Title: General Relativity for Cosmology - Lecture 14A

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Abstract:

GR for Cosmology, Achim Kempf, Fall 09, Lecture 15

The tetrad formulation of GR

Why?

1. Need a formalism that is suited for study of local versus global entities:

E.g. a binary star system, or a galaxy, far from other matter

E.g.: For suitably isolated local systems there should exist a notion of global energy and momentum conservation, even in the absence of Killing vector fields.

Recall: since antisymmetric, they transform like the Jacobian of the integral measure $d^n x$.

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2. Need a formalism that is suited for the study of fermions:

E.g.: In special relativity, the Dirac equation (for electrons and quarks etc) has the form:

"Spinor"

$$\left(i\gamma^\mu \frac{\partial}{\partial x^\mu} - m\mathbb{1}\right)\Psi = 0 \quad \text{with } \Psi = \begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix}$$

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$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu} = 2\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$$

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→ Strategy: At each $p\in M$ choose an orthonormal basis $\{\theta^i\}$ of $T_p(M)$, and dual basis $\{e_i\}$ so that $g_{\mu\nu} = g(e_\mu, e_\nu) = \begin{pmatrix}-1 & & 0\\ & 1 & \\ 0 & & 1\end{pmatrix}$.

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3. Important benefit:

General relativity takes the form of a so-called "gauge theory", which is analogous to the gauge theories of electromagnetism, the weak and the strong force. ⟶ Useful for Quantum Gravity?

Recall the math:

□ Frames $\{\theta^\mu\}, \{e_\mu\}$:

Often, one uses as the bases of $T_p(M)$, and $T_p(M)'$ the canonical bases $\{dx^\mu\}$ and $\{\frac{\partial}{\partial x^\mu}\}$ respectively, which suggest themselves when one chooses coordinates, say $(x^0,\dots,x^3)$.

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Important: The only reason why the components of a tensor can (but don't have to) change when we change coordinates is that we can (and often do) change basis in the (co-) tangent spaces, namely

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$\longrightarrow$

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Conversely: Even staying with one coordinate system, we can freely change our choice of basis in the (co-)tangent spaces:

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- The curvature form: $\Omega'^\mu{}_\nu = A^\mu{}_a (A^{-1})_\nu{}^b \Omega^a{}_b$

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(Matrix notation: $\omega' = A\,\omega\,A^{-1} - (dA)\,A^{-1}$)

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How to specify frames?

In an arbitrary coordinate system, we may specify the bases in terms of the canonical bases:

$$\theta^{i}(x) = \lambda^{i}{}_{j}(x)\,dx^{j}$$

(Another possibility: Take n scalar functions $f^{(1)},\dots,f^{(n)}$ and define $\theta^{i} := df^{(i)}$. For generic functions these $\{\theta^{i}\}$ will be linearly independent almost everywhere)

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Note: the $\lambda^i{}_j(x)$ change nontrivially when changing the coordinate system!

Our choice now: orthonormal frames, i.e. "Tetrads":

- We say that a frame $\{\theta^\mu\}, \{e_\mu\}$ is orthonormal if in this frame, for all $p \in M$:

$$g(e_\mu, e_\nu) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}_{\mu\nu} = \eta_{\mu\nu} \quad \text{i.e. if:} \quad g = -\theta^0 \otimes \theta^0 + \sum_{i=1}^{3} \theta^i \otimes \theta^i$$

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- Existence? Always: At each $p \in M$ may choose e.g. $\theta^\mu = dx^\mu$ where dx^μ are canonical ON basis at centre of a geodesic cds.

- Uniqueness?

For a given space-time, (M, g), any ON frame yields a new ON frame by transforming the bases through

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recall: this is the defining equation for Lorentz transformations.

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$\Rightarrow$ Frames are unique up to local Lorentz transformations.

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Can now answer an old question:

Q: The connection 1-forms $\omega^\mu{}_\nu$ are not, we know, tensor-valued 1-forms. Wherein do they take their values?

A: The connection 1-forms take values in the set of infinitesimal Lorentz transformations!

Intuition?

The connection yields the change under infinitesimal

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The connection yields the change under infinitesimal parallel transport - and parallel transport preserves the metric, i.e. it preserves the lengths of vectors, i.e. the change can only be an infinitesimal "rotation", i.e. an infinitesimal Lorentz transformation.

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Infinitesimal Lorentz transformations are of this form:

$\varepsilon^2 = 0$ assumed

$$A^\mu{}_a = \delta^\mu{}_a + \varepsilon^\mu{}_a$$

(technically, we are going from the Lorentz group to the Lorentz Lie algebra)

Now, (*), i.e. $A^\mu{}_a A^\nu{}_b \eta_{\mu\nu} = \eta_{ab}$ reads:

$$(\delta^\mu{}_a + \varepsilon^\mu{}_a)(\delta^\nu{}_b + \varepsilon^\nu{}_b)\eta_{\mu\nu} = \eta_{ab}$$

i.e. $$\varepsilon^\mu{}_a \eta_{\mu b} + \varepsilon^\nu{}_b \eta_{a\nu} = 0$$

$\Rightarrow$ Infinitesimal Lorentz transformations are given by all $A^\mu{}_a = \delta^\mu{}_a + \varepsilon^\mu{}_a$

which obey: $$\boxed{\varepsilon_{ba} + \varepsilon_{ab} = 0}$$

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Proposition:

In orthonormal frames, the 1-form $\omega_{\mu\nu}$ obeys

$$\omega_{\mu\nu} + \omega_{\nu\mu} = 0$$

i.e. it takes values that are infinitesimal Lorentz transformations.

Proof:

Recall: Absolute exterior derivative: (an anti-derivation)

$$Dt^{a..b}{}_{c..d} = dt^{a..b}{}_{c..d} + \omega^{a}{}_{i} \wedge t^{i..b}{}_{c..d} + \dots - \omega^{i}{}_{c} \wedge t^{a..b}{}_{i..d} - \dots$$

any tensor-valued differential form.

play the role of the $\Gamma^{a}{}_{bc}$

Thus:

(0,2) tensor-valued 0 form

$= 0$ because $g_{\mu\nu} = \eta_{\mu\nu} = \text{const}$

can drop the $\wedge$ because g is a 0-form.

