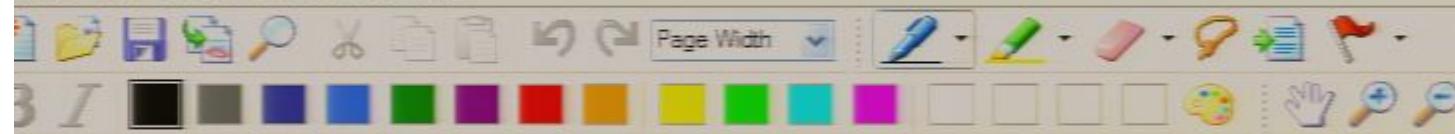


Title: General Relativity for Cosmology - Lecture 15

Date: Nov 09, 2009 04:00 PM

URL: <http://pirsa.org/09110003>

Abstract:



GR for Cosmology, Achim Kempf, Fall 2009, Lecture 17

Causal structure

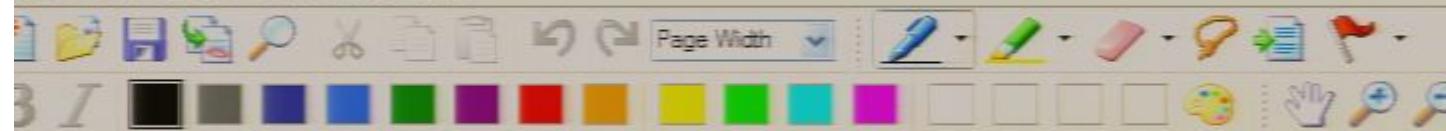
(will be important also
for singularity theorems)

A) Local causal structure

The metric, g , not only defines the "shape" of a pseudo-Riemannian manifold, it also defines what is causal and what is acausal: (by defining what is space-, null- or time-like)

Preparation: • Consider an arbitrary point $p \in M$ and an arbitrary "convex normal neighbourhood" of p , i.e. a set $U \subset M$ with $p \in U$ for which holds:

$\forall x, y \in U \rightarrow$ there exists a unique geodesic connecting x and y



GR for Cosmology, Achim Kempf, fall 2009, Lecture 1+

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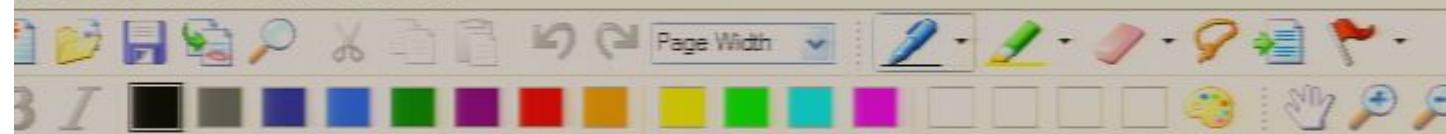
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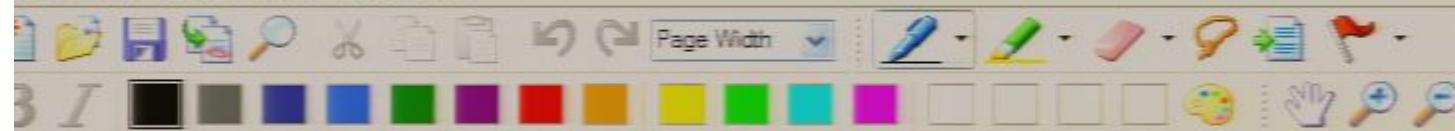
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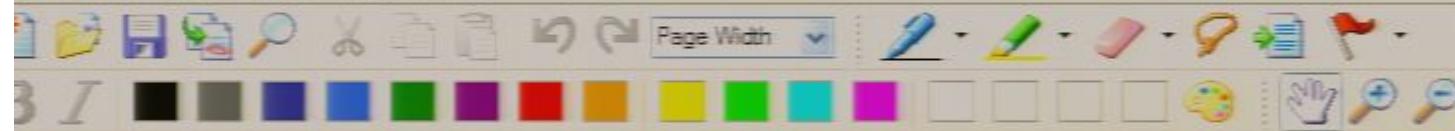
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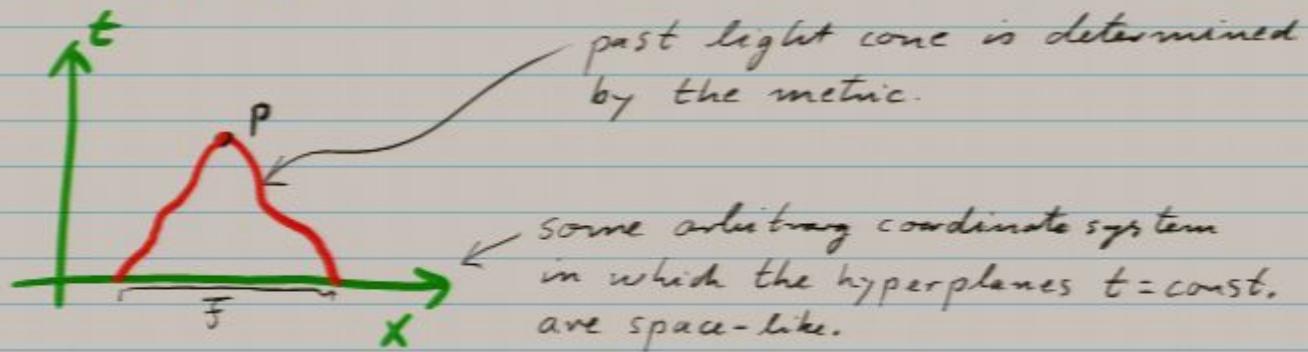
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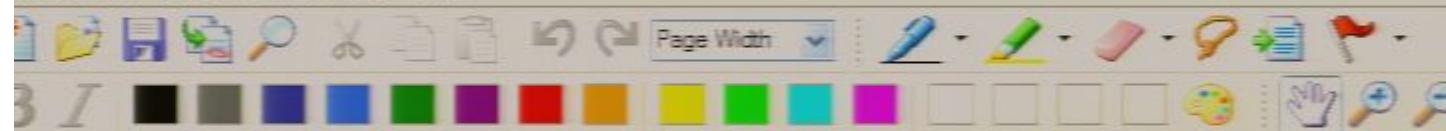


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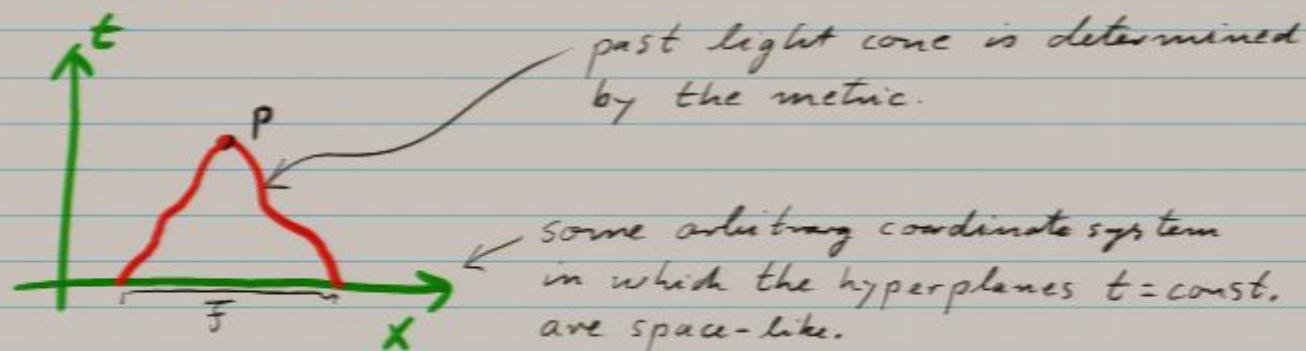


□ Definition: In order for the laws of matter fields Ψ to be called "locally causal" (and therefore reasonable), their equations of motion must allow one to calculate $\Psi(p)$ from only the values $\Psi(q)$ and finite order derivatives $\Psi^{(n)}$, for all $q \in \mathcal{F}$.

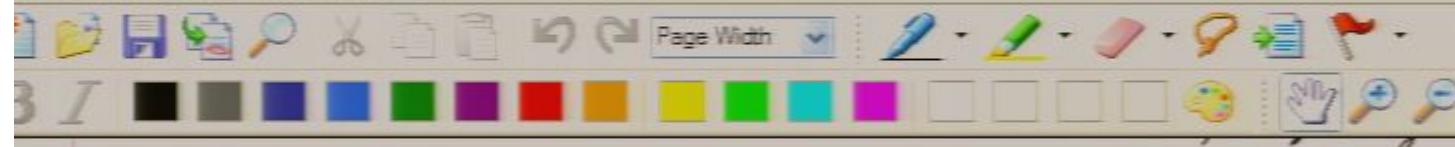


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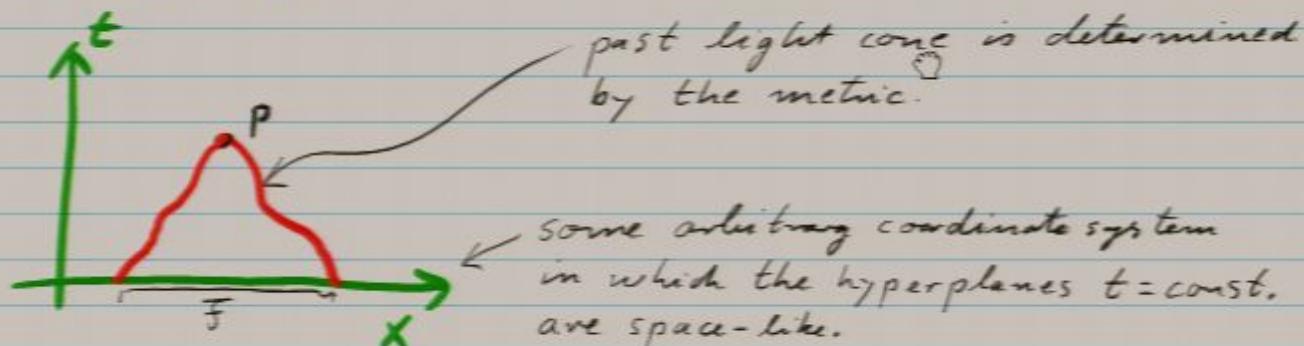


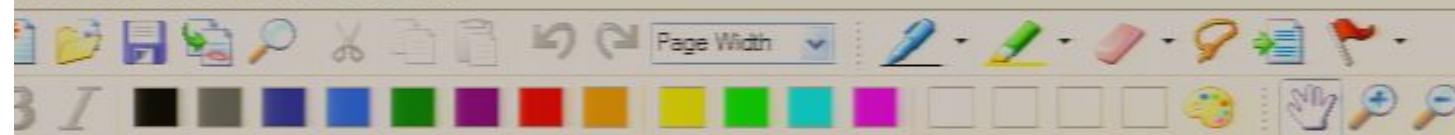
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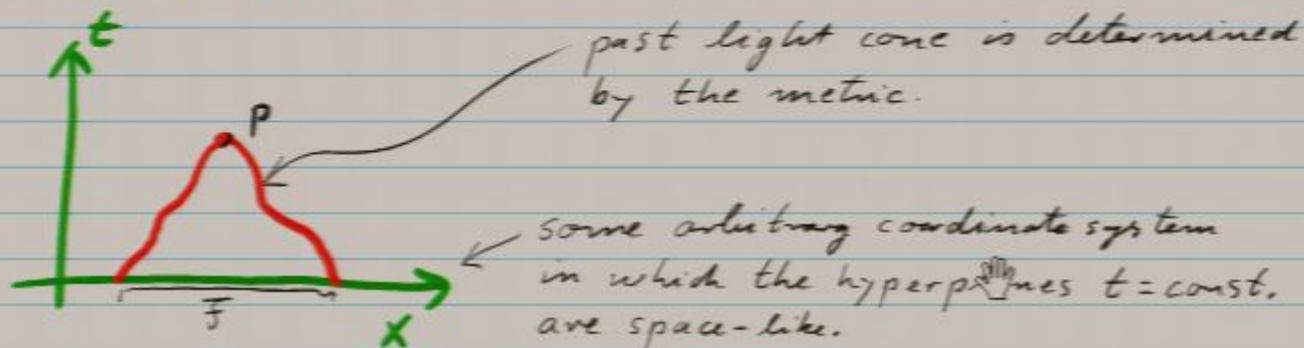
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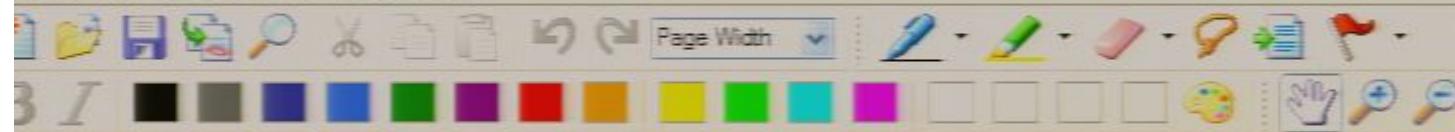




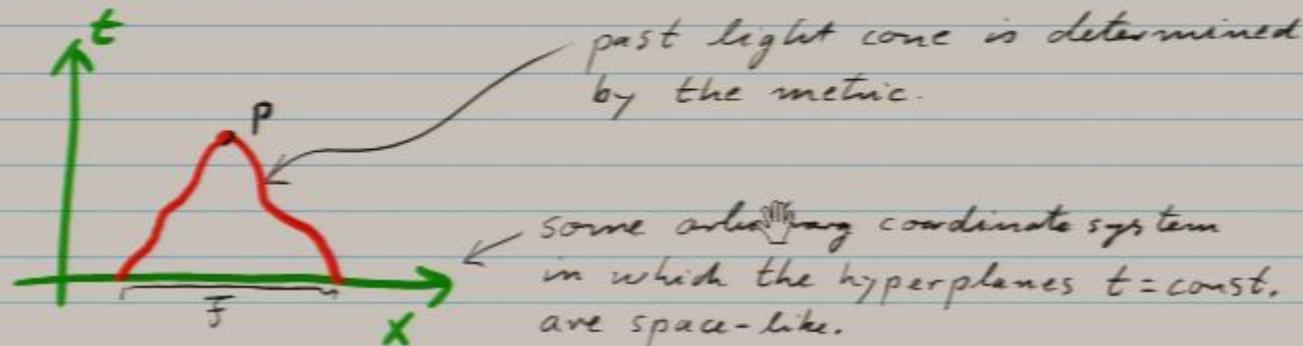
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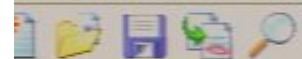
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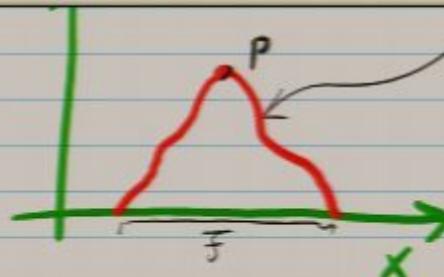
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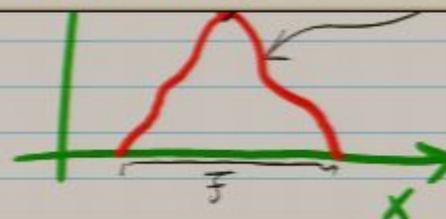
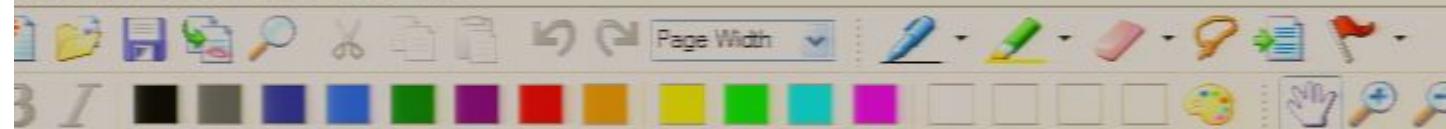


