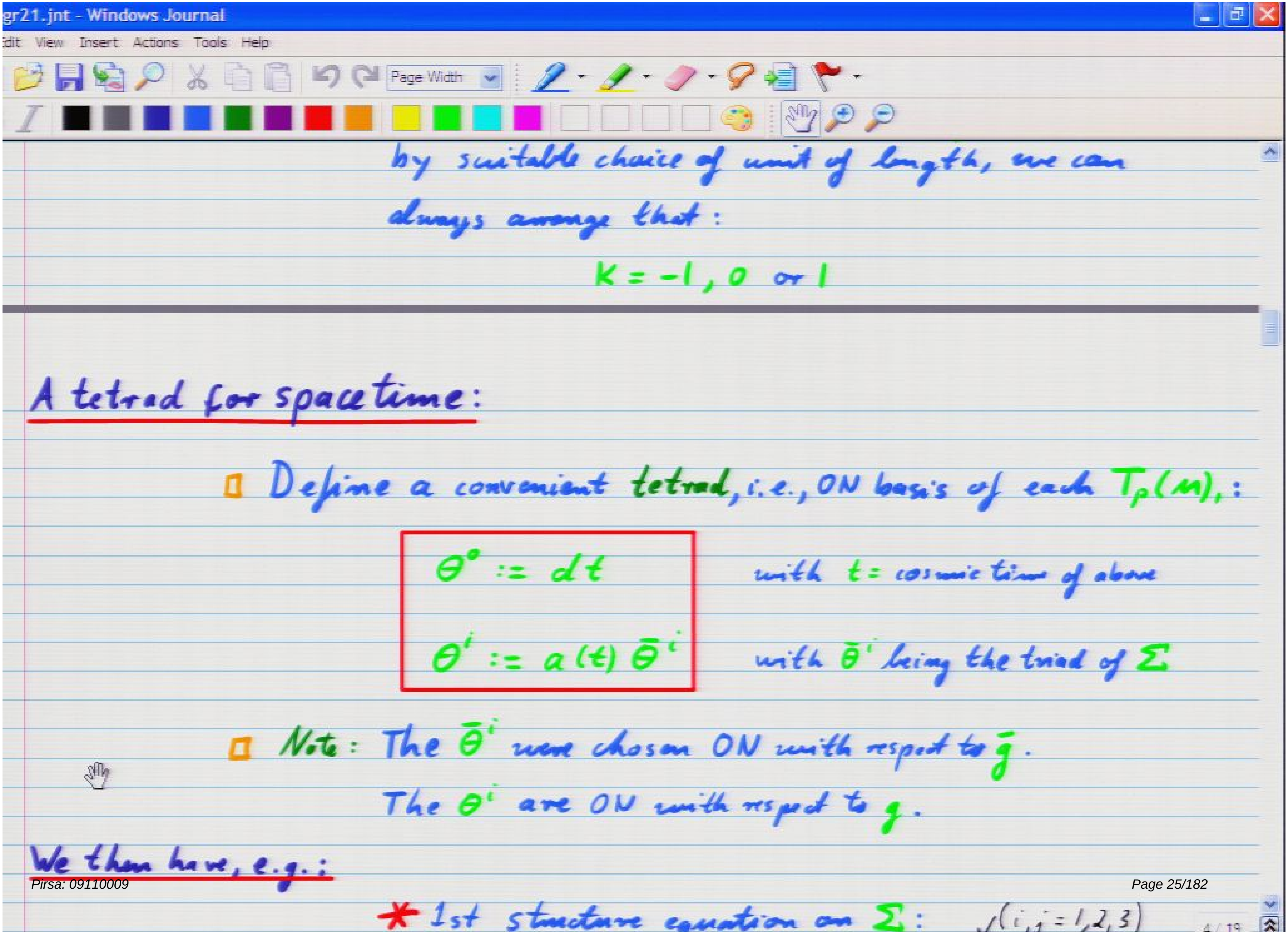

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A tetrad for spacetime:

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$$\boxed{\begin{aligned} \theta^0 &:= dt \\ \theta^i &:= a(t)\,\bar{\theta}^i \end{aligned}}$$

with t = cosmic time of above

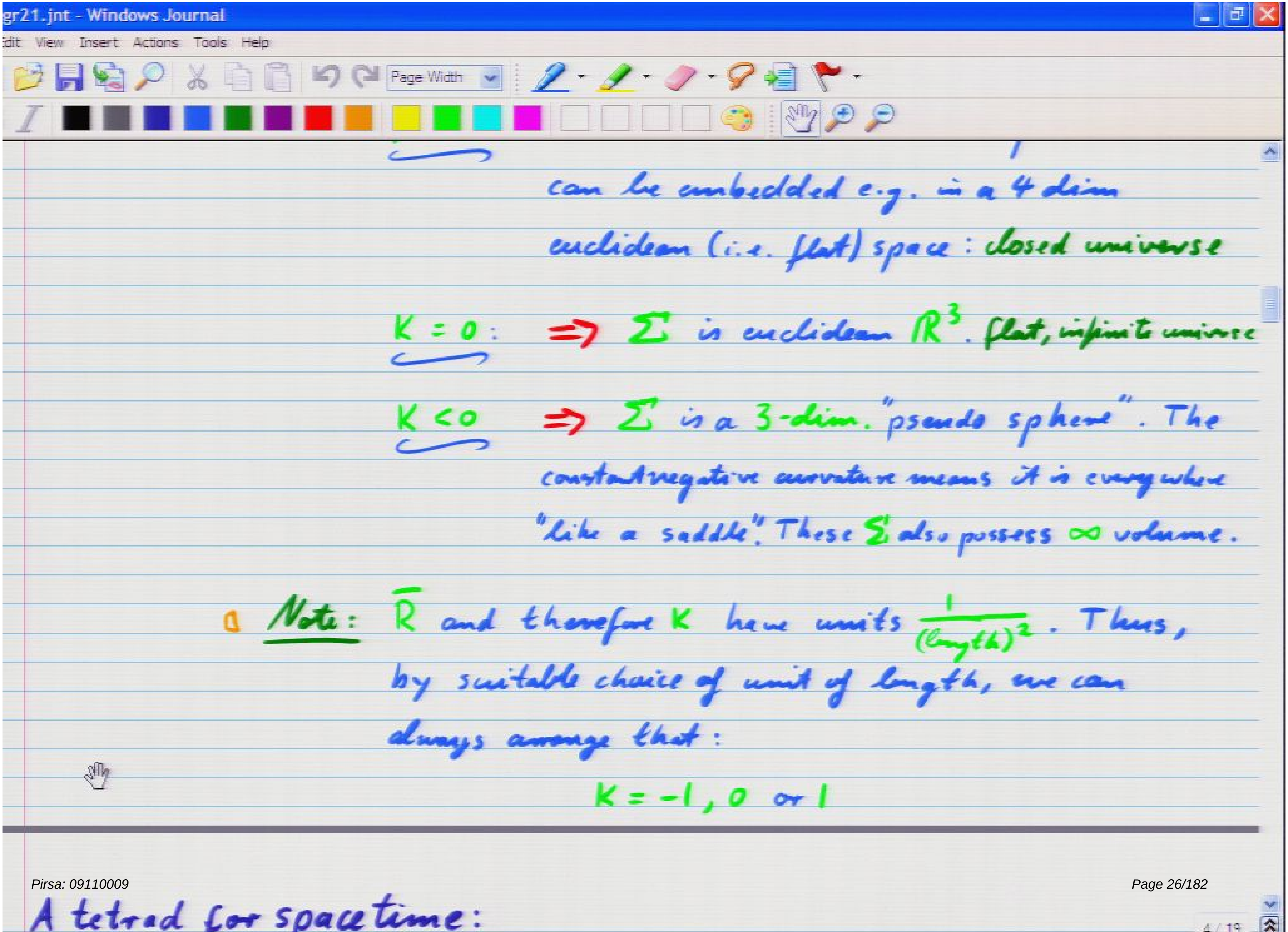
with $\bar{\theta}^i$ being the triad of Σ

☐ Note: The $\bar{\theta}^i$ were chosen ON with respect to $\bar{g}$.

The θ^i are ON with respect to g.

We then have, e.g.:

* 1st structure equation on Σ: $(i,j = 1,2,3)$



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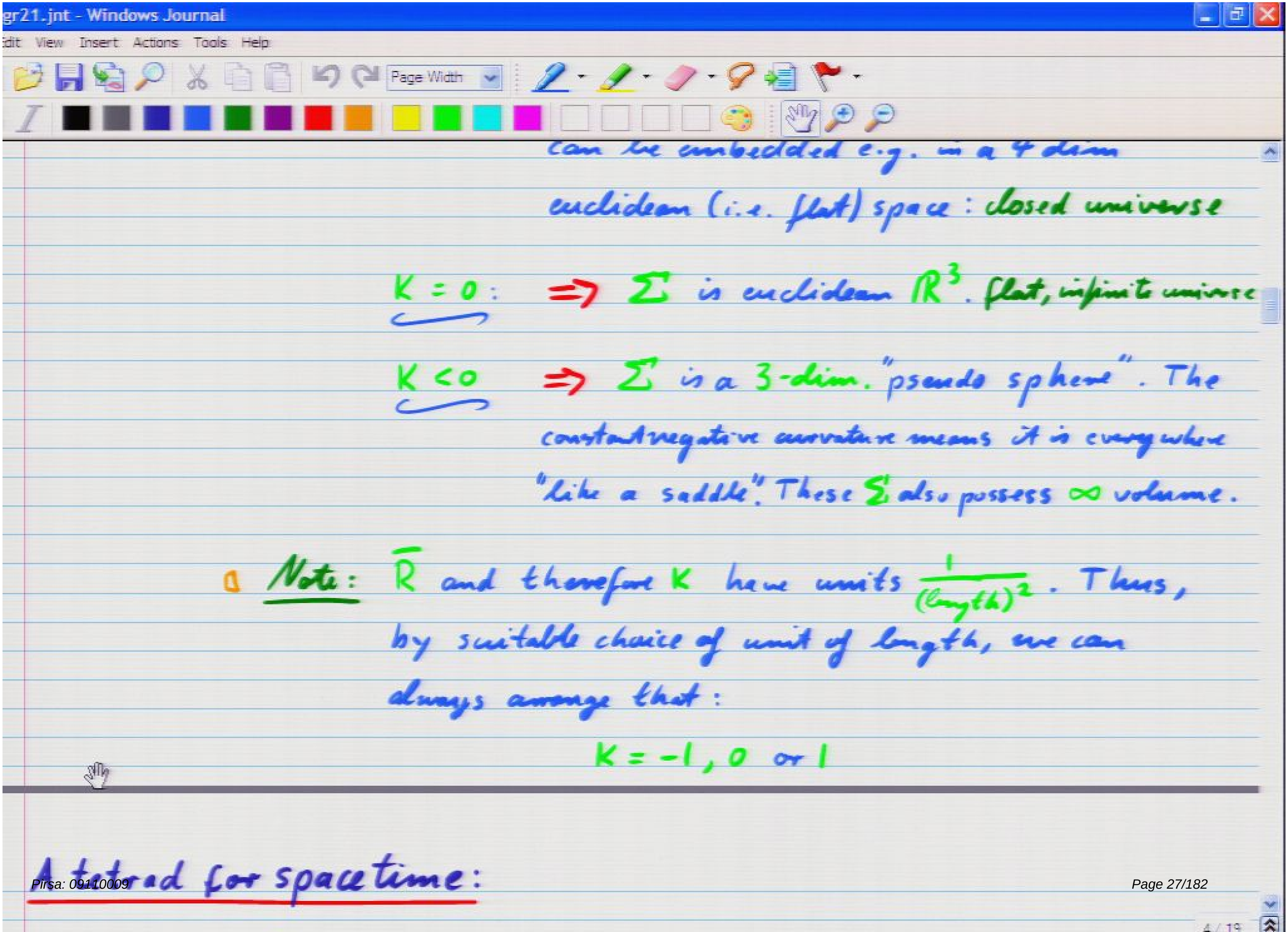

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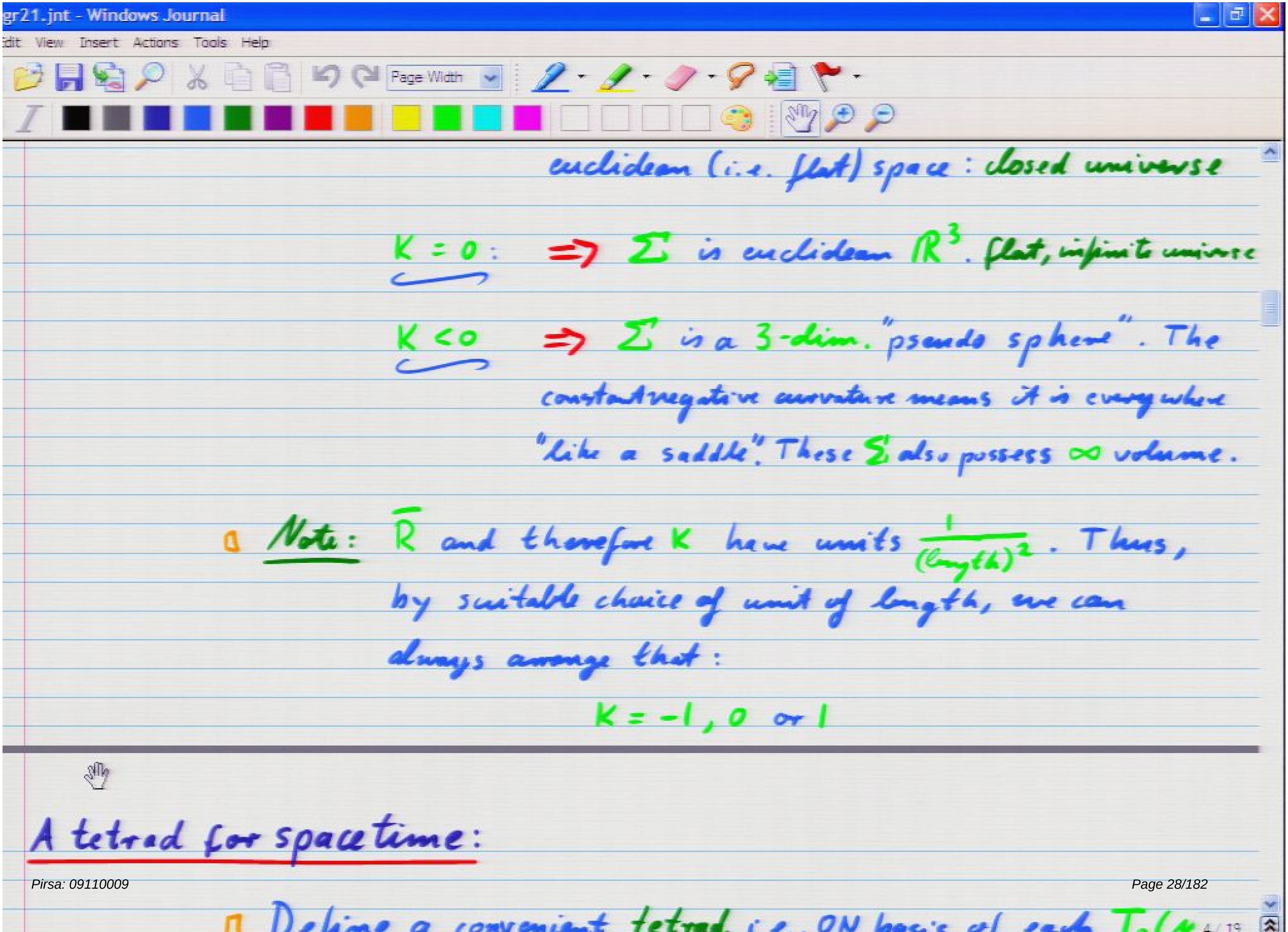

euclidean (i.e. flat) space : closed universe

$\underline{K = 0}$: $\Rightarrow$ Σ is euclidean $\mathbb{R}^3$. flat, infinite universe

$\underline{K < 0}$ $\Rightarrow$ Σ is a 3-dim. "pseudo sphere". The constant negative curvature means it is everywhere "like a saddle". These Σ also possess ∞ volume.

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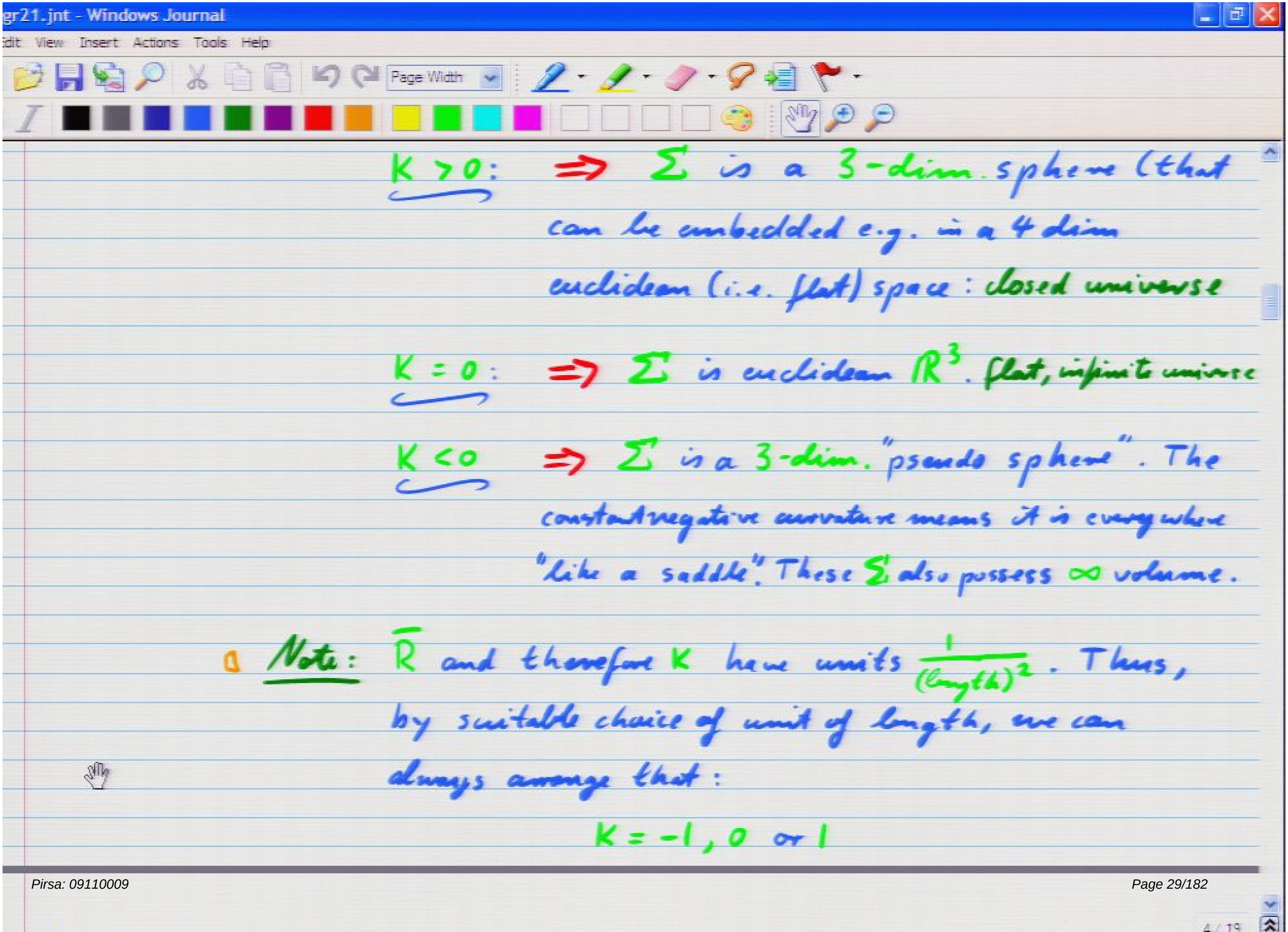

$\underline{K > 0}$: $\Rightarrow$ Σ is a 3-dim. sphere (that can be embedded e.g. in a 4 dim euclidean (i.e. flat) space: closed universe

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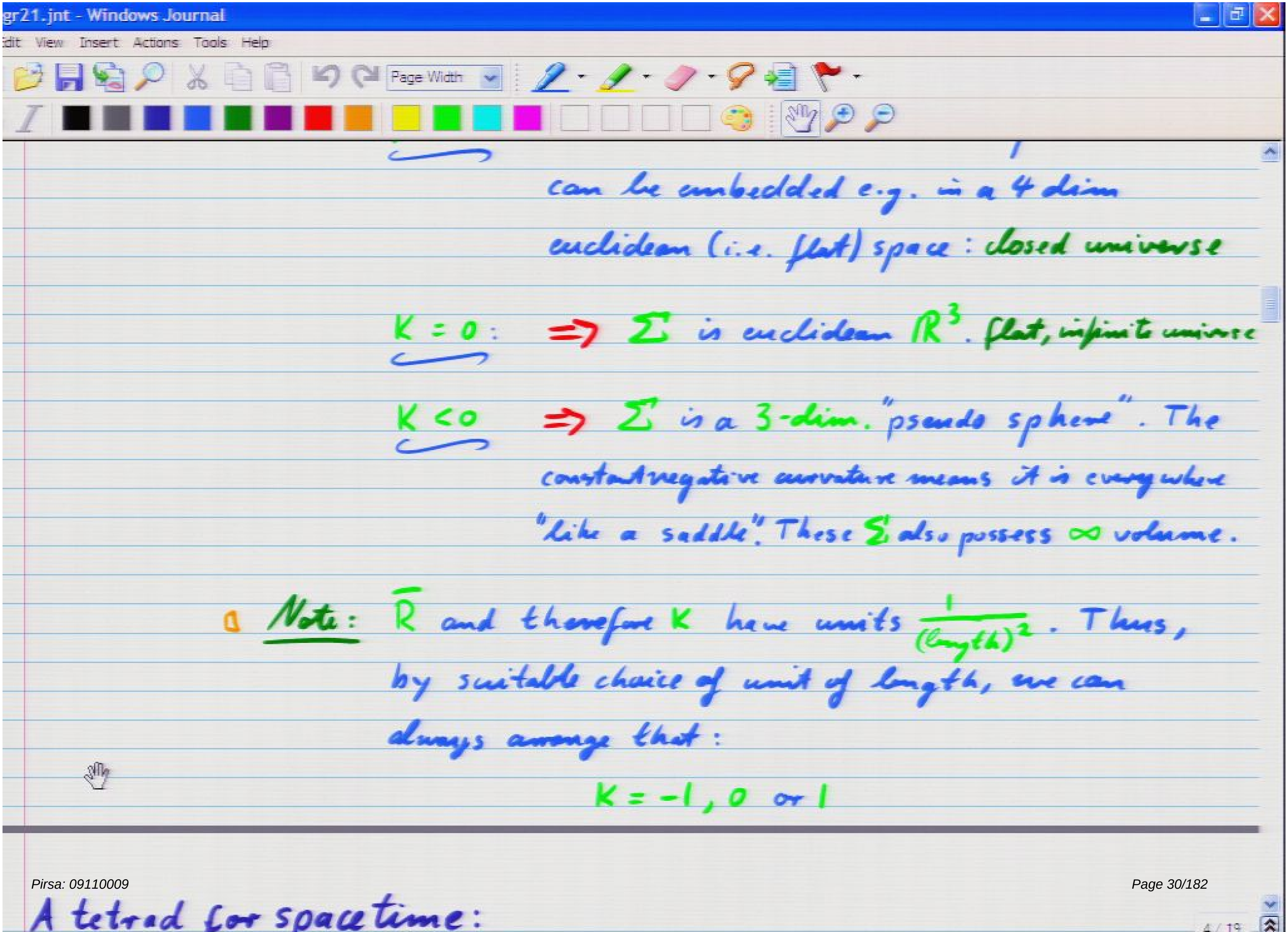

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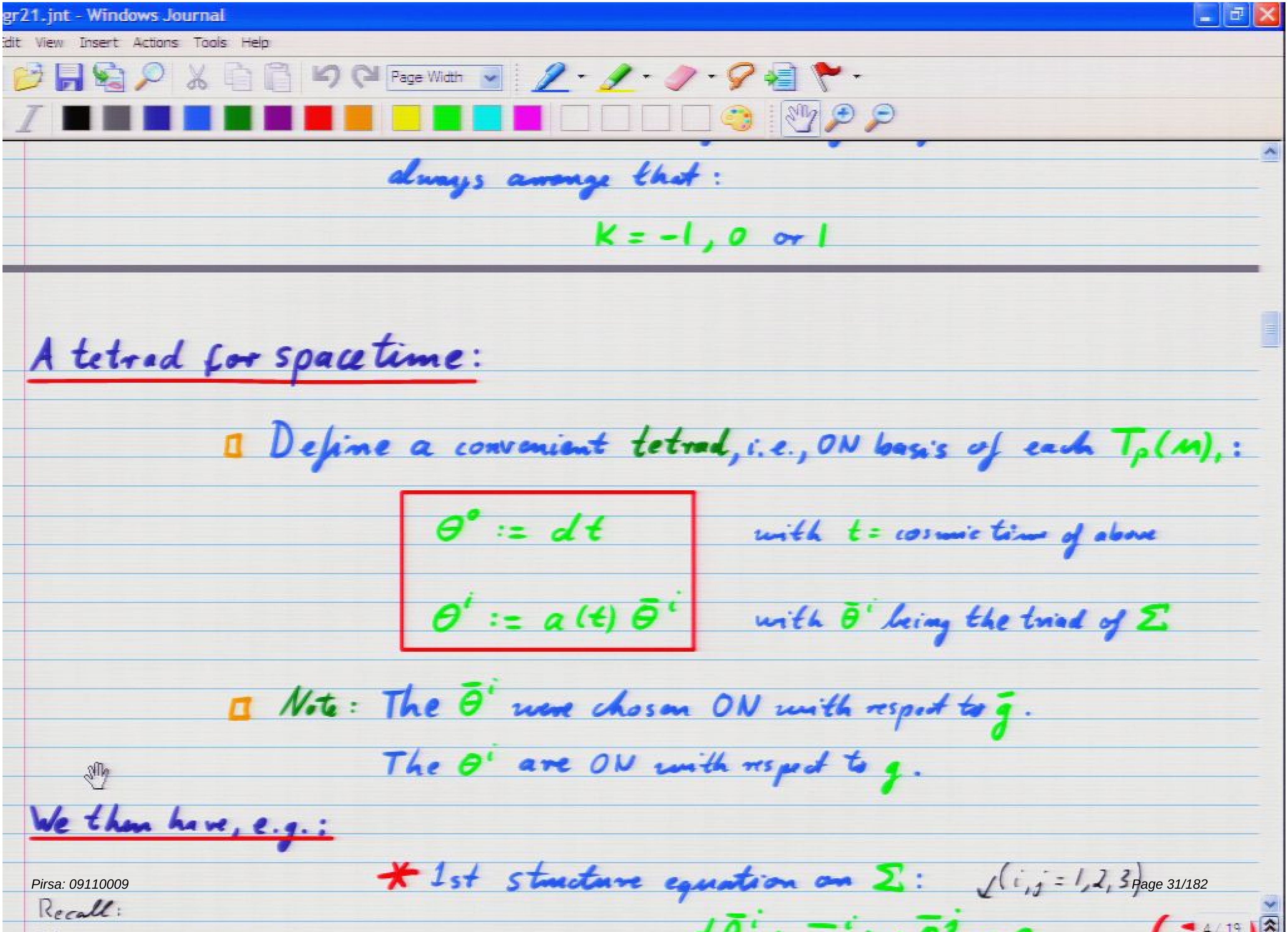

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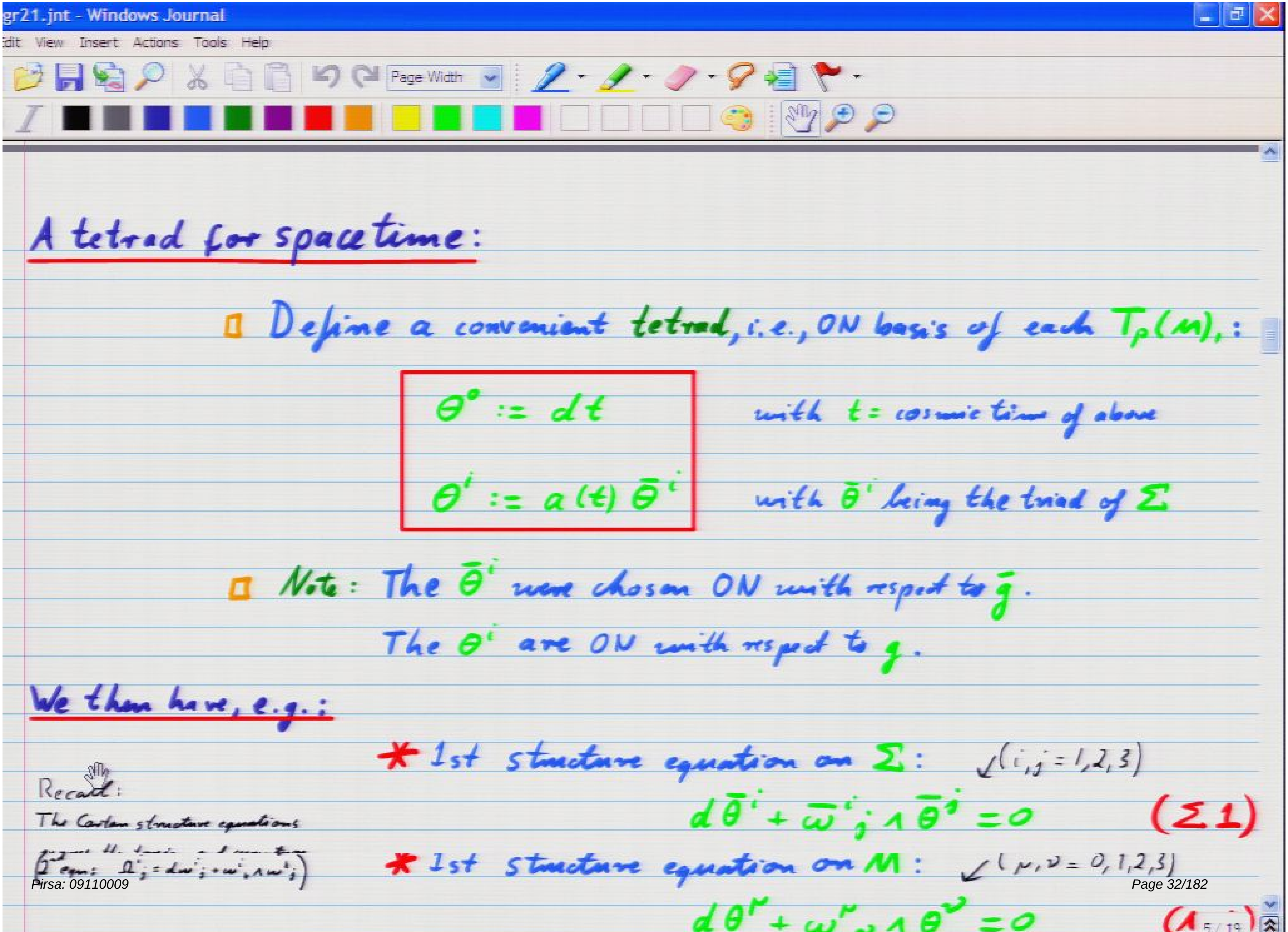

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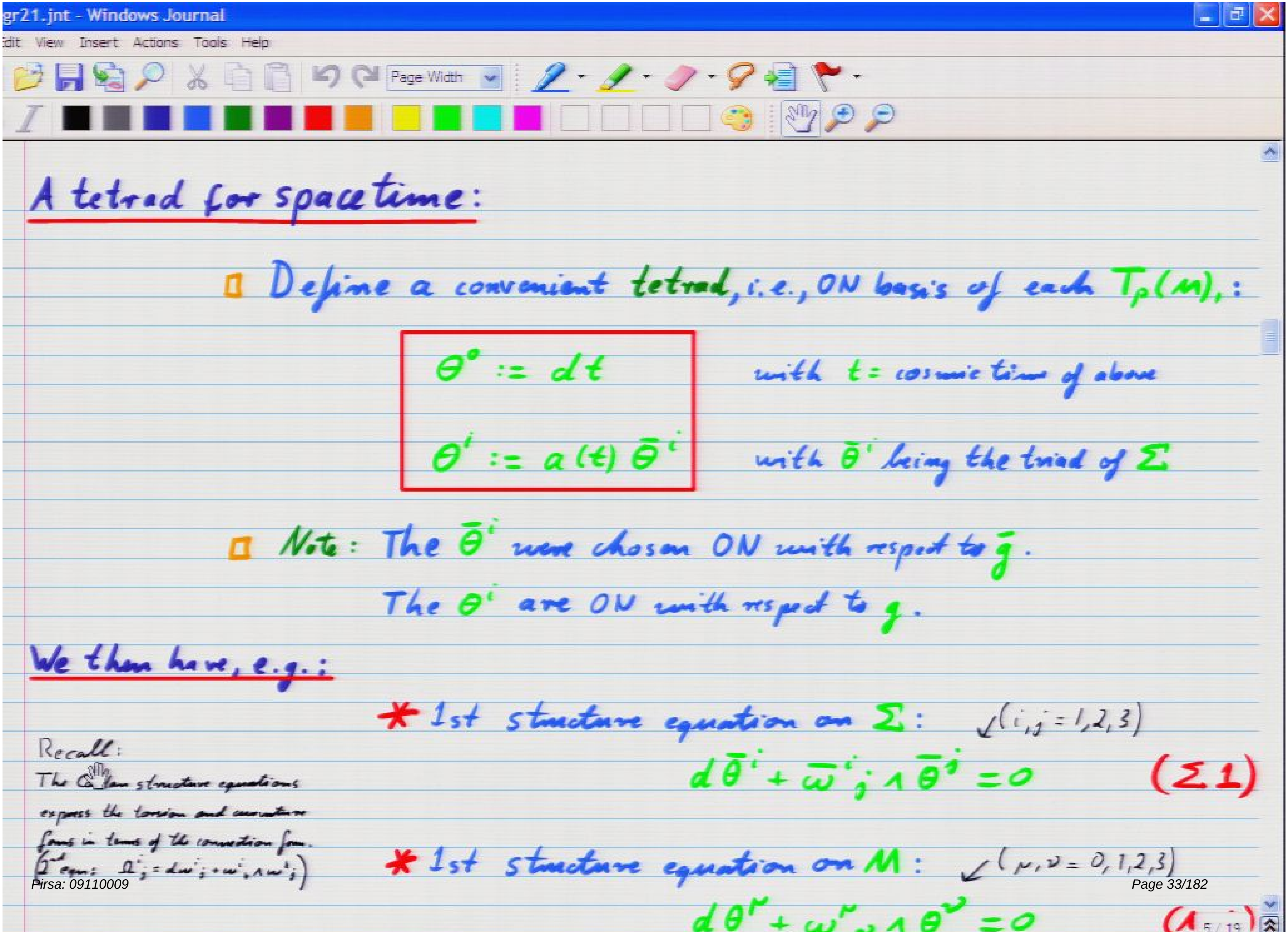

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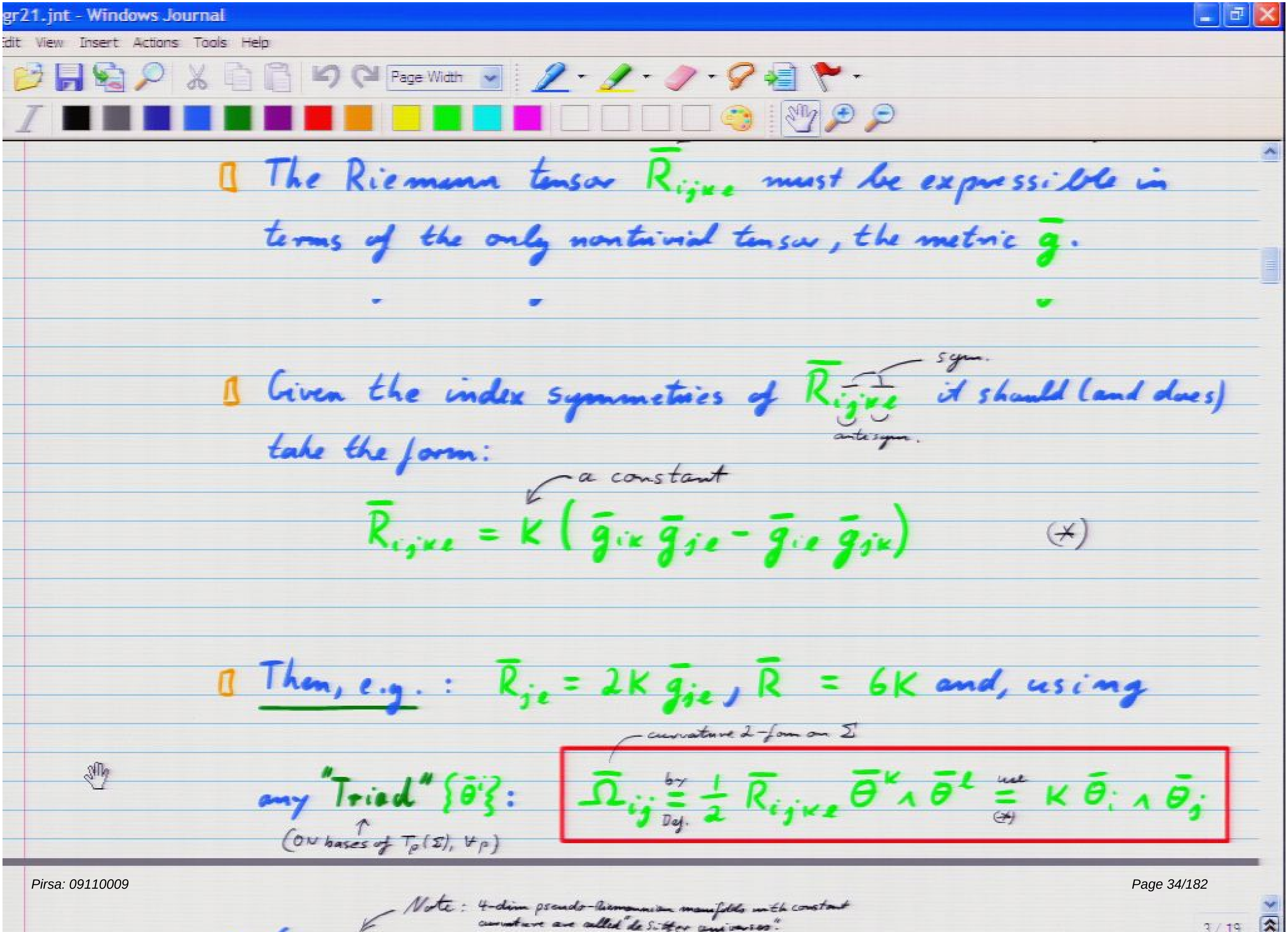


- The Riemann tensor $\bar{R}_{ijkl}$ must be expressible in terms of the only nontrivial tensor, the metric $\bar{g}$.

- Given the index symmetries of $\overset{\text{sym.}}{\bar{R}_{\underbrace{ij}_{\text{antisym.}}\underbrace{kl}}}$ it should (and does) take the form:

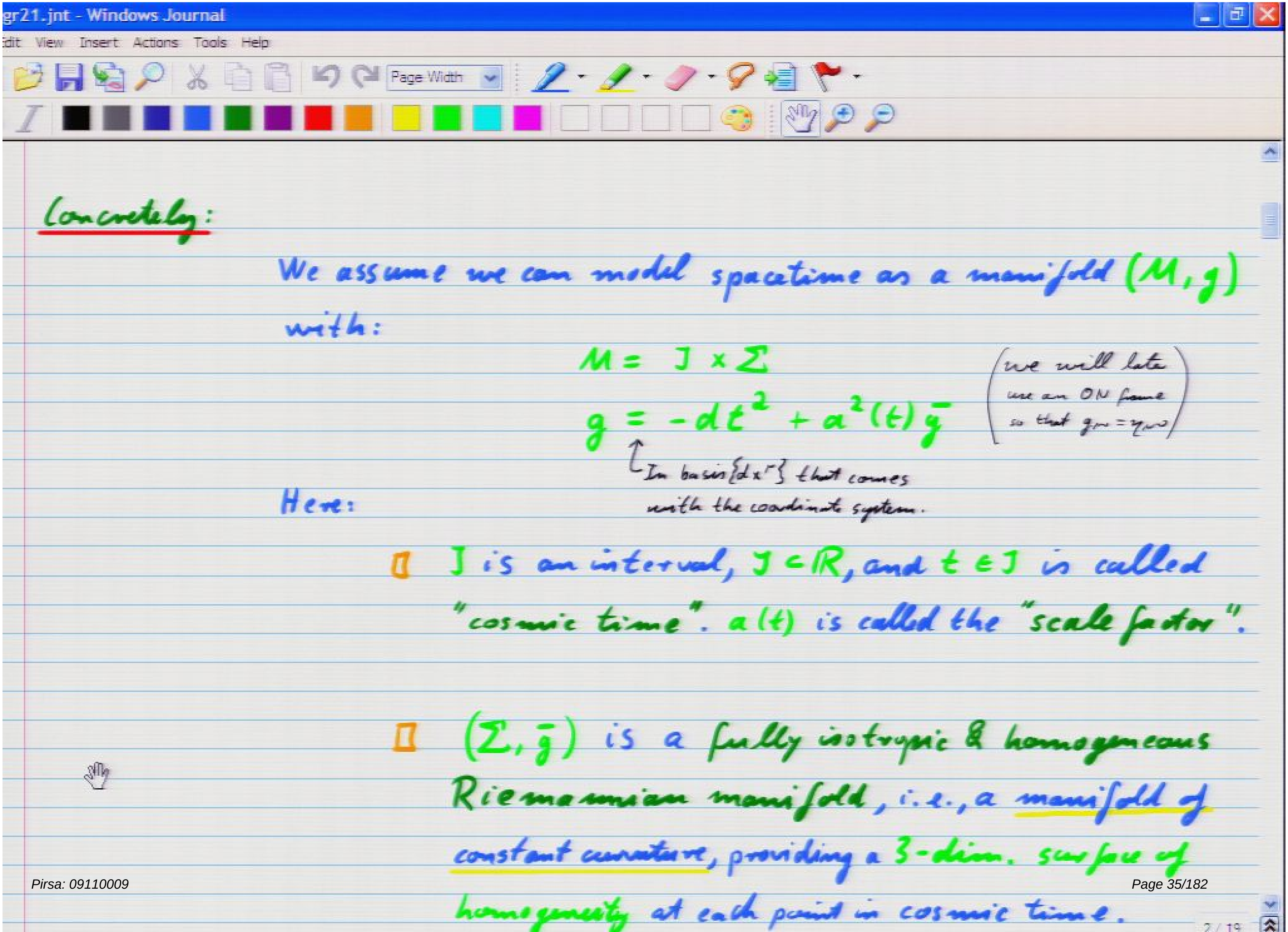
$$\bar{R}_{ijkl} = \overset{\text{a constant}}{K}\left(\bar{g}_{ik}\bar{g}_{jl} - \bar{g}_{il}\bar{g}_{jk}\right) \qquad (*)$$

- Then, e.g.: $\bar{R}_{jl} = 2K\,\bar{g}_{jl}$, $\bar{R} = 6K$ and, using any "Triad" $\{\bar{\theta}^i\}$:

(ON bases of $T_p(\Sigma)$, $\forall p$)

$$\boxed{\overset{\text{curvature 2-form on }\Sigma}{\bar{\Omega}_{ij}} \underset{\text{Def.}}{\overset{\text{by}}{=}} \frac{1}{2}\bar{R}_{ijkl}\,\bar{\theta}^k \wedge \bar{\theta}^l \underset{(*)}{\overset{\text{use}}{=}} K\,\bar{\theta}_i \wedge \bar{\theta}_j}$$

Note: 4-dim pseudo-Riemannian manifolds with constant curvature are called "de Sitter universes".



Concretely:

We assume we can model spacetime as a manifold (M, g) with:

$$M = J \times \Sigma$$

$$g = -dt^2 + a^2(t)\,\bar{g}$$

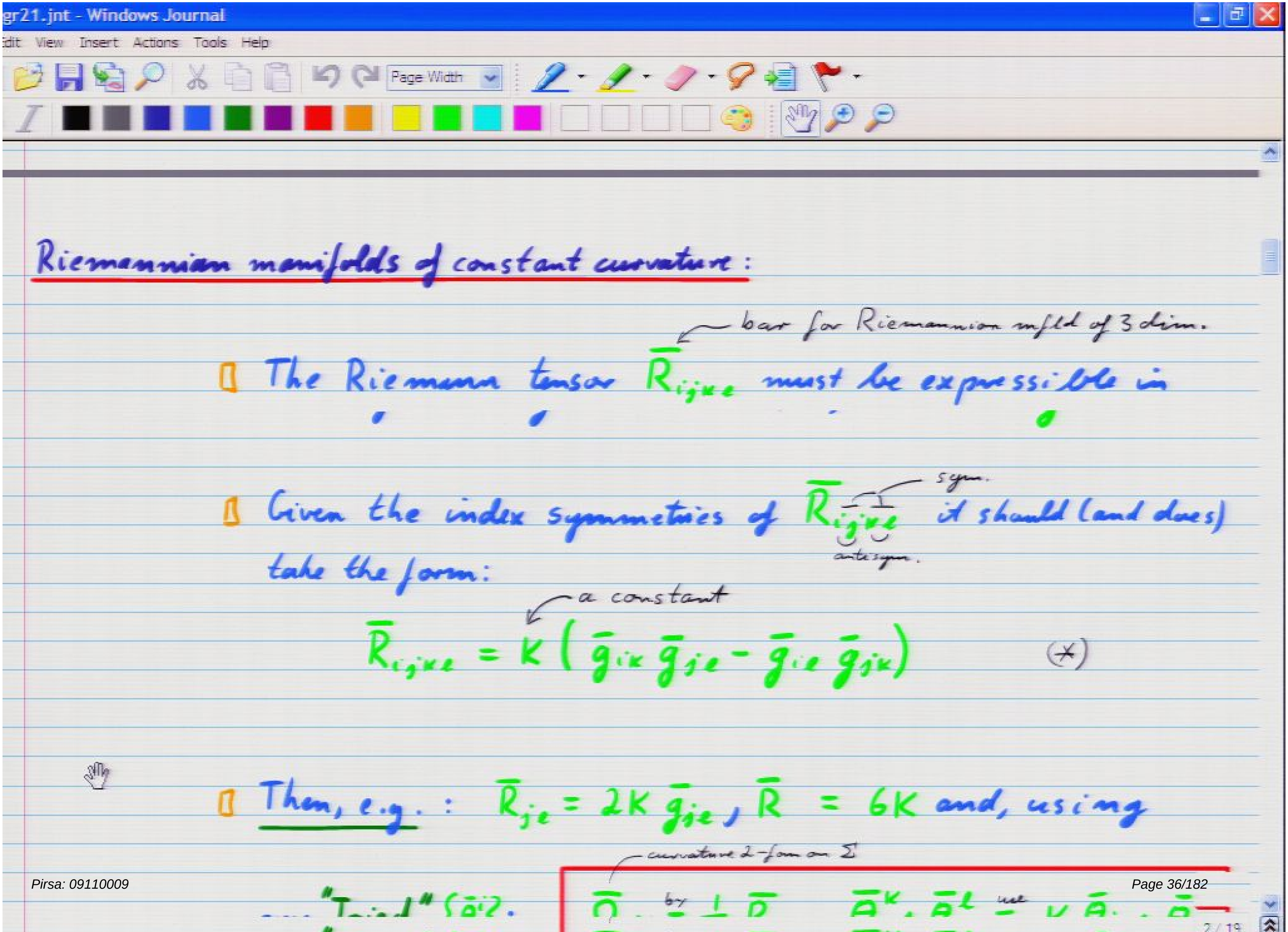
$\uparrow$ In basis $\{dx^\mu\}$ that comes with the coordinate system.

(we will late use an ON frame so that $g_{\mu\nu} = \eta_{\mu\nu}$)

Here:

- J is an interval, $J \subset \mathbb{R}$, and $t \in J$ is called "cosmic time". $a(t)$ is called the "scale factor".

- $(\Sigma, \bar{g})$ is a fully isotropic & homogeneous Riemannian manifold, i.e., a manifold of constant curvature, providing a 3-dim. surface of homogeneity at each point in cosmic time.



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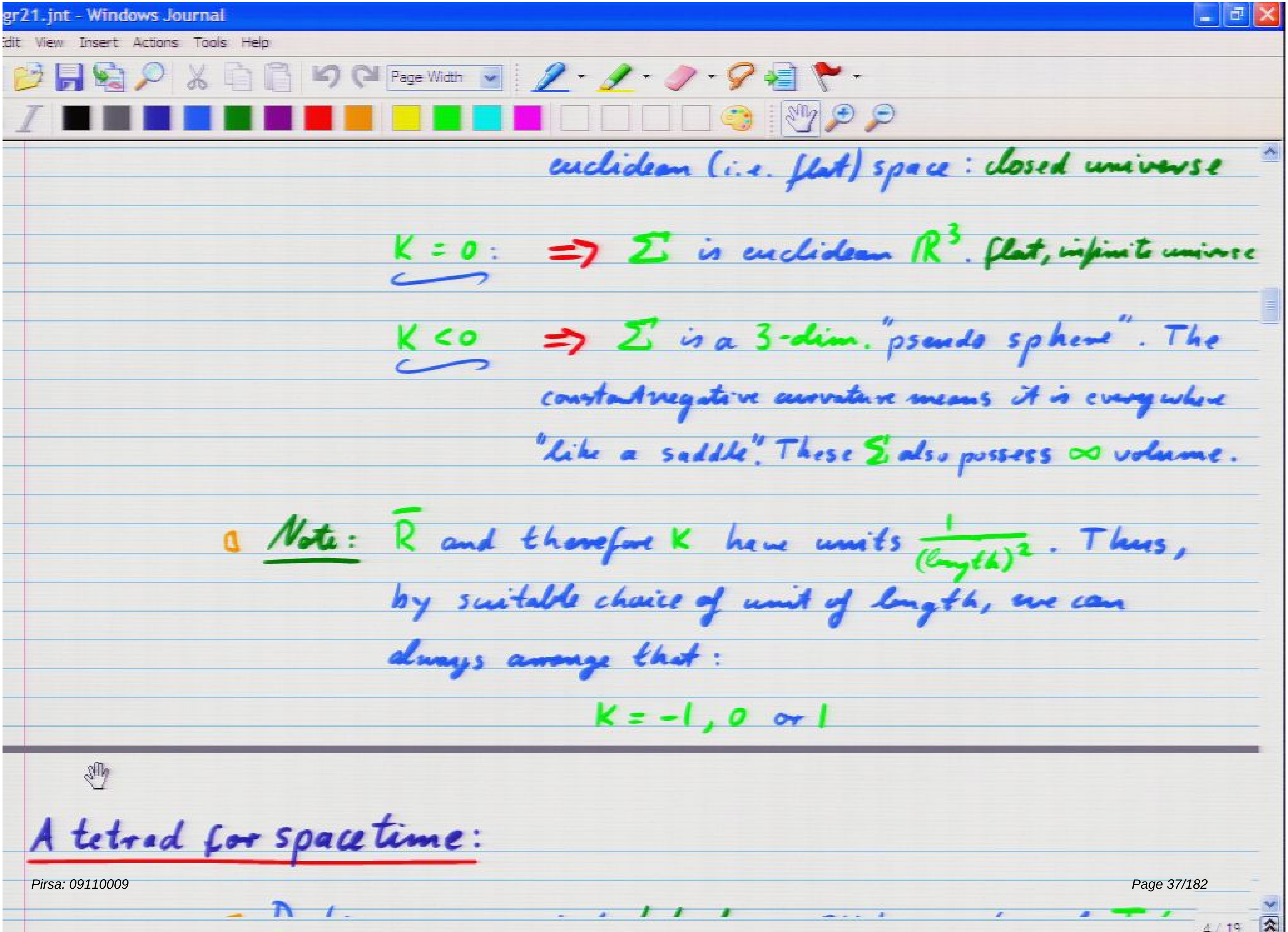

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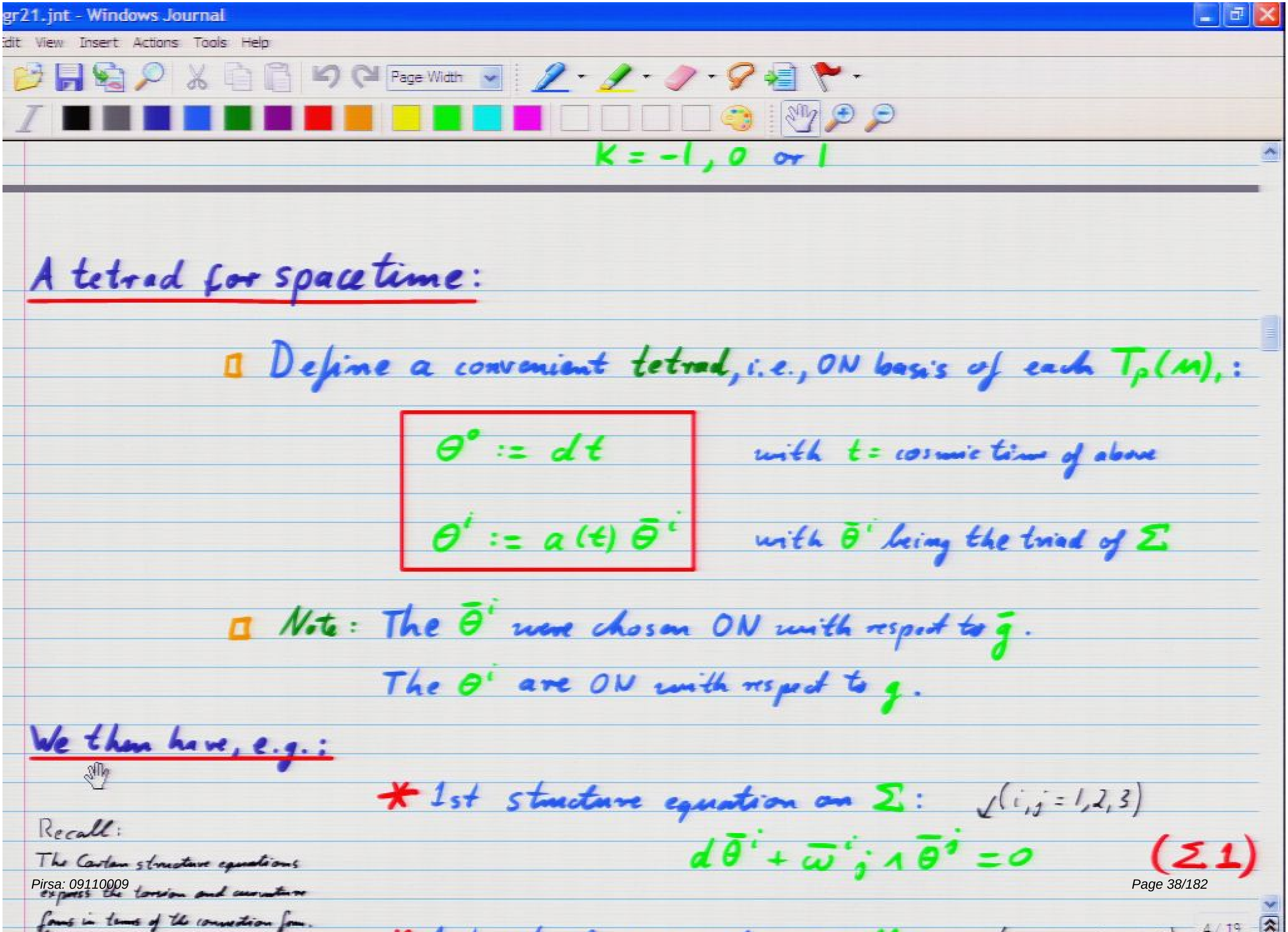

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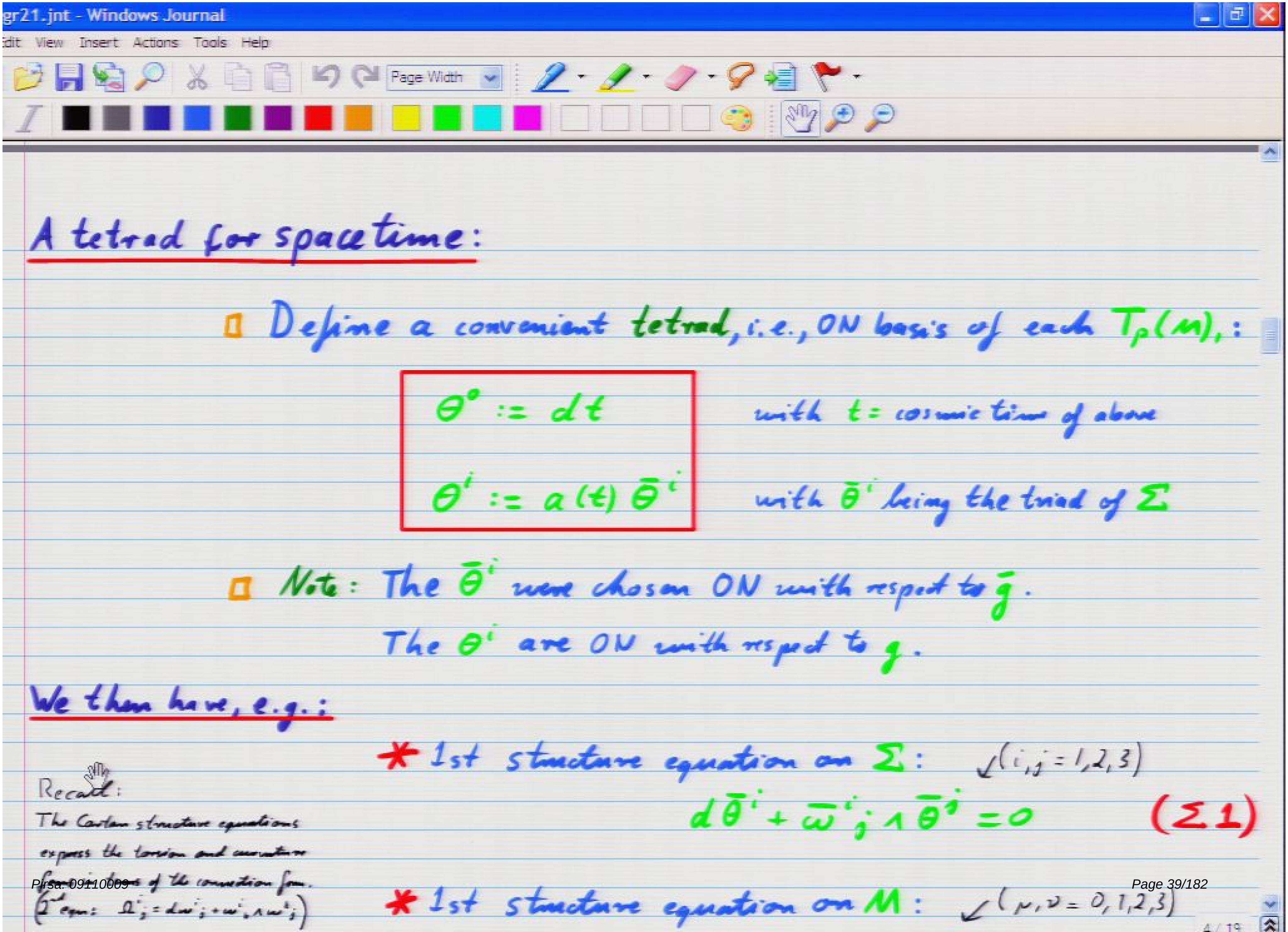

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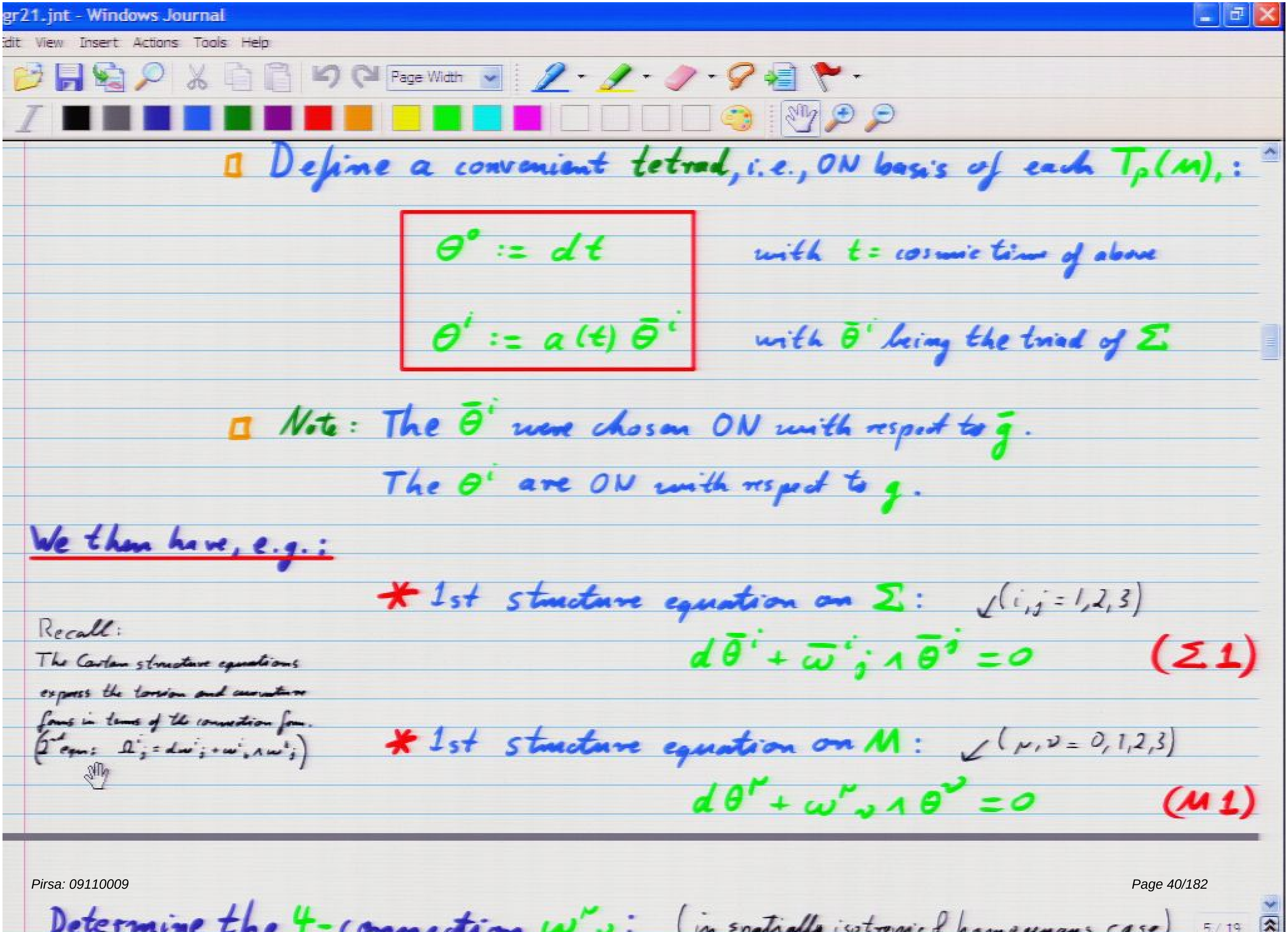

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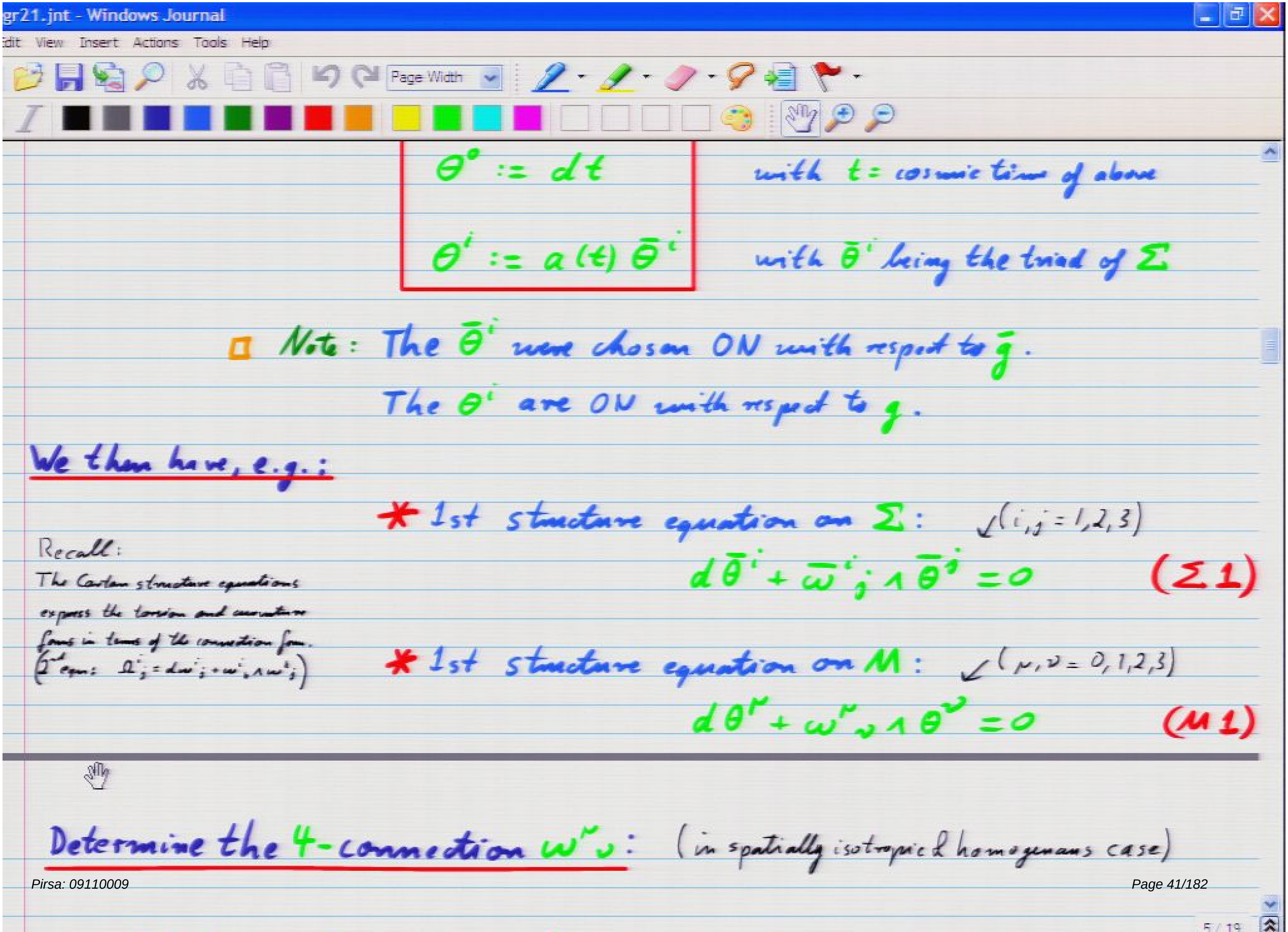

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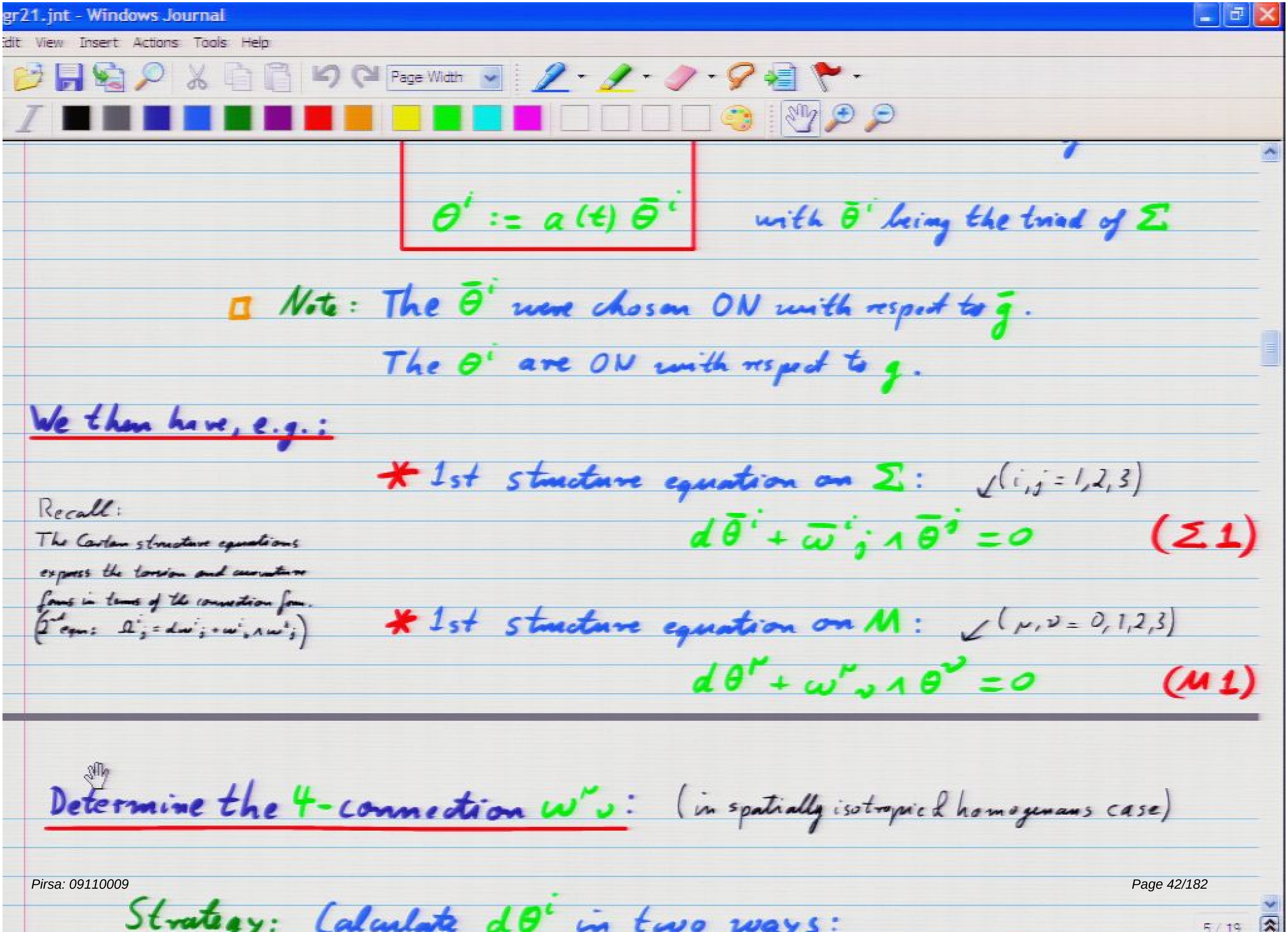

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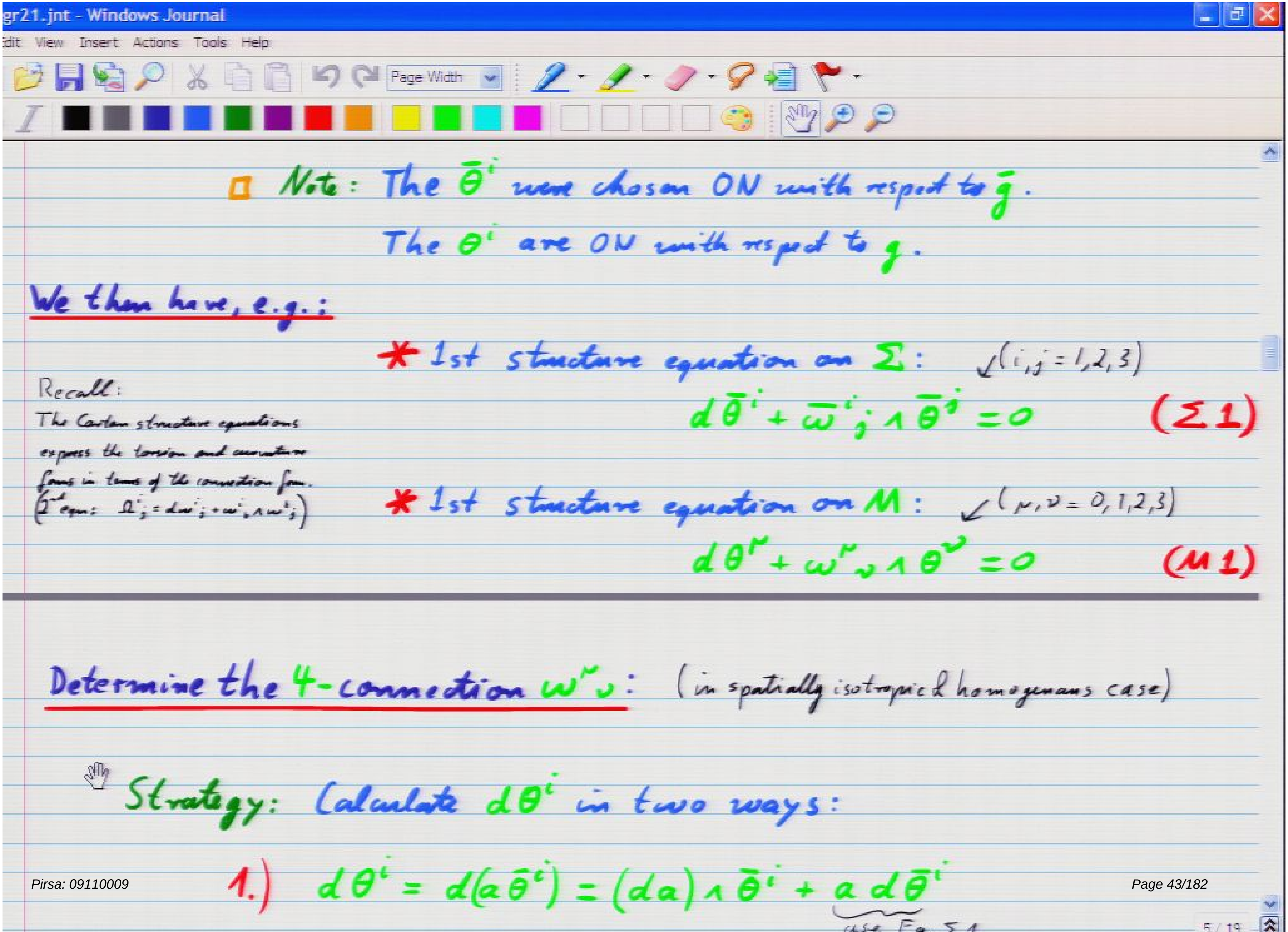

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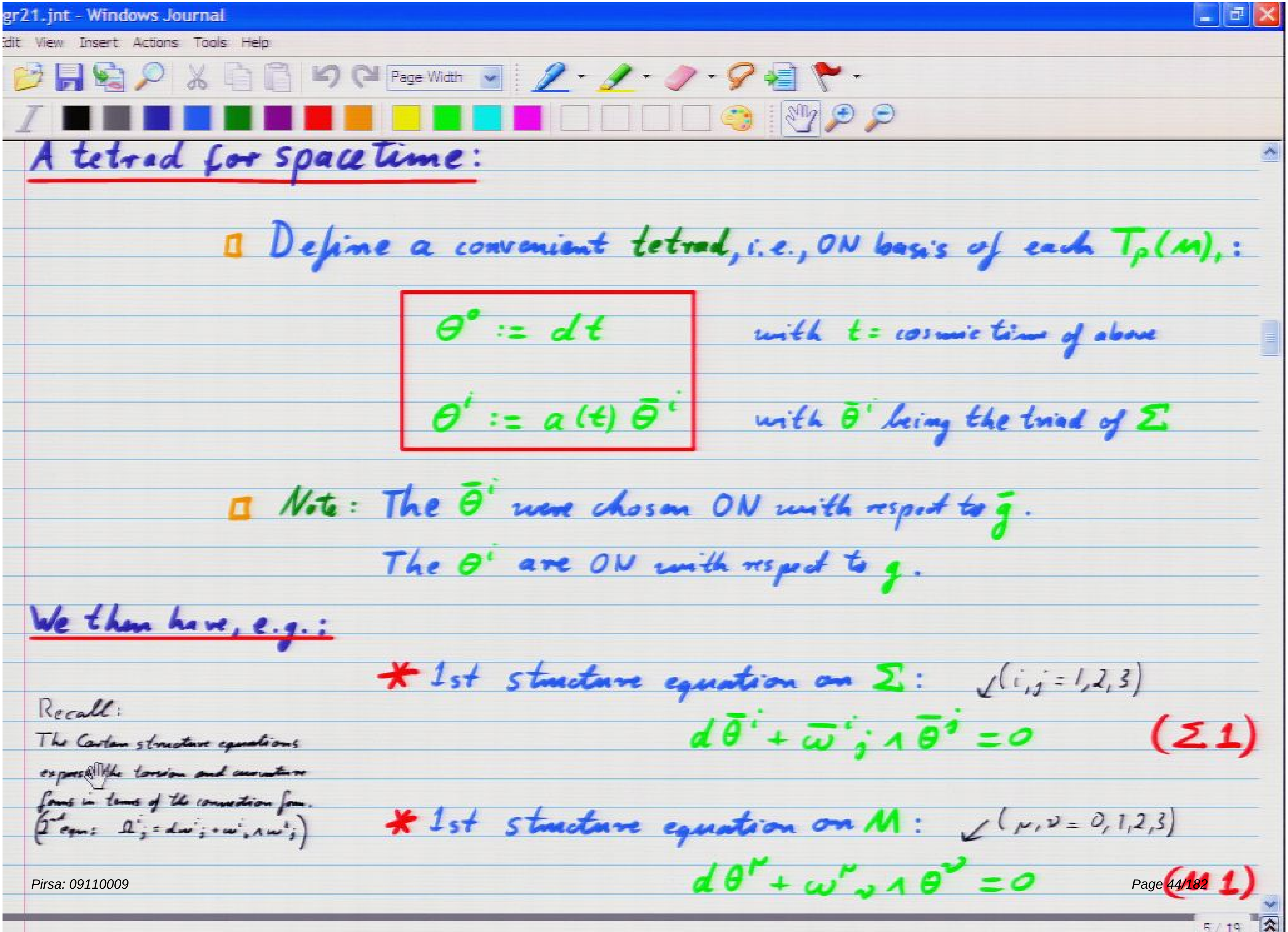