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Recall that by using a tetrad, we achieved that $g_{\mu\nu} = \begin{pmatrix} -1 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix} = \eta_{\mu\nu}$ everywhere!

i.e. $0 = \omega_{\nu\mu} + \omega_{\mu\nu}$ ✓

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Proposition:

In orthonormal frames, the 1-form $\omega_{\mu\nu}$ obeys

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i.e. it takes values that are infinitesimal Lorentz transformations.

Proof:

Recall: Absolute exterior derivative: (an anti-derivation)

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Then ... (0,2) tensor-valued 0-form ... = 0 because $g_{\mu\nu} = \eta_{\mu\nu} = \text{const}$

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$\varepsilon^2 = 0$ assumed

$$A^\mu{}_a = \delta^\mu{}_a + \varepsilon^\mu{}_a$$

(technically, we are going from the Lorentz group to the Lorentz Lie algebra)

Now, (*), i.e. $A^\mu{}_a A^\nu{}_b \eta_{\mu\nu} = \eta_{ab}$ reads:

$$(\delta^\mu{}_a + \varepsilon^\mu{}_a)(\delta^\nu{}_b + \varepsilon^\nu{}_b)\eta_{\mu\nu} = \eta_{ab}$$

i.e. $\varepsilon^\mu{}_a \eta_{\mu b} + \varepsilon^\nu{}_b \eta_{a\nu} = 0$

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Tetrad formulation of GR:

- Redefine the degrees of freedom:
  - We used to specify space-times

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Tetrad formulation of GR:

- Redefine the degrees of freedom:
  - We used to specify space-times through these data: $(\mathcal{M}, g)$
  - Now, let us specify space-times, equivalently, through data $(\mathcal{M}, \{\theta^\mu\})$:

Namely:

Assume the $\{\theta^i\}$ are given w. resp. to a basis $\{dx^\mu\}$, through functions $A^\mu{}_\nu$ so that:

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"Gauge principle of GR":

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Remark:

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□ **Remark:**

The Schrödinger and Dirac equations allow

$$\psi(x) \to e^{i\alpha}\psi(x)$$ "global" phase transformation

trivially. Indeed, also local transformations

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work: At the cost of introducing a field $F_{\mu\nu}(x)$. Thus, the electromagnetic field is derivable this way. Similar for weak and strong force. Main difference to gravity: Connection yes, but no to gravity: Connection yes, but no metric.

The action principle: (in terms of $\{\theta^i\}$ and $\Omega^i{}_j$)

Consider the action, for now, without cosmological constant and without matter:

$$S_{grav} := \frac{1}{16\pi G} \int_B \overset{\text{0-form}}{R} \sqrt{g}\, d^4x$$

Recall Hodge $*$: If $\nu = \frac{1}{p!} \nu_{i_1 \ldots i_p} \theta^{i_1} \wedge \ldots \wedge \theta^{i_p}$

then $*\nu = \frac{1}{p!} \sqrt{g}\, \underbrace{\varepsilon_{i_1 \ldots i_n}}_{= \pm 1,\text{ totally anti-symmetric}} \nu^{i_1 \ldots i_p}\, \theta^{i_{p+1}} \wedge \ldots \wedge \theta^{i_n}$

i.e. $* : \Lambda^p \rightarrow \Lambda^{n-p}$

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Consider the action, for now, without cosmological constant and without matter:

$$S'_{grav} := \frac{1}{16\pi G}\int_B \overbrace{R}^{\text{0-form}} \sqrt{g}\, d^4x$$

Recall Hodge $*$: If $\quad v = \frac{1}{p!} v_{i_1 \ldots i_p}\, \theta^{i_1}\wedge \ldots \wedge \theta^{i_p}$

then $\quad *v = \frac{1}{p!}\sqrt{g}\, \overbrace{\varepsilon_{i_1 \ldots i_n}}^{= \pm 1,\ \text{totally anti-symmetric}}\, v^{i_1 \ldots i_p}\, \theta^{i_{p+1}} \wedge \ldots \wedge \theta^{i_n}$

i.e. $\quad * : \Lambda^p \to \Lambda^{n-p}$

Thus:

$$S'_{grav} = \frac{1}{16\pi G}\int_B \underbrace{*R}_{\text{4-form}}$$

Aim now: Re-express S' in terms of $\{\theta^\mu\}$ and Ω^μ ...

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and without matter:

$$S'_{grav} := \frac{1}{16\pi G}\int_B \underbrace{R}_{0\text{-form}} \sqrt{g}\, d^4x$$

Recall Hodge $*$: If $\quad v = \frac{1}{p!} v_{i_1 \ldots i_p} \theta^{i_1} \wedge \ldots \wedge \theta^{i_p}$

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Thus:

$$S'_{grav} = \frac{1}{16\pi G}\int_B \underbrace{*R}_{4\text{-form}}$$

Aim now: Re-express S'_{grav} in terms of $\{\theta^r\}$ and $\Omega^r{}_v$.

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$$S'_{grav} := \frac{1}{16\pi G}\int_B R\sqrt{g}\, d^4x$$

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Aim now: Re-express S'_{grav} in terms of $\{\theta^r\}$ and $\Omega^r{}_v$.

Define: "capital r" is a (0,2) tensor-valued 2-form

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and matter matter:

$$S'_{grav} := \frac{1}{16\pi G} \int_B \overset{0\text{-form}}{R} \sqrt{g}\, d^4x$$

Recall Hodge $*$:

If $\quad v = \frac{1}{p!} v_{i_1 \ldots i_p} \theta^{i_1} \wedge \ldots \wedge \theta^{i_p}$

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$\varepsilon = \pm 1$, totally anti-symmetric

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Thus:

$$S'_{grav} = \frac{1}{16\pi G} \int_B \underbrace{*R}_{4\text{-form}}$$

Aim now: Re-express S'_{grav} in terms of $\{\theta^\mu\}$ and $\Omega^\mu{}_\nu$.

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$$S'_{grav} = \frac{1}{16\pi G} \int_B \underbrace{*R}_{4\text{-form}}$$

Aim now: Re-express S'_{grav} in terms of $\{\theta^\mu\}$ and $\Omega^\mu{}_\nu$.

