Title: General Relativity for Cosmology - Lecture 17

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Abstract:

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GR for Cosmology, Achim Kempf, Fall 2009, Lecture 20

Note Title 11/27/2005

Classification of cosmological models

Recall:

- The task is to solve the equations of motion of matter, jointly with the Einstein equation:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

- In practice, this problem must be simplified, i.e., the number of to-be-determined functions must be reduced.

- Q: How? A: By making symmetry assumptions.

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- Potential problem (with symmetry assumptions):

  - The so-obtained highly symmetric

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Potential problem (with symmetry assumptions):

- The so-obtained highly symmetric solutions may possess properties that are peculiar to high symmetry.

- We questioned, in particular, the robustness of the Friedmann Lemaître model's prediction of a big bang singularity.

- But singularity theorems confirm the robustness under certain conditions (such as strong energy condition).

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→ More confidence in significance of the properties of highly symmetric solutions.

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of highly symmetric solutions.

Strategy:

- Classify cosmological models $(M,g), T_{\mu\nu}$ by the amount and type of symmetry assumed.
- For each amount and type of symmetry assumed, try to find exact solutions or at least (asymptotic) properties of exact solutions.
- Focus on (M,g) with those symmetries that appear to hold in good approximation in nature → $\underbrace{\text{F.L. models}}_{\text{Friedmann Lemaitre}}$
- Notice, regarding the other high symmetry models:
  - They come arbitrarily close to F.L. at finite time

Why? All (M,g) of a symmetry equivalence

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- Notice, regarding the other high symmetry models:
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  - Quantum gravity: all (M,g) contribute to quantum path integral but highly symmetric ones contribute most

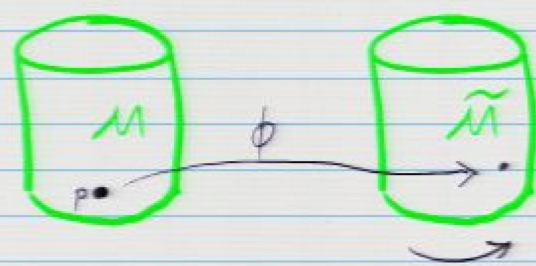
Why? All (M,g) of a symmetry equivalence class have the same action, i.e. same probability amplitude ⇒ constructive interference

Recall: Symmetries & Killing vector fields

- Two spacetimes $(\mathcal{M}, g)$, $(\tilde{\mathcal{M}}, \tilde{g})$ are isometric (and therefore of exactly identical shape) if there is a diffeomorphism $\phi: \mathcal{M} \to \tilde{\mathcal{M}}$ so that the image of the metric g in $\tilde{\mathcal{M}}$ is $\tilde{g}$: $Tg = \tilde{g}$.

- A space-time has a symmetry, if we find such a ϕ for $\tilde{\mathcal{M}} = \mathcal{M}$.

- Example:



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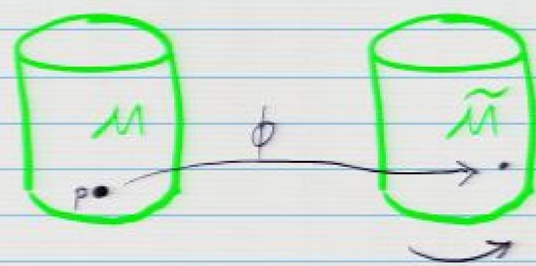

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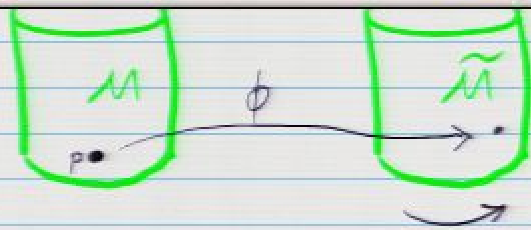

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Definition: p has an "orbit" under the action of a symmetry group.