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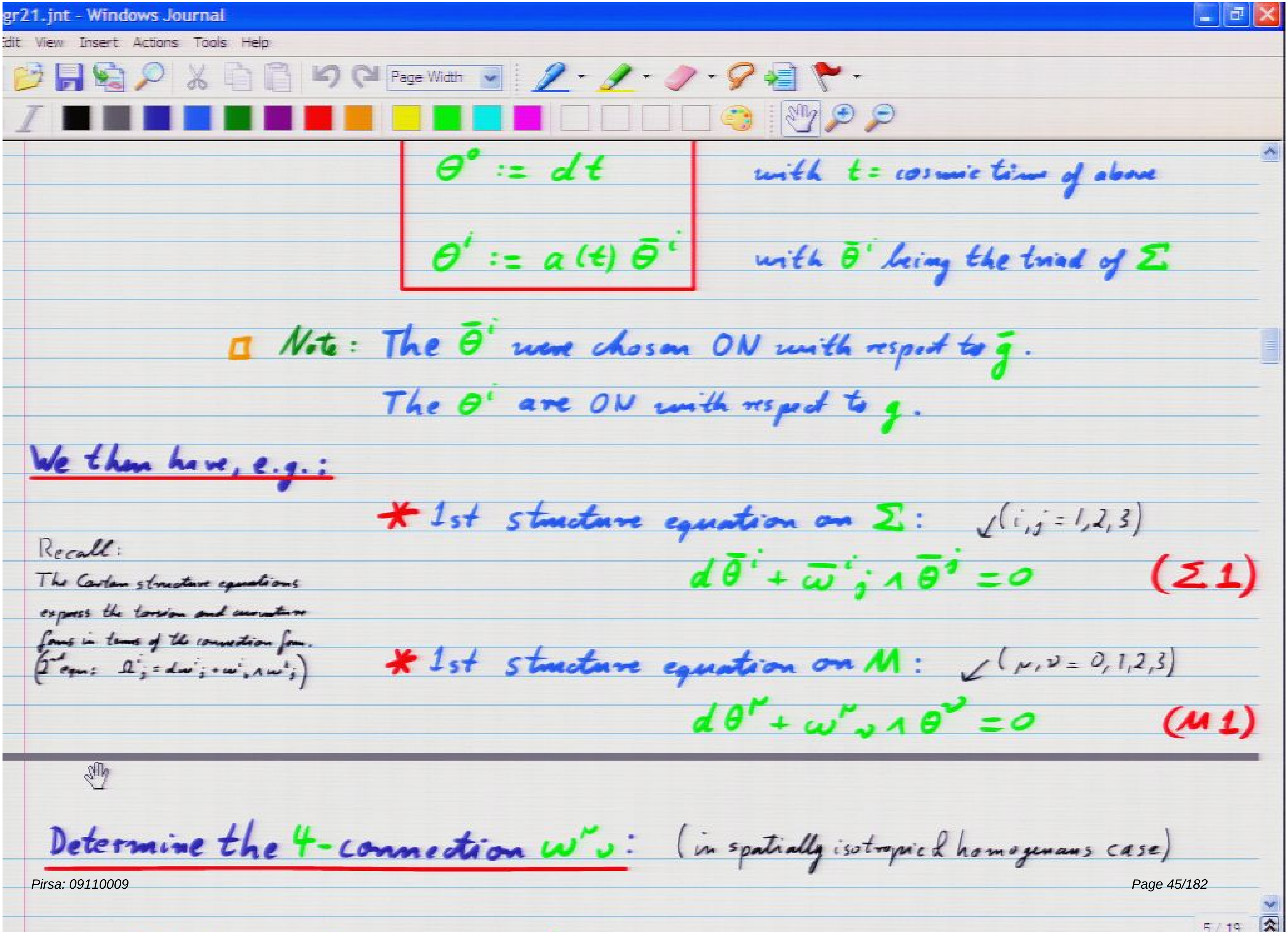

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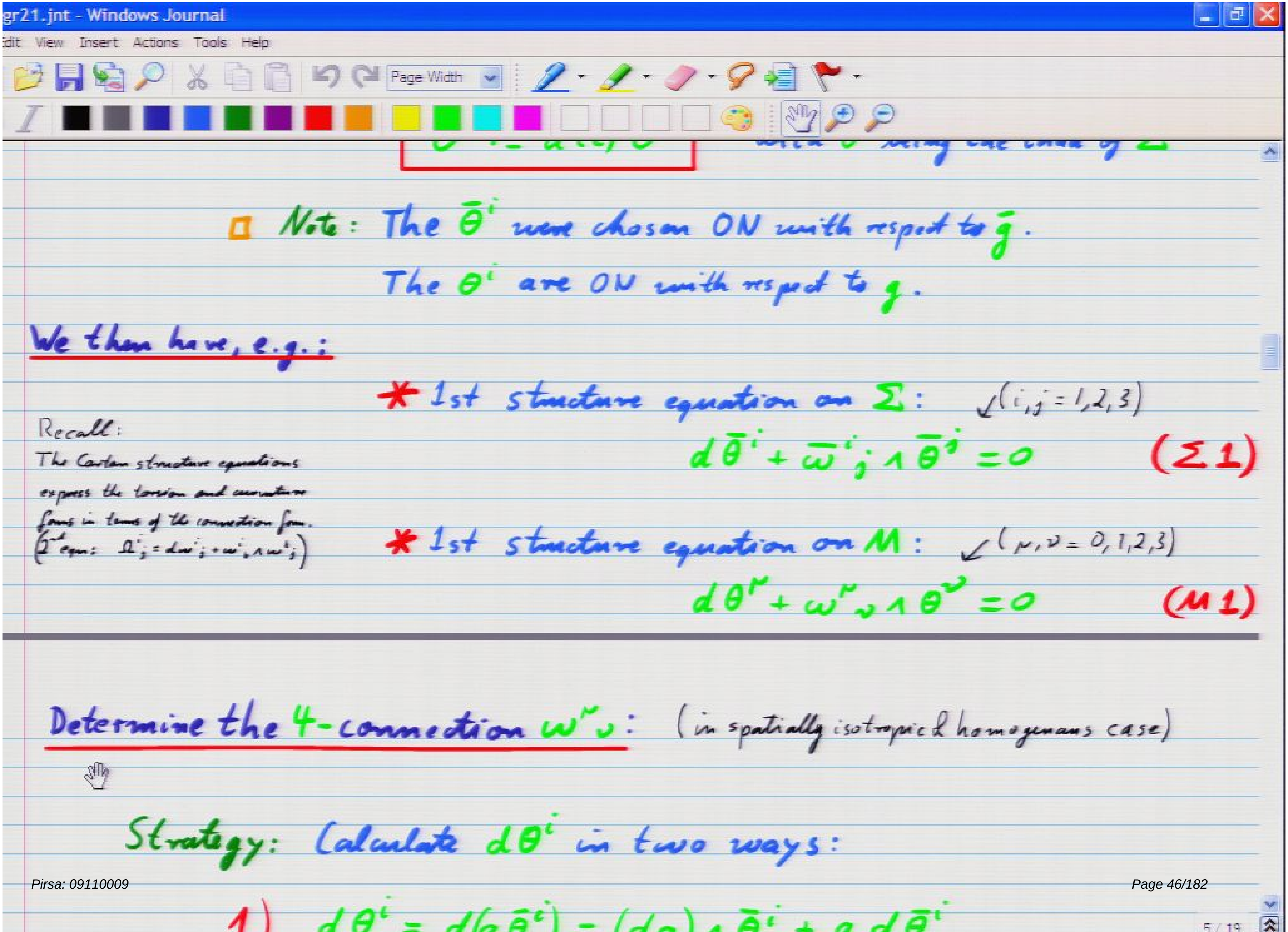

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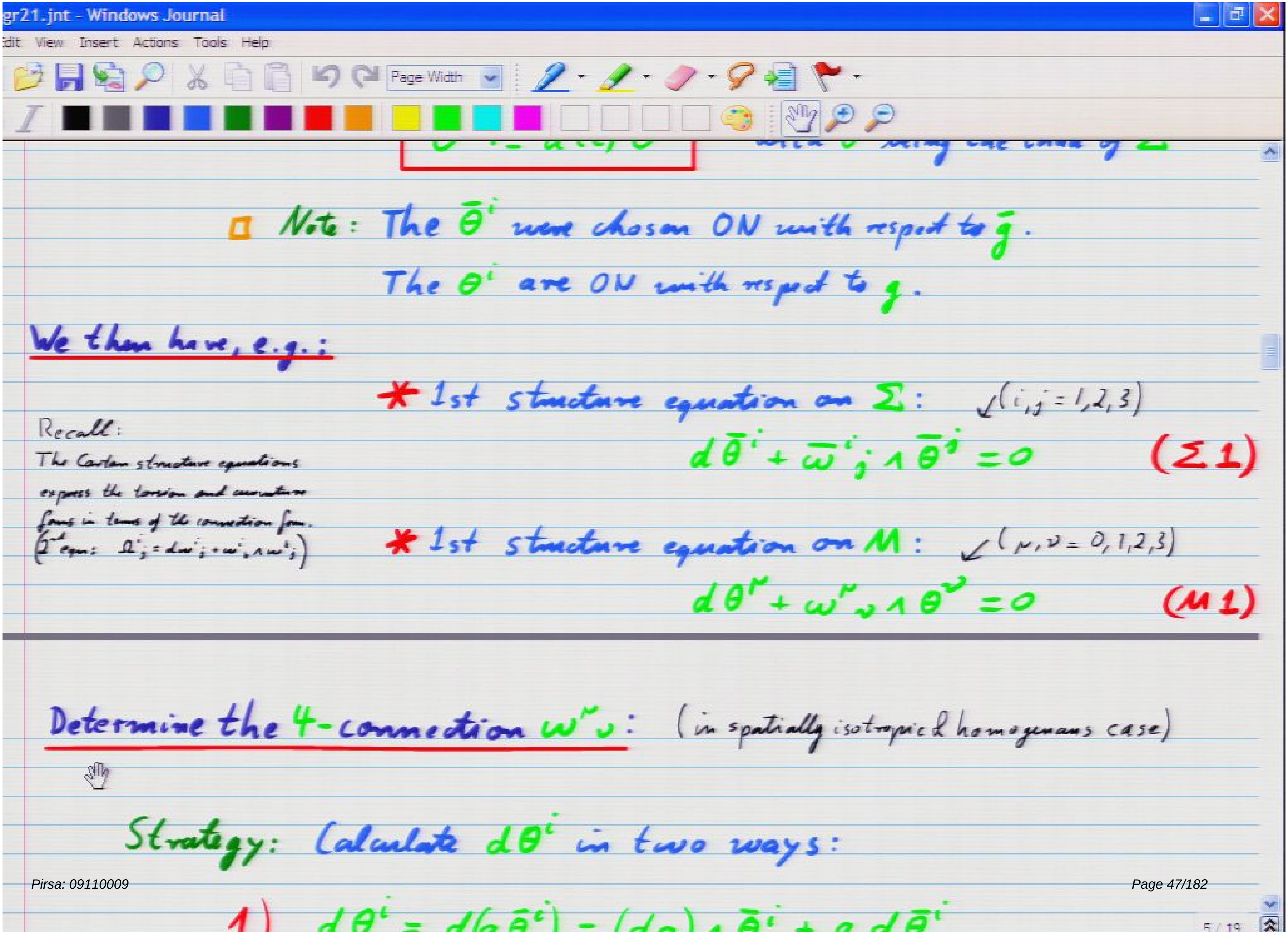

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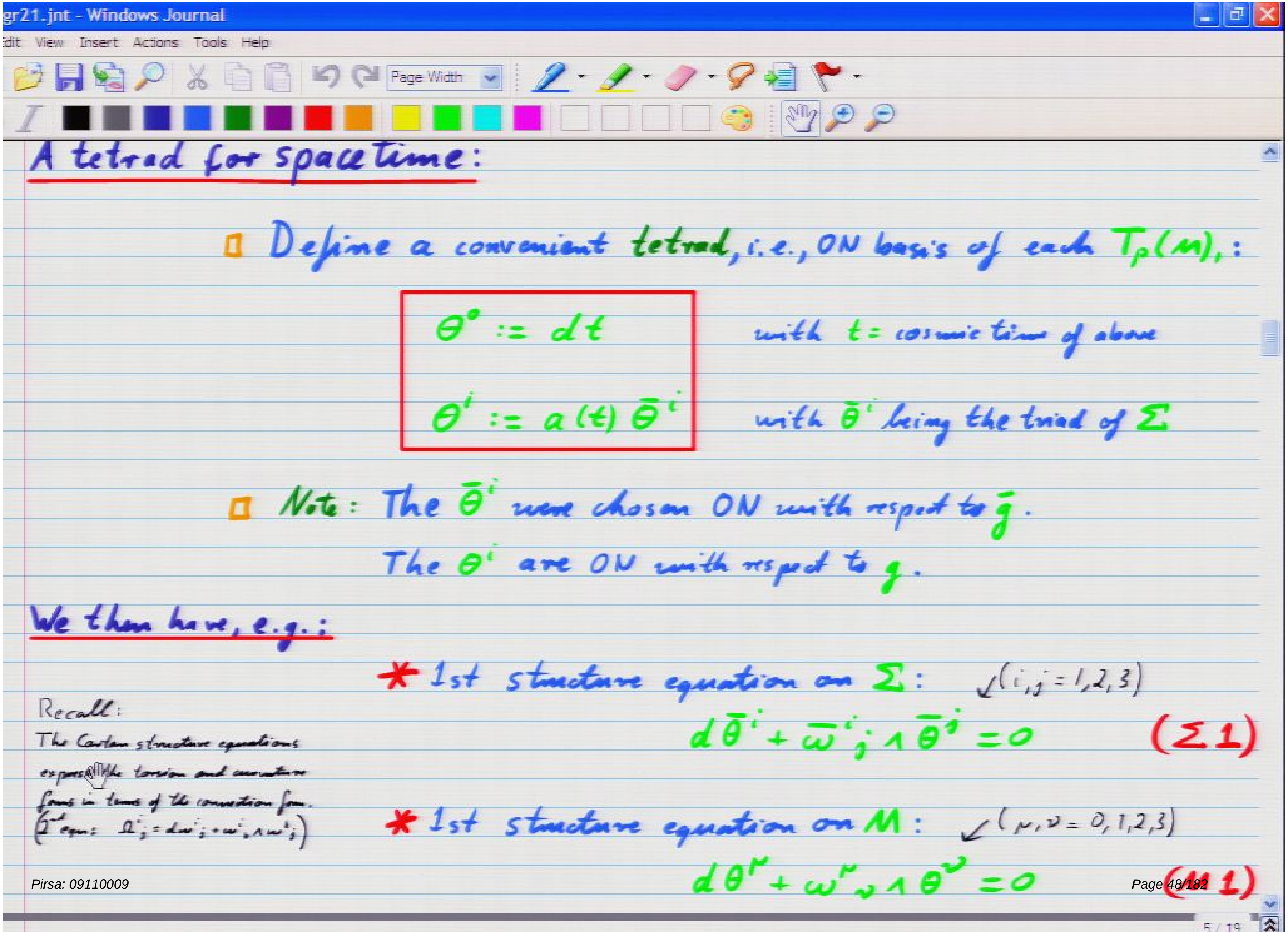

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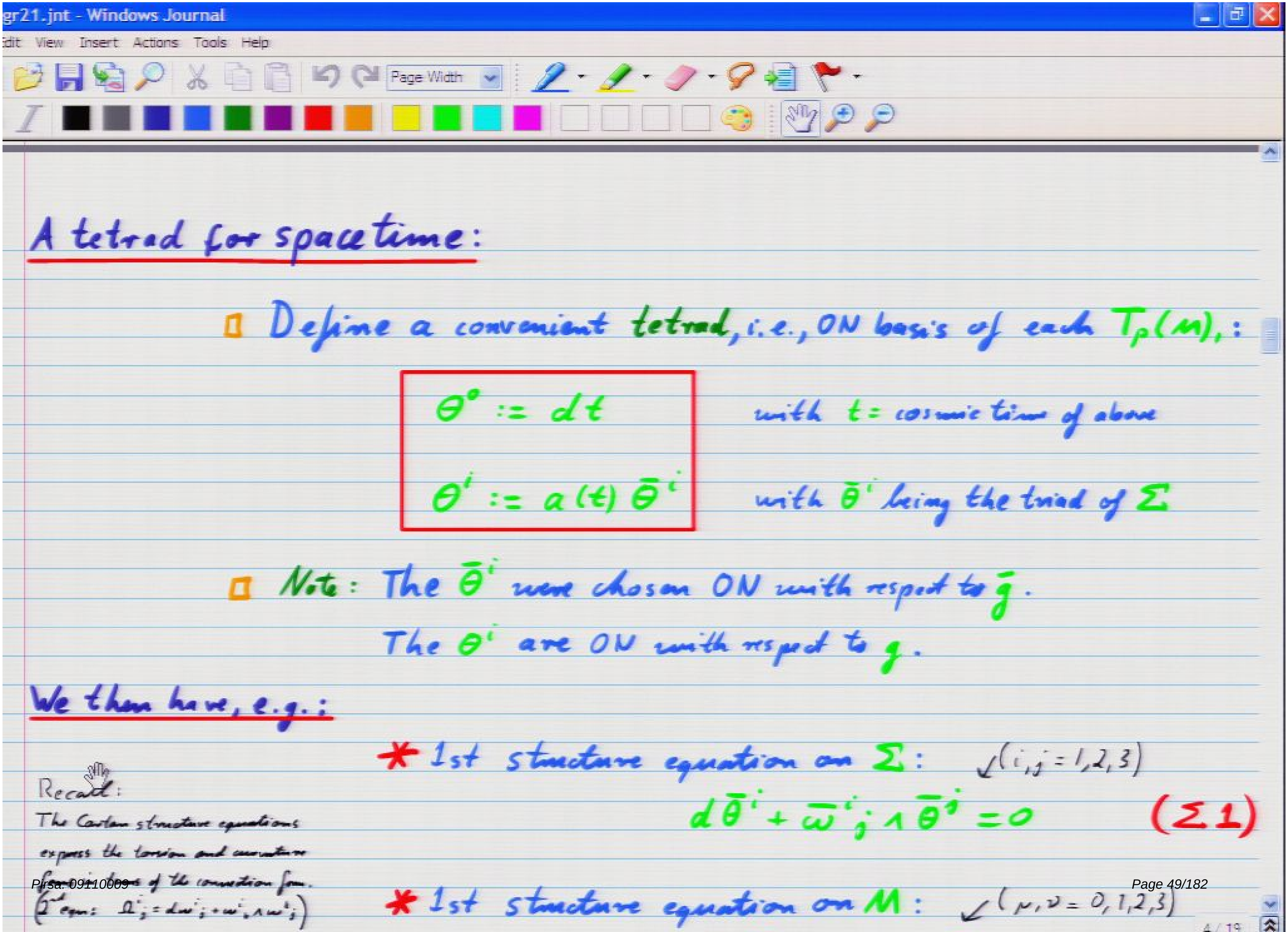

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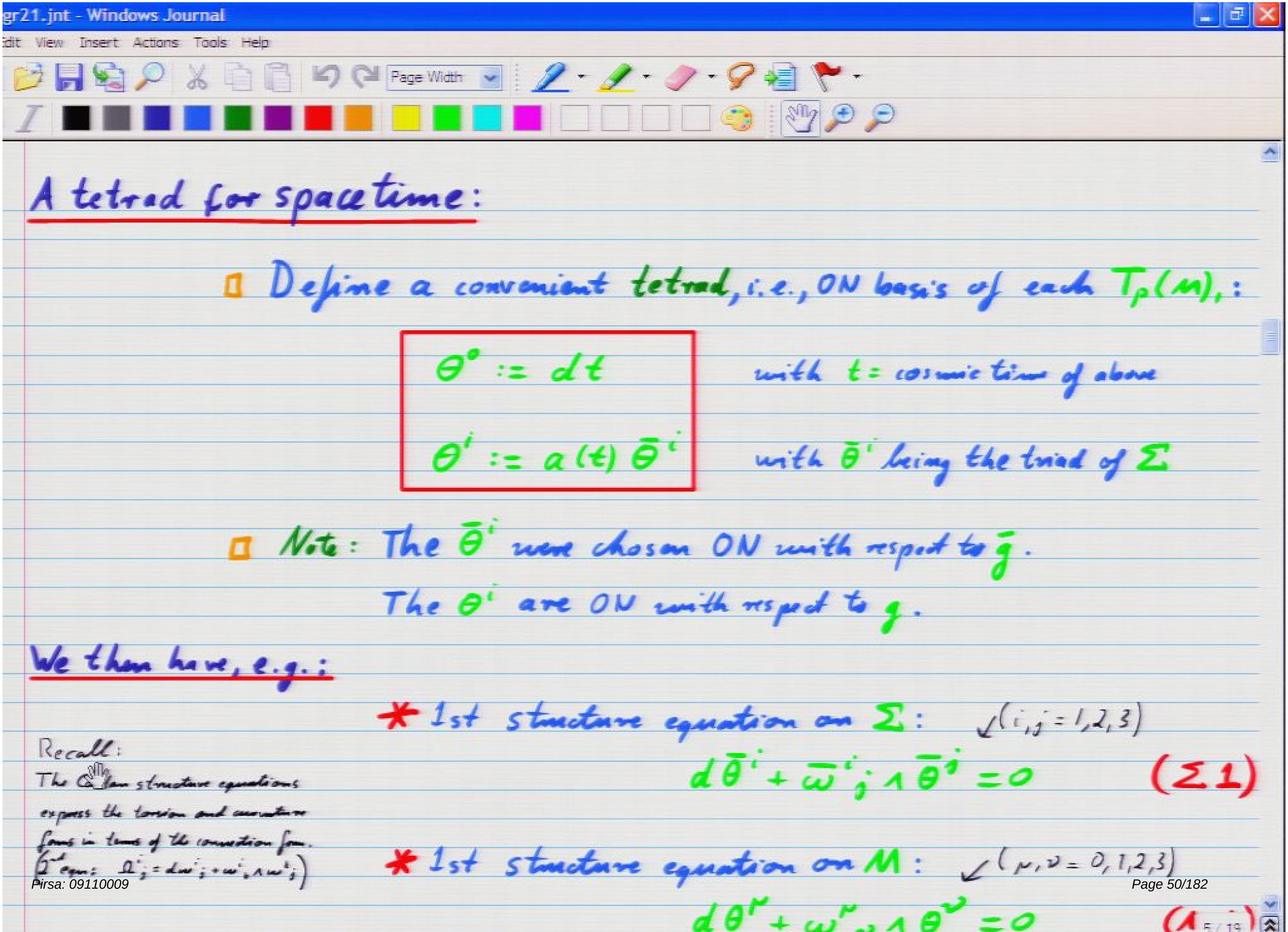

A tetrad for spacetime:

☐ Define a convenient tetrad, i.e., ON basis of each $T_p(M)$, :

$$\boxed{\begin{aligned}\theta^0 &:= dt\\ \theta^i &:= a(t)\,\bar{\theta}^i\end{aligned}}$$

with $t =$ cosmic time of above

with $\bar{\theta}^i$ being the triad of Σ

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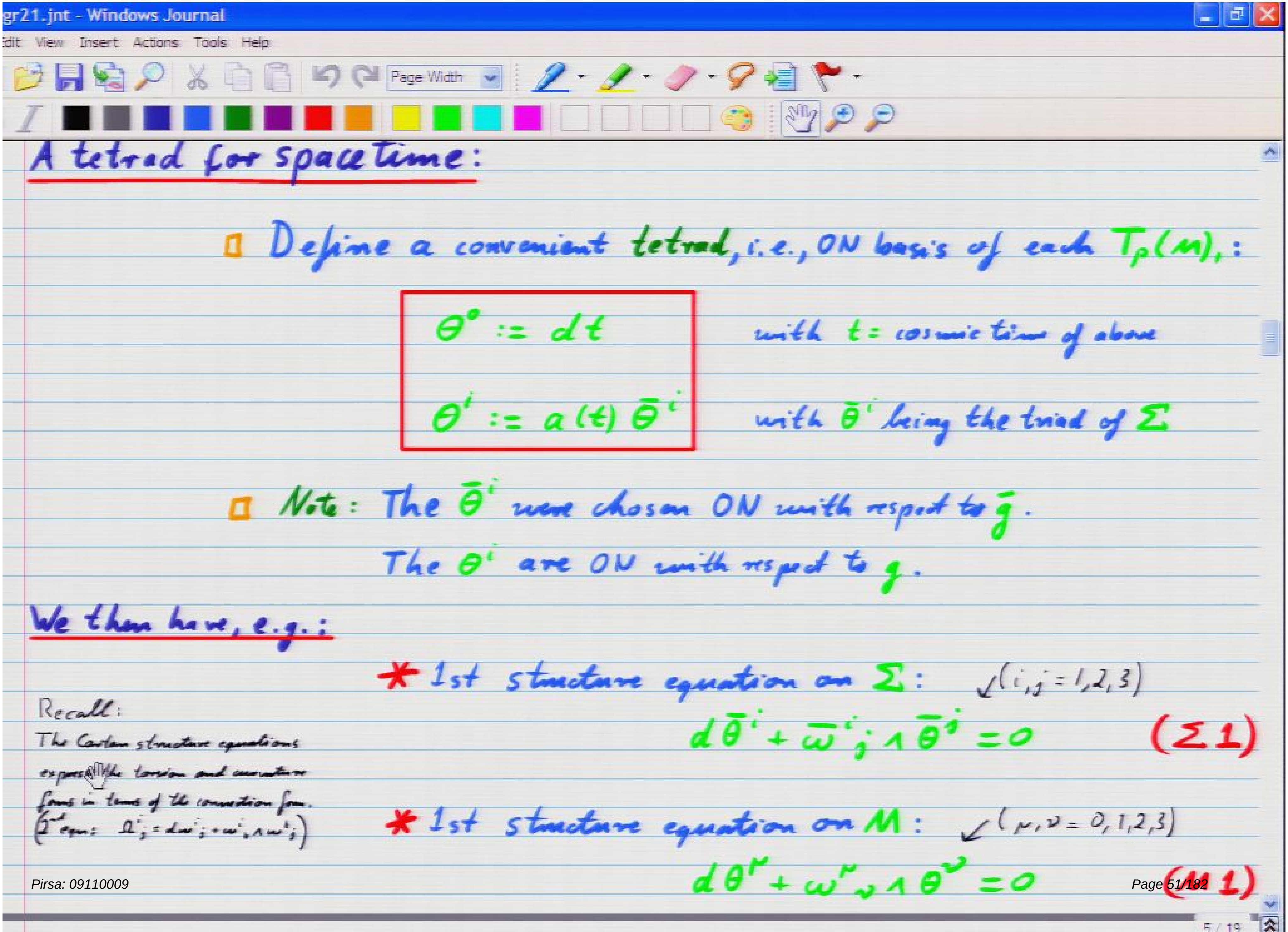

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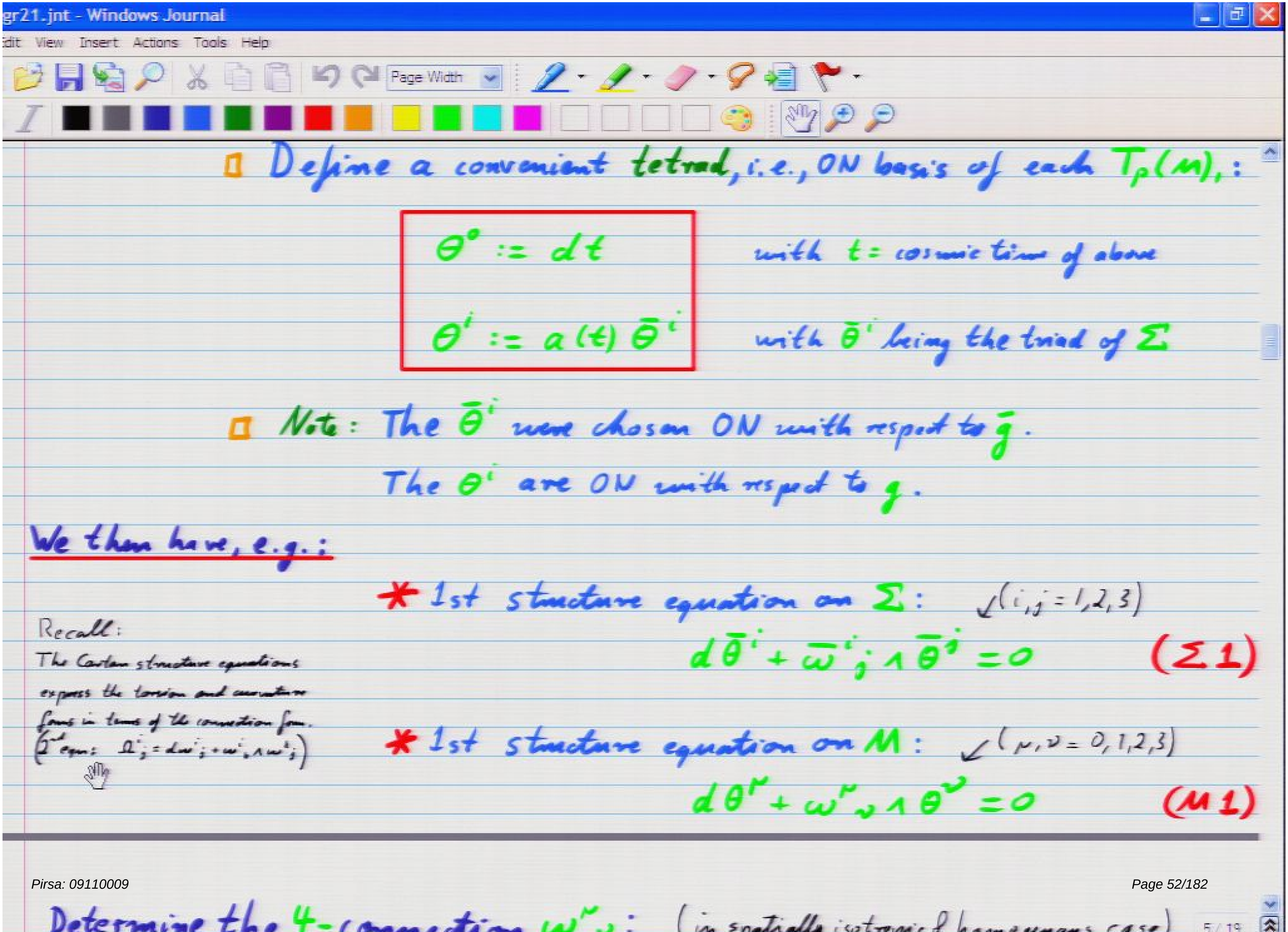

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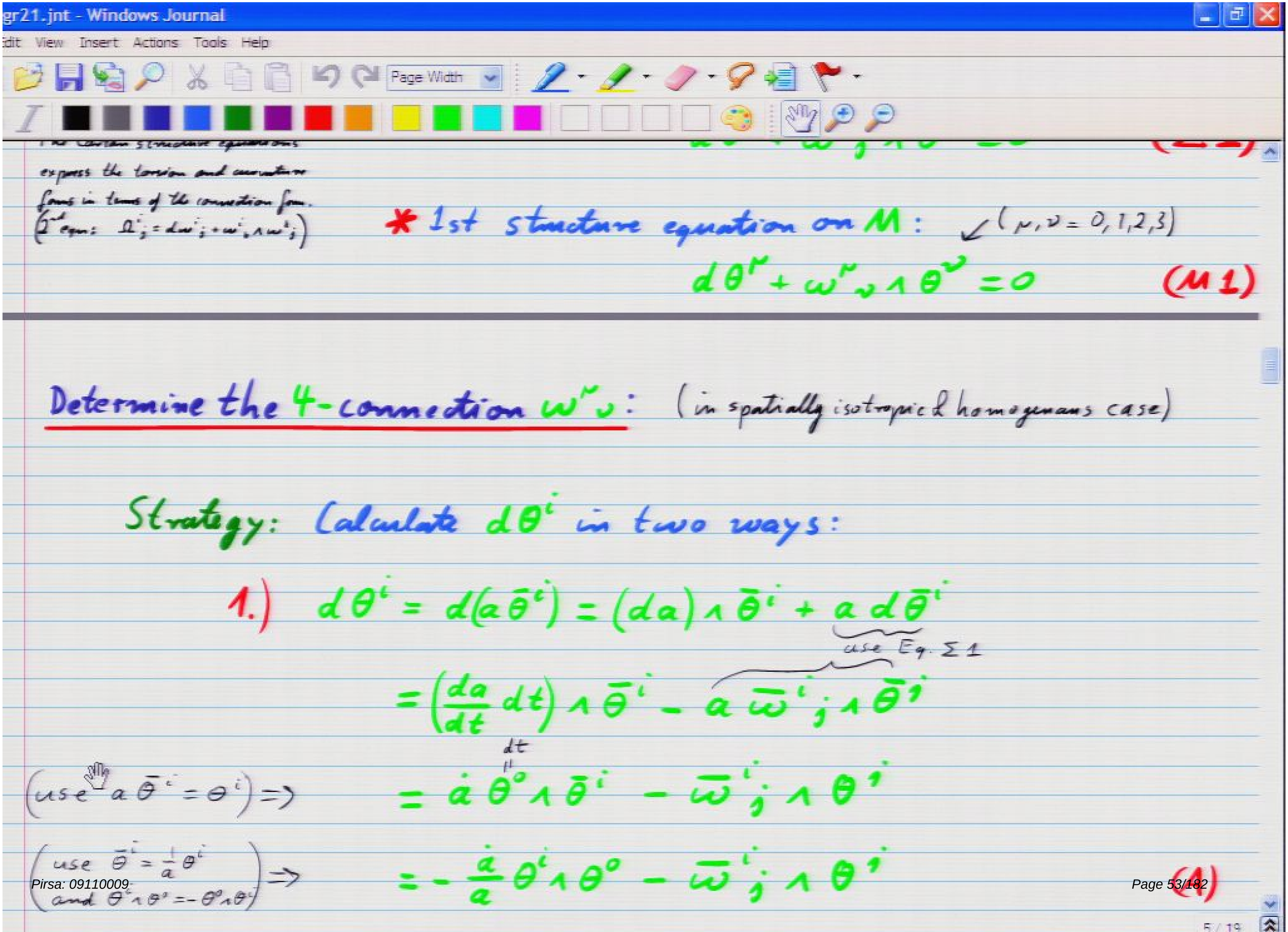
Determine the 4-connection $\omega^\mu{}_\nu$: (in spatially isotropic & homogeneous case)



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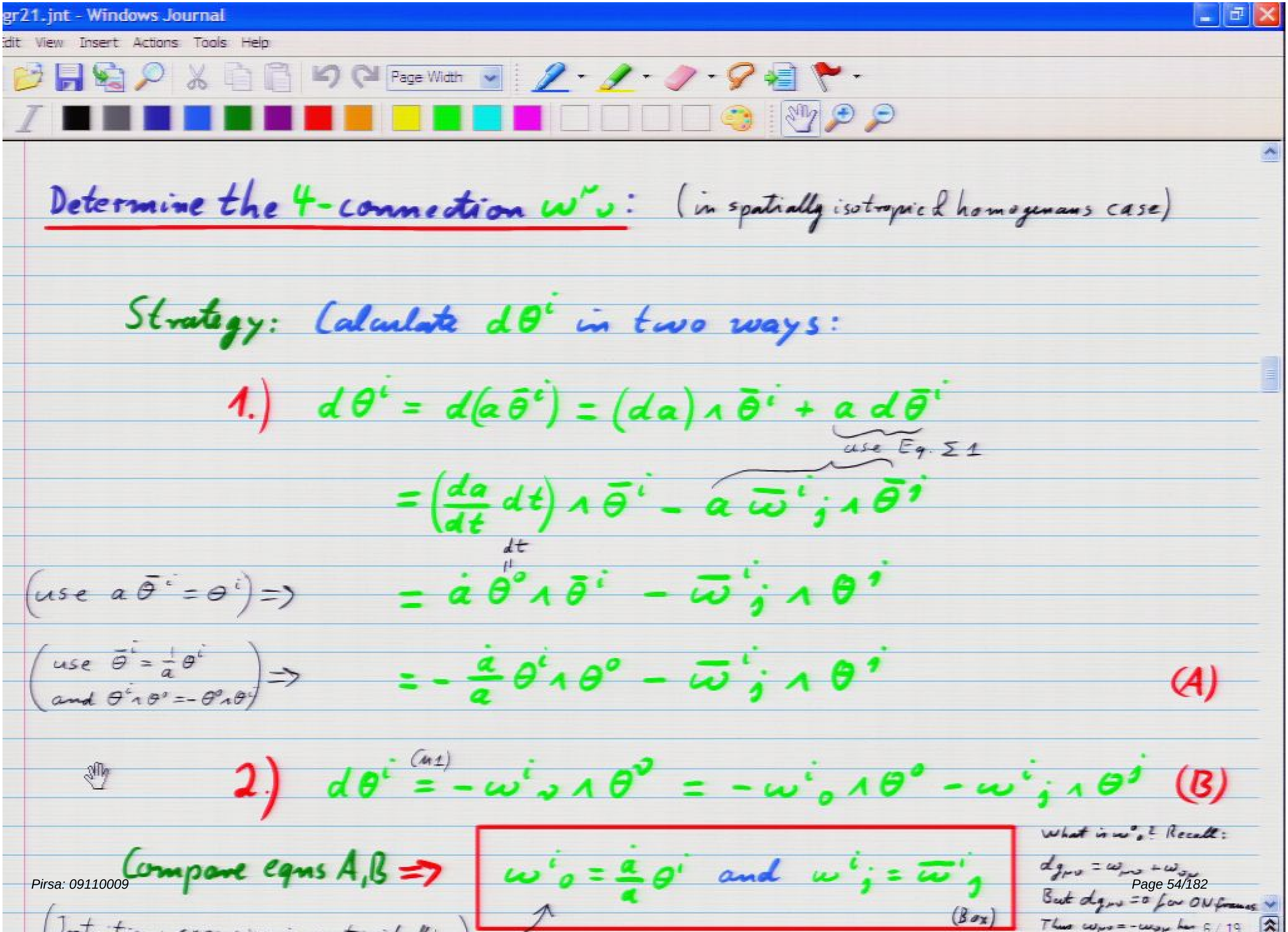

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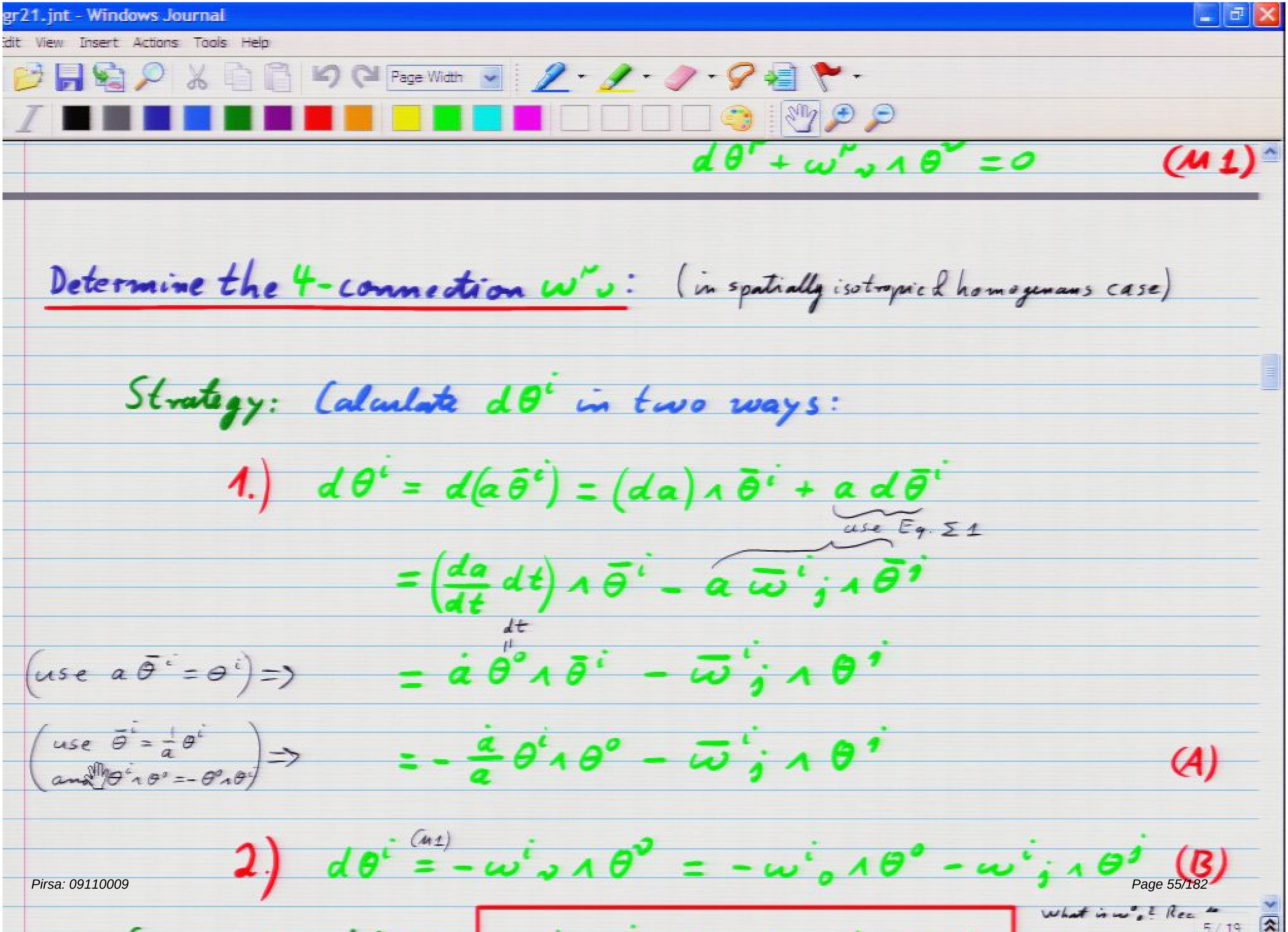

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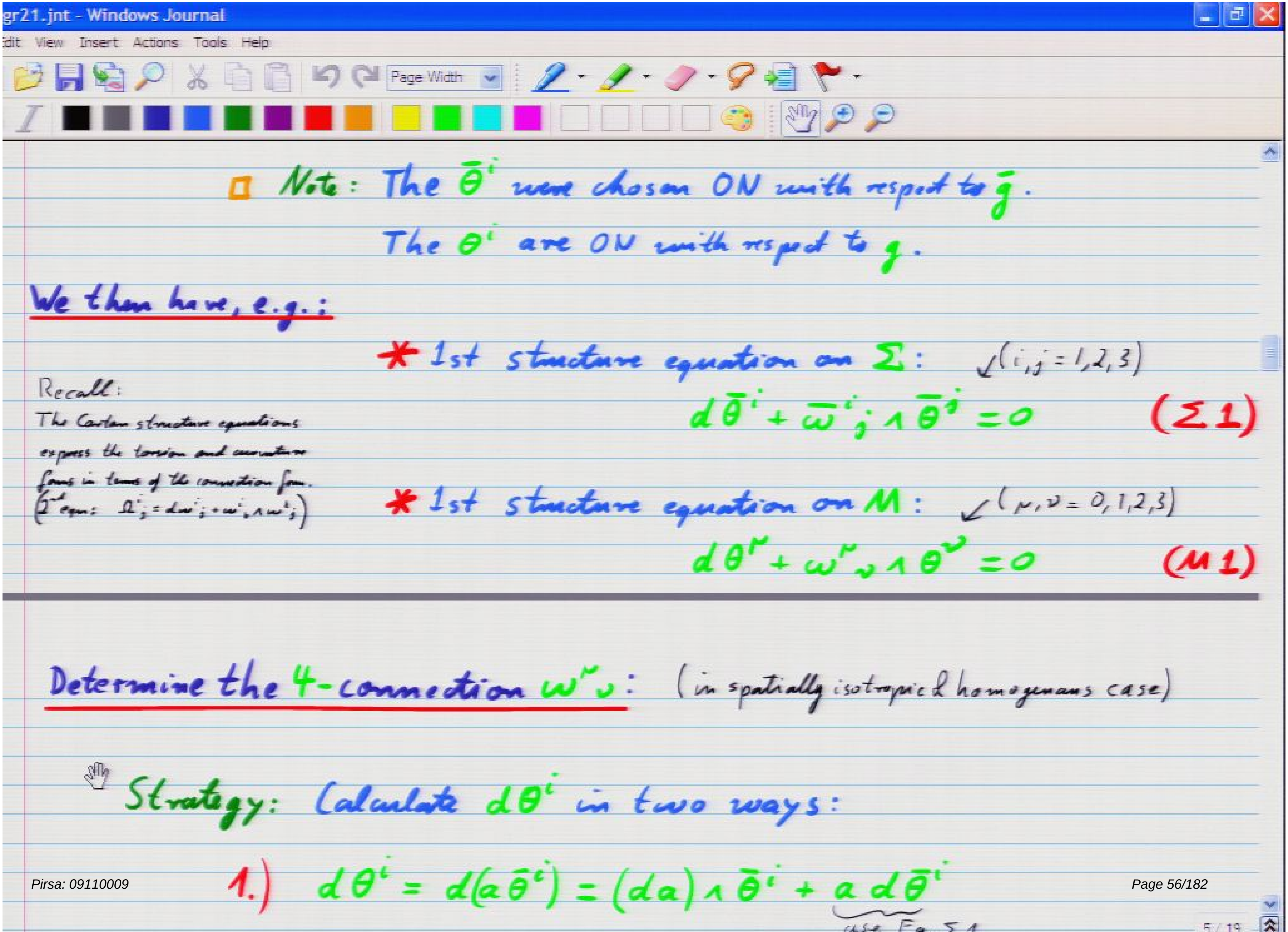

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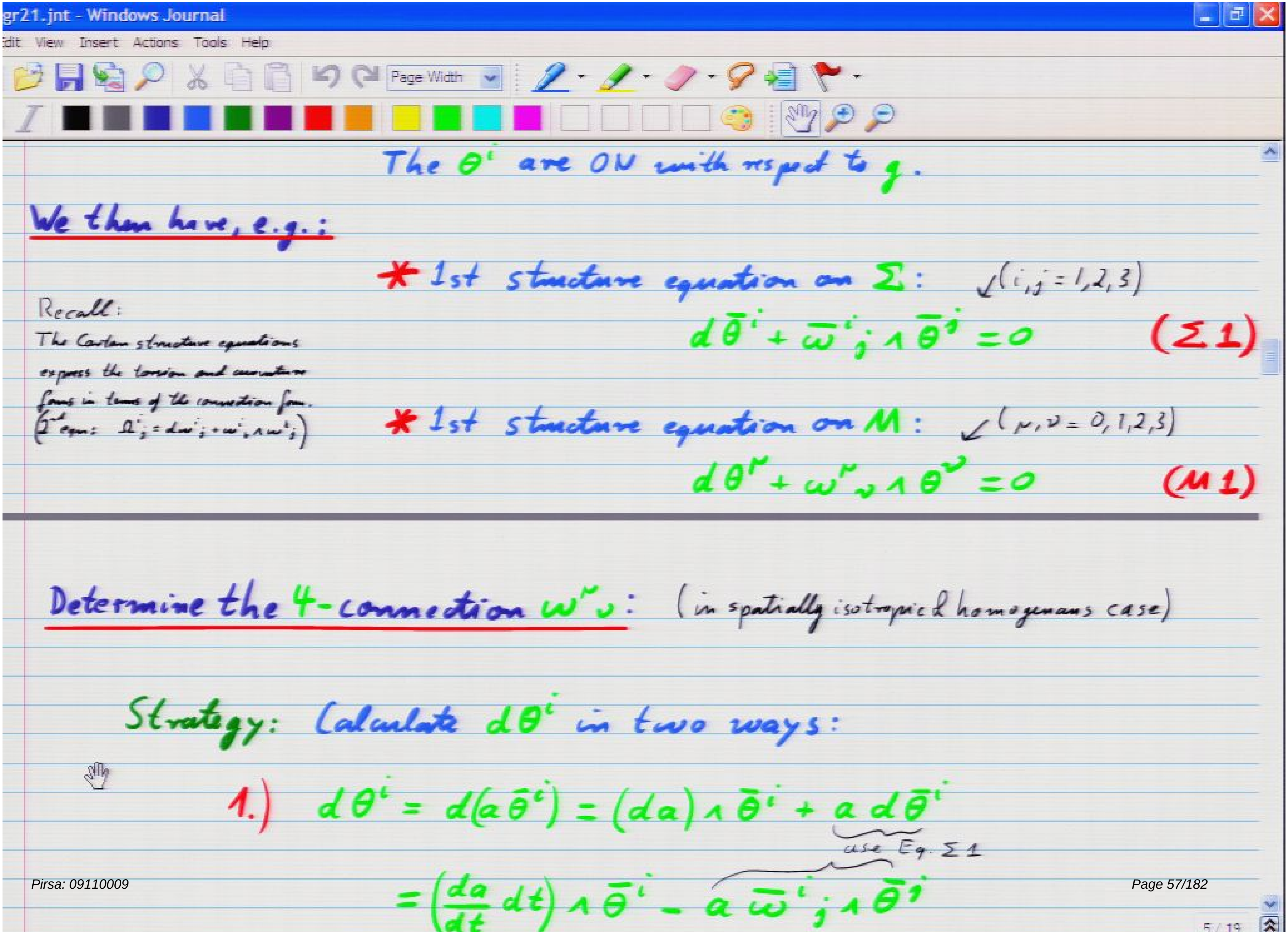

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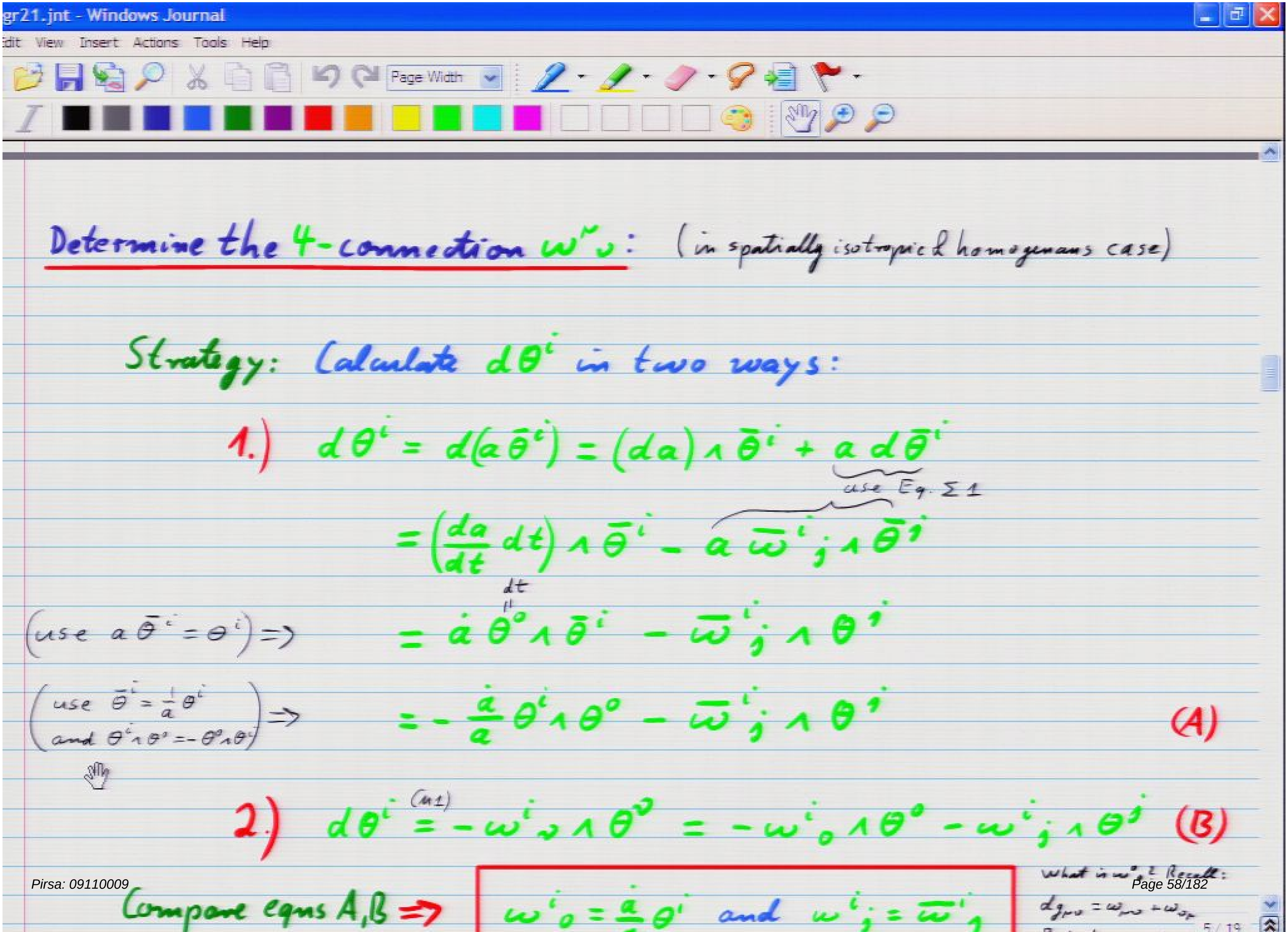

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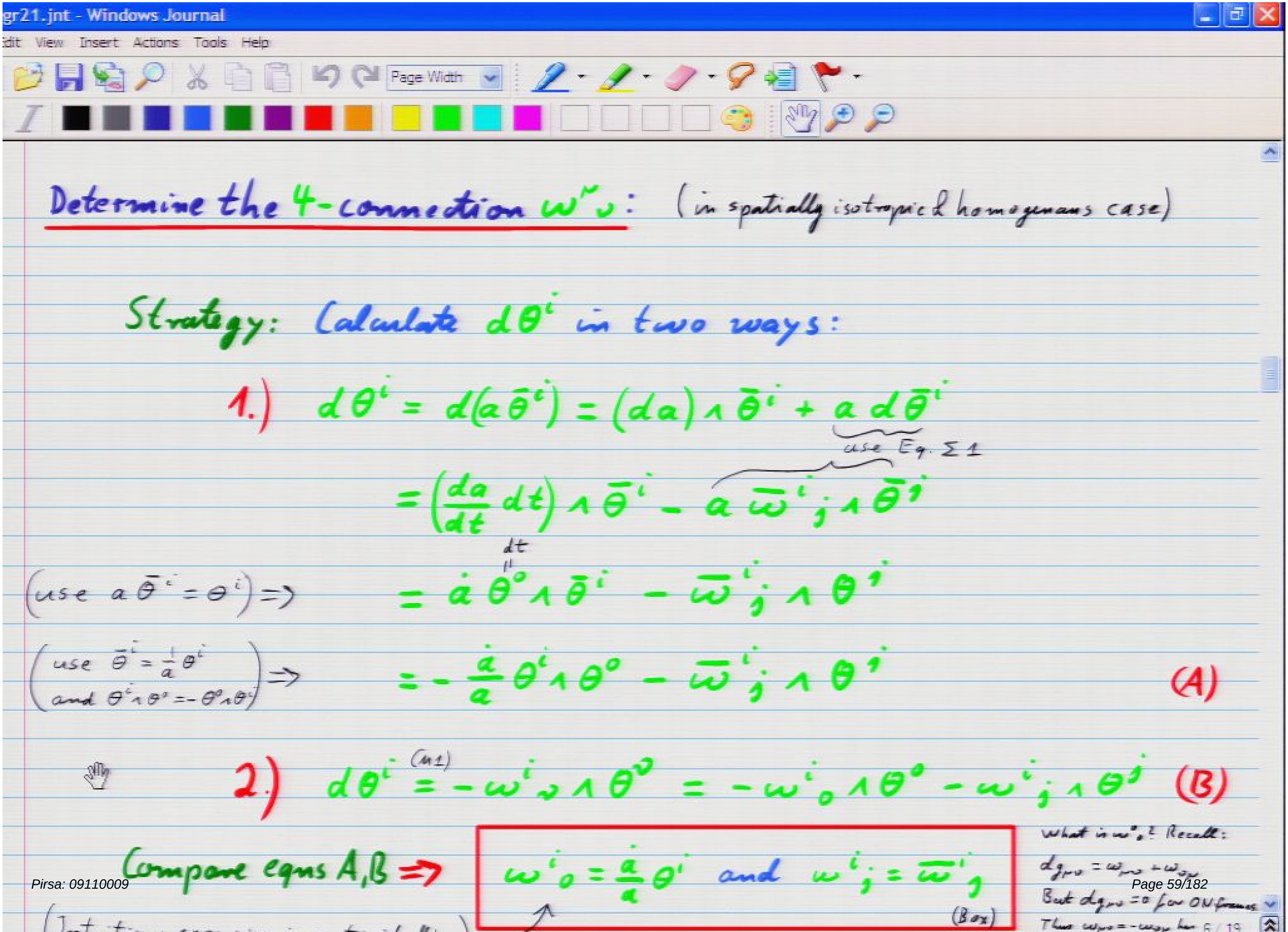

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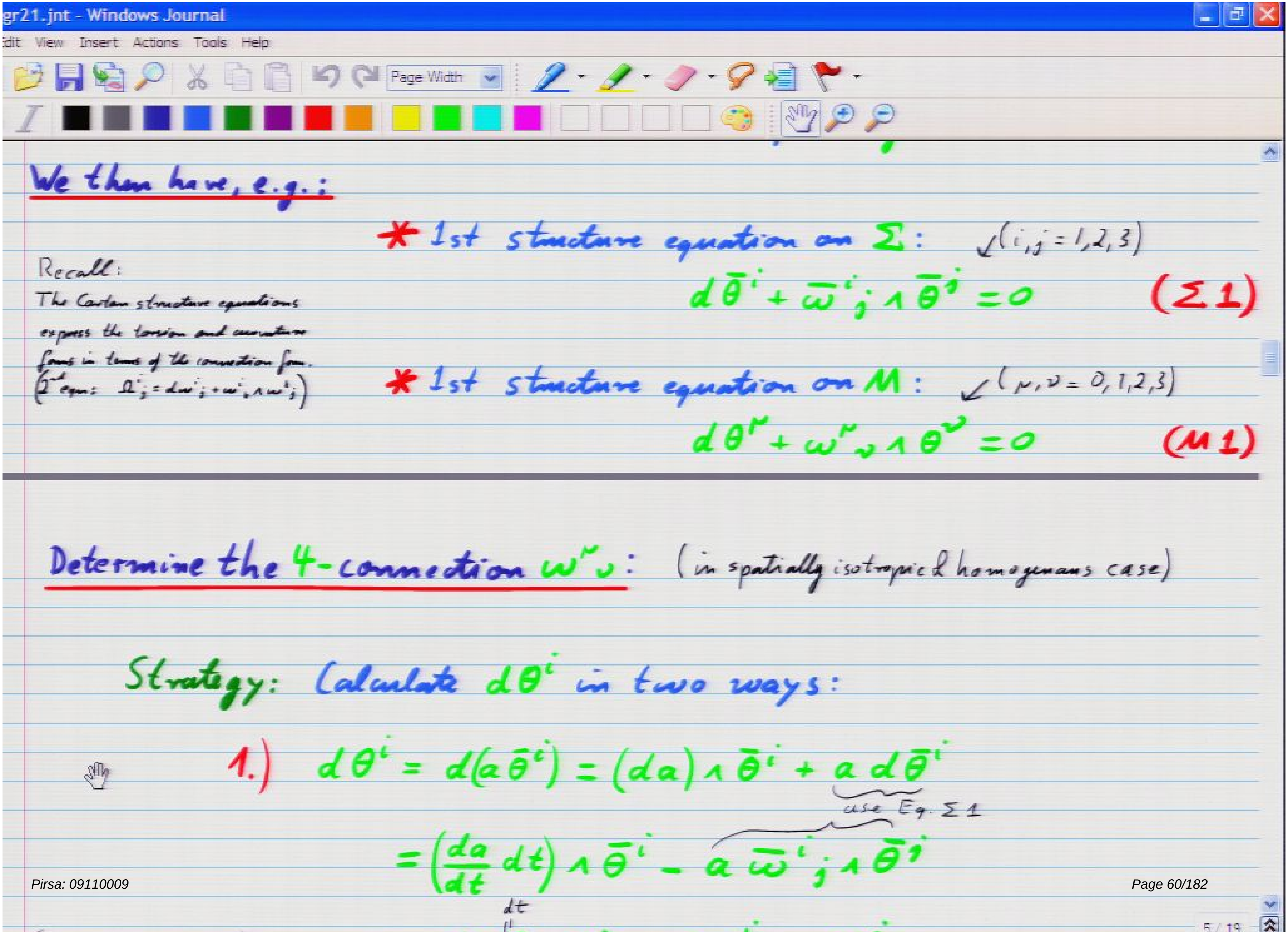

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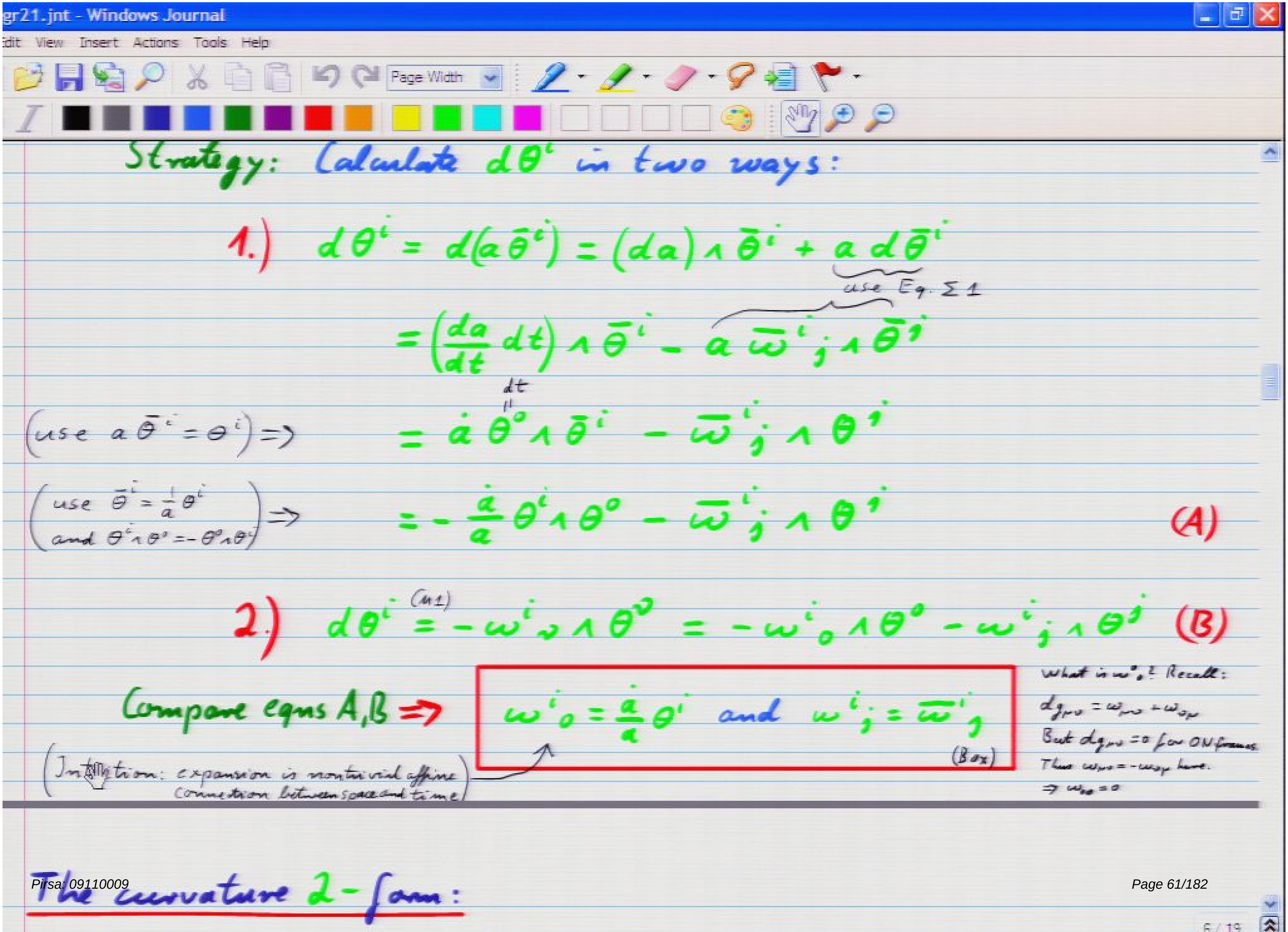
dt



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(Intuition: expansion is nontrivial affine connection between space and time)

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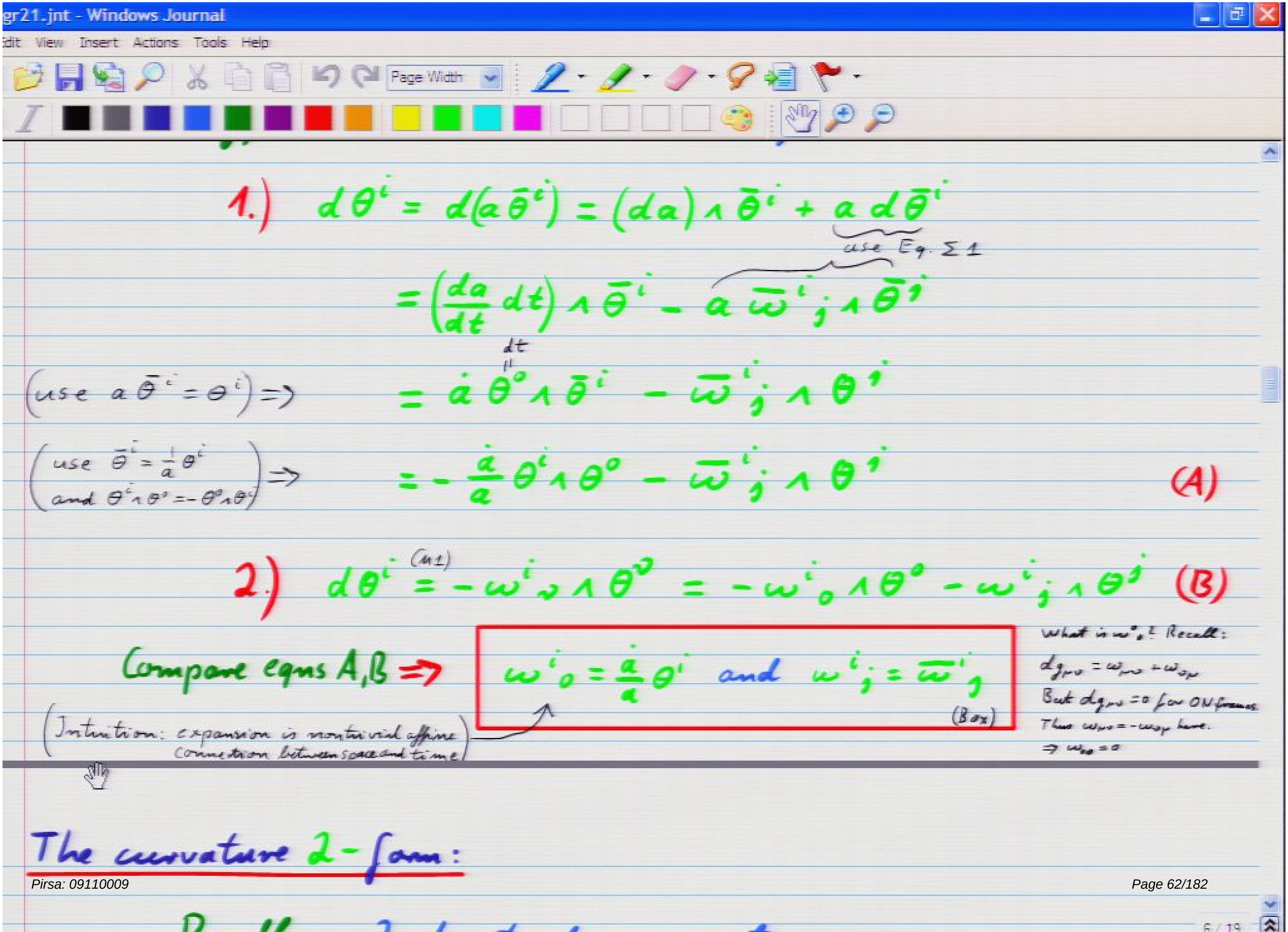
The curvature 2-form:



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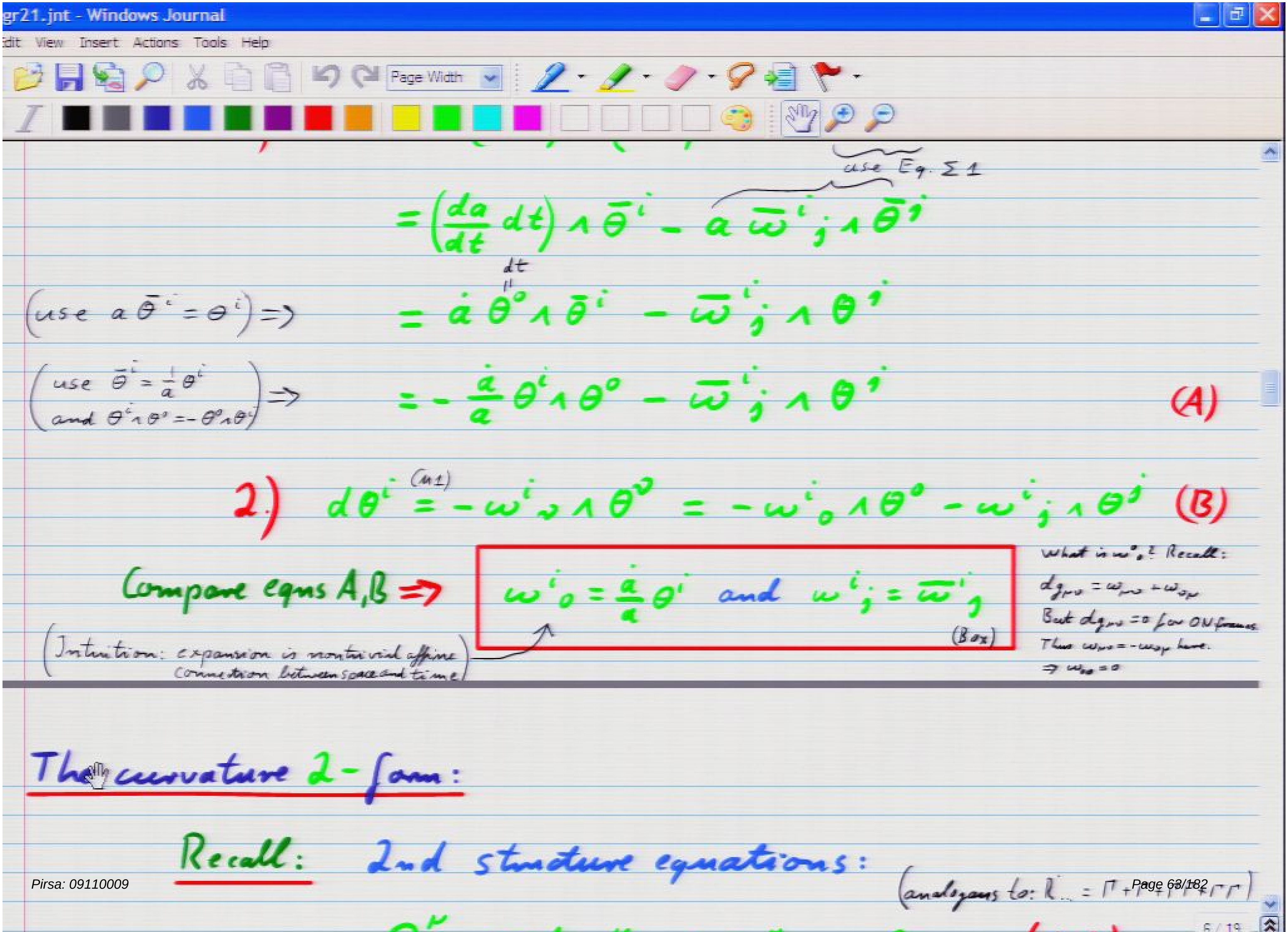

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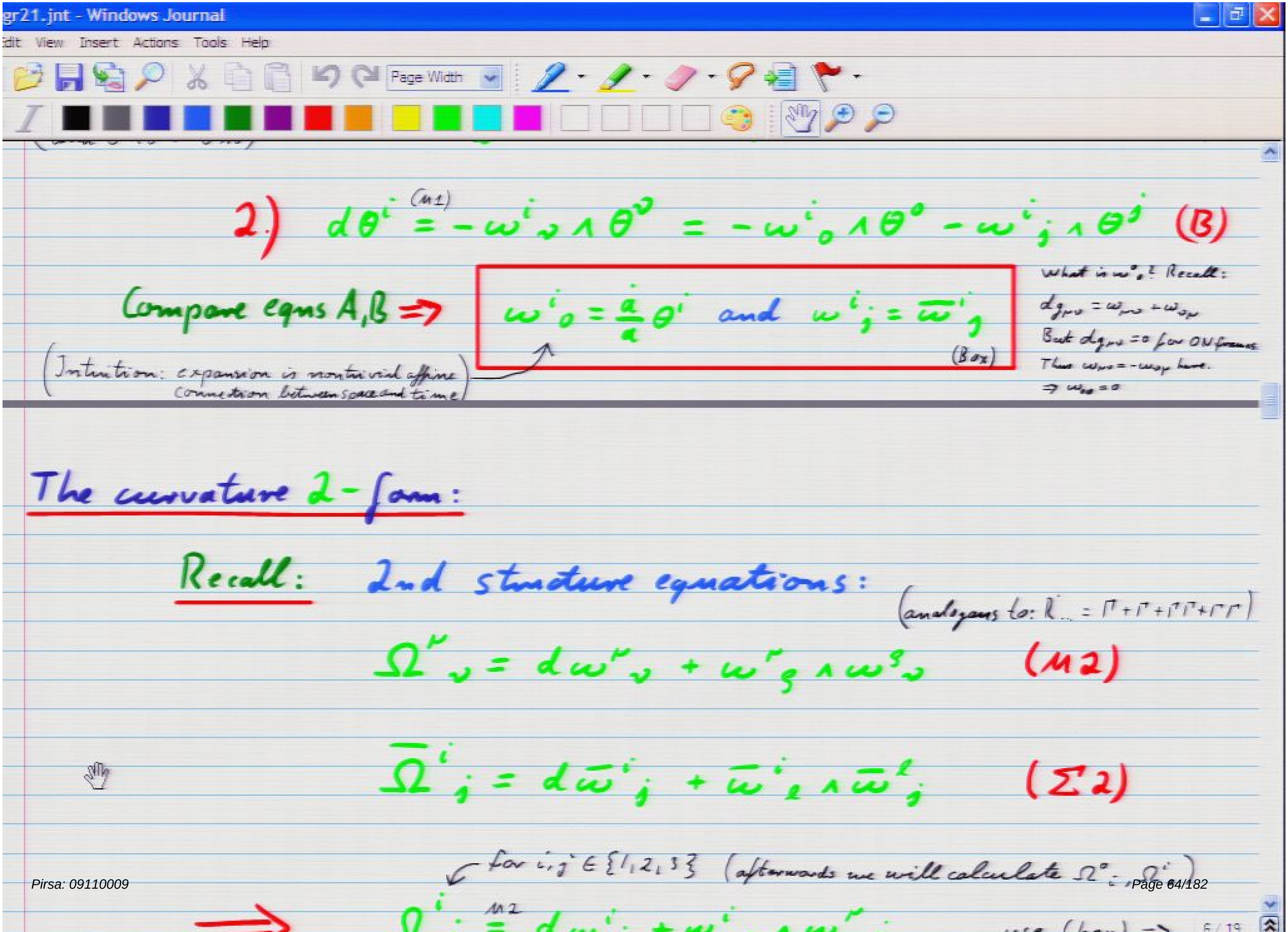
Recall: 2nd structure equations:

(analogous to: $R_{\dots} = \Gamma + \Gamma + \Gamma\Gamma + \Gamma\Gamma$)



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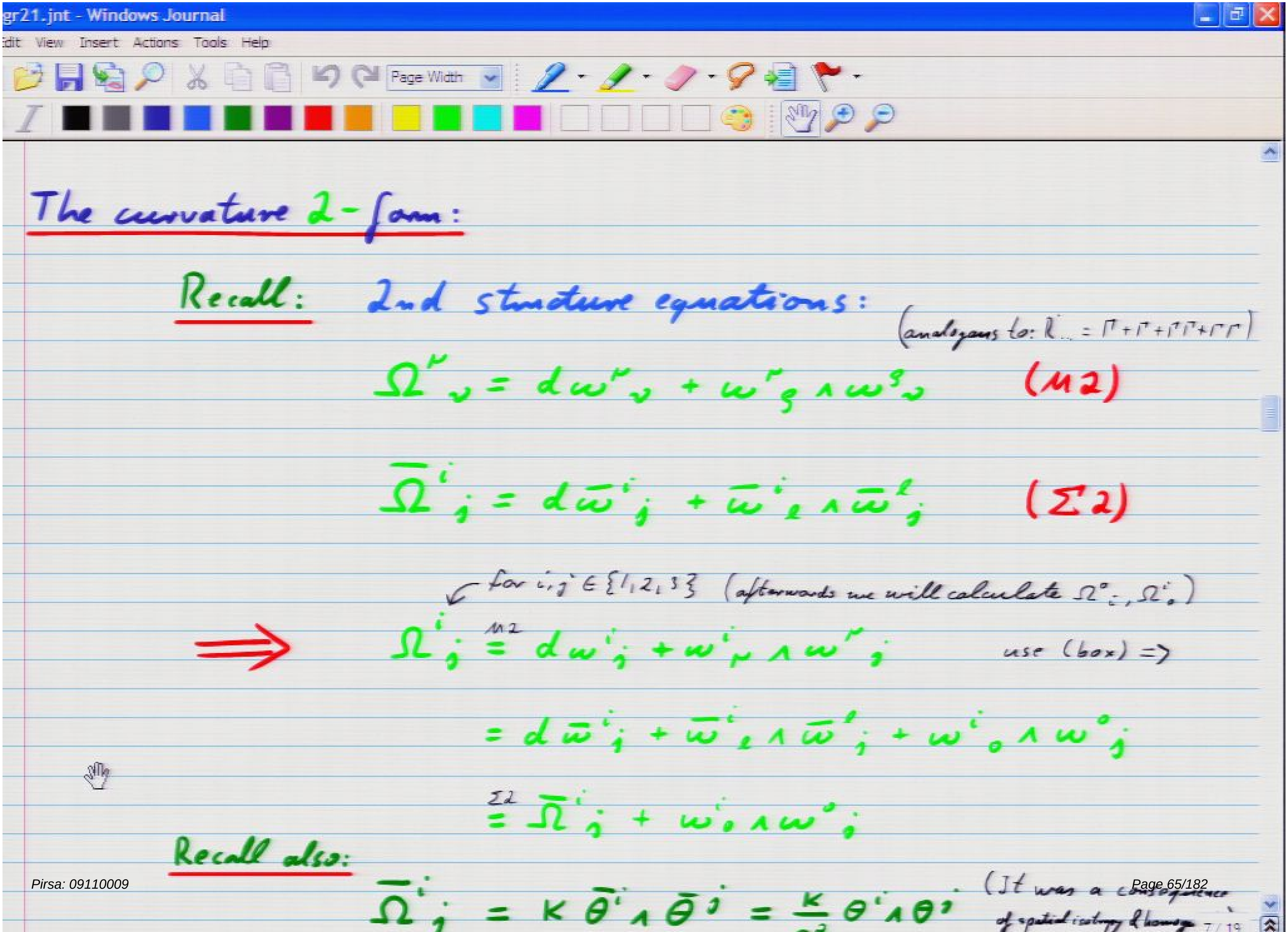
$$\Omega^\mu{}_\nu = d\omega^\mu{}_\nu + \omega^\mu{}_\rho \wedge \omega^\rho{}_\nu \qquad (M2)$$

$$\bar\Omega^i{}_j = d\bar\omega^i{}_j + \bar\omega^i{}_l \wedge \bar\omega^l{}_j \qquad (\Sigma 2)$$

for $i,j \in \{1,2,3\}$ (afterwards we will calculate $\Omega^0{}_i, \Omega^i{}_0$)

$\Longrightarrow \quad \Omega^i{}_j \overset{M2}{=} d\omega^i{}_j + \omega^i{}_\mu \wedge \omega^\mu{}_j$ use (box) $\rightarrow$



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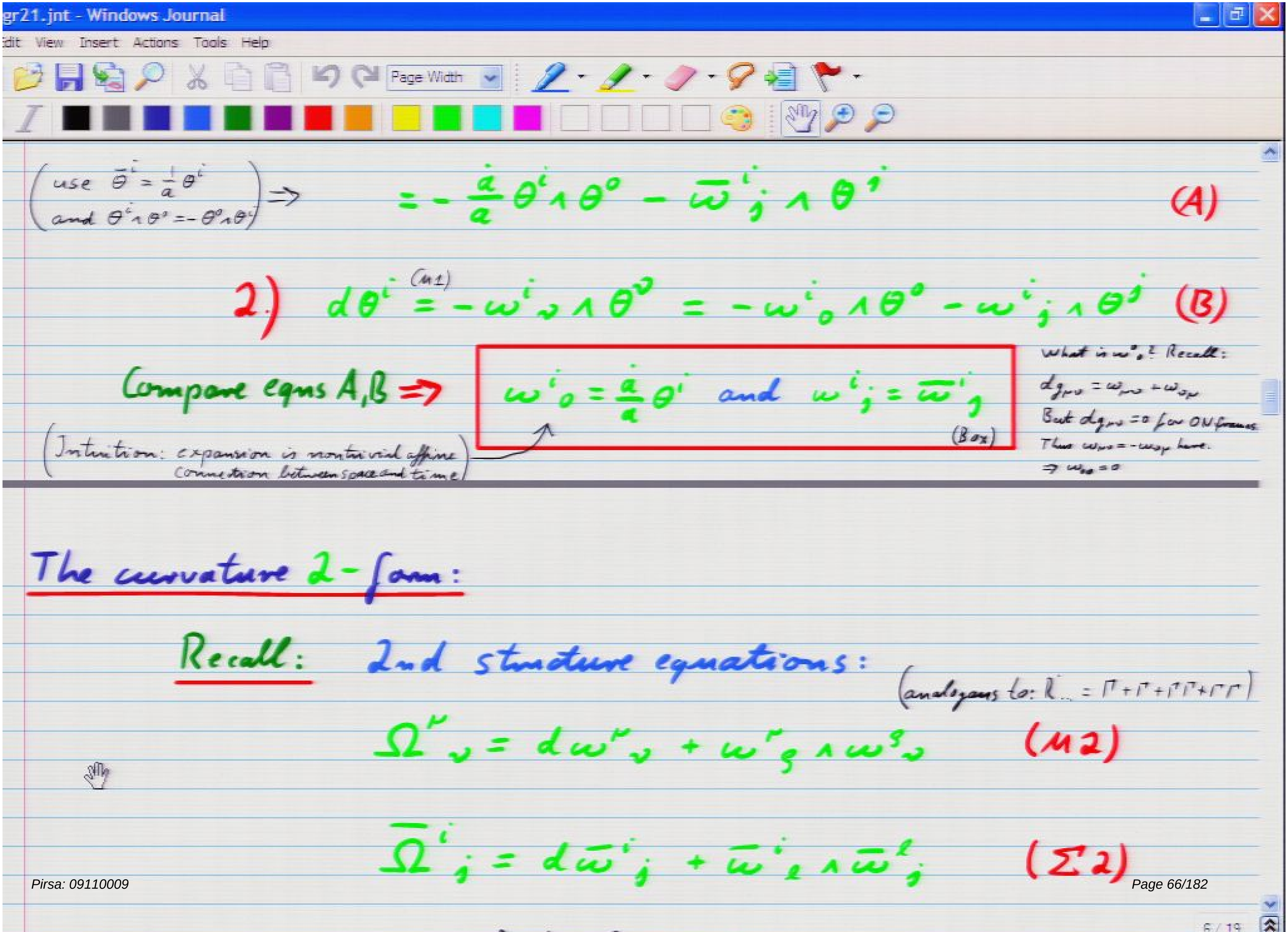

$$\left(\begin{array}{l}\text{use } \bar{\theta}^i = \frac{1}{a}\theta^i \\ \text{and } \theta^i\wedge\theta^0 = -\theta^0\wedge\theta^i\end{array}\right) \Rightarrow \quad = -\frac{\dot a}{a}\,\theta^i\wedge\theta^0 - \bar{\omega}^i{}_j\wedge\theta^j \qquad (A)$$

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Recall: 2nd structure equations:

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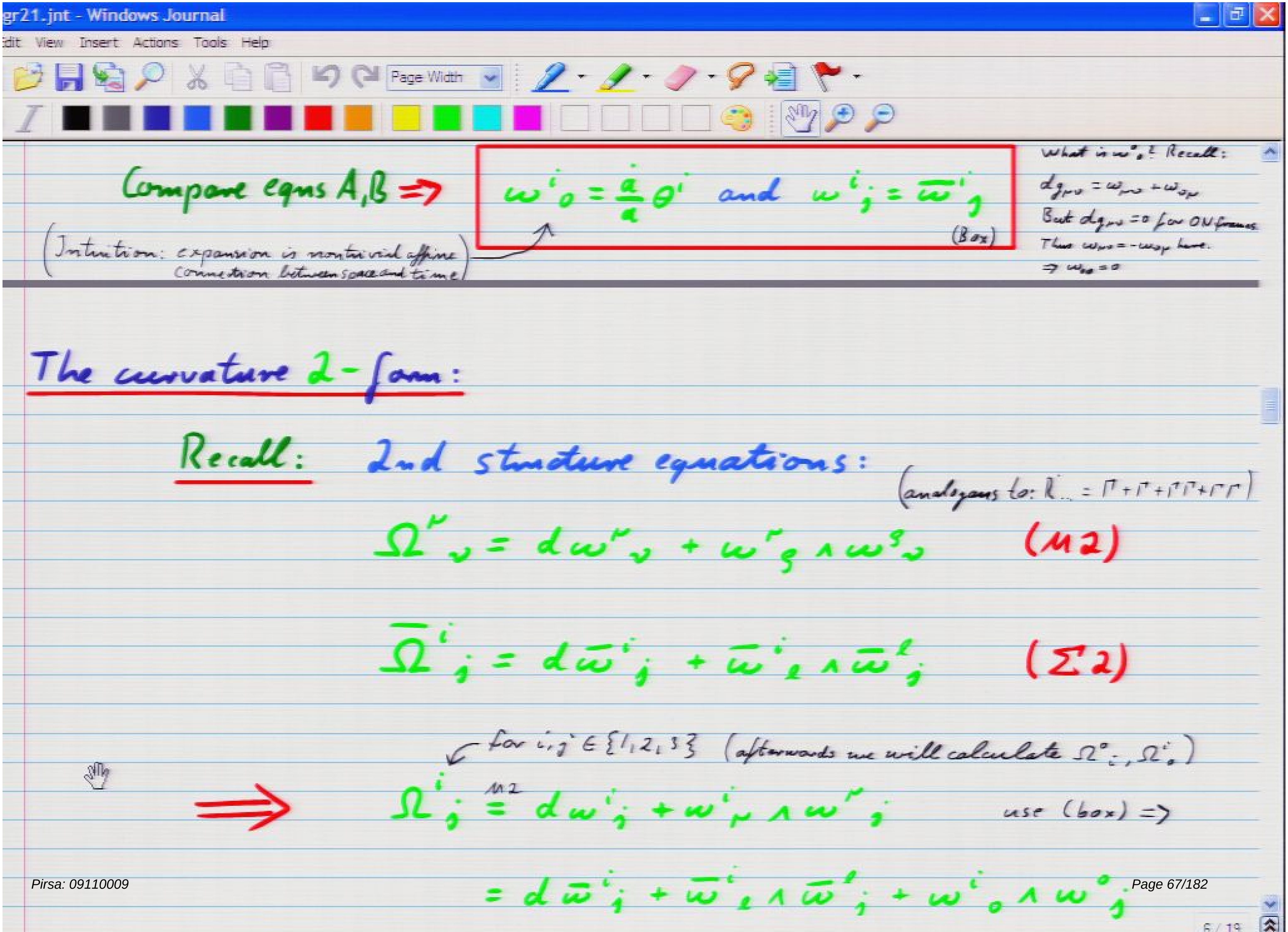

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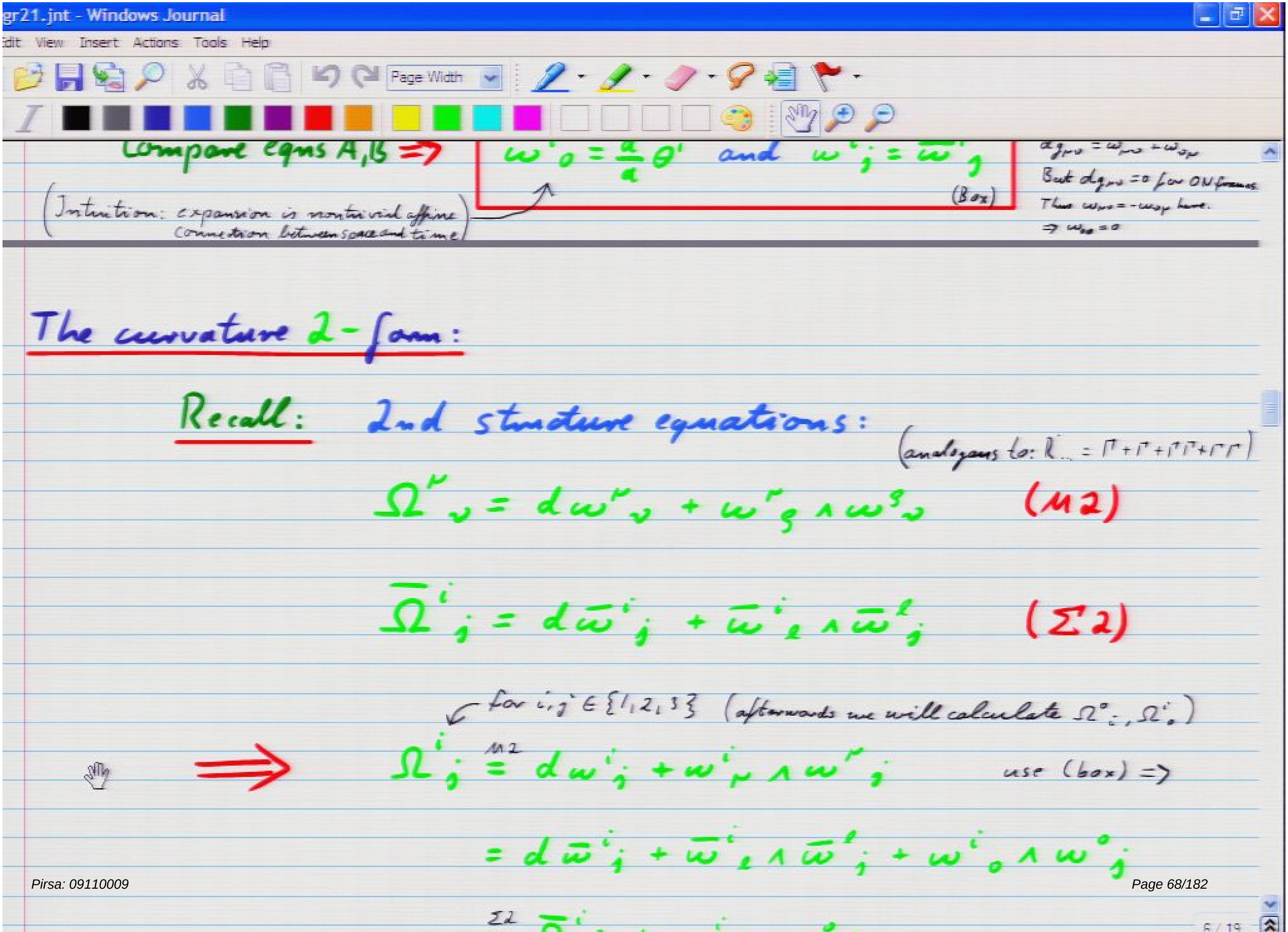

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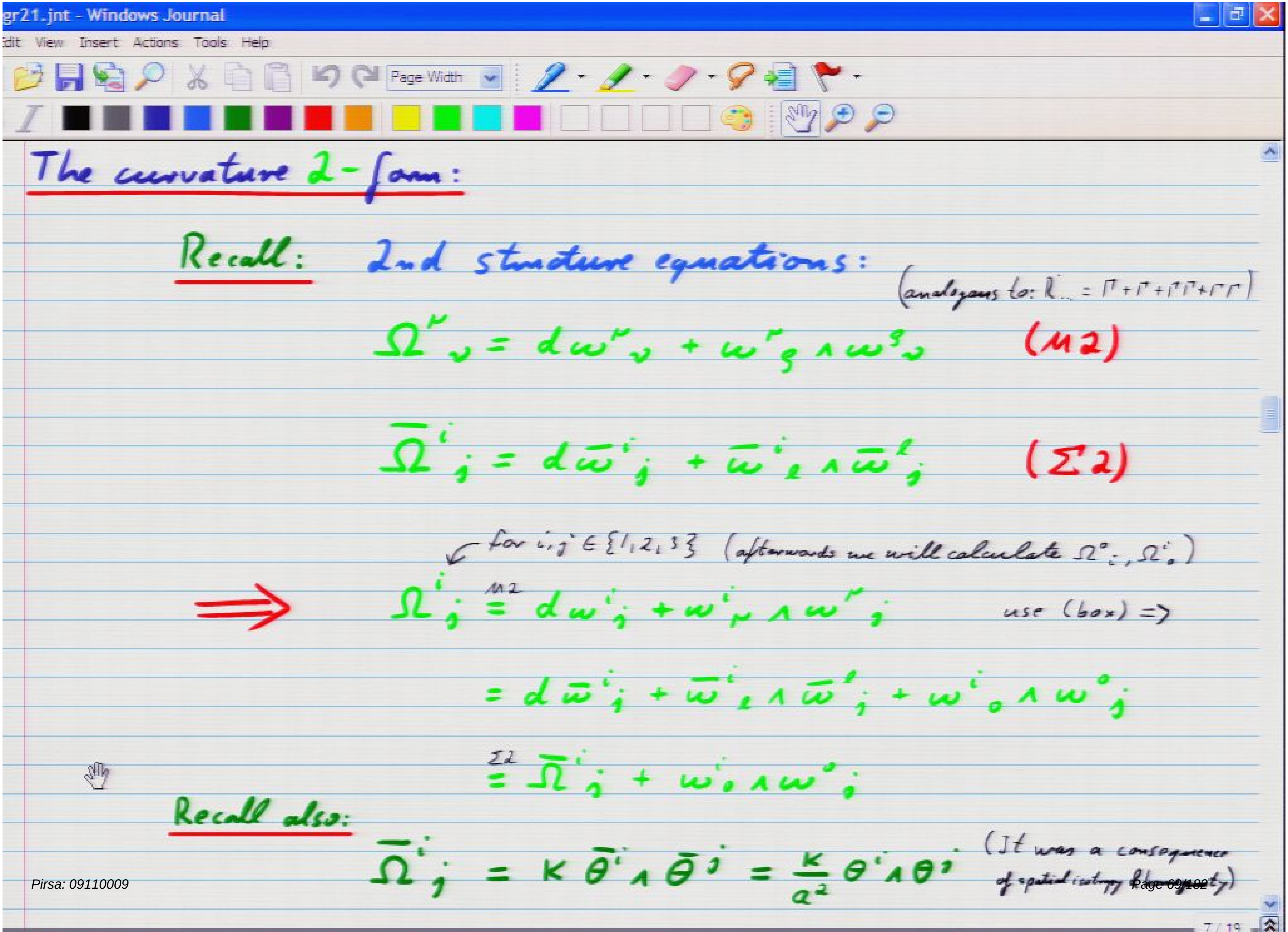

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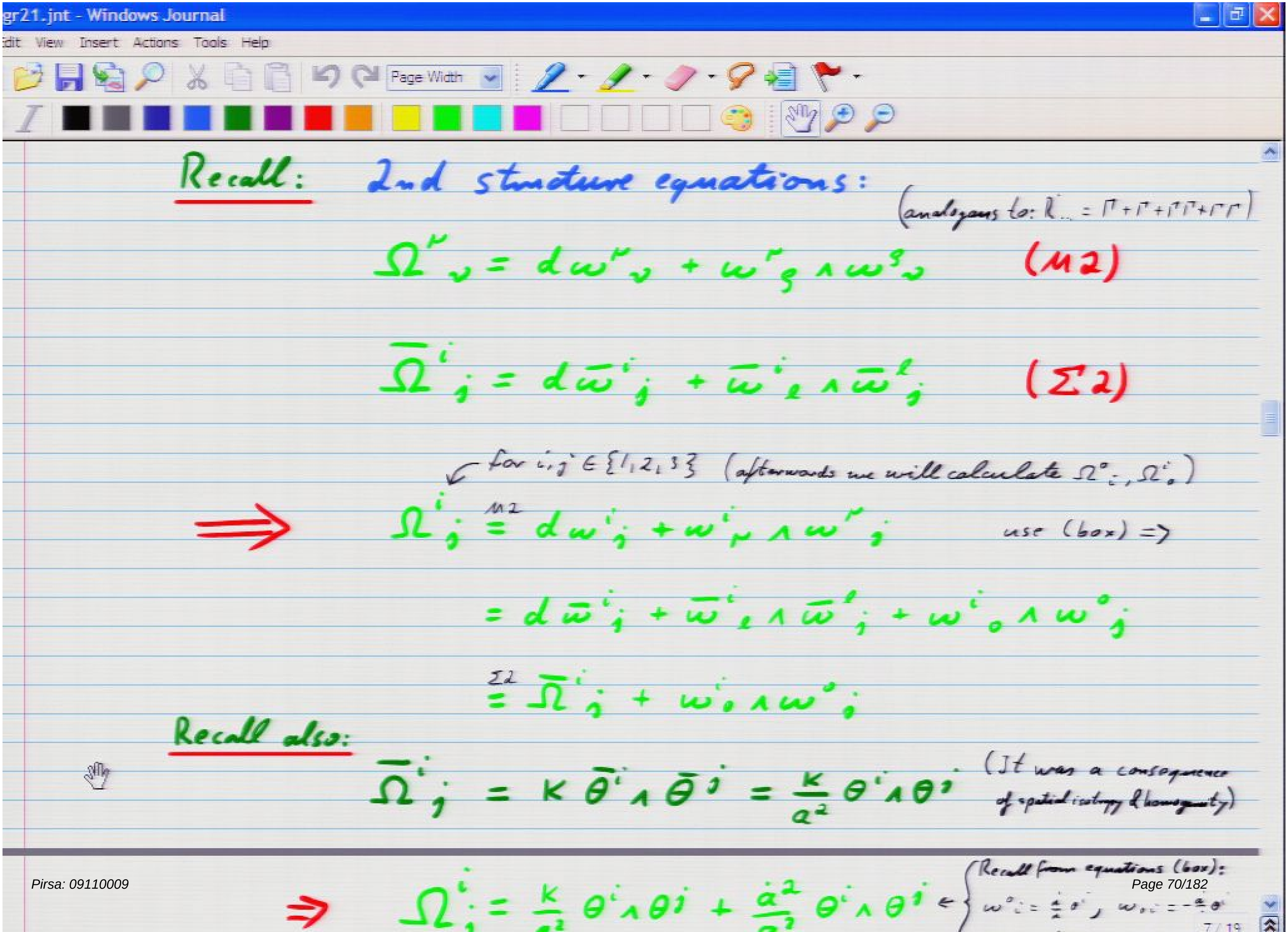

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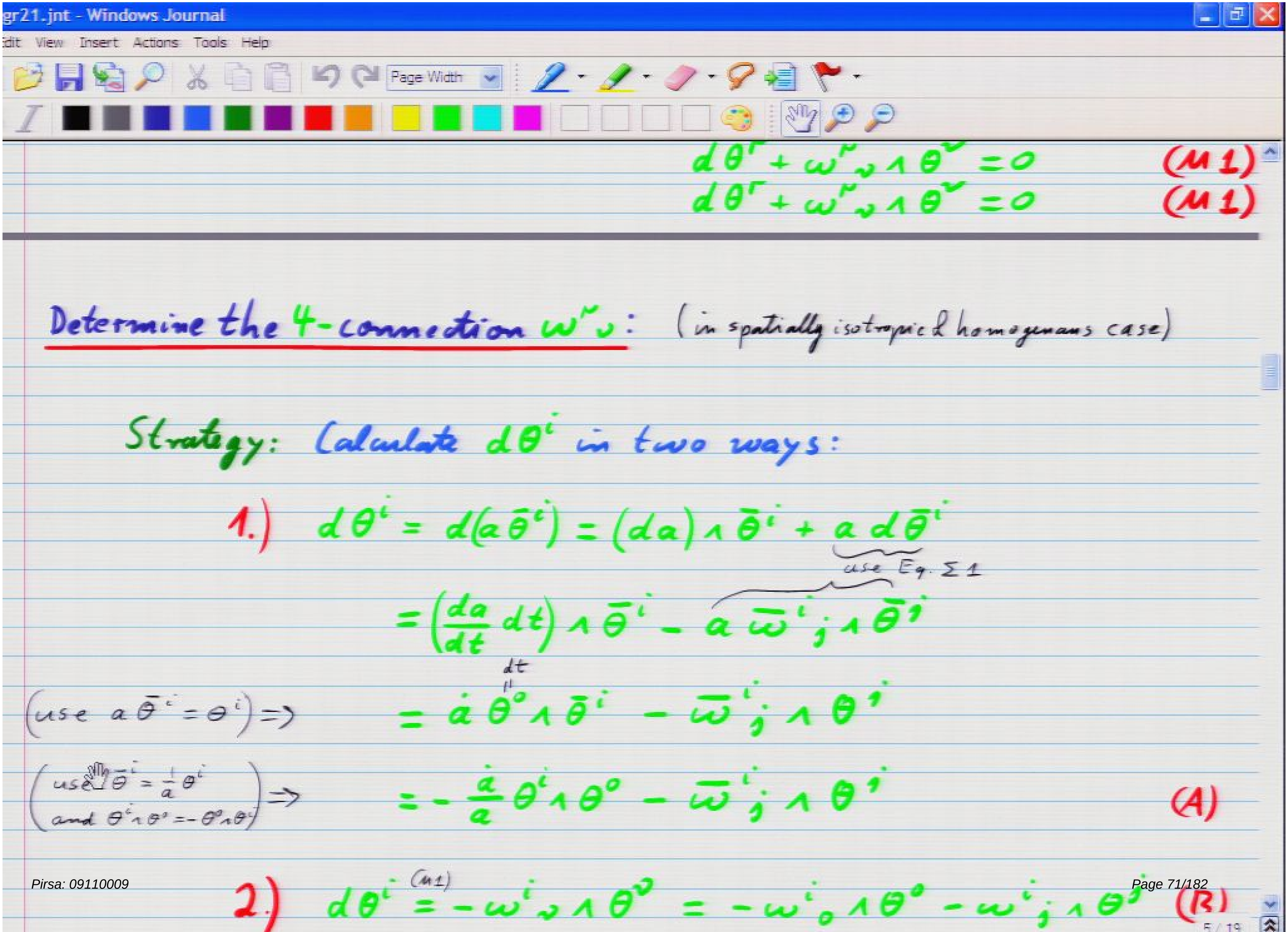
$$\Rightarrow \quad \Omega^i{}_j = \frac{k}{a^2} \theta^i \wedge \theta^j + \frac{\dot{a}^2}{a^2} \theta^i \wedge \theta^j \leftarrow \begin{cases} \text{Recall from equations (box):} \\ \omega^0{}_i = \frac{\dot{a}}{a} \theta^i, \quad \omega_{0i} = -\frac{\dot{a}}{a} \theta^i \end{cases}$$



$$d\theta^\mu + \omega^\mu{}_\nu \wedge \theta^\nu = 0 \qquad (M1)$$

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Determine the 4-connection $\omega^\mu{}_\nu$: (in spatially isotropic & homogenous case)

Strategy: Calculate $d\theta^i$ in two ways:

1.) $$d\theta^i = d(a\bar\theta^i) = (da)\wedge\bar\theta^i + \underbrace{a\, d\bar\theta^i}_{\text{use Eq. } \Sigma 1}$$

$$= \left(\frac{da}{dt}dt\right)\wedge\bar\theta^i \overbrace{- a\,\bar\omega^i{}_j \wedge \bar\theta^j}$$

(use $a\bar\theta^i = \theta^i$) $\Rightarrow$ $$= \dot a\, \overset{dt}{\overset{\shortparallel}{\theta^0}} \wedge \bar\theta^i - \bar\omega^i{}_j \wedge \theta^j$$

(use $\bar\theta^i = \frac{1}{a}\theta^i$ and $\theta^i\wedge\theta^0 = -\theta^0\wedge\theta^i$) $\Rightarrow$ $$= -\frac{\dot a}{a}\theta^i\wedge\theta^0 - \bar\omega^i{}_j\wedge\theta^j \qquad (A)$$

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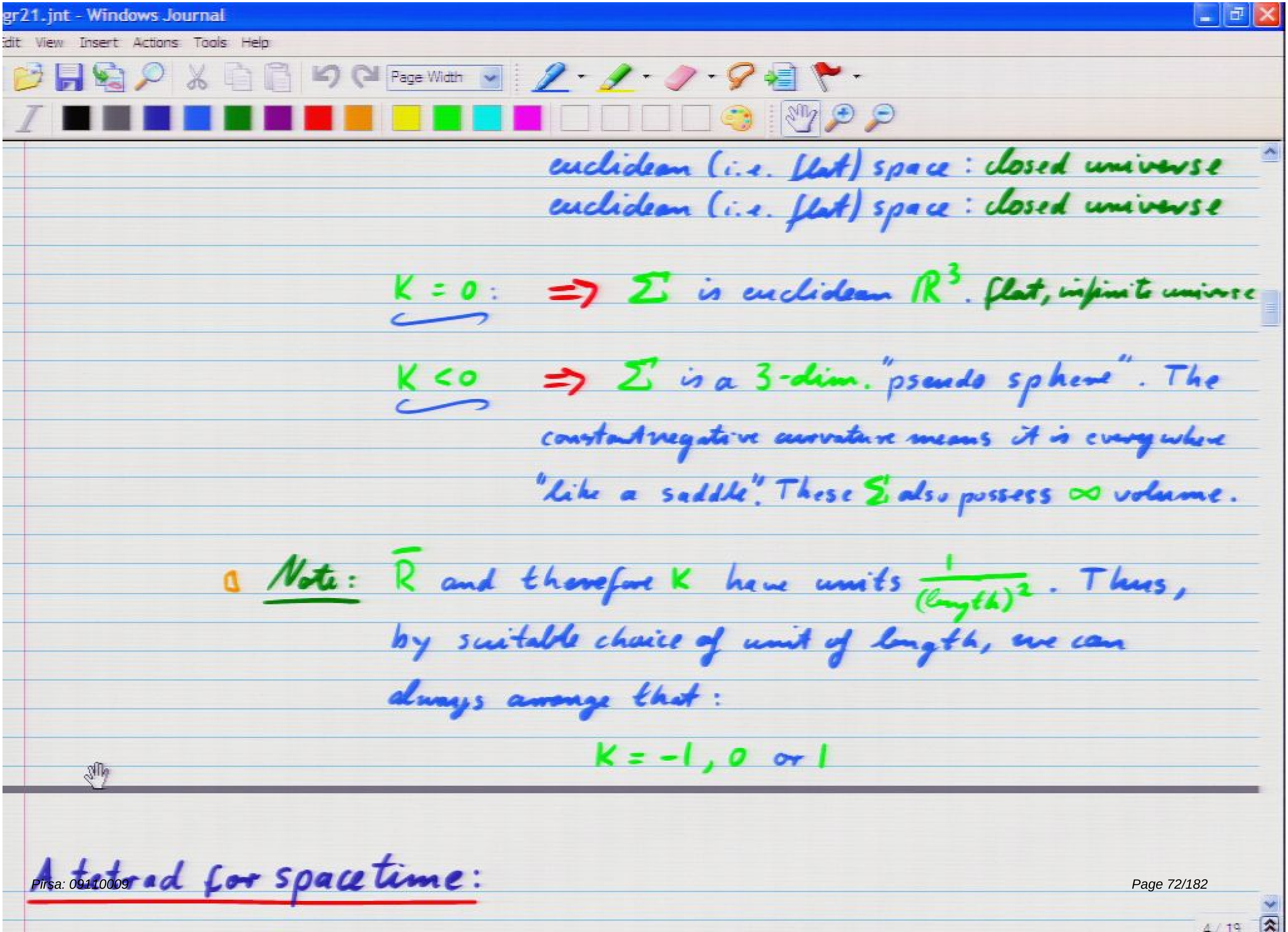

euclidean (i.e. flat) space : closed universe

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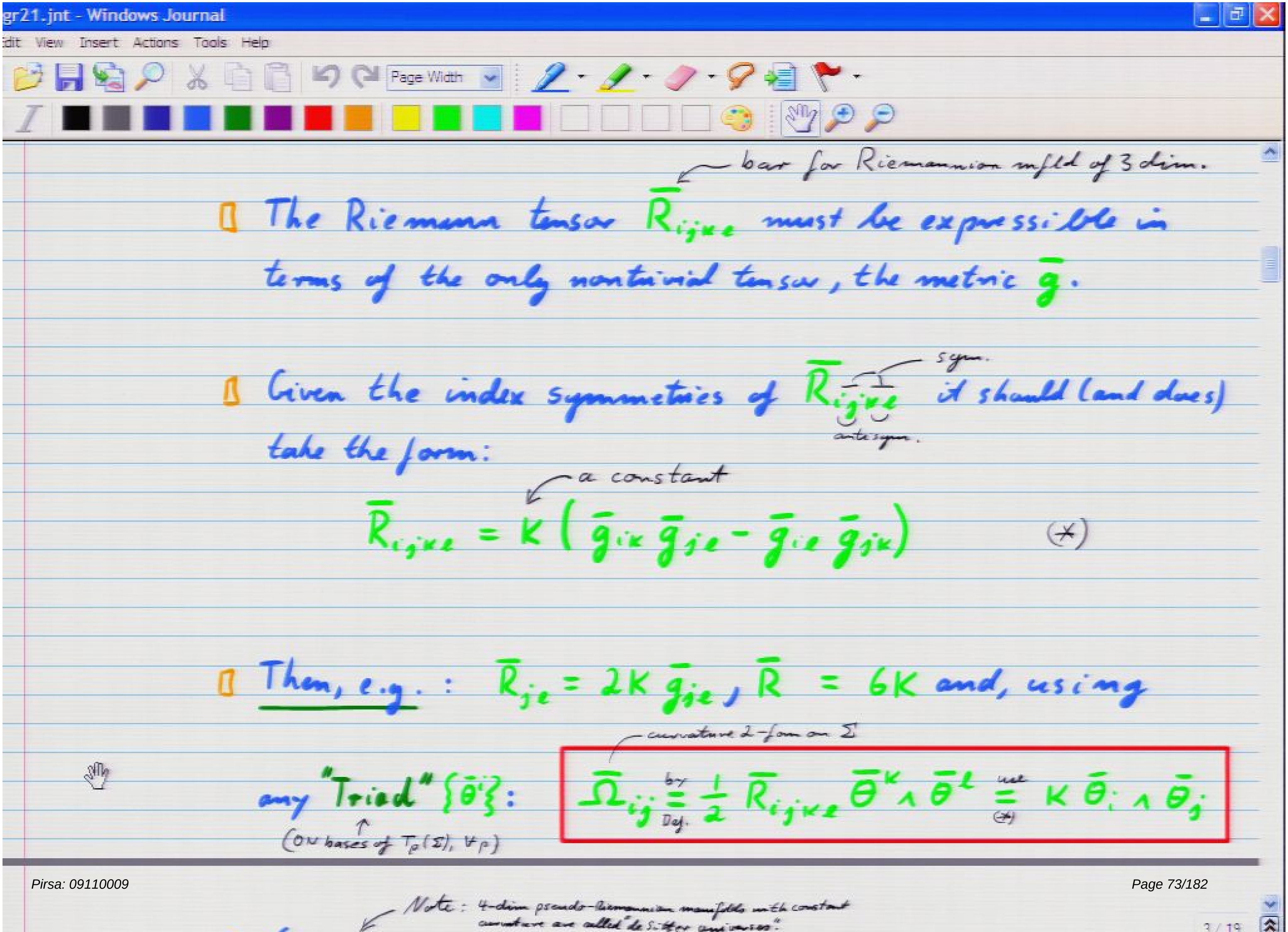
$K = 0$: $\Rightarrow$ Σ is euclidean $\mathbb{R}^3$. flat, infinite universe

$K < 0$ $\Rightarrow$ Σ is a 3-dim. "pseudo sphere". The constant negative curvature means it is everywhere "like a saddle". These Σ also possess ∞ volume.

Note: $\bar{R}$ and therefore K have units $\frac{1}{(\text{length})^2}$. Thus, by suitable choice of unit of length, we can always arrange that:

$$K = -1, 0 \text{ or } 1$$

A tetrad for spacetime:



bar for Riemannian mfld of 3 dim.

- The Riemann tensor $\bar{R}_{ijkl}$ must be expressible in terms of the only nontrivial tensor, the metric $\bar{g}$.

- Given the index symmetries of $\bar{R}_{\underbrace{ij}_{\text{antisym.}}\underbrace{kl}}$ (sym.) it should (and does) take the form:

a constant

$$\bar{R}_{ijkl} = K\left(\bar{g}_{ik}\bar{g}_{jl} - \bar{g}_{il}\bar{g}_{jk}\right) \qquad (*)$$

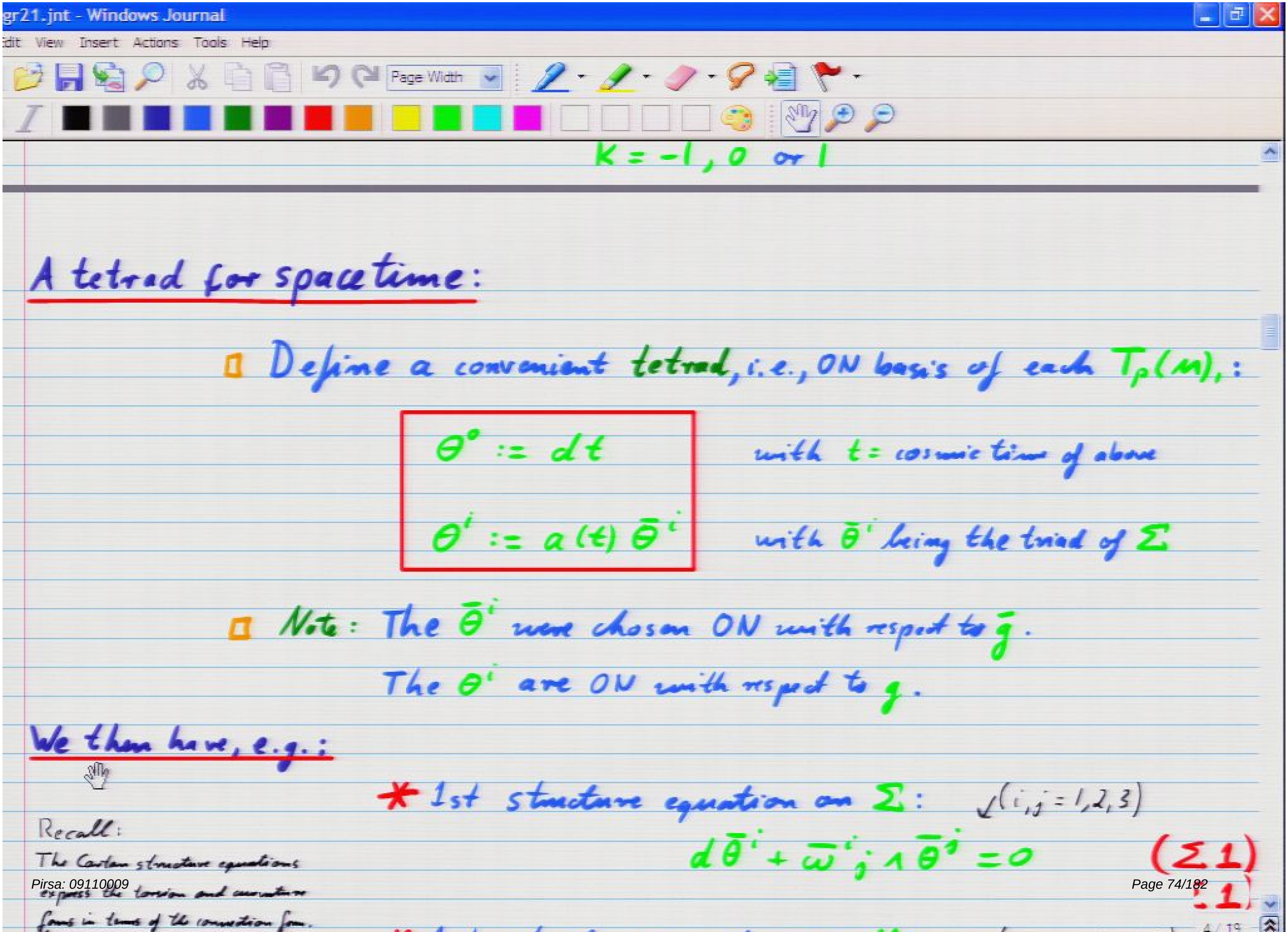
- Then, e.g.: $\bar{R}_{jl} = 2K\,\bar{g}_{jl}$, $\bar{R} = 6K$ and, using any "Triad" $\{\bar{\theta}^i\}$:

(ON bases of $T_p(\Sigma)$, $\forall p$)

curvature 2-form on Σ

$$\boxed{\bar{\Omega}_{ij} \overset{\text{by}}{\underset{\text{Def.}}{=}} \frac{1}{2}\bar{R}_{ijkl}\,\bar{\theta}^k \wedge \bar{\theta}^l \overset{\text{use}}{\underset{(*)}{=}} K\,\bar{\theta}_i \wedge \bar{\theta}_j}$$

Note: 4-dim pseudo-Riemannian manifolds with constant curvature are called "de Sitter universes".



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A tetrad for spacetime:

☐ Define a convenient tetrad, i.e., ON basis of each $T_p(M)$, :

$$\boxed{\begin{array}{l} \theta^0 := dt \\ \theta^i := a(t)\,\bar{\theta}^i \end{array}}$$

with t = cosmic time of above

with $\bar{\theta}^i$ being the triad of Σ

☐ Note: The $\bar{\theta}^i$ were chosen ON with respect to $\bar{g}$.

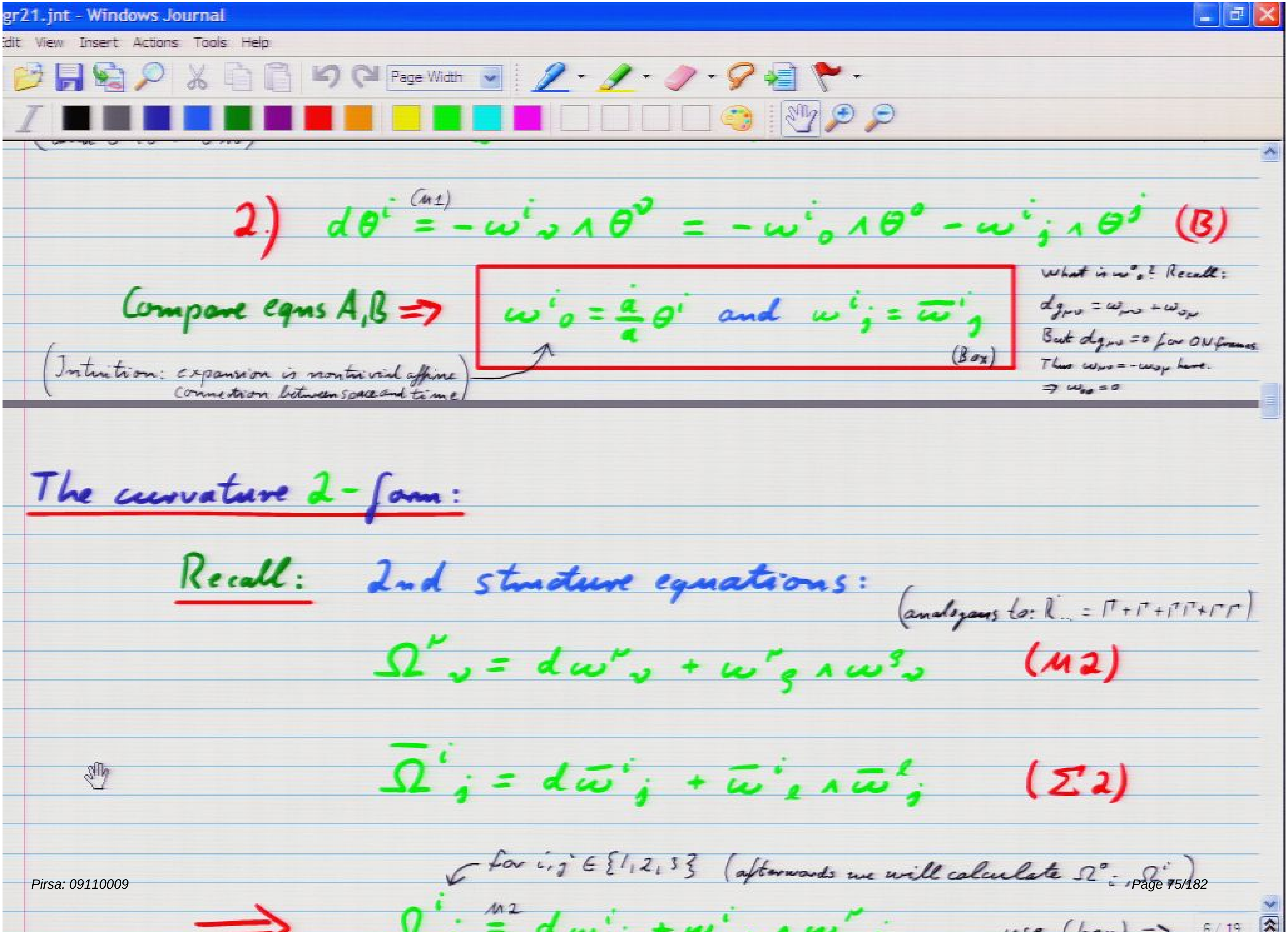
The θ^i are ON with respect to g.

We then have, e.g.:

* 1st structure equation on Σ: $\swarrow (i,j = 1,2,3)$

$$d\bar{\theta}^i + \bar{\omega}^i{}_j \wedge \bar{\theta}^j = 0 \qquad (\Sigma 1)$$

Recall:
The Cartan structure equations express the torsion and curvature forms in terms of the connection form.



2) $d\theta^i \overset{(M1)}{=} -\omega^i{}_\nu \wedge \theta^\nu = -\omega^i{}_0 \wedge \theta^0 - \omega^i{}_j \wedge \theta^j$ (B)

Compare eqns A,B $\Rightarrow$ $\boxed{\omega^i{}_0 = \frac{\dot{a}}{a}\theta^i \text{ and } \omega^i{}_j = \bar{\omega}^i{}_j}$ (Box)

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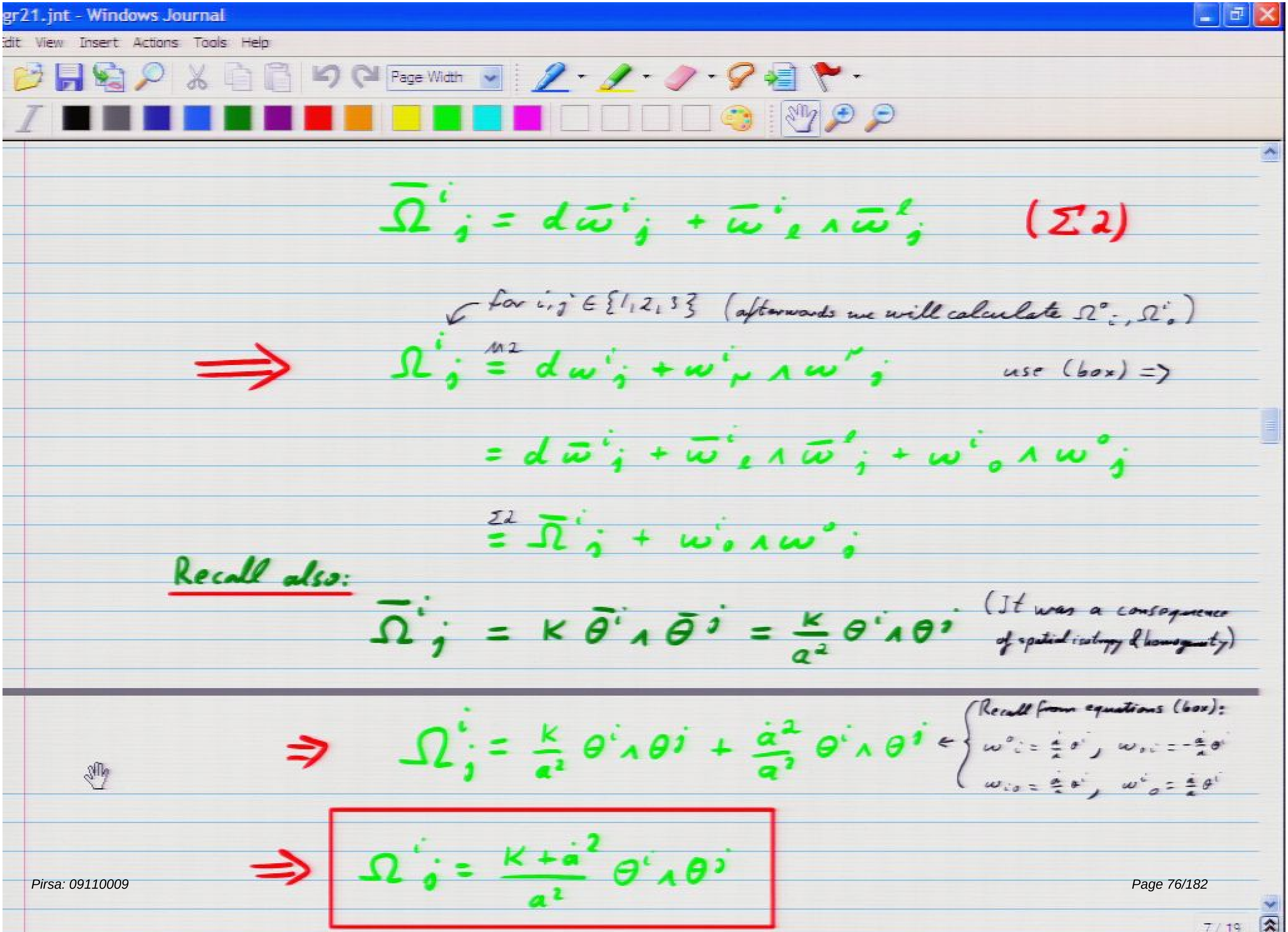

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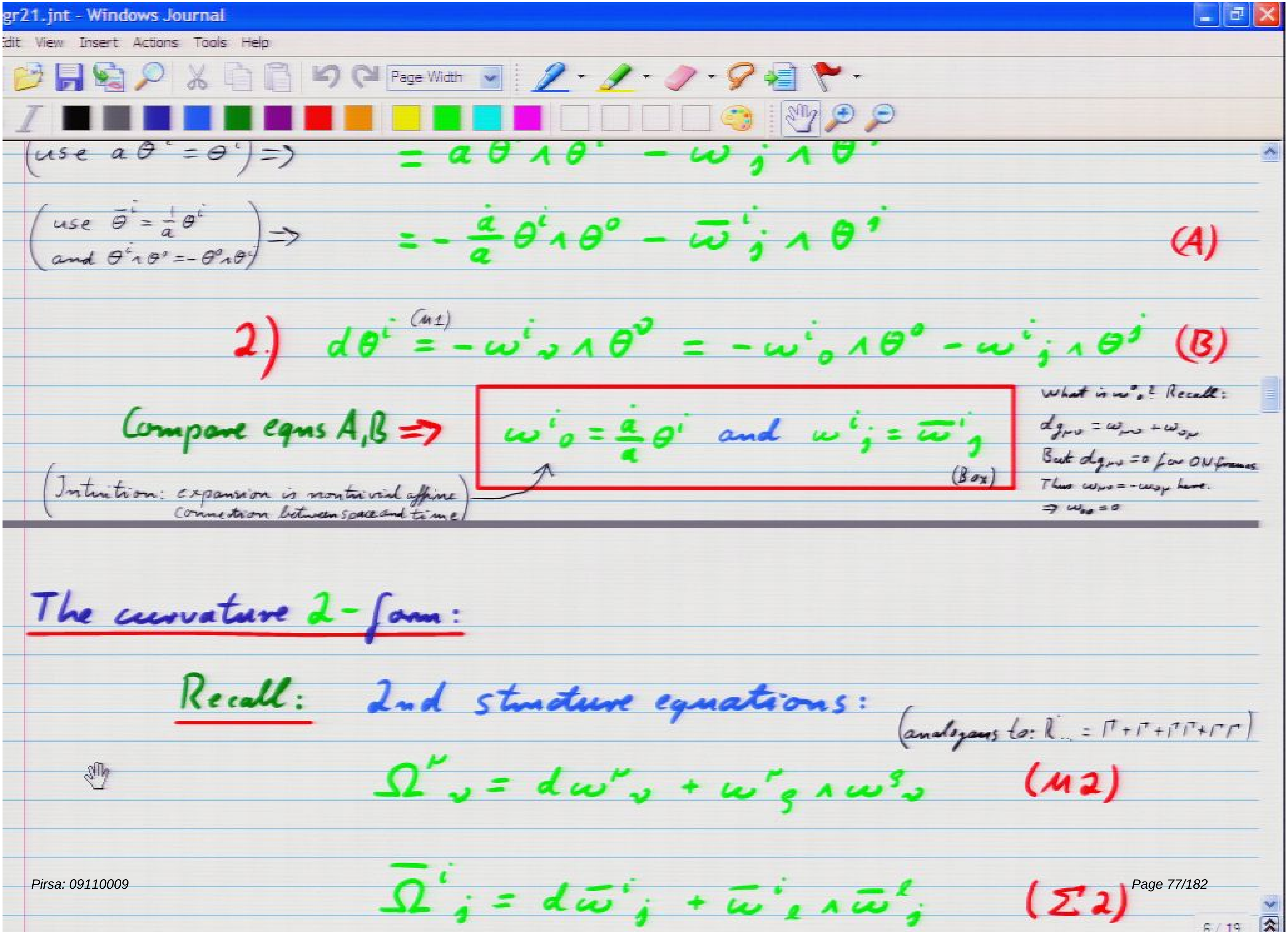

(use $a\theta^i = \theta^i$) ⇒ $= a\,\theta \wedge \theta^i - \omega_j \wedge \theta$

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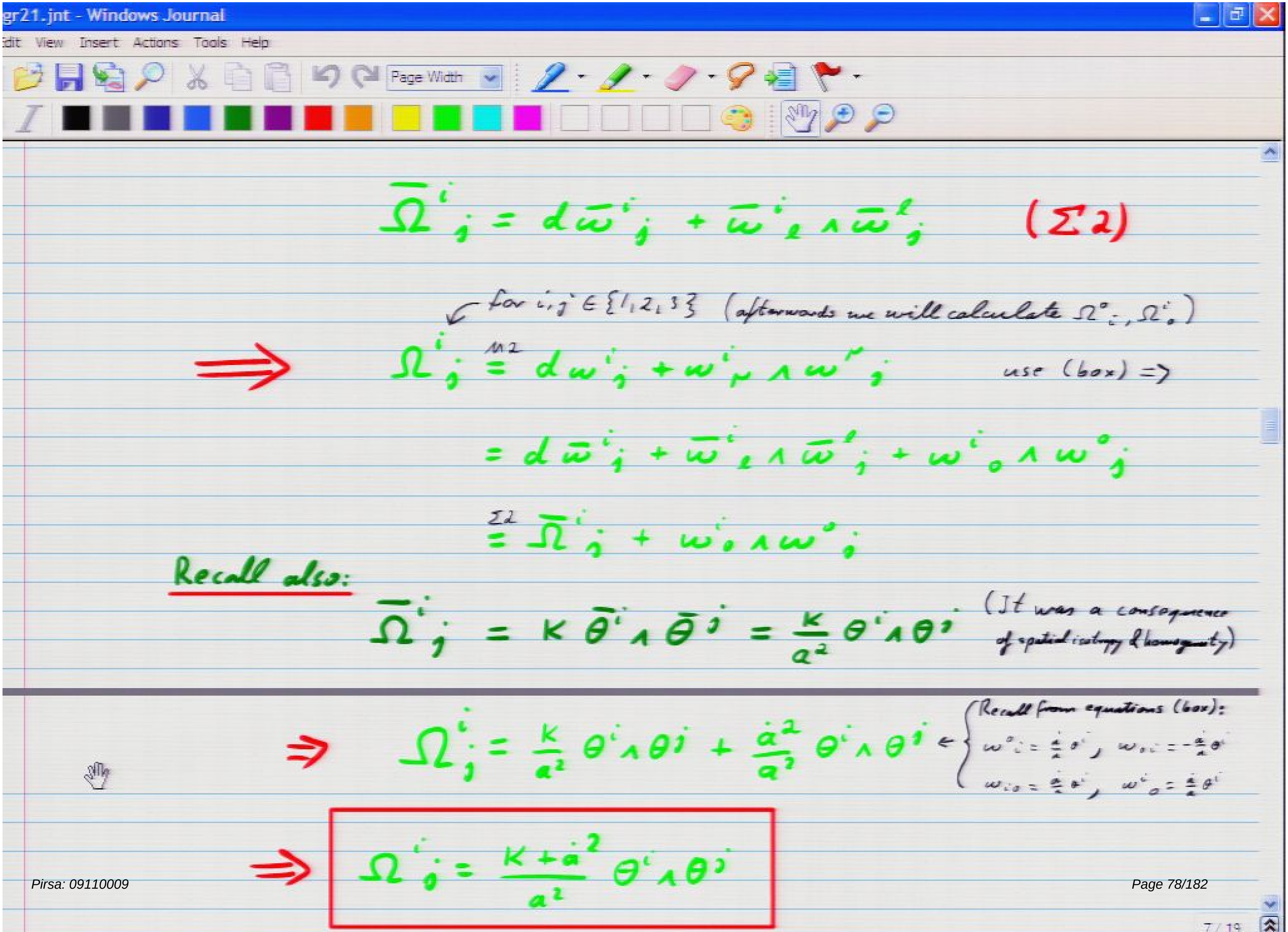

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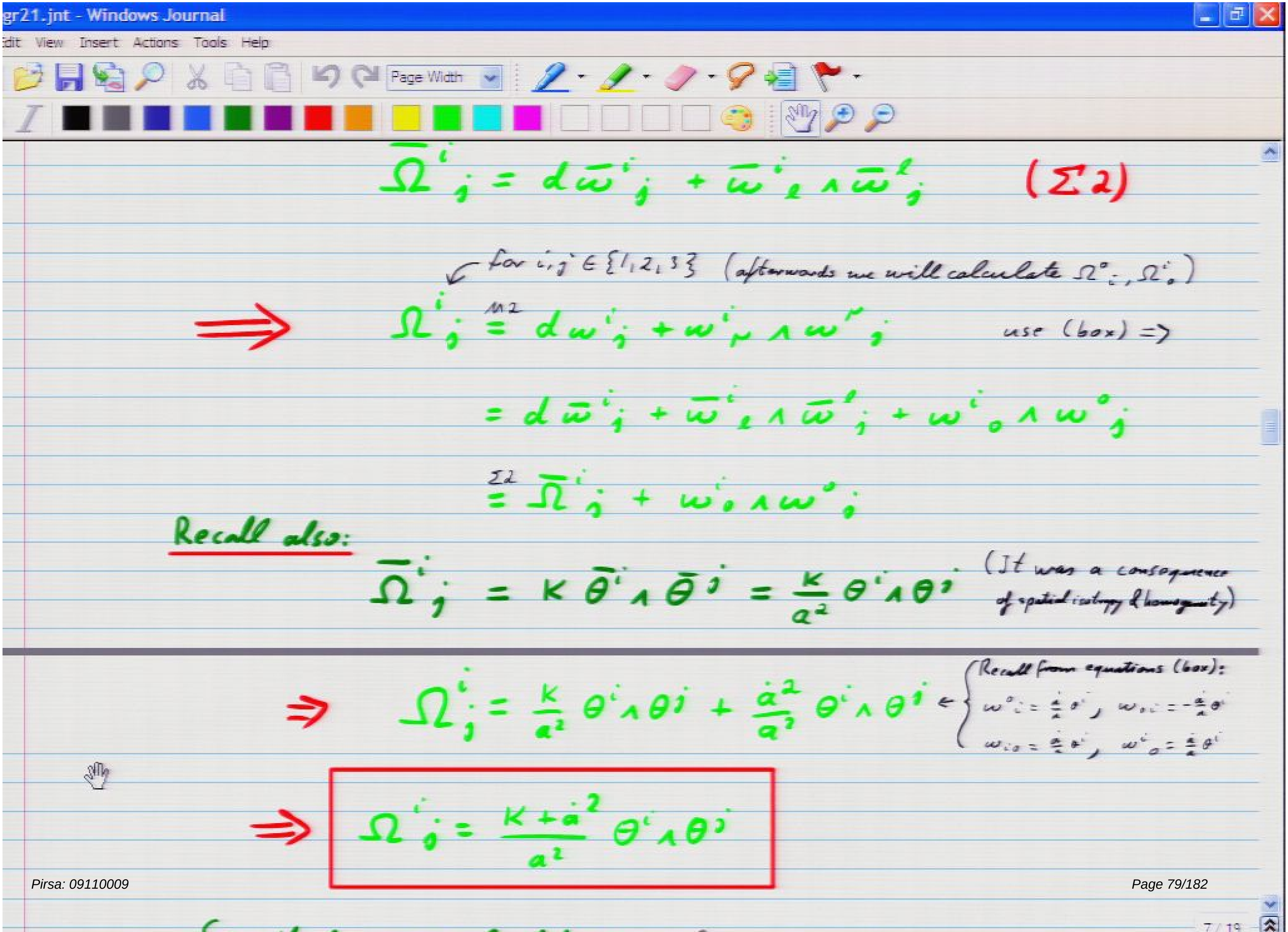

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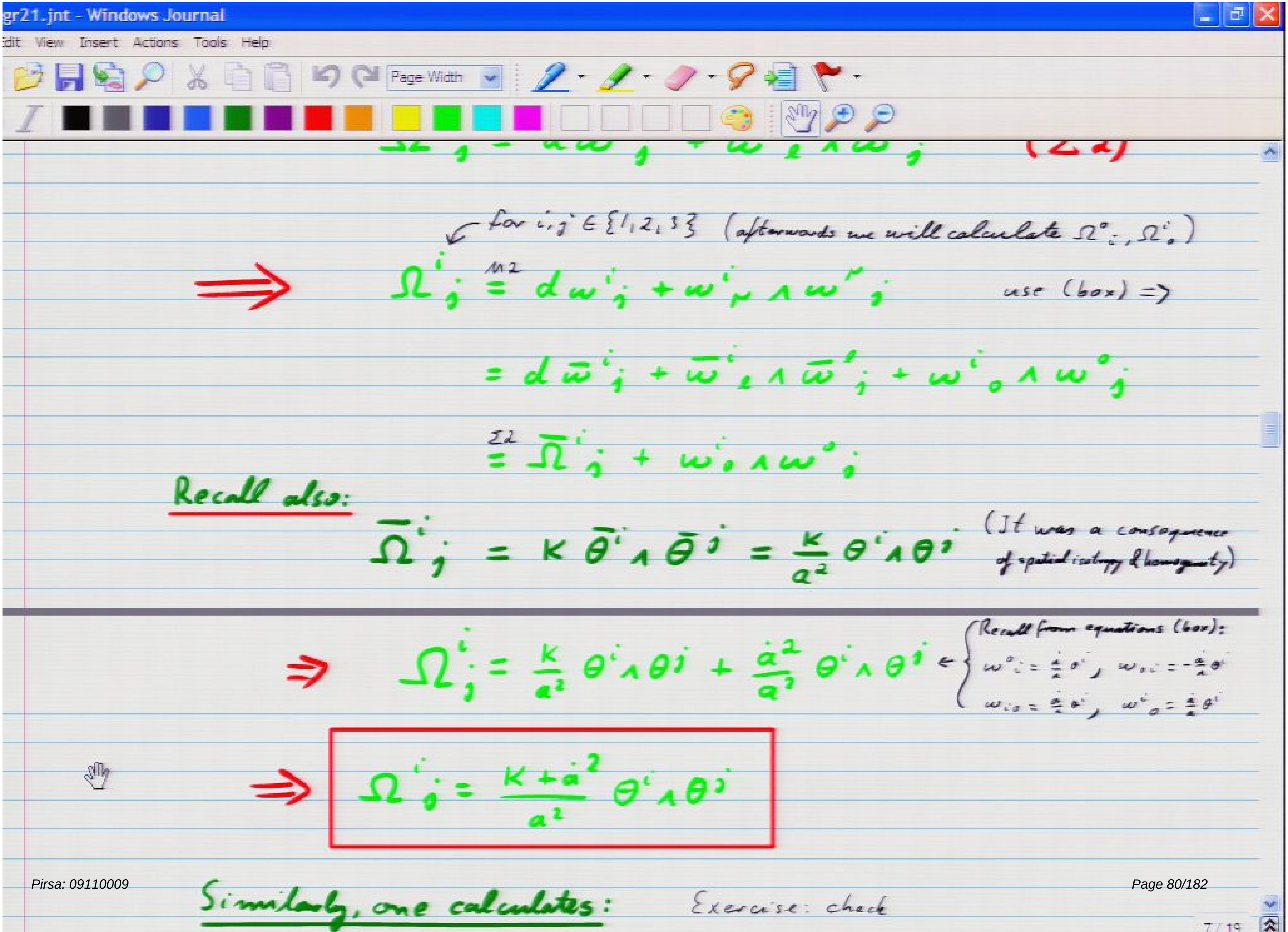

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⇒ $$\Omega^i{}_j = \frac{K}{a^2}\theta^i \wedge \theta^j + \frac{\dot{a}^2}{a^2}\theta^i \wedge \theta^j$$ ← Recall from equations (box): $\omega^0{}_i = \frac{\dot{a}}{a}\theta^i$, $\omega_{0i} = -\frac{\dot{a}}{a}\theta^i$, $\omega_{i0} = \frac{\dot{a}}{a}\theta^i$, $\omega^i{}_0 = \frac{\dot{a}}{a}\theta^i$

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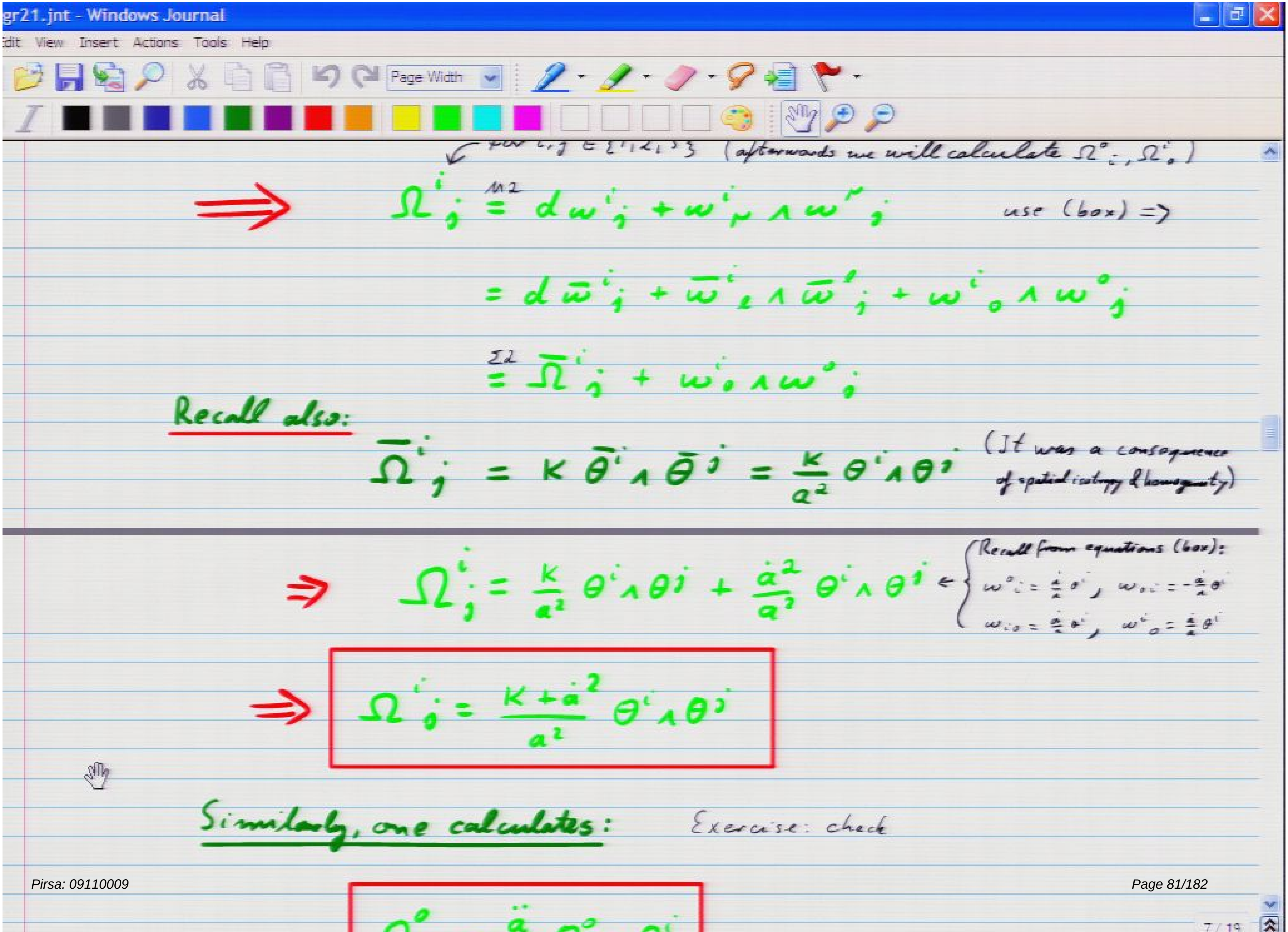
Similarly, one calculates: Exercise: check



for $i,j \in \{1,2,3\}$ (afterwards we will calculate $\Omega^0{}_i, \Omega^i{}_0$)

$$\Longrightarrow \quad \Omega^i{}_j \overset{M2}{=} d\omega^i{}_j + \omega^i{}_\mu \wedge \omega^\mu{}_j$$ use (box) =>

$$= d\bar{\omega}^i{}_j + \bar{\omega}^i{}_l \wedge \bar{\omega}^l{}_j + \omega^i{}_0 \wedge \omega^0{}_j$$

$$\overset{\Sigma 2}{=} \bar{\Omega}^i{}_j + \omega^i{}_0 \wedge \omega^0{}_j$$

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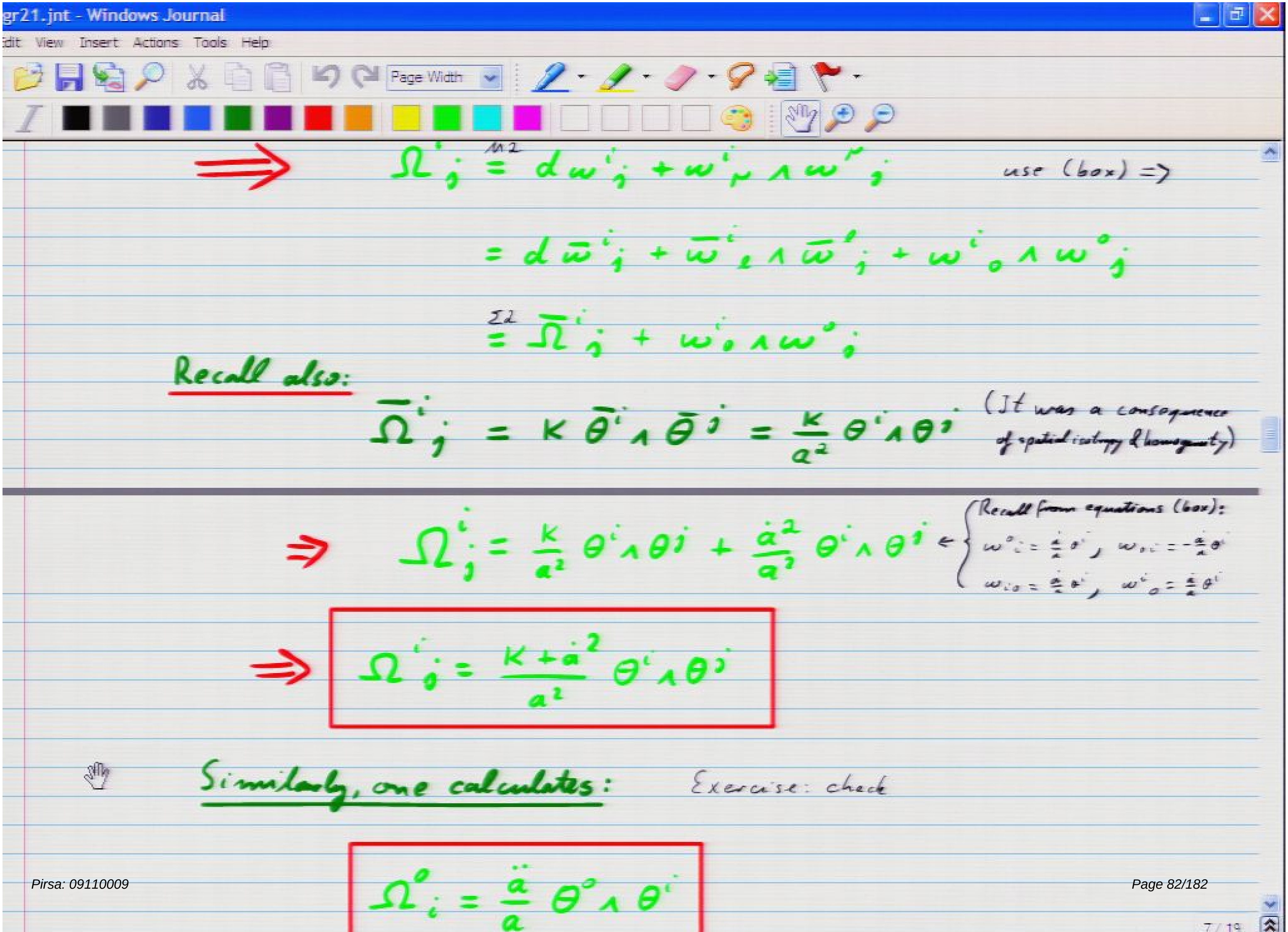

$$\Rightarrow \quad \Omega^i{}_j \overset{\Omega 2}{=} d\omega^i{}_j + \omega^i{}_\mu \wedge \omega^\mu{}_j$$