Define: "capital η" is a $(0,2)$ tensor-valued 2-form

$H_{\alpha\beta} := *(\theta^\alpha \wedge \theta^\beta) = \frac{1}{2}\sqrt{g}\,\varepsilon \dots$

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Aim now: Re-express $S'_{\mu\nu}$ in terms of $\{\theta^\mu\}$ and $\Omega^\mu{}_\nu$.

Define:

"capital η" is a $(0,2)$ tensor-valued 2-form

$$H_{\alpha\beta} := *(\theta^\alpha \wedge \theta^\beta) = \frac{1}{2}\sqrt{g}\,\varepsilon_{\alpha\beta\gamma\delta}\,\theta^\gamma \wedge \theta^\delta$$

$$H_{\alpha\beta\gamma} := *(\theta^\alpha \wedge \theta^\beta \wedge \theta^\gamma) = \frac{1}{2}\sqrt{g}\,\varepsilon_{\alpha\beta\gamma\delta}\,\theta^\delta$$

a $(0,3)$ tensor-valued 1-form.

Proposition:

$$\boxed{*R = H_{\mu\nu} \wedge \Omega^{\mu\nu}}$$

(it is a $(0,0)$ tensor-valued 4-form)

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Aim now: Re-express $S'_{\mu\nu}$ in terms of $\{\theta^\mu\}$ and $\Omega^\mu{}_\nu$.

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Aim now: Re-express S'_{grav} in terms of $\{\theta^\mu\}$ and $\Omega^\mu{}_\nu$.

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$$H_{\alpha\beta} := *\left(\theta^\alpha \wedge \theta^\beta\right) = \frac{1}{2}\sqrt{g}\, \varepsilon_{\alpha\beta\gamma\delta}\, \theta^\gamma \wedge \theta^\delta$$

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Proposition:

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Proof:

Use $\Omega^\mu{}_\nu = \frac{1}{2} R^\mu{}_{\nu\kappa\lambda}\,\theta^\kappa \wedge \theta^\lambda \Rightarrow$

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$$H_{\alpha\beta\gamma} := *\left(\theta^\alpha \wedge \theta^\beta \wedge \theta^\gamma\right) = \frac{1}{2}\sqrt{g}\,\varepsilon_{\alpha\beta\gamma\delta}\,\theta^\delta$$

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Use $\Omega^\mu{}_\nu = \frac{1}{2} R^\mu{}_{\nu\kappa\lambda}\,\theta^\kappa \wedge \theta^\lambda \Rightarrow$

$$H_{\mu\nu} \wedge \Omega^{\mu\nu} = \frac{1}{2\cdot 2}\sqrt{g}\,\varepsilon_{\mu\nu\gamma\delta}\,R^{\mu\nu}{}_{\kappa\lambda}\,\theta^\gamma \wedge \theta^\delta \wedge \theta^\kappa \wedge \theta^\lambda$$

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Proof:

Use $\Omega^{\mu}{}_{\nu} = \frac{1}{2} R^{\mu}{}_{\nu\kappa\lambda}\, \theta^{\kappa} \wedge \theta^{\lambda}$ $\Rightarrow$

$$H_{\mu\nu} \wedge \Omega^{\mu\nu} = \frac{1}{2 \cdot 2} \sqrt{g}\, \varepsilon_{\mu\nu\gamma\delta} R^{\mu\nu}{}_{\kappa\lambda} \underbrace{\theta^{\gamma} \wedge \theta^{\delta} \wedge \theta^{\kappa} \wedge \theta^{\lambda}}_{= \varepsilon_{\gamma\delta\kappa\lambda}\, \theta^{\gamma} \otimes \theta^{\delta} \otimes \theta^{\kappa} \otimes \theta^{\lambda}}$$

Use also: $\varepsilon_{\gamma\delta\kappa\lambda}\, \varepsilon_{\mu\nu\gamma\delta} = 2\left(\delta_{\kappa\mu} \delta_{\lambda\nu} - \delta_{\kappa\nu} \delta_{\lambda\mu} \right)$ $\Rightarrow$

$$H_{\mu\nu} \wedge \Omega^{\mu\nu} = \frac{4}{4} R^{\mu\nu}{}_{\mu\nu} \sqrt{g}\, \theta^1 \wedge \theta^2 \wedge \theta^3 \wedge \theta^4 = {*R} \quad \checkmark$$

(need later for derivation of the Einstein equation)

Proposition: $D H_{\mu\nu} = 0$

Recall the "first structure equation": $D\theta^{\sigma} = 0$

Proof: $$D H_{\mu\nu} = D\Big(\overbrace{\tfrac{1}{2}\sqrt{g}\, \varepsilon_{\mu\nu\rho\sigma}}^{\text{constant because ON basis}} \theta^{\rho} \wedge \theta^{\sigma} \Big) = \frac{1}{2}\sqrt{g}\, \varepsilon_{\mu\nu\rho\sigma} \big(\overset{0}{D\theta^{\rho}} \wedge \theta^{\sigma} + \theta^{\rho} \wedge \overset{0}{D\theta^{\sigma}} \big)$$

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Proof:

Use $\Omega^{\mu}{}_{\nu} = \frac{1}{2} R^{\mu}{}_{\nu\kappa\lambda}\, \theta^{\kappa} \wedge \theta^{\lambda} \implies$

$$H_{\mu\nu} \wedge \Omega^{\mu\nu} = \frac{1}{2\cdot 2} \sqrt{g}\, \varepsilon_{\mu\nu\gamma\delta} R^{\mu\nu}{}_{\kappa\lambda} \underbrace{\theta^{\gamma} \wedge \theta^{\delta} \wedge \theta^{\kappa} \wedge \theta^{\lambda}}_{= \varepsilon_{\gamma\delta\kappa\lambda}\, \theta^{\gamma} \otimes \theta^{\delta} \otimes \theta^{\kappa} \otimes \theta^{\lambda}}$$

Use also: $\varepsilon_{\gamma\delta\kappa\lambda} \varepsilon_{\mu\nu\gamma\delta} = 2\left(\delta_{\kappa\mu}\delta_{\lambda\nu} - \delta_{\kappa\nu}\delta_{\lambda\mu}\right) \implies$