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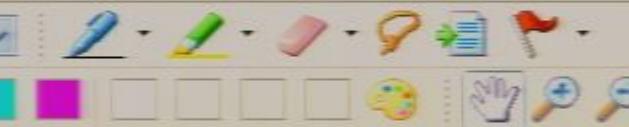
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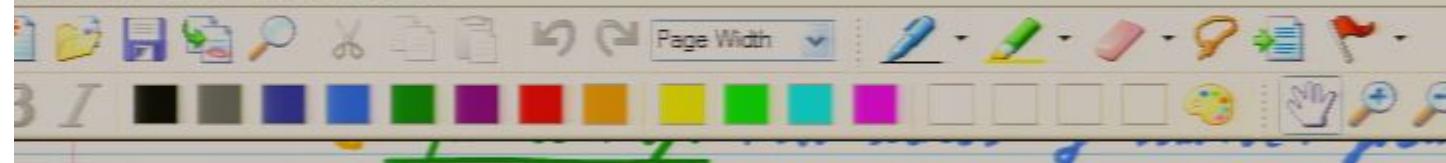


□ Equivalently: The laws of matter fields are locally causal if signals can be sent between events $q, p \in \mathcal{U}$ only iff there is a curve $\gamma \subset \mathcal{U}$ with $\gamma(t_1) = q, \gamma(t_2) = p$ whose tangents are non-space-like:

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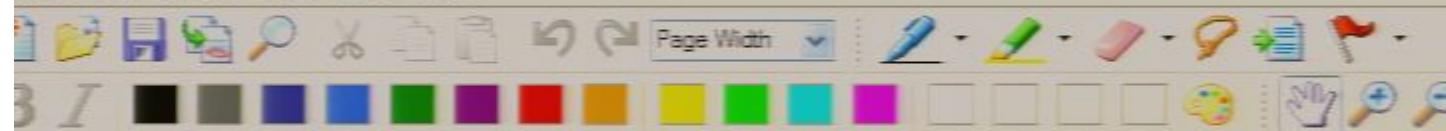
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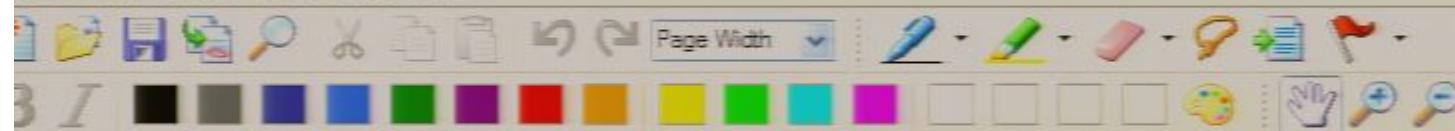
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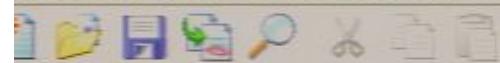
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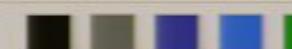
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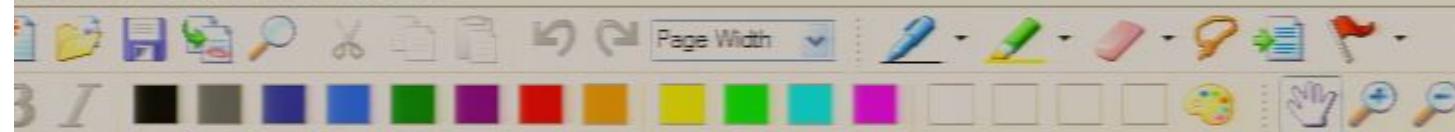
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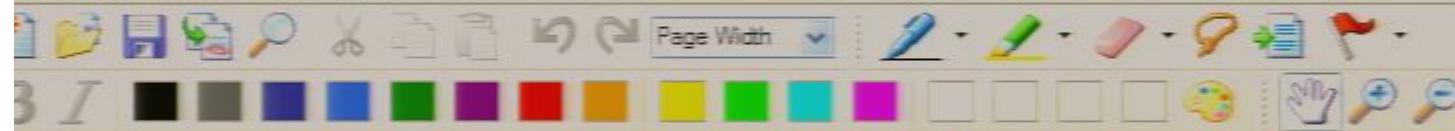
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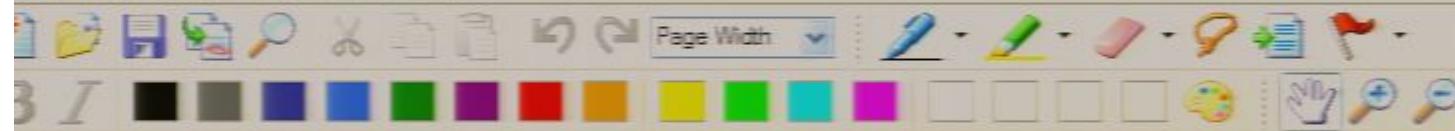
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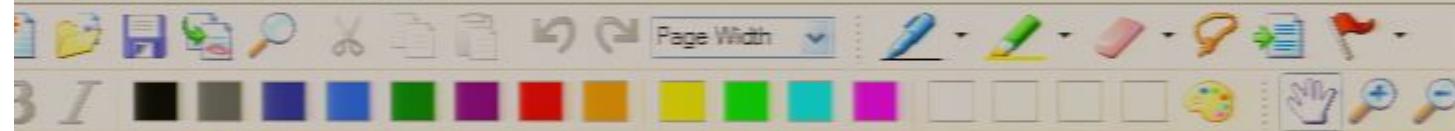
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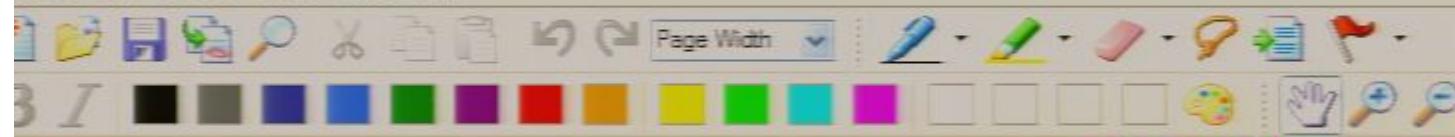


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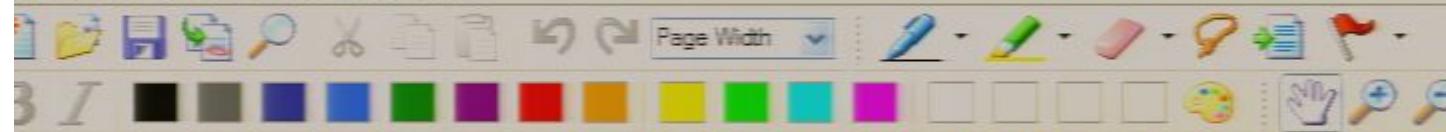
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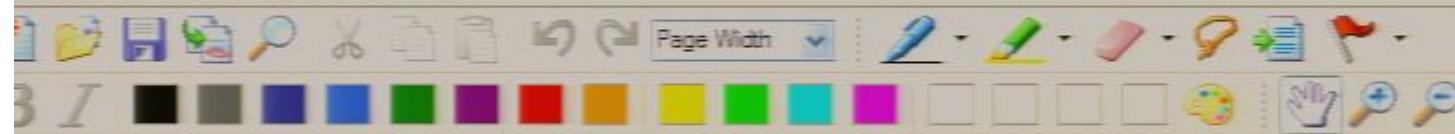
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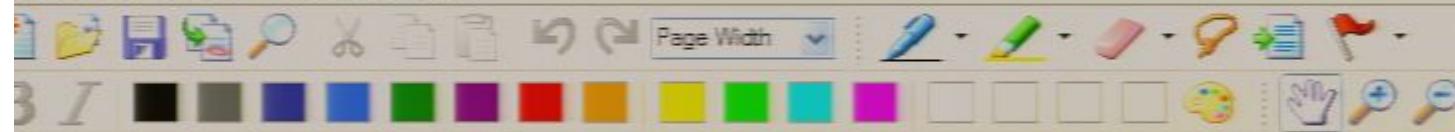
$$\cos(\varphi(\xi, \eta)) = \frac{g(\eta, \eta)}{\sqrt{g(\xi, \xi) g(\eta, \eta)}}$$

Proof: □ Consider a timelike ξ and a spacelike η .

Are there linear combinations

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that are light-like? If yes, we can assume that we know these λ from knowing the causal structure!



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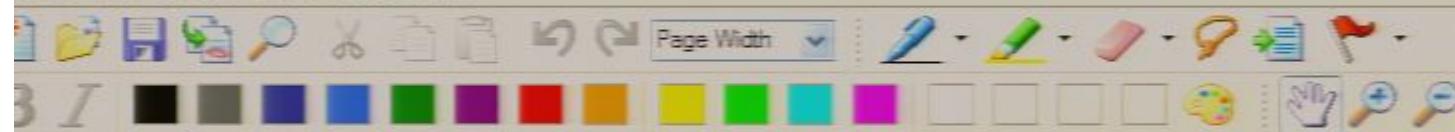
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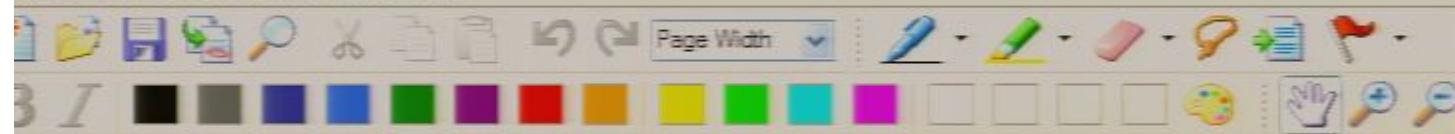
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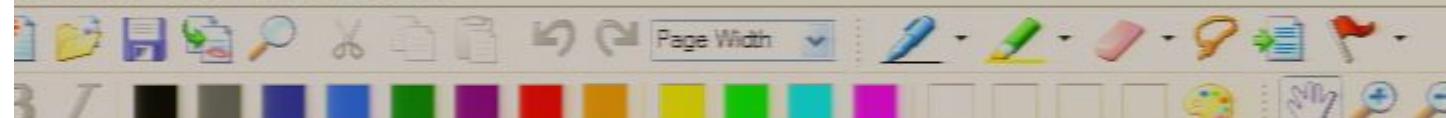
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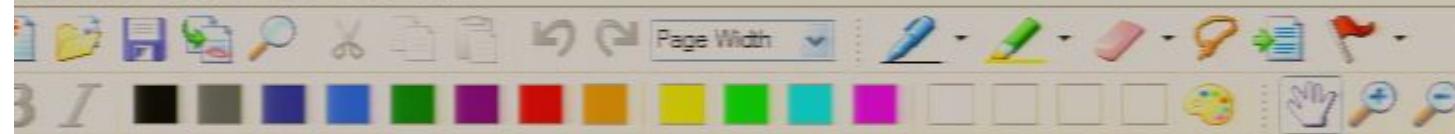
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□ Need to solve this quadratic equation in λ :

$$f(\lambda) = g(\xi + \lambda \eta, \xi + \lambda \eta) = 0 \quad (*)$$

$$\text{i.e.: } g^{\mu\nu} (\xi_\mu + \lambda \eta_\mu)(\xi_\nu + \lambda \eta_\nu) = 0$$

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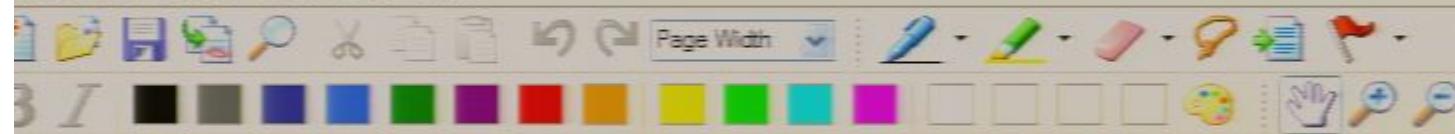
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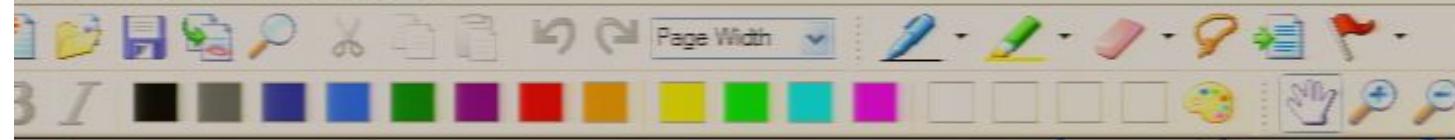
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ξ timelike $\Rightarrow f(\lambda) \leq 0$



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$$\text{i.e.: } f^*(\xi_r + \lambda\gamma_r, \xi_s + \lambda\gamma_s) = 0$$

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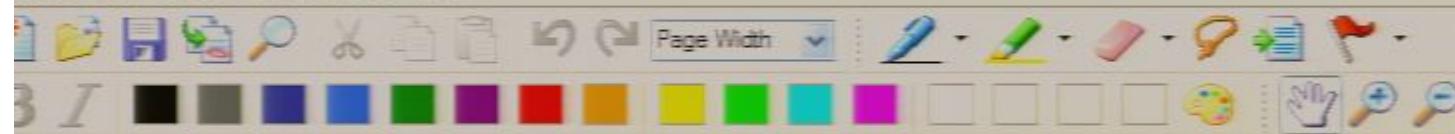
Yes:

ξ timelike $\Rightarrow f(0) < 0$

γ spacelike $\Rightarrow f(\lambda) > 0$ for large enough λ

$\Rightarrow f(\lambda) = 0$ has one real root

\Rightarrow Both roots, λ_1, λ_2 , of $f(\lambda) = 0$ are real.