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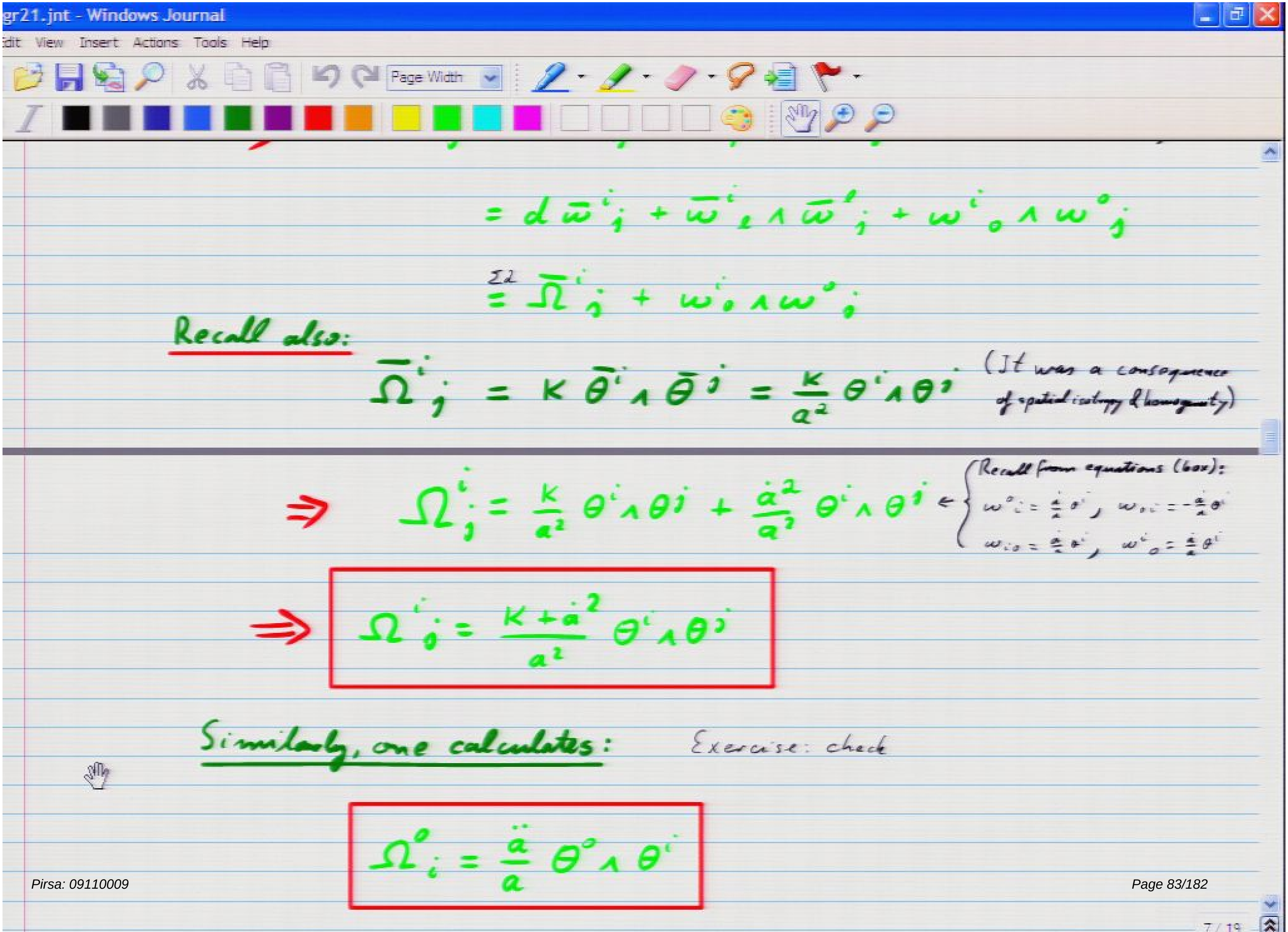

$$= d\bar{\omega}^i{}_j + \bar{\omega}^i{}_l \wedge \bar{\omega}^l{}_j + \omega^i{}_0 \wedge \omega^0{}_j$$

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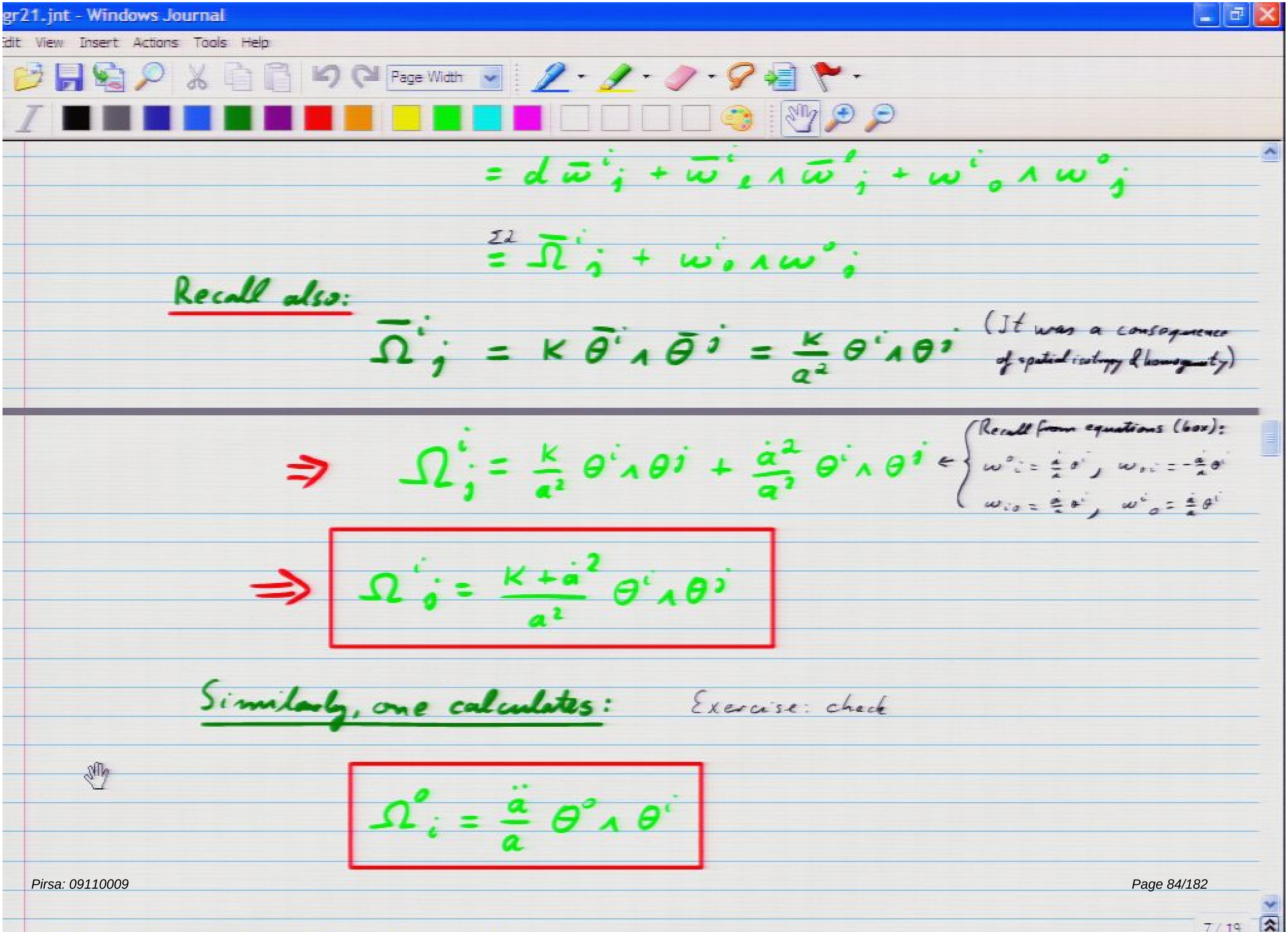

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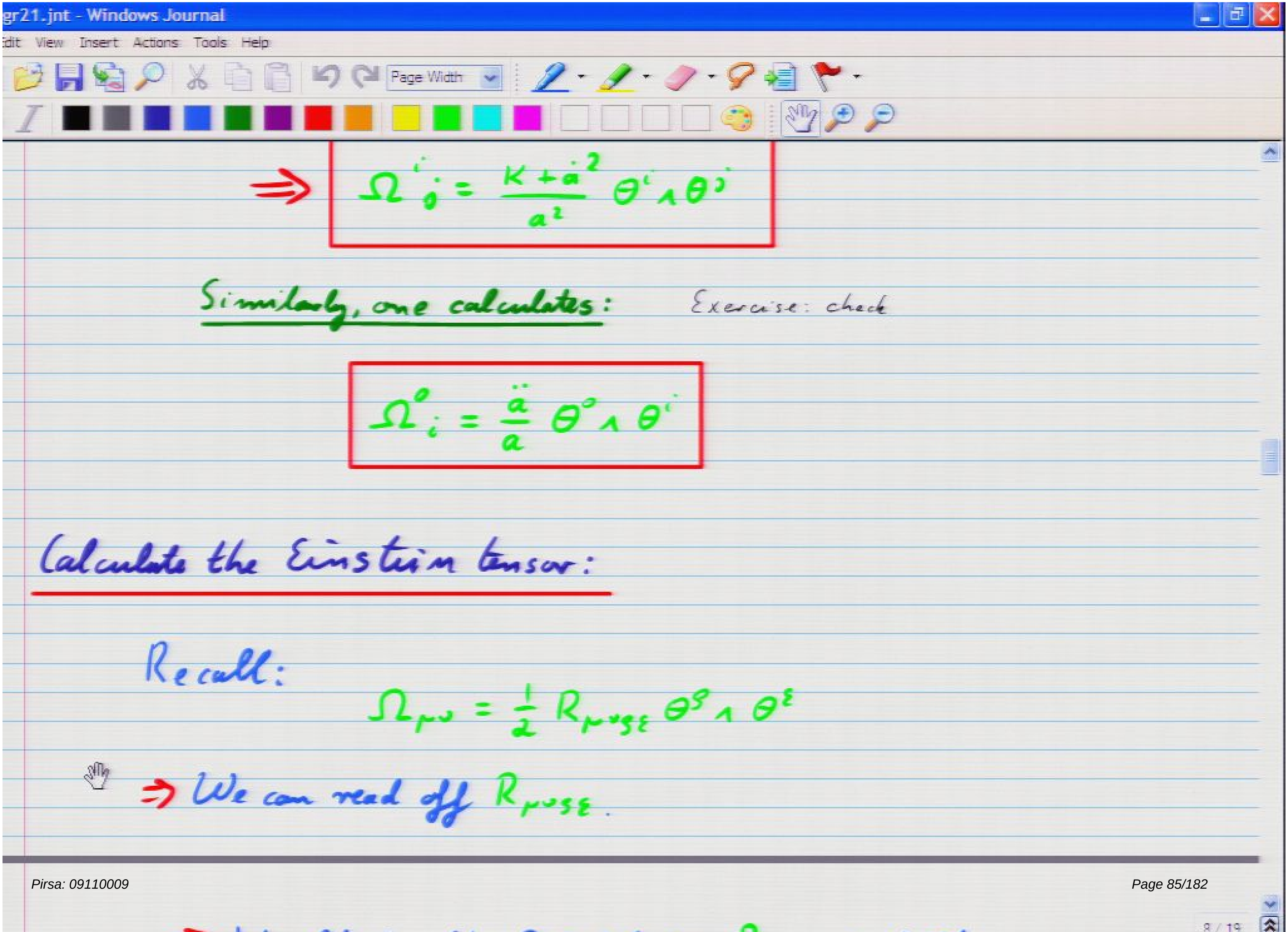

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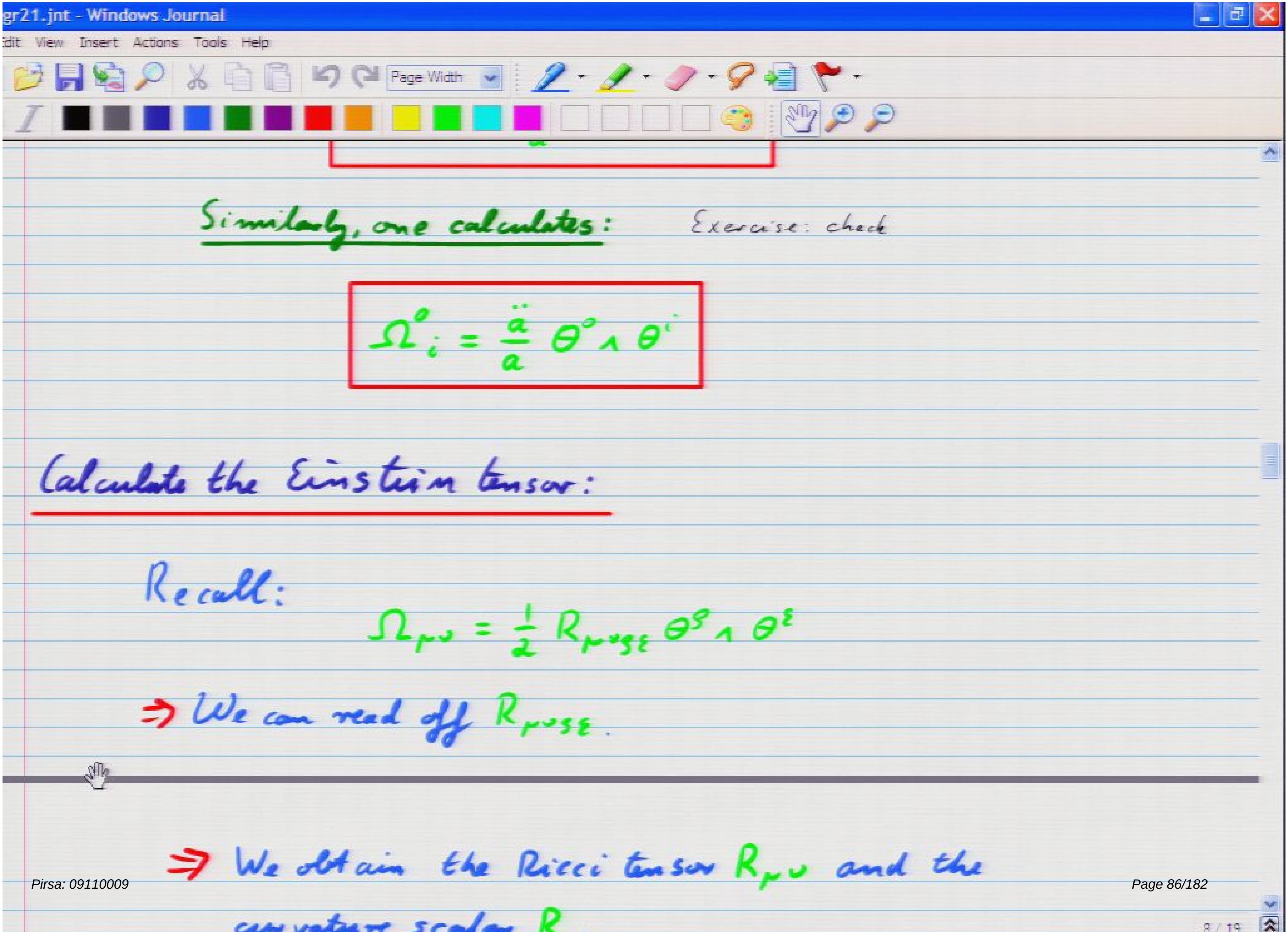
$$\Omega_{\mu\nu} = \frac{1}{2} R_{\mu\nu\rho\varepsilon}\,\theta^\rho\wedge\theta^\varepsilon$$

⇒ We can read off $R_{\mu\nu\rho\varepsilon}$.



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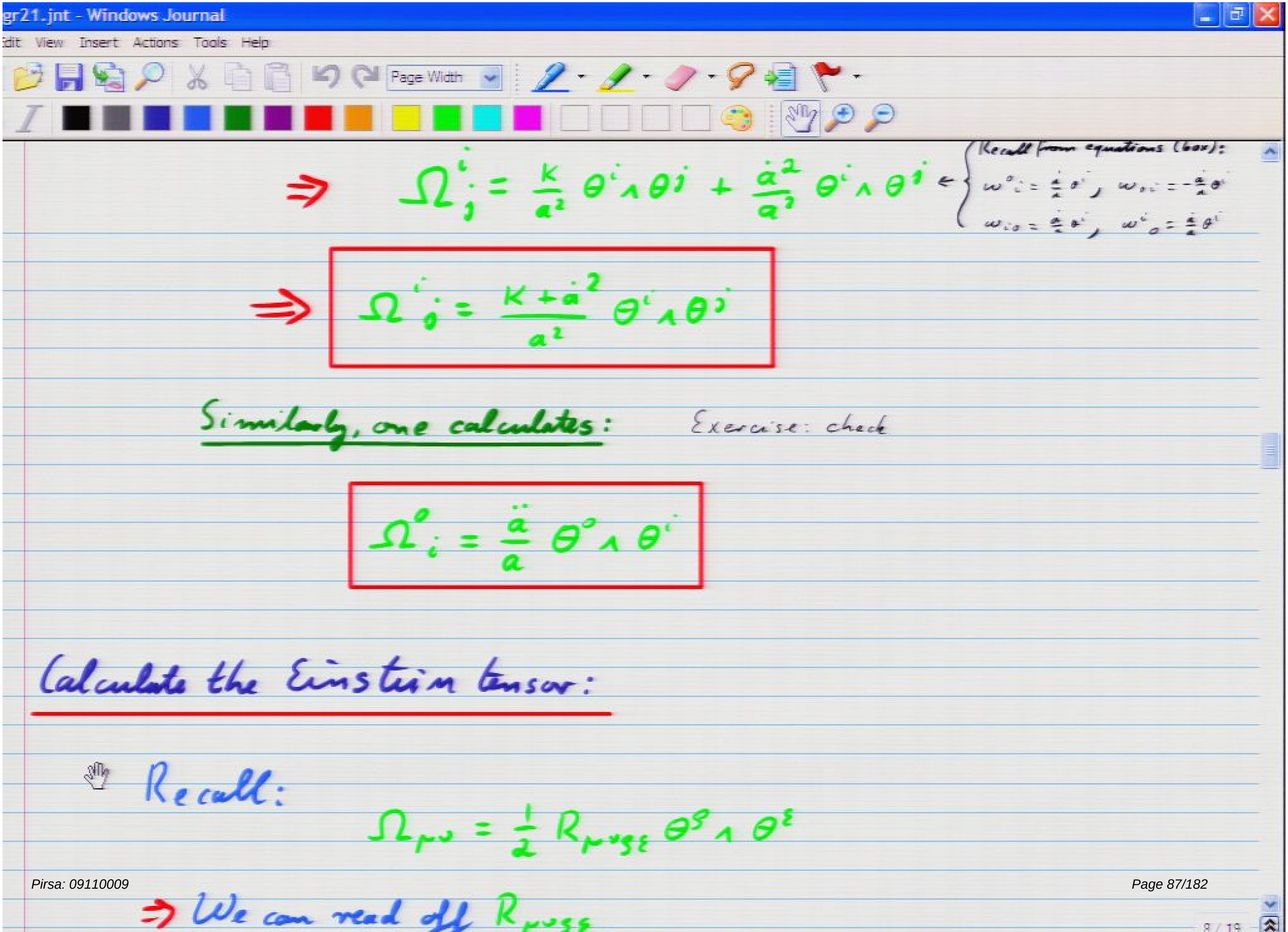

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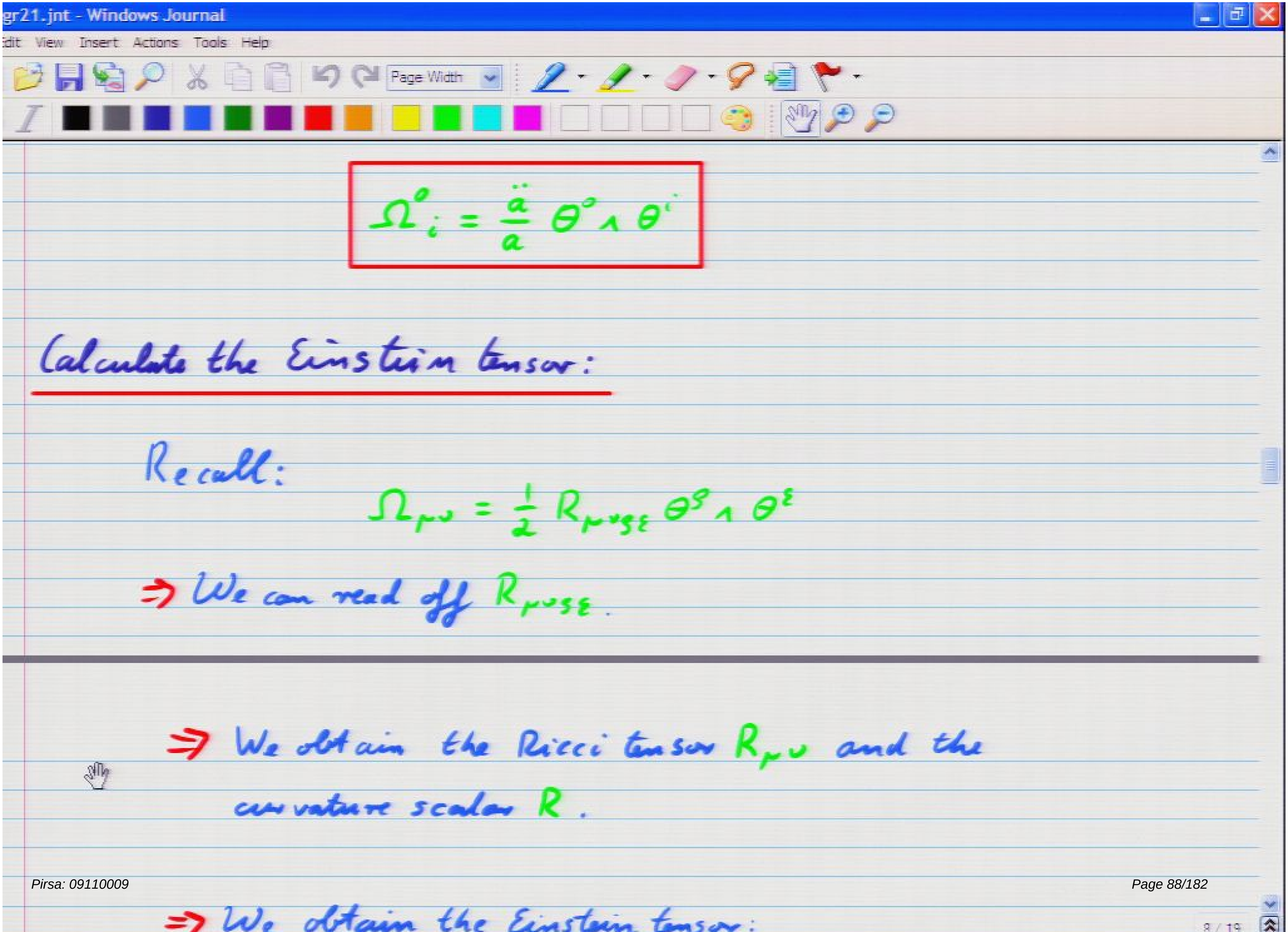

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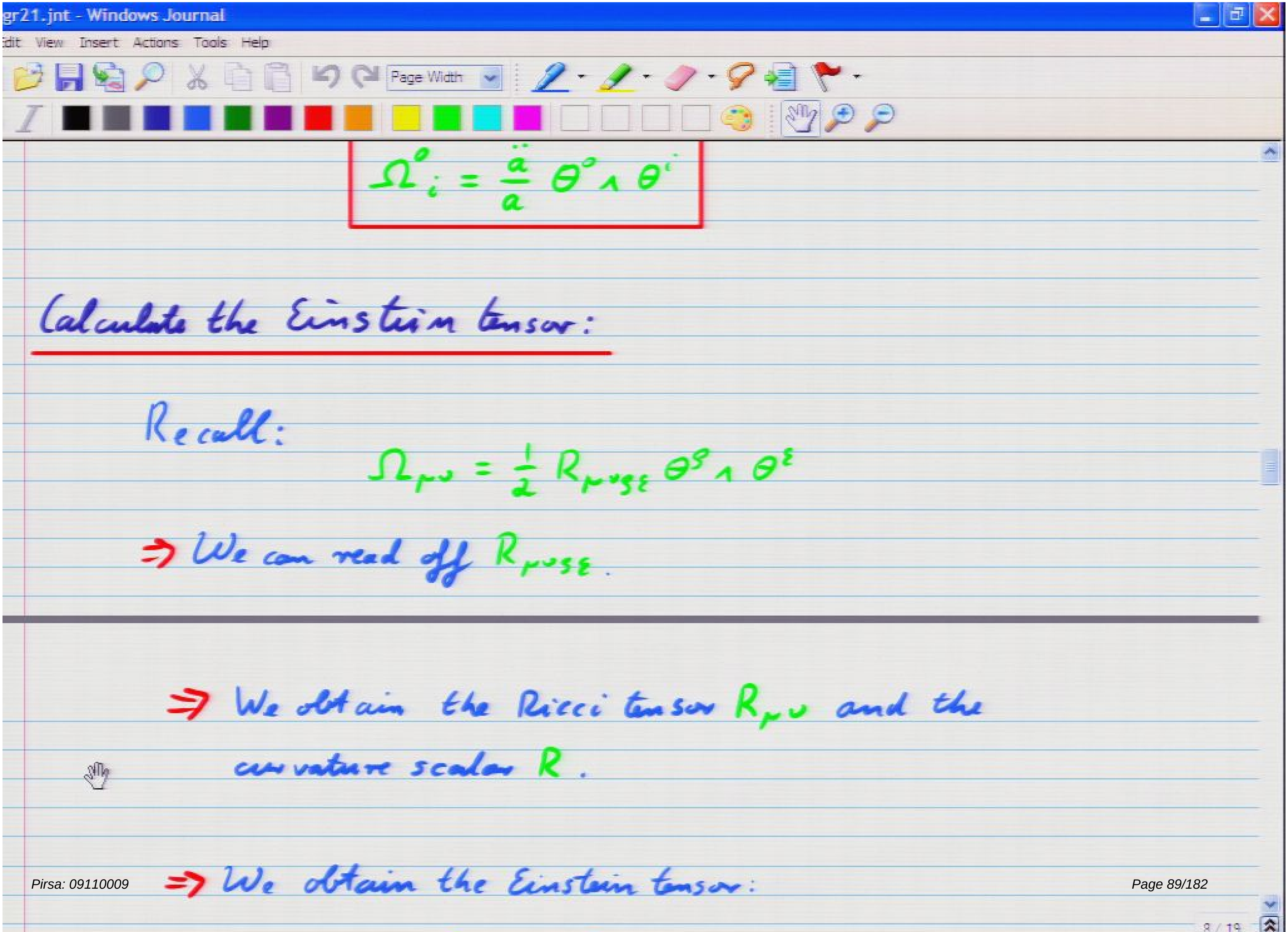

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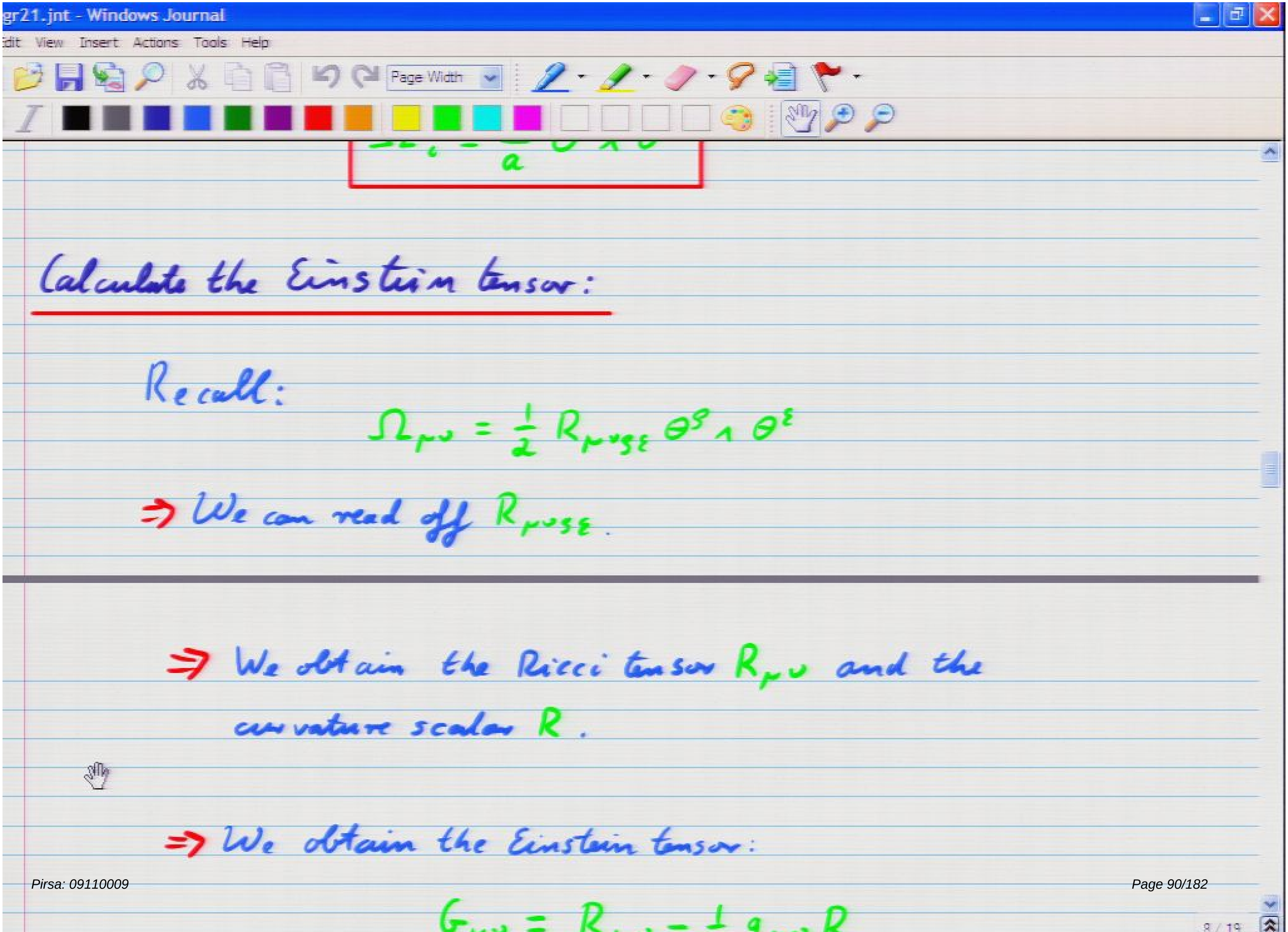

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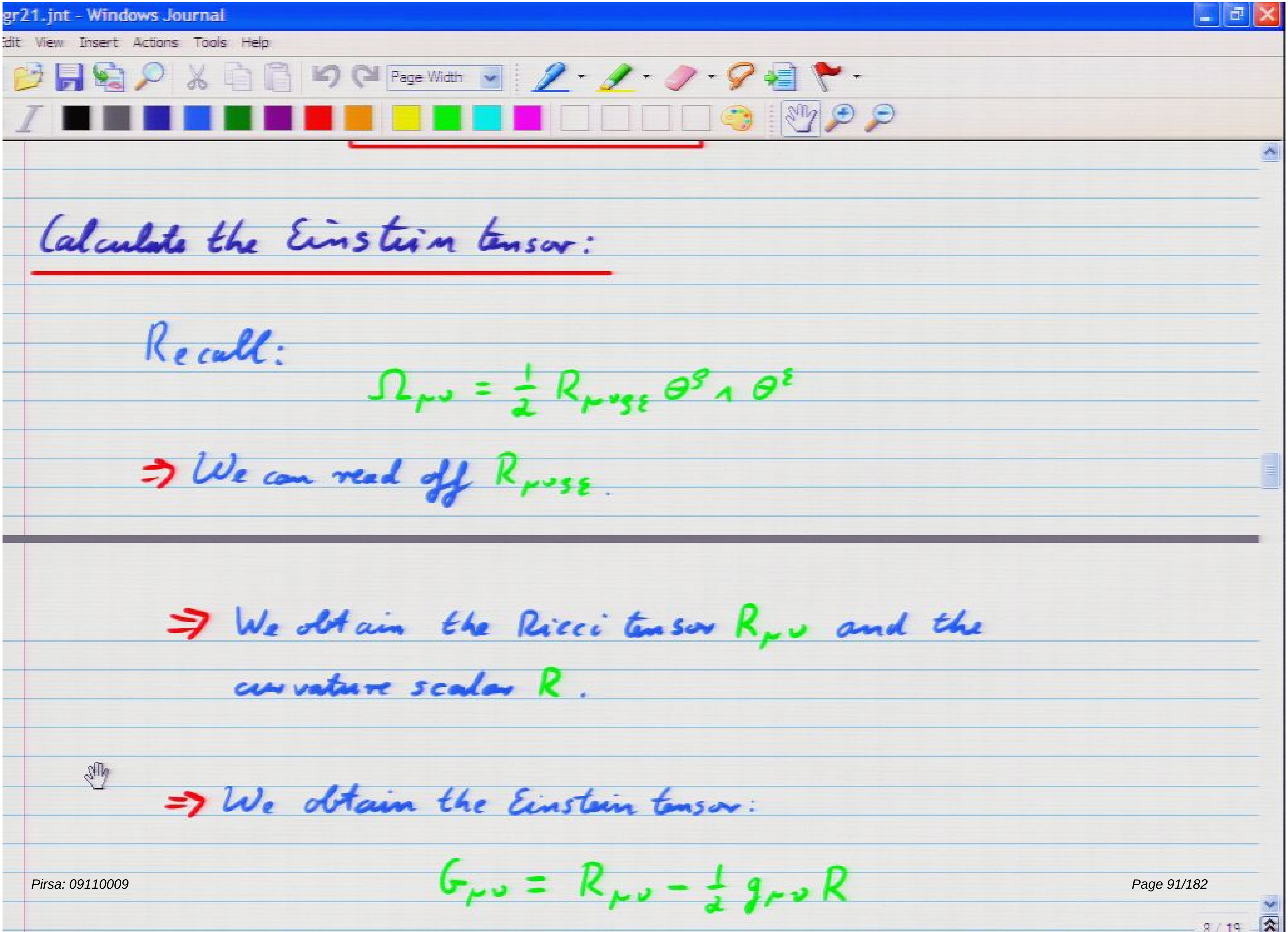

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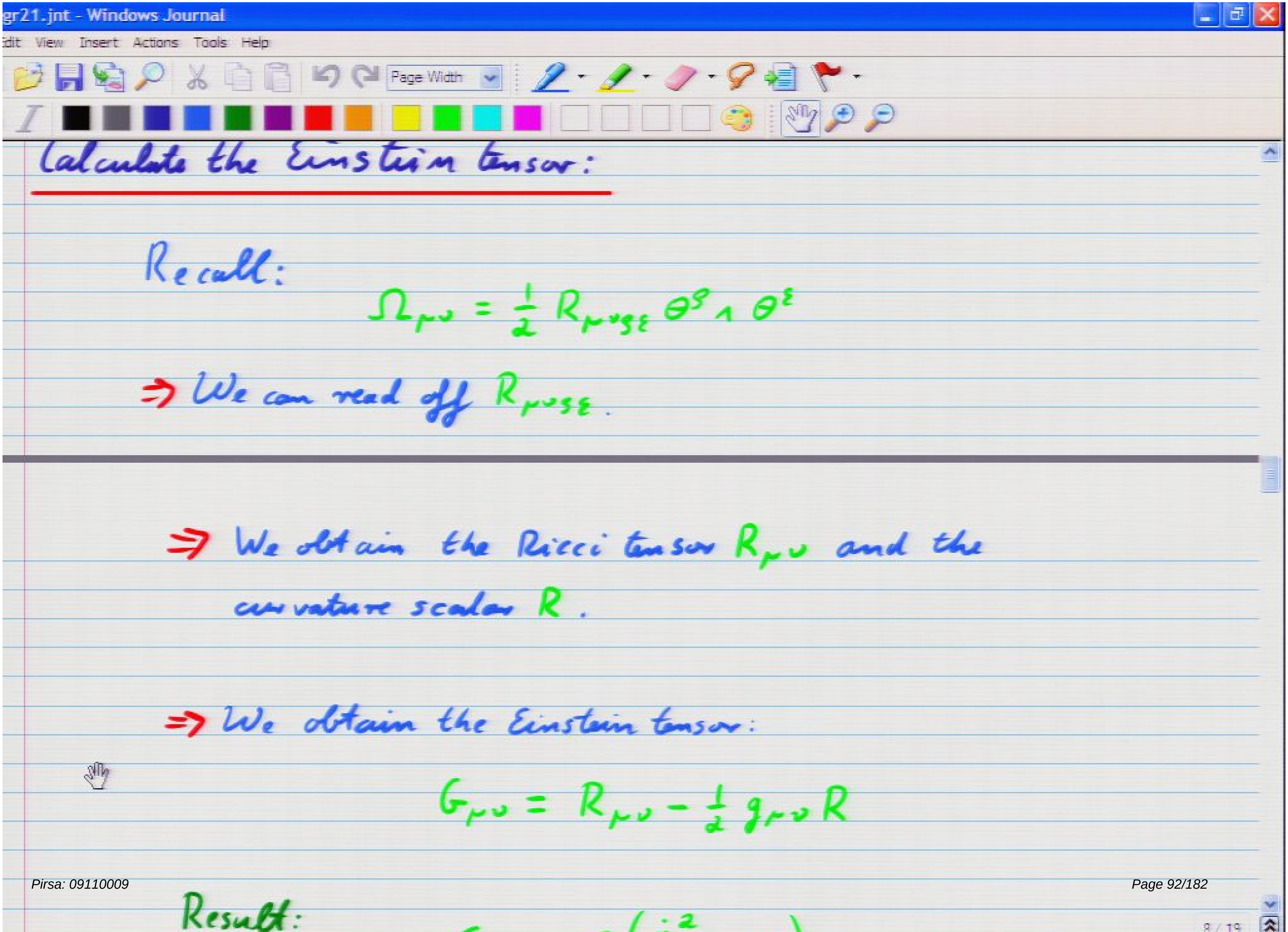

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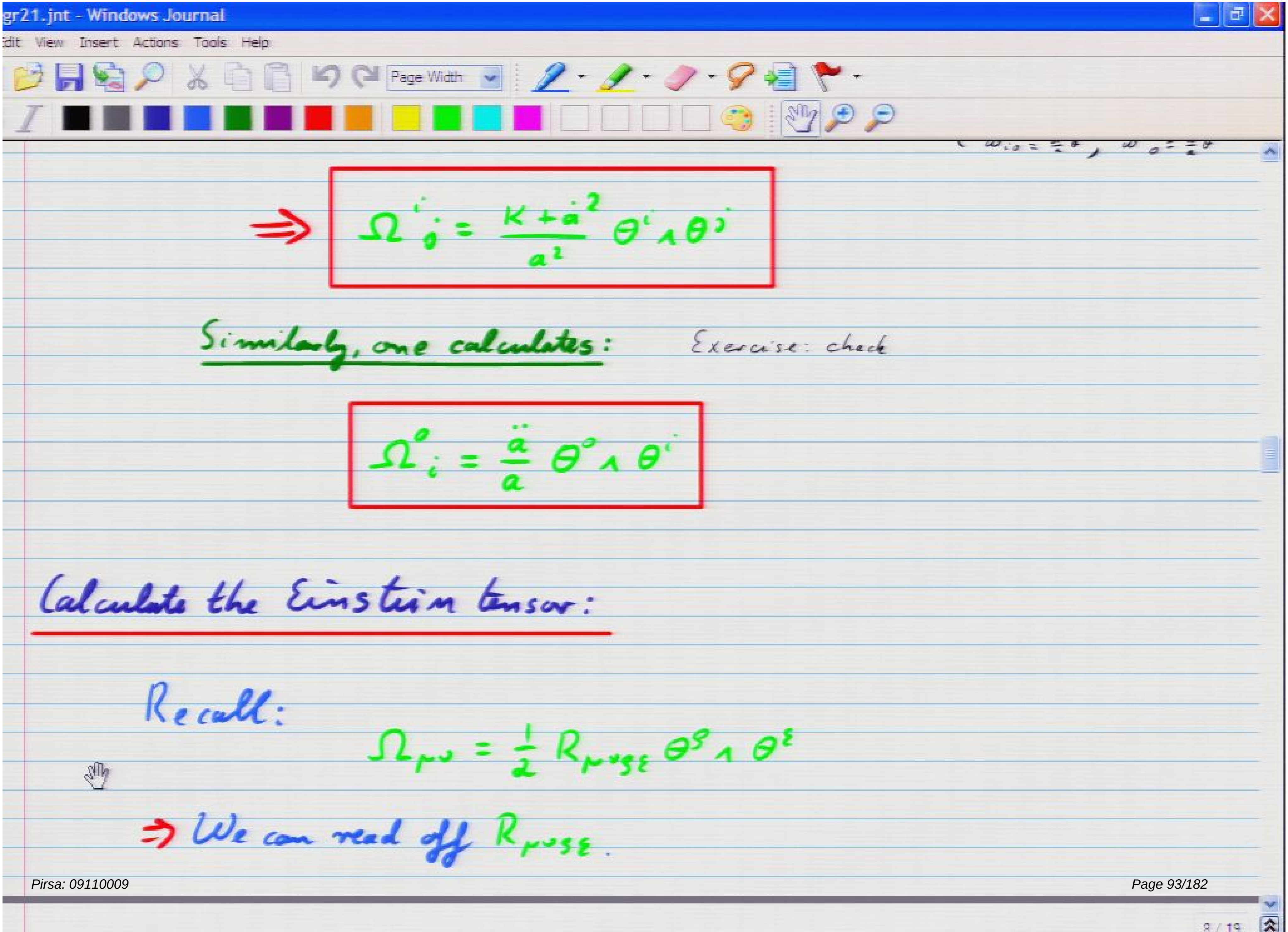

$$\Rightarrow \boxed{\Omega^{i}{}_{j} = \frac{k + \dot{a}^{2}}{a^{2}}\, \theta^{i} \wedge \theta^{j}}$$

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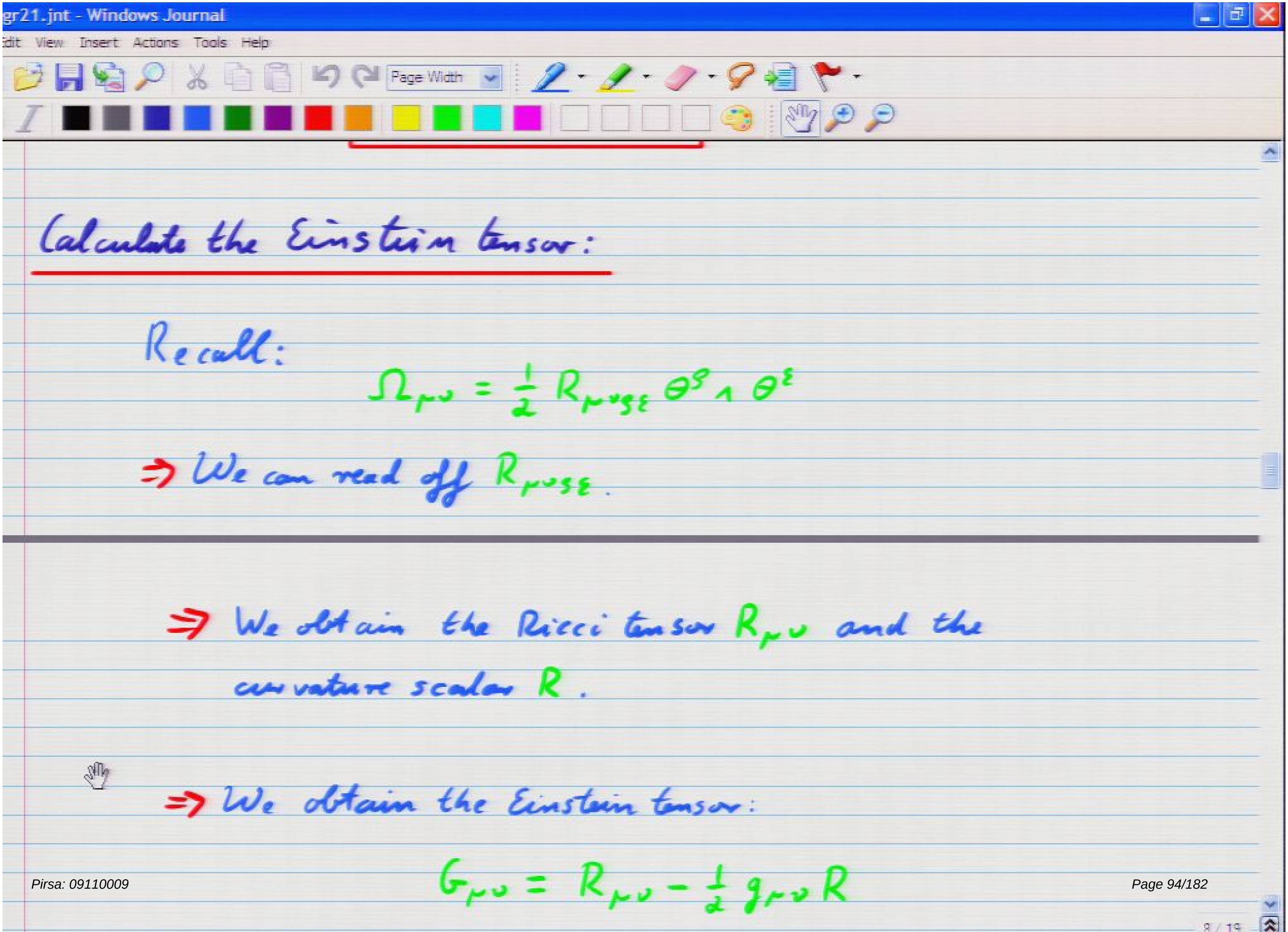

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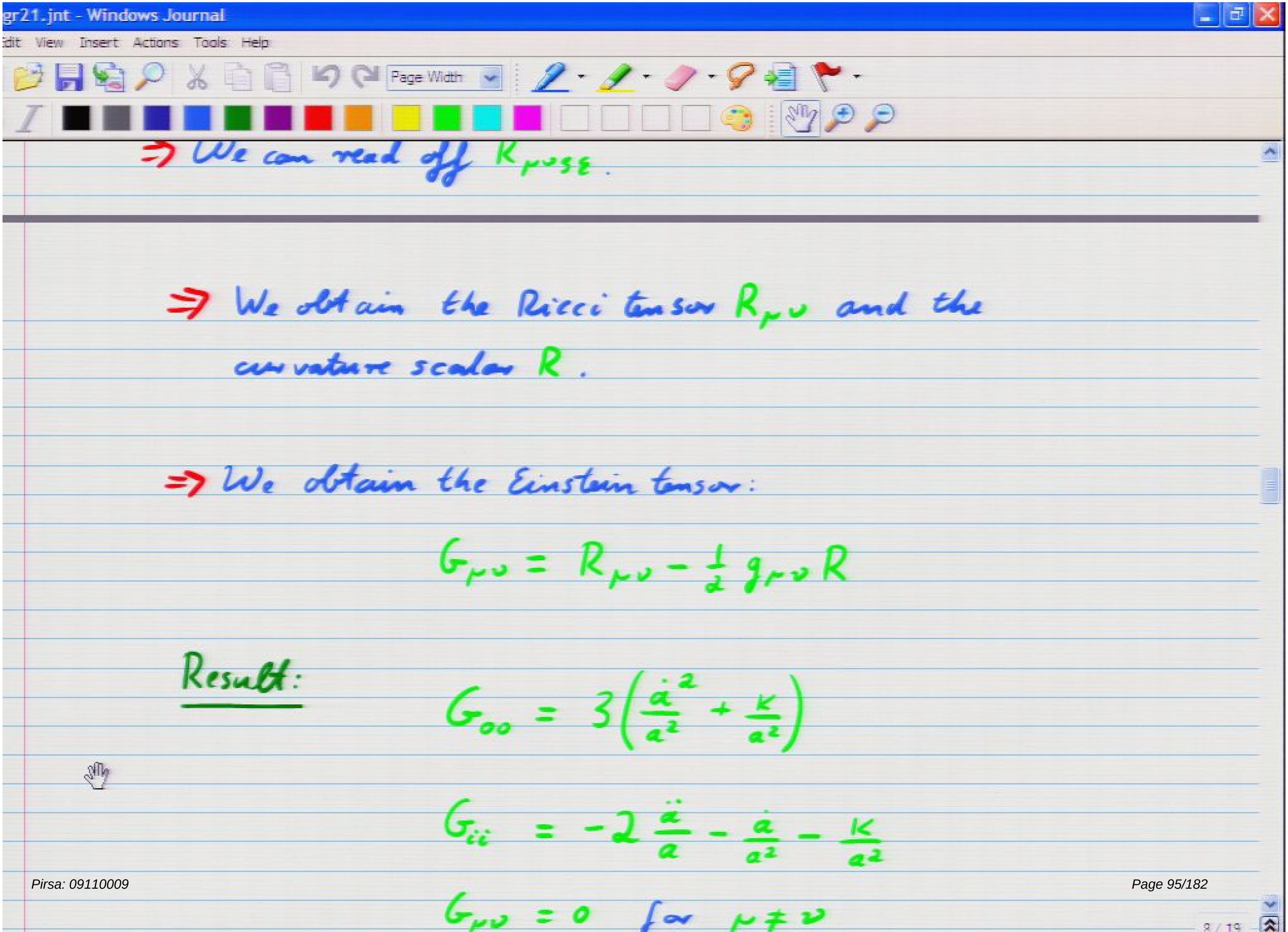

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$$G_{00} = 3\left(\frac{\dot{a}^2}{a^2} + \frac{K}{a^2}\right)$$

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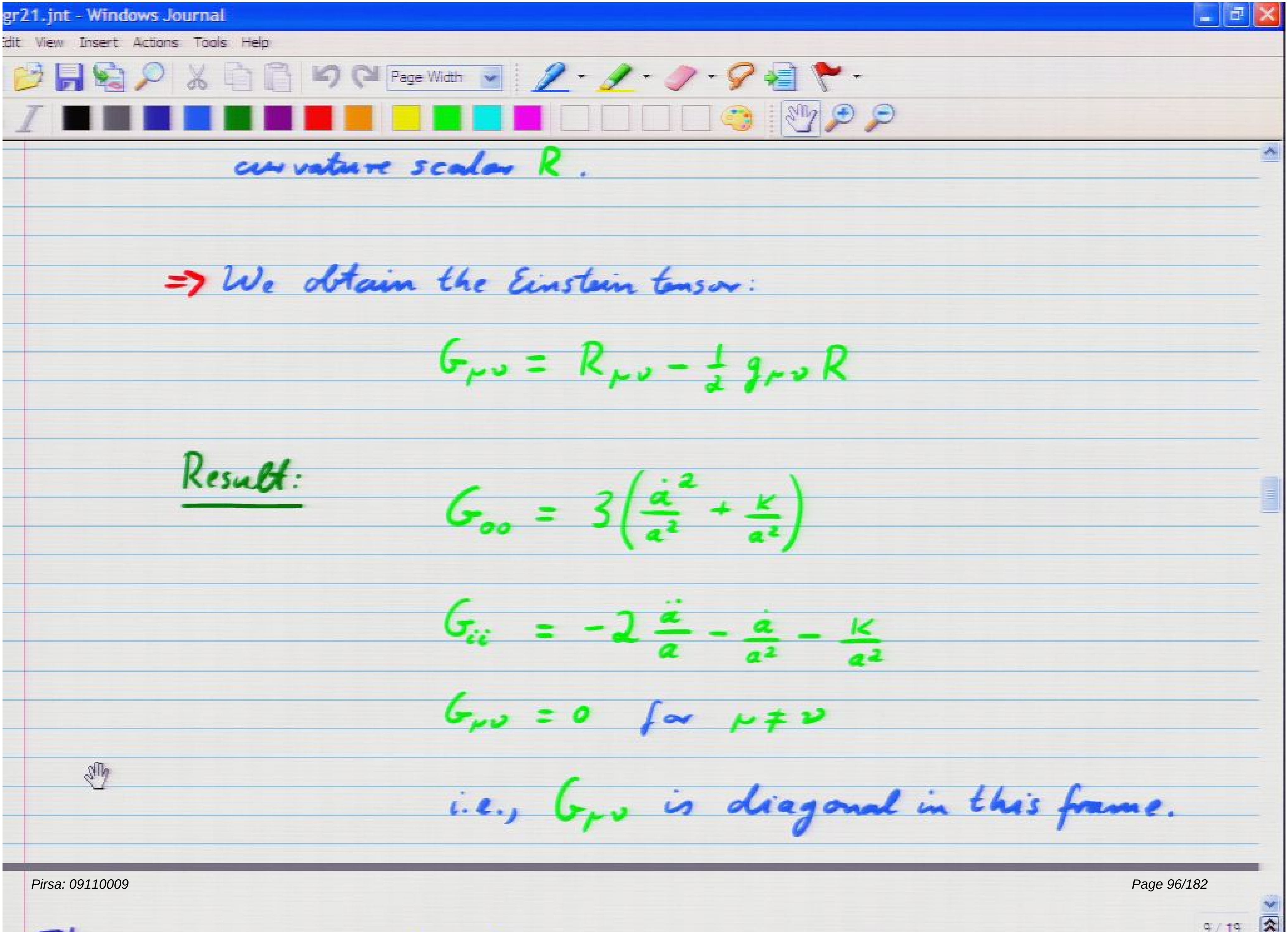
$G_{\mu\nu} = 0$ for $\mu \neq \nu$



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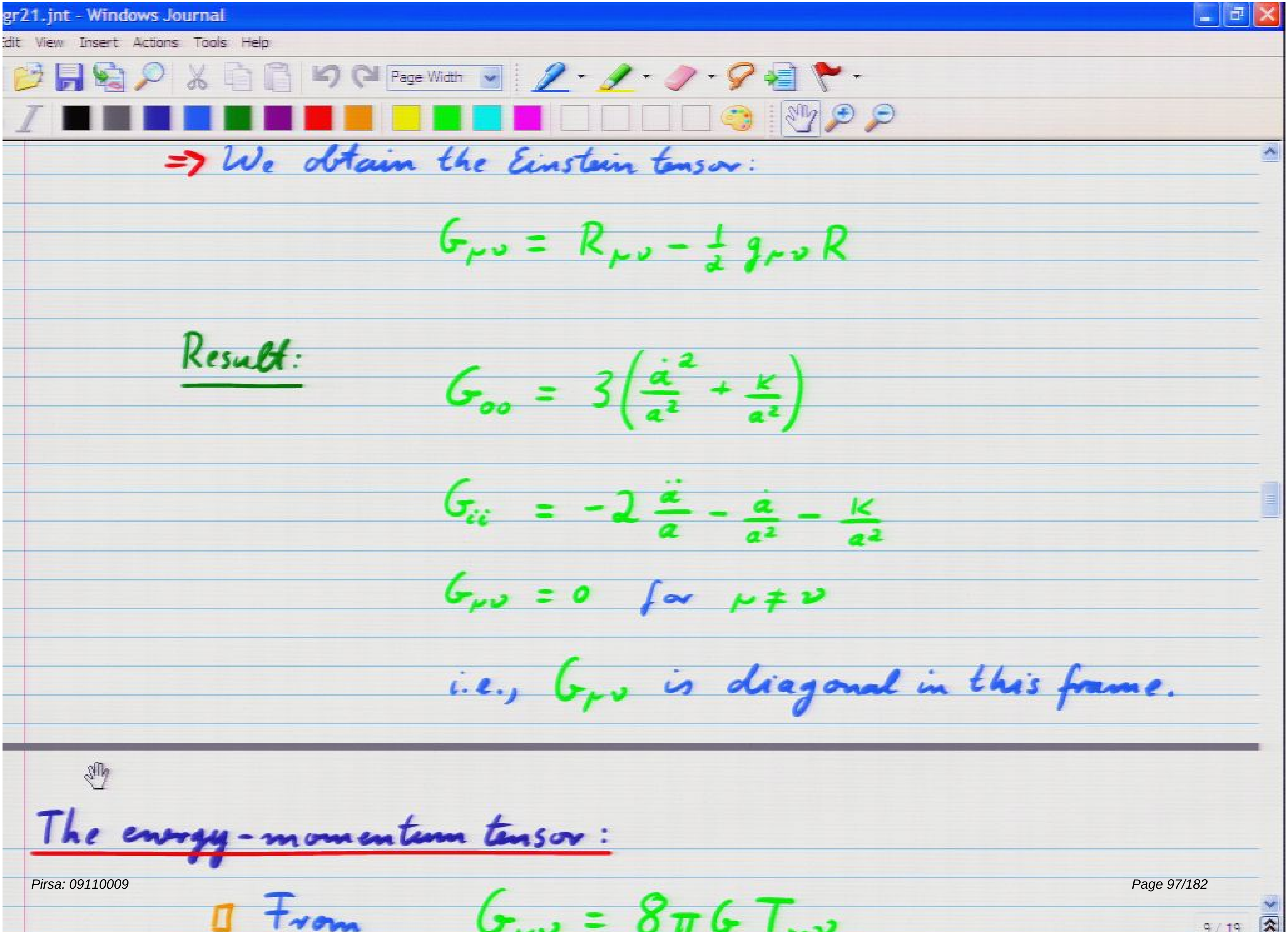

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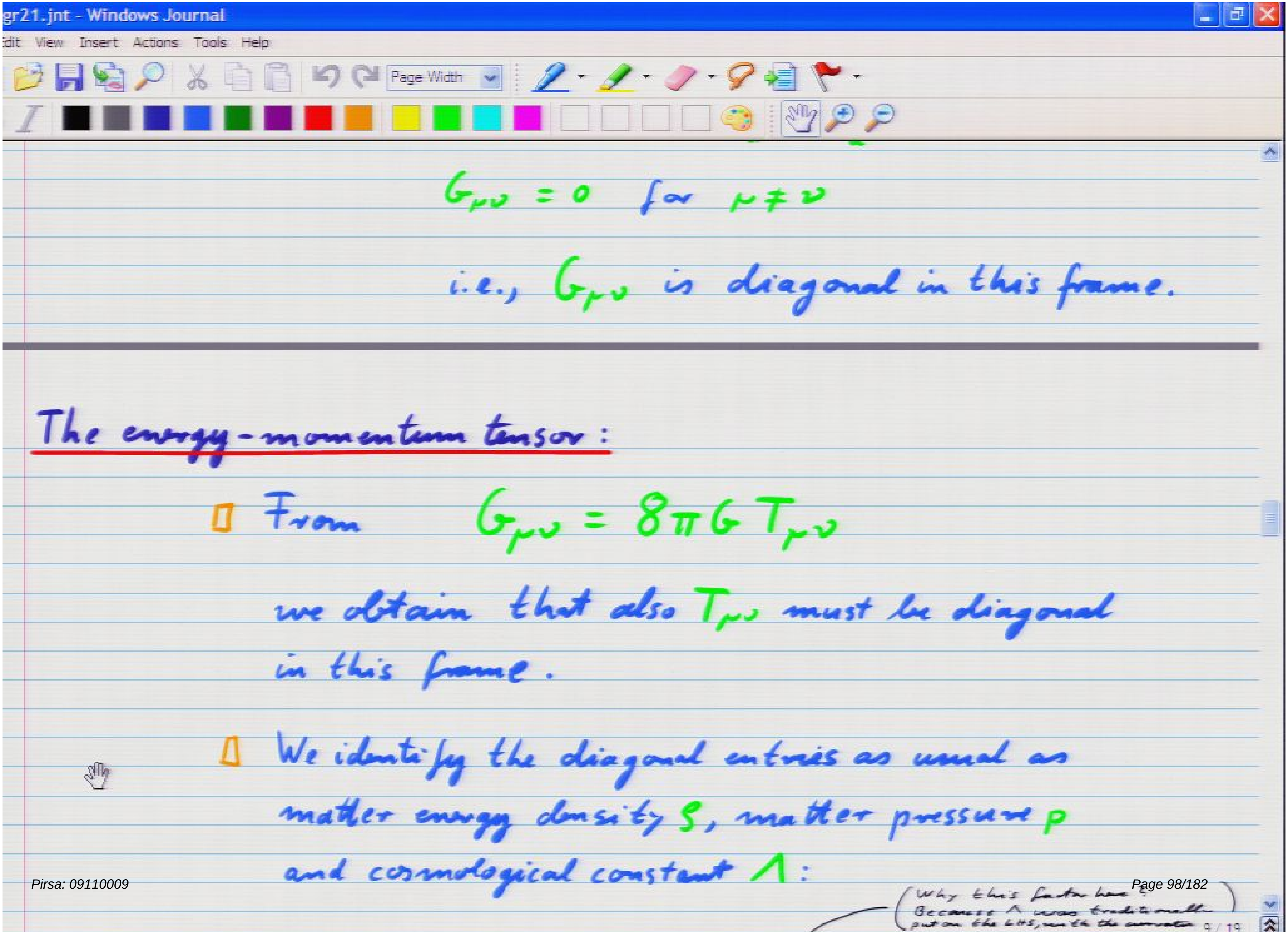
The energy-momentum tensor:

▯ From $G_{\mu\nu} = 8\pi G T_{\mu\nu}$



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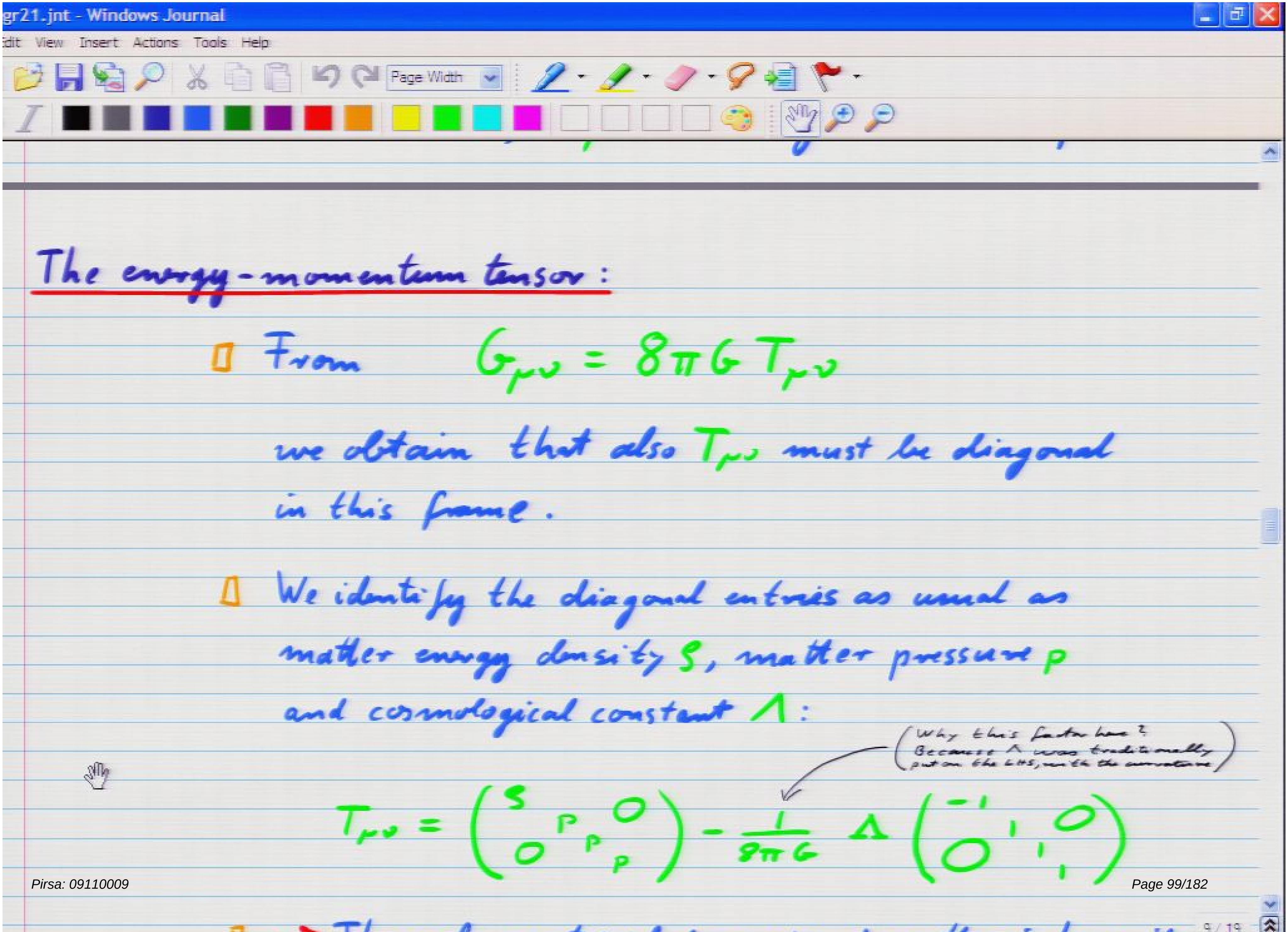
The energy-momentum tensor:

- From $G_{\mu\nu} = 8\pi G\, T_{\mu\nu}$

  we obtain that also $T_{\mu\nu}$ must be diagonal in this frame.

- We identify the diagonal entries as usual as matter energy density ρ, matter pressure p and cosmological constant Λ:

(Why this factor here? Because Λ was traditionally put on the LHS, with the curvature



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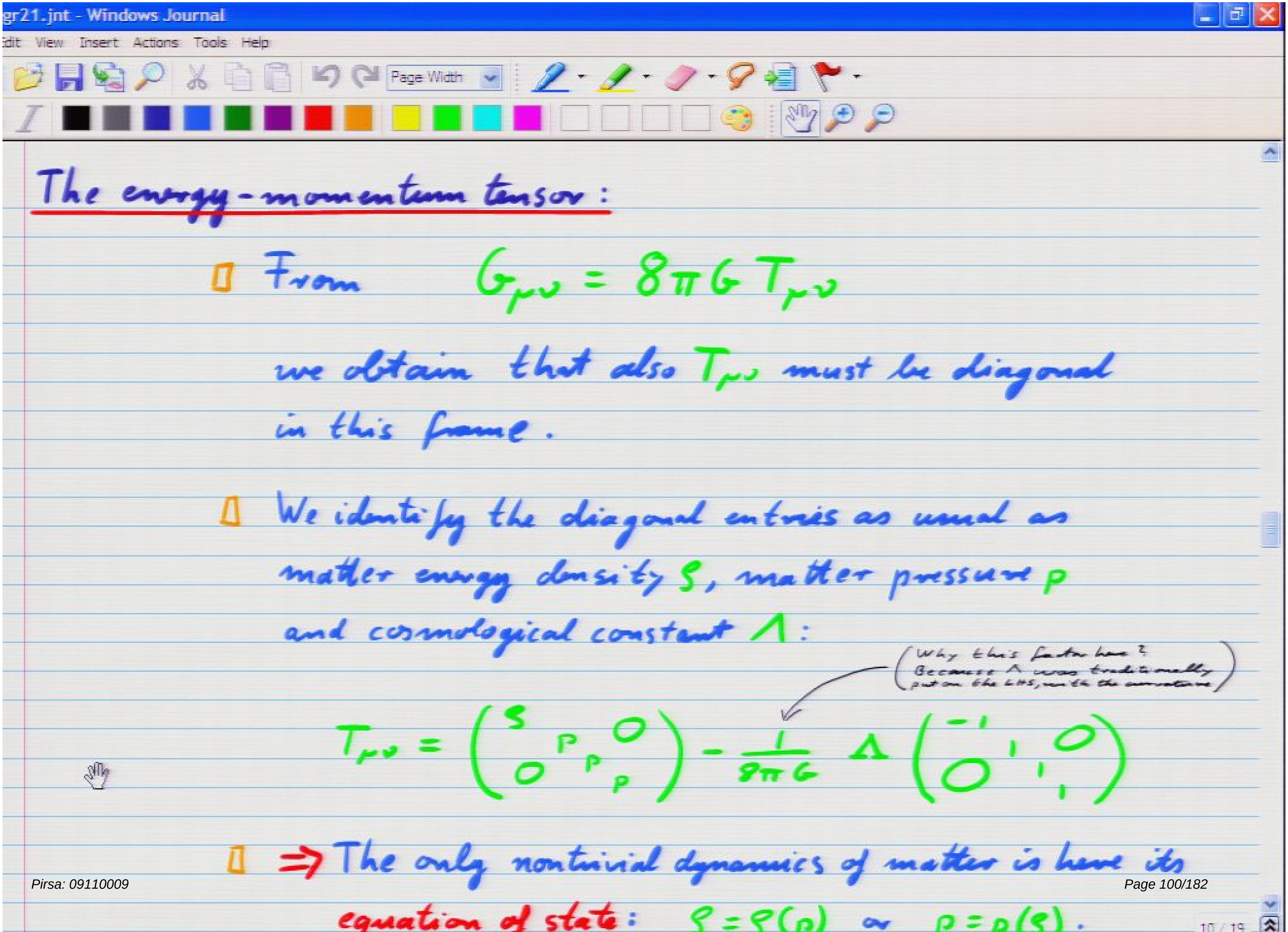

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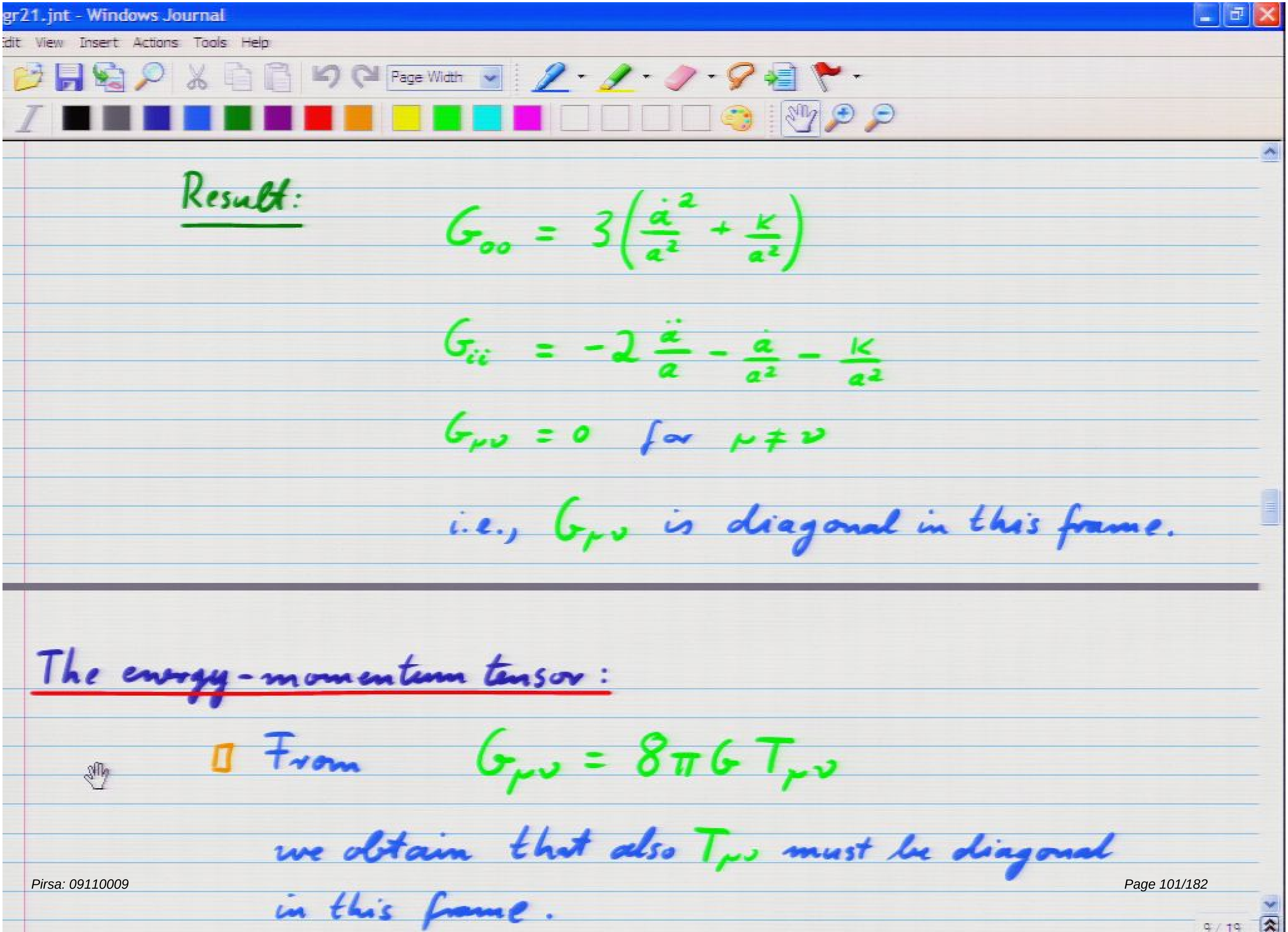

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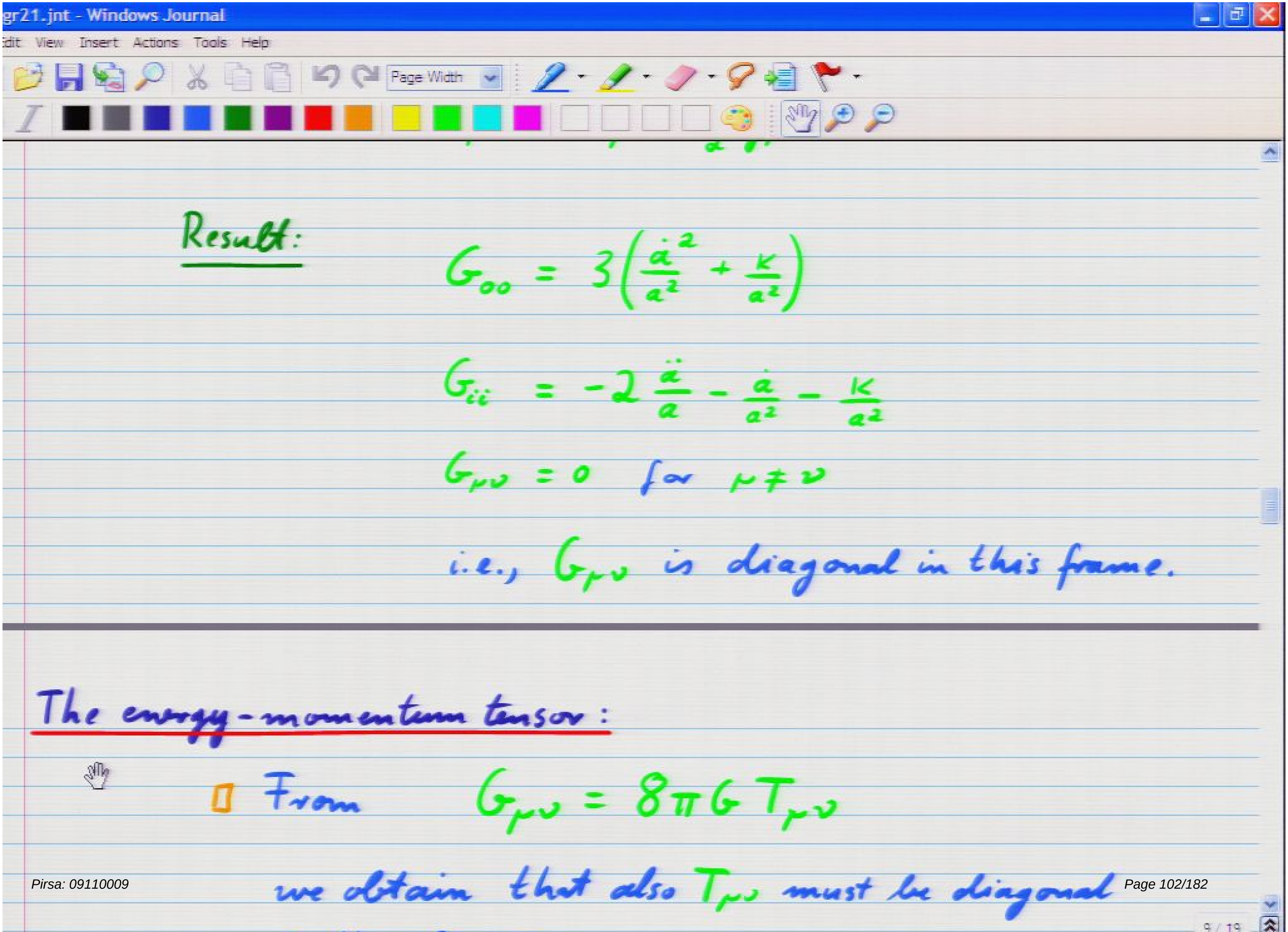

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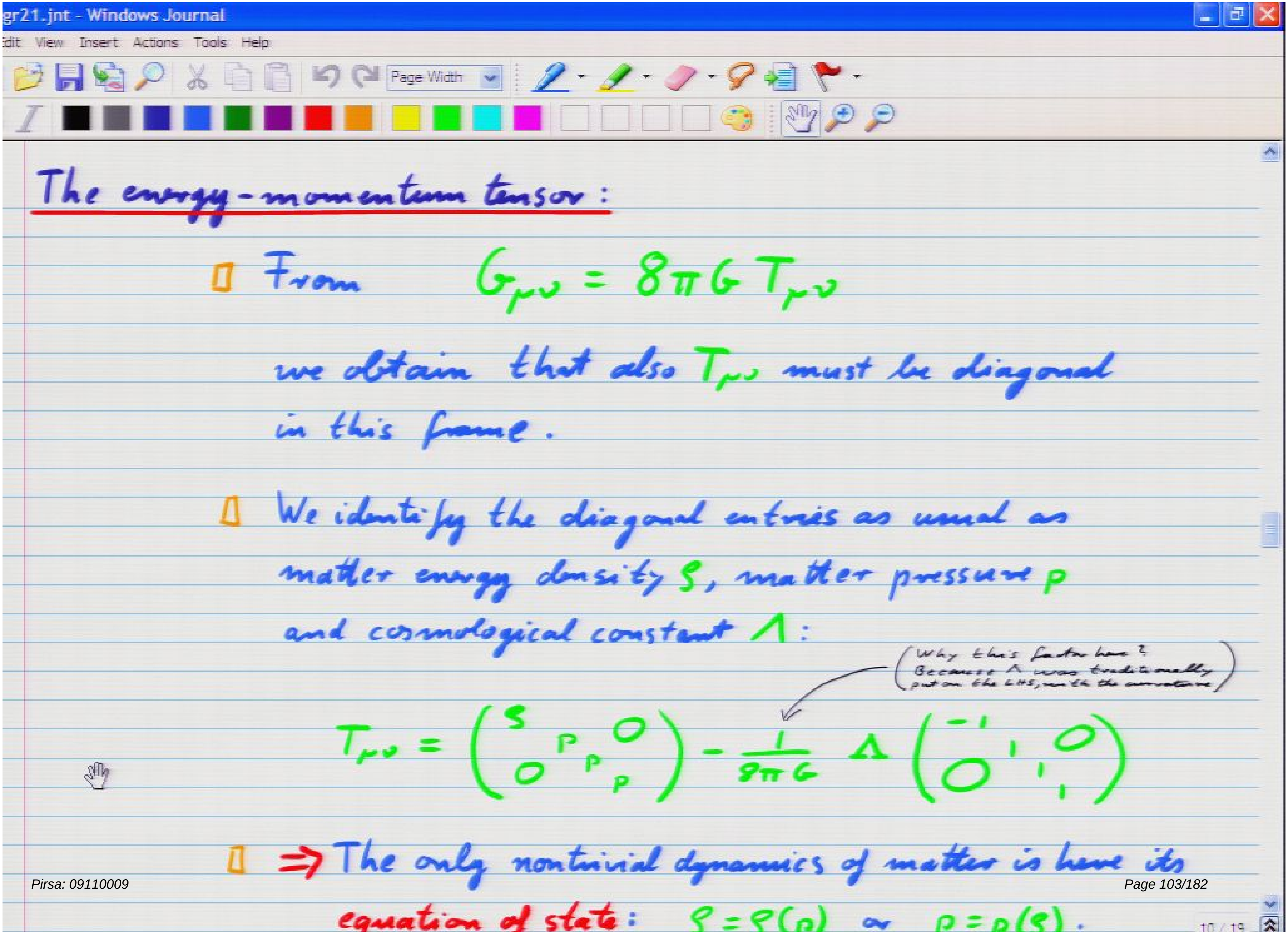

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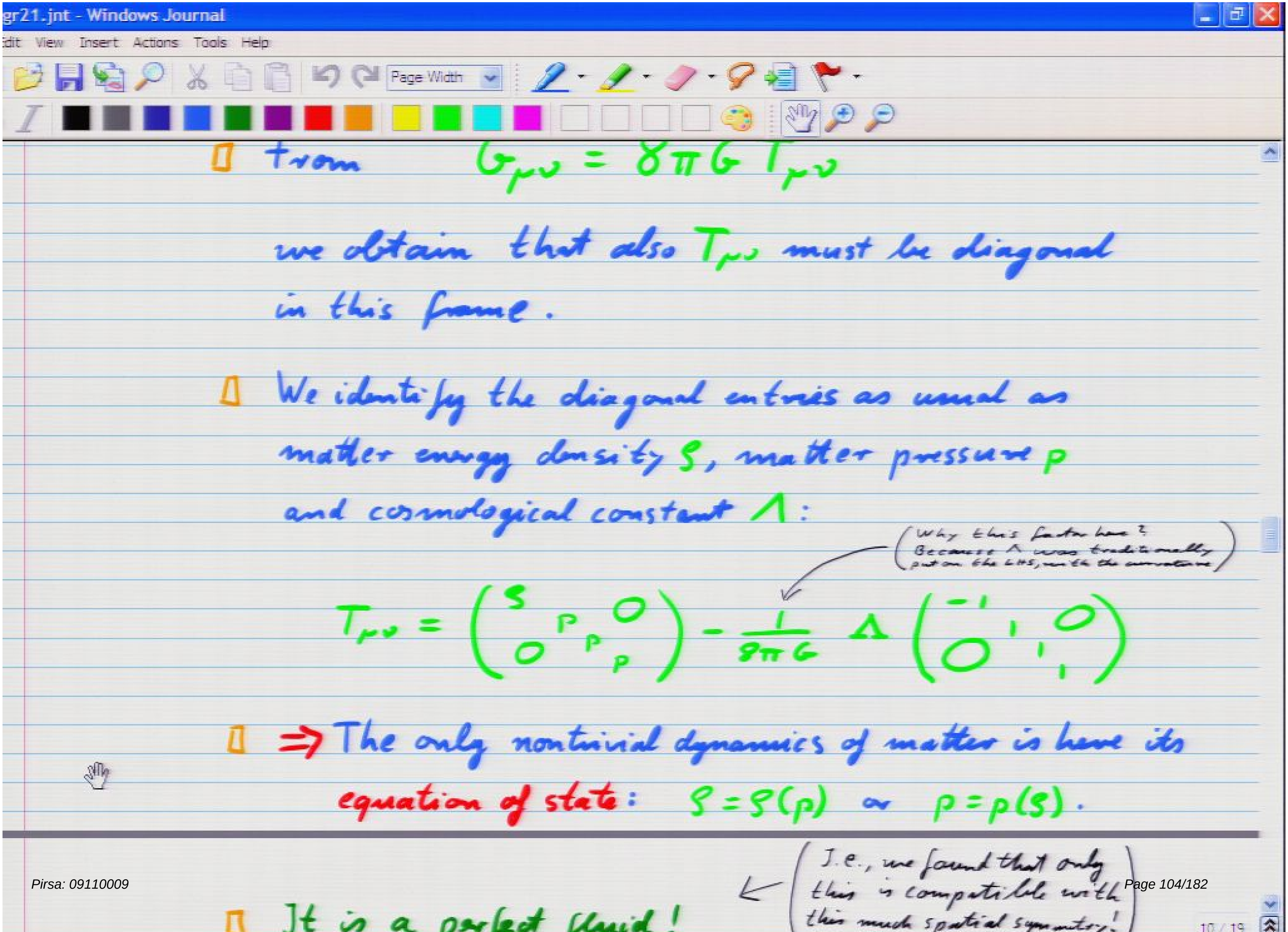

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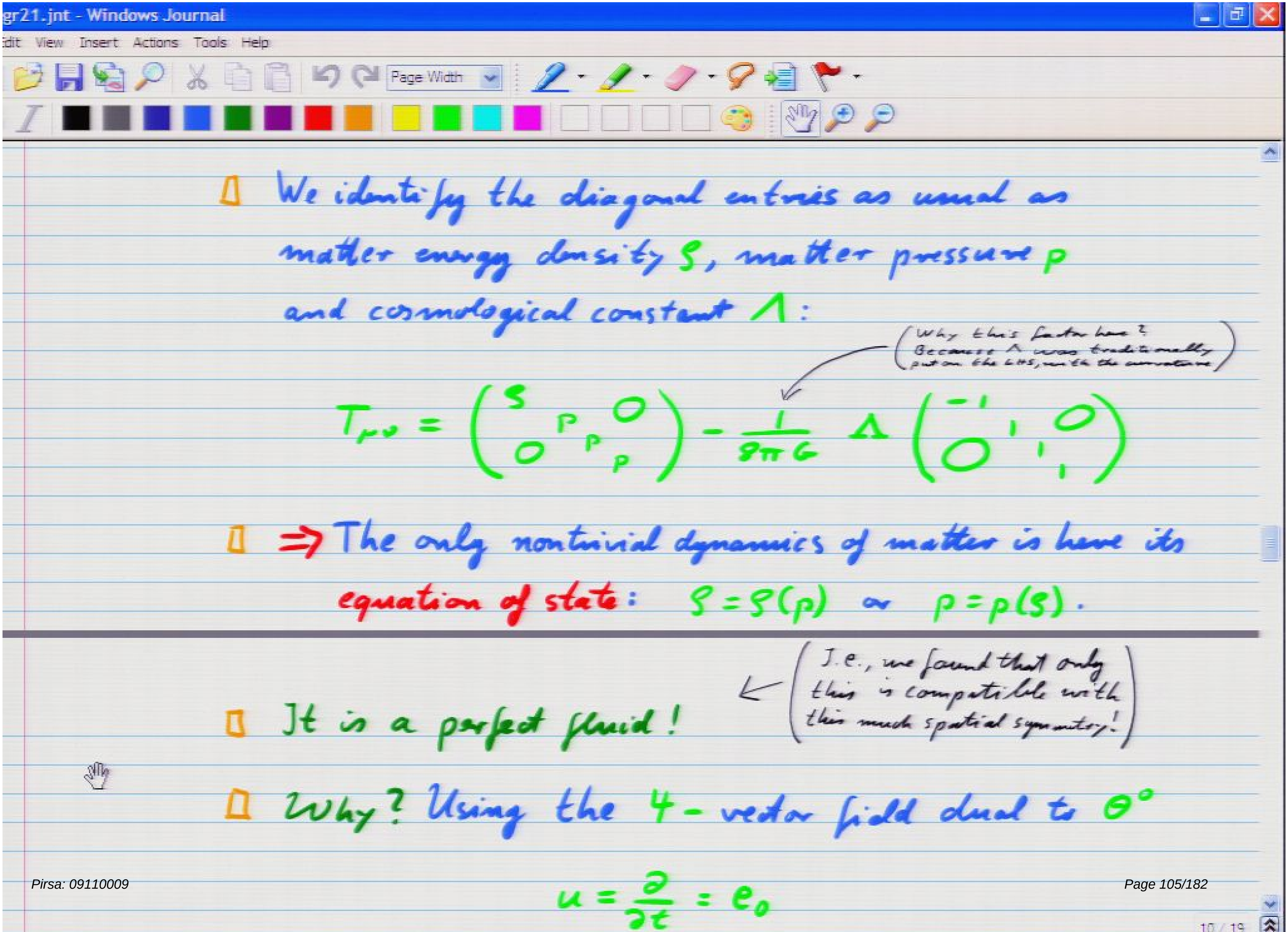
- It is a perfect fluid!