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(need later for derivation of the Einstein equation)

Proposition: $DH_{\mu\nu} = 0$

Recall the "first structure equation": $D\theta^{\sigma} = 0$

constant because ON basis

Proof:

$$DH_{\mu\nu} = D\Big(\frac{1}{2}\underbrace{\sqrt{g}\, \varepsilon_{\mu\nu\varsigma\sigma}}_{\text{constant because ON basis}}\, \theta^{\varsigma} \wedge \theta^{\sigma}\Big) = \frac{1}{2}\sqrt{g}\, \varepsilon_{\mu\nu\varsigma\sigma}\Big(\underbrace{D\theta^{\varsigma}}_{=0} \wedge \theta^{\sigma} + \theta^{\varsigma} \wedge \underbrace{D\theta^{\sigma}}_{=0}\Big)$$

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$$\boxed{*R = H_{\mu\nu} \wedge \Omega^{\mu\nu}}$$

(it is a (0,0) tensor-valued 4-form)

Proof:

Use $\Omega^{\mu}{}_{\nu} = \frac{1}{2} R^{\mu}{}_{\nu\kappa\lambda}\, \theta^{\kappa} \wedge \theta^{\lambda}$ $\Rightarrow$

$$H_{\mu\nu} \wedge \Omega^{\mu\nu} = \frac{1}{2\cdot 2} \sqrt{g}\, \varepsilon_{\mu\nu\gamma\delta} R^{\mu\nu}{}_{\kappa\lambda} \underbrace{\theta^{\gamma} \wedge \theta^{\delta} \wedge \theta^{\kappa} \wedge \theta^{\lambda}}_{= \ \varepsilon_{\gamma\delta\kappa\lambda}\, \theta^{\gamma} \otimes \theta^{\delta} \otimes \theta^{\kappa} \otimes \theta^{\lambda}}$$

Use also: $\varepsilon_{\gamma\delta\kappa\lambda}\, \varepsilon_{\mu\nu\gamma\delta} = 2(\delta_{\kappa\mu}\delta_{\lambda\nu} - \delta_{\kappa\nu}\delta_{\lambda\mu})$ $\Rightarrow$

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Define: "capital η" is a $(0,2)$ tensor-valued 2-form

$$H_{\alpha\beta} := *(\theta^\alpha \wedge \theta^\beta) = \frac{1}{2}\sqrt{g}\,\varepsilon_{\alpha\beta\gamma\delta}\,\theta^\gamma \wedge \theta^\delta$$

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Proposition:

$$\boxed{*R = H_{\mu\nu} \wedge \Omega^{\mu\nu}}$$

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Proof:

Use $\Omega^{\mu}{}_{\nu} = \frac{1}{2} R^{\mu}{}_{\nu\kappa\lambda}\, \theta^{\kappa}\wedge\theta^{\lambda} \Longrightarrow$

$$H_{\mu\nu}\wedge\Omega^{\mu\nu} = \frac{1}{2\cdot 2}\sqrt{g}\,\varepsilon_{\mu\nu\gamma\delta}\, R^{\mu\nu}{}_{\kappa\lambda}\, \underbrace{\theta^{\gamma}\wedge\theta^{\delta}\wedge\theta^{\kappa}\wedge\theta^{\lambda}}_{=\ \varepsilon_{\gamma\delta\kappa\lambda}\,\theta^{\gamma}\otimes\theta^{\delta}\otimes\theta^{\kappa}\otimes\theta^{\lambda}}$$

Use also: $\varepsilon_{\gamma\delta\kappa\lambda}\,\varepsilon_{\mu\nu\gamma\delta} = 2\left(\delta_{\kappa\mu}\delta_{\lambda\nu} - \delta_{\kappa\nu}\delta_{\lambda\mu}\right) \Longrightarrow$

(need later for derivation of the Einstein equation) ↳

$$H_{\mu\nu}\wedge\Omega^{\mu\nu} = \frac{4}{4} R^{\mu\nu}{}_{\mu\nu}\,\sqrt{g}\,\theta^{1}\wedge\theta^{2}\wedge\theta^{3}\wedge\theta^{4} = *R \quad ✓$$

Proposition: $DH_{\mu\nu} = 0$

Recall the "first structure equation": $D\theta^{\sigma} = 0$

Proof: $DH_{\mu\nu} = D\Big(\overbrace{\tfrac{1}{2}\sqrt{g}\,\varepsilon_{\mu\nu\varrho\sigma}}^{\text{constant because ON basis}}\,\theta^{\varrho}\wedge\theta^{\sigma}\Big) = \frac{1}{2}\sqrt{g}\,\varepsilon_{\mu\nu\varrho\sigma}\Big(\overbrace{D\theta^{\varrho}}^{0}\wedge\theta^{\sigma} + \theta^{\varrho}\wedge\overbrace{D\theta^{\sigma}}^{0}\Big)$

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Proof:

Use $\Omega^{\mu}{}_{\nu} = \frac{1}{2} R^{\mu}{}_{\nu\kappa\lambda}\, \theta^{\kappa} \wedge \theta^{\lambda} \Rightarrow$

$$H_{\mu\nu} \wedge \Omega^{\mu\nu} = \frac{1}{2\cdot 2} \sqrt{g}\, \varepsilon_{\mu\nu\gamma\delta} R^{\mu\nu}{}_{\kappa\lambda} \underbrace{\theta^{\gamma} \wedge \theta^{\delta} \wedge \theta^{\kappa} \wedge \theta^{\lambda}}_{= \varepsilon_{\gamma\delta\kappa\lambda}\, \theta^{\gamma} \otimes \theta^{\delta} \otimes \theta^{\kappa} \otimes \theta^{\lambda}}$$

Use also: $\varepsilon_{\gamma\delta\kappa\lambda}\, \varepsilon_{\mu\nu\gamma\delta} = 2\left(\delta_{\kappa\mu}\delta_{\lambda\nu} - \delta_{\kappa\nu}\delta_{\lambda\mu}\right) \Rightarrow$

(need later for derivation of the Einstein equation)

$$H_{\mu\nu} \wedge \Omega^{\mu\nu} = \frac{4}{4} R^{\mu\nu}{}_{\mu\nu} \sqrt{g}\, \theta^{1} \wedge \theta^{2} \wedge \theta^{3} \wedge \theta^{4} = * R \;\checkmark$$