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$$\text{i.e.: } g^{\mu\nu}(\xi_\mu + \lambda\gamma_\mu)(\xi_\nu + \lambda\gamma_\nu) = 0$$

□ Eq. (#) has two roots λ_1, λ_2 . Are they real?

Yes:

ξ timelike $\Rightarrow f(0) < 0$

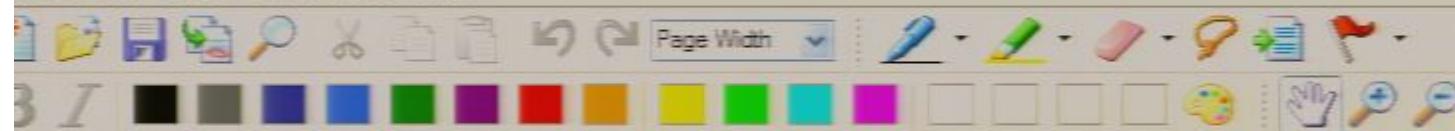
γ spacelike $\Rightarrow f(\lambda) > 0$ for large enough λ

$\Rightarrow f(\lambda) = 0$ has one real root

\Rightarrow Both roots, λ_1, λ_2 of $f(\lambda) = 0$ are real.

□ Since by assumption we can identify all null vectors

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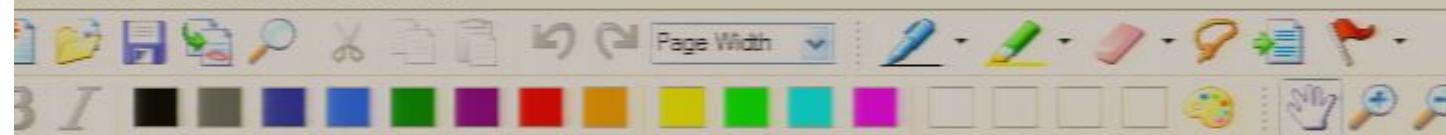
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Are there linear combinations

$$\xi + \lambda \gamma$$

that are light-like? If yes, we can assume
that we know these λ from knowing the causal structure!

□ Need to solve this quadratic equation in λ :

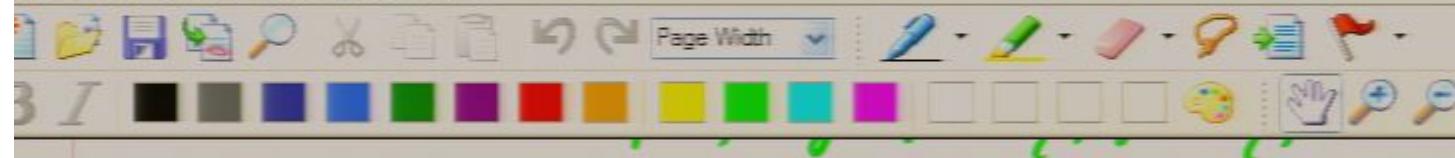
$$f(\lambda) = g(\xi + \lambda \gamma, \xi + \lambda \gamma) = 0 \quad \text{Hand} \quad (\star)$$

$$\text{i.e.: } f'(\xi_v + \lambda \gamma_v)(\xi_v + \lambda \gamma_v) = 0$$

□ Eq. (\star) has two roots λ_1, λ_2 . Are they real?

Yes:

$$\xi \text{ timelike} \Rightarrow f(0) < 0$$



$$\text{i.e.: } f''(\xi_r + 2\eta_r)(\xi_r + 2\eta_r) = 0$$

□ Eq. (A) has two roots λ_1, λ_2 . Are they real?

Yes:

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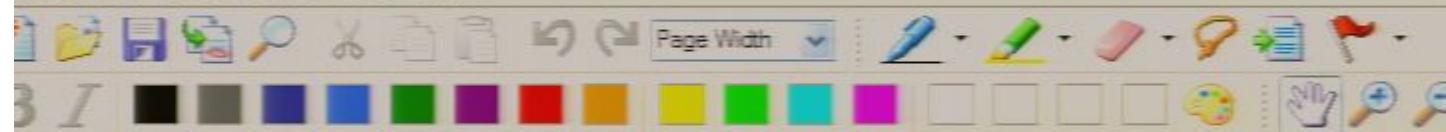
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□ Lemma:



$$\text{i.e.: } f(\xi_r - 2\eta_r)(\xi_r + 2\eta_r) = 0$$

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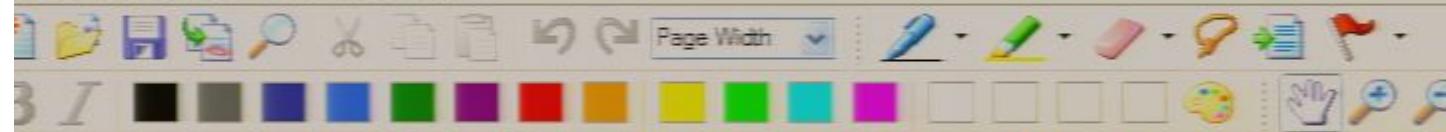
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$$g(\xi, \xi)$$



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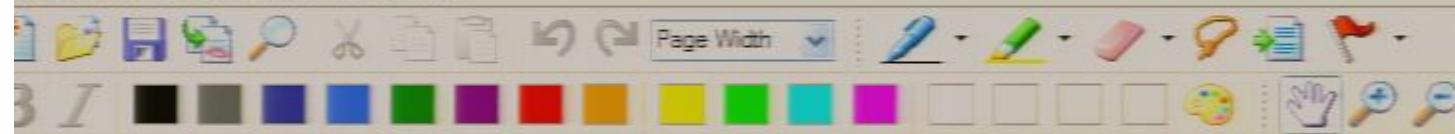
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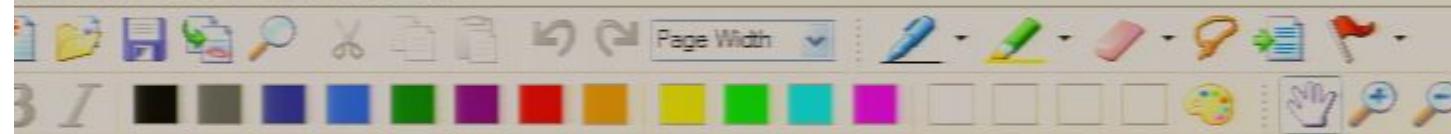
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$$\frac{g(\xi, \xi)}{g(\gamma, \gamma)} = \lambda_1 \lambda_2$$



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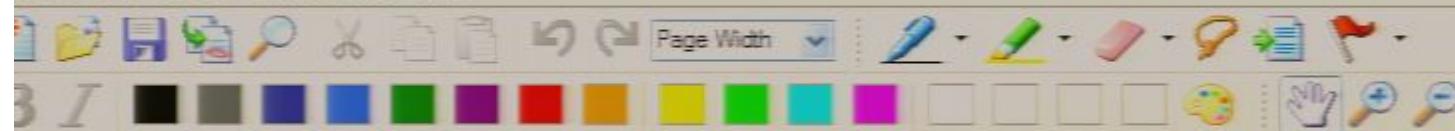
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$$\frac{g(g,g)}{g(\gamma,\gamma)} = \lambda_1 \lambda_2$$



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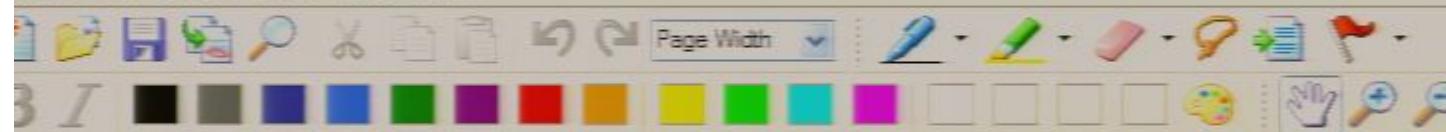
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□ Lemma:

$$\frac{g(\xi, \xi)}{g(\eta, \eta)} = \lambda_1 \lambda_2$$



Thus, the ratio $\frac{g(\xi, \xi)}{g(\eta, \eta)}$ can be assumed known
 for all timelike ξ and all spacelike η .



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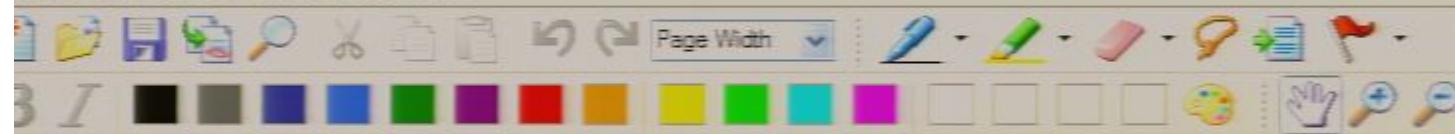
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Proof: From $g(\xi + \lambda_{1,2}\eta, \xi + \lambda_{1,2}\eta) = 0$



Lemma:

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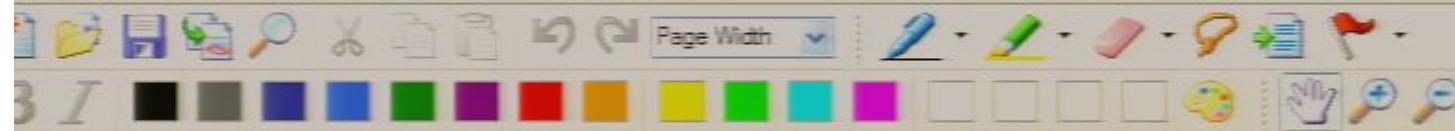
Thus, the ratio $\frac{g(\xi, \xi)}{g(\eta, \eta)}$ can be assumed known
for all timelike ξ and all spacelike η .

Proof: From $g(\xi + \lambda_{1,2} \eta, \xi + \lambda_{1,2} \eta) = 0$

we have: $g(\xi, \xi) + 2\lambda_1 g(\xi, \eta) + \lambda_1^2 g(\eta, \eta) = 0$

and: $g(\xi, \xi) + 2\lambda_2 g(\xi, \eta) + \lambda_2^2 g(\eta, \eta) = 0$

Eliminate $g(\xi, \eta)$ \Rightarrow $\frac{g(\xi, \xi)}{g(\eta, \eta)} = \lambda_1 \lambda_2$



Lemma:

$$\frac{g(\xi, \xi)}{g(\eta, \eta)} = \lambda_1, \lambda_2$$

Thus, the ratio $\frac{g(\xi, \xi)}{g(\eta, \eta)}$ can be assumed known
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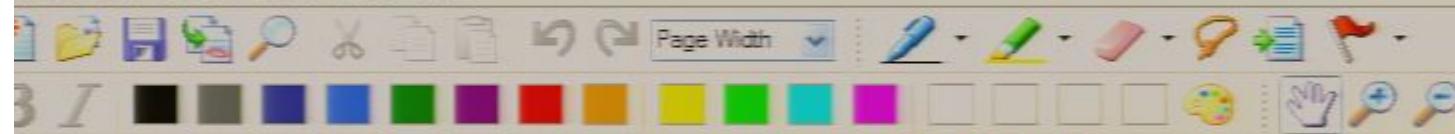
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$$\text{we have: } g(\xi, \xi) + 2\lambda_1 g(\xi, \eta) + \lambda_1^2 g(\eta, \eta) = 0$$

$$\text{and: } g(\xi, \xi) + 2\lambda_2 g(\xi, \eta) + \lambda_2^2 g(\eta, \eta) = 0$$

Eliminate $g(\xi, \eta)$ \Rightarrow $\frac{g(\xi, \xi)}{g(\eta, \eta)} = \lambda_1, \lambda_2$

Exercise: show this



$$\frac{g(\xi, \xi)}{g(\eta, \eta)} = \lambda_1 \lambda_2$$

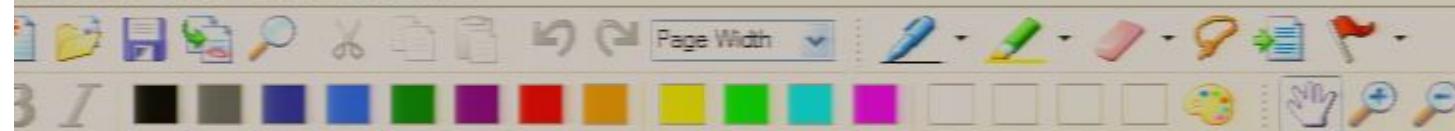
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Eliminate $g(\xi, \eta)$ \Rightarrow $\frac{g(\xi, \xi)}{g(\eta, \eta)} = \lambda_1 \lambda_2$ ✓
Exercise: show this



Thus, the ratio $\frac{g(\xi, \xi)}{g(\eta, \eta)}$ can be assumed known for all timelike ξ and all spacelike η .

Proof: From $g(\xi + \lambda_{1,2}\eta, \xi + \lambda_{1,2}\eta) = 0$

$$\text{we have: } g(\xi, \xi) + 2\lambda_1 g(\xi, \eta) + \lambda_1^2 g(\eta, \eta) = 0$$

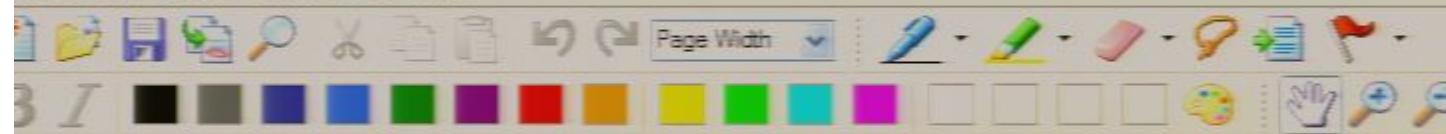
$$\text{and: } g(\xi, \xi) + 2\lambda_2 g(\xi, \eta) + \lambda_2^2 g(\eta, \eta) = 0$$

Eliminate $g(\xi, \eta)$ \Rightarrow $\frac{g(\xi, \xi)}{g(\eta, \eta)} = \lambda_1 \lambda_2$ ✓

Exercise: show this

□ Corollary:

Also the ratios $\frac{g(\xi, \xi)}{g(\xi', \xi')}$ for ξ, ξ' both timelike



□ Corollary:

Also the ratios $\frac{g(\xi, \xi)}{g(\xi', \xi')}$ for ξ, ξ' both timelike
(or both spacelike) can be assumed known:

$$\frac{g(\xi, \xi)}{g(\eta, \eta)} = \lambda_1 \cdot \lambda_2 \text{ ; } \frac{g(\xi', \xi')}{g(\eta, \eta)} = \lambda'_1 \cdot \lambda'_2 \Rightarrow \frac{g(\xi', \xi')}{g(\xi, \xi)} = \frac{\lambda'_1 \cdot \lambda'_2}{\lambda_1 \cdot \lambda_2}$$

□ Corollary:



Consider arbitrary non-null vectors α, β .