Symmetry groups

Definition: A "group" G is a set, with an operation, say "$\circ$",

$$\circ : G \times G \to G$$

and a "neutral element", say "e", $e \in G$, such that

$$(a \circ b) \circ c = a \circ (b \circ c) \qquad \forall a, b, c \in G$$

$$a \circ e = e \circ a = a \qquad \forall a \in G$$

$$\exists a^{-1} : \; a^{-1} \circ a = a \circ a^{-1} = e \qquad \forall a \in G$$

↑ "there exists" ↑ "for all"

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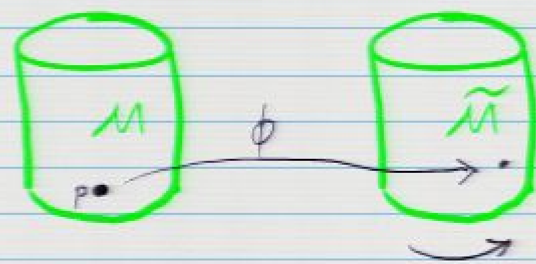

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Example: The sets of translations and rotations form Lie groups.

Remark: Often, symmetries form a group or even Lie group.

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Recall: The Lie derivative,

$$\mathcal{L}_\xi Q^{a..b}{}_{c..d} = Q^{a..b}{}_{c..d;k}\,\xi^k$$

$$- Q^{k..b}{}_{c..d}\,\xi^a{}_{;k} - \dots - Q^{a..k}{}_{c..d}\,\xi^b{}_{;k}$$

$$+ Q^{a..b}{}_{k..d}\,\xi^k{}_{;c} + \dots + Q^{a..b}{}_{c..k}\,\xi^k{}_{;d}$$

yields the rate of change of a tensor Q along the flow of diffeomorphisms ϕ generated by a vector field ξ.

→ Here, can use $\mathcal{L}_\xi$ to differentiate along symmetry group orbits.

Thus, if

$$\mathcal{L}_\xi g_{\mu\nu} = 0$$

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then ξ generates isometries $\phi: M \to M$, $g \to \tilde{g} = g$.

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□ But $L_\xi g_{\mu\nu} = 0 = \xi^{\kappa} \underbrace{g_{\mu\nu;\kappa}}_{\overset{0}{\shortparallel} \text{ always for } \Gamma, g \text{ compatibility}} + g_{\kappa\nu}\,\xi^{\kappa}{}_{;\mu} + g_{\mu\kappa}\,\xi^{\kappa}{}_{;\nu}$

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$\Rightarrow$ A vector field ξ generates a symmetry of spacetime if it obeys

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Q: Maximum number of Killing vector fields?

A: $\underline{10}$. Namely, there are 2 ways to obey Eq. (X):

a) $\xi_{\mu;\nu} = 0$ i.e. $\nabla\xi = 0$

Can have maximally 4 such indep. vectors

b) $\nabla\xi \neq 0$, but then $K_{\mu\nu} := \xi_{\mu;\nu}$ is antisymmetric

Can have at most 6 indep. such cases.

generally, in n-dimensional manifolds, by same reasoning:

$d_{max} = n + (n^2 - n)/2$
$= n(n+1)/2$

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What is a Lie algebra?

Idea: It is to be the set of infinitesimal generators of a Lie group.

Example: 3-dim rotations are generated by the angular momentum

Here: Killing vector fields generate isometries, i.e., isometry groups.

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Definition:

- A Lie algebra is a vector space A, with an operation $\{,\}$

$$\{,\}: A \times A \to A \qquad \text{"Lie bracket"}$$

obeying $\{r,s\} = -\{s,r\} \quad \forall r,s \in A$

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and $\{\{r,s\},t\} + \{\{t,r\},s\} + \{\{s,t\},r\} = 0$

(a_1, a_2, a_3)

$\frac{\partial}{\partial x}$

$$e^{a_i \frac{\partial}{\partial x_i}} f(x_1, x_2, x_3) = f(x_1+a_1, x_2+a_2, x_3+a_3)$$

by Taylor.

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Exercise: Prove this, i.e., show the following:

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1. The symmetries of any (M,g) form a group: they can be concatenated associatively, and all possess an inverse. The symmetries are differentiable parametrized by the flow $\Rightarrow$ the symmetries form a Lie group.

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3. We see here that the Killing vector fields indeed form a Lie algebra.

4. One can show that every Lie algebra generates a Lie group.

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parametrized by the flow ⇒ the symmetries form a Lie group.