Proposition: $D H_{\mu\nu} = 0$

Recall the "first structure equation": $D\theta^{\sigma} = 0$

Proof: $D H_{\mu\nu} = D\Big(\frac{1}{2} \underbrace{\sqrt{g}\, \varepsilon_{\mu\nu\rho\sigma}}_{\text{constant because ON basis}} \theta^{\rho} \wedge \theta^{\sigma}\Big) = \frac{1}{2}\sqrt{g}\, \varepsilon_{\mu\nu\rho\sigma}\Big(\underbrace{D\theta^{\rho}}_{=0} \wedge \theta^{\sigma} + \theta^{\rho} \wedge \underbrace{D\theta^{\sigma}}_{=0}\Big)$

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$\square$ Proof:

Use $\Omega^{\mu}{}_{\nu} = \frac{1}{2} R^{\mu}{}_{\nu\kappa\lambda}\, \theta^{\kappa} \wedge \theta^{\lambda} \Rightarrow$

$$H_{\mu\nu} \wedge \Omega^{\mu\nu} = \frac{1}{2\cdot 2} \sqrt{g}\, \varepsilon_{\mu\nu\gamma\delta} R^{\mu\nu}{}_{\kappa\lambda} \underbrace{\theta^{\gamma} \wedge \theta^{\delta} \wedge \theta^{\kappa} \wedge \theta^{\lambda}}_{\substack{\shortparallel \\ \varepsilon_{\gamma\delta\kappa\lambda}\, \theta^{\gamma} \otimes \theta^{\delta} \otimes \theta^{\kappa} \otimes \theta^{\lambda}}}$$

Use also: $\varepsilon_{\gamma\delta\kappa\lambda}\, \varepsilon_{\mu\nu\gamma\delta} = 2\left(\delta_{\kappa\mu}\delta_{\lambda\nu} - \delta_{\kappa\nu}\delta_{\lambda\mu}\right) \Rightarrow$

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(need later for derivation of the Einstein equation)

$\square$ Proposition: $D H_{\mu\nu} = 0$

Recall the "first structure equation": $D\theta^{\sigma} = 0$

$\square$ Proof: $D H_{\mu\nu} = D\Big(\overbrace{\tfrac{1}{2}\sqrt{g}\, \varepsilon_{\mu\nu\varsigma\sigma}}^{\text{constant because ON basis}}\, \theta^{\varsigma} \wedge \theta^{\sigma}\Big) = \tfrac{1}{2}\sqrt{g}\, \varepsilon_{\mu\nu\varsigma\sigma}\big(\underbrace{D\theta^{\varsigma}}_{=0} \wedge \theta^{\sigma} + \theta^{\varsigma} \wedge \underbrace{D\theta^{\sigma}}_{=0}\big)$

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Proof:

Use $\Omega^\mu{}_\nu = \frac{1}{2} R^\mu{}_{\nu\kappa\lambda}\, \theta^\kappa \wedge \theta^\lambda \implies$

$$H_{\mu\nu} \wedge \Omega^{\mu\nu} = \frac{1}{2\cdot 2}\sqrt{g}\, \varepsilon_{\mu\nu\gamma\delta} R^{\mu\nu}{}_{\kappa\lambda} \underbrace{\theta^\gamma \wedge \theta^\delta \wedge \theta^\kappa \wedge \theta^\lambda}_{= \varepsilon_{\gamma\delta\kappa\lambda}\, \theta^\gamma \otimes \theta^\delta \otimes \theta^\kappa \otimes \theta^\lambda}$$

Use also: $\varepsilon_{\gamma\delta\kappa\lambda}\, \varepsilon_{\mu\nu\gamma\delta} = 2(\delta_{\kappa\mu}\delta_{\lambda\nu} - \delta_{\kappa\nu}\delta_{\lambda\mu}) \implies$

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(need later for derivation of the Einstein equation)

Proposition: $DH_{\mu\nu} = 0$

Recall the "first structure equation": $D\theta^\mu = 0$

constant because ON basis

Proof:

Use $\Omega^{\mu}{}_{\nu} = \frac{1}{2} R^{\mu}{}_{\nu\kappa\lambda}\, \theta^{\kappa} \wedge \theta^{\lambda} \Rightarrow$

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$$H_{\mu\nu} \wedge \Omega^{\mu\nu} = \frac{1}{2\cdot 2}\sqrt{g}\, \varepsilon_{\mu\nu\gamma\delta} R^{\mu\nu}{}_{\kappa\lambda} \underbrace{\theta^\gamma \wedge \theta^\delta \wedge \theta^\kappa \wedge \theta^\lambda}_{\varepsilon_{\gamma\delta\kappa\lambda}\, \theta^\gamma \otimes \theta^\delta \otimes \theta^\kappa \otimes \theta^\lambda}$$

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(need later for derivation of the Einstein equation) $\hookrightarrow$

$$H_{\mu\nu} \wedge \Omega^{\mu\nu} = \frac{4}{4} R^{\mu\nu}{}_{\mu\nu} \sqrt{g}\, \theta^1 \wedge \theta^2 \wedge \theta^3 \wedge \theta^4 = * R \ ✓$$

□ Proposition: $DH_{\mu\nu} = 0$

Recall the "first structure equation": $D\theta^\sigma = 0$

□ Proof: $DH_{\mu\nu} = D\Big(\underbrace{\tfrac{1}{2}\sqrt{g}\,\varepsilon_{\mu\nu\rho\sigma}}_{\text{constant because ON basis}}\theta^\rho \wedge \theta^\sigma\Big) = \frac{1}{2}\sqrt{g}\,\varepsilon_{\mu\nu\rho\sigma}\big(\overbrace{D\theta^\rho}^{=0} \wedge \theta^\sigma + \theta^\rho \wedge \overbrace{D\theta^\sigma}^{=0}\big)$

Recall important identities: (torsionless case)