Then

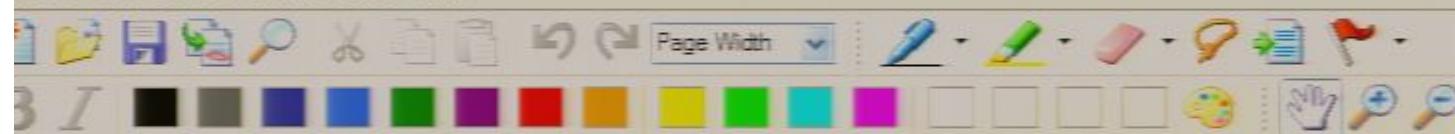
$$g(\alpha, \beta) = \frac{-1}{2} \left[g(\alpha, \alpha) + g(\beta, \beta) - g(\alpha + \beta, \alpha + \beta) \right]$$

and thus:

By lemma, all these ratios can be assumed known

We can consider $g(\xi, \xi)$ to
be a fixed, unknown scalar

$$\frac{g(\alpha, \beta)}{g(\xi, \xi)} = \frac{-1}{2} \left[\underbrace{\frac{g(\alpha, \alpha)}{g(\xi, \xi)}}_{\lambda_1} + \underbrace{\frac{g(\beta, \beta)}{g(\xi, \xi)}}_{\lambda_2} - \underbrace{\frac{g(\alpha + \beta, \alpha + \beta)}{g(\xi, \xi)}}_{\lambda_3} \right]$$



$$\frac{g(\xi, \xi)}{g(\eta, \eta)} = \lambda_1 \cdot \lambda_1 \text{ ; } \frac{g(\eta', \eta')}{g(\xi, \xi)} = \lambda_2 \cdot \lambda_2 \Rightarrow \frac{g(\xi, \xi')}{g(\eta, \eta)} = \frac{\lambda_1 \cdot \lambda_1'}{\lambda_2 \cdot \lambda_2}$$

□ Corollary:

Consider arbitrary non-null vectors α, β .

Then

$$g(\alpha, \beta) = \frac{-1}{2} [g(\alpha, \alpha) + g(\beta, \beta) - g(\alpha + \beta, \alpha + \beta)]$$

and thus:

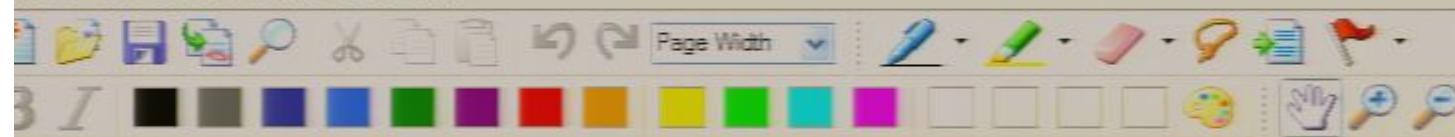
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function.

$$\frac{g(\alpha, \beta)}{g(\xi, \xi)} = \frac{-1}{2} \left[\underbrace{\frac{g(\alpha, \alpha)}{g(\xi, \xi)}}_{\text{fixed}} + \underbrace{\frac{g(\beta, \beta)}{g(\xi, \xi)}}_{\text{fixed}} - \underbrace{\frac{g(\alpha + \beta, \alpha + \beta)}{g(\xi, \xi)}}_{\text{unknown}} \right]$$

□ Conclusion:

Therefore, if it is known which vectors are timelike,



□ Corollary:

Consider arbitrary non-null vectors α, β .

Then

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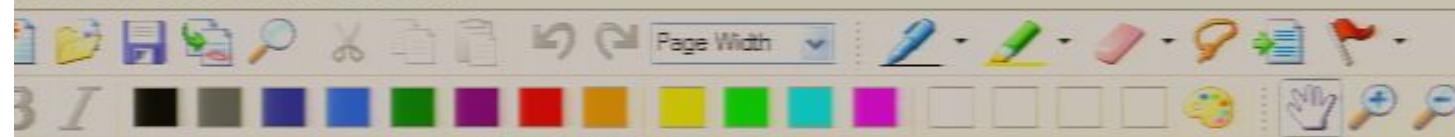
$$\frac{g(\alpha, \beta)}{g(\xi, \xi)} = \frac{-1}{2} \left[\underbrace{\frac{g(\alpha, \alpha)}{g(\xi, \xi)}}_{\text{fixed}} + \underbrace{\frac{g(\beta, \beta)}{g(\xi, \xi)}}_{\text{fixed}} - \underbrace{\frac{g(\alpha + \beta, \alpha + \beta)}{g(\xi, \xi)}}_{\text{fixed}} \right]$$



□ Conclusion:

Therefore, if it is known which vectors are timelike,
spacelike or null, then it is possible to calculate

$g(\alpha, \beta)$ at all $\alpha \in M$ for all $\beta \in T(M)$



Consider arbitrary non-null vectors α, β .

Then

$$g(\alpha, \beta) = \frac{-1}{2} [g(\alpha, \alpha) + g(\beta, \beta) - g(\alpha + \beta, \alpha + \beta)]$$

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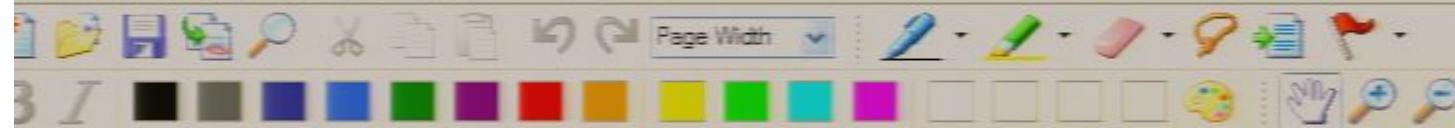
$$\frac{g(\alpha, \beta)}{g(\xi, \xi)} = \frac{-1}{2} \left[\frac{g(\alpha, \alpha)}{g(\xi, \xi)} + \frac{g(\beta, \beta)}{g(\xi, \xi)} - \frac{g(\alpha + \beta, \alpha + \beta)}{g(\xi, \xi)} \right]$$



Conclusion:

Therefore, if it is known which vectors are timelike,
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$g(\alpha, \beta)$ at all $p \in M$ for all $\alpha, \beta \in T_p(M)$



Consider arbitrary non-null vectors α, β .

Then

$$g(\alpha, \beta) = \frac{-1}{2} [g(\alpha, \alpha) + g(\beta, \beta) - g(\alpha + \beta, \alpha + \beta)]$$

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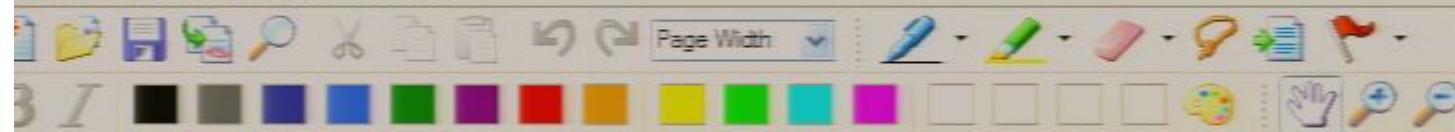
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□ Conclusion:

Therefore, if it is known which vectors are timelike, spacelike or null, then it is possible to calculate

$g(d, \beta)$ at all $\rho \in M$ for all $d, \beta \in T_p(M)$

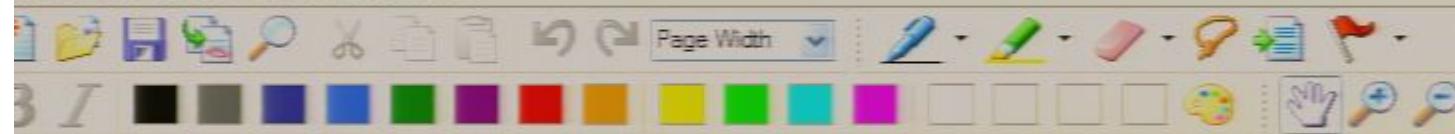
up to a scalar prefactor. \Rightarrow Proof of Theorem complete.

□ Interpretation:



The causal structure alone already determines:

- the "angles" between vectors precisely
- the "lengths" of vectors up to a scalar function.



□ Conclusion:

Therefore, if it is known which vectors are timelike, spacelike or null, then it is possible to calculate

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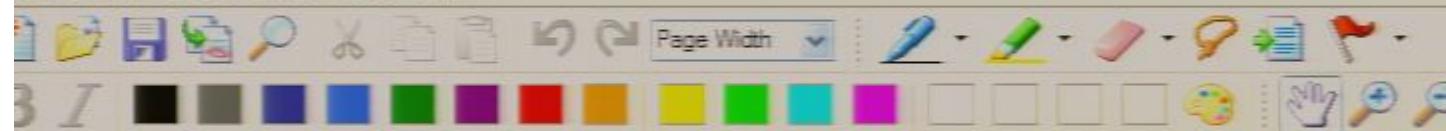
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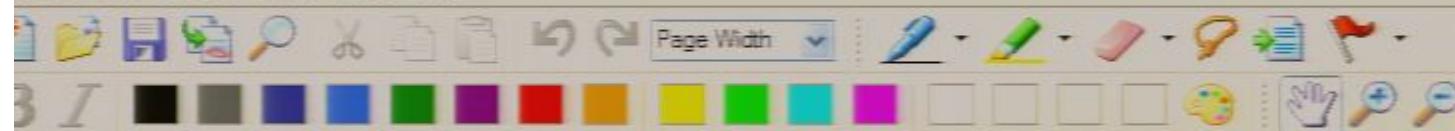
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Therefore, if it is known which vectors are timelike,
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We can consider $g(\xi, \xi)$ to
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$$\frac{g(\alpha, \beta)}{g(\xi, \xi)} = \frac{-1}{2} \left[\underbrace{\frac{g(\alpha, \alpha)}{g(\xi, \xi)} + \frac{g(\beta, \beta)}{g(\xi, \xi)}}_{\text{known}} - \underbrace{\frac{g(\alpha + \beta, \alpha + \beta)}{g(\xi, \xi)}}_{\text{unknown}} \right]$$

□ Conclusion:

Therefore, if it is known which vectors are timelike,
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$g(\alpha, \beta)$ at all $\rho \in M$ for all $\alpha, \beta \in T_p(M)$

up to a scalar prefactor. \Rightarrow Proof of Theorem complete.

□ Interpretation:



We can consider $g(\xi, \xi)$ to be a fixed, unknown scalar function.

and others:

By lemma, all these values can be assumed known.

$$\frac{g(\alpha, \beta)}{g(\xi, \xi)} = \frac{1}{2} \left[\frac{g(\alpha, \alpha)}{g(\xi, \xi)} + \frac{g(\beta, \beta)}{g(\xi, \xi)} - \frac{g(\alpha + \beta, \alpha + \beta)}{g(\xi, \xi)} \right]$$

□ Conclusion:

Therefore, if it is known which vectors are timelike, spacelike or null, then it is possible to calculate



$g(\alpha, \beta)$ at all $\rho \in M$ for all $\alpha, \beta \in T_p(M)$

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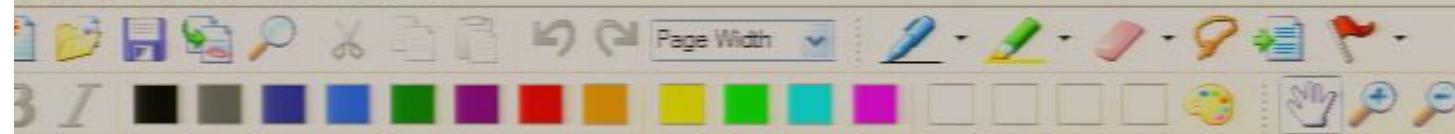
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The causal structure alone already determines:

- the "angles" between vectors precisely
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Remarks



$g(\alpha, \beta)$ at all $\rho \in M$ for all $\alpha, \beta \in T_p(M)$

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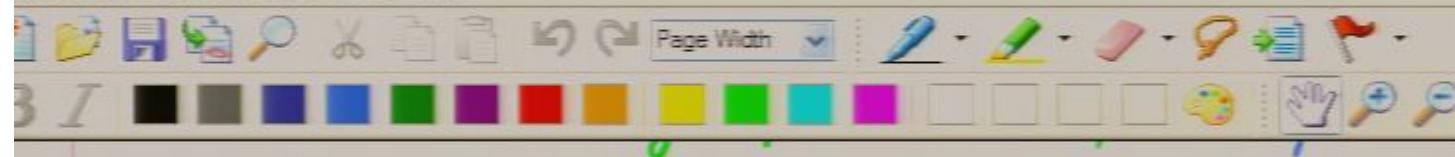
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Remarks

□ Spacetimes (M, g) and (M, \tilde{g}) for which



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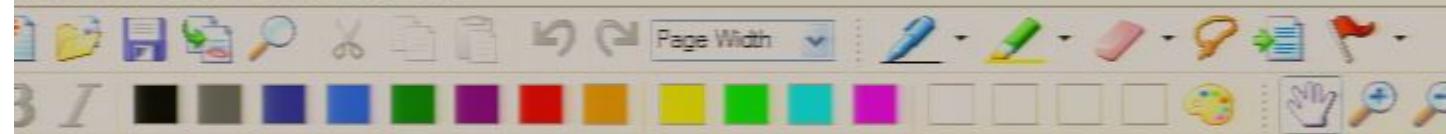
Remarks

- Spacetimes (M, g) and (M, \tilde{g}) for which

$$\tilde{g} = \phi g$$

(if $\phi > 0$ then \tilde{g} not invertible)
(if $\phi < 0$ then change signature)

some positive scalar function



up to a scalar prefactor. \Rightarrow Proof of Theorem complete.