(I.e., we found that only this is compatible with this much spatial symmetry!)



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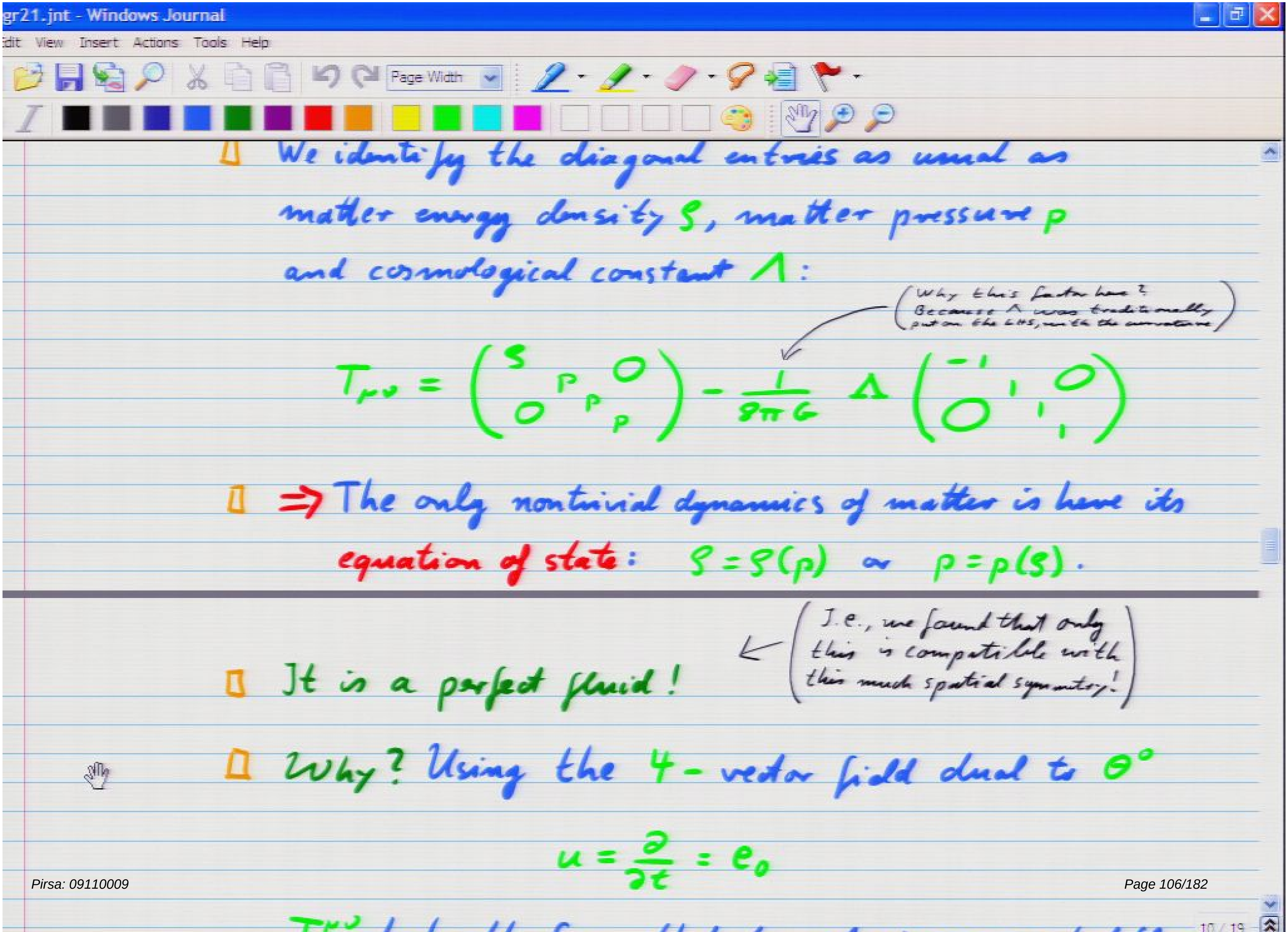

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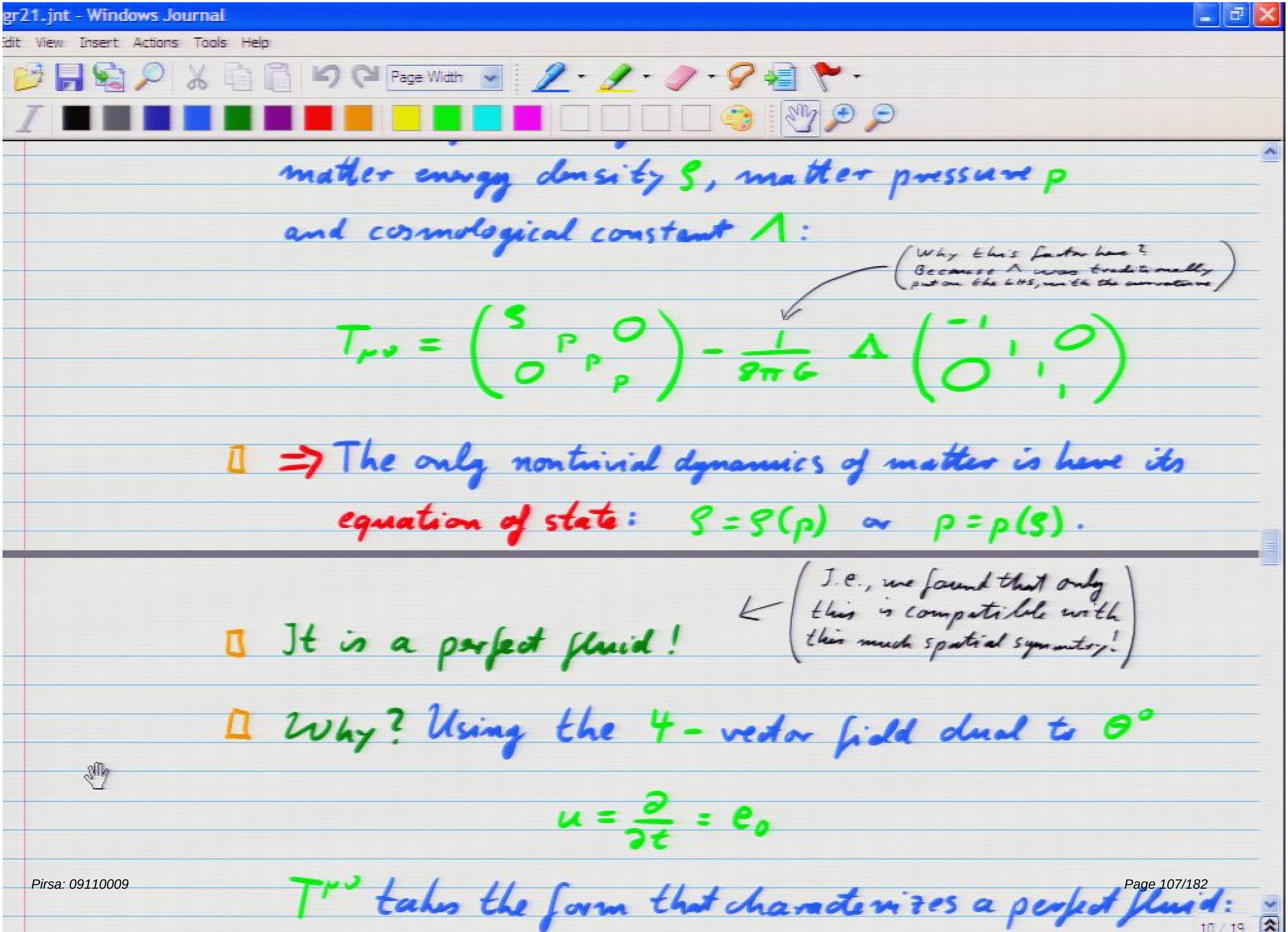

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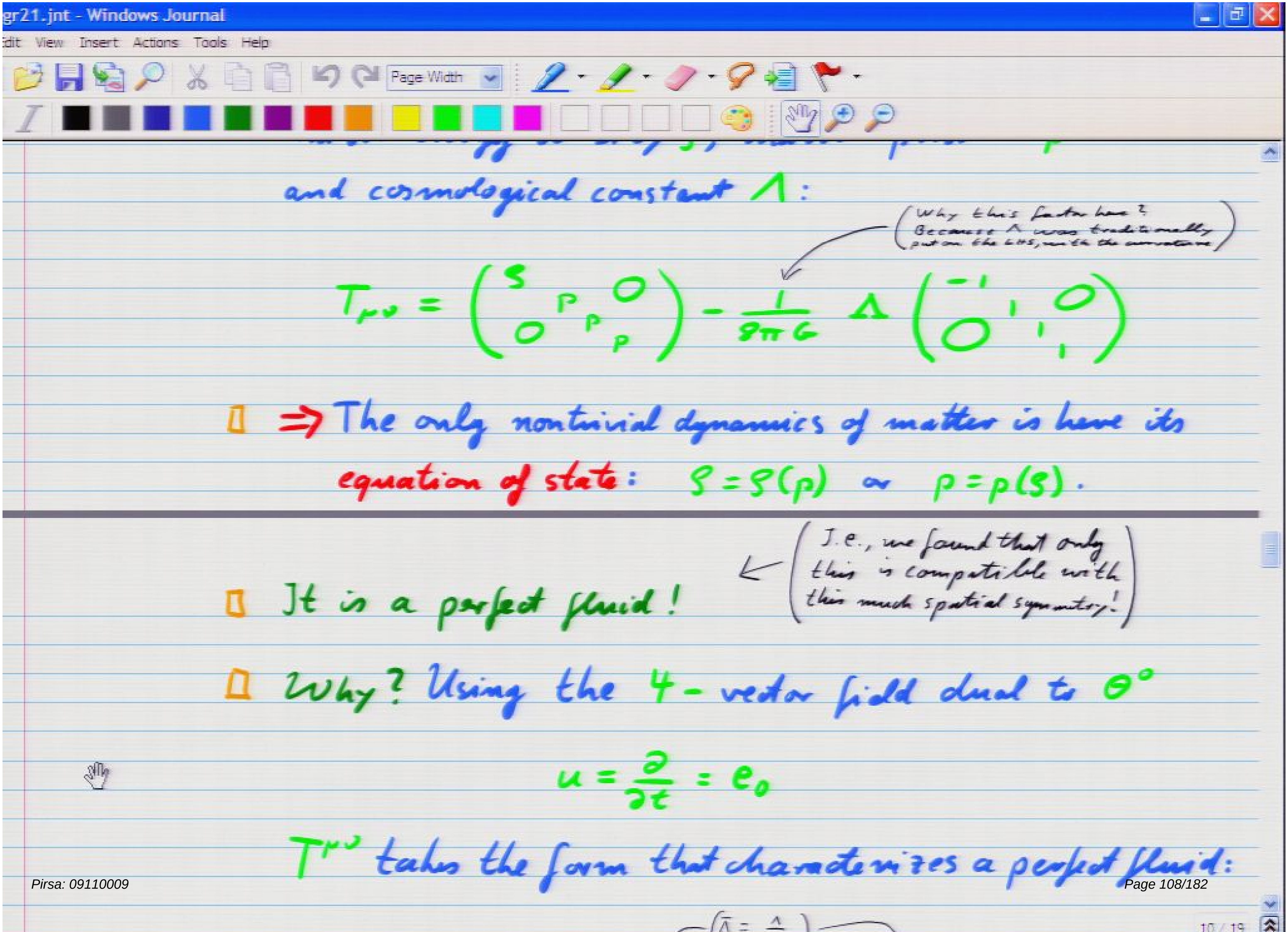

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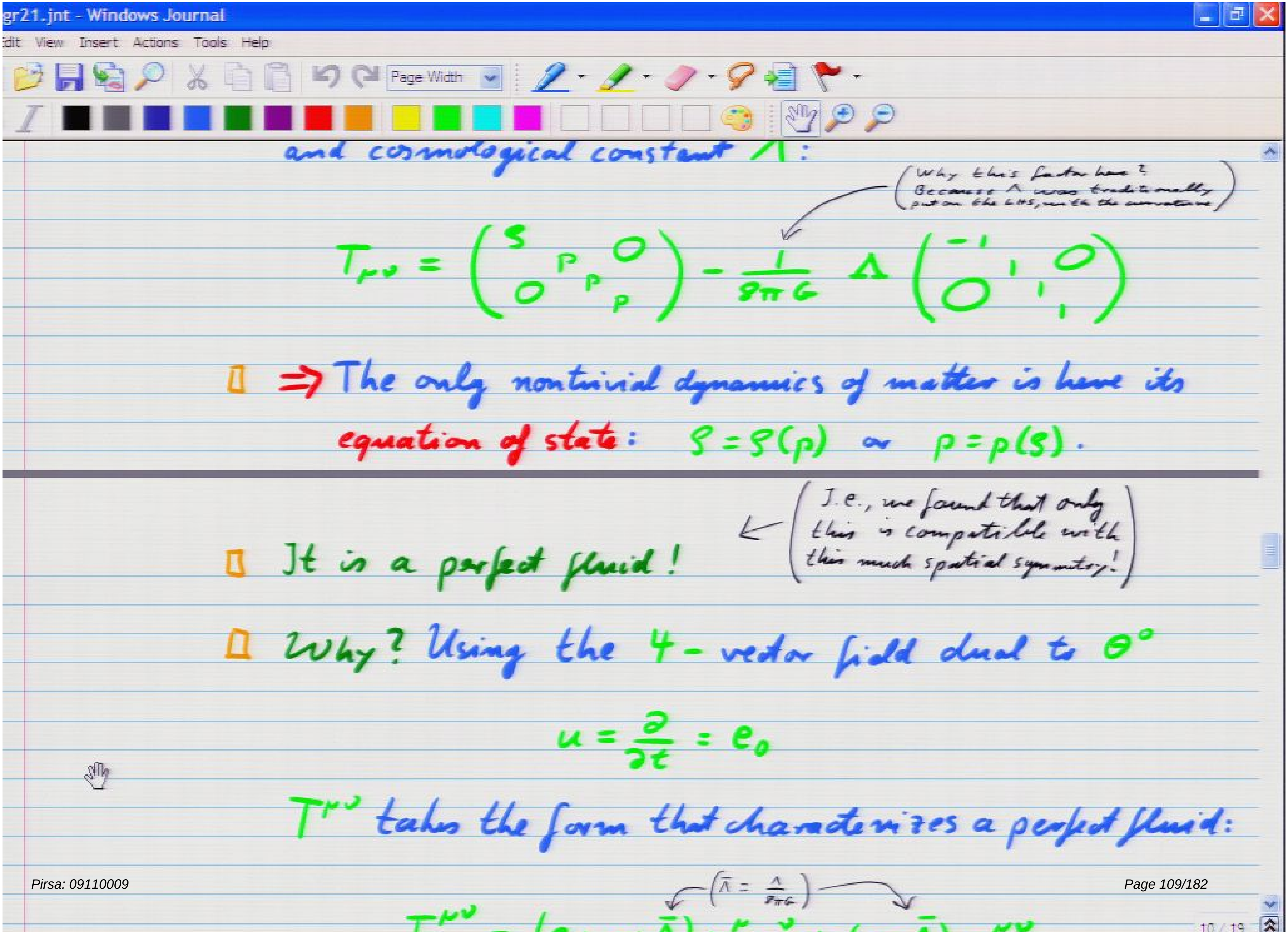

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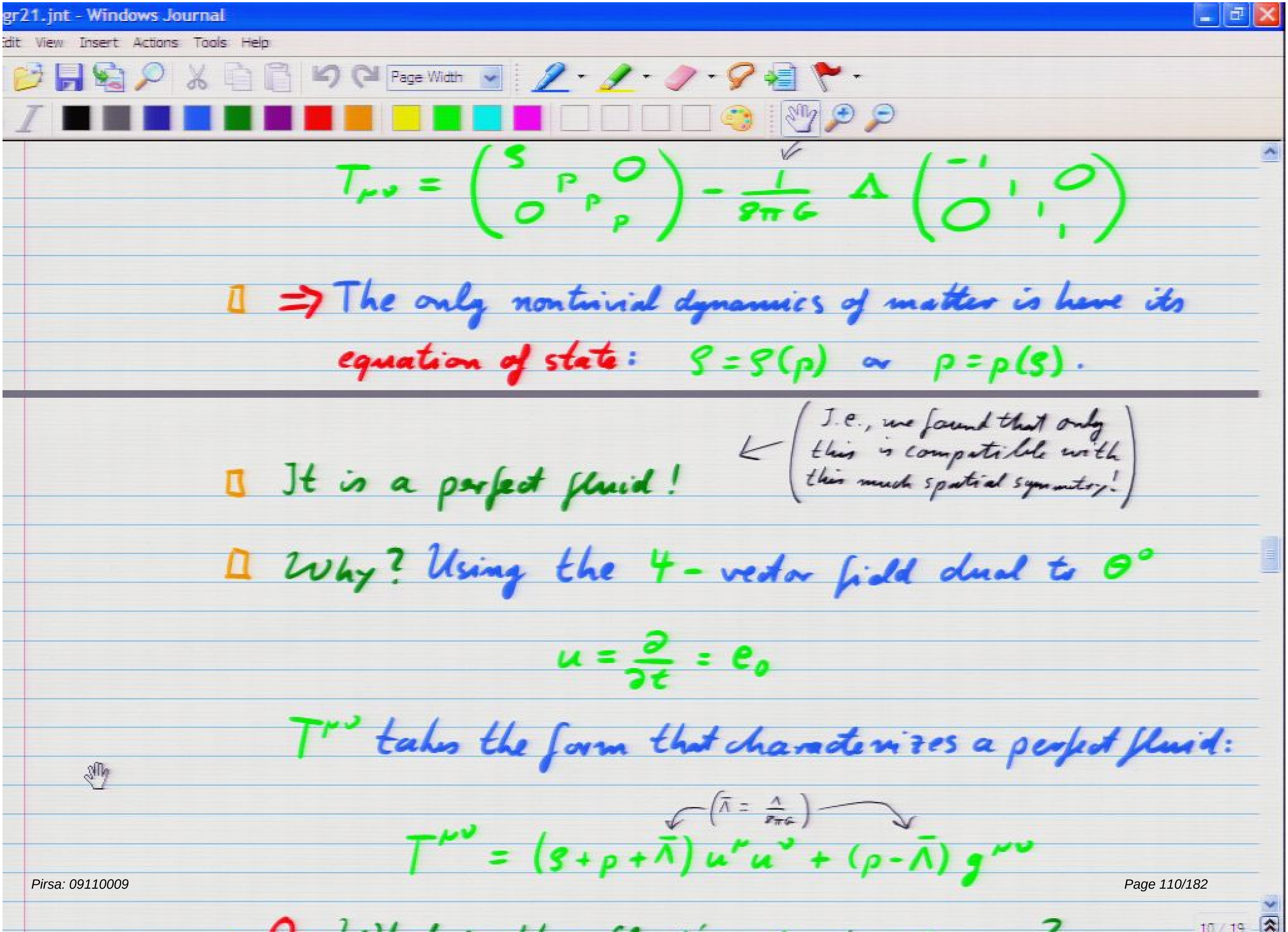

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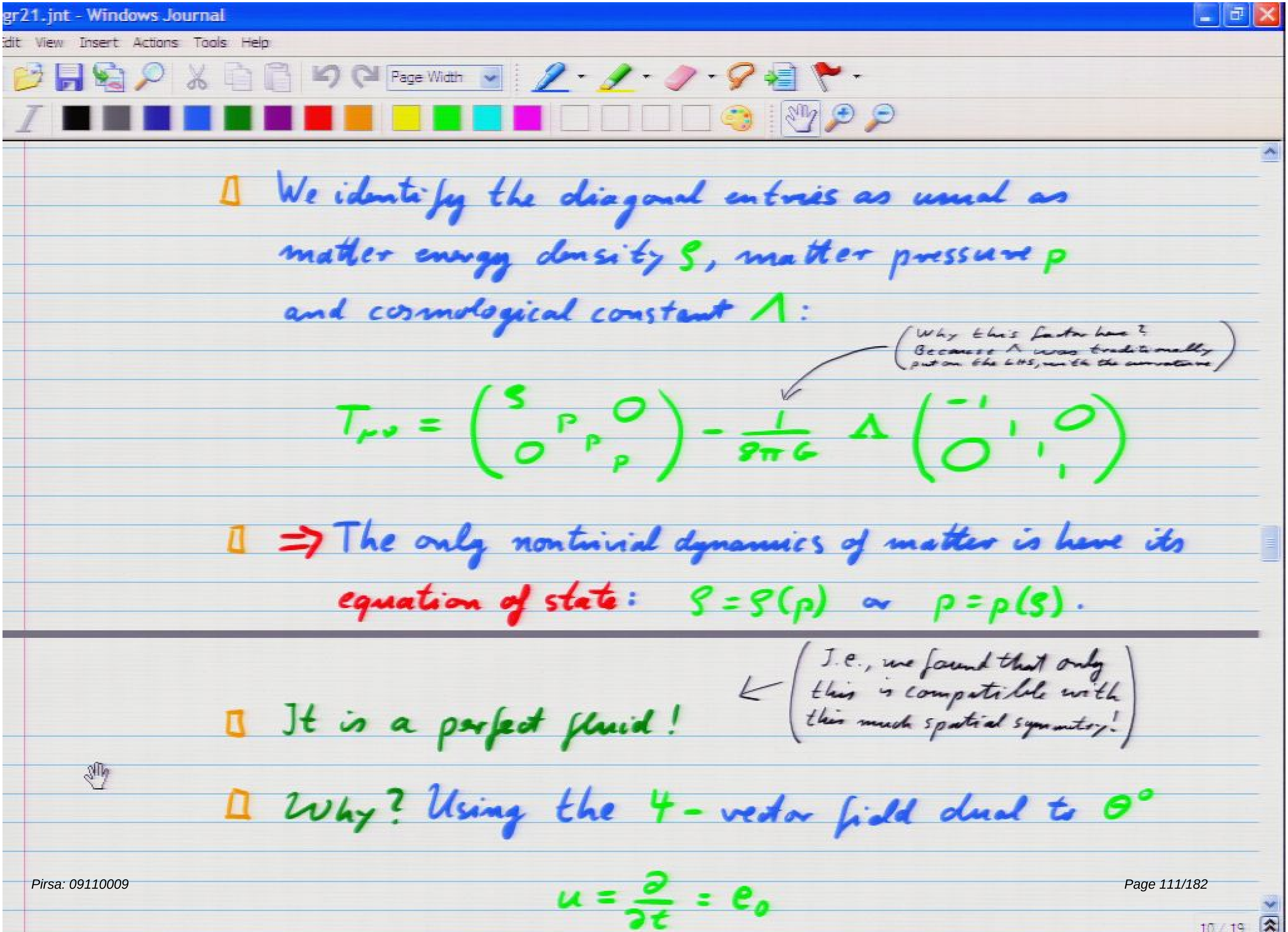

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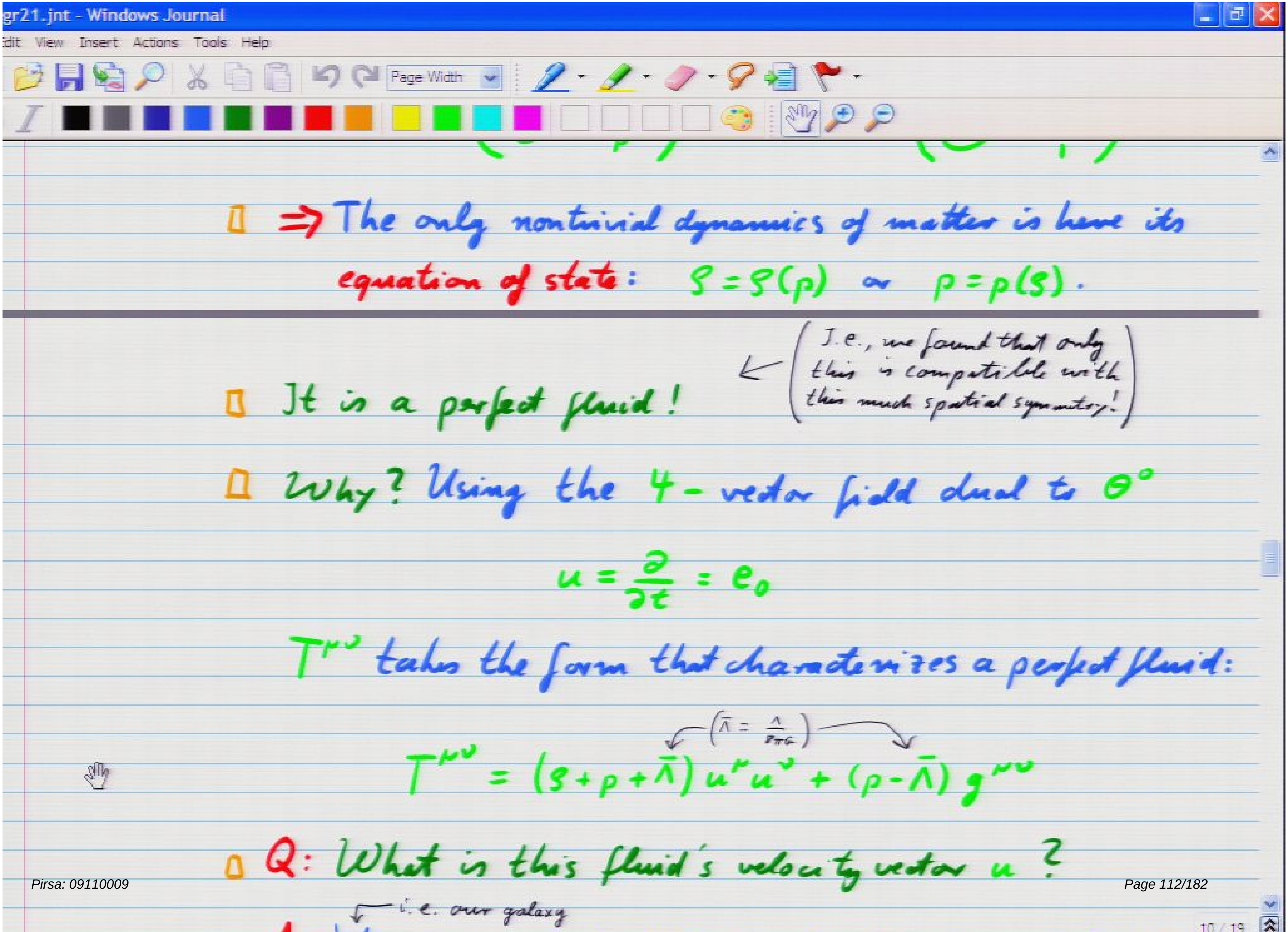

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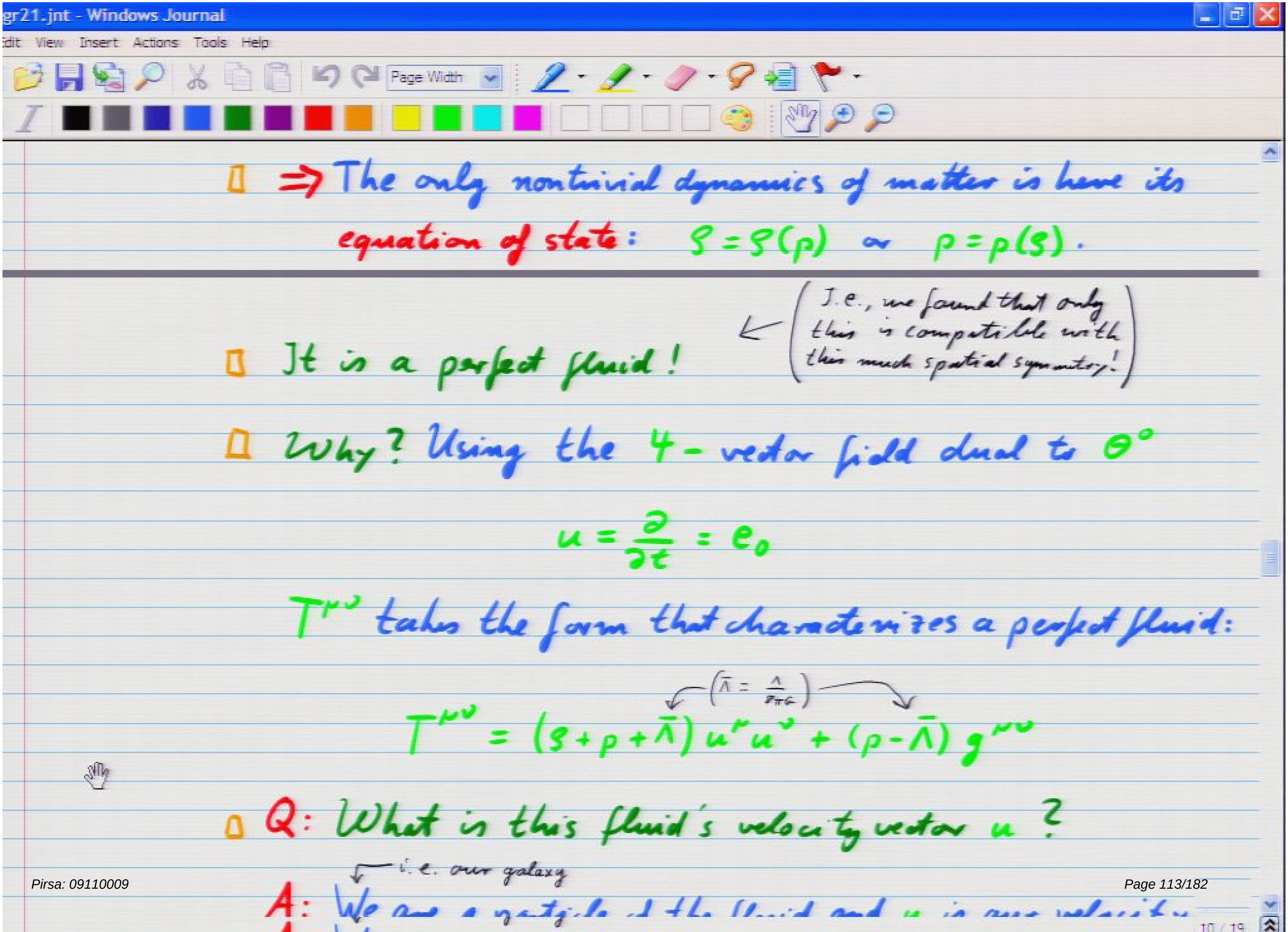

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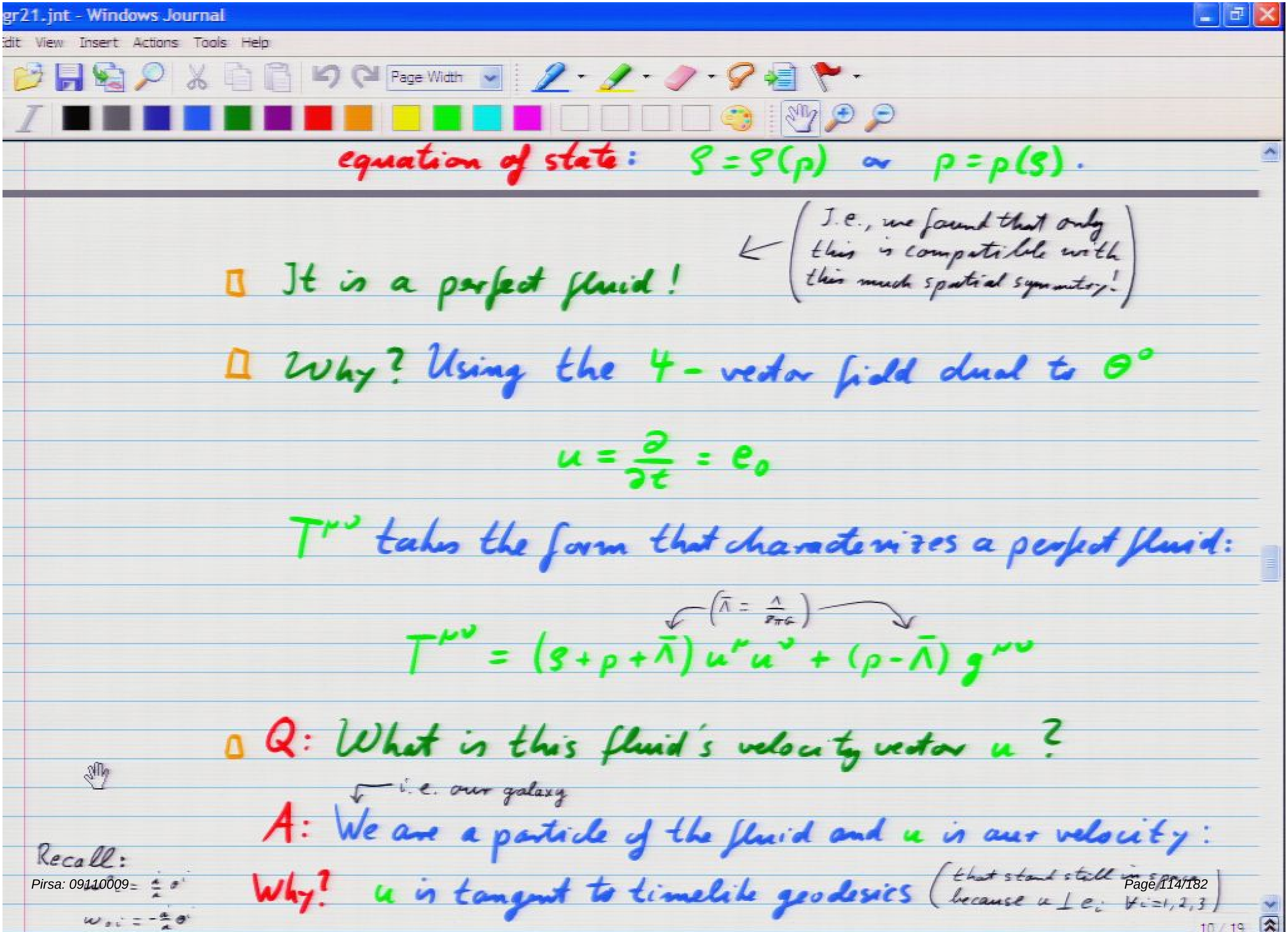

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Why? u is tangent to timelike geodesics (that stand still in space because $u \perp e_i$ $\forall i = 1,2,3$)

Recall:

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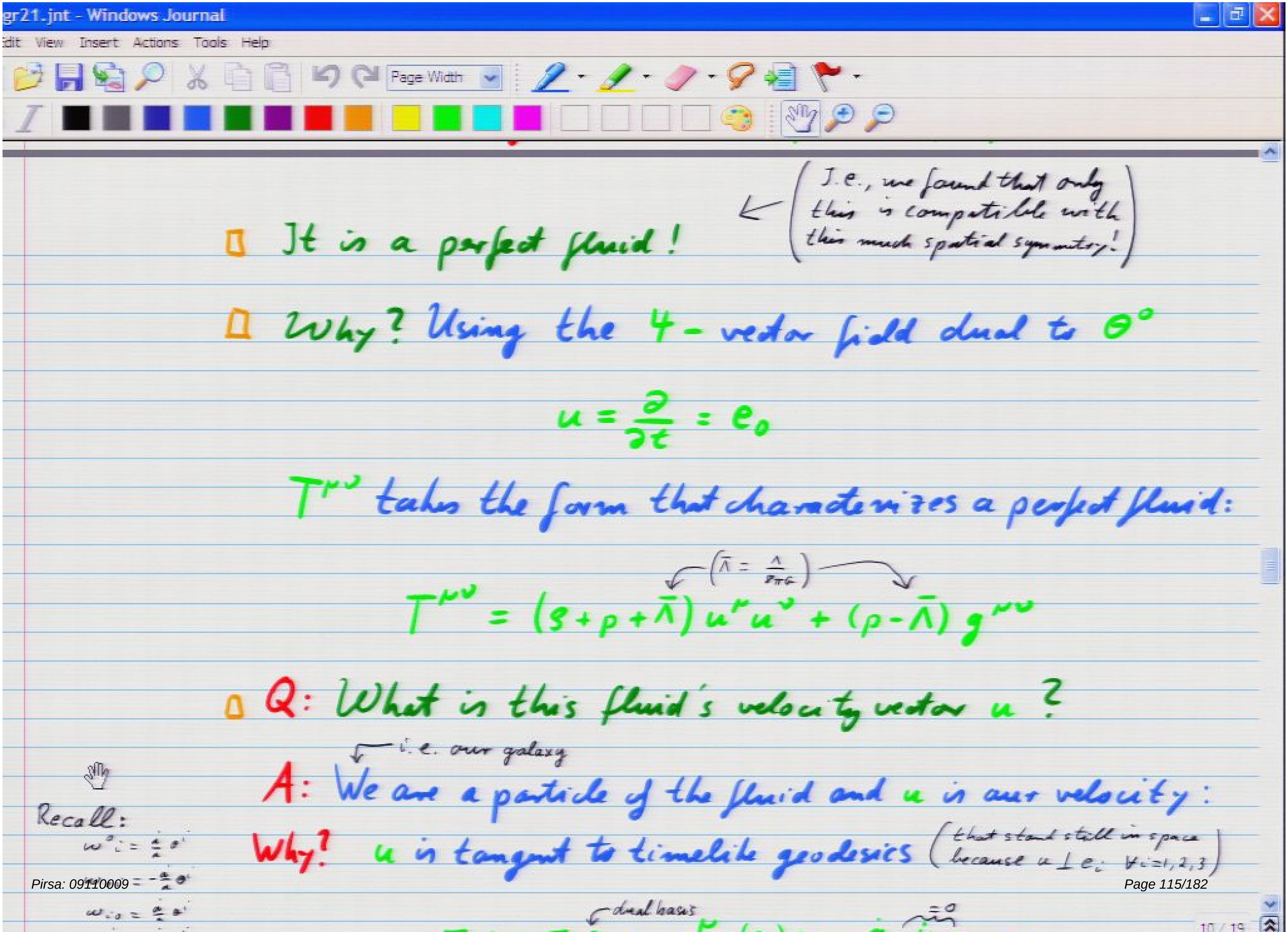

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dual basis

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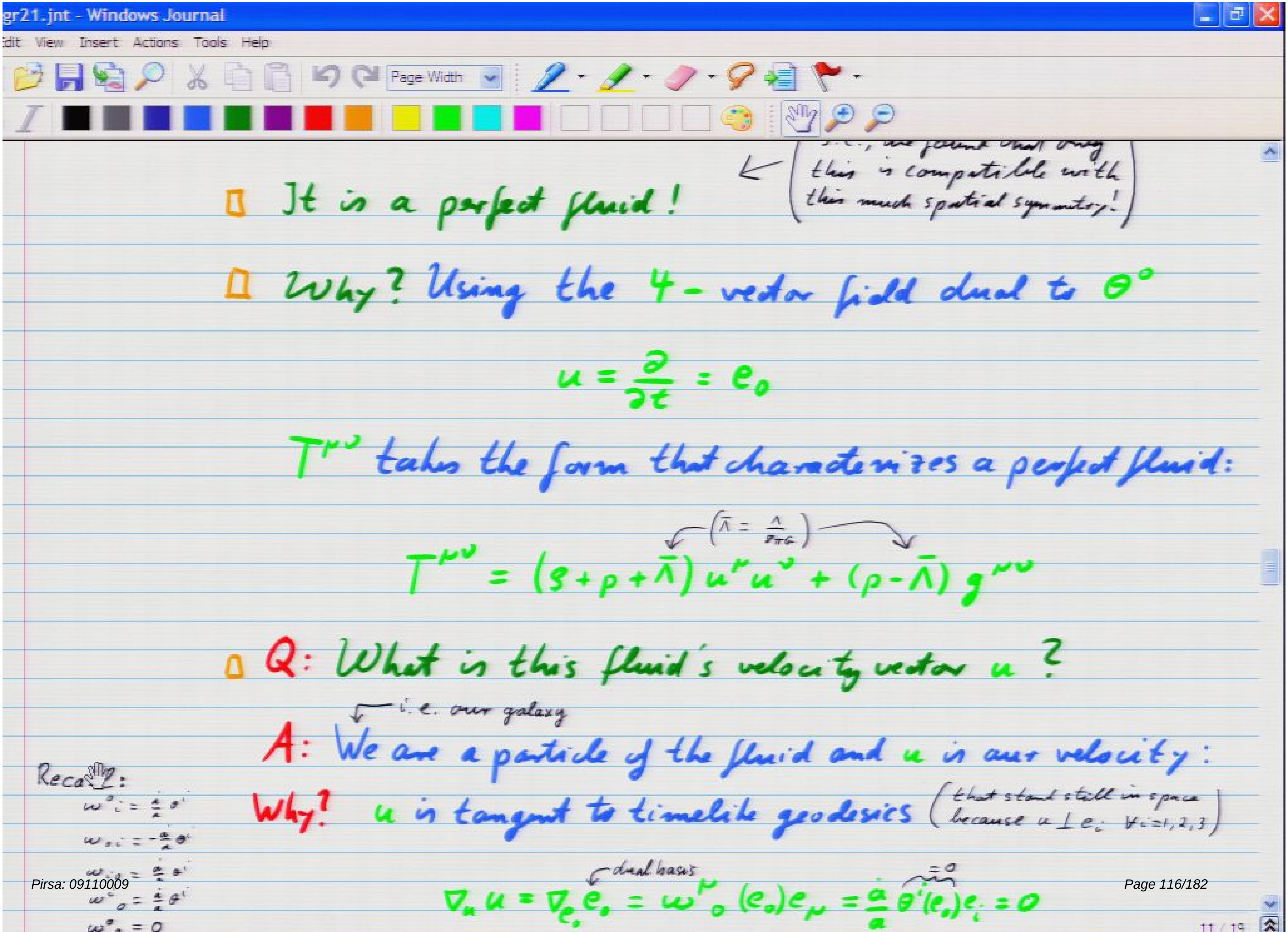

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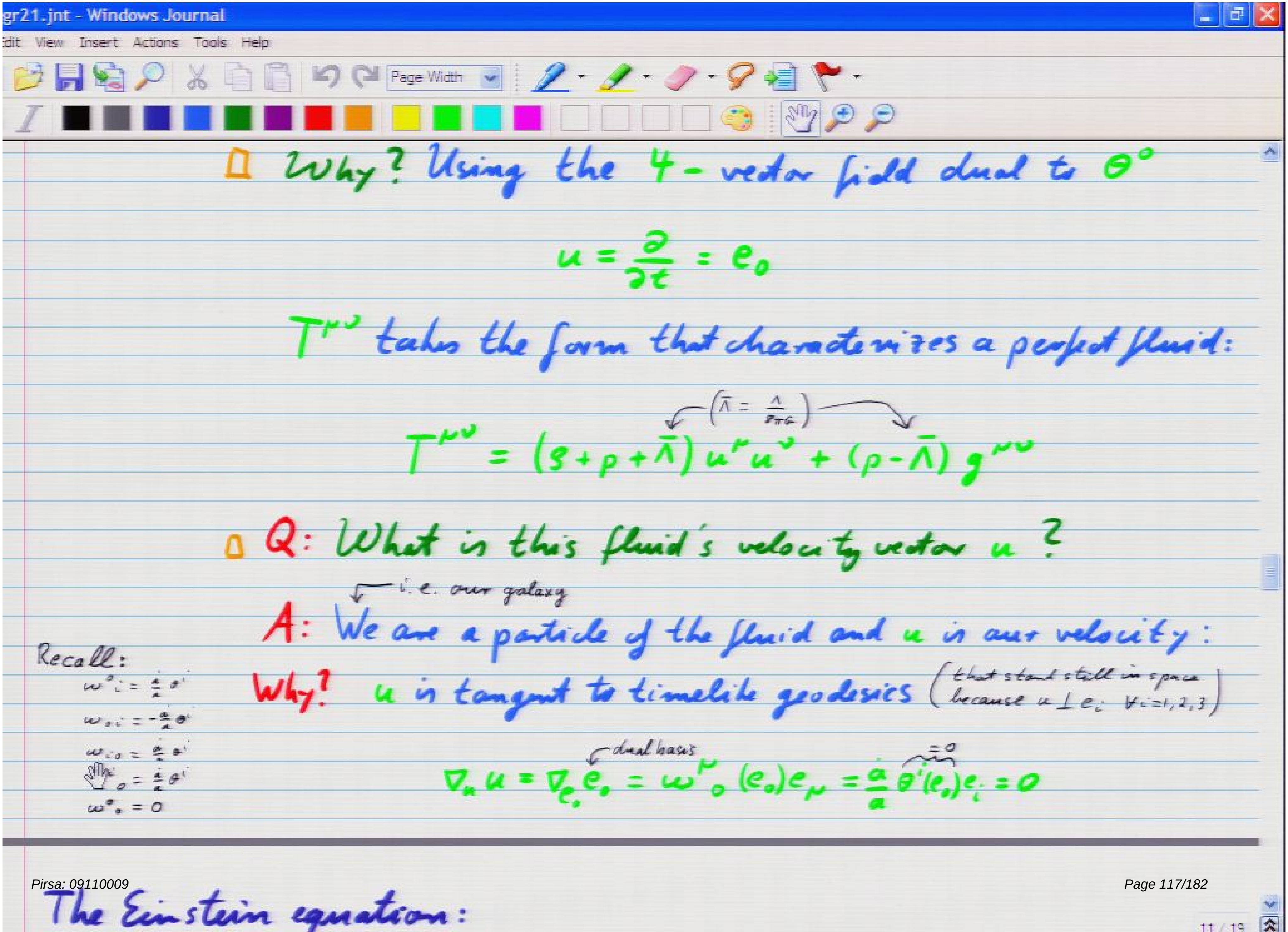

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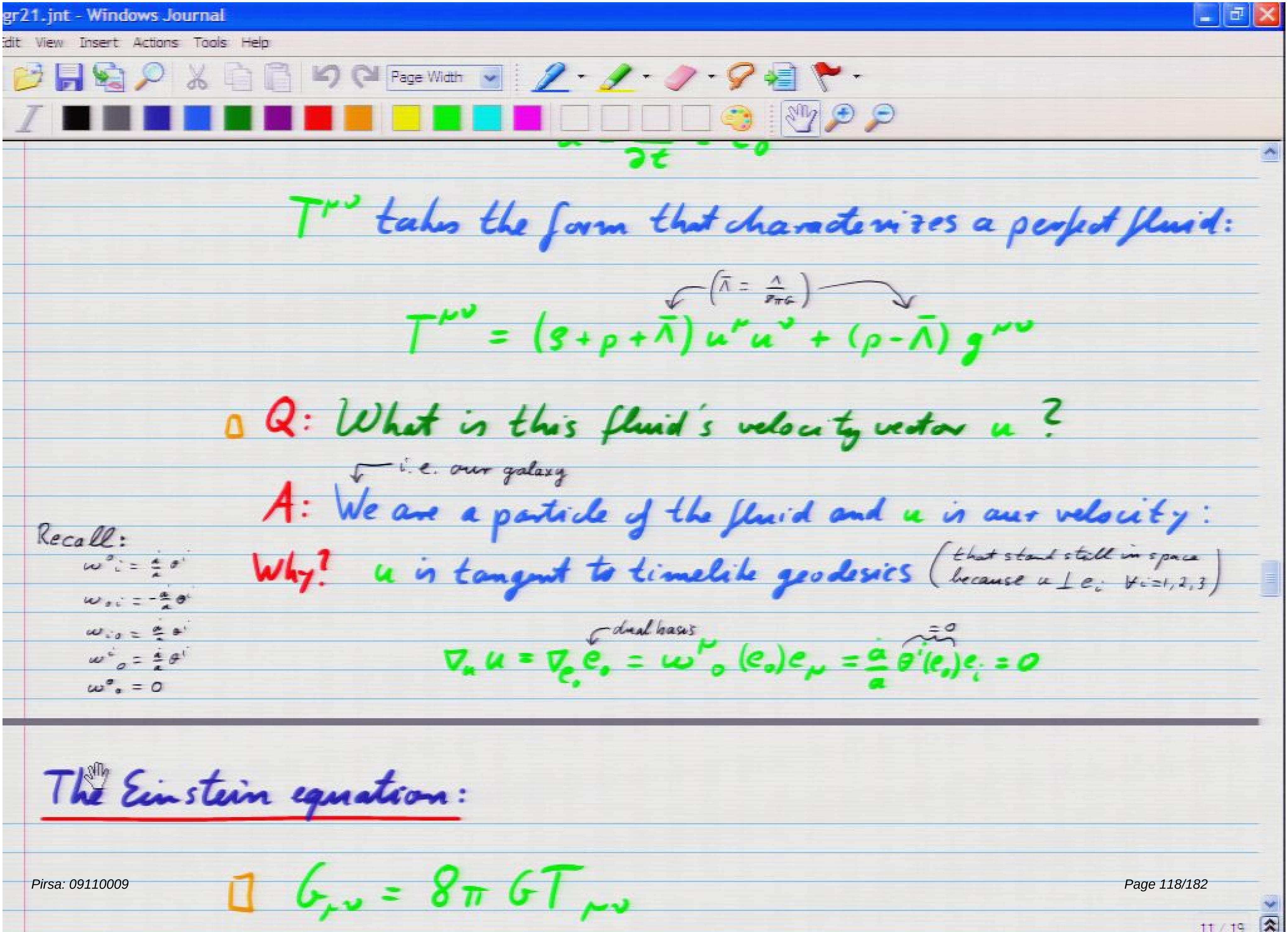
The Einstein equation:



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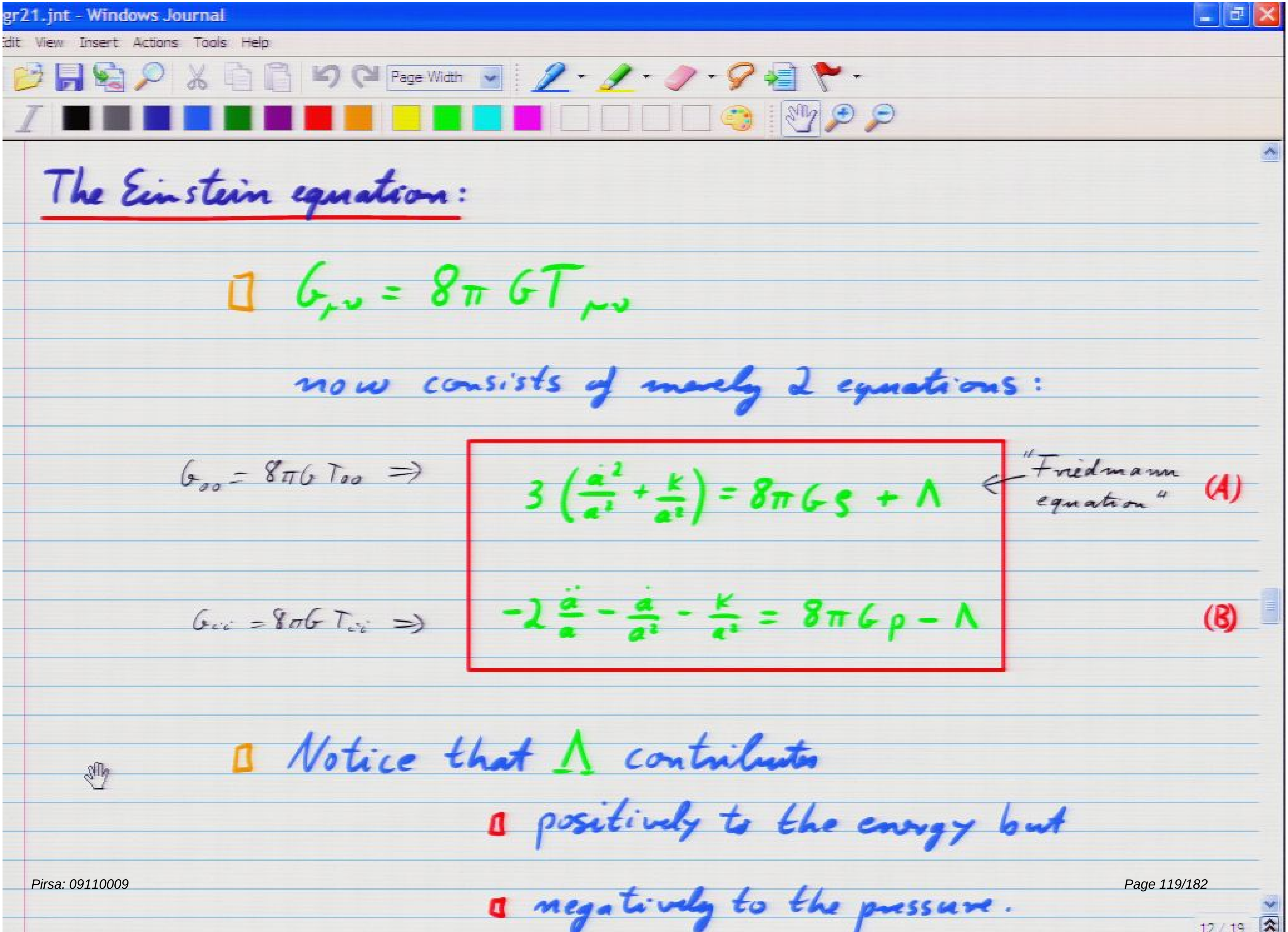
The Einstein equation:

☐ $G_{\mu\nu} = 8\pi G T_{\mu\nu}$



The Einstein equation:

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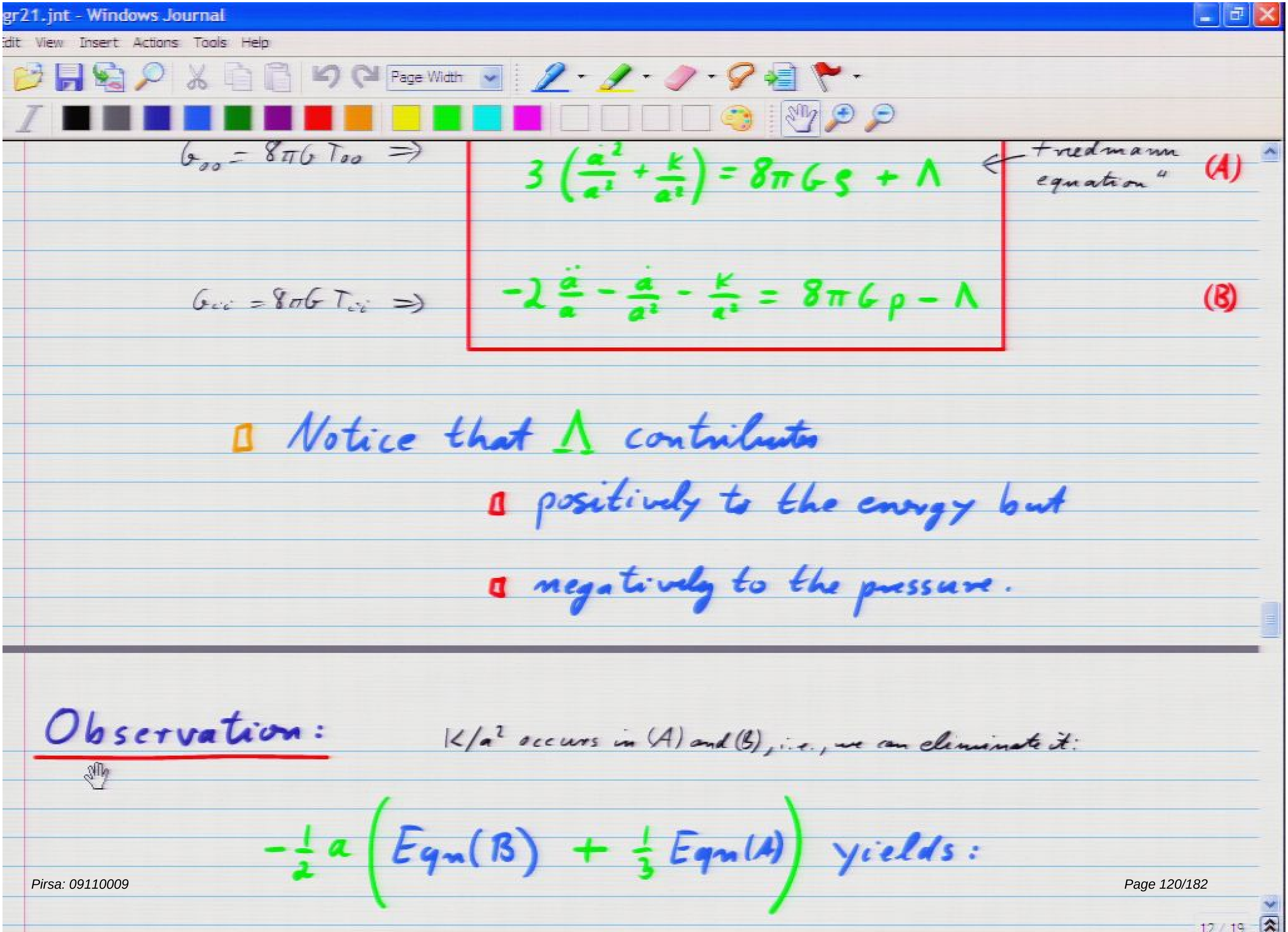
now consists of merely 2 equations:

$G_{00} = 8\pi G T_{00} \Rightarrow$ $$3\left(\frac{\dot a^2}{a^2} + \frac{k}{a^2}\right) = 8\pi G \rho + \Lambda$$ ← "Friedmann equation" (A)

$G_{ii} = 8\pi G T_{ii} \Rightarrow$ $$-2\frac{\ddot a}{a} - \frac{\dot a}{a^2} - \frac{k}{a^2} = 8\pi G p - \Lambda$$ (B)

☐ Notice that Λ contributes

- ☐ positively to the energy but
- ☐ negatively to the pressure.



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- Notice that $\underline{\Lambda}$ contributes
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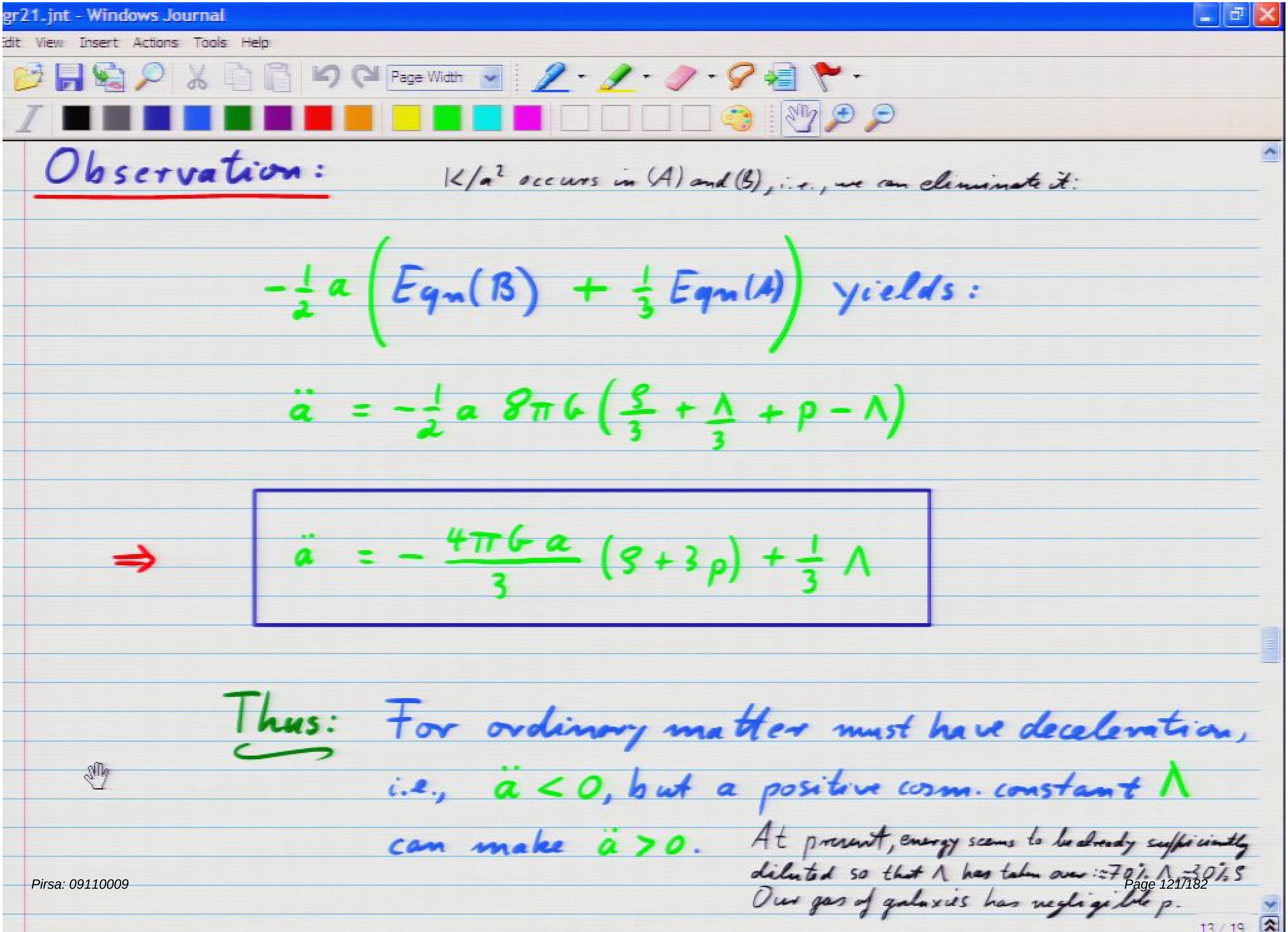
Observation:

K/a^2 occurs in (A) and (B), i.e., we can eliminate it:

$$-\frac{1}{2}a\left(\text{Eqn}(B) + \frac{1}{3}\text{Eqn}(A)\right) \text{ yields:}$$



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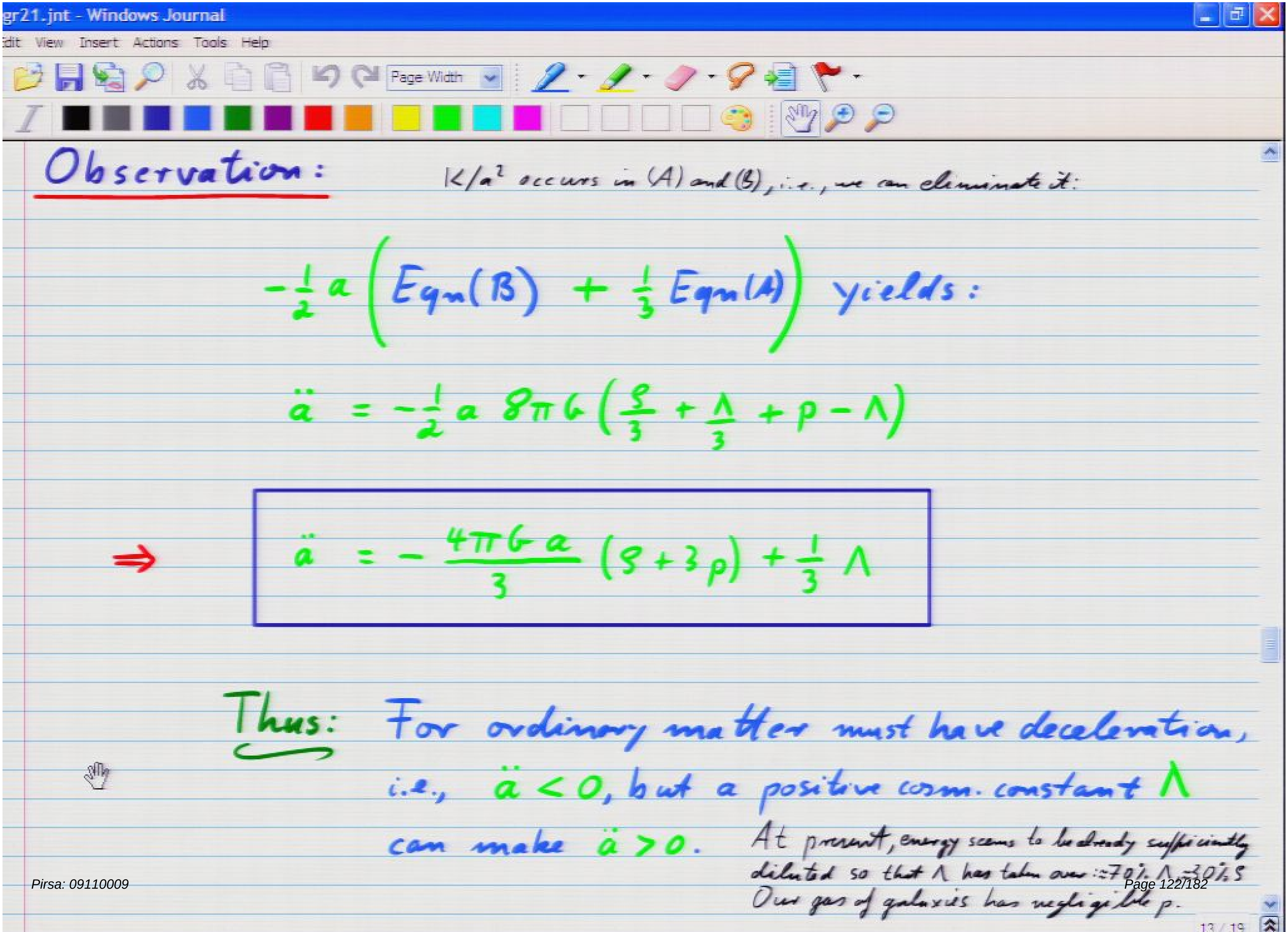
$$\Rightarrow \quad \boxed{\ddot{a} = -\frac{4\pi G a}{3}(\rho+3p)+\frac{1}{3}\Lambda}$$

Thus: For ordinary matter must have deceleration, i.e., $\ddot{a}<0$, but a positive cosm. constant Λ can make $\ddot{a}>0$.

At present, energy seems to be already sufficiently diluted so that Λ has taken over: $\approx 70\%\ \Lambda$, $\approx 30\%\ \rho$. Our gas of galaxies has negligible p.