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$$H_{\mu\nu}\wedge\Omega^{\mu\nu} = \frac{1}{2\cdot 2}\sqrt{g}\,\varepsilon_{\mu\nu\gamma\delta}\,R^{\mu\nu}{}_{\kappa\lambda}\,\underbrace{\theta^{\gamma}\wedge\theta^{\delta}\wedge\theta^{\kappa}\wedge\theta^{\lambda}}_{\varepsilon_{\gamma\delta\kappa\lambda}\,\theta^{\gamma}\otimes\theta^{\delta}\otimes\theta^{\kappa}\otimes\theta^{\lambda}}$$

Use also: $\varepsilon_{\gamma\delta\kappa\lambda}\,\varepsilon_{\mu\nu\gamma\delta} = 2\left(\delta_{\kappa\mu}\delta_{\lambda\nu} - \delta_{\kappa\nu}\delta_{\lambda\mu}\right) \Longrightarrow$

(need later for derivation of the Einstein equation)

$$H_{\mu\nu}\wedge\Omega^{\mu\nu} = \frac{4}{4}R^{\mu\nu}{}_{\mu\nu}\sqrt{g}\,\theta^1\wedge\theta^2\wedge\theta^3\wedge\theta^4 = *R \;✓$$

□ Proposition: $DH_{\mu\nu} = 0$

Recall the "first structure equation": $D\theta^{\alpha} = 0$

□ Proof: $DH_{\mu\nu} = D\Big(\overbrace{\tfrac{1}{2}\sqrt{g}\,\varepsilon_{\mu\nu\varrho\sigma}}^{\text{constant because ON basis}}\,\theta^{\varrho}\wedge\theta^{\sigma}\Big) = \tfrac{1}{2}\sqrt{g}\,\varepsilon_{\mu\nu\varrho\sigma}\Big(\overbrace{D\theta^{\varrho}}^{0}\wedge\theta^{\sigma} + \theta^{\varrho}\wedge\overbrace{D\theta^{\sigma}}^{0}\Big)$

Recall important identities: (torsionless case)

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$$\varepsilon_{\gamma\delta\kappa\lambda}\,\theta^{\gamma}\otimes\theta^{\delta}\otimes\theta^{\kappa}\otimes\theta^{\lambda}$$

Use also: $\varepsilon_{\gamma\delta\kappa\lambda}\,\varepsilon_{\mu\nu\gamma\delta} = 2\left(\delta_{\kappa\mu}\delta_{\lambda\nu} - \delta_{\kappa\nu}\delta_{\lambda\mu}\right) \Longrightarrow$

(need later for derivation of the Einstein equation)

$$H_{\mu\nu}\wedge\Omega^{\mu\nu} = \frac{4}{4}R^{\mu\nu}{}_{\mu\nu}\sqrt{g}\,\theta^1\wedge\theta^2\wedge\theta^3\wedge\theta^4 = *R \checkmark$$

□ Proposition: $DH_{\mu\nu} = 0$

Recall the "first structure equation": $D\theta^{\sigma} = 0$

□ Proof: $DH_{\mu\nu} = D\Big(\underbrace{\tfrac{1}{2}\sqrt{g}\,\varepsilon_{\mu\nu\varsigma\sigma}}_{\text{constant because ON basis}}\theta^{\varsigma}\wedge\theta^{\sigma}\Big) = \frac{1}{2}\sqrt{g}\,\varepsilon_{\mu\nu\varsigma\sigma}\Big(\overbrace{D\theta^{\varsigma}}^{0}\wedge\theta^{\sigma} + \theta^{\varsigma}\wedge\overbrace{D\theta^{\sigma}}^{0}\Big)$

Recall important identities: (torsionless case)

□ Structure eqn. I: used it just now

$$\varepsilon_{\gamma\delta\kappa\lambda}\, \theta^\gamma \otimes \theta^\delta \otimes \theta^\kappa \otimes \theta^\lambda$$

Use also: $\varepsilon_{\gamma\delta\kappa\lambda}\,\varepsilon_{\mu\nu\gamma\delta} = 2\left(\delta_{\kappa\mu}\delta_{\lambda\nu} - \delta_{\kappa\nu}\delta_{\lambda\mu}\right) \Longrightarrow$

(need later for derivation of the Einstein equation)

$$H_{\mu\nu}\wedge\Omega^{\mu\nu} = \frac{4}{4} R^{\mu\nu}{}_{\mu\nu}\sqrt{g}\,\theta^1\wedge\theta^2\wedge\theta^3\wedge\theta^4 = *R \checkmark$$

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Recall important identities: (torsionless case)

Structure eqn. I: used it just now

$$D\theta^i = d\theta^i + \omega^i{}_j\wedge\theta^j = 0$$

(need later for derivation of the Einstein equation)

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$$\Omega^i{}_j = d\omega^i{}_j + \omega^i{}_k\wedge\omega^k{}_j$$

(Recall: $R^{\cdot}{}_{\cdots} = \Gamma + \Gamma + \Gamma\Gamma + \Gamma\Gamma$

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constant because ON basis

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And, in the case of ON frames:

$$\omega_{\mu\nu} + \omega_{\nu\mu} = 0$$

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The main proposition: *variation, not co-derivative*

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$$\omega_{\mu\nu} + \omega_{\nu\mu} = 0$$

The main proposition:

variation, not co-derivative

Variation of the action with respect to $\delta\theta^\mu$ yields:

$$\delta(*R) = (\delta\theta^\mu) \wedge H_{\mu\nu\rho} \wedge \Omega^{\nu\rho} + d(\text{something})$$

Stokes: $\int_B df = \int_{\partial B} f$

It implies:

$$16\pi G\, \delta S_{grav} = \int_B \delta\theta^\mu \wedge H_{\mu\nu\rho} \wedge \Omega^{\nu\rho} + \int_{\partial B} (\text{something})$$

← require variation to vanish at boundary ∂B, so: $= 0$

Definition: The "energy-momentum 1-form" T_μ is defined as the solution to:

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variation, not co-derivative

Variation of the action with respect to $\delta\theta^\mu$ yields:

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Definition: The "energy-momentum 1-form" T_μ is defined as the solution to:

$$\delta S_{matter} =: \int_B \delta\theta^\mu\wedge(*T_\mu)$$

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variation, not co-derivative

Variation of the action with respect to $\delta\theta^\mu$ yields:

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Variation of the action with respect to $\delta\theta^r$ yields:

$$\delta(*R) = (\delta\theta^r) \wedge H_{r\nu s} \wedge \Omega^{\nu s} + d(\text{something})$$

Stokes: $\int_B df = \int_{\partial B} f$

<u>It implies:</u>

$$16\pi G\, \delta S_{grav} = \int_B \delta\theta^r \wedge H_{r\nu s} \wedge \Omega^{\nu s} + \int_{\partial B} (\text{something})$$

← require variation to vanish at boundary ∂B, so: $= 0$

<u>Definition:</u> The "energy-momentum 1-form" T_r is defined as the solution to:

$$\delta S_{matter} =: \int_B \delta\theta^r \wedge (*T_r)$$

⇒ The equation of motion, i.e., the Einstein equation

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$$\int_B \qquad\qquad \int_{\partial B}$$

← require variation to vanish at boundary ∂B, so: $= 0$

Definition: The "energy-momentum 1-form" T_μ is defined as the solution to:

$$\delta S_{matter} =: \int_B \delta\theta^\mu \wedge (*T_\mu)$$

⇒ The equation of motion, i.e., the Einstein equation,

$$\frac{\delta(S_{grav} + S_{matter})}{\delta\theta^\mu} = 0$$

becomes:

$$\boxed{-\frac{1}{2} H_{\mu\nu\rho} \wedge \Omega^{\nu\rho} = 8\pi G *T_\mu}$$

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Definition: The "energy-momentum 1-form" T_μ is defined as the solution to:

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$$\delta S_{matter} =: \int_B \delta\theta^\mu \wedge (*T_\mu)$$

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Exercise: add the cosmological constant.

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$\Rightarrow$ The equation of motion, i.e., the Einstein equation,

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becomes:

$$\boxed{-\frac{1}{2} H_{\mu\nu\rho} \wedge \Omega^{\nu\rho} = 8\pi G \, {*T_\mu}}$$

Exercise: add the cosmological constant.

(it is a (0,1) tensor-valued 1 form)

Remark: The Einstein form $G_\mu := G_{\mu\nu}\,\theta^\nu$ obeys

$$*G_\mu = -\frac{1}{2} H_{\mu\nu\rho} \wedge \Omega^{\nu\rho}$$

$\Rightarrow$ $\boxed{G_\mu = 8\pi G\, T_\mu}$

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Proof of the main proposition:

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Proof of the main proposition:

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Consider the first term:

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by definition of $H_{\mu\nu\rho}$ above:

$$H_{\mu\nu\rho} := \tfrac{1}{2}\sqrt{g}\,\varepsilon_{\alpha\beta\gamma\delta}\,\theta^\delta$$

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by definition of $H_{\mu\nu\rho}$ above: $H_{\mu\nu\rho} := \frac{1}{2}\sqrt{g}\, \varepsilon_{\alpha\beta\gamma\delta}\, \theta^\delta$

$$\Rightarrow \quad \delta(*R) = (\delta\theta^\mu) \wedge H_{\mu\nu\rho} \wedge \Omega^{\nu\rho} + \underbrace{H_{\mu\nu} \wedge \delta\Omega^{\mu\nu}}_{\text{examine this term:}}$$

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$$\delta H_{\mu\nu} = \delta \underbrace{\tfrac{1}{2}\sqrt{g}\,\varepsilon_{\mu\nu\rho\sigma}}^{\text{const.}}\,\theta^\rho \wedge \theta^\sigma$$

$$= (\delta\theta^\mu) \wedge H_{\mu\nu\rho}$$

by definition of $H_{\mu\nu\rho}$ above: $H_{\mu\nu\rho} := \frac{1}{2}\sqrt{g}\,\varepsilon_{\alpha\beta\gamma\delta}\,\theta^\delta$

$$\Rightarrow \quad \delta(*R) = (\delta\theta^\mu) \wedge H_{\mu\nu\rho} \wedge \Omega^{\nu\rho} + \underbrace{H_{\mu\nu} \wedge \delta\Omega^{\mu\nu}}_{\text{examine this term:}}$$

2nd structure equation

$$\delta\Omega^{\mu\nu} = \delta(d\omega^{\mu\nu} + \omega^\mu{}_\rho \wedge \omega^{\rho\nu})$$

$$= d\delta\omega^{\mu\nu} + \delta\omega^\mu{}_\rho \wedge \omega^{\rho\nu} + \omega^\mu{}_\rho \wedge \delta\omega^{\rho\nu}$$

$$\Rightarrow \quad H_{\mu\nu} \wedge \delta\Omega^{\mu\nu} = d(H_{\mu\nu} \wedge \delta\omega^{\mu\nu}) - (dH_{\mu\nu}) \wedge \delta\omega^{\mu\nu}$$

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Consider the first term:

$$\delta H_{\mu\nu} = \delta \underbrace{\tfrac{1}{2}\sqrt{g}\,\varepsilon_{\mu\nu\rho\sigma}}_{\text{const.}}\,\theta^\rho \wedge \theta^\sigma$$

$$= (\delta\theta^\rho) \wedge H_{\mu\nu\rho}$$

by definition of $H_{\mu\nu\rho}$ above: $H_{\mu\nu\rho} := \frac{1}{2}\sqrt{g}\,\varepsilon_{\alpha\beta\gamma\delta}\,\theta^\delta$

$$\Rightarrow \quad \delta({*R}) = (\delta\theta^\rho) \wedge H_{\mu\nu\rho} \wedge \Omega^{\nu\rho} + \underbrace{H_{\mu\nu} \wedge \delta\Omega^{\mu\nu}}_{\text{examine this term:}}$$

2nd structure equation

$$\delta\Omega^{\mu\nu} = \delta(d\omega^{\mu\nu} + \omega^\mu{}_\rho \wedge \omega^{\rho\nu})$$