Interpretation:

The causal structure alone already determines:

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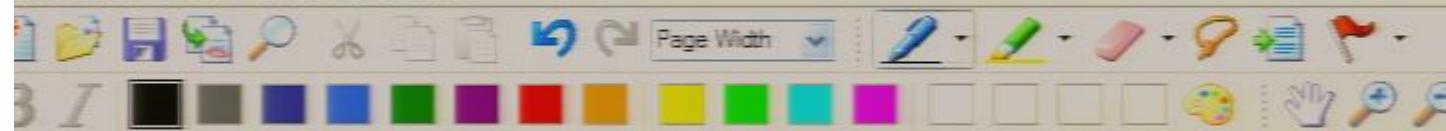
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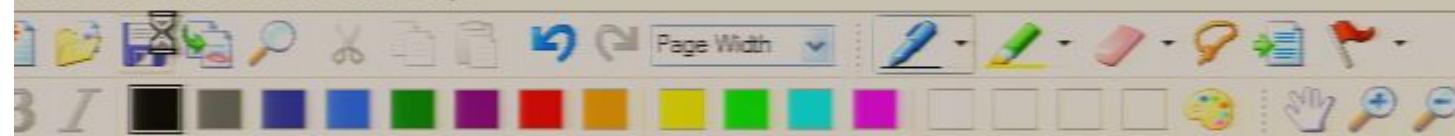
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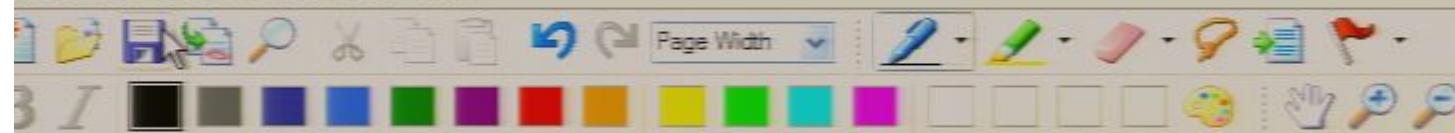
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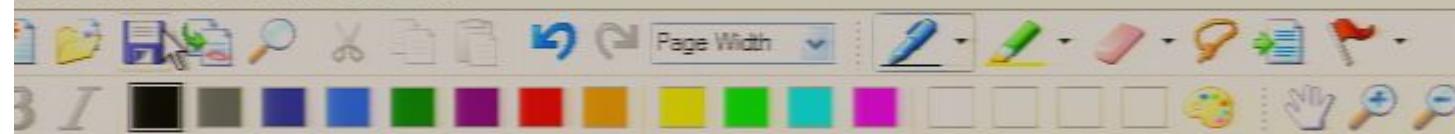
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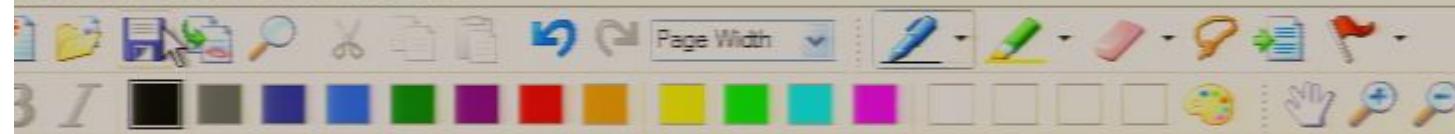


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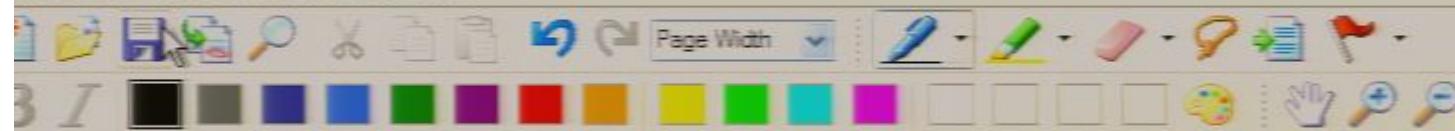
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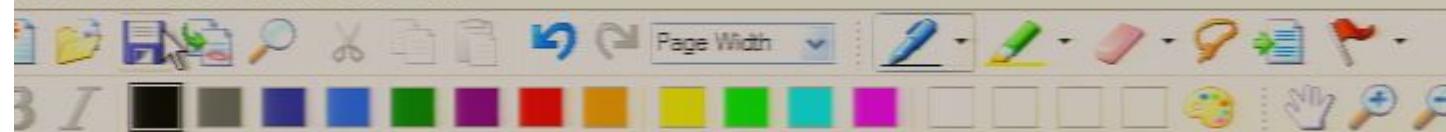
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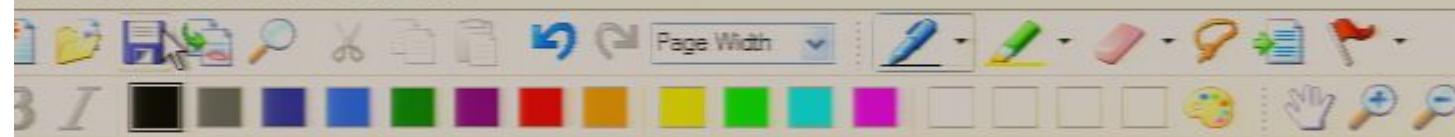
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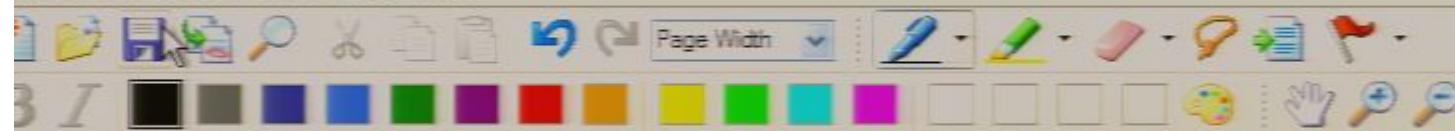
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$$R_{sq}^{am} = C_{sq}^{am} + \frac{1}{2} (g_s^i R_{iq}^m + g_i^m R_{sq}^i - g_i^q R_{sq}^i - g_q^s R_{is}^m) - \frac{1}{6} (g_s^i g_i^m - g_i^q g_q^s) R$$

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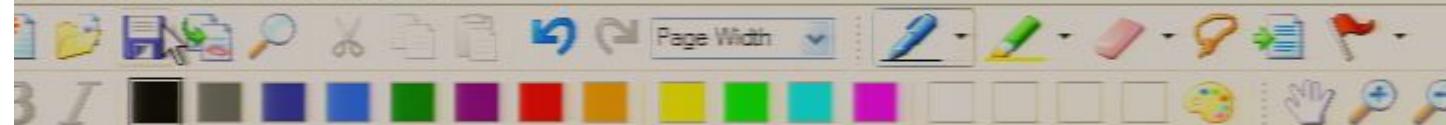
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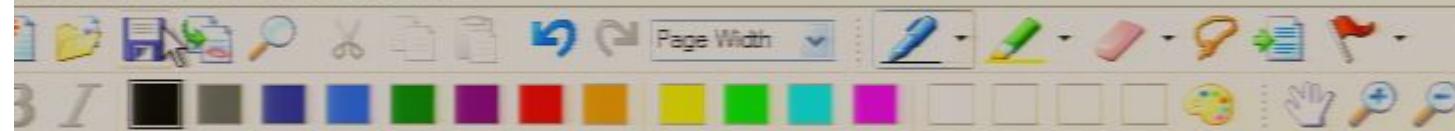
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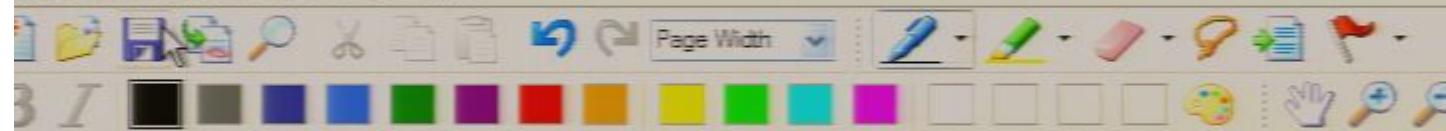
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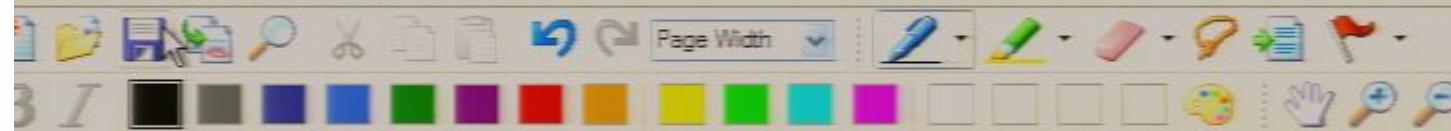
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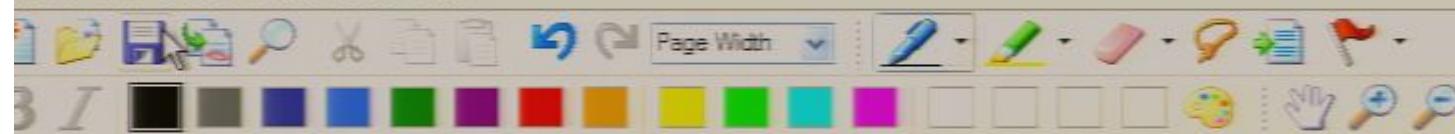
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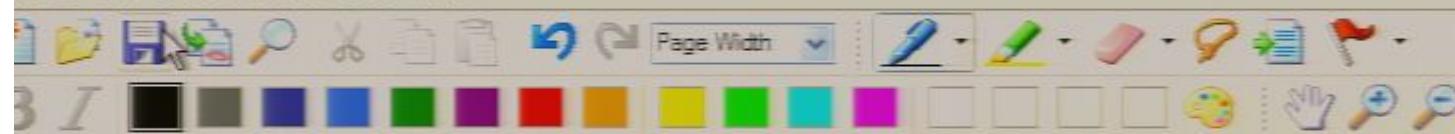
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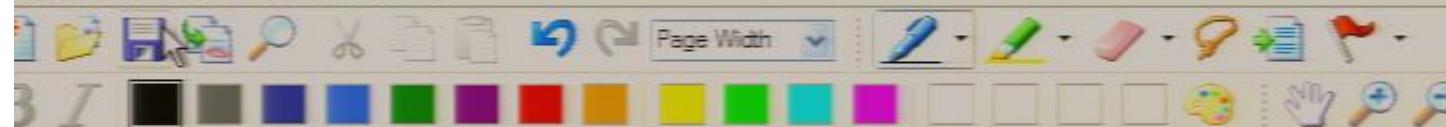
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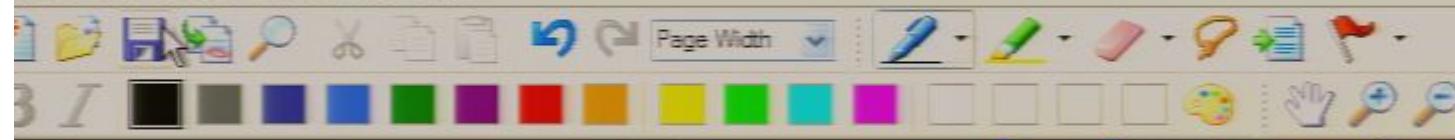
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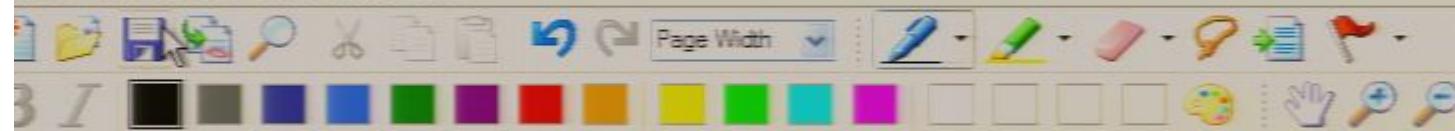
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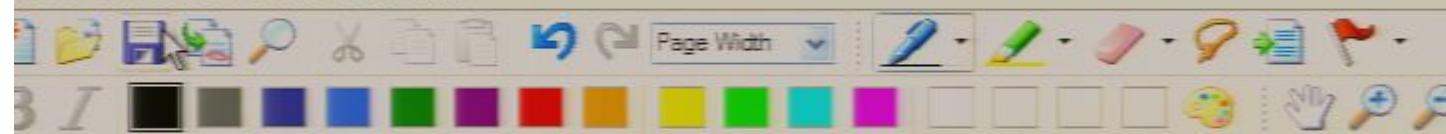
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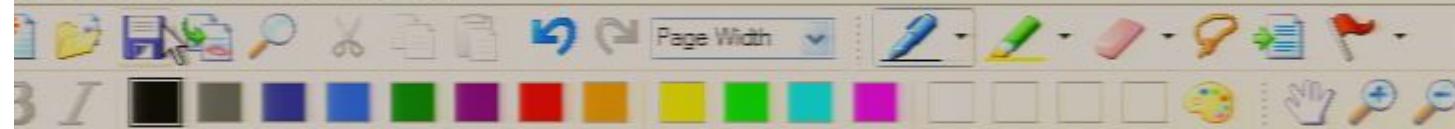
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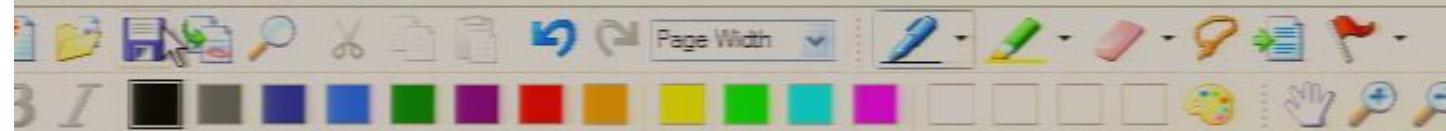
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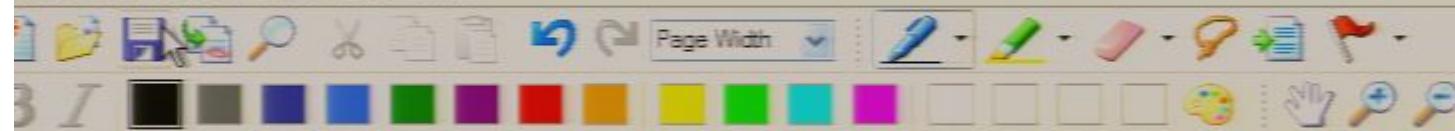
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$$R = 8\pi G T^{\mu}_{\mu}$$