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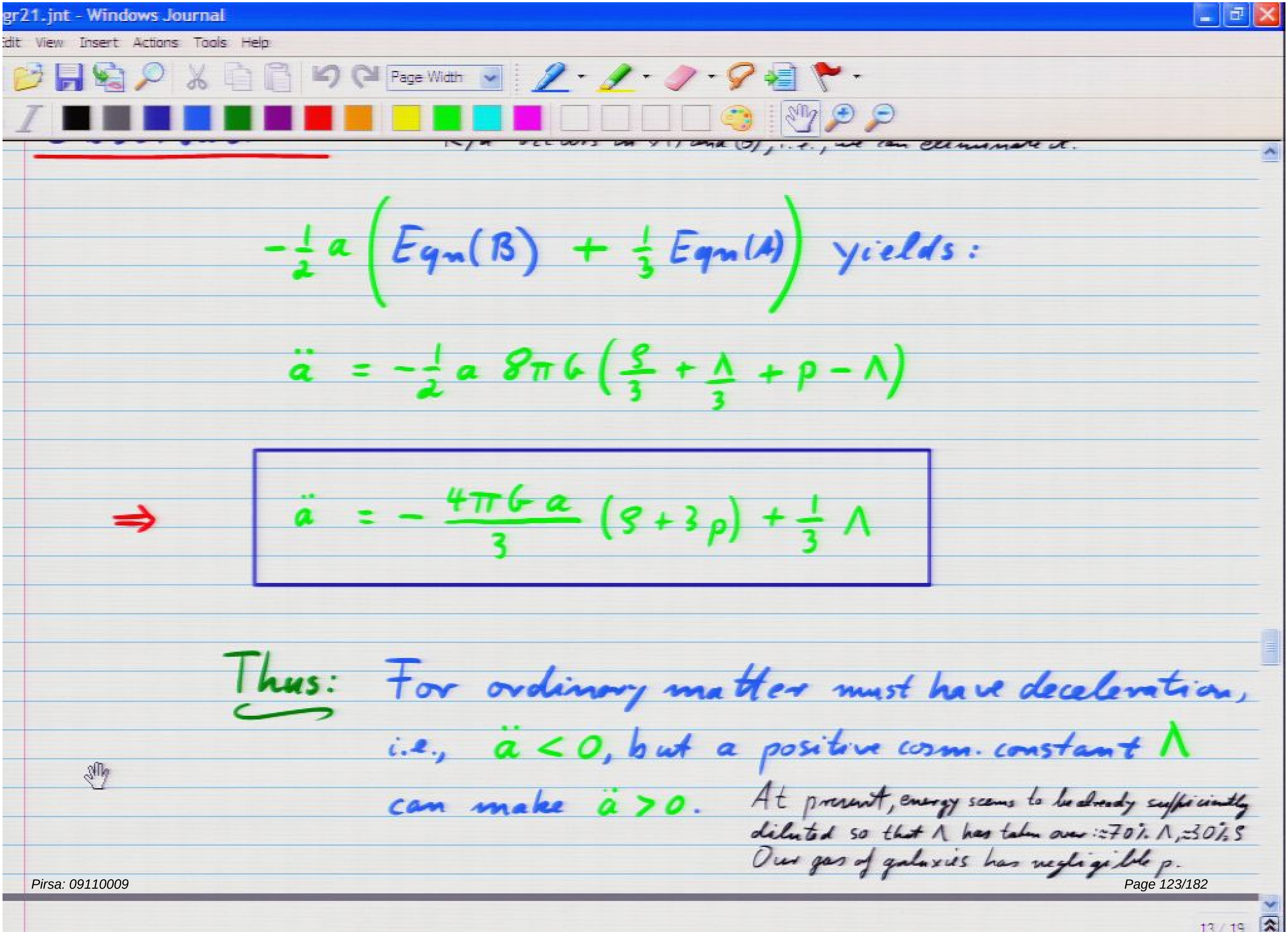

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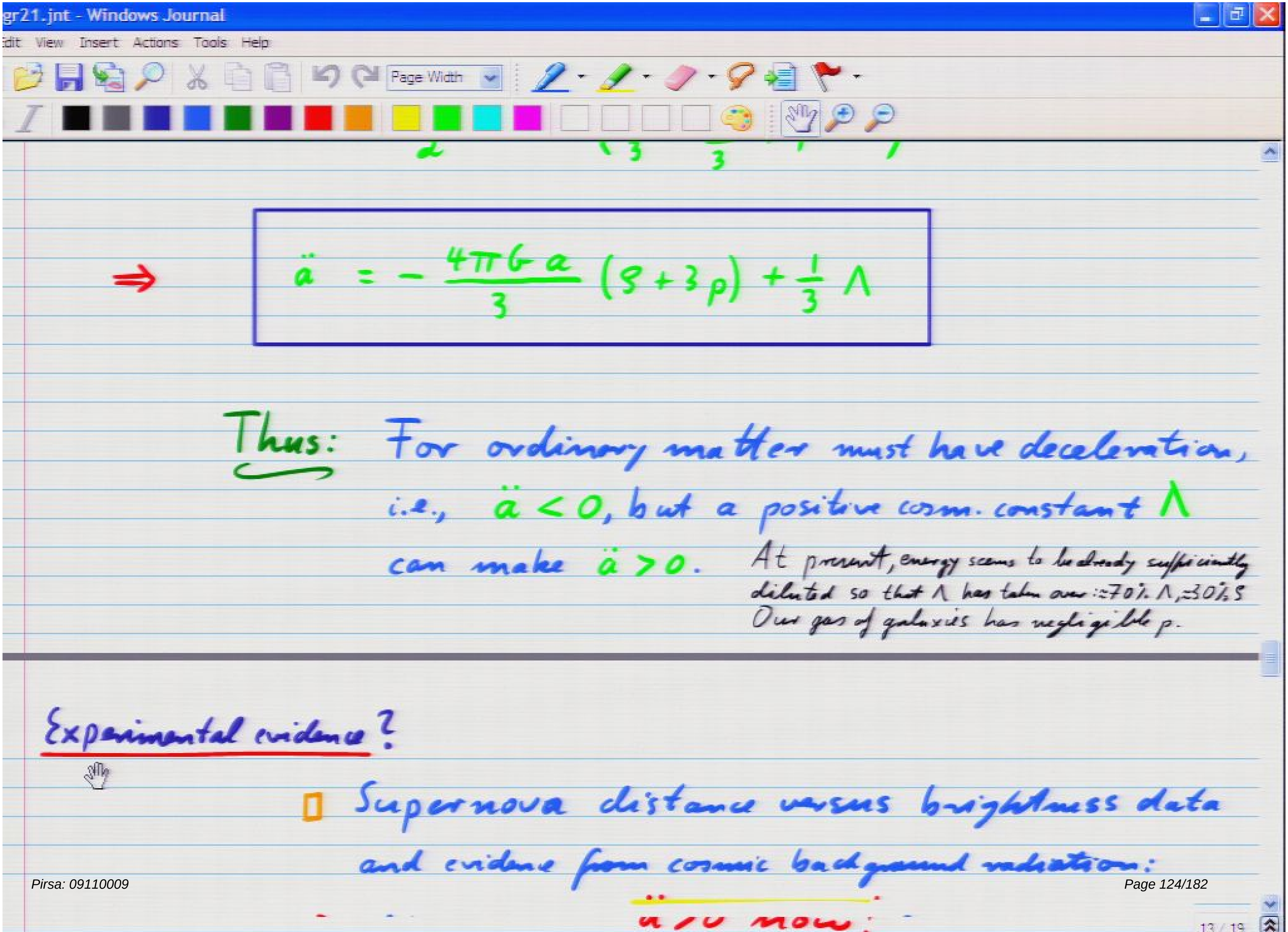

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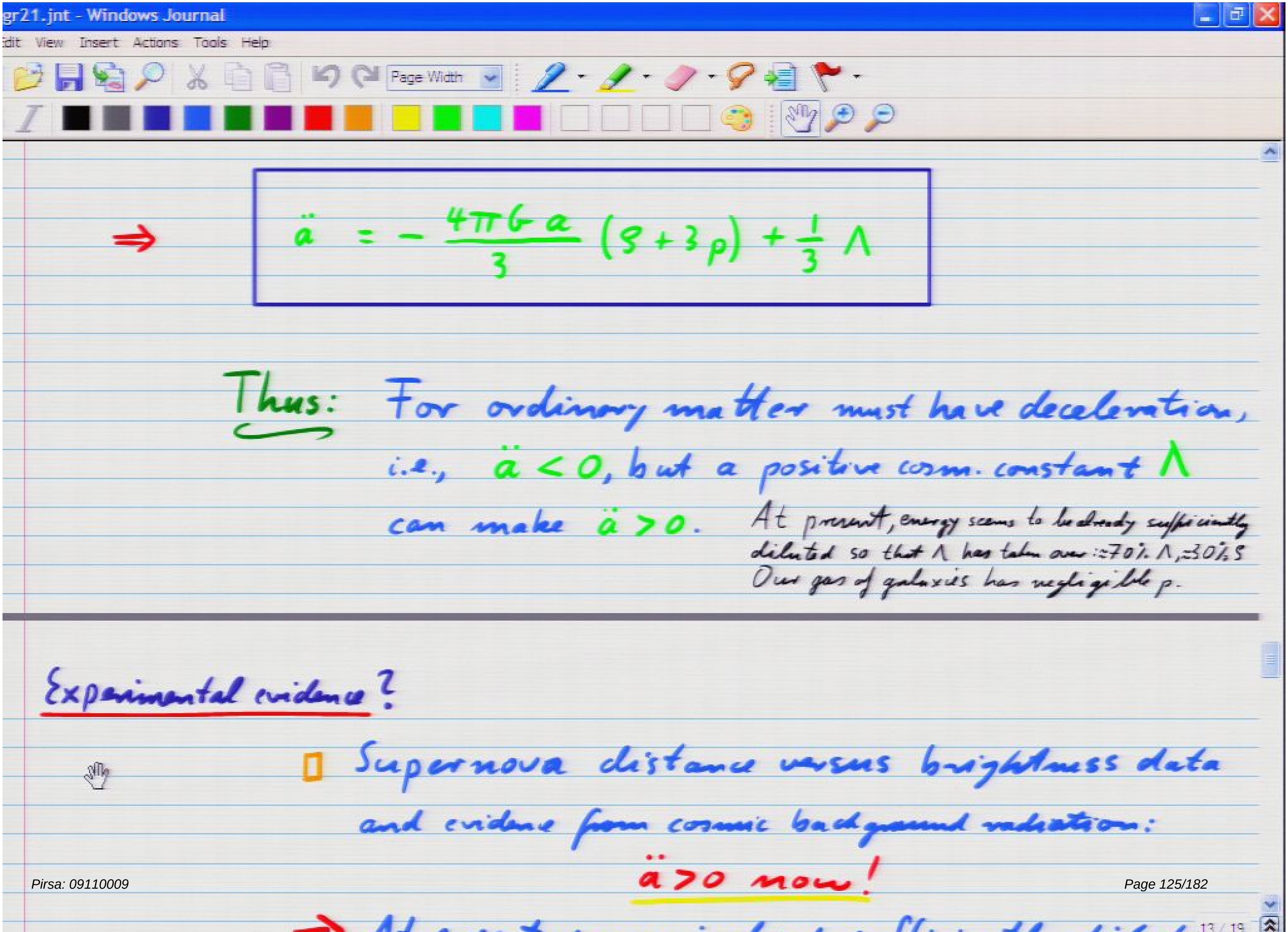
Experimental evidence?

- Supernova distance versus brightness data and evidence from cosmic background radiation:



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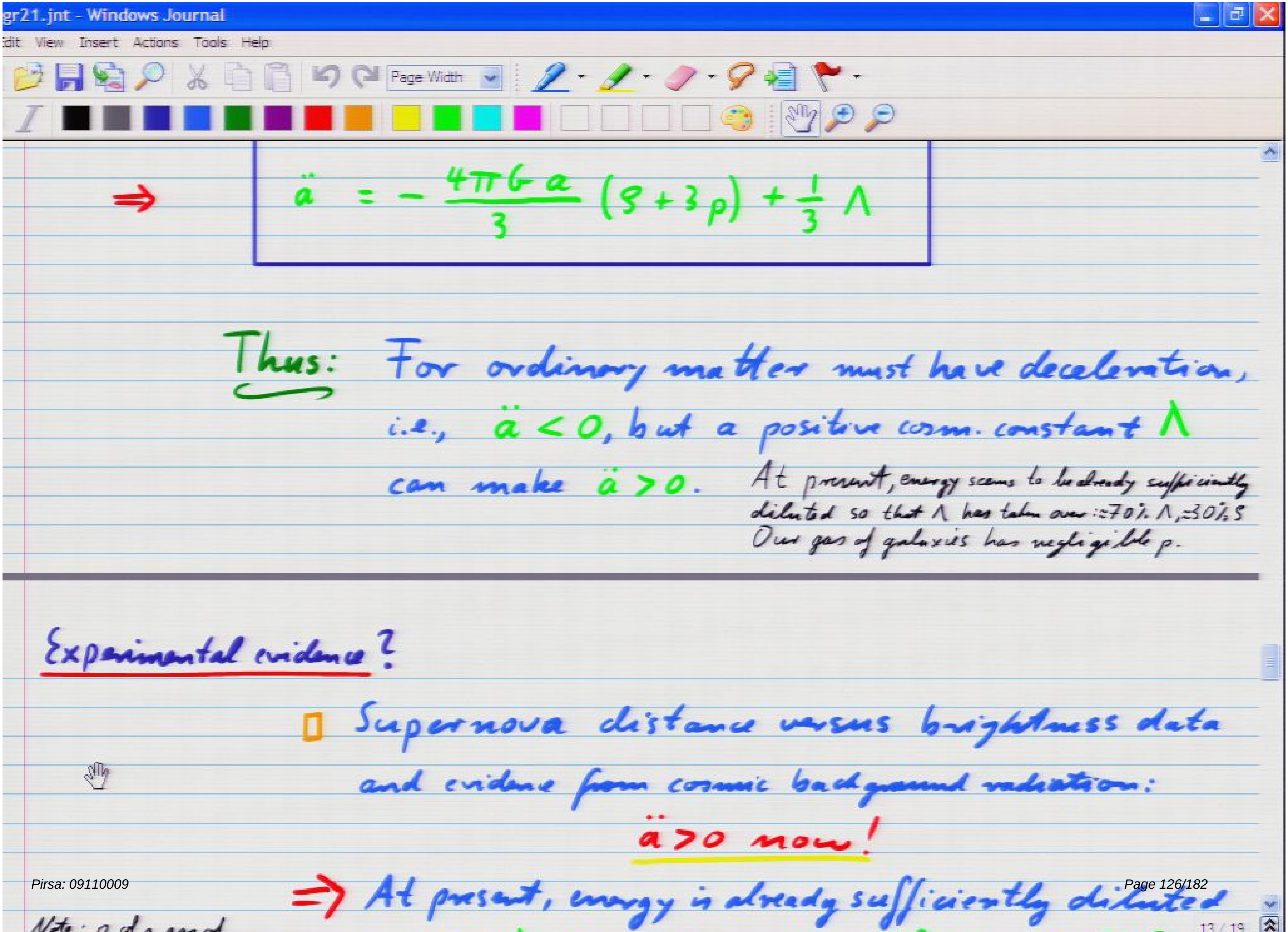
$\ddot{a} > 0$ now!



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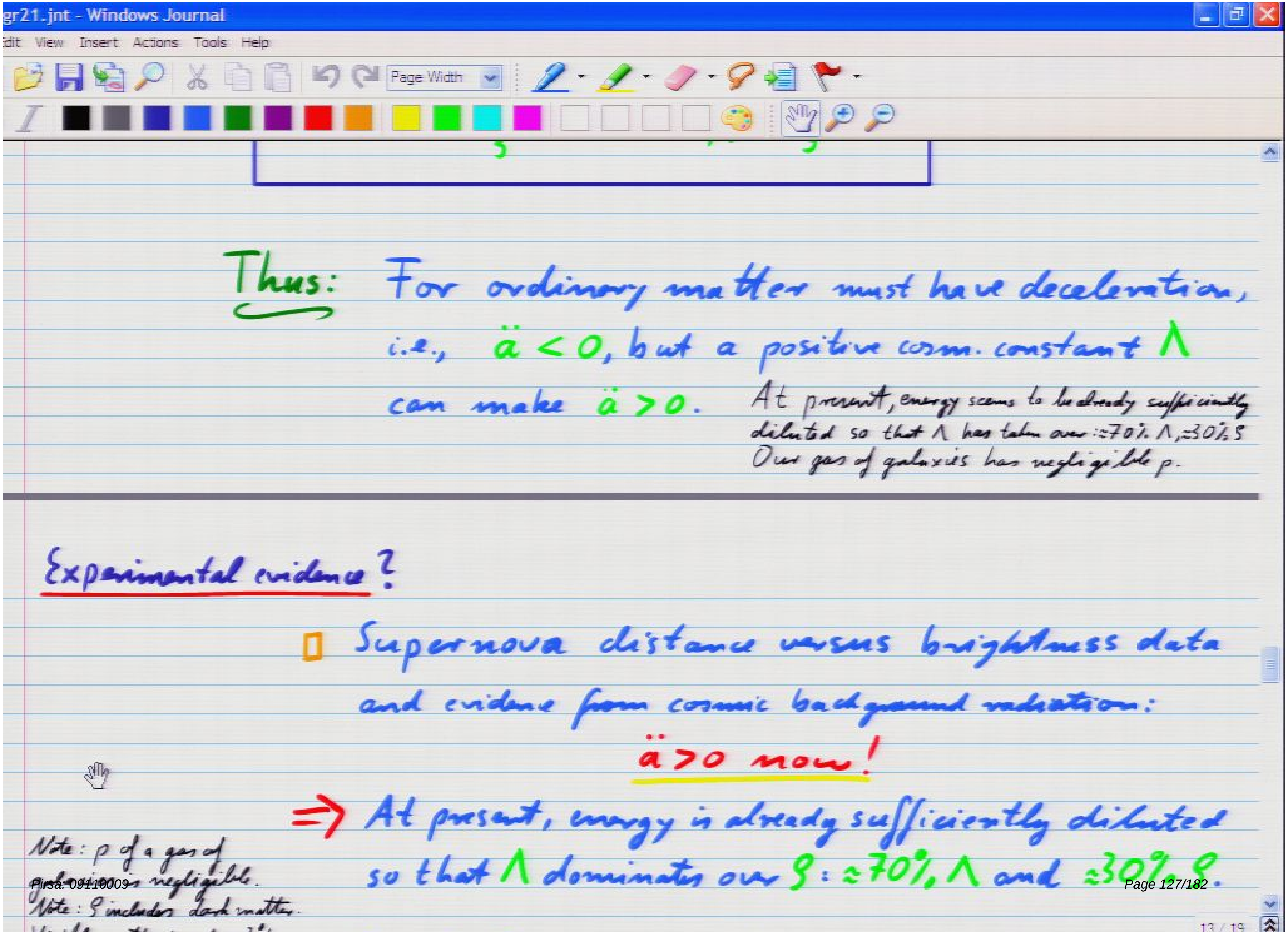
$\Rightarrow$ At present, energy is already sufficiently diluted

Note:



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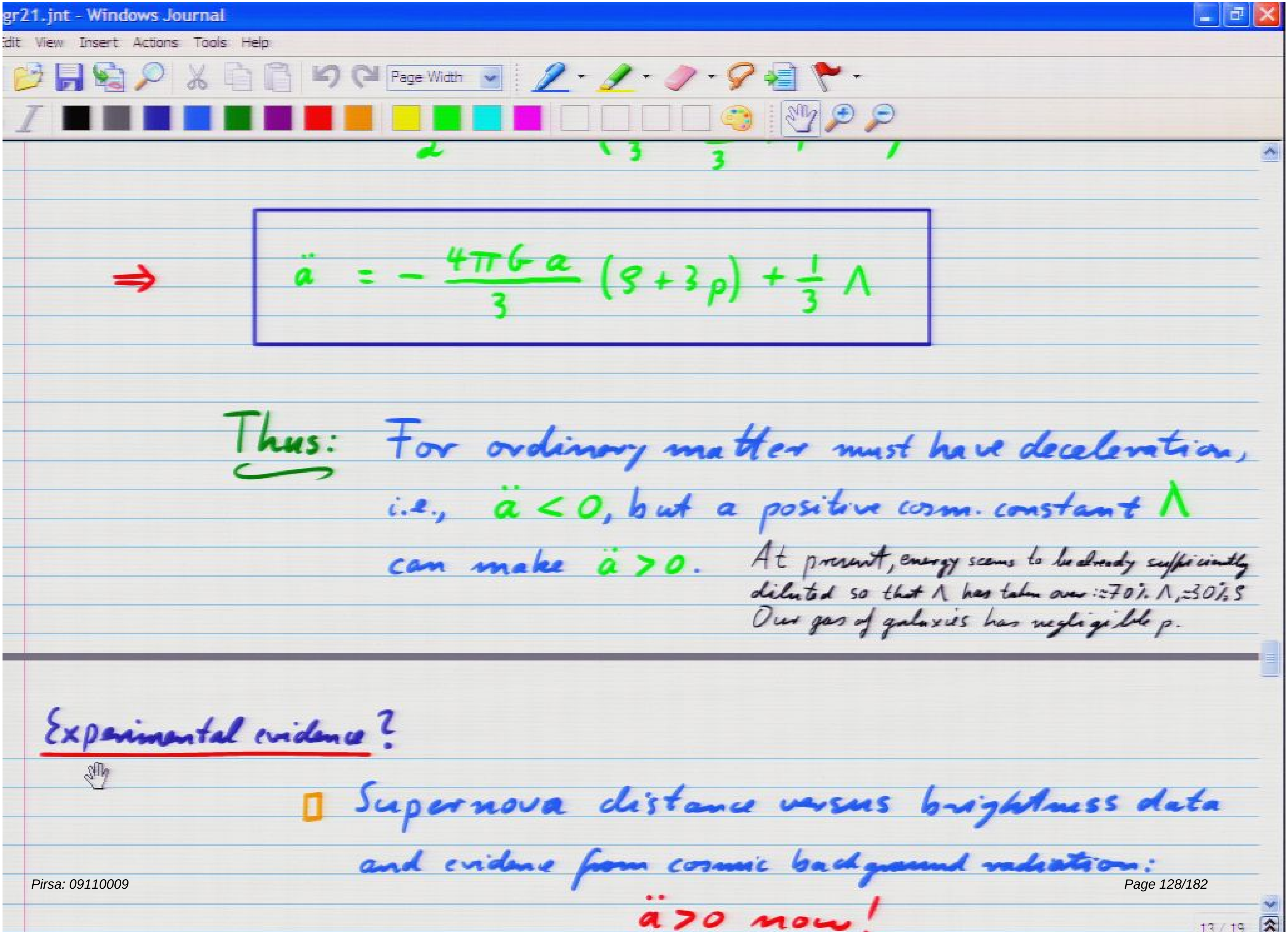
Experimental evidence?

☐ Supernova distance versus brightness data and evidence from cosmic background radiation:

$$\ddot{a} > 0 \text{ now!}$$

$\Rightarrow$ At present, energy is already sufficiently diluted so that Λ dominates over ρ: $\simeq 70\%\ \Lambda$ and $\simeq 30\%\ \rho$.

Note: p of a gas of galaxies is negligible.
Note: ρ includes dark matter.



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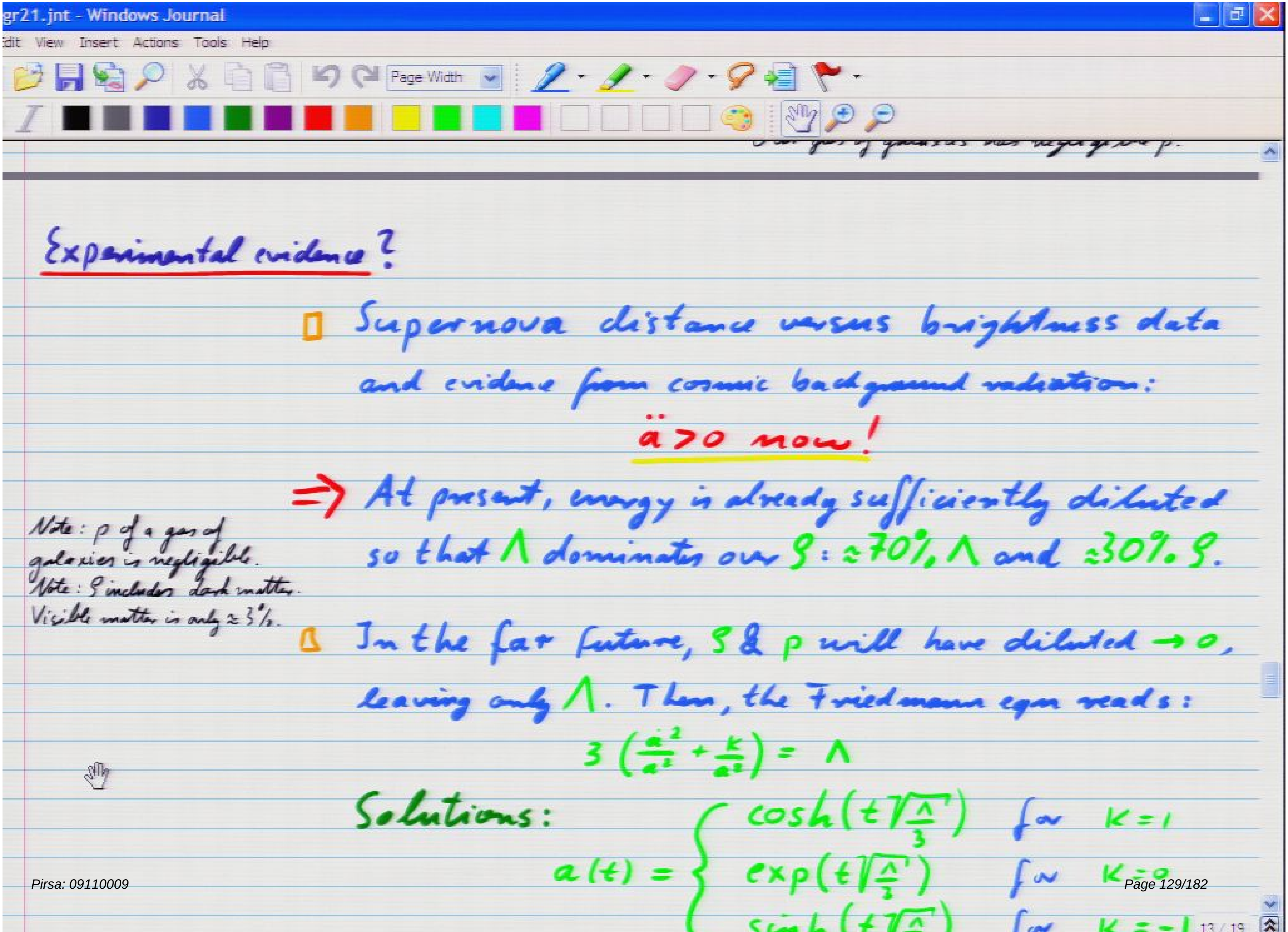
Thus: For ordinary matter must have deceleration, i.e., $\ddot{a} < 0$, but a positive cosm. constant Λ can make $\ddot{a} > 0$.

At present, energy seems to be already sufficiently diluted so that Λ has taken over: $\simeq 70\%\ \Lambda$, $\simeq 30\%\ \rho$. Our gas of galaxies has negligible p.

Experimental evidence?

- ▯ Supernova distance versus brightness data and evidence from cosmic background radiation:

$\ddot{a} > 0$ now!



Experimental evidence?

- Supernova distance versus brightness data and evidence from cosmic background radiation:

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⇒ At present, energy is already sufficiently diluted so that Λ dominates over ρ: $\approx 70\%\ \Lambda$ and $\approx 30\%\ \rho$.

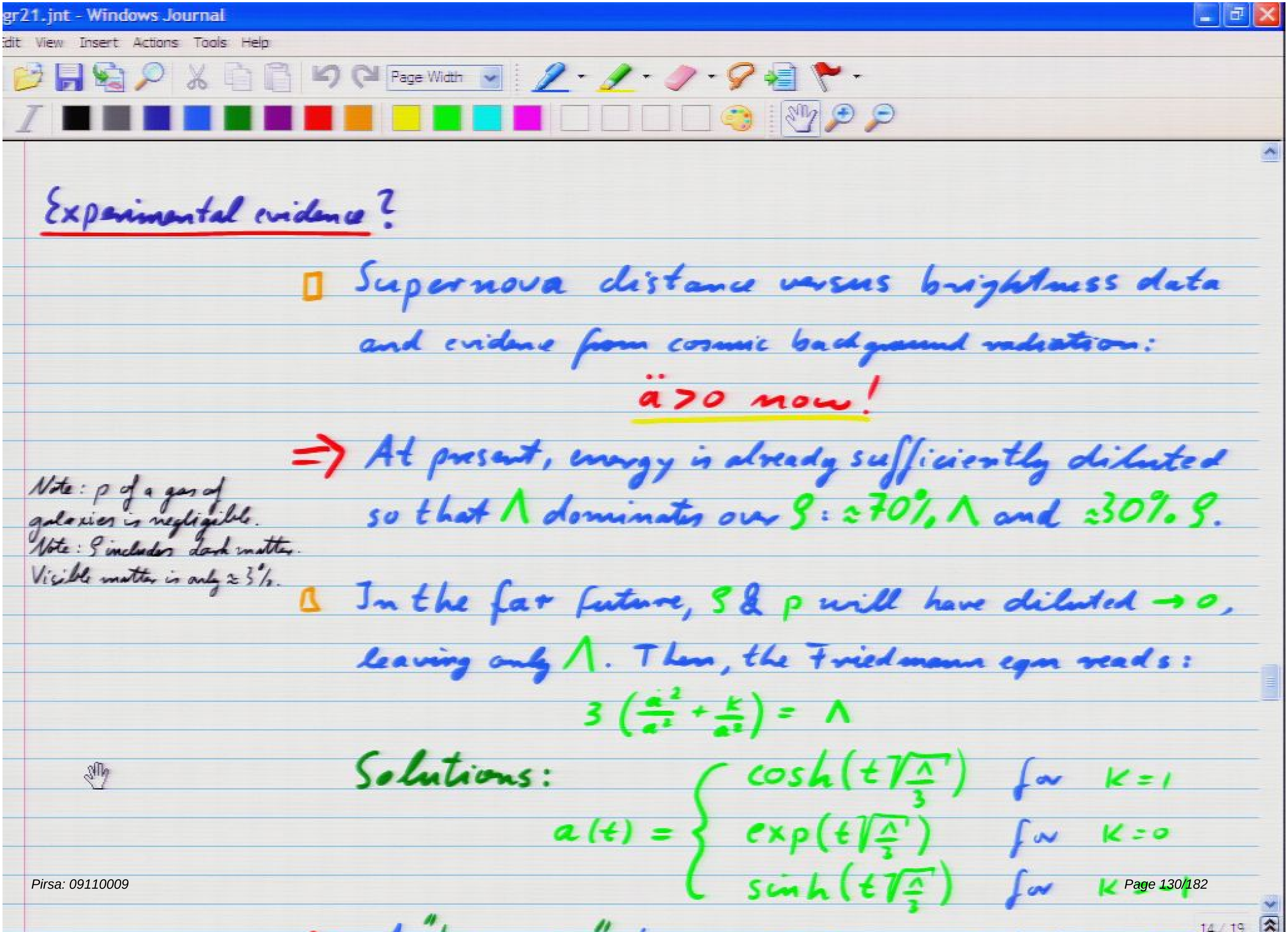
Note: p of a gas of galaxies is negligible.
Note: ρ includes dark matter.
Visible matter is only $\approx 3\%$.

- In the far future, ρ & p will have diluted → 0, leaving only Λ. Then, the Friedmann eqn reads:

$$3\left(\frac{\dot{a}^2}{a^2}+\frac{k}{a^2}\right) = \Lambda$$

Solutions:

$$a(t) = \begin{cases} \cosh\left(t\sqrt{\frac{\Lambda}{3}}\right) & \text{for } k=1 \\ \exp\left(t\sqrt{\frac{\Lambda}{3}}\right) & \text{for } k=0 \\ \sinh\left(t\sqrt{\frac{\Lambda}{3}}\right) & \text{for } k=-1 \end{cases}$$



Experimental evidence?

Supernova distance versus brightness data and evidence from cosmic background radiation:

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⇒ At present, energy is already sufficiently diluted so that Λ dominates over ϱ: ≈70% Λ and ≈30% ϱ.

Note: p of a gas of galaxies is negligible.
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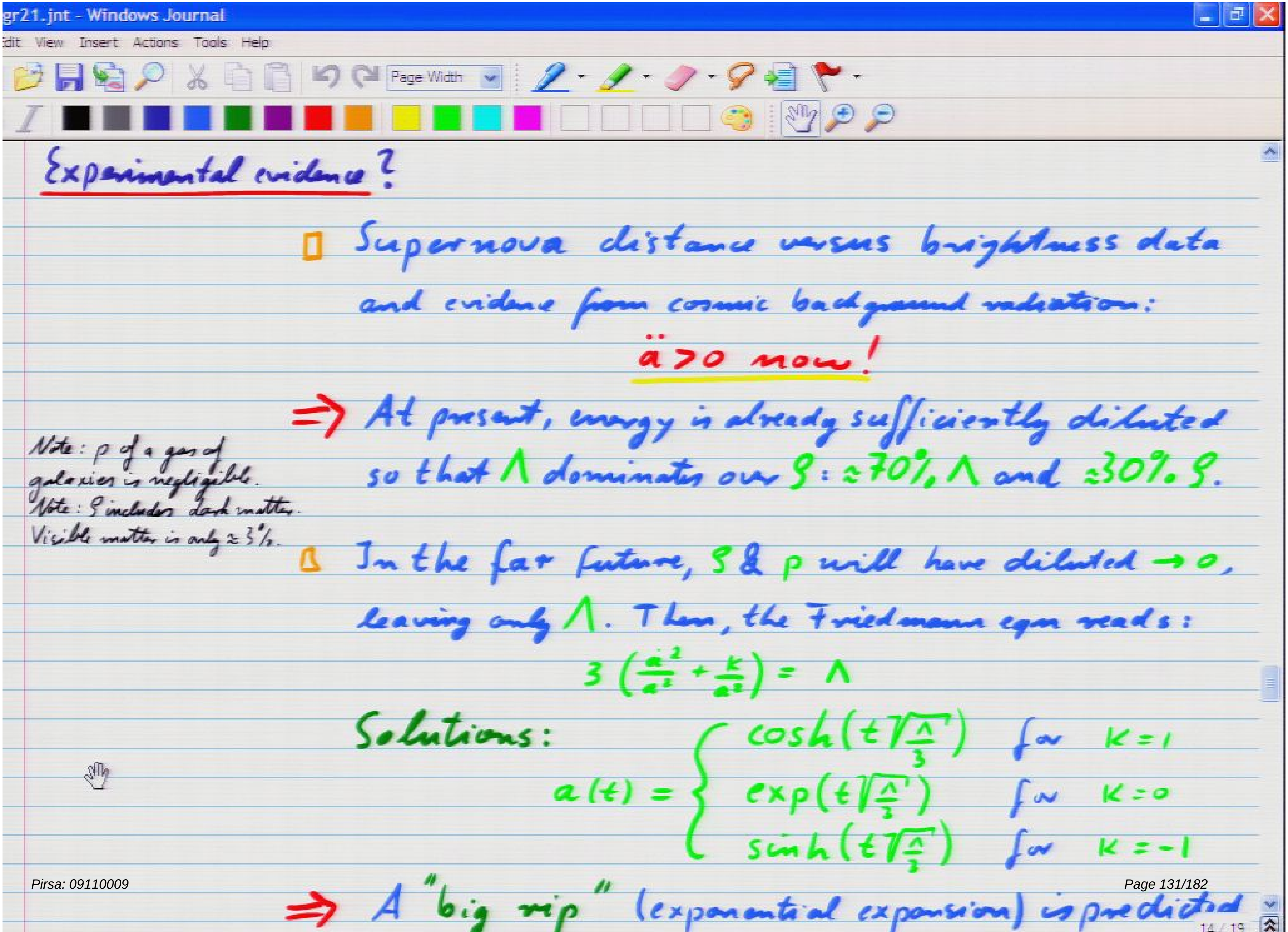

Experimental evidence?

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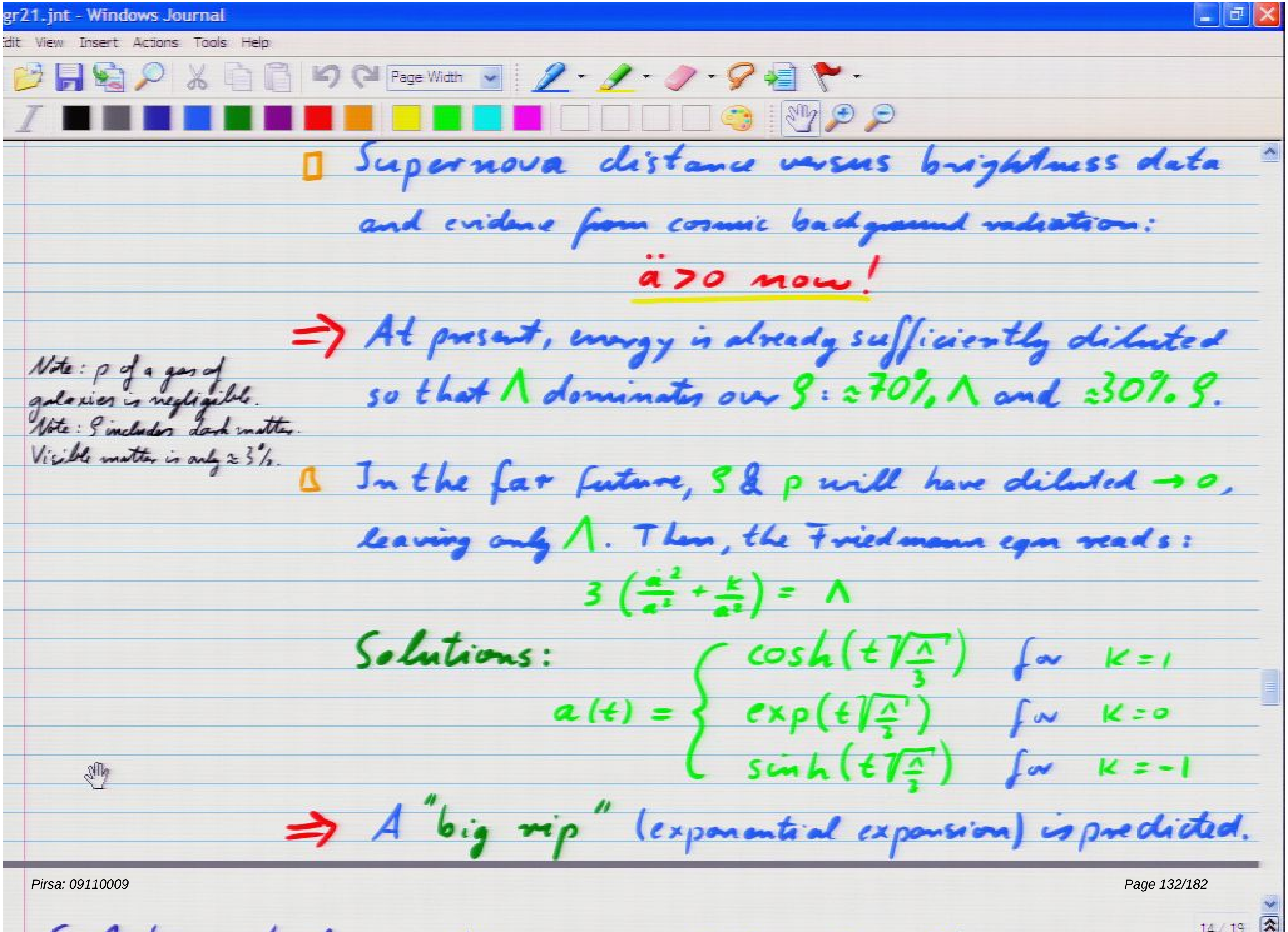
⇒ A "big rip" (exponential expansion) is predicted



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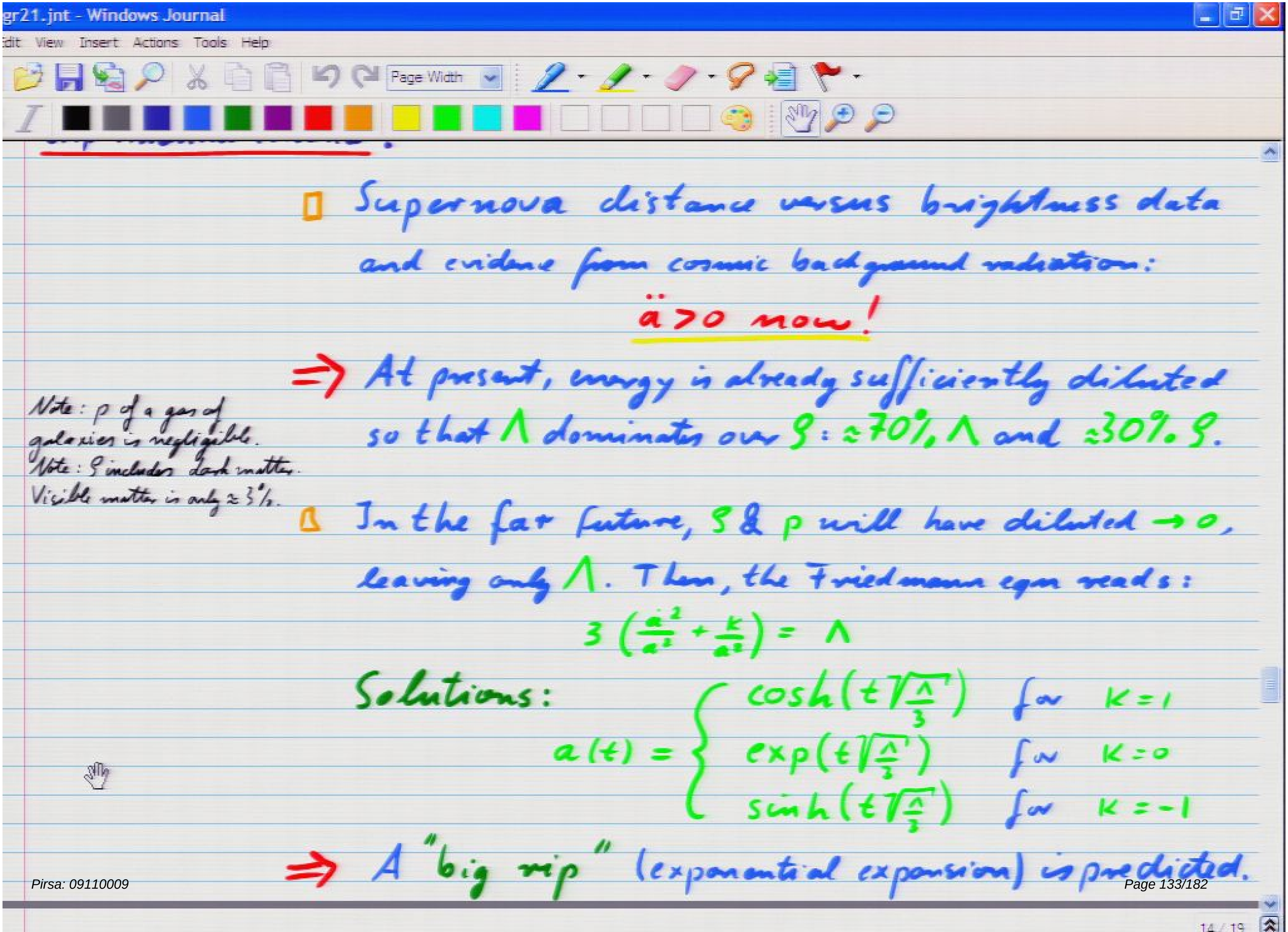

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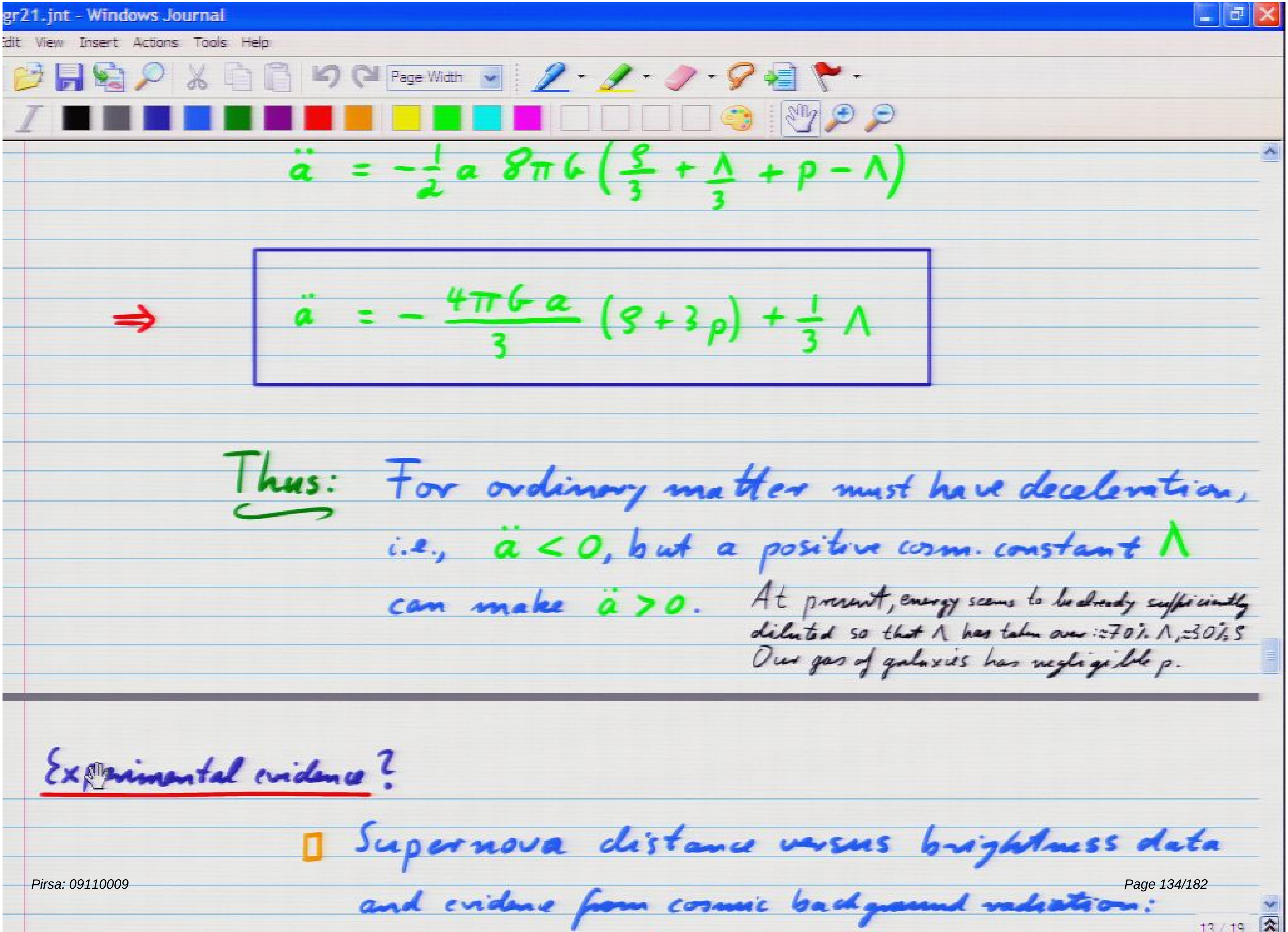

$$\ddot{a} = -\frac{1}{2} a\, 8\pi G \left(\frac{\rho}{3} + \frac{\Lambda}{3} + p - \Lambda \right)$$

$$\Rightarrow \quad \boxed{\ddot{a} = -\frac{4\pi G a}{3}(\rho + 3p) + \frac{1}{3}\Lambda}$$

Thus: For ordinary matter must have deceleration, i.e., $\ddot{a} < 0$, but a positive cosm. constant Λ can make $\ddot{a} > 0$.

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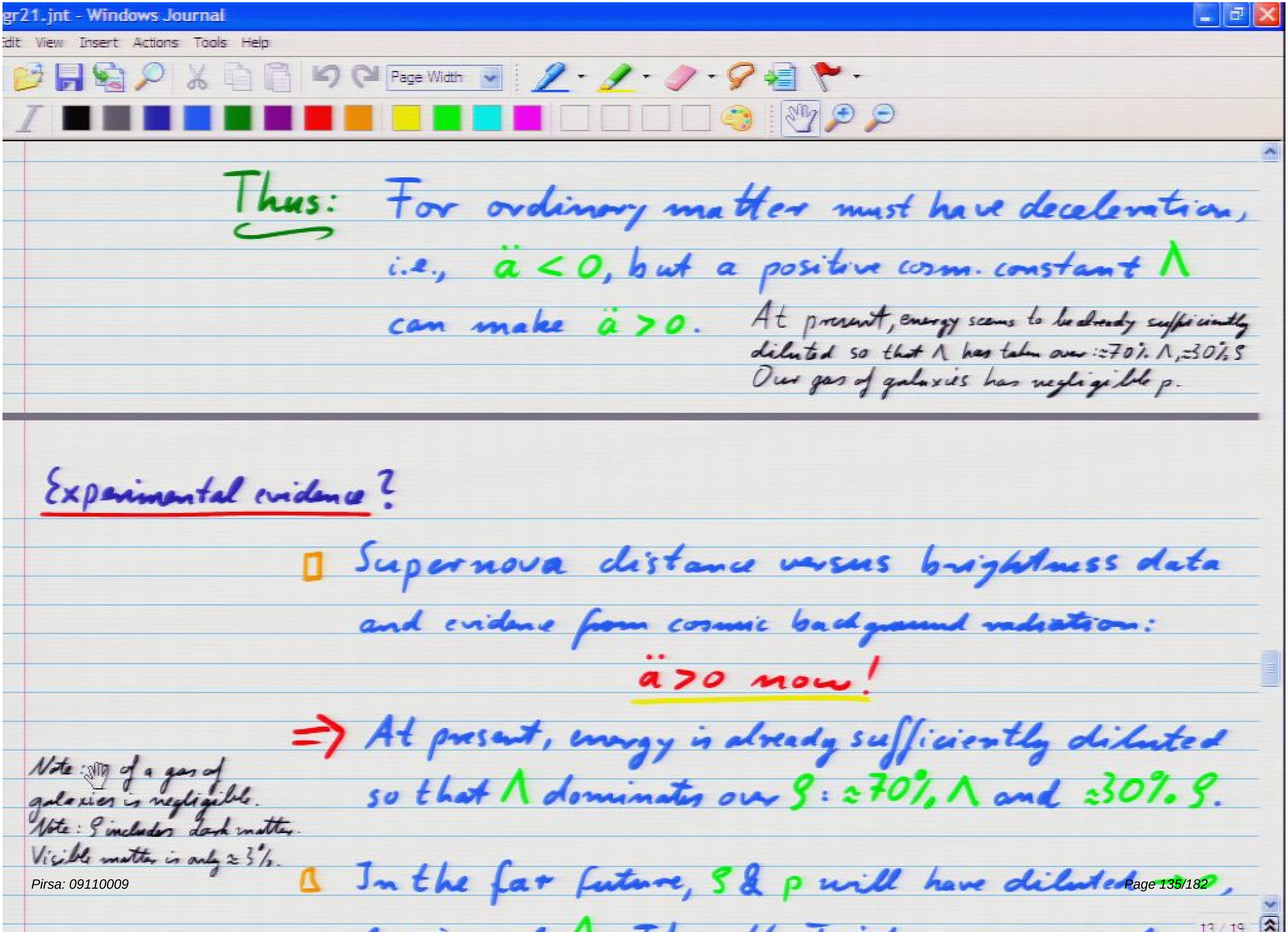

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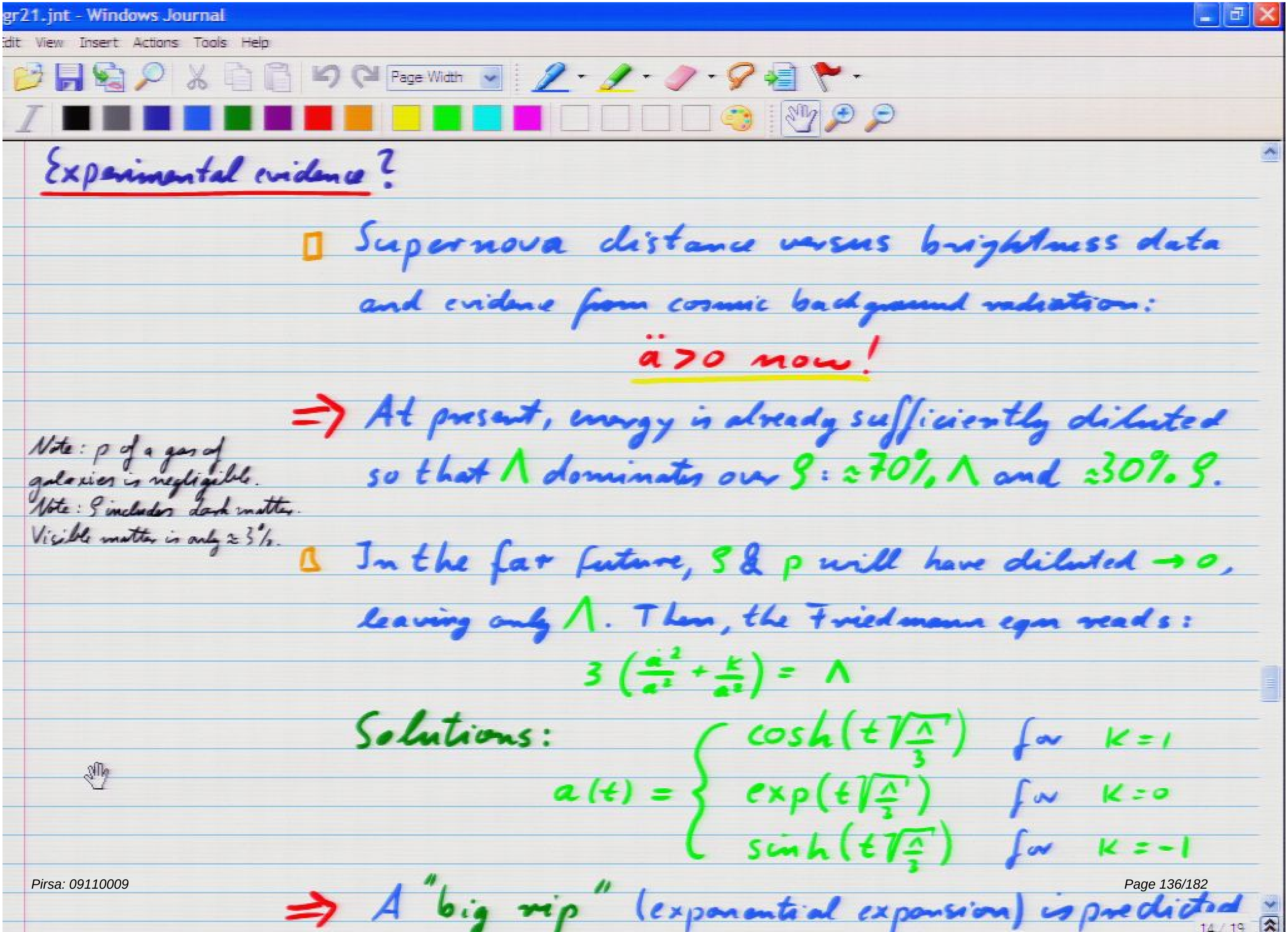

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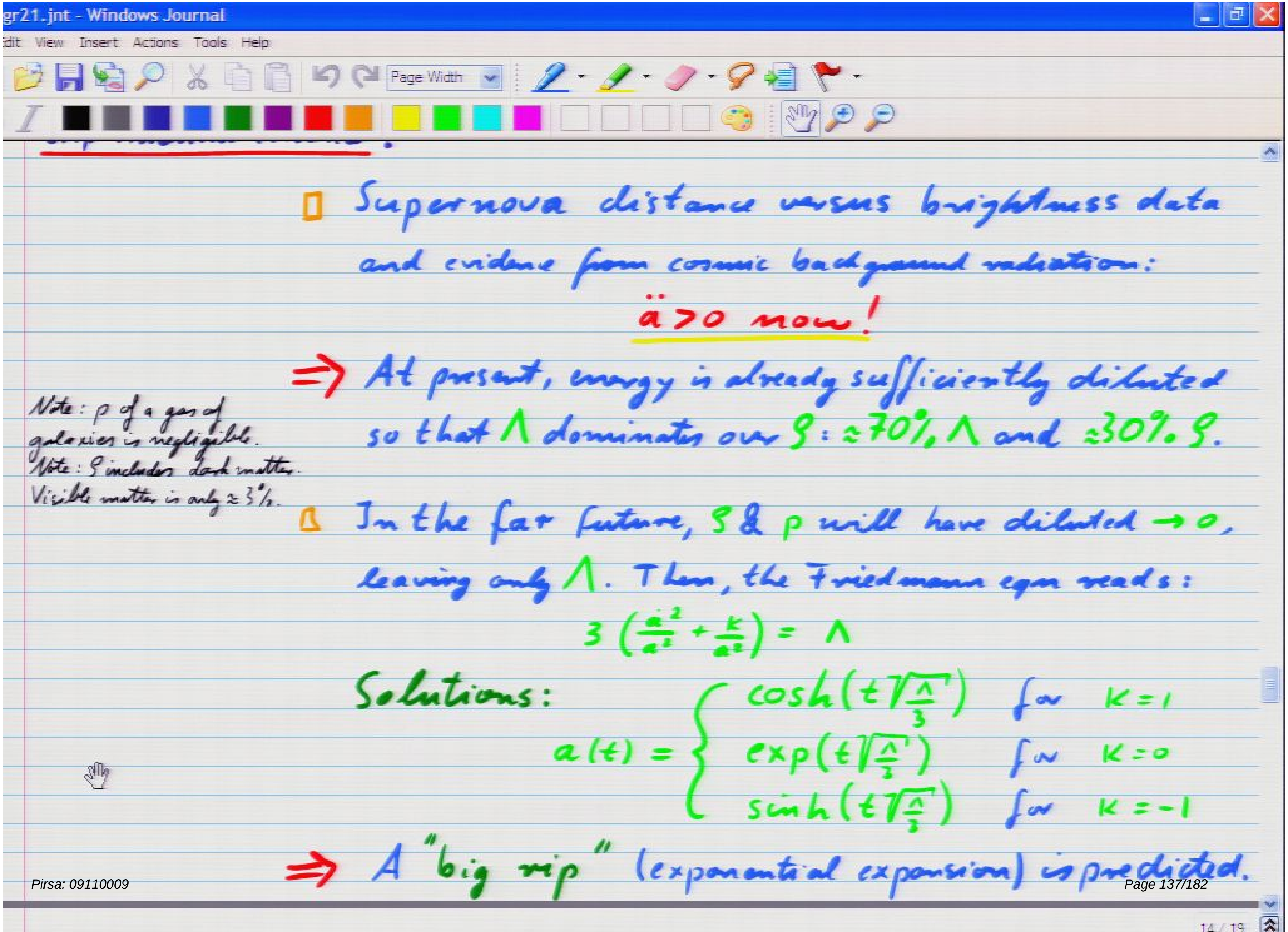

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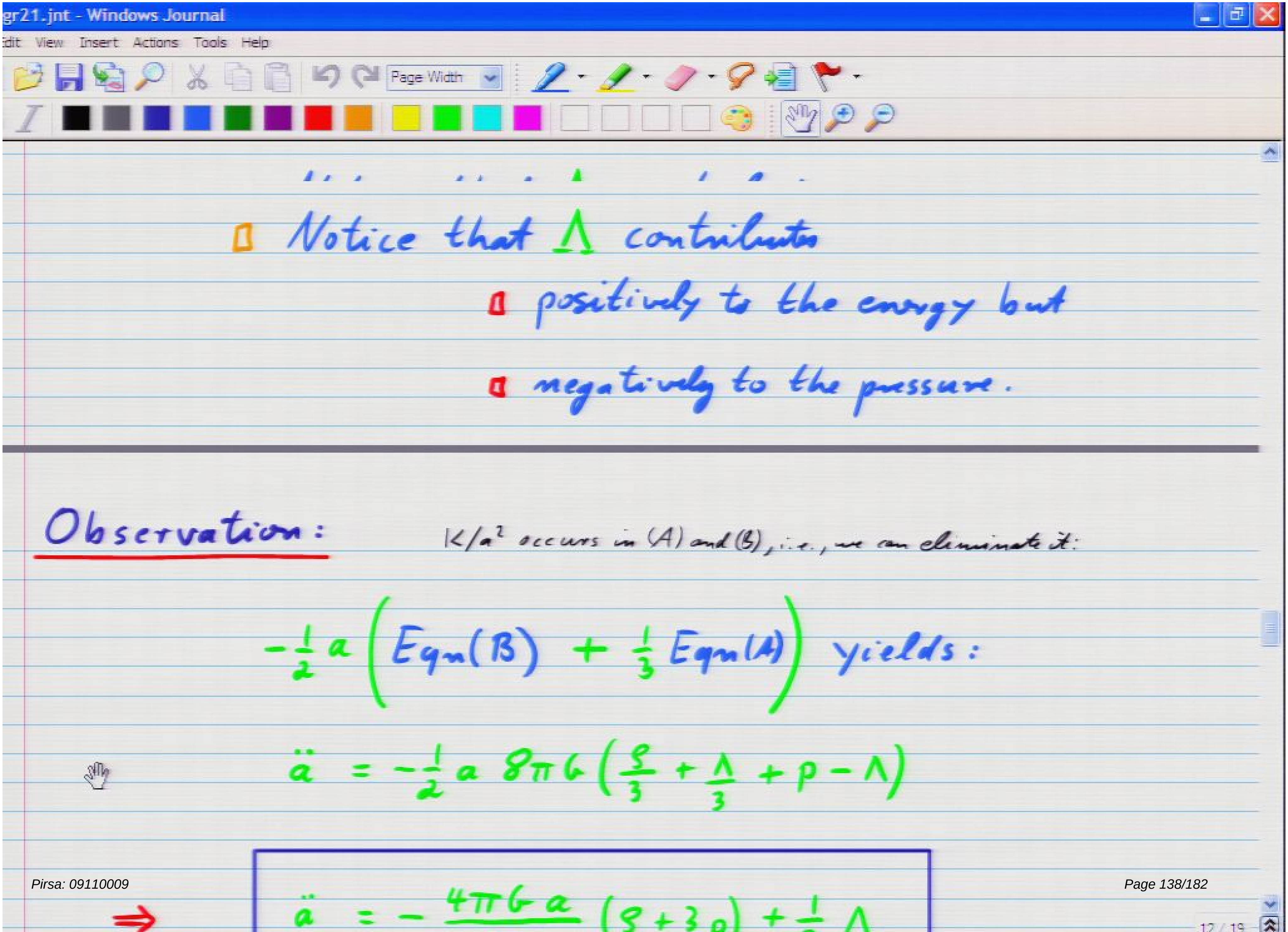


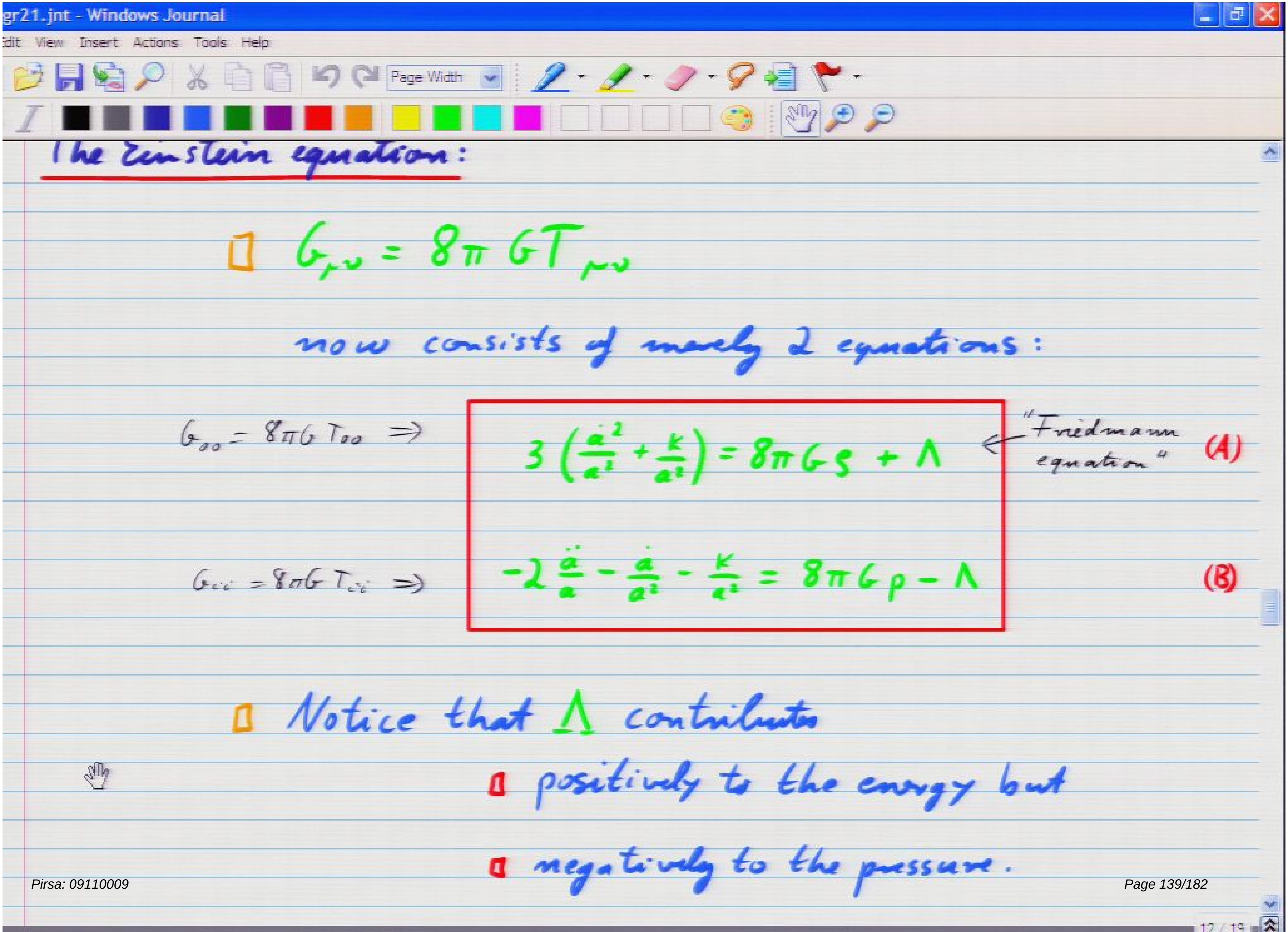
- Notice that Λ contributes
  - positively to the energy but
  - negatively to the pressure.

Observation: K/a^2 occurs in (A) and (B), i.e., we can eliminate it:

$$-\frac{1}{2}a\left(\text{Eqn}(B) + \frac{1}{3}\text{Eqn}(A)\right) \text{ yields:}$$

$$\ddot{a} = -\frac{1}{2}a\, 8\pi G\left(\frac{\rho}{3} + \frac{\Lambda}{3} + p - \Lambda\right)$$

$$\Rightarrow \quad \ddot{a} = -\frac{4\pi G a}{}(\rho + 3p) + \frac{1}{}\Lambda$$



The Einstein equation:

- $G_{\mu\nu} = 8\pi G T_{\mu\nu}$

now consists of merely 2 equations:

$$G_{00} = 8\pi G T_{00} \Rightarrow \quad \boxed{3\left(\frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right) = 8\pi G \rho + \Lambda} \leftarrow \text{"Friedmann equation"} \quad (A)$$

$$G_{ii} = 8\pi G T_{ii} \Rightarrow \quad \boxed{-2\frac{\ddot{a}}{a} - \frac{\dot{a}}{a^2} - \frac{k}{a^2} = 8\pi G p - \Lambda} \quad (B)$$

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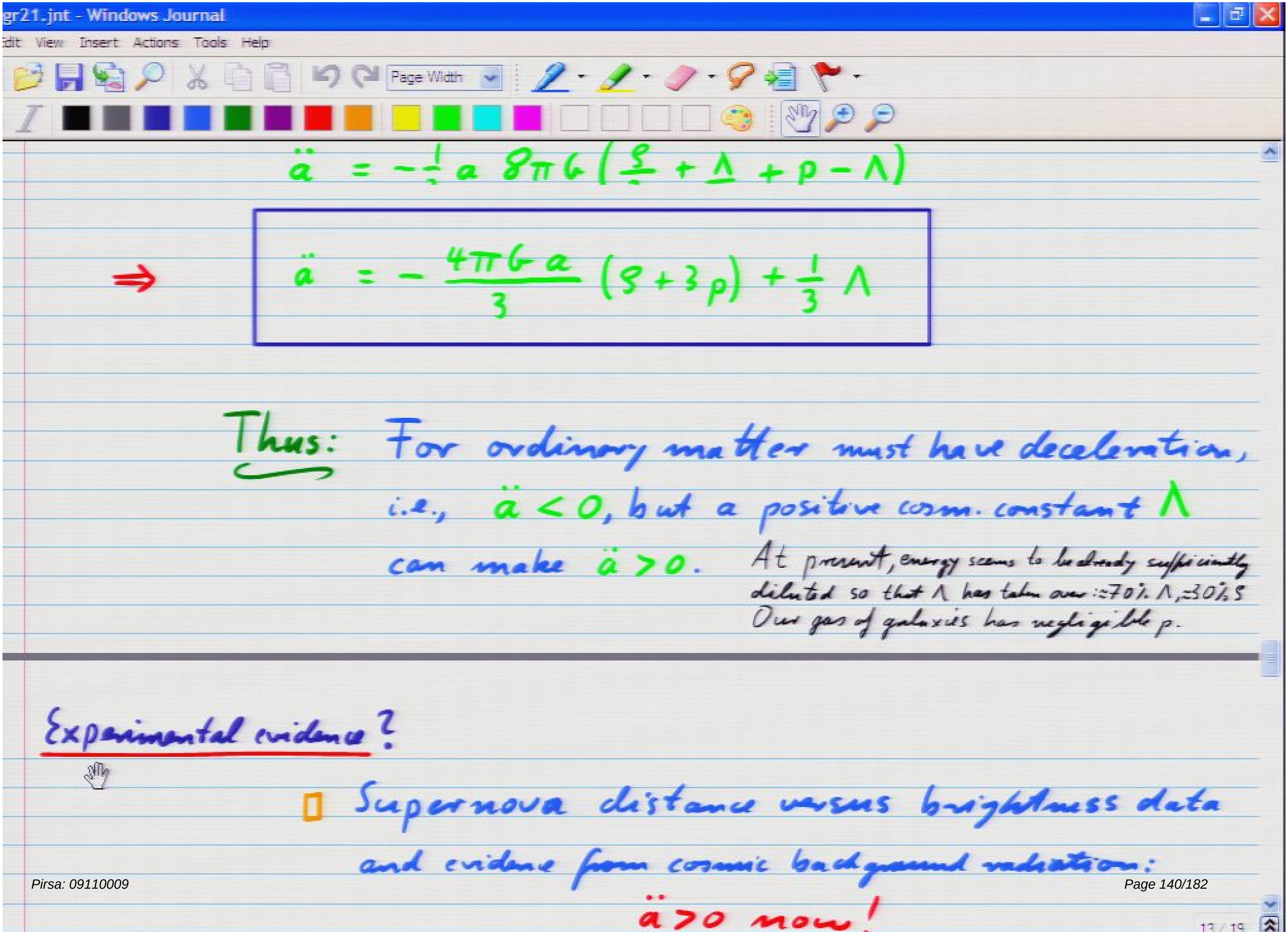

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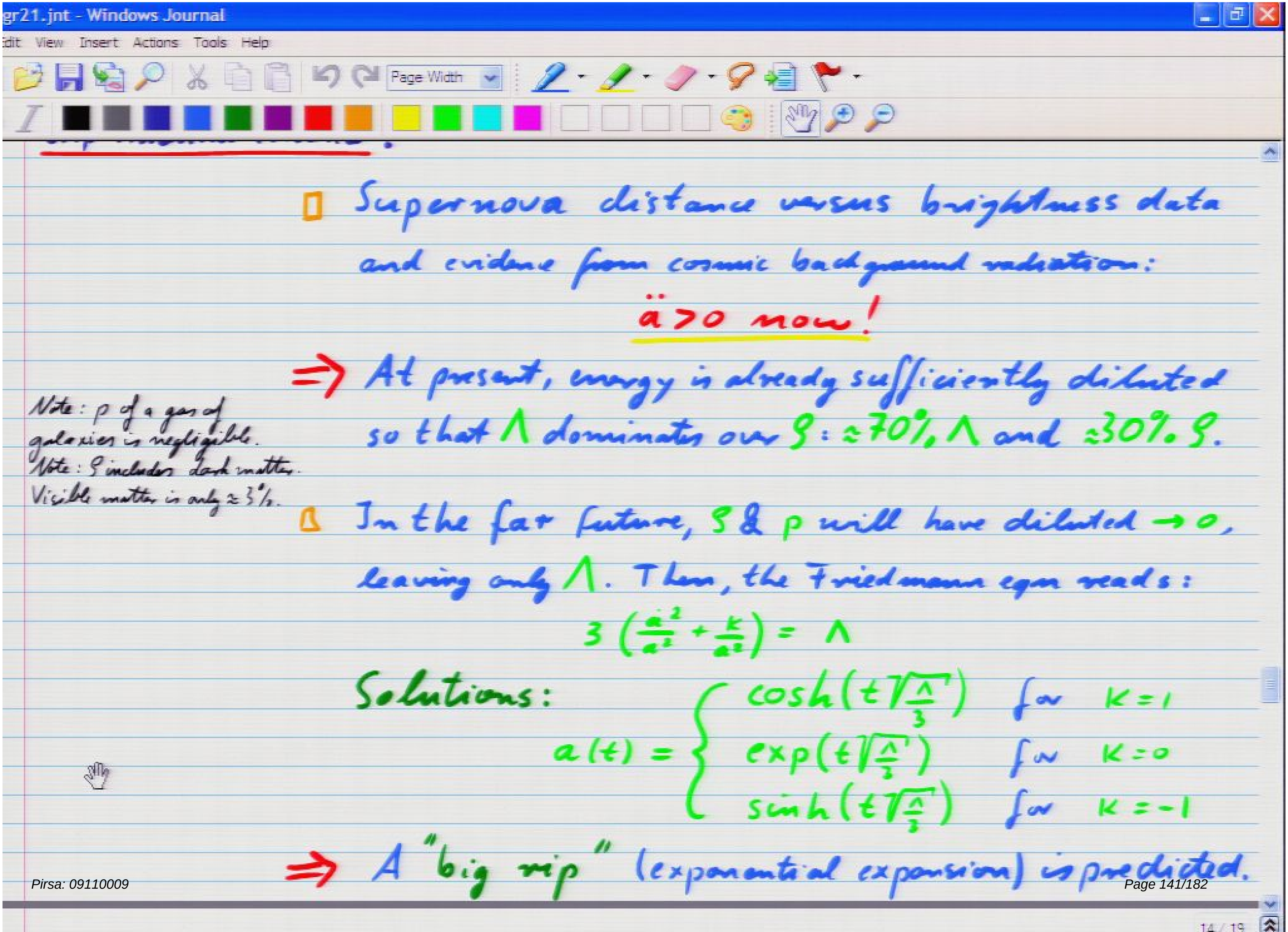

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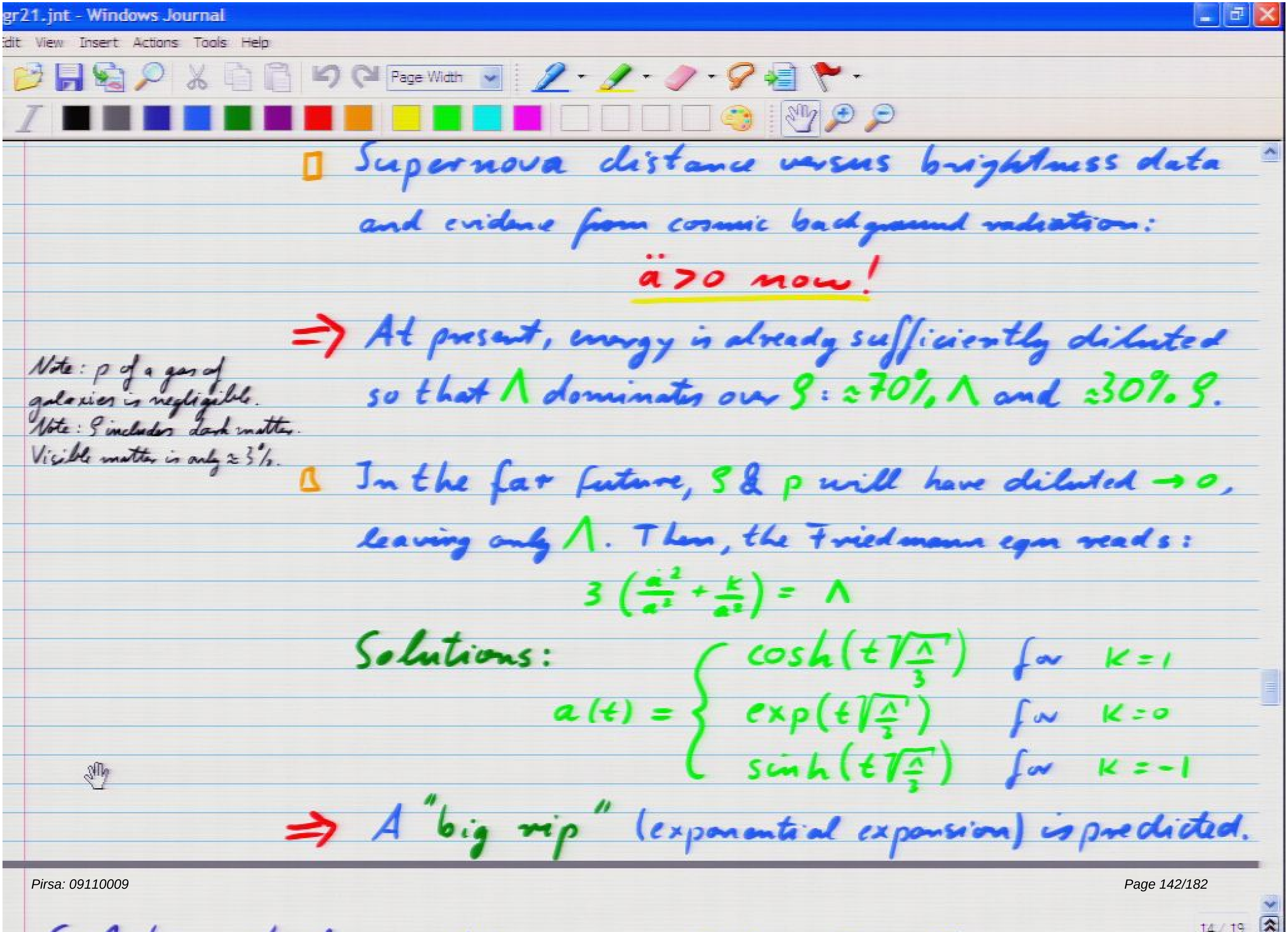