2. Each Killing vector field is the infinitesimal generator of a flow of isometric diffeomorphisms. i.e. of a symmetry.

3. We see here that the Killing vector fields indeed form a Lie algebra.

4. One can show that every Lie algebra generates a Lie group.

Surfaces of homogeneity and the isotropy subgroup:

□ Definition:

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Let r be the dimension of the Lie algebra, i.e., also the dimension of the Lie group of symmetries.

- Definition:

  - Consider the set of points $O(p)$ that a point p can flow to along the Killing vector fields.

  - $O(p)$ is called the orbit of $p \in M$ under the action of the symmetry group. We denote the dimension of the orbit by s.

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- Clearly:

The dimension of an orbit cannot be larger than the dimension of the symmetry group, i.e.

$$s \leq r,$$

but $s < r$ easily happens:

- Example:
  - Consider flat $\mathbb{R}^2$ and $p = (0,0)$.
  - Then $r = r_{max} = n(n+1)/2 = \underline{\underline{3}}$ is dim. of sym. group. ($n = 2$)
  - $\Longrightarrow$ The three-dimensional Lie algebra of Killing vector fields is spanned by three Killing vector fields:

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they can be concatenated associatively, and all possess an inverse. The symmetries are differentiable parametrized by the flow $\Rightarrow$ the symmetries form a Lie group.

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Same reasoning:

$d_{max} = n + (n^2 - n)/2$

$= n(n+1)/2$

Can have maximally 4 such indep. vectors

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b) $\nabla \xi \neq 0$, but then $K_{\mu\nu} := \xi_{\mu;\nu}$ is antisymmetric

Can have at most 6 indep. such cases.

From a symmetry Lie group to a "symmetry Lie algebra":

What is a Lie algebra?

- **Idea:** It is to be the set of infinitesimal generators of a Lie group.

Example: 3-dim rotations are generated by the angular momentum

Here: Killing vector fields generate isometries, i.e., isometry groups.

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if it obeys

$$\xi_{\mu;\nu} + \xi_{\nu;\mu} = 0 \qquad (X)$$

i.e., if it is a Killing vector field.

Q: Maximum number of Killing vector fields?

A: $\underline{10}$. Namely, there are 2 ways to obey Eq. (X):

generally, in n-dimensional manifolds, by same reasoning: $d_{max} = n + (n^2 - n)/2 = n(n+1)/2$

a) $\xi_{\mu;\nu} = 0$ i.e. $\nabla\xi = 0$

Can have maximally 4 such indep. vectors

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Can have at most 6 indep. such cases.

From a symmetry Lie group to a "symmetry Lie algebra":

What is a Lie algebra?

Proposition: The set of Killing vector fields $\xi^{(i)}$ of (M,g) is a Lie algebra.

Exercise: Prove this, i.e., show the following:

Assume $\xi^{(1)}, \xi^{(2)}$ are Killing vector fields of (M,g) and $\alpha, \beta \in \mathbb{R}$.

Then: $\alpha \xi^{(1)} + \beta \xi^{(2)}$ (i.e., they form a vector space)

and $\{\xi^{(1)}, \xi^{(2)}\} := \xi^{(1)}\xi^{(2)} - \xi^{(2)}\xi^{(1)}$

are also Killing vector fields,

and the $\xi^{(i)}$ obey the Jacobi identity.

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$$e^{a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}} f(x,y) = f(x+a, y+b)$$

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- Orbit of $p = (0,0)$:

$\mathcal{O}(p) = \mathbb{R}^2$ (because generators $\frac{\partial}{}$ and $\frac{\partial}{}$