No gravity was
here because
 $T^{\mu}_{\nu} = 0$ (Minkowski)

- Equivalence principle ok.
- Light bending & Mercury perihelion shift wrong

In electromagnetism $T^{(EM)}_{\mu\nu} = 0$
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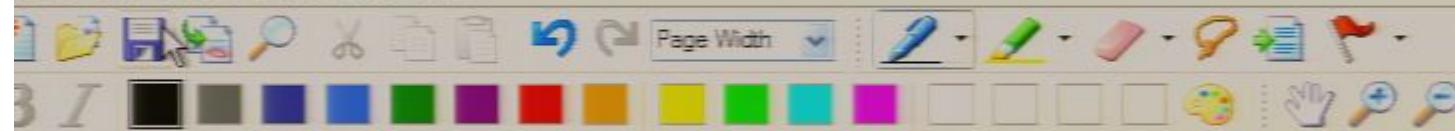
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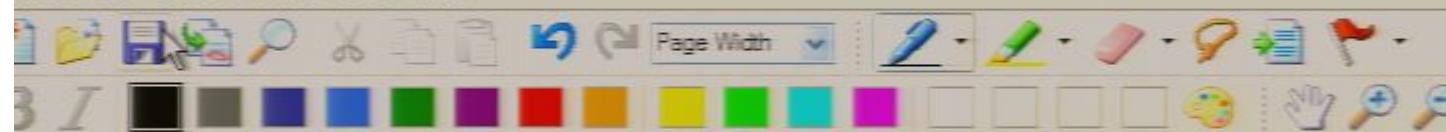
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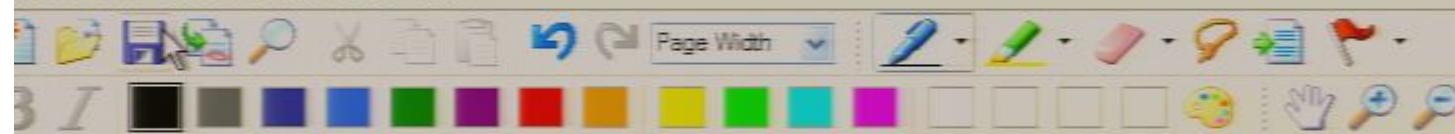
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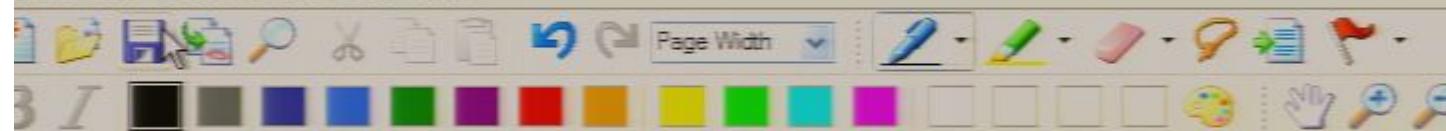
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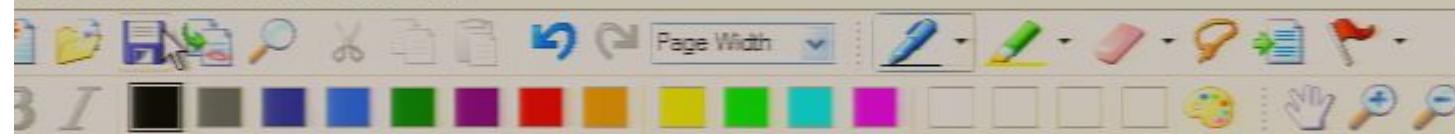
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if at each point $p \in M$ if we can separate the
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Newton gravity
does come out
correctly as a
limiting case!

Then, $S = \int_M g \sqrt{g} d^4x + \int_{\text{matter}} L_m \sqrt{g} d^4x$ and $\frac{\delta S}{\delta g_{\mu\nu}} = 0$
yield:

$$R = 8\pi G T^{\mu}_{\mu}$$

No gravity was
here because
 $C^{ab}_{cd} = C^{ab}_{cd}(\text{Minkowski})$
 $= 0$

Equivalence principle ok.

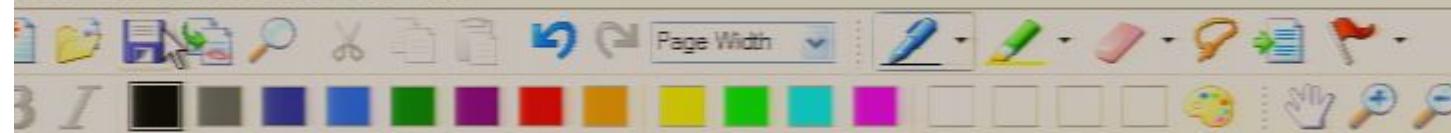
In electromagnetism $T^{(EM)}_{\mu\nu} = 0$
i.e. EM fields would not gravitate.

Light bending & Mercury perihelion shift wrong.

B) Global causal structure

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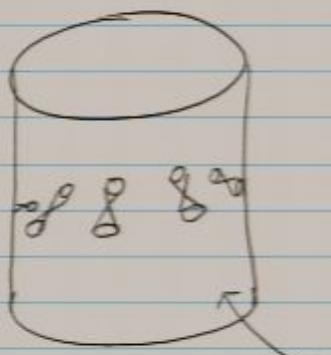


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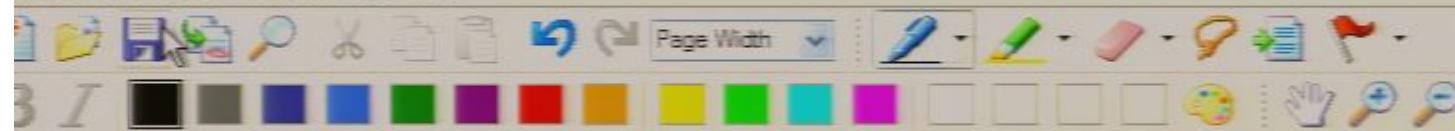
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into two classes, which will be called future-directed and past-directed so that this separation is continuous in M .

Consider e.g. such a spacetime, which is the outside of a cylinder.

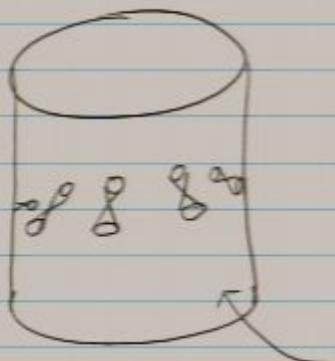
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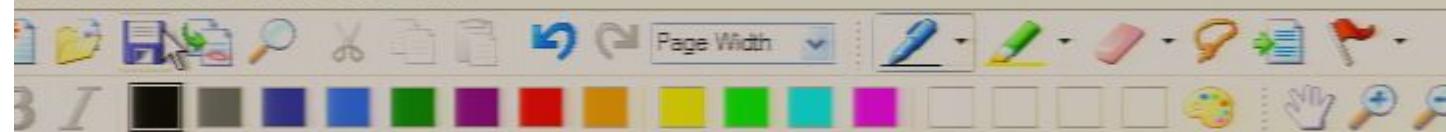
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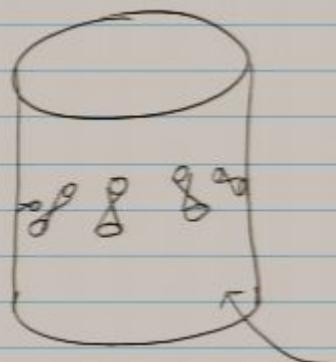
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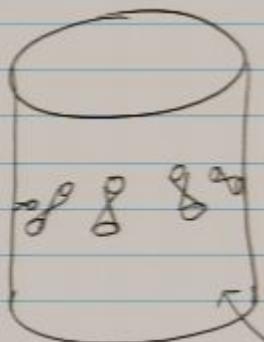
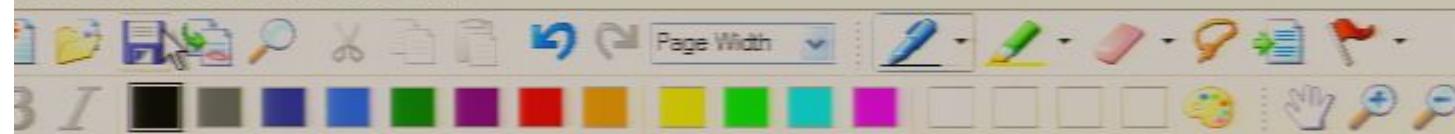
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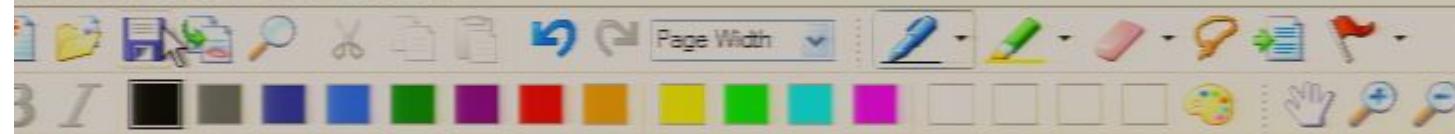
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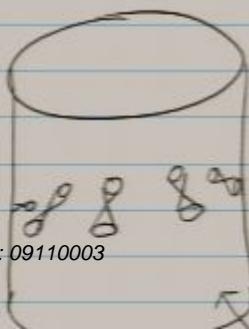


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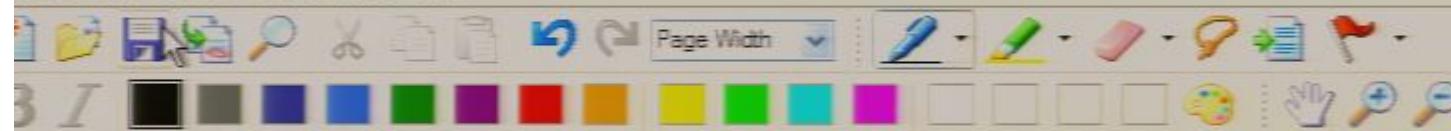
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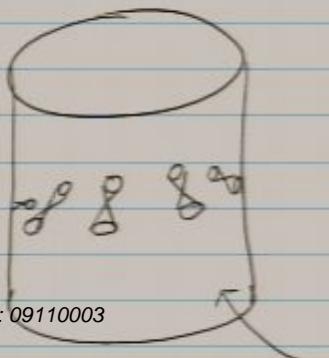