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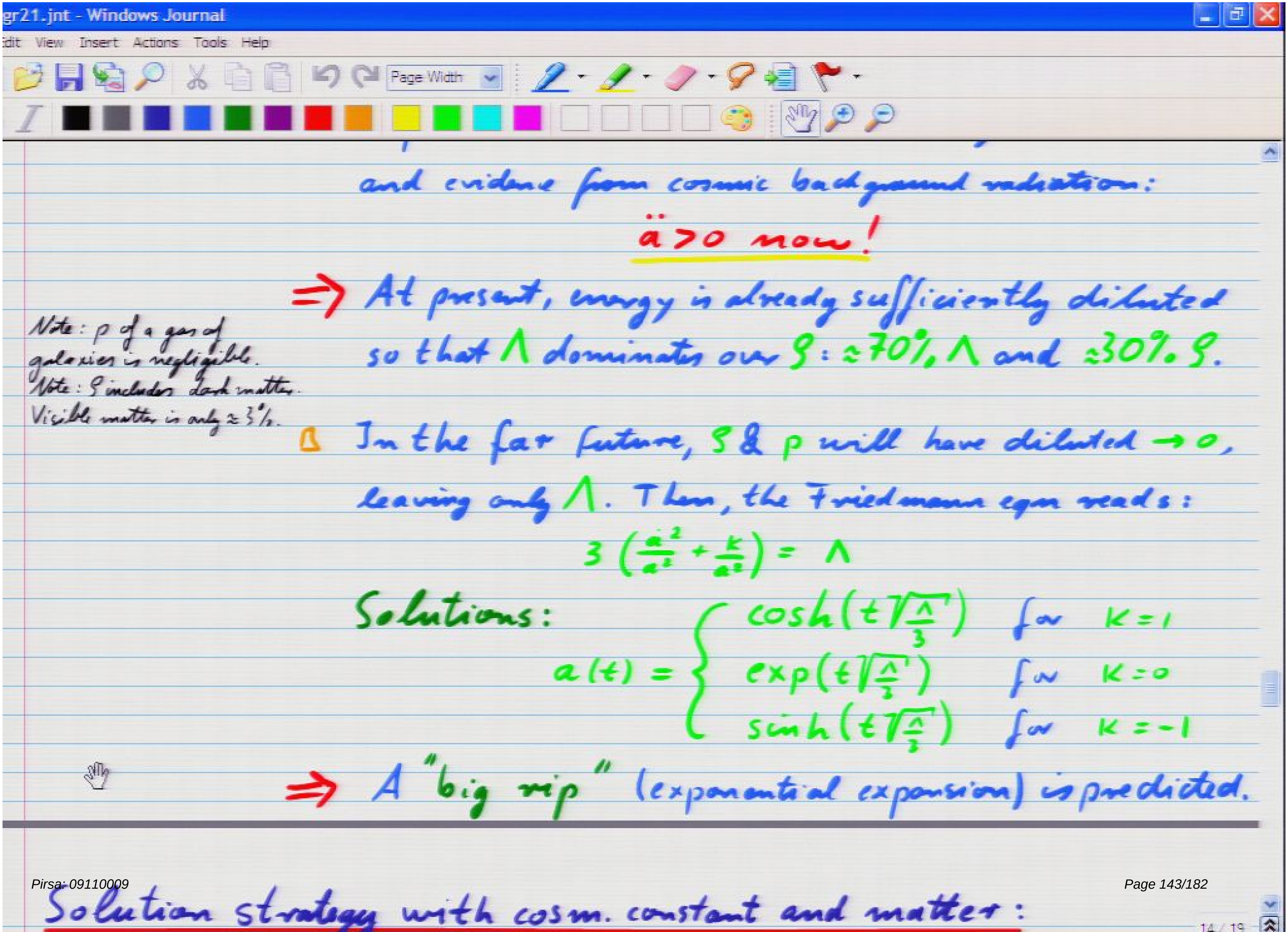

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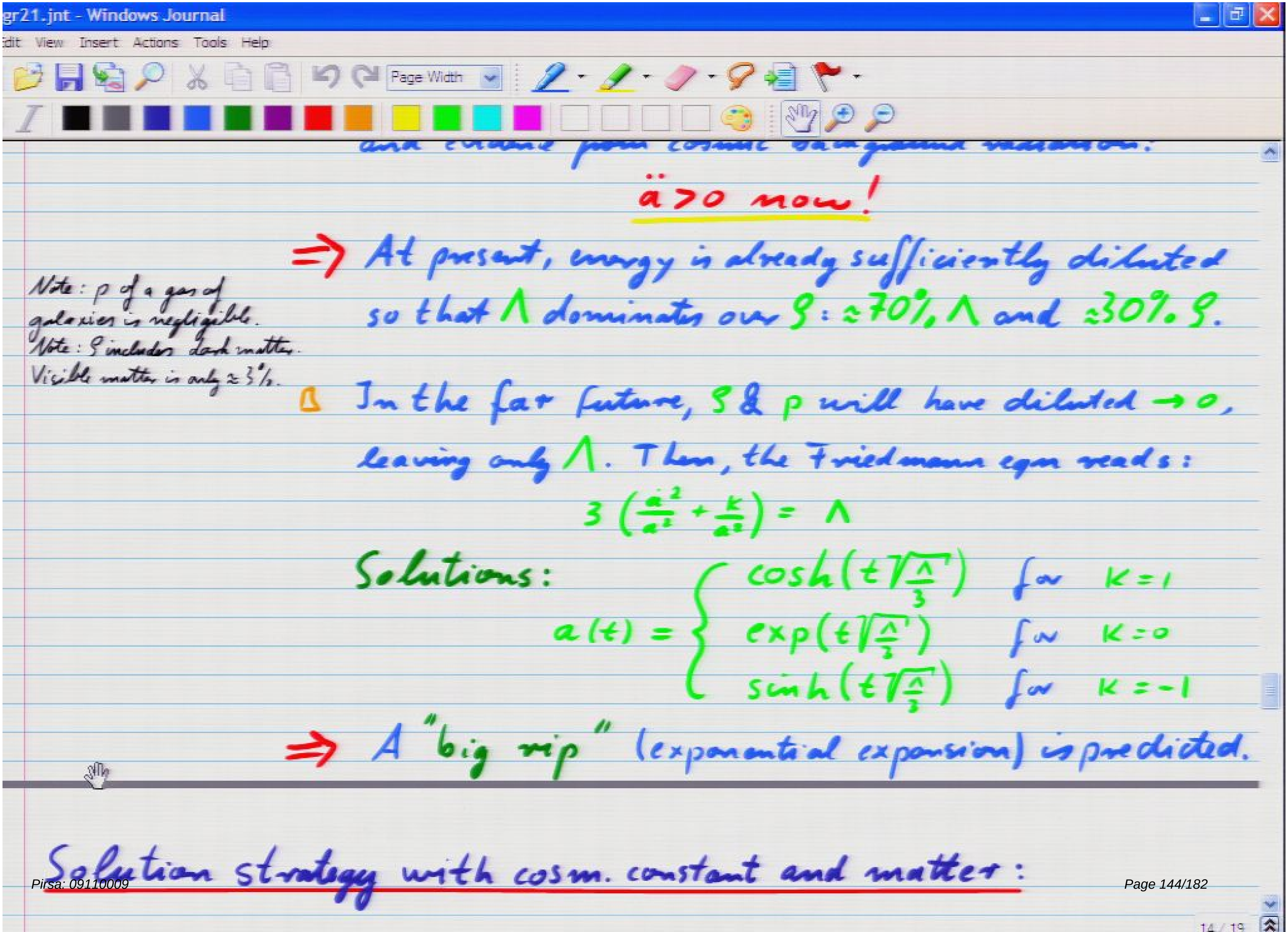
Solution strategy with cosm. constant and matter:



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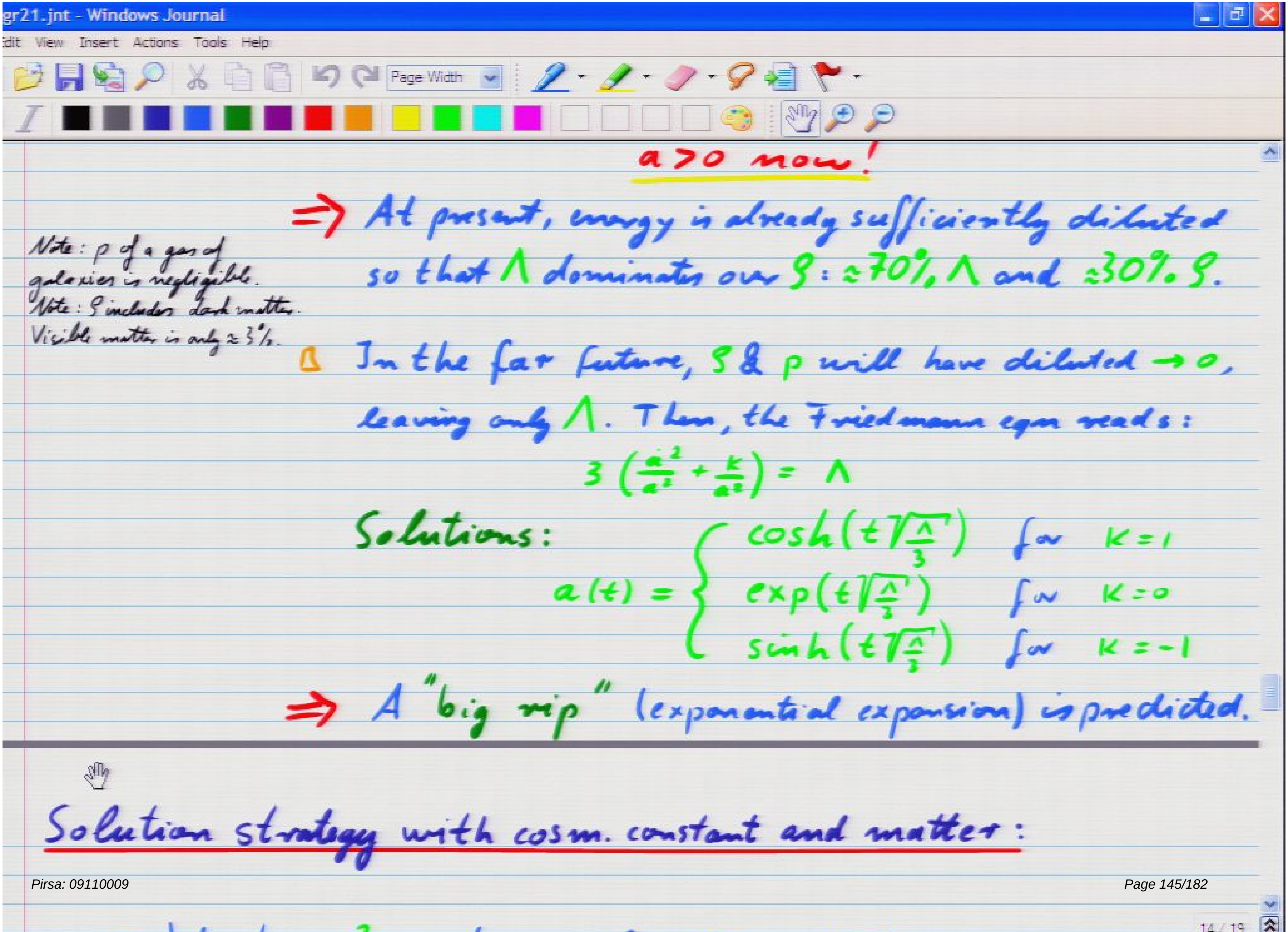

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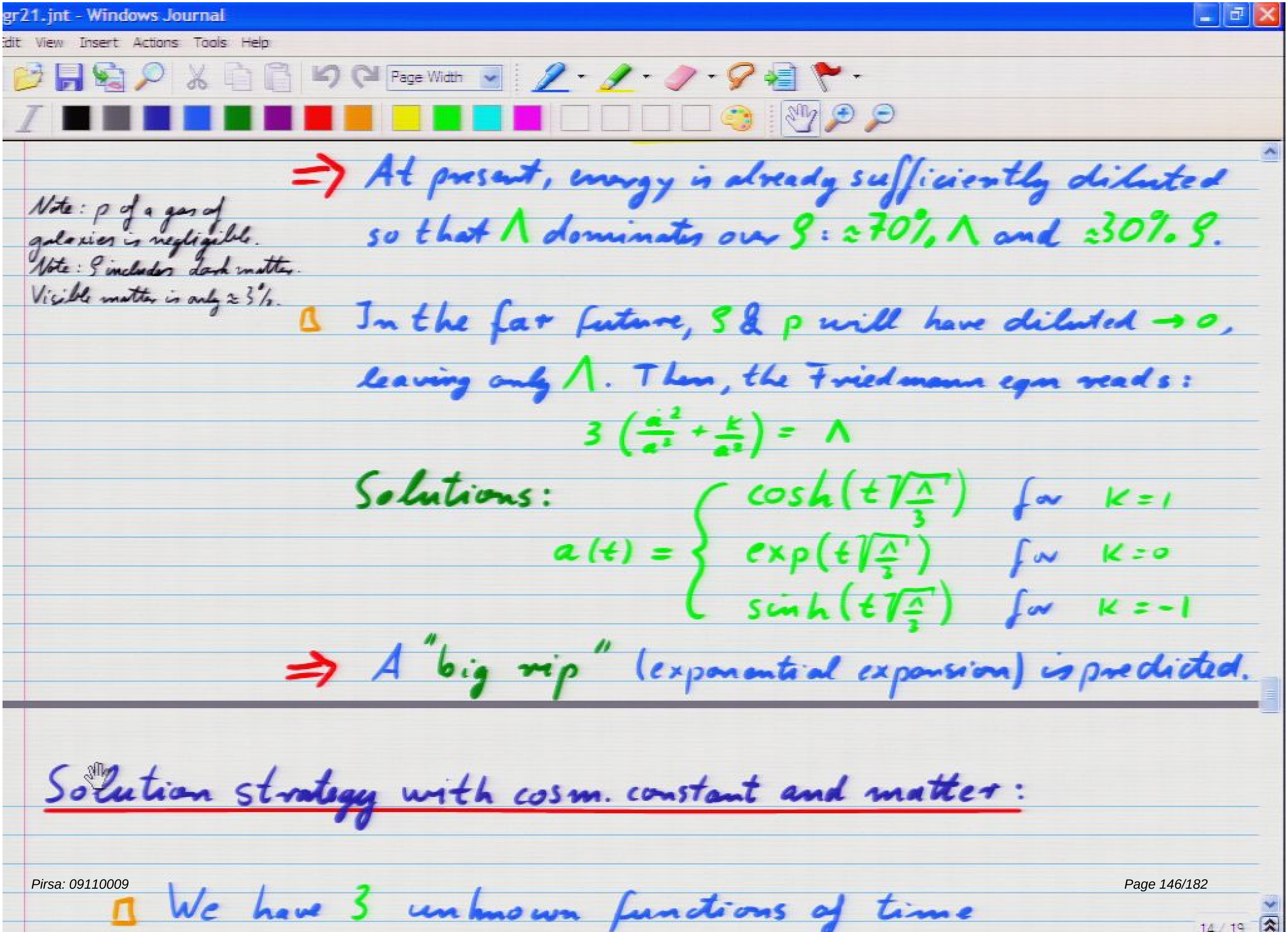

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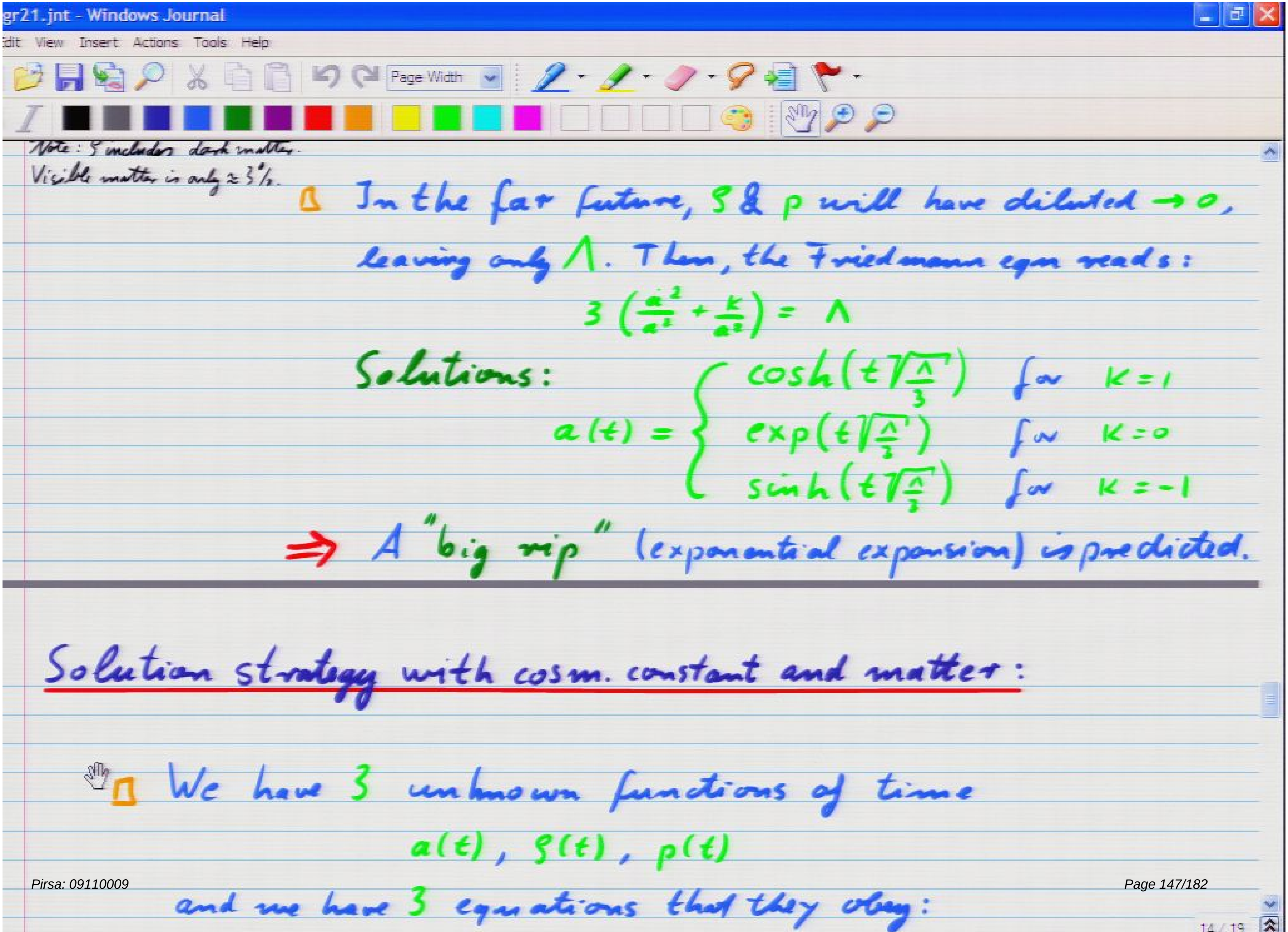
Solution strategy with cosm. constant and matter:

We have 3 unknown functions of time



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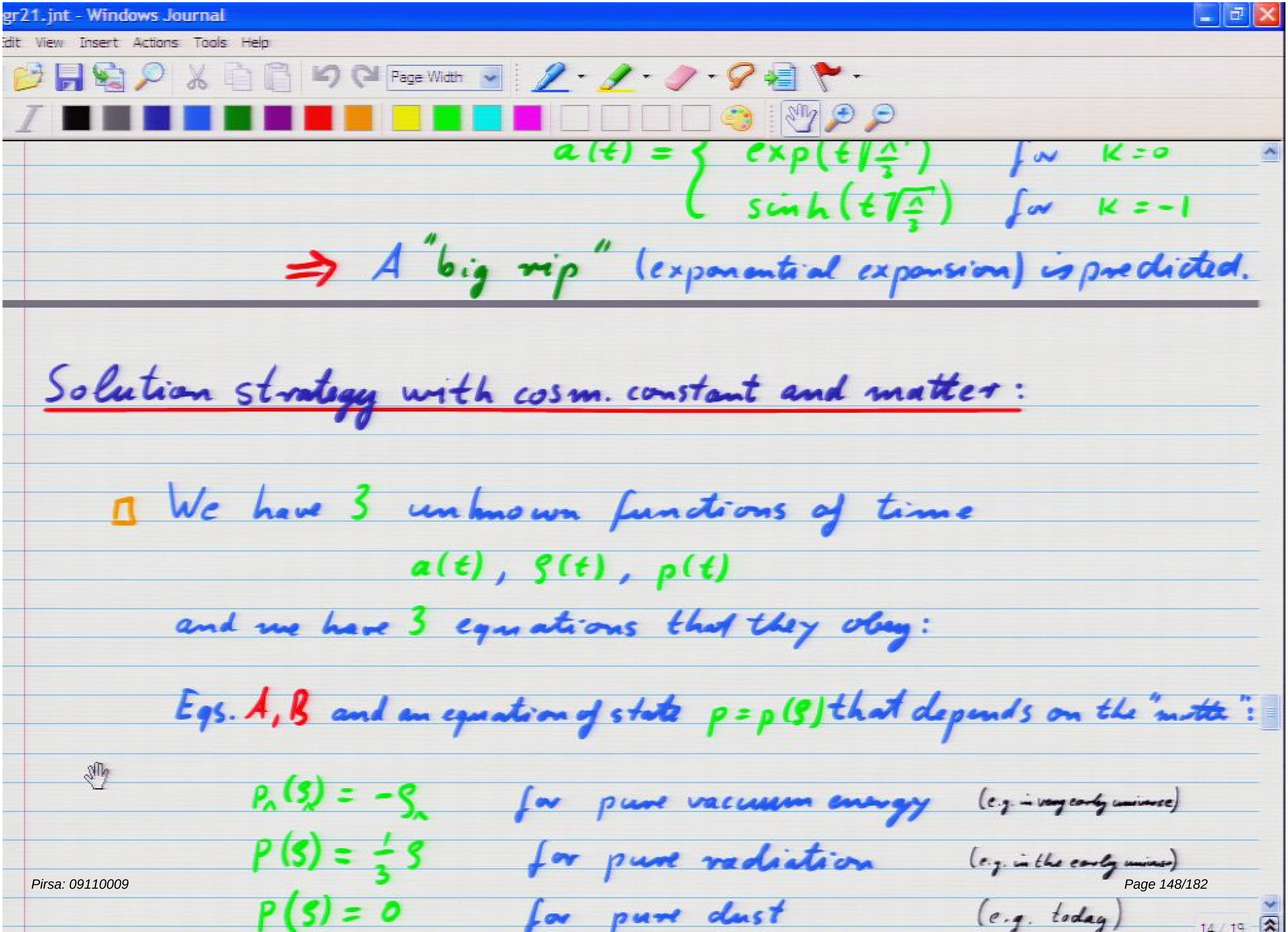

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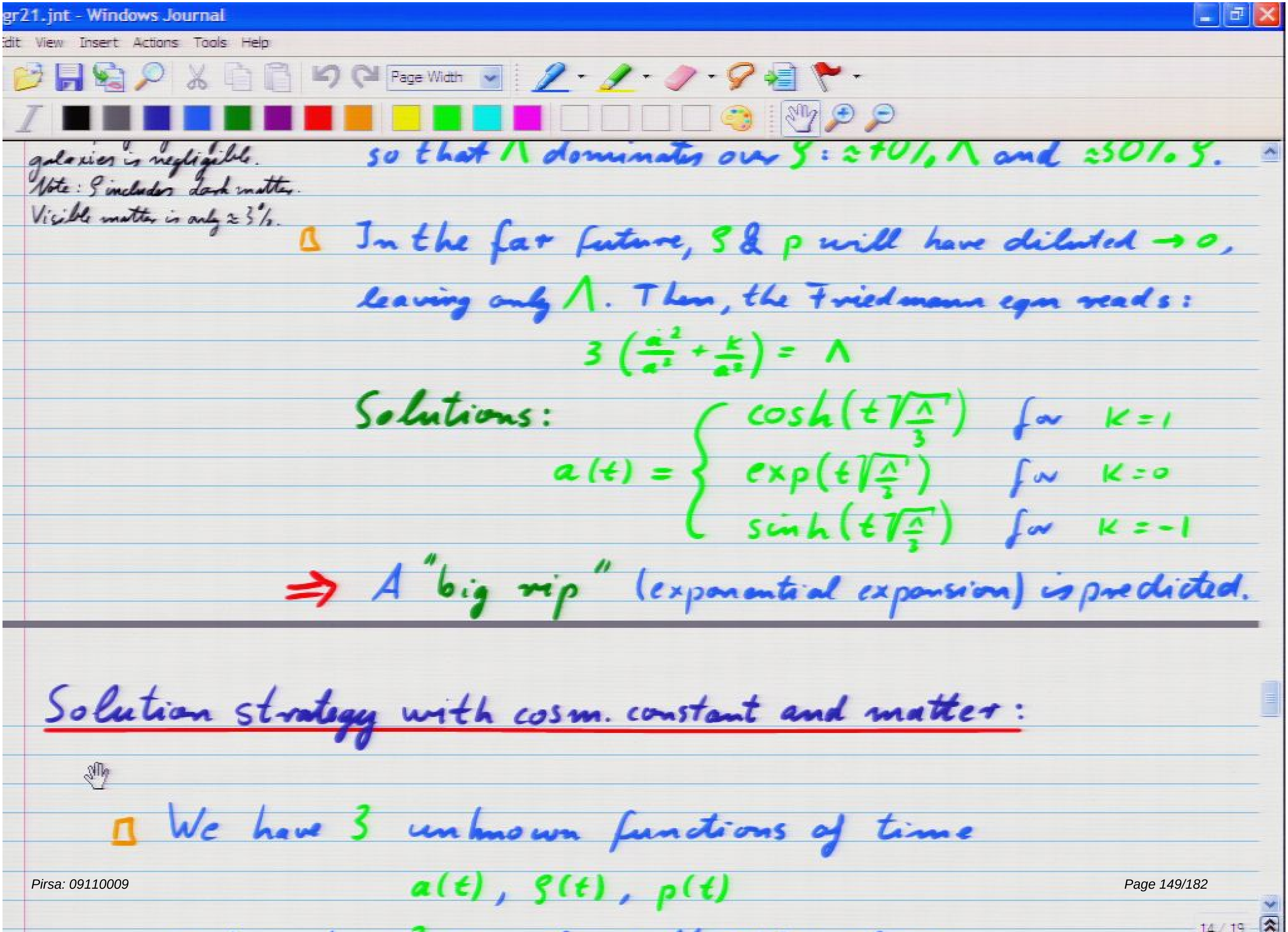
Eqs. A, B and an equation of state $p = p(\rho)$ that depends on the "matter":

$p_\Lambda(\rho_\Lambda) = -\rho_\Lambda$ for pure vacuum energy (e.g. in very early universe)

$p(\rho) = \frac{1}{3}\rho$ for pure radiation (e.g. in the early universe)

$p(\rho) = 0$ for pure dust (e.g. today)



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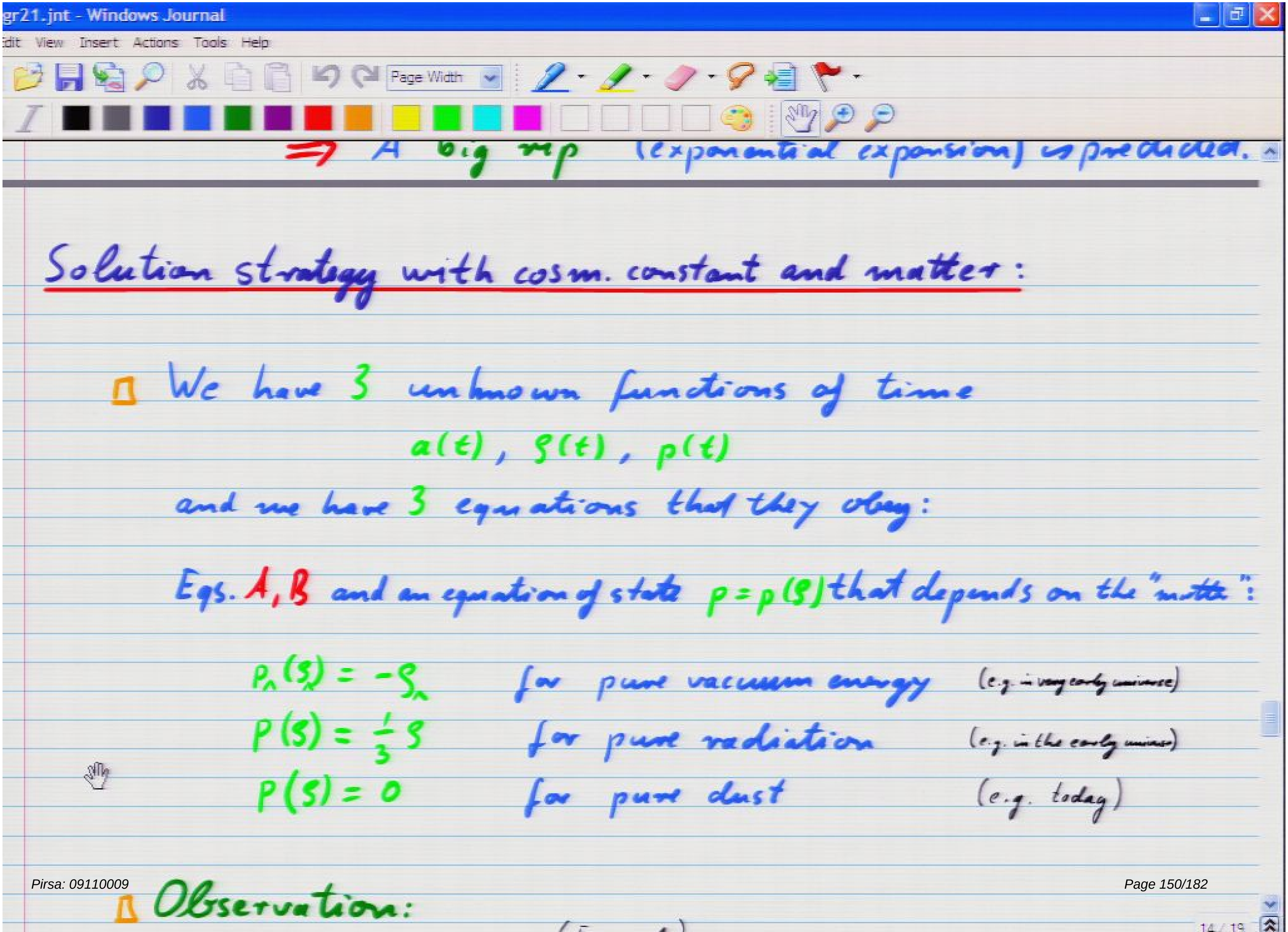

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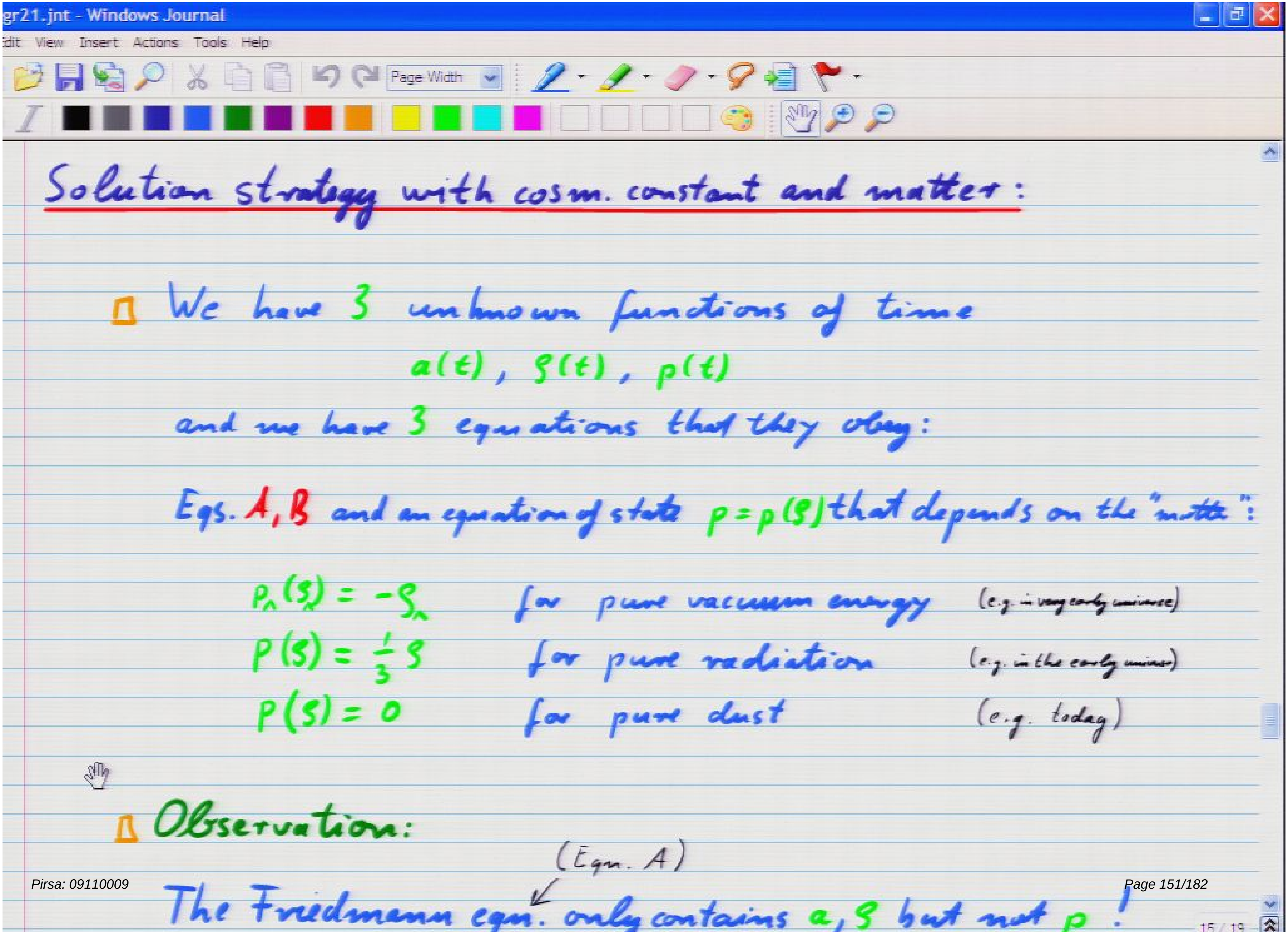

Solution strategy with cosm. constant and matter:

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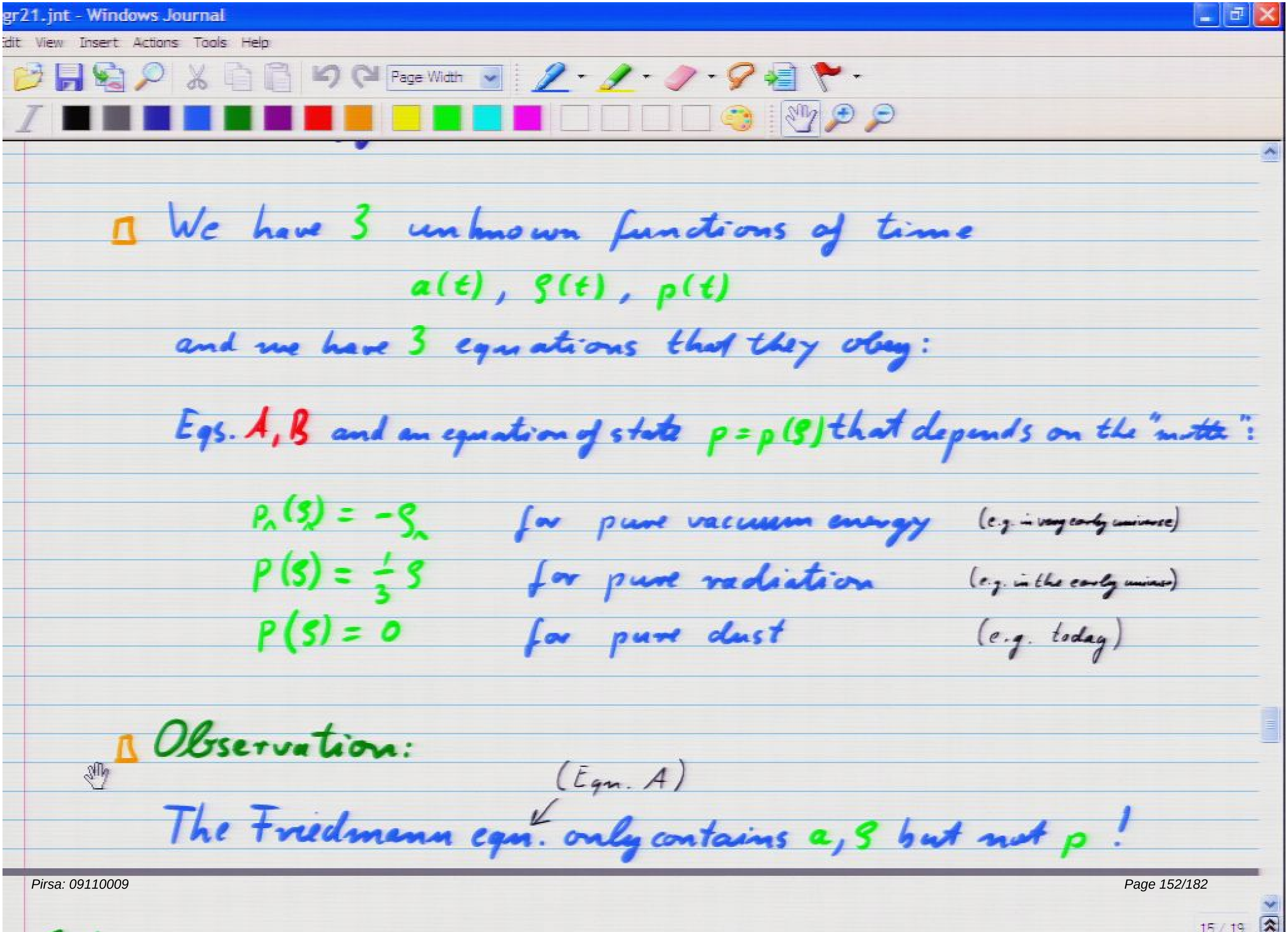

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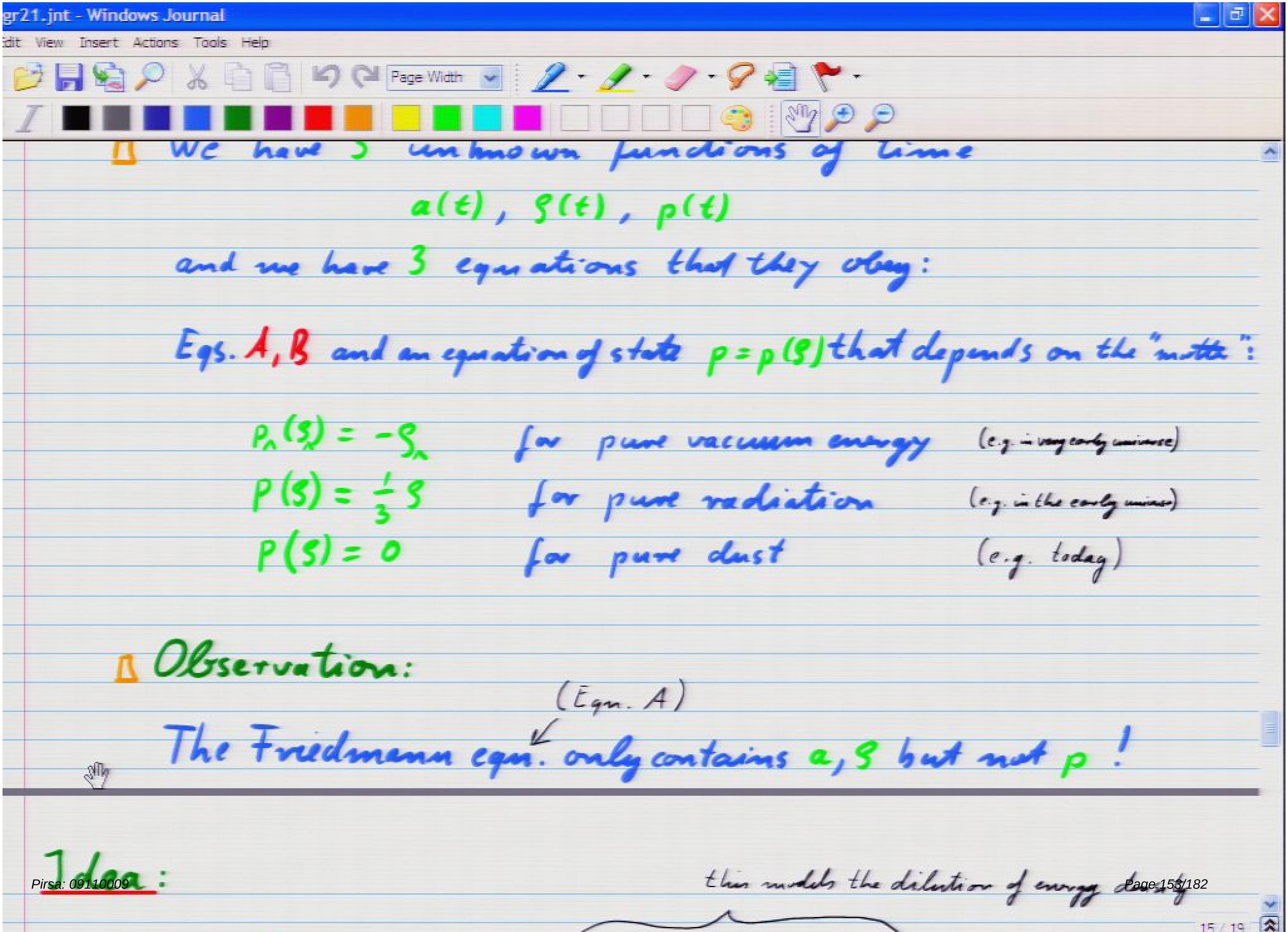

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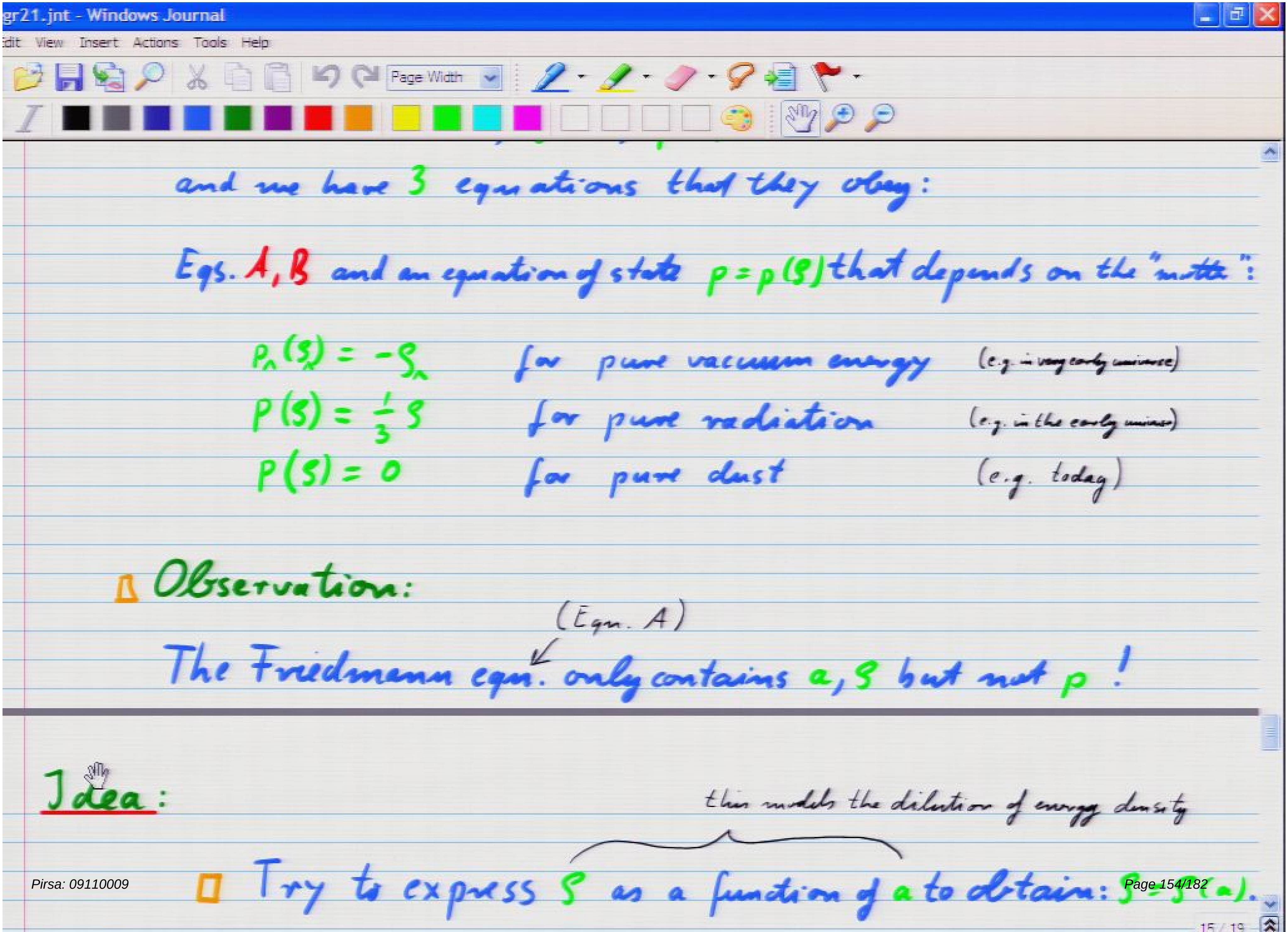

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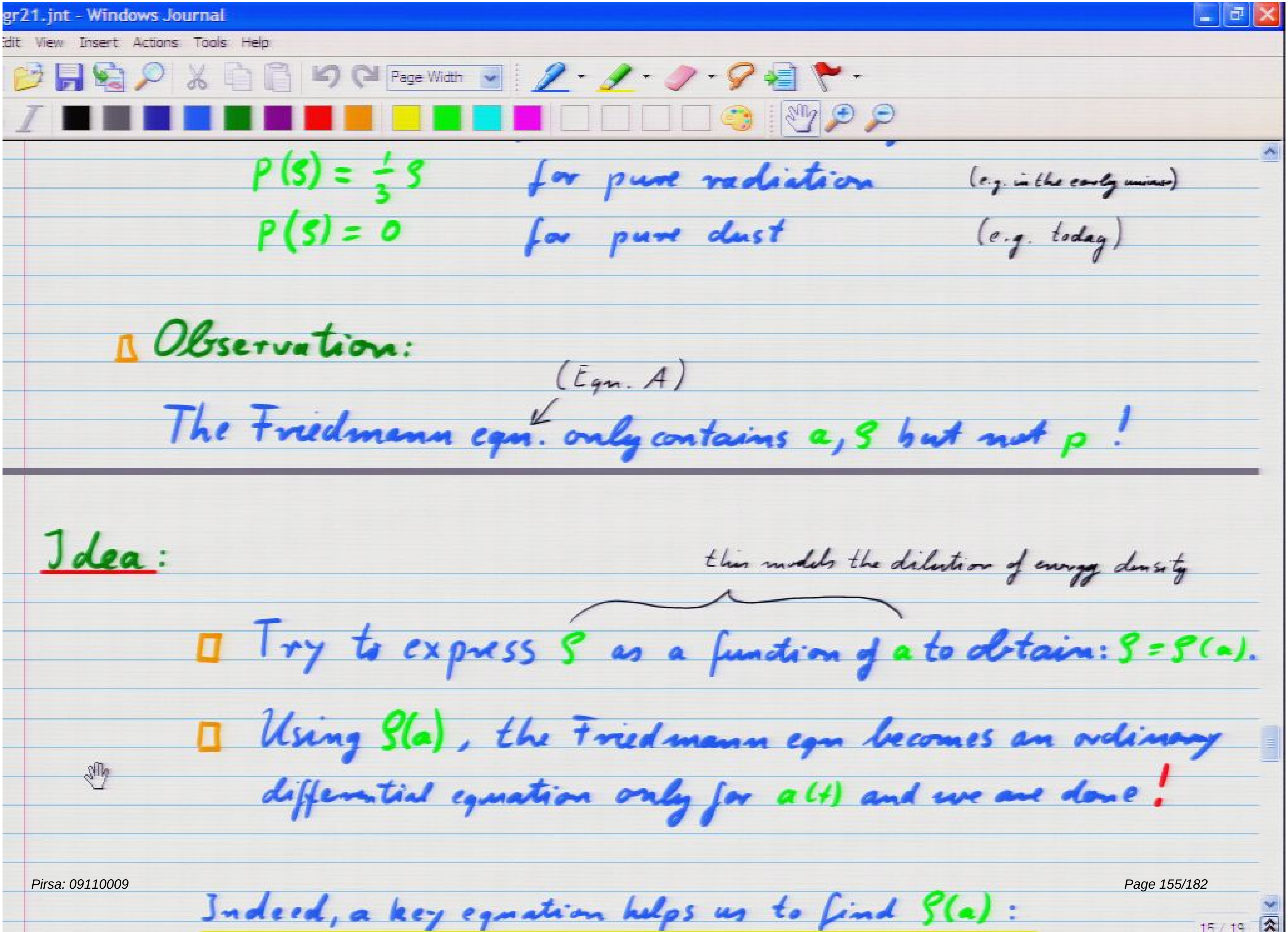

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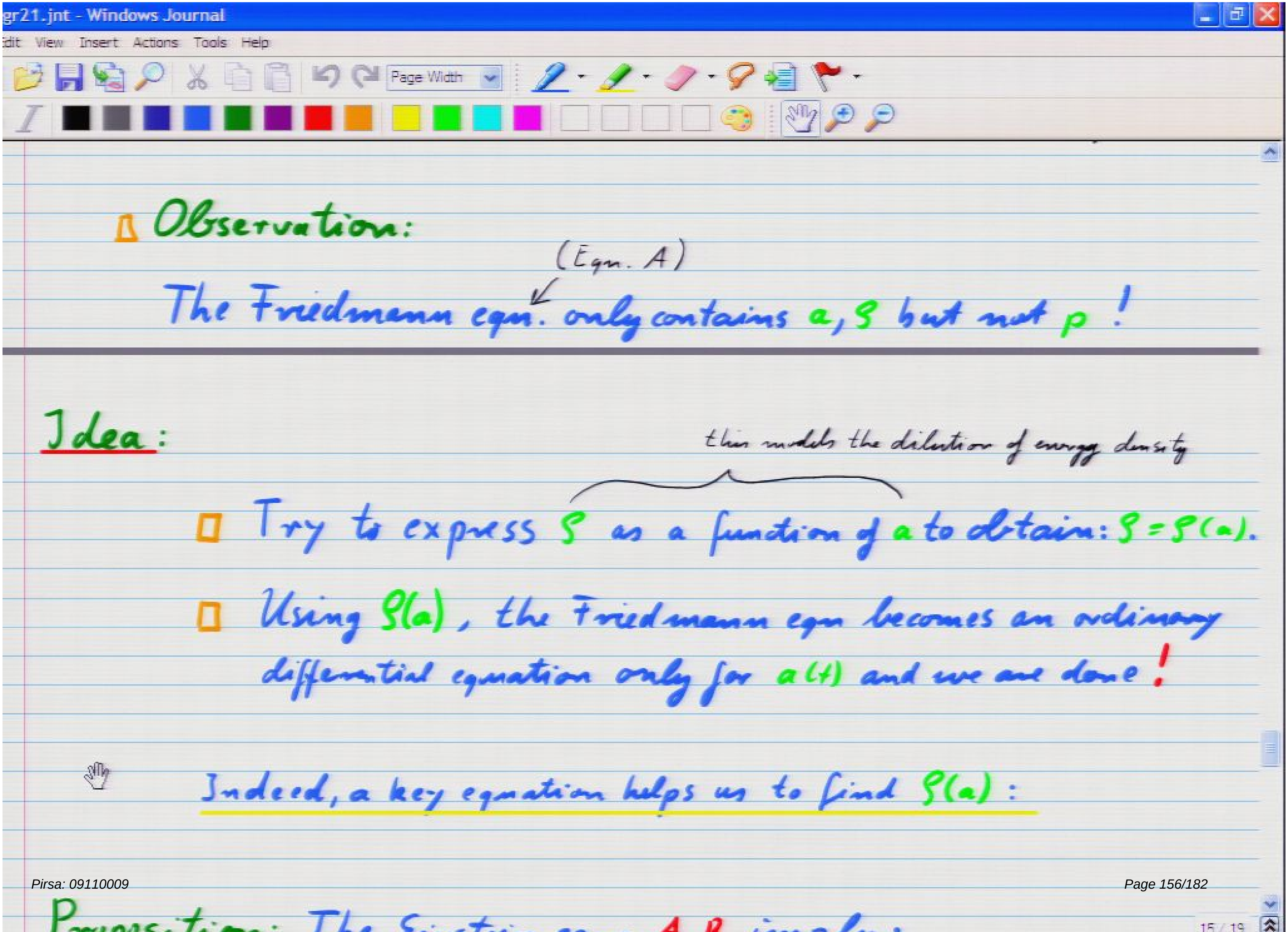

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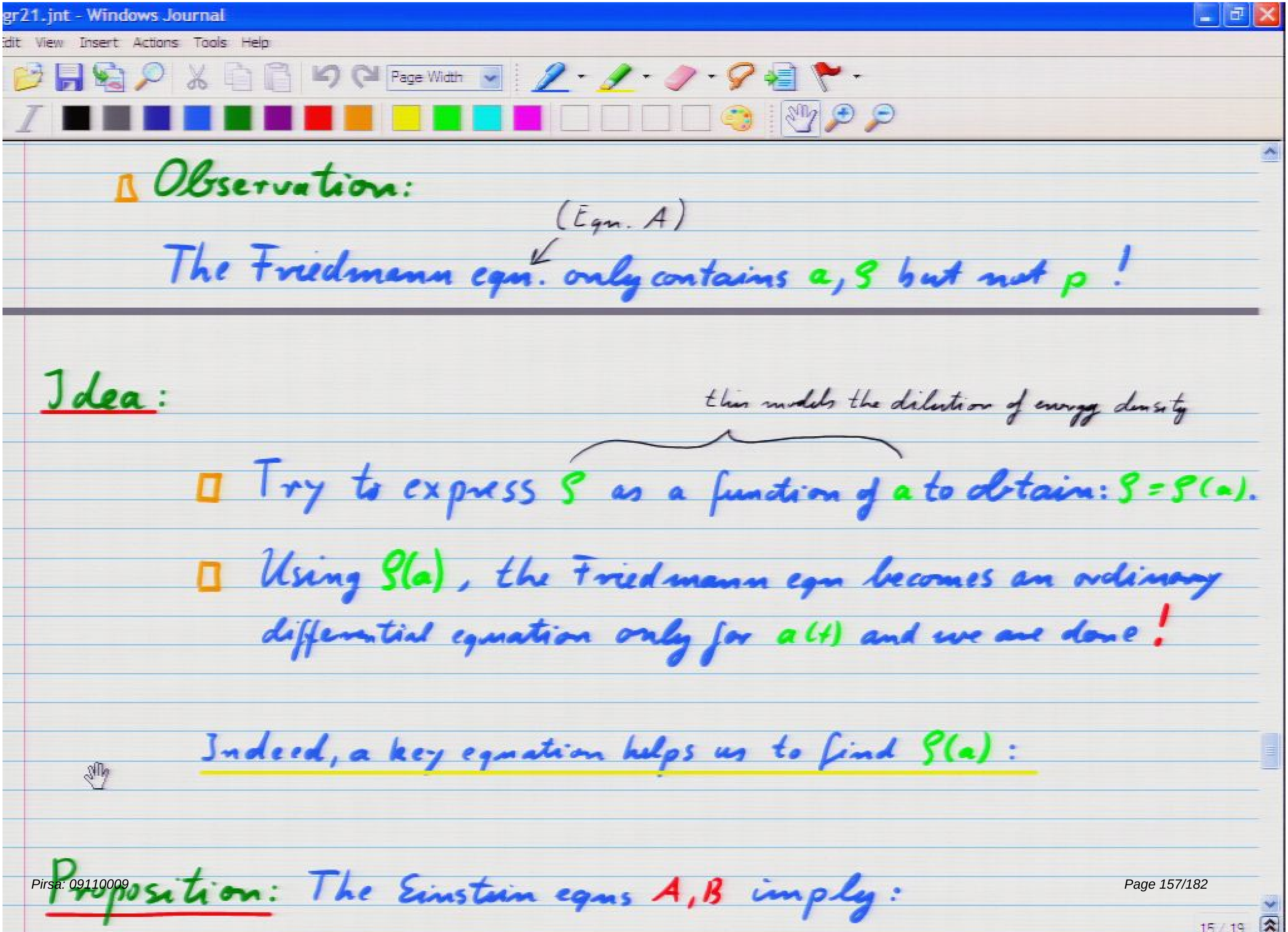

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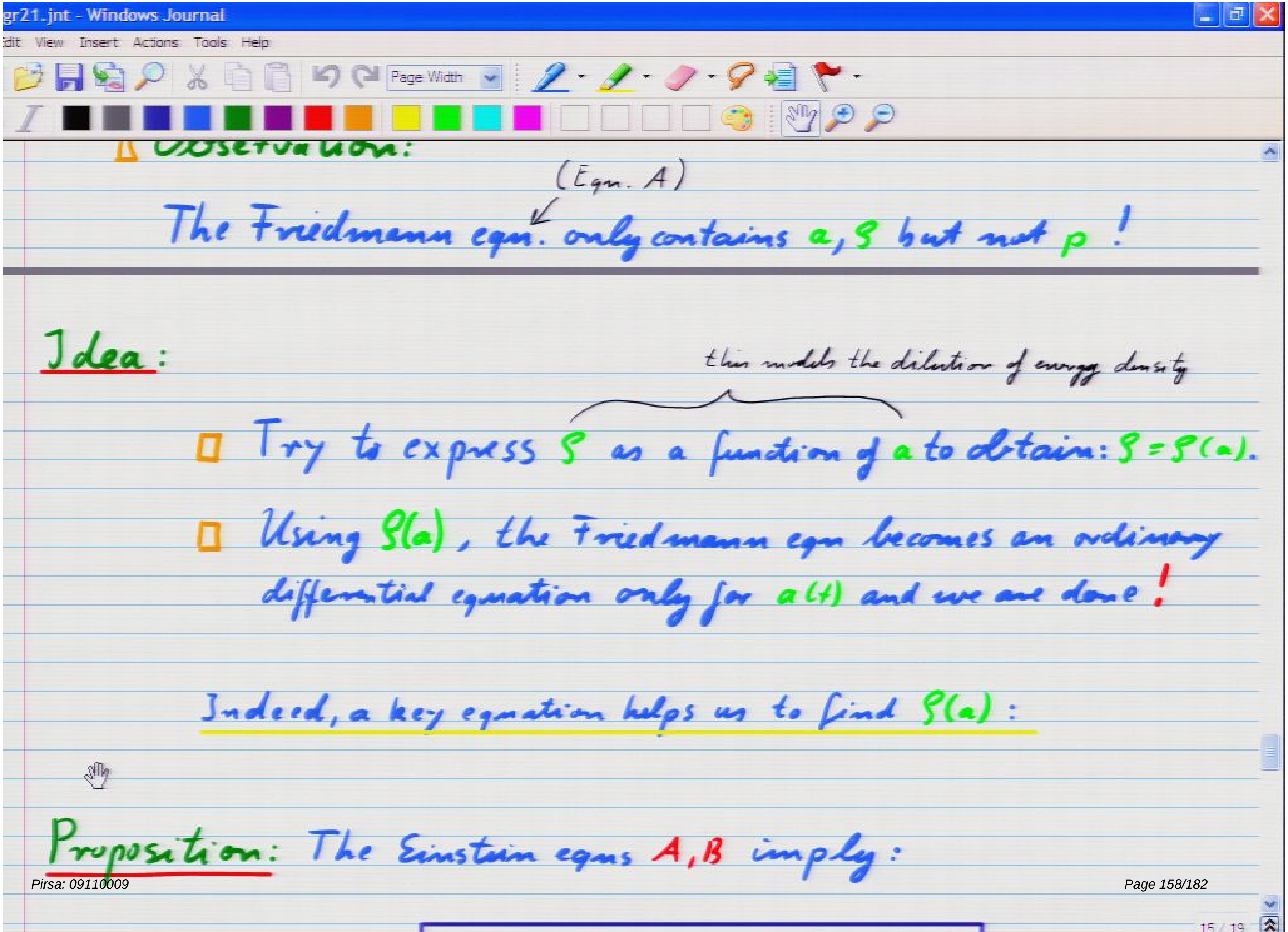

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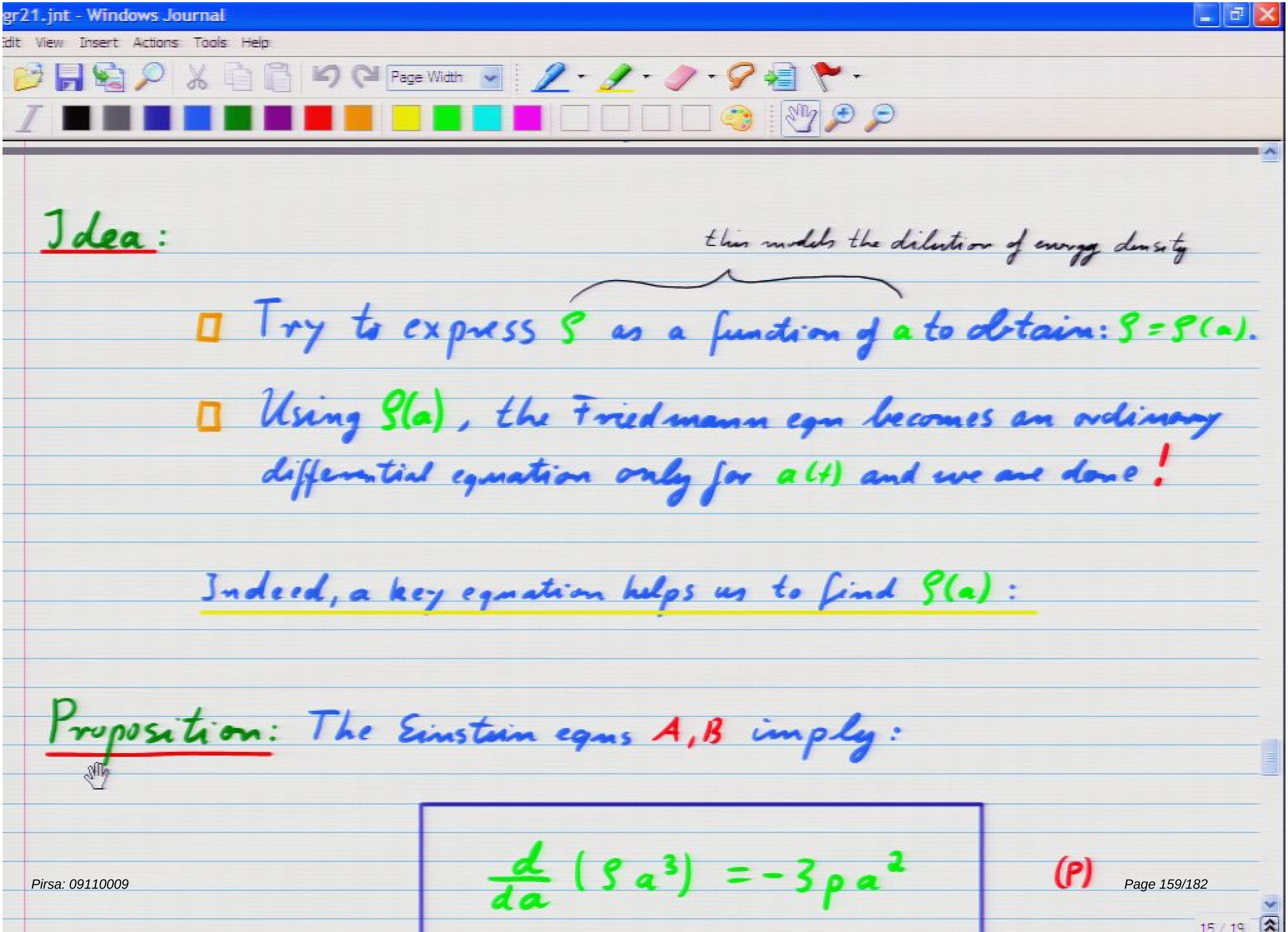

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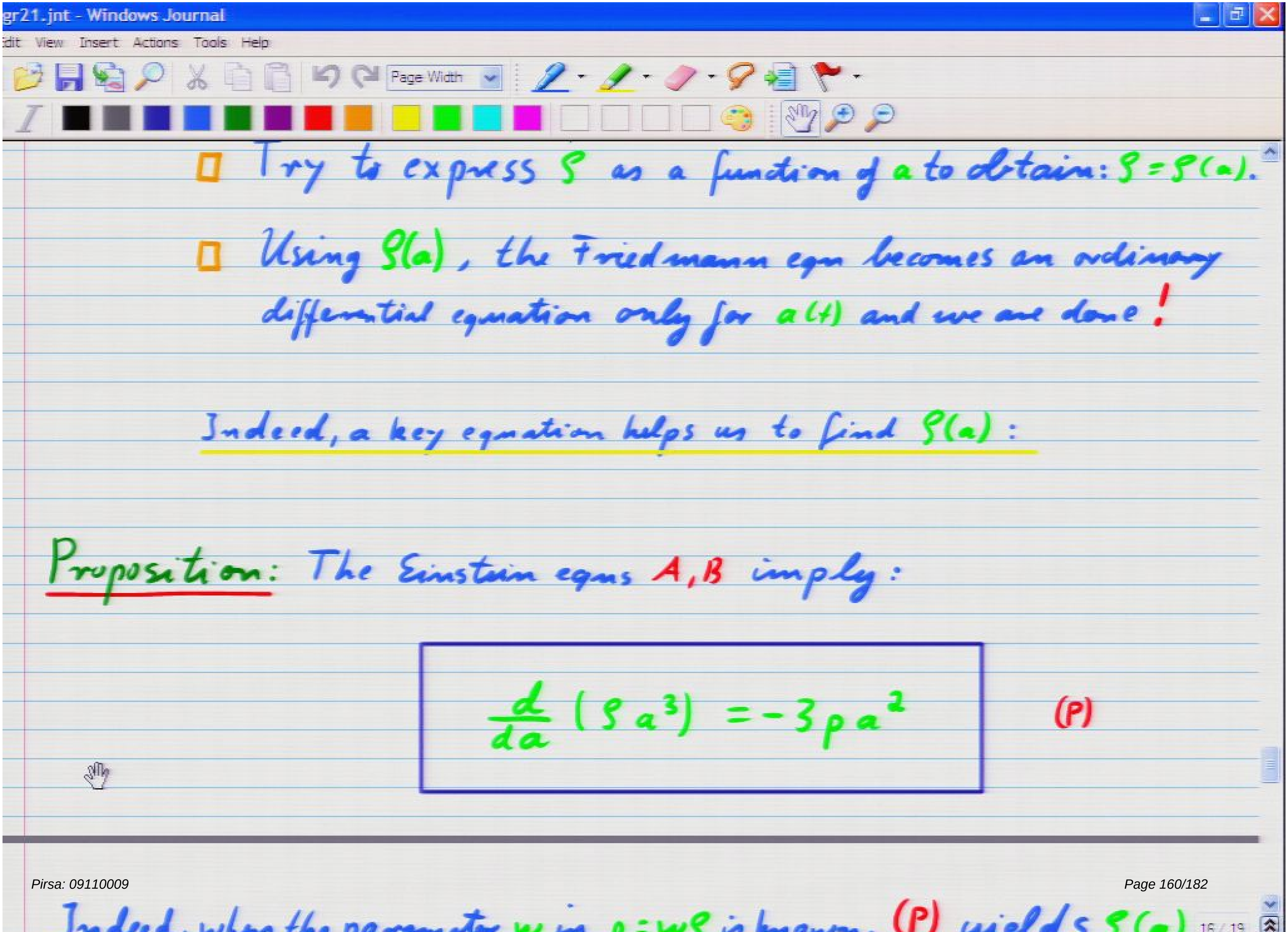

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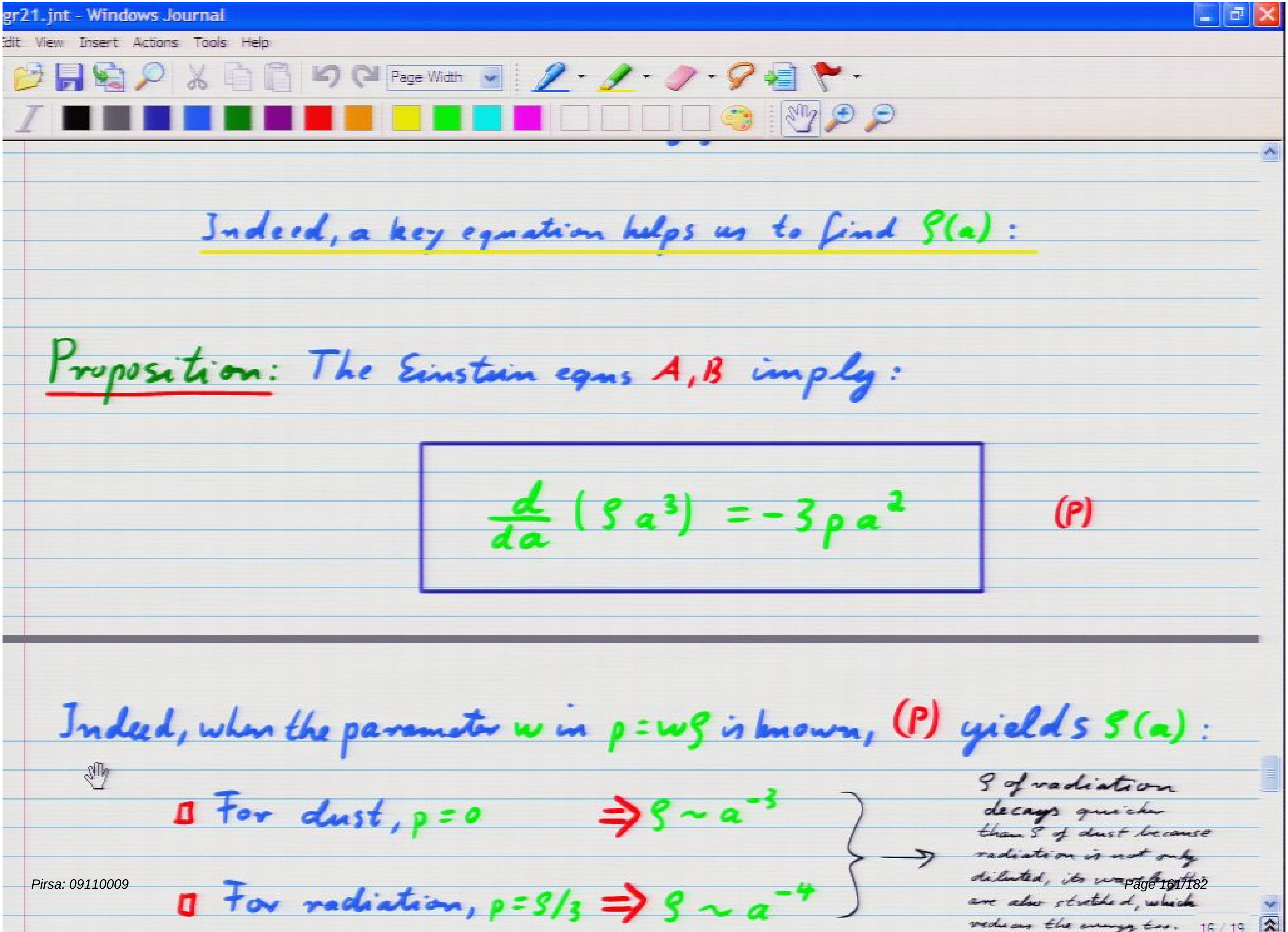

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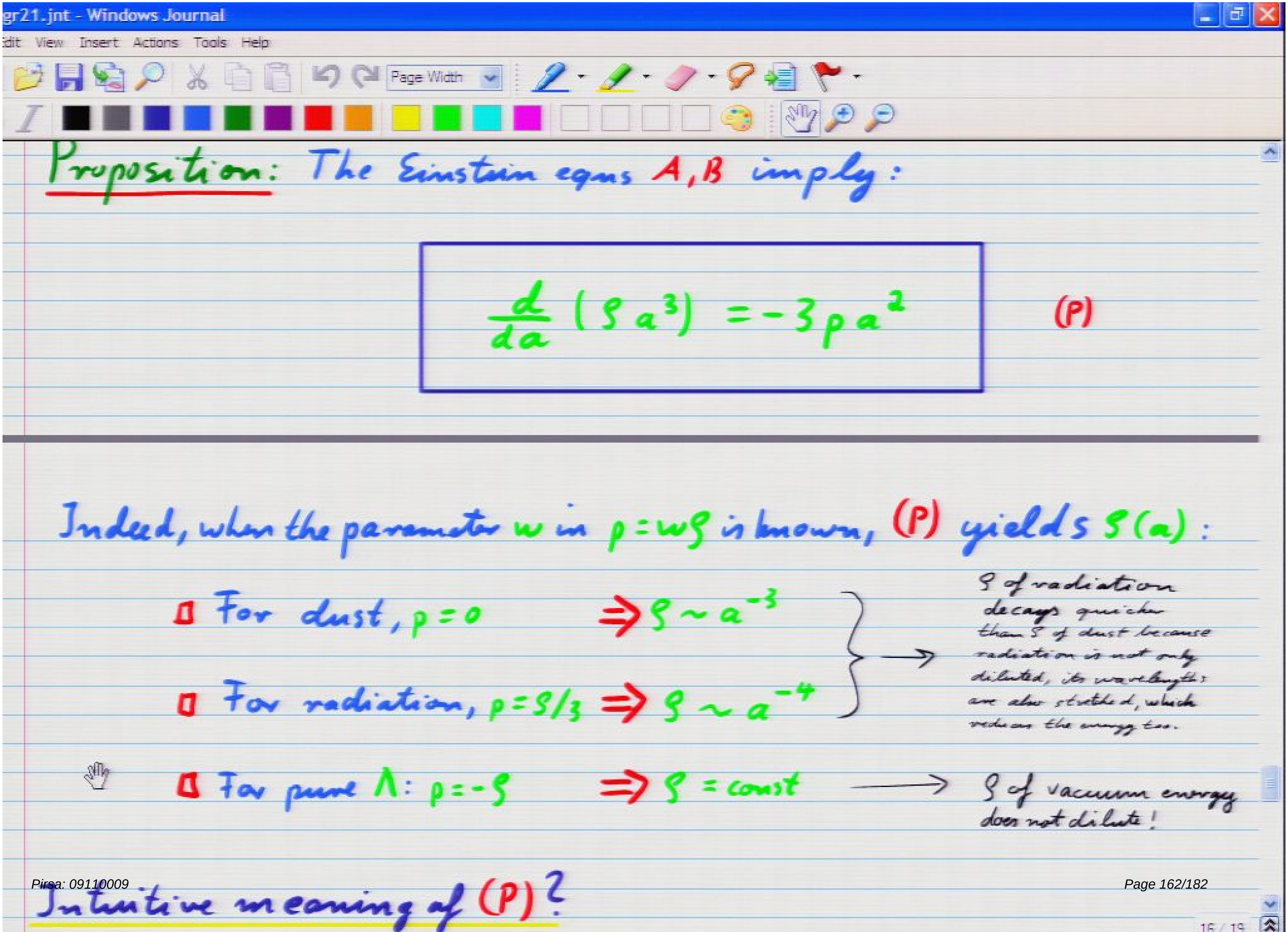