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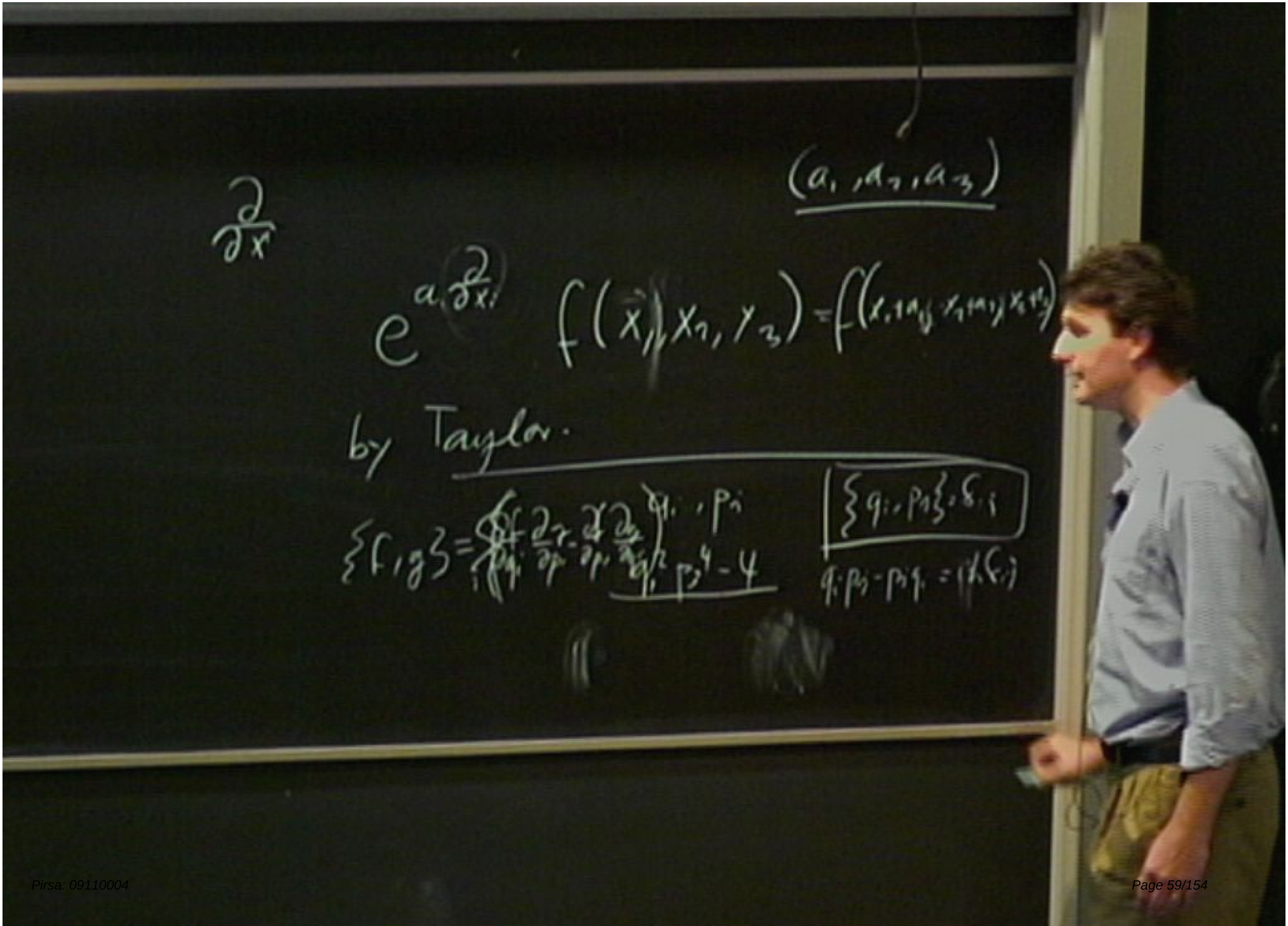
(a₁, a₂, a₃)