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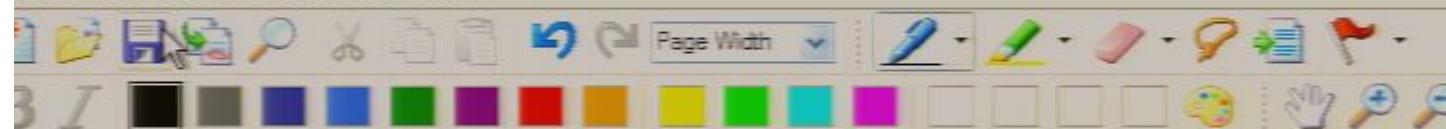
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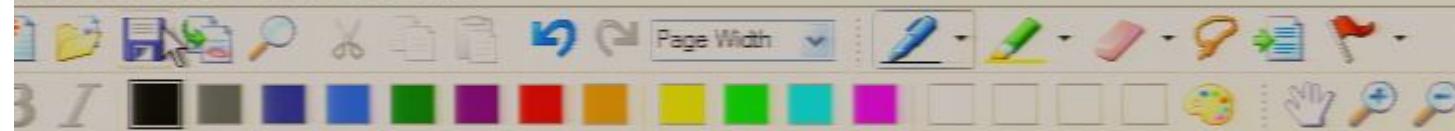
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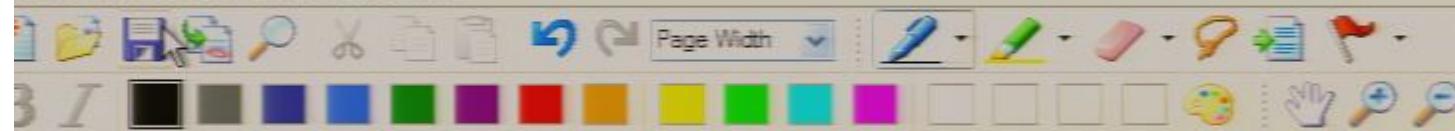
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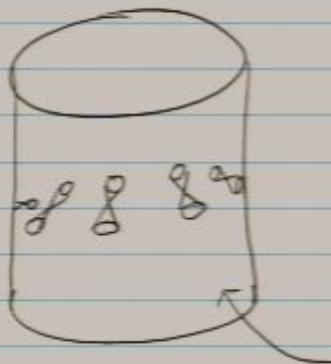
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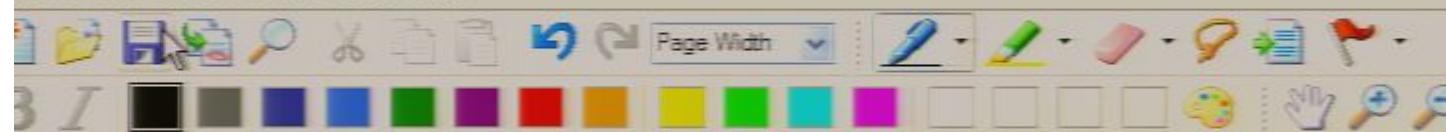
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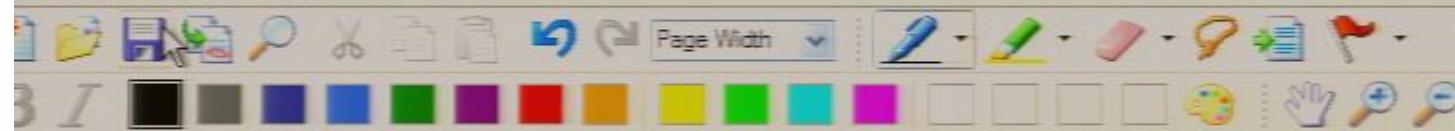
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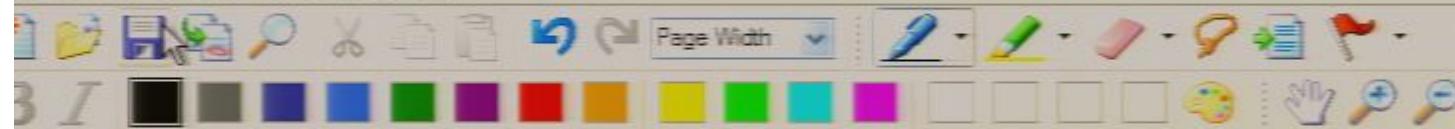
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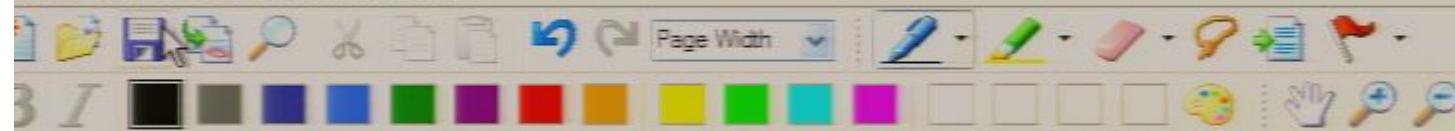
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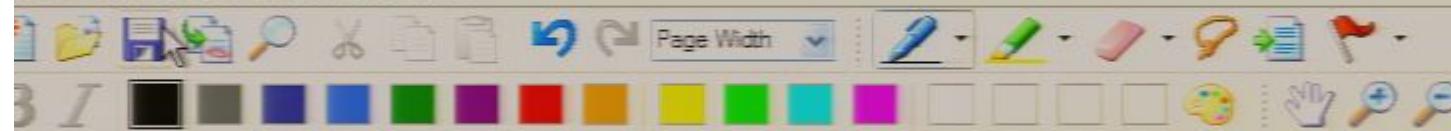
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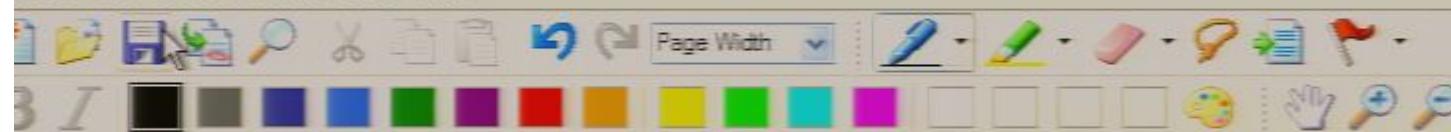
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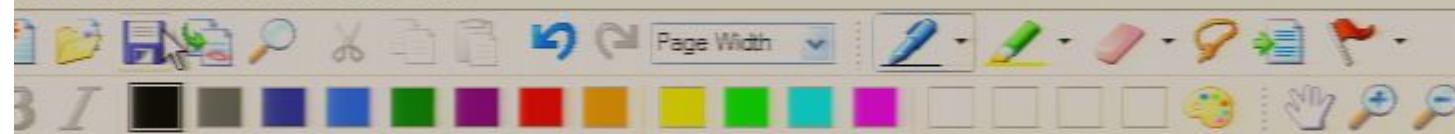
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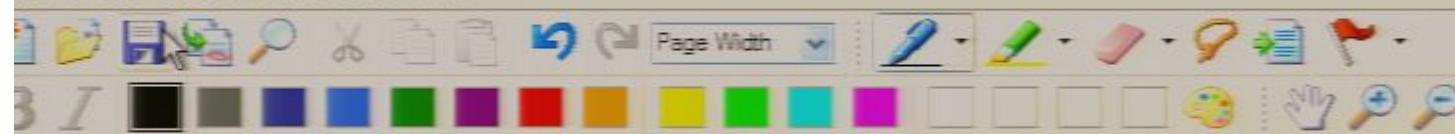
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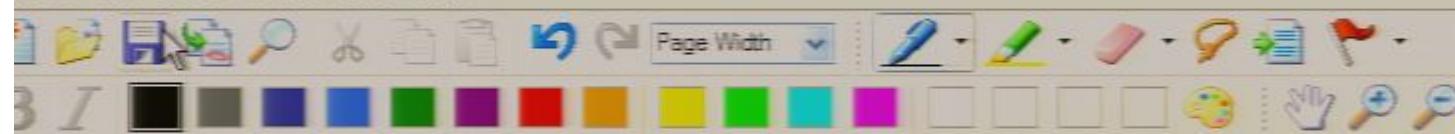
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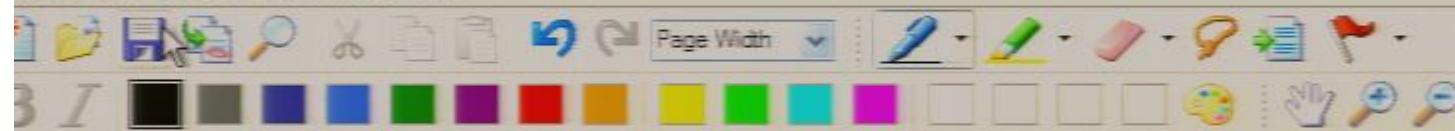
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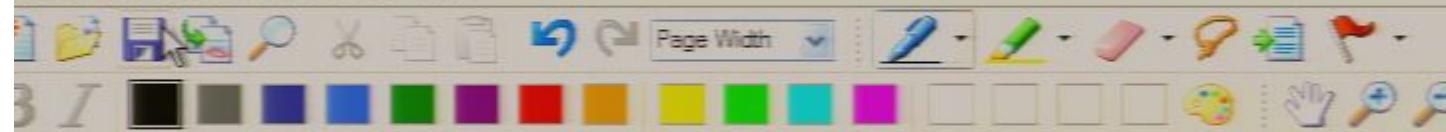
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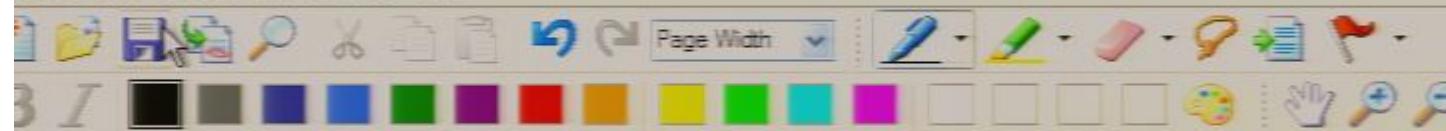
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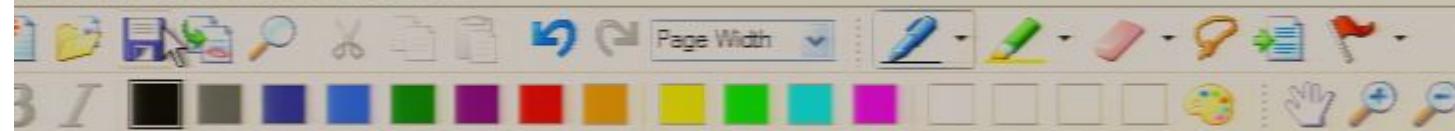
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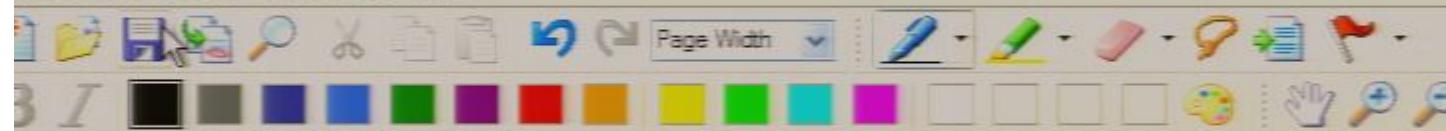
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The past & future of an event



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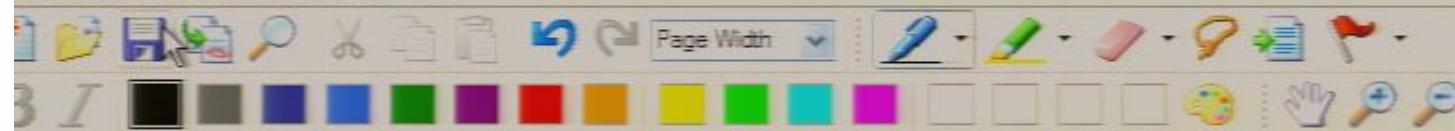
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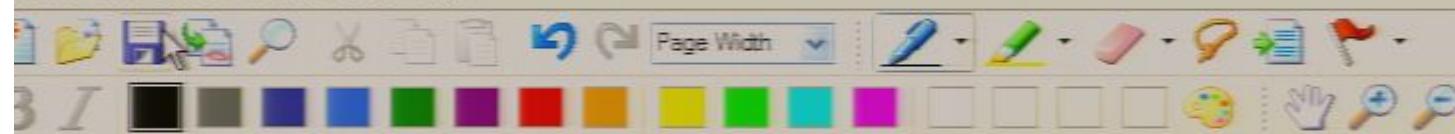
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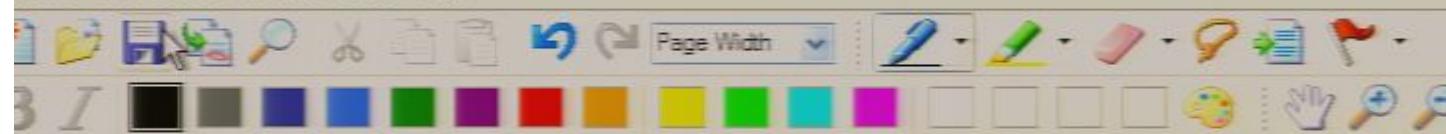
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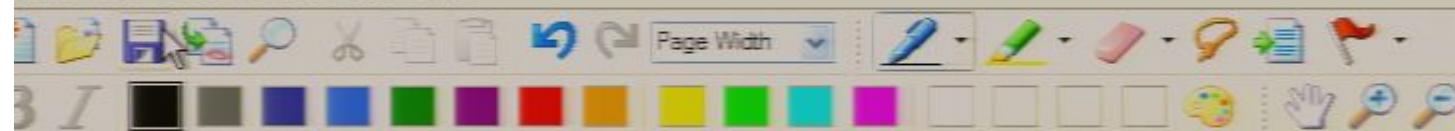
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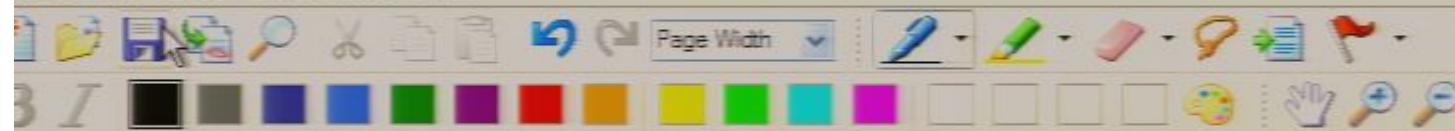
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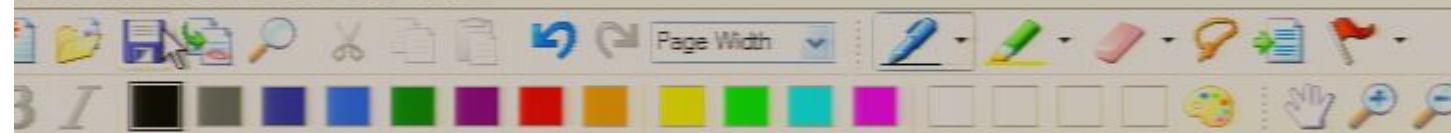
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$$\mathcal{I}^+(\rho)$$

Mnemonic help:
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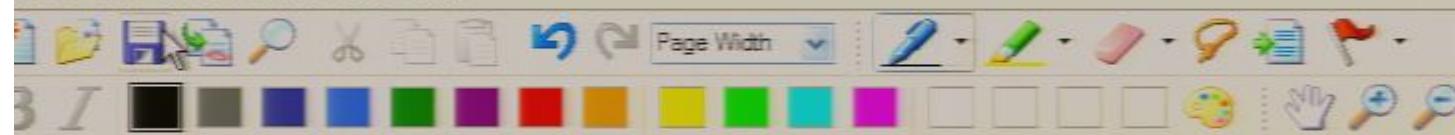
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Mnemonic help:
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The past & future of an event

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The "chronological future" of a point $p \in M$ is the set