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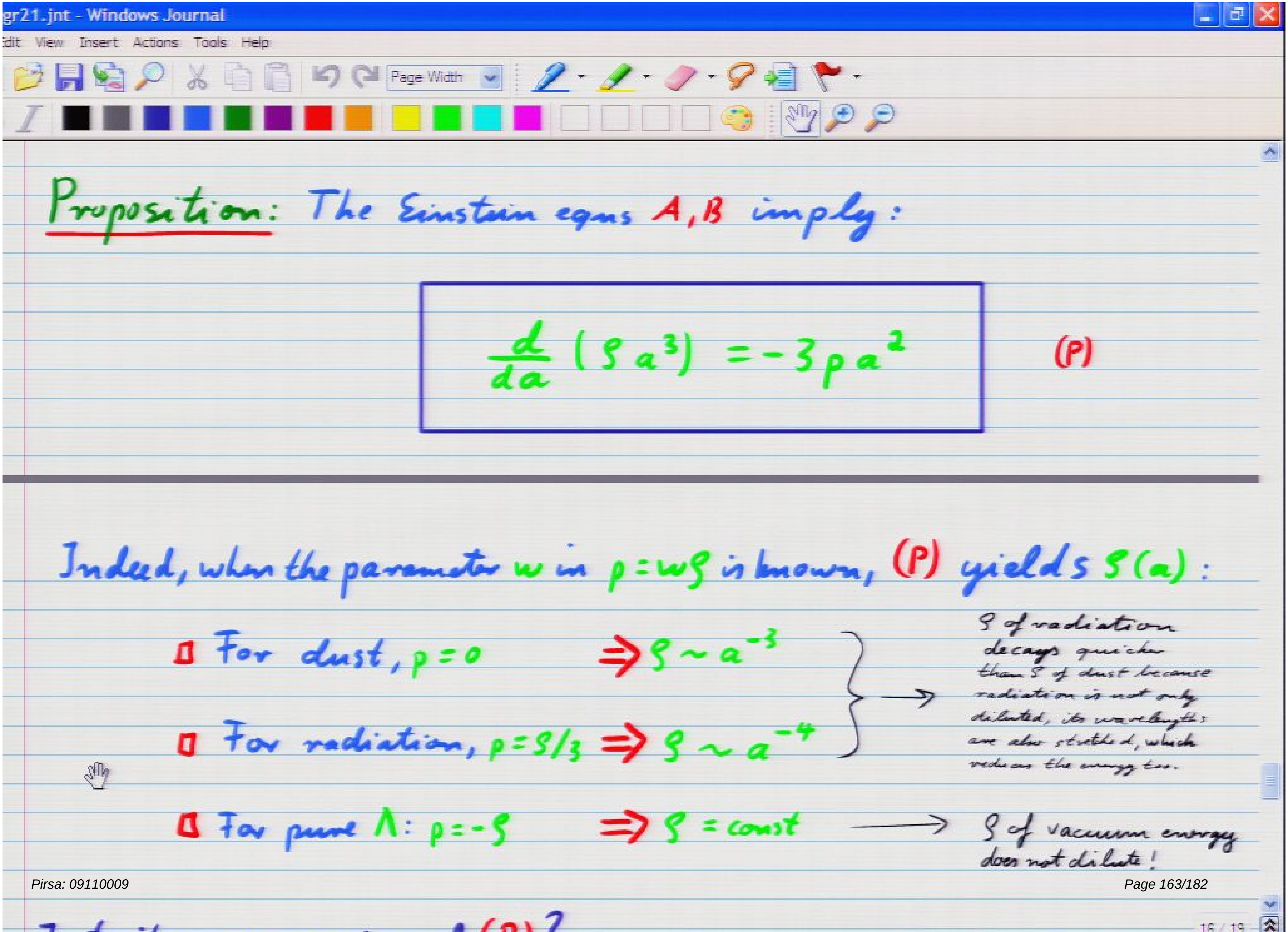

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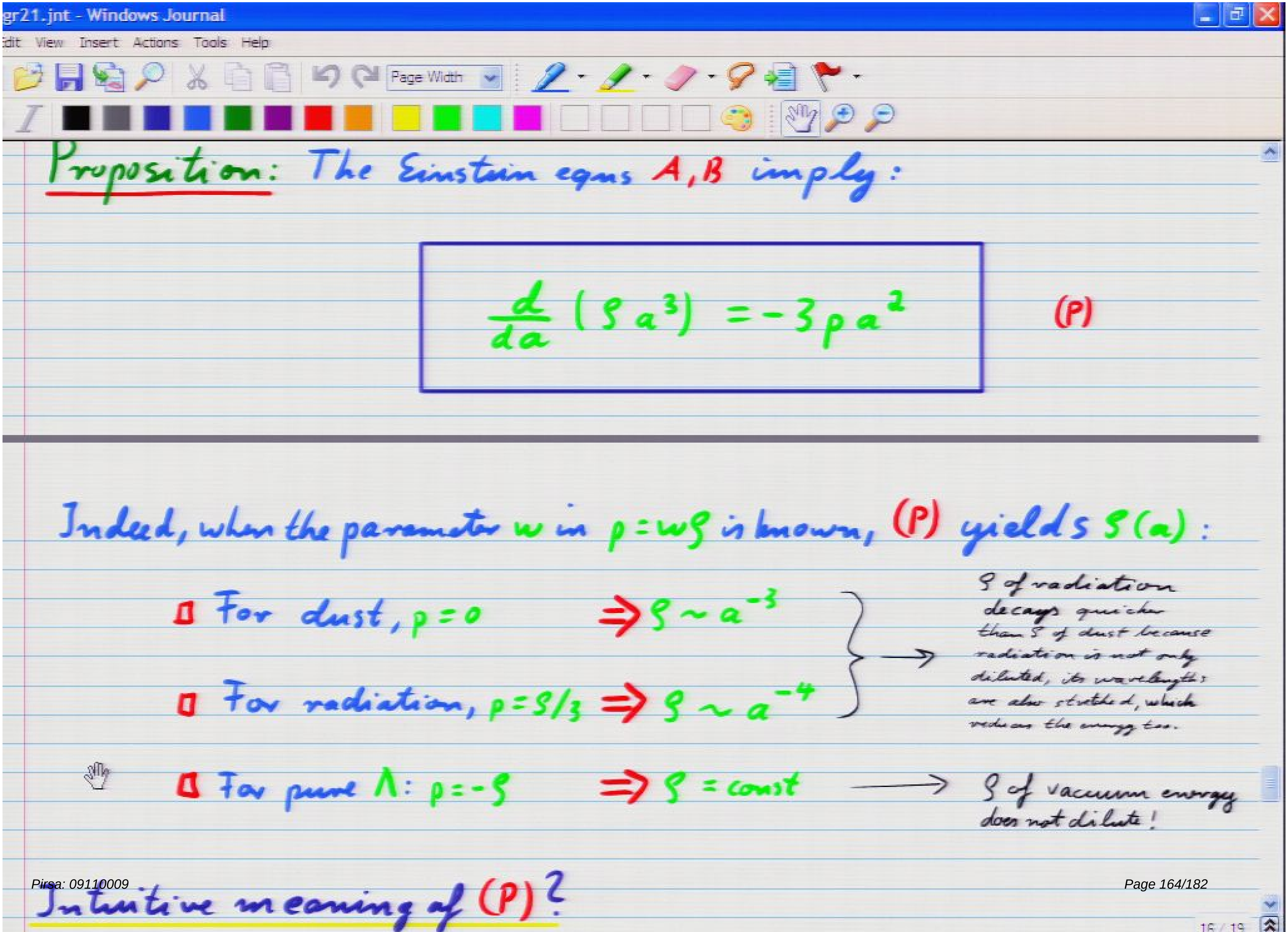

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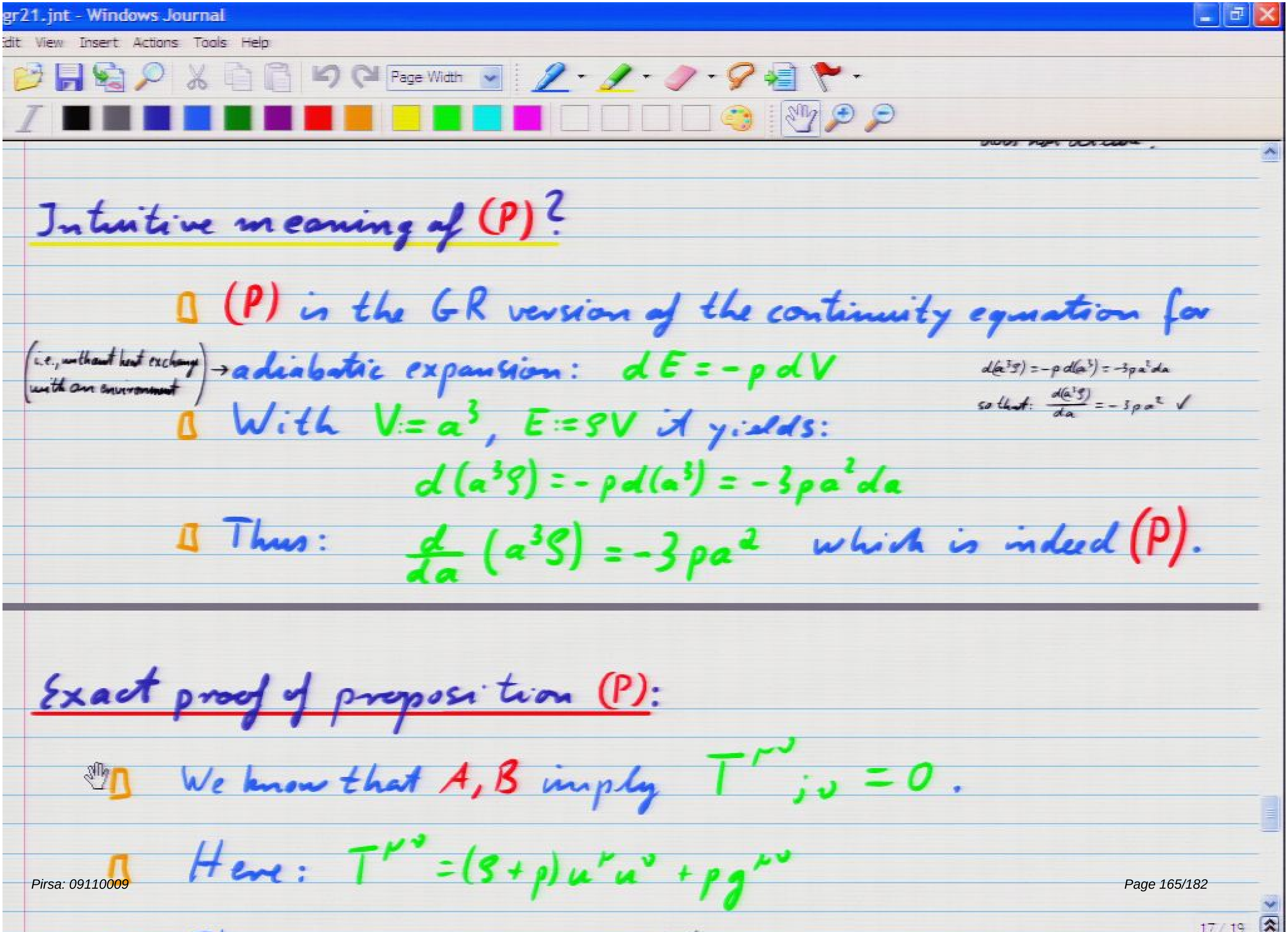

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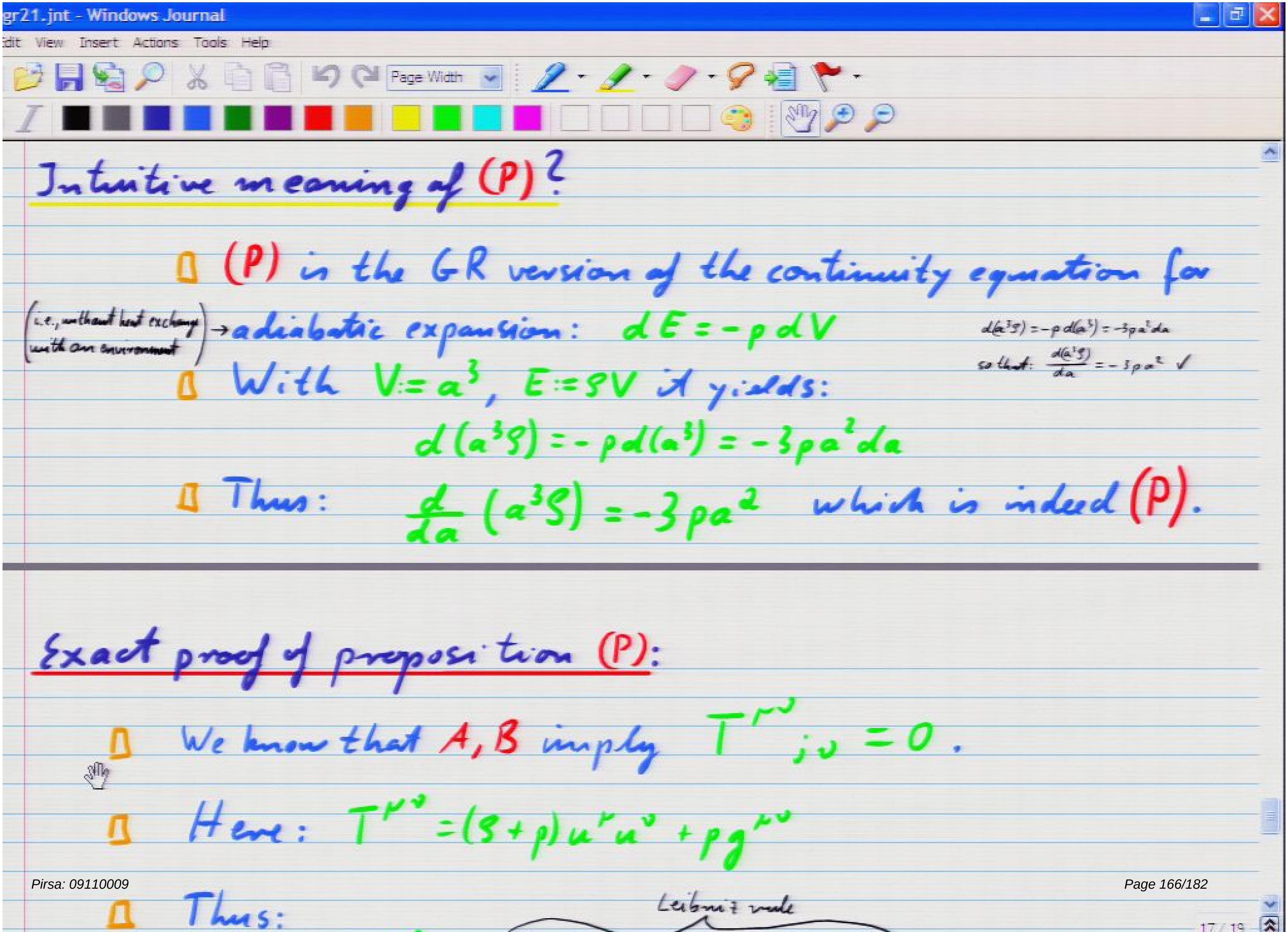

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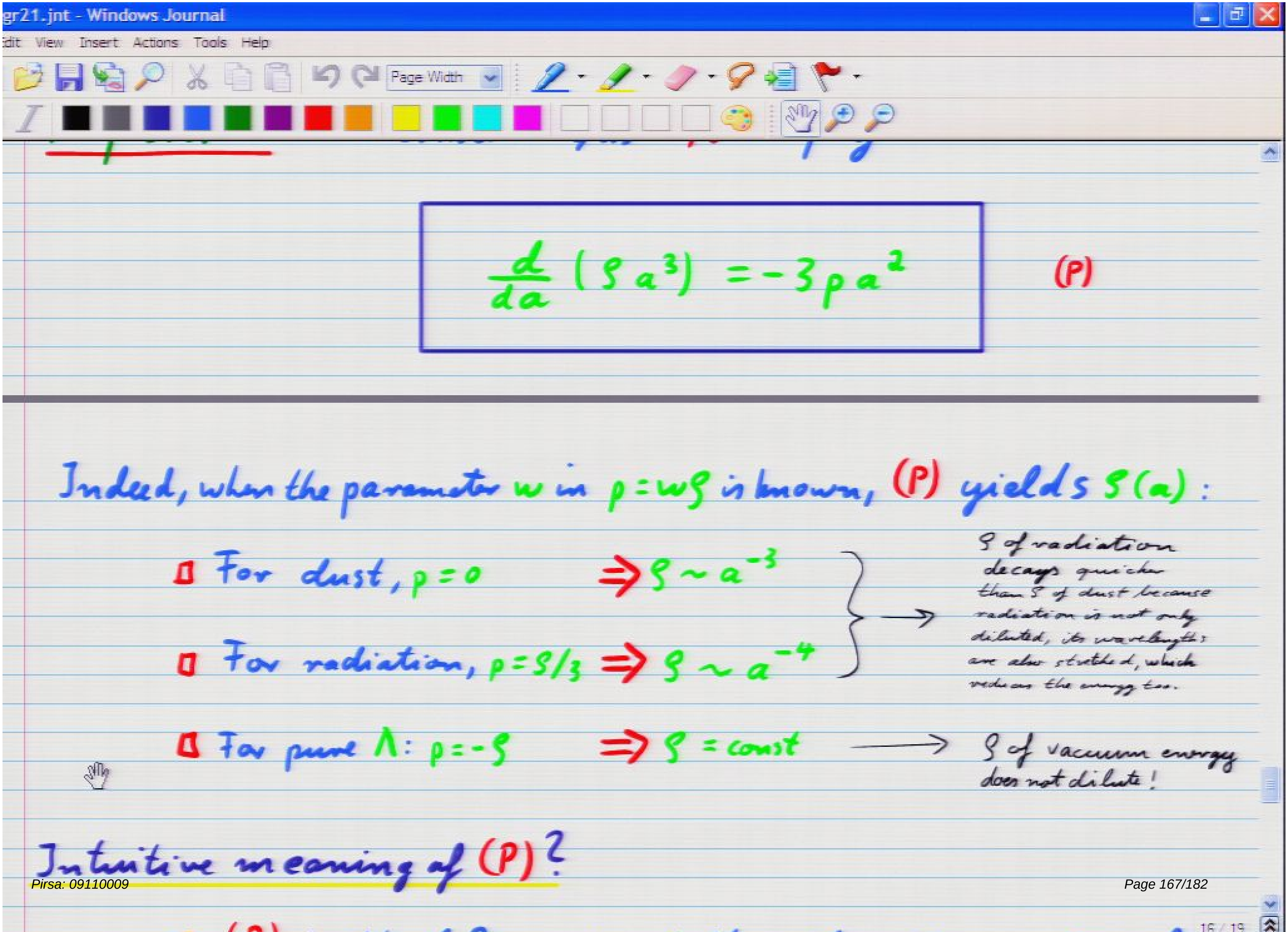

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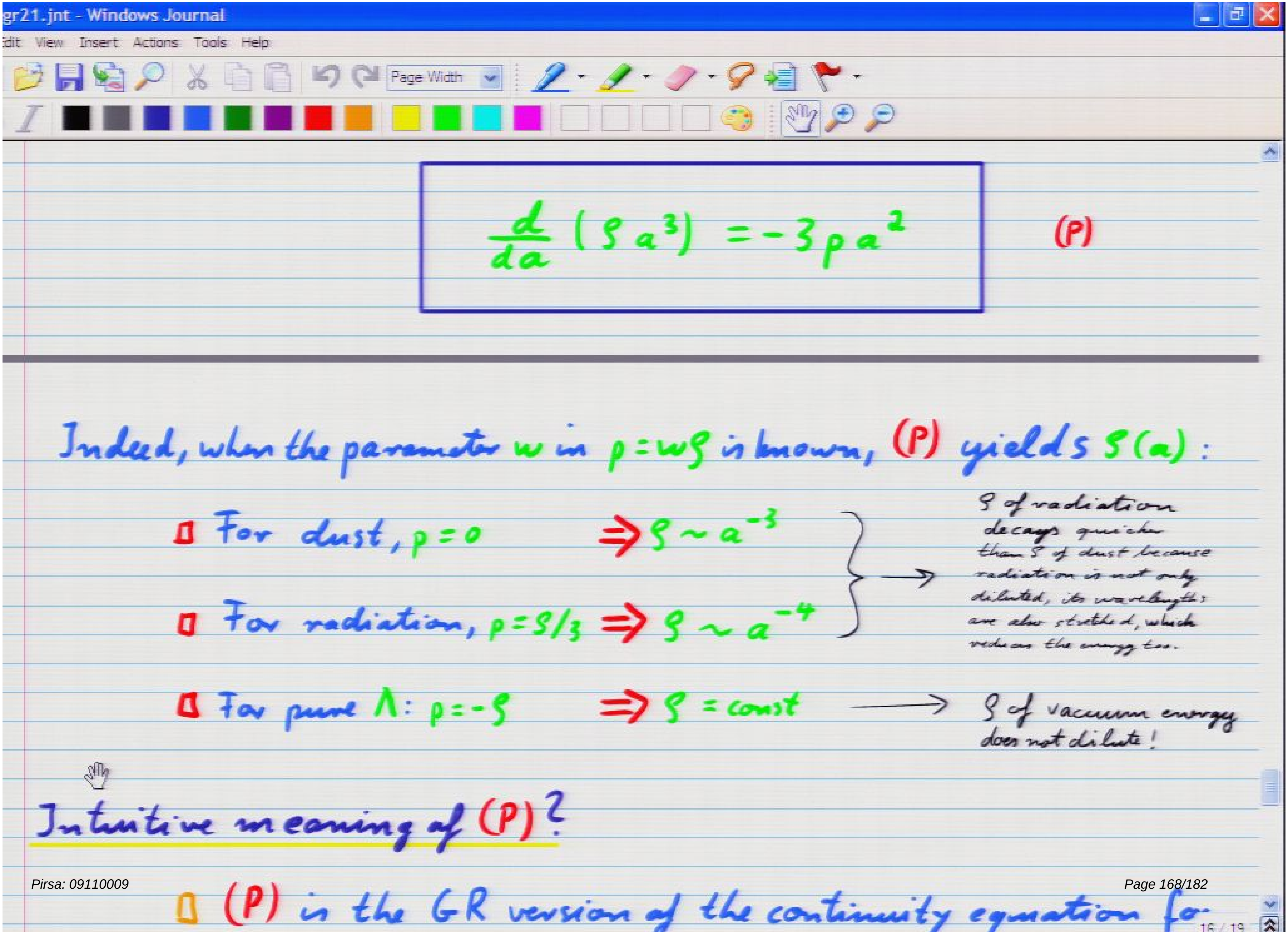

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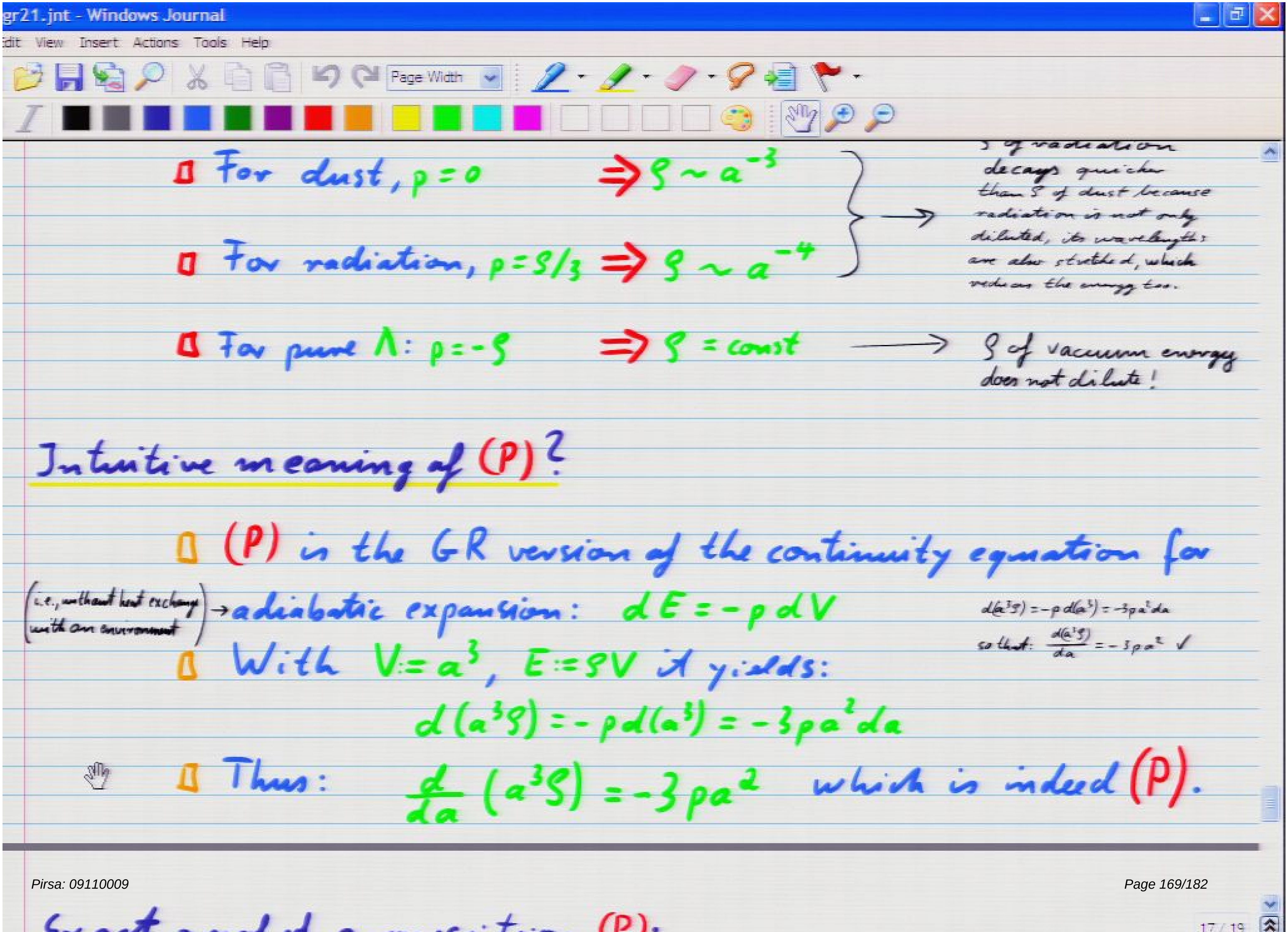

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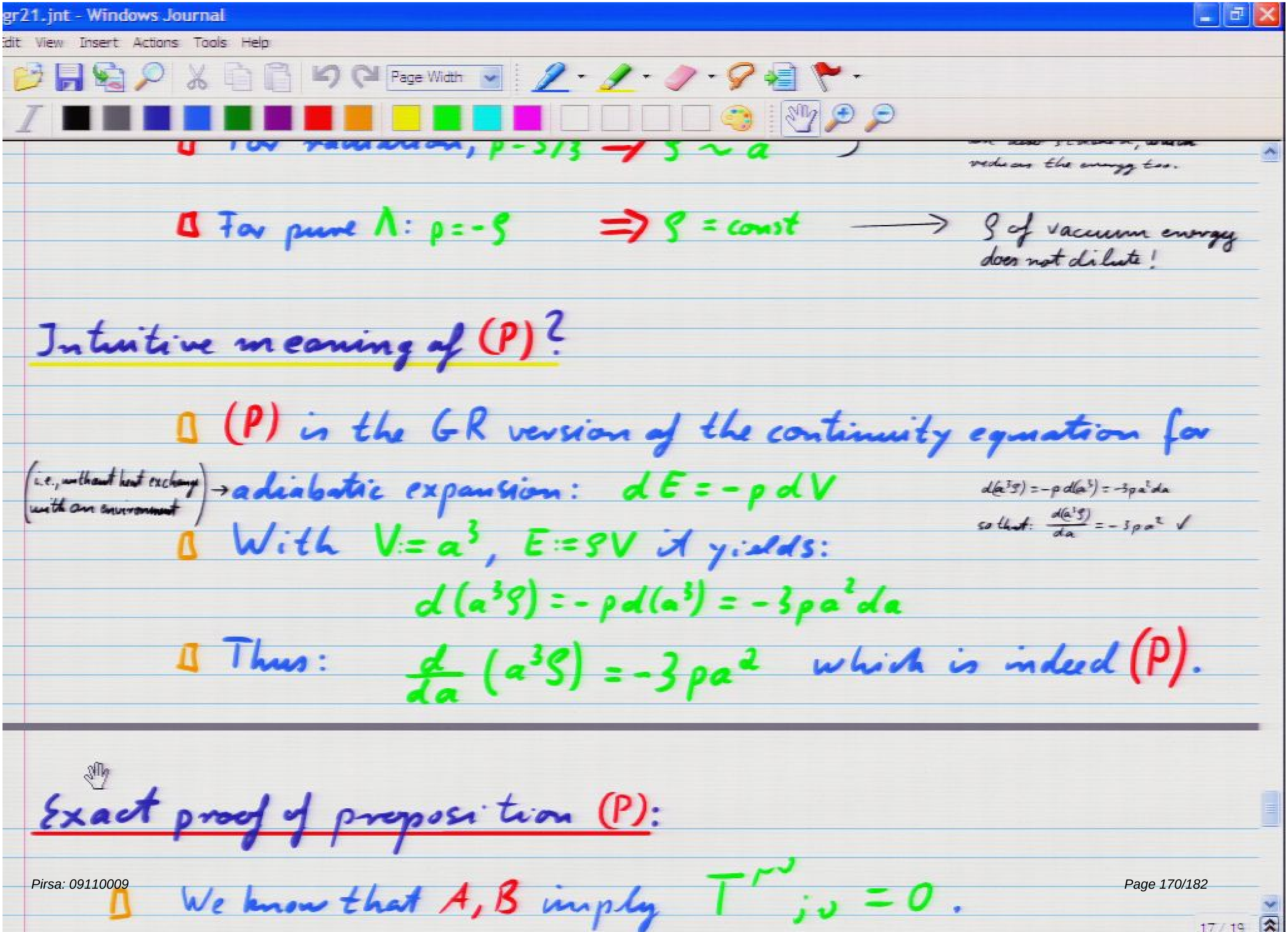

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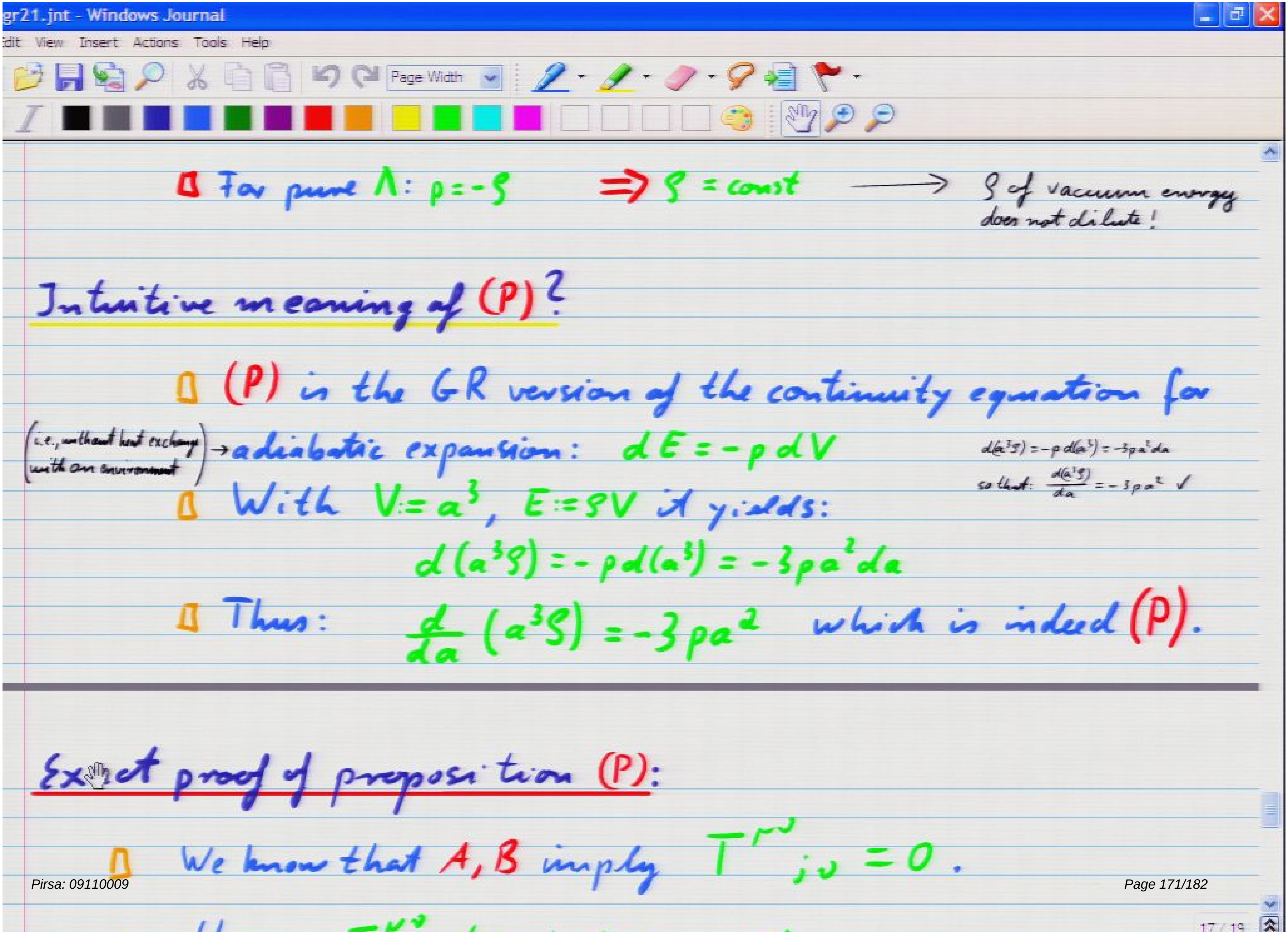

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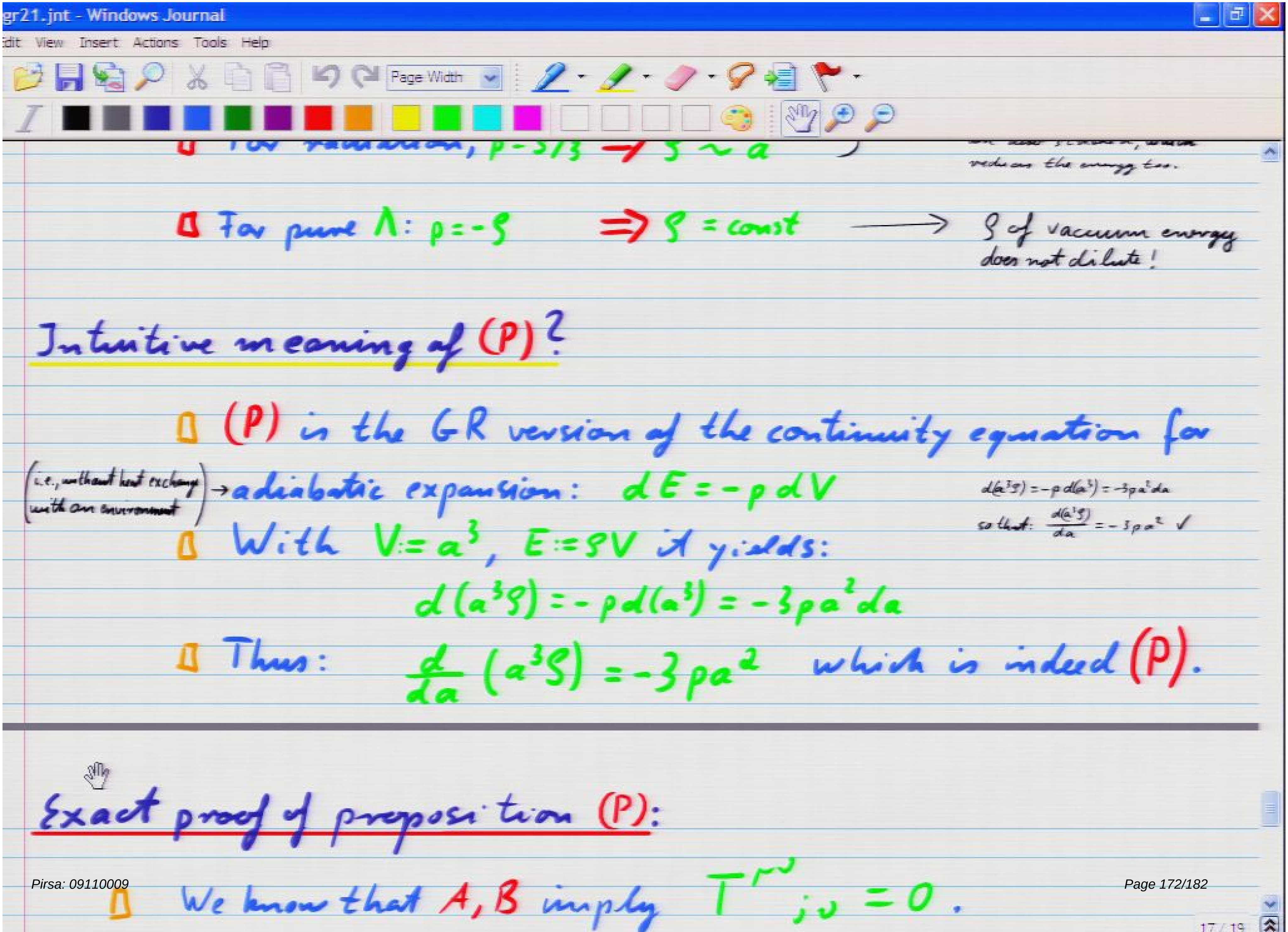

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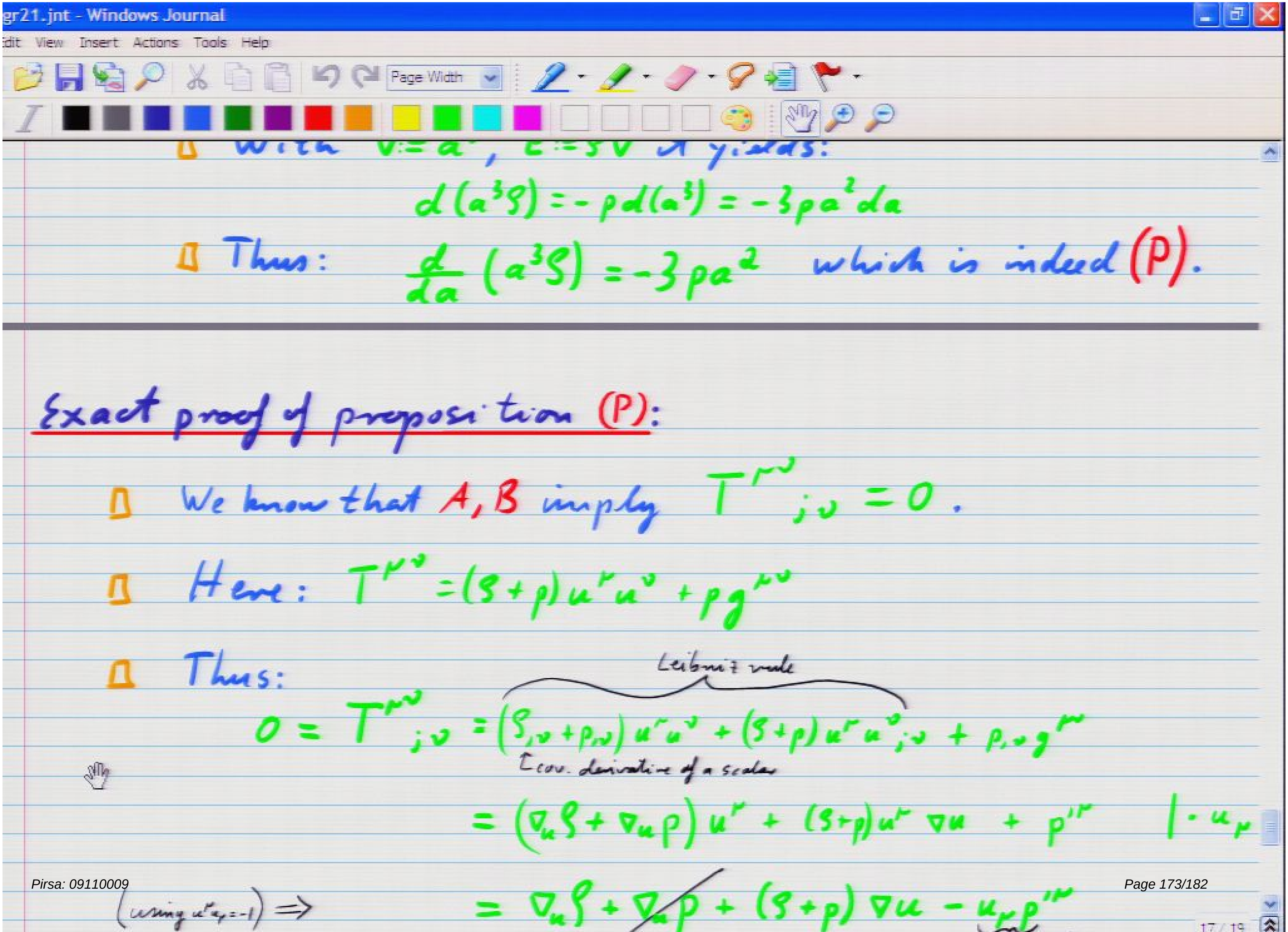

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[cov. derivative of a scalar

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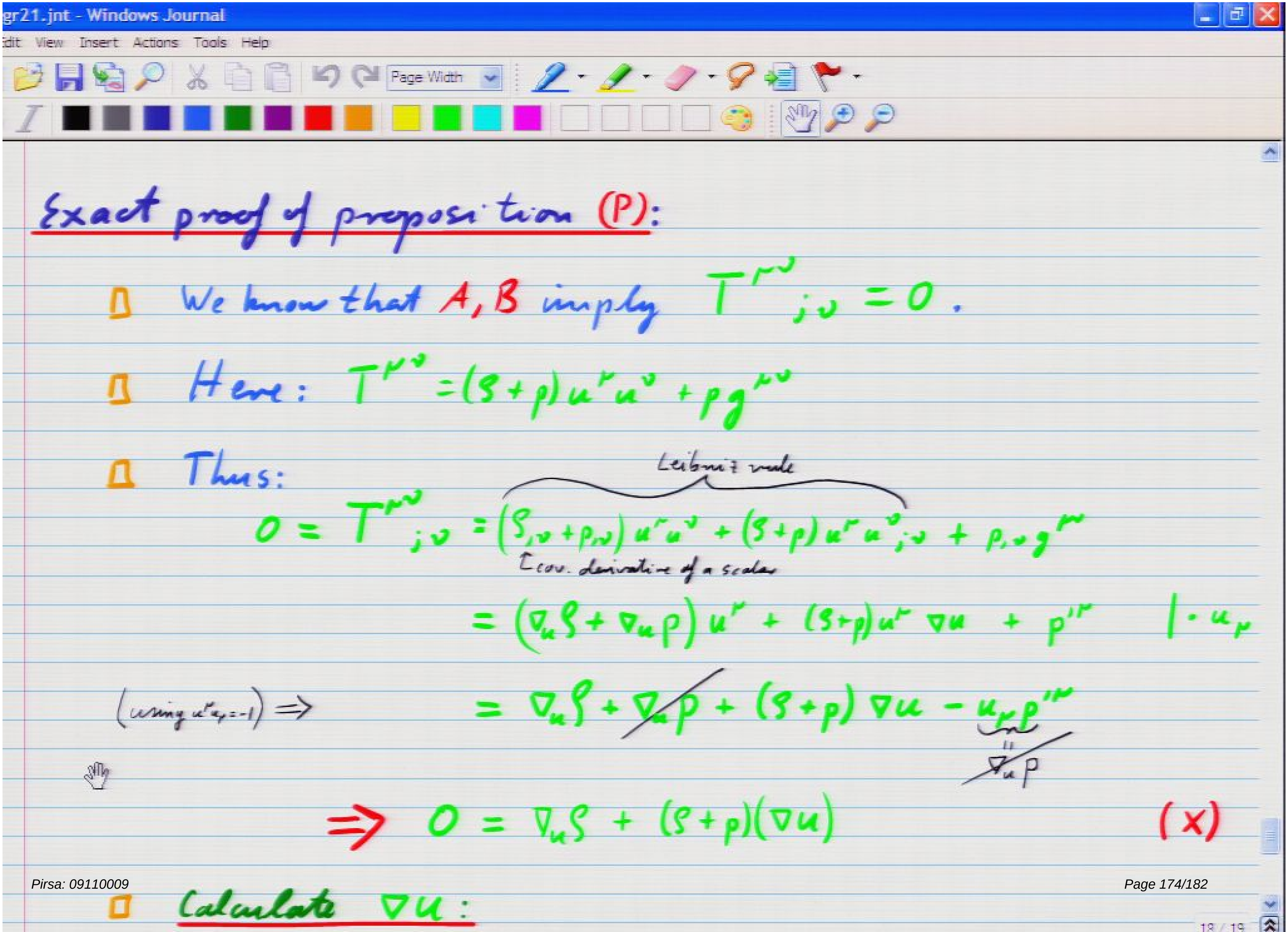

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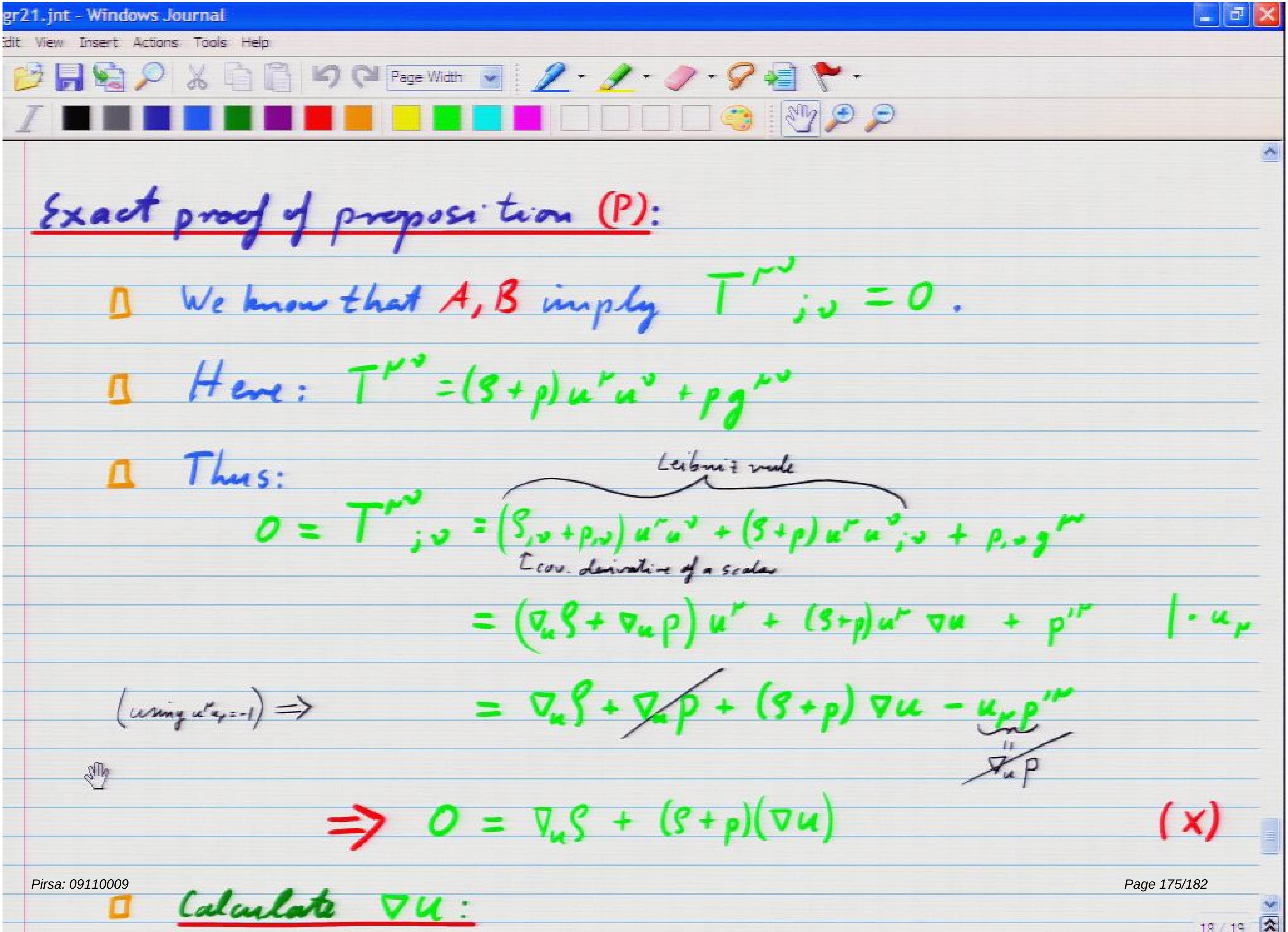

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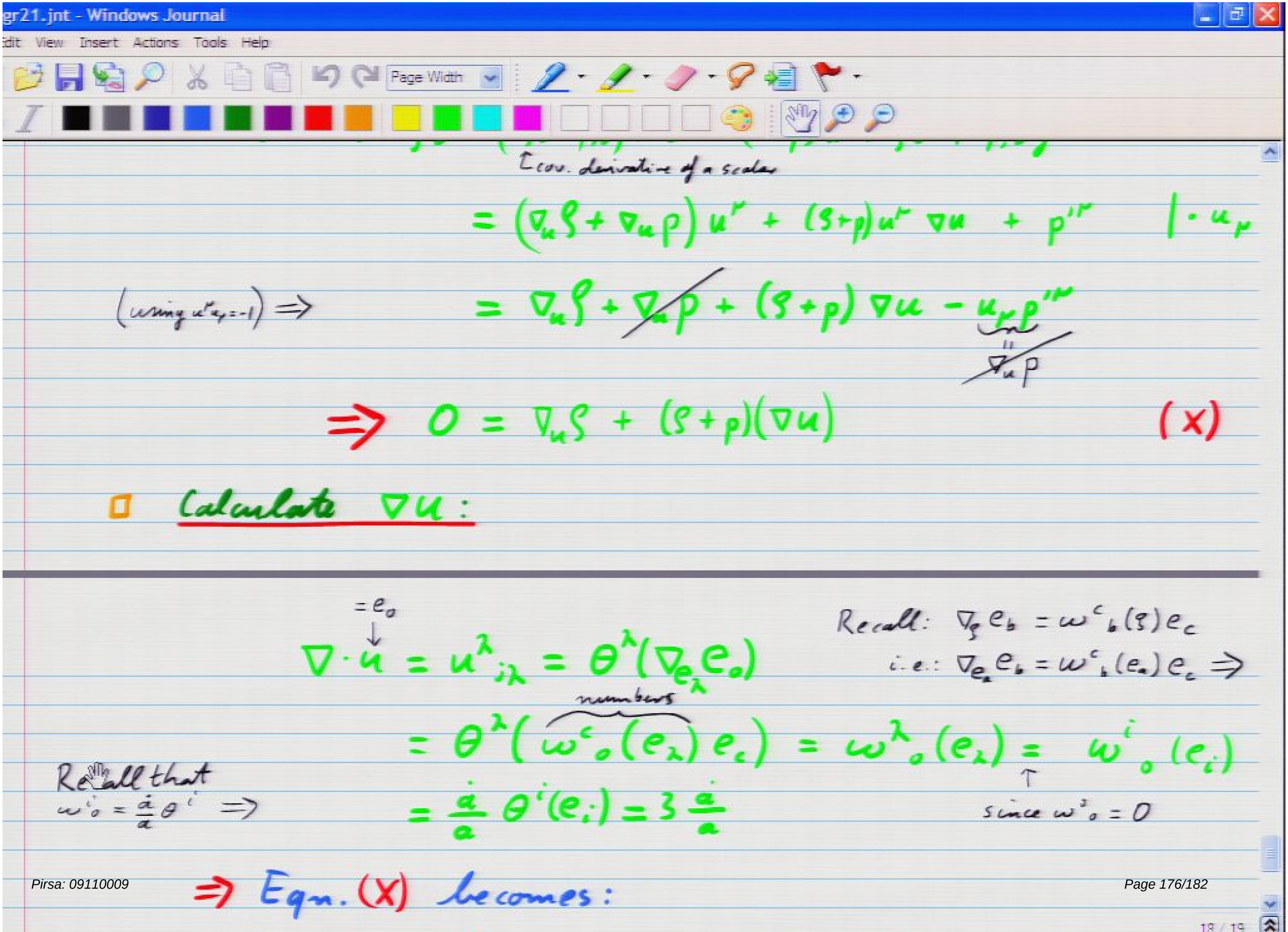


↑ cov. derivative of a scalar

$$= (\nabla_u \rho + \nabla_u p) u^\mu + (\rho + p) u^\mu \nabla u + p^{;\mu} \qquad | \cdot u_\mu$$

(using $u^\mu u_\mu = -1$) ⇒

$$= \nabla_u \rho + \cancel{\nabla_u p} + (\rho + p) \nabla u - \underbrace{u_\mu p^{;\mu}}_{\cancel{\nabla_u p}}$$

$$\Rightarrow \quad 0 = \nabla_u \rho + (\rho + p)(\nabla u) \qquad (X)$$

☐ **Calculate ∇u:**

Recall: $\nabla_\xi e_b = \omega^c{}_b(\xi) e_c$

i.e.: $\nabla_{e_a} e_b = \omega^c{}_b(e_a) e_c$ ⇒

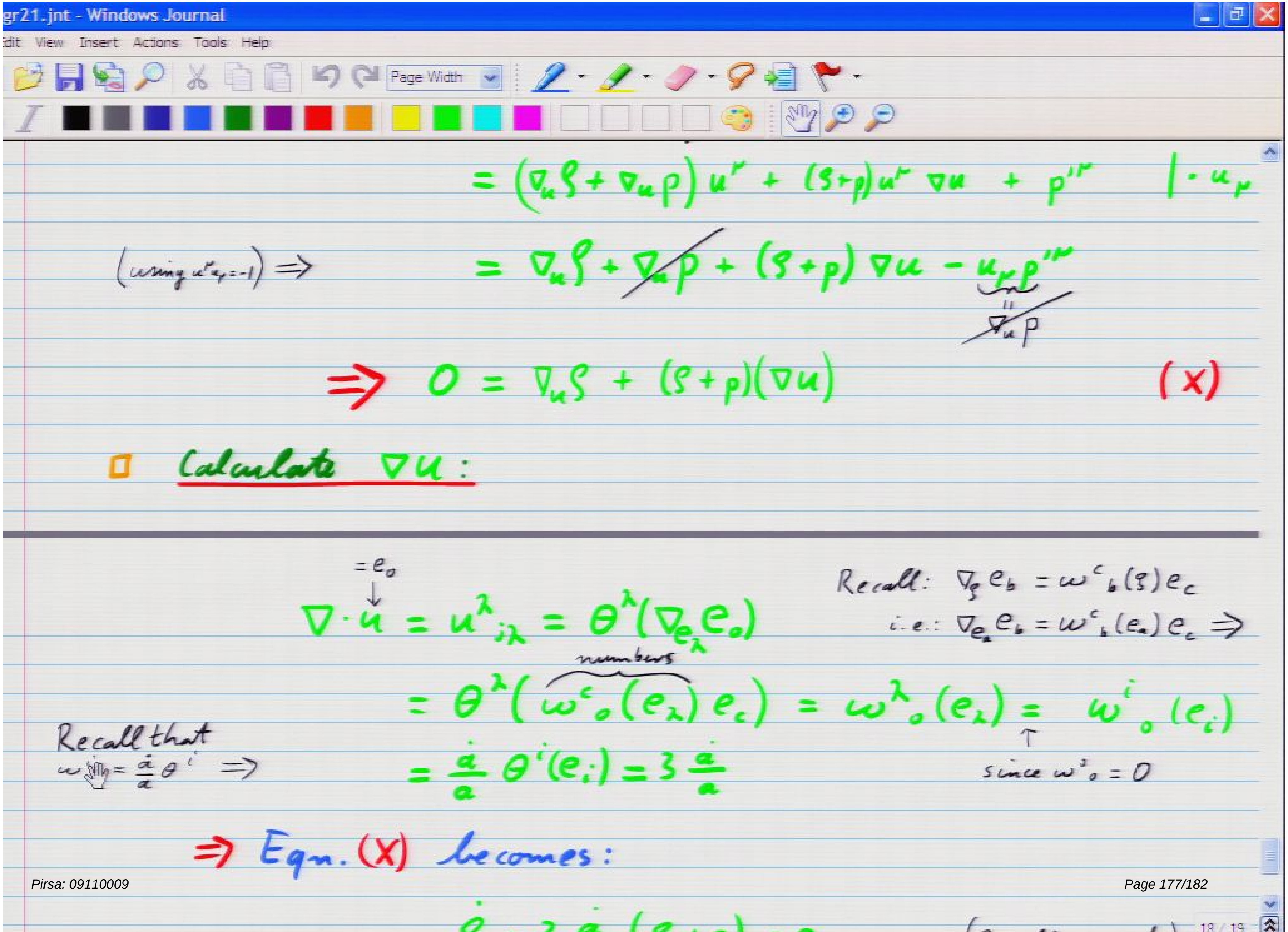
$$\nabla \cdot \overset{= e_0}{u} = u^\lambda{}_{;\lambda} = \theta^\lambda(\nabla_{e_\lambda} e_0)$$

$$= \theta^\lambda(\overbrace{\omega^c{}_0(e_\lambda)}^{\text{numbers}} e_c) = \omega^\lambda{}_0(e_\lambda) \underset{\substack{\uparrow \\ \text{since } \omega^0{}_0 = 0}}{=} \omega^i{}_0(e_i)$$

Recall that $\omega^i{}_0 = \frac{\dot{a}}{a} \theta^i$ ⇒

$$= \frac{\dot{a}}{a} \theta^i(e_i) = 3 \frac{\dot{a}}{a}$$

⇒ Eqn. (X) becomes:



$$= (\nabla_u \rho + \nabla_u p) u^\mu + (\rho + p) u^\mu \nabla u + p^{;\mu} \qquad \Big| \cdot u_\mu$$

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since $\omega^0{}_0 = 0$

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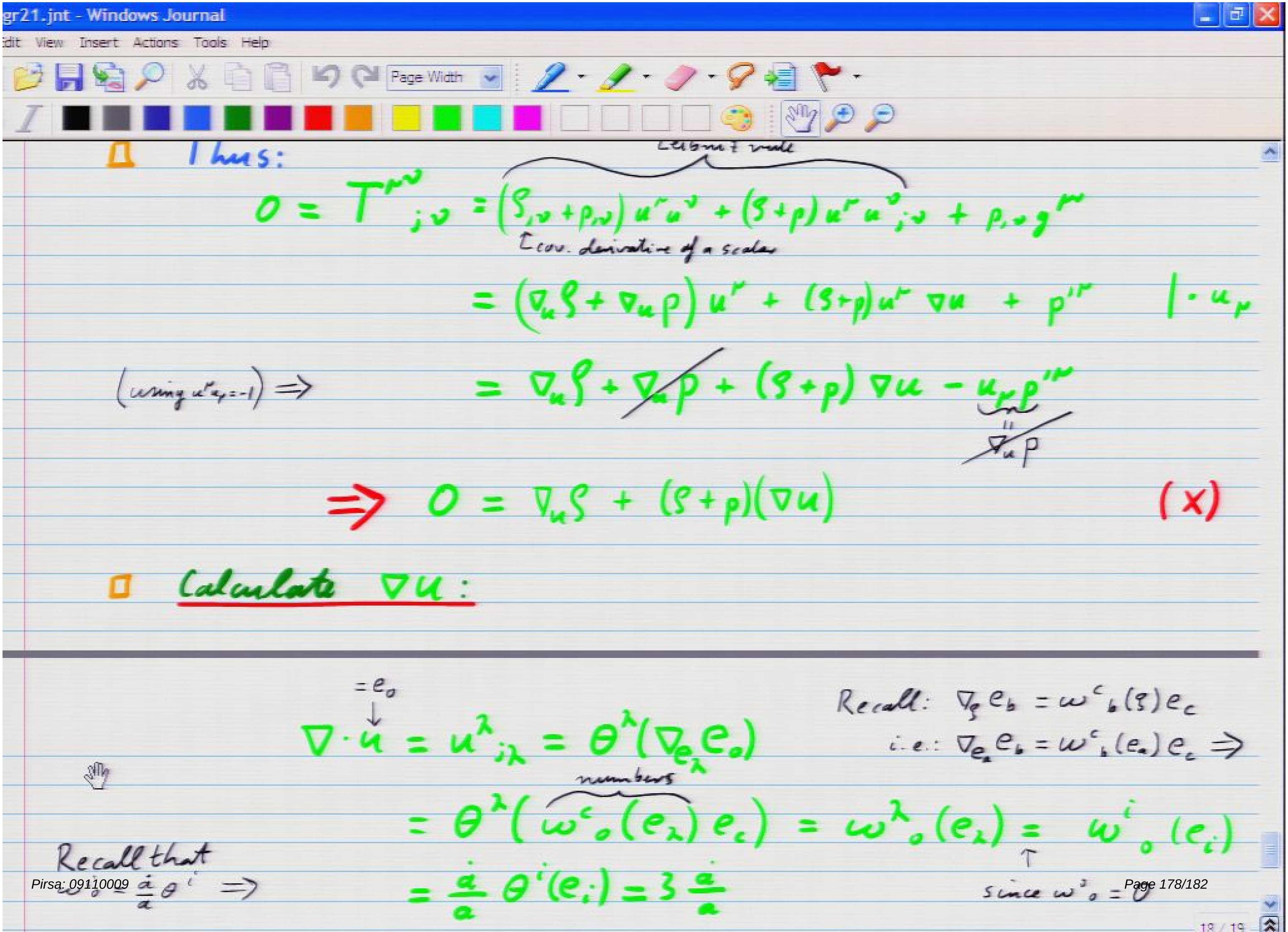

Thus:

$$0 = T^{\mu\nu}{}_{;\nu} = \overbrace{\underset{\uparrow \text{cov. derivative of a scalar}}{(\rho_{,\nu} + p_{,\nu})} u^\mu u^\nu + (\rho + p) u^\mu u^\nu{}_{;\nu}}^{\text{Leibniz rule}} + p_{,\nu} g^{\mu\nu}$$

$$= (\nabla_u \rho + \nabla_u p) u^\mu + (\rho + p) u^\mu \nabla u + p^{,\mu} \qquad \Big| \cdot u_\mu$$

(using $u^\mu u_\mu = -1$) ⇒

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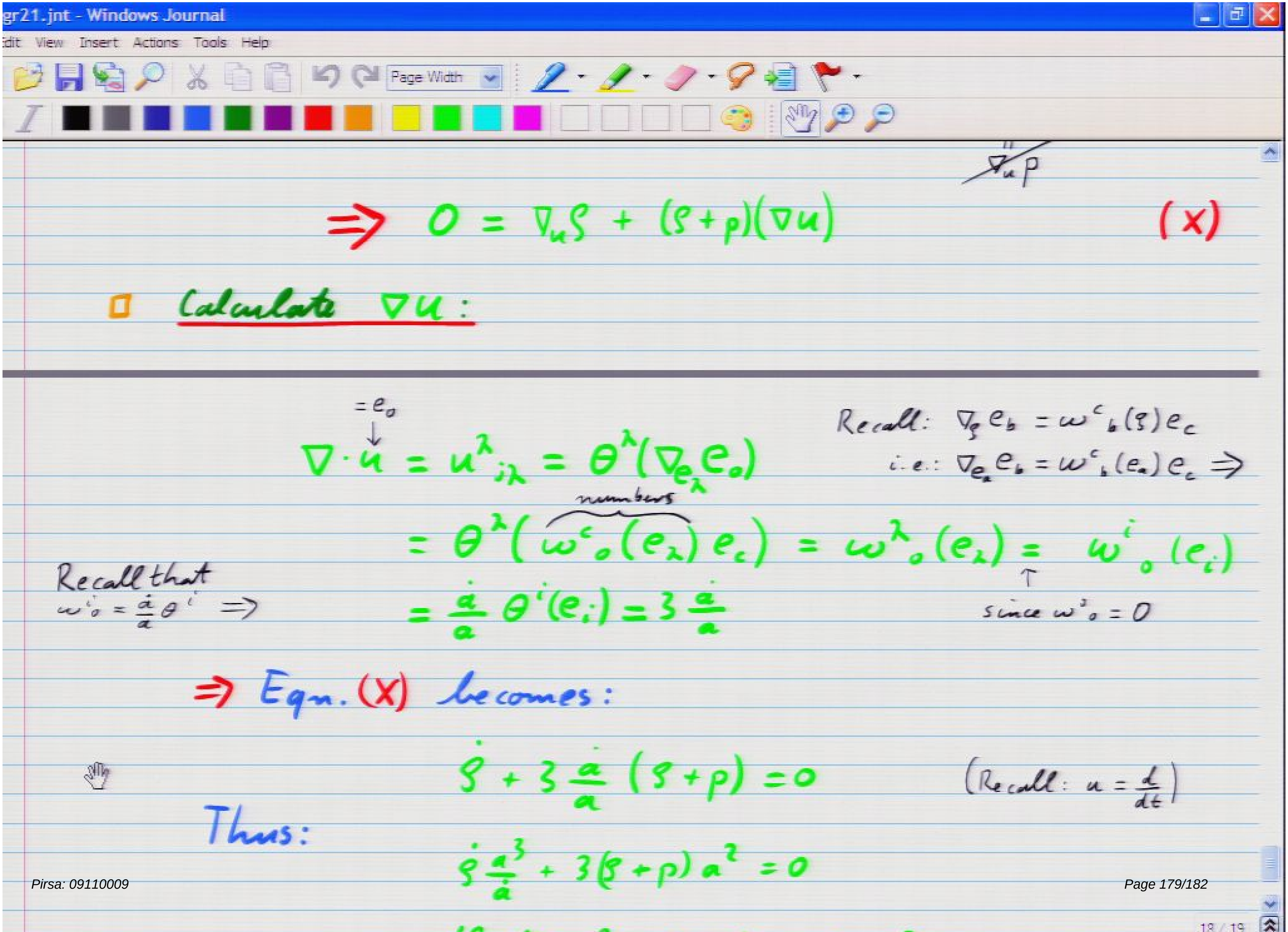

$\overset{"}{\cancel{\nabla_u p}}$

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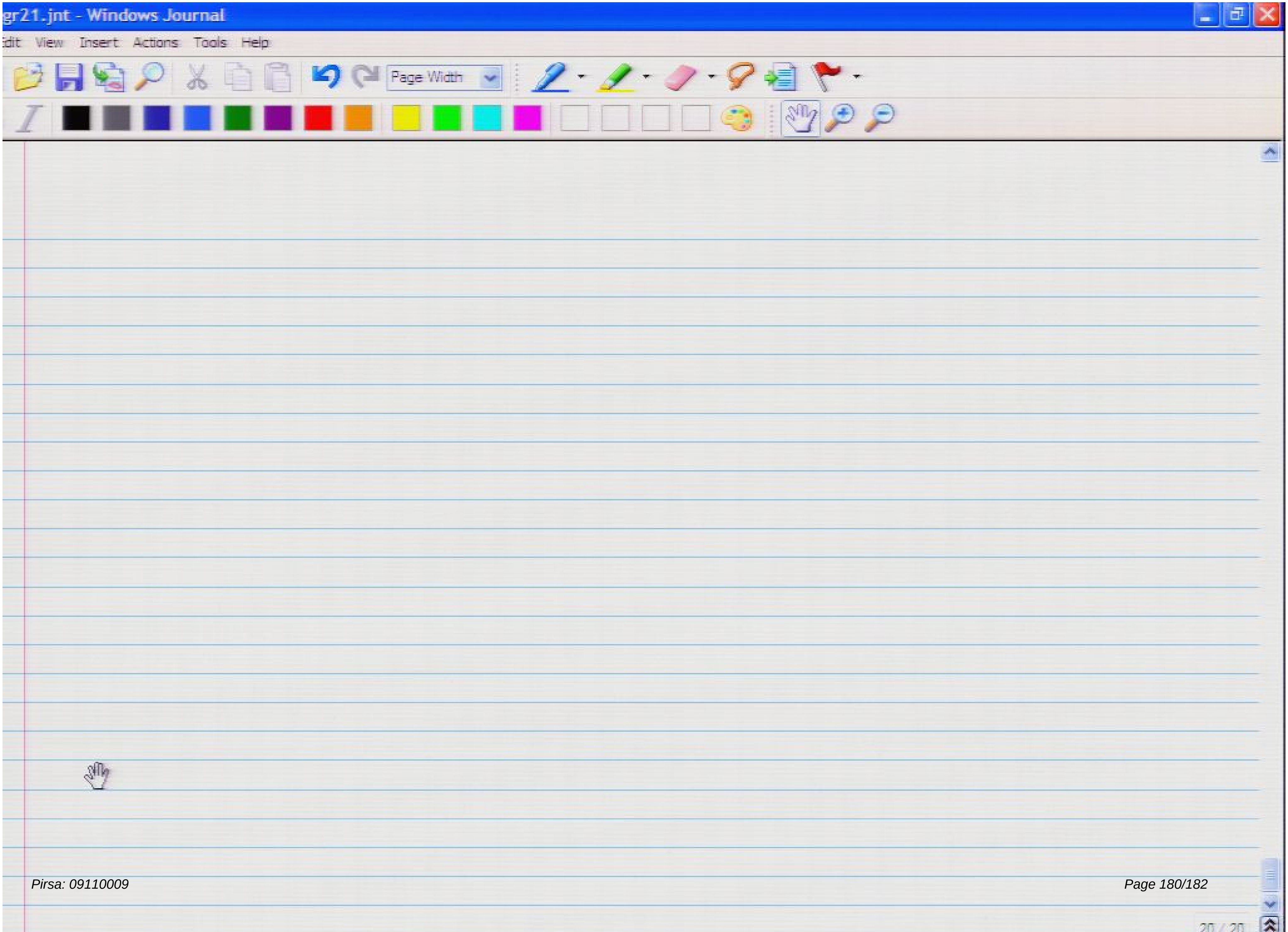
$\Rightarrow$ Eqn. (X) becomes:

$$\dot\rho + 3\frac{\dot a}{a}(\rho + p) = 0$$

(Recall: $u = \frac{d}{dt}$)

Thus:

$$\dot\rho \frac{a^3}{\dot a} + 3(\rho + p)a^2 = 0$$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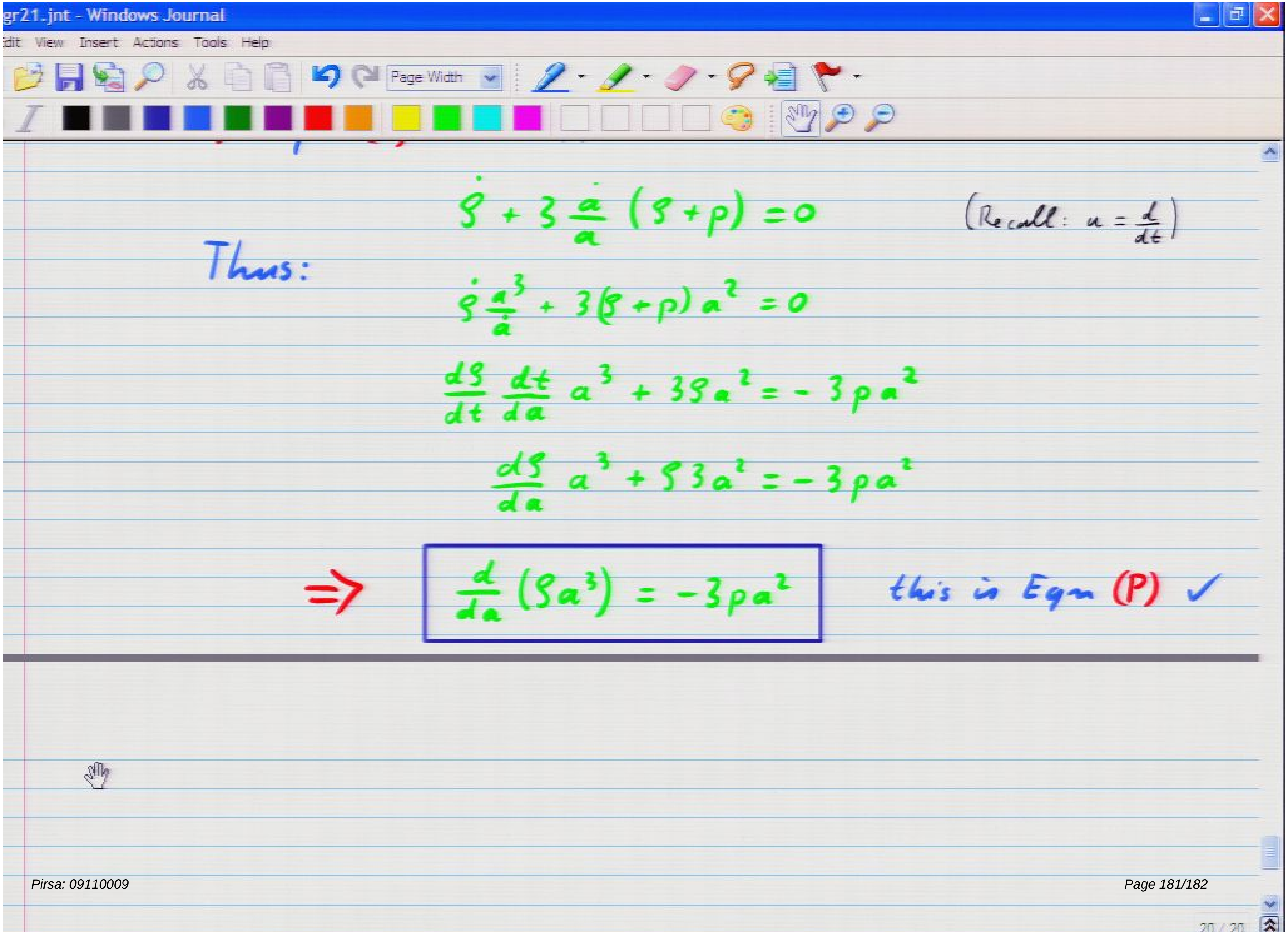


$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0$$

(Recall: $u = \frac{d}{dt}$)

Thus:

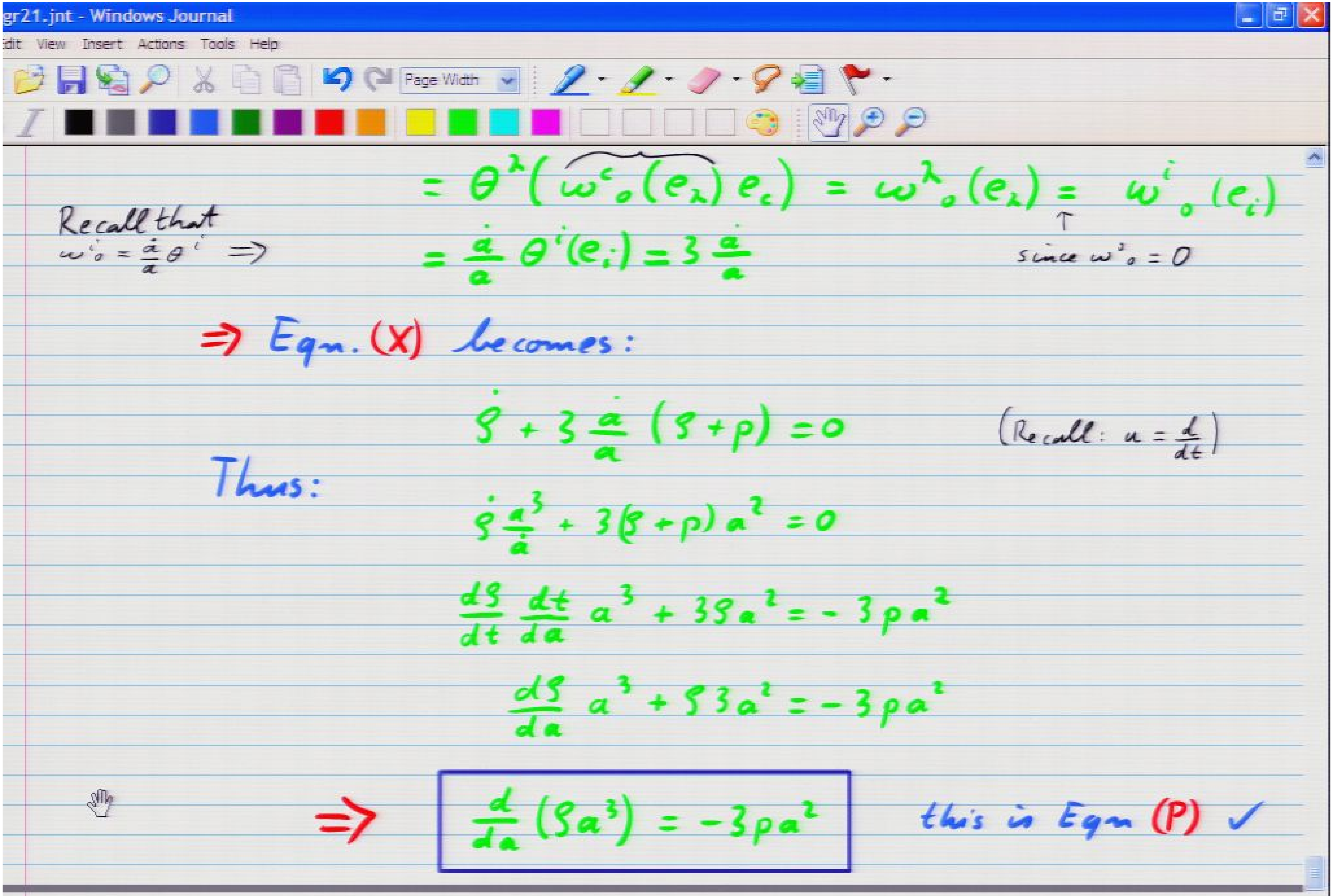
$$\dot{\rho}\frac{a^3}{\dot{a}} + 3(\rho + p)a^2 = 0$$

$$\frac{d\rho}{dt}\frac{dt}{da}a^3 + 3\rho a^2 = -3pa^2$$

$$\frac{d\rho}{da}a^3 + \rho 3a^2 = -3pa^2$$

$$\Rightarrow \boxed{\frac{d}{da}(\rho a^3) = -3pa^2}$$

this is Eqn (P) ✓



$$= \theta^\lambda(\overbrace{\omega^c{}_0(e_\lambda)}\, e_c) = \omega^\lambda{}_0(e_\lambda) \underset{\uparrow}{=} \omega^i{}_0(e_i)$$

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