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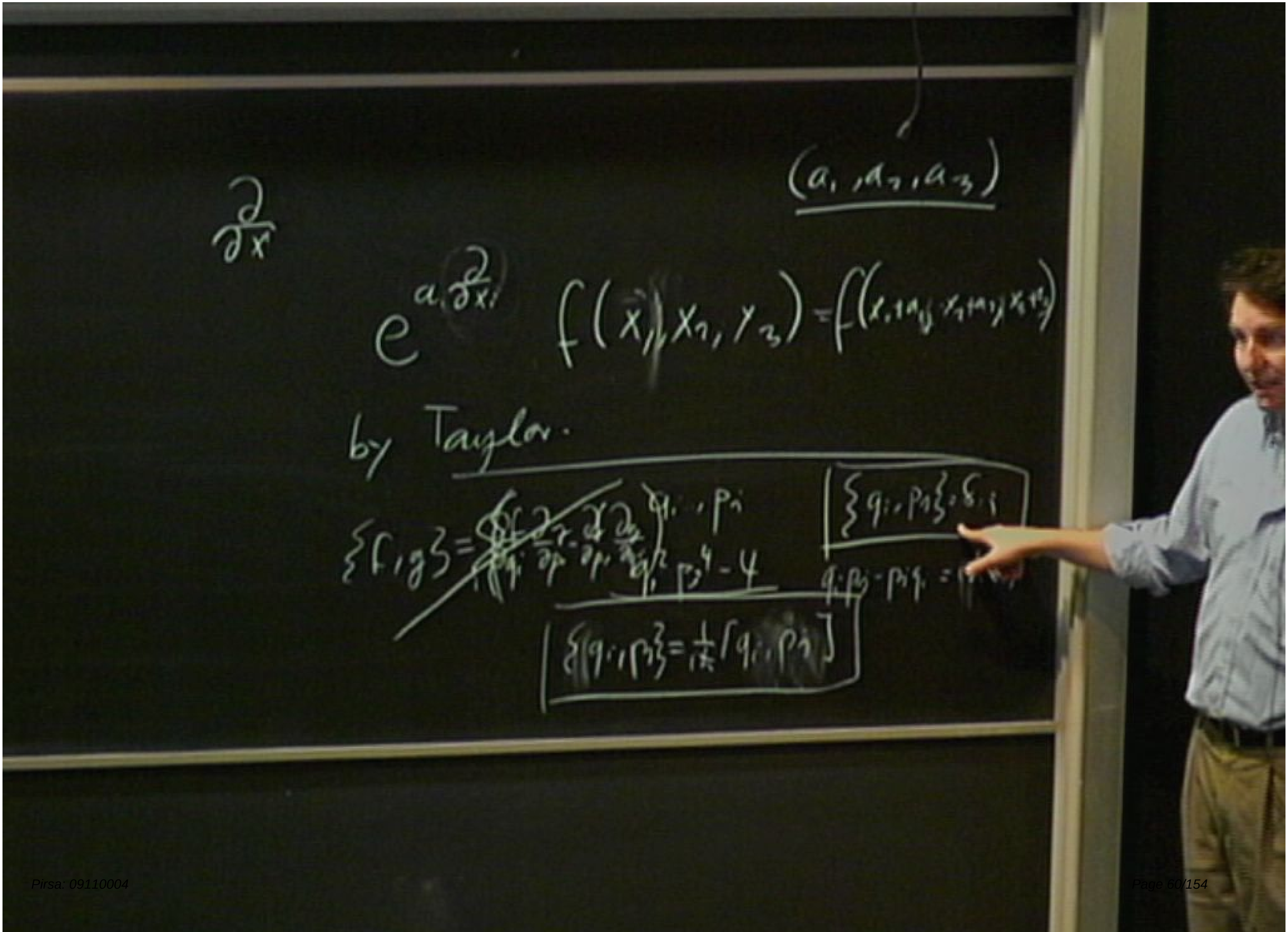