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2nd structure equation

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$$= d\delta\omega^{\mu\nu} + \delta\omega^{\mu}{}_{\rho} \wedge \omega^{\rho\nu} + \omega^{\mu}{}_{\rho} \wedge \delta\omega^{\rho\nu}$$

$$\Rightarrow H_{\mu\nu} \wedge \delta\Omega^{\mu\nu} = d(H_{\mu\nu} \wedge \delta\omega^{\mu\nu}) - (dH_{\mu\nu}) \wedge \delta\omega^{\mu\nu}$$

$$+ H_{\mu\nu} \wedge \delta\omega^{\mu}{}_{\rho} \wedge \omega^{\rho\nu} + H_{\mu\nu} \wedge \omega^{\mu}{}_{\rho} \wedge \delta\omega^{\rho\nu}$$

$$\overset{\text{by Def. of } D}{=} (\delta\omega^{\mu\nu}) \wedge \underbrace{D H_{\mu\nu}}_{\substack{\text{recall: } = 0 \\ \text{by Prop. above.}}} + d(H_{\mu\nu} \wedge \delta\omega^{\mu\nu})$$

$\Rightarrow$ Indeed:

$$\delta(*R) = (\delta\theta^{\mu}) \wedge H_{\mu\nu\rho} \wedge \Omega^{\nu\rho} + d(H_{\mu\nu} \wedge \delta\omega^{\mu\nu}) \quad \checkmark$$

$\Rightarrow$ The Einstein eqn. indeed follows from local Lorentz invarian

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by Def. of D:

$$= (\delta\omega^{\mu\nu}) \wedge DH_{\mu\nu} + d(H_{\mu\nu} \wedge \delta\omega^{\mu\nu})$$

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2nd structure equation

$$\delta\Omega^{\mu\nu} = \delta\left(d\omega^{\mu\nu} + \omega^{\mu}{}_{\rho}\wedge\omega^{\rho\nu}\right)$$

$$= d\,\delta\omega^{\mu\nu} + \delta\omega^{\mu}{}_{\rho}\wedge\omega^{\rho\nu} + \omega^{\mu}{}_{\rho}\wedge\delta\omega^{\rho\nu}$$

$$\Rightarrow \quad H_{\mu\nu}\wedge\delta\Omega^{\mu\nu} = d\left(H_{\mu\nu}\wedge\delta\omega^{\mu\nu}\right) - \left(dH_{\mu\nu}\right)\wedge\delta\omega^{\mu\nu}$$

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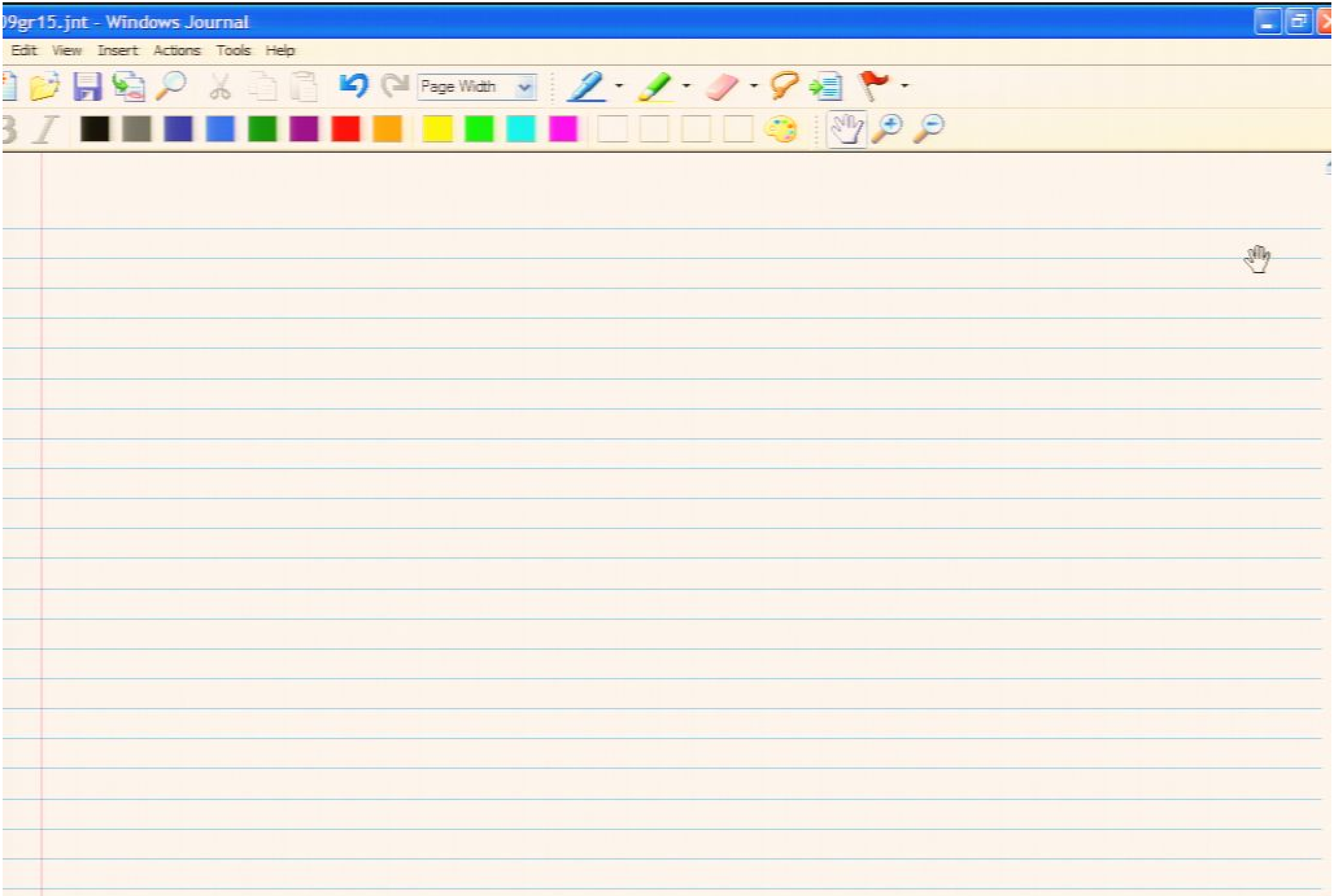
$$\delta(*R) = \left(\delta\theta^{\mu}\right)\wedge H_{\mu\nu\rho}\wedge\Omega^{\nu\rho} + d\left(H_{\mu\nu}\wedge\delta\omega^{\mu\nu}\right) \quad \checkmark$$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$$\delta\Omega = \delta(d\omega' + \omega'_{\rho} \wedge \omega'^{\,})$$

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