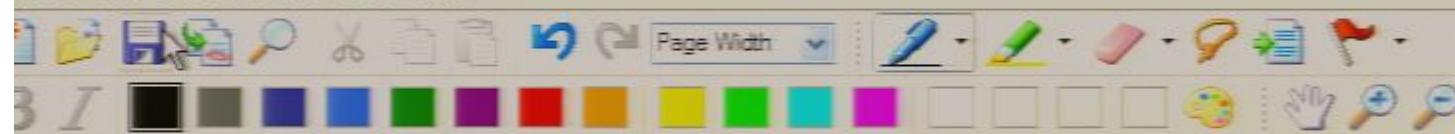
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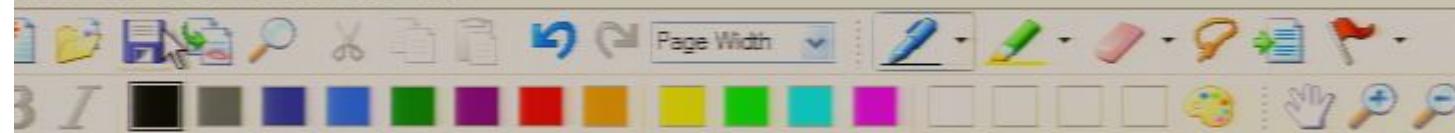
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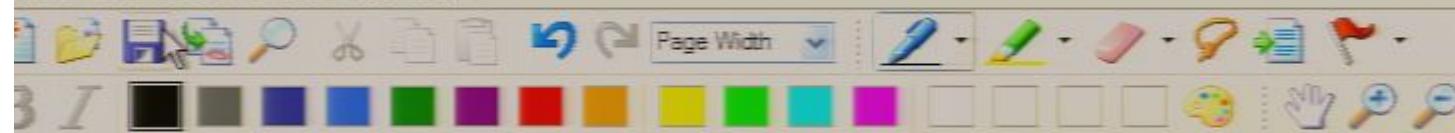
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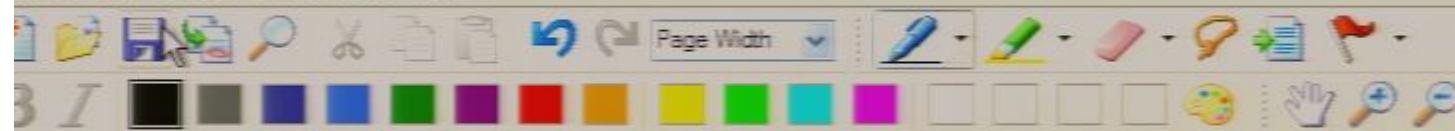
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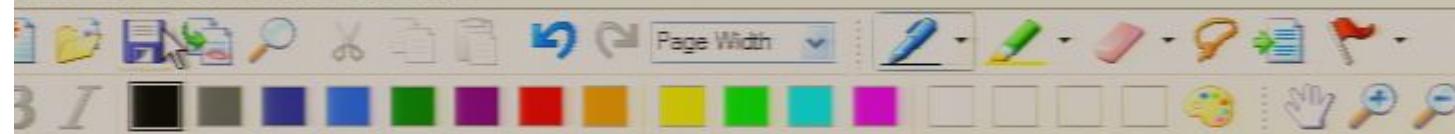
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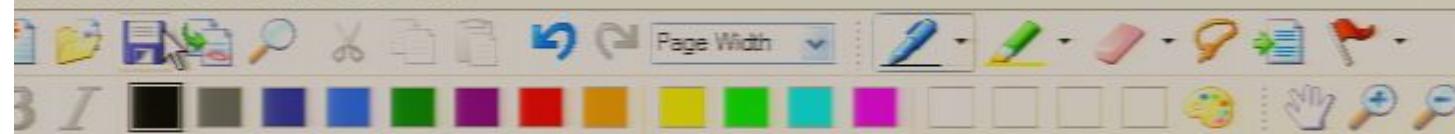
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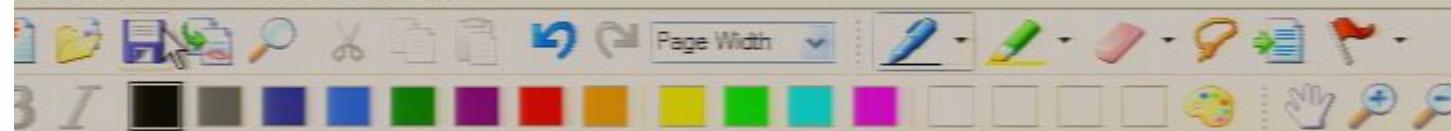
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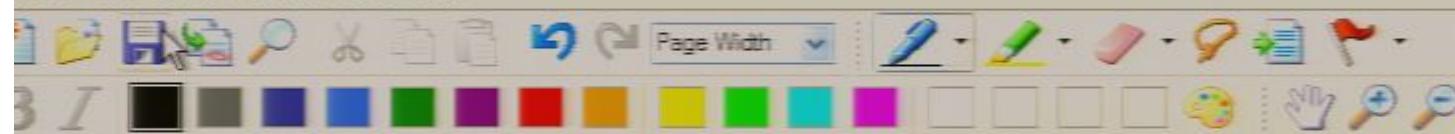
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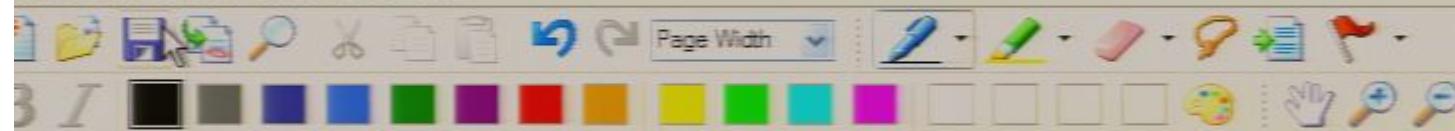
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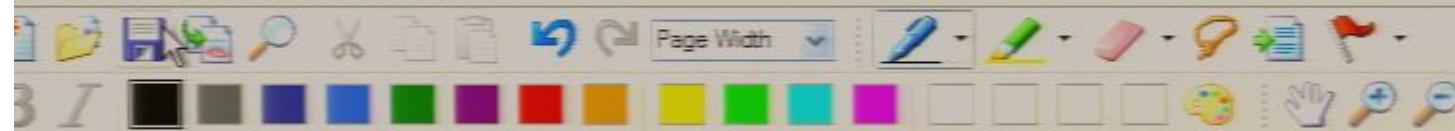
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□ Analogously, one defines the chronological past $I^-(p)$ and the causal past $J^-(p)$.

□ One defines the pasts and futures of a set S of events through:

$$I^\pm(S) := \bigcup_{p \in S} I^\pm(p), \quad J^\pm(S) := \bigcup_{p \in S} J^\pm(p)$$

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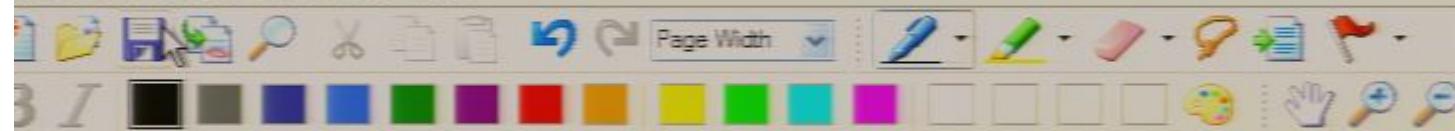
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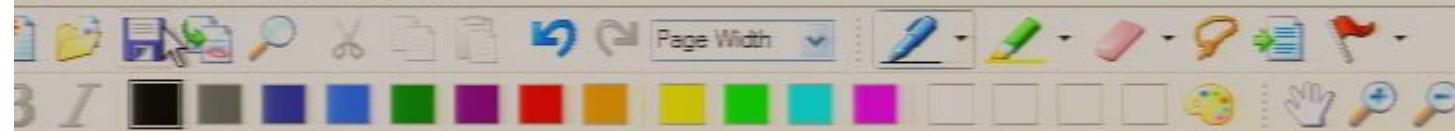
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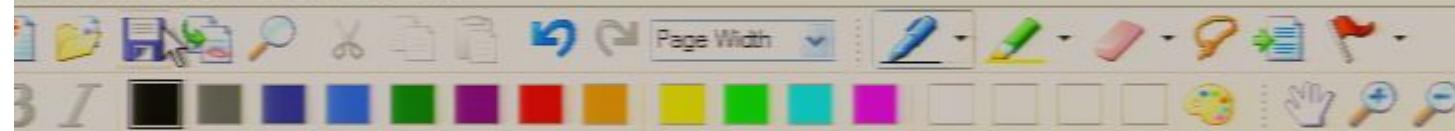


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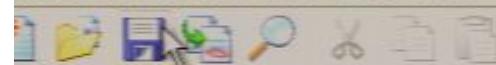
- The above definitions do not refer to geodesics.
□ For Minkowski space, clear:

$I^+(p)$ = set of events reachable by future-directed timelike geodesics from p .

$\dot{I}^+(p)$ = set of events reachable by future-directed null geodesics from p .

But this is not true in general spacetimes!

Intuition: E.g. singularities can be in the way of a geodesic.



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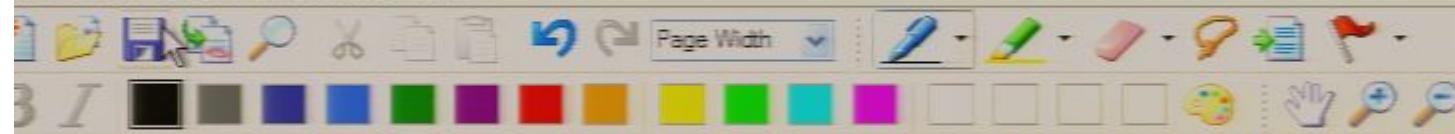
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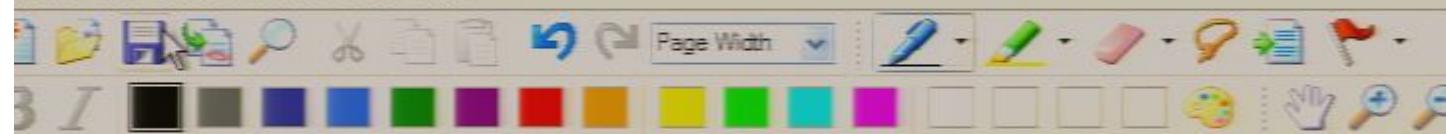
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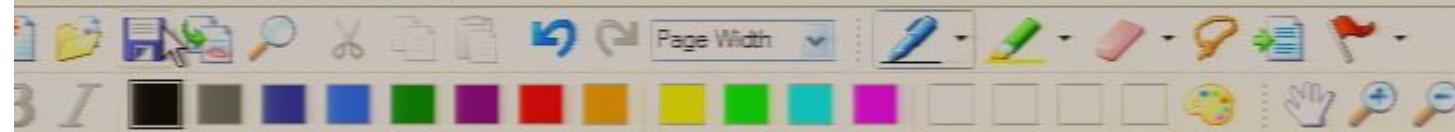
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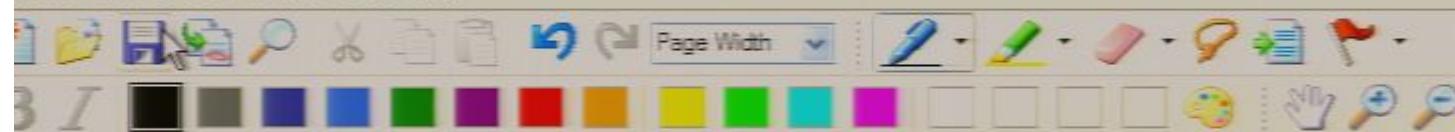
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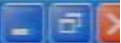
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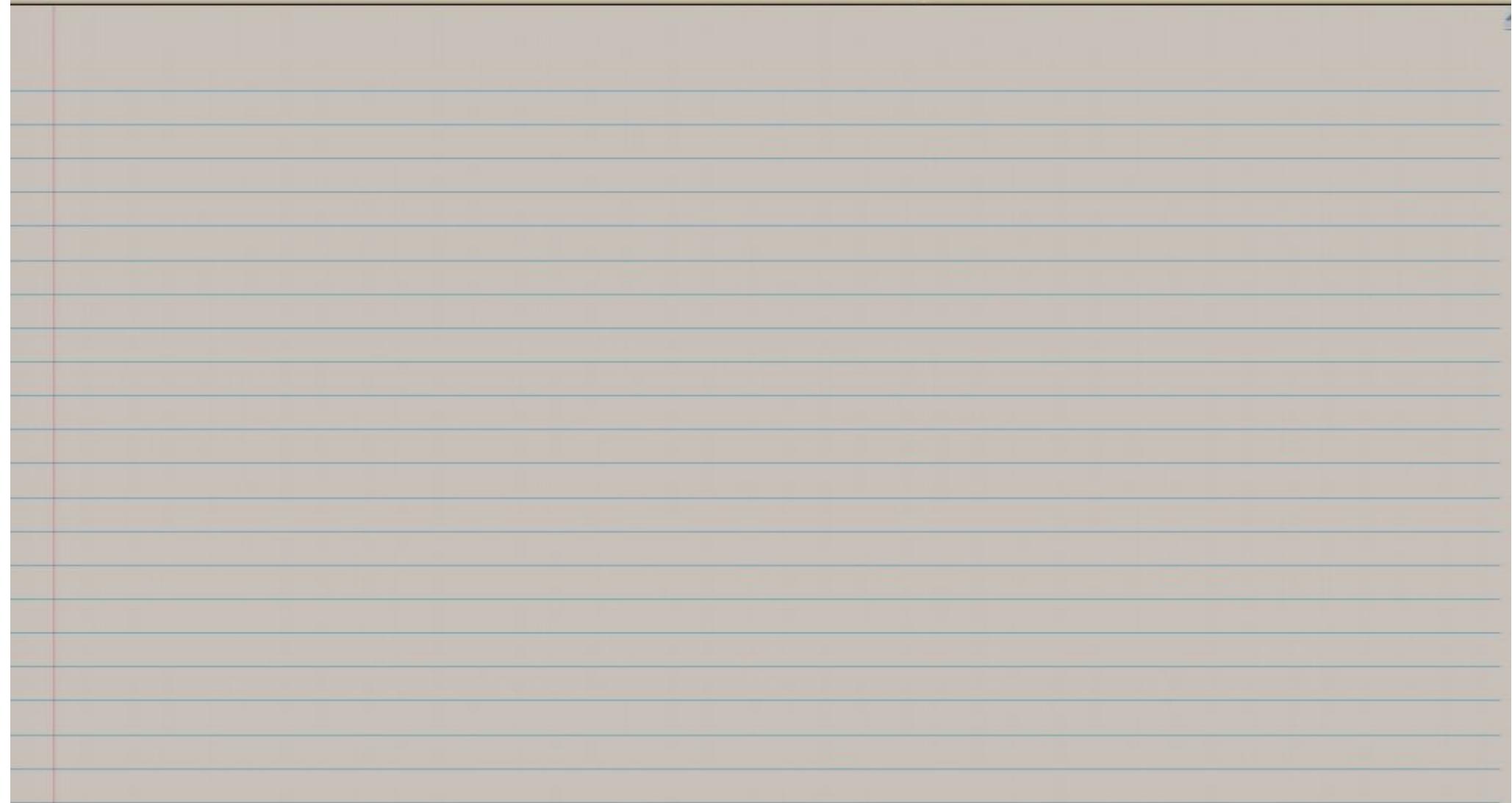
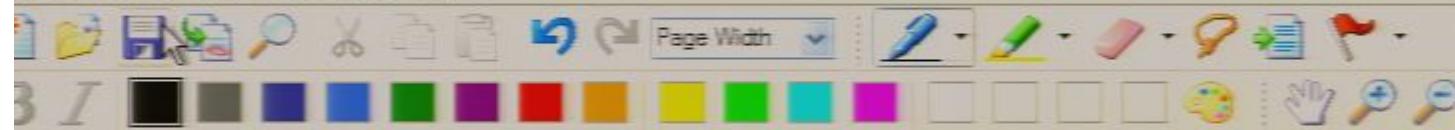
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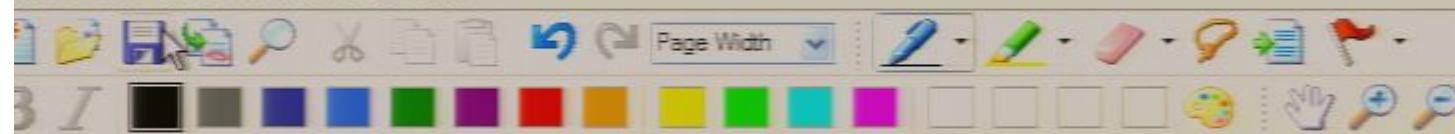
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