(a1, a2, a3)
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$$\frac{\partial}{\partial x}$$

$$\underline{(a_1, a_2, a_3)}$$

$$e^{a_i \frac{\partial}{\partial x_i}} f(x_1, x_2, x_3) = f(x_1+a_1, x_2+a_2, x_3+a_3)$$

by Taylor.

$$\{f, g\} = \cancel{\left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}\right)} \qquad q_i, p_i$$

$$q_1^2 p_2^4 - 4$$

$$\boxed{\{q_i, p_j\} = \delta_{ij}}$$

$$q_i p_j - p_j q_i = i\hbar\,\delta_{ij}$$

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at most 2 Killing vectors can be linearly independent at p.

- Rôle of $K^{(3)}$?

($K^{(3)}$ is the angular momentum and it of course generates rotations: $e^{\alpha K^{(3)}} f(x,y) = f(x\cos\alpha - y\sin\alpha,\ x\sin\alpha + y\cos\alpha)$)

The flow generated by $K^{(3)}$ leaves p fixed and rotates everything around p.

- Definition:

We say that those Killing vector fields which do not generate a homogeneity surface, i.e., which do not generate a nontrivial group orbit, are generating the isotropy subgroup (of the full symmetry group generated by all Killing vectors).

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Clearly: $d = r - s$

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- Examples: homogeneity isotropy

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- **Examples:**

  - homogeneity → s, isotropy → d

s	d	
4	3	Einstein's static model
4	1	Gödel's model
4	0	Ozsvath - Kerr models
3	3	Friedmann Lemaitre models
3	1	spatially hom & locally one rot. sym axis
3	0	Bianchi models

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For every physical energy momentum tensor $T_{\mu\nu}$ there exists a unique timelike vector field u so that $T_{\mu\nu}$ takes this standard form:

$$T_{ab} = \mu\, u_a u_b + q_a u_b + q_b u_a + p\,(g_{ab} + u_a u_b) + \pi_{ab}$$

where q and π are a vector field and a tensor field obeying:

$$q_a u^a = 0, \quad \pi_{ab} u^b = 0, \quad \pi_a{}^a = 0, \quad \pi_{ab} = \pi_{ba}$$

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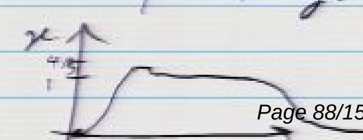

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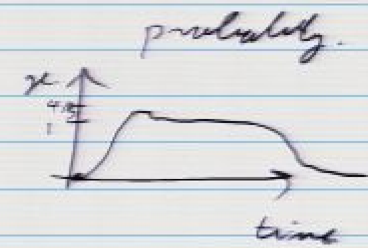

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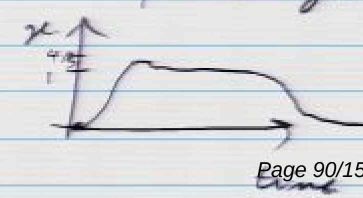

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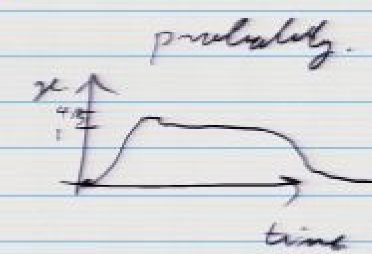

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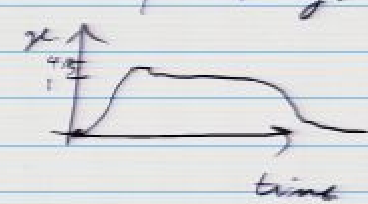

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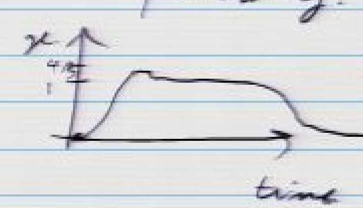

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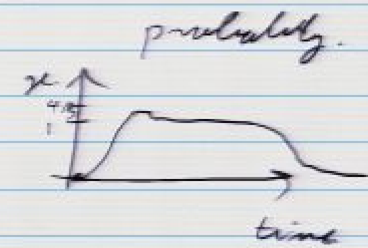

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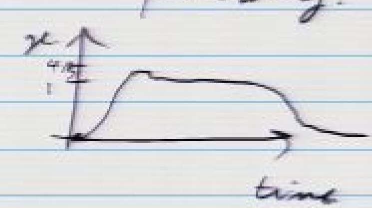

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type LIGO aims to detect. Like light, their strength decays $\sim \frac{1}{r}$ from the source.

Type I: Longitudinal gravitational waves

These waves cause a shear effect. However, they decay fast: $\sim \frac{1}{r^2}$

Why? Gravitational waves, when small enough, travel with speed of light. Like light, they then cannot oscillate longitudinally.

Types I, II: Mixtures of the above.

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We define a rate of expansion tensor that includes shear:

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- **Definition:** $H_i := \frac{1}{3}\theta_i$ Local Hubble expansion function in direction e_i.

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The l_i, l are defined as the solutions to:

$$\frac{\dot{l}_i}{l_i} = H_i$$

$$\frac{\dot{l}}{l} = H$$

Here, the time derivative is defined as:

recall: u is timelike.

$$\dot{l} = u(l) = u^\mu \frac{\partial}{\partial x^\mu} l$$

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Explicit examples are known where e.g:

- All $\ell_i \to 0$ as in FL cosmologies
- $\ell_1, \ell_2 \to 0$, $\ell_3 \to \infty$ "cigar singularity"
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