Title: General Relativity for Cosmology - Lecture 12B

Date: Oct 29, 2009 06:00 PM

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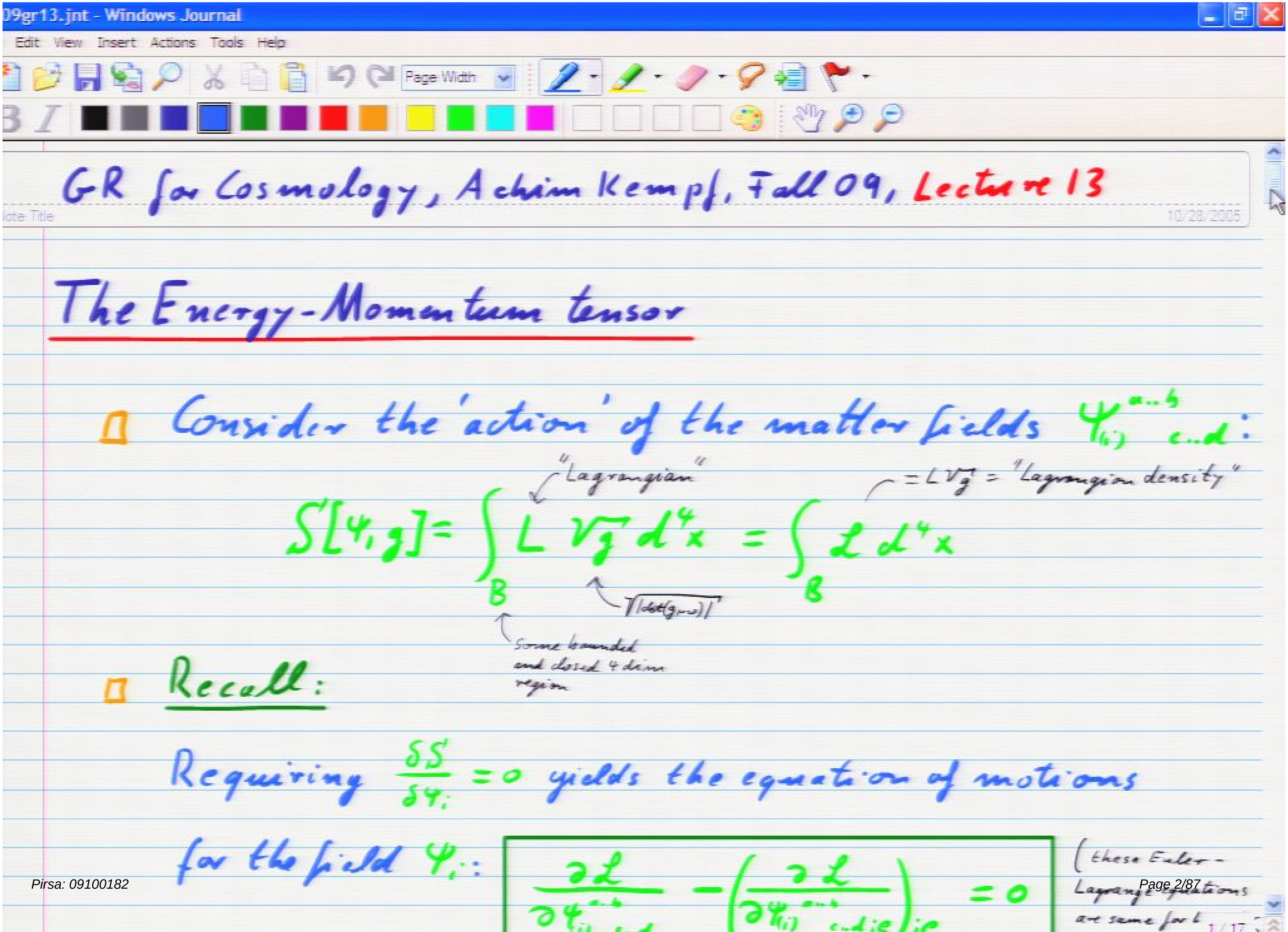
Abstract:



GR for Cosmology, Achim Kempf, Fall 09, Lecture 13

The Energy-Momentum tensor

☐ Consider the 'action' of the matter fields $\psi_{(i)}{}^{a..b}{}_{c..d}$:

$$S[\psi, g] = \int_B \overset{\text{"Lagrangian"}}{L} \underbrace{\sqrt{g}}_{\sqrt{|\det(g_{\mu\nu})|}} d^4x = \int_B \overset{= L\sqrt{g} = \text{"Lagrangian density"}}{\mathcal{L}} d^4x$$

B: Some bounded and closed 4 dim region

☐ Recall:

Requiring $\frac{\delta S}{\delta \psi_i} = 0$ yields the equation of motions

for the field ψ_i:

$$\boxed{\frac{\partial \mathcal{L}}{\partial \psi_{(i)}{}^{a..b}{}_{c..d}} - \left(\frac{\partial \mathcal{L}}{\partial \psi_{(i)}{}^{a..b}{}_{c..d;e}}\right)_{;e} = 0}$$

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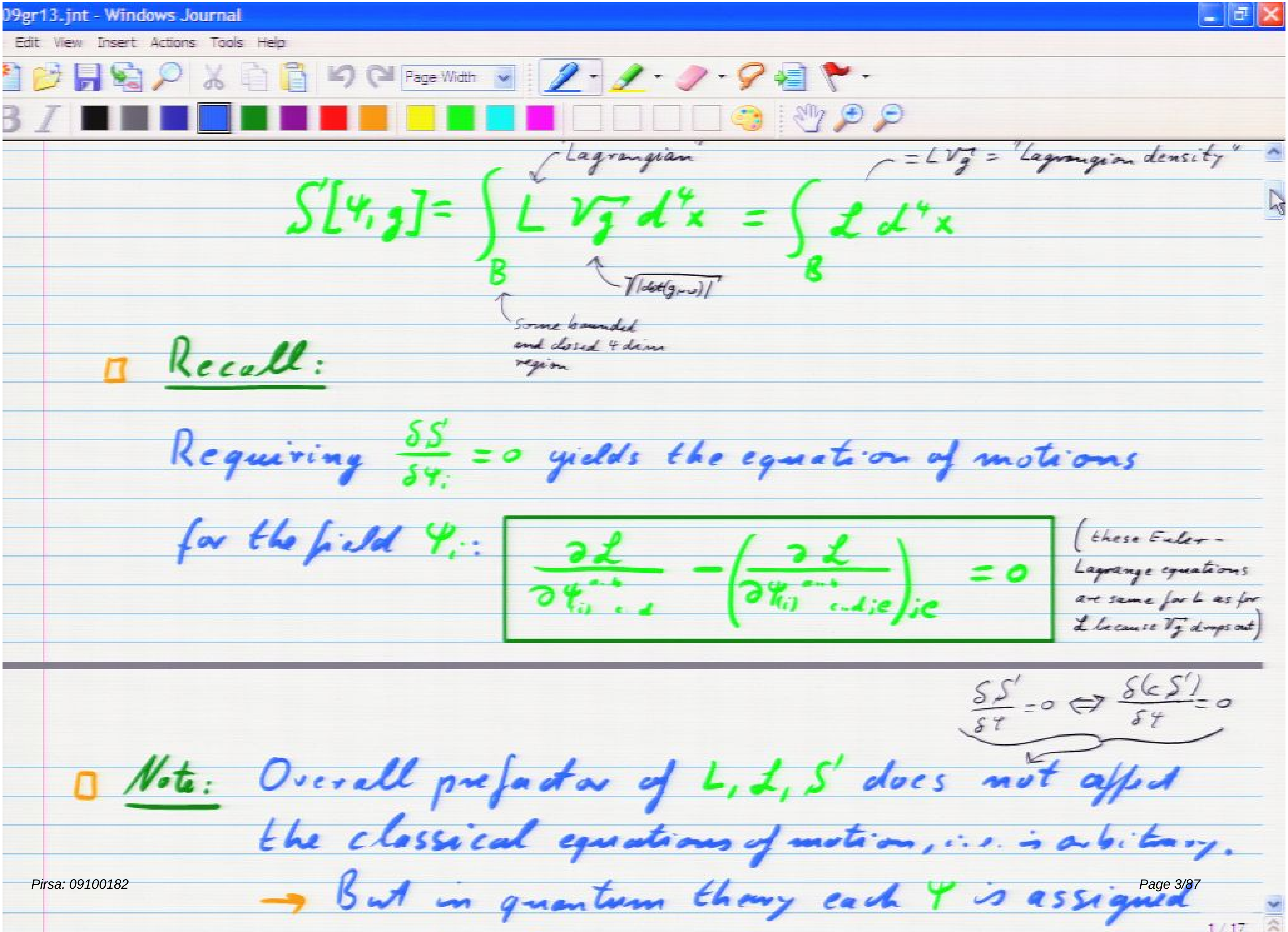

"Lagrangian"

$$S'[\psi, g] = \int_B L \sqrt{g}\, d^4x = \int_B \mathcal{L}\, d^4x$$

$= L\sqrt{g} =$ "Lagrangian density"

$\sqrt{|\det(g_{\mu\nu})|}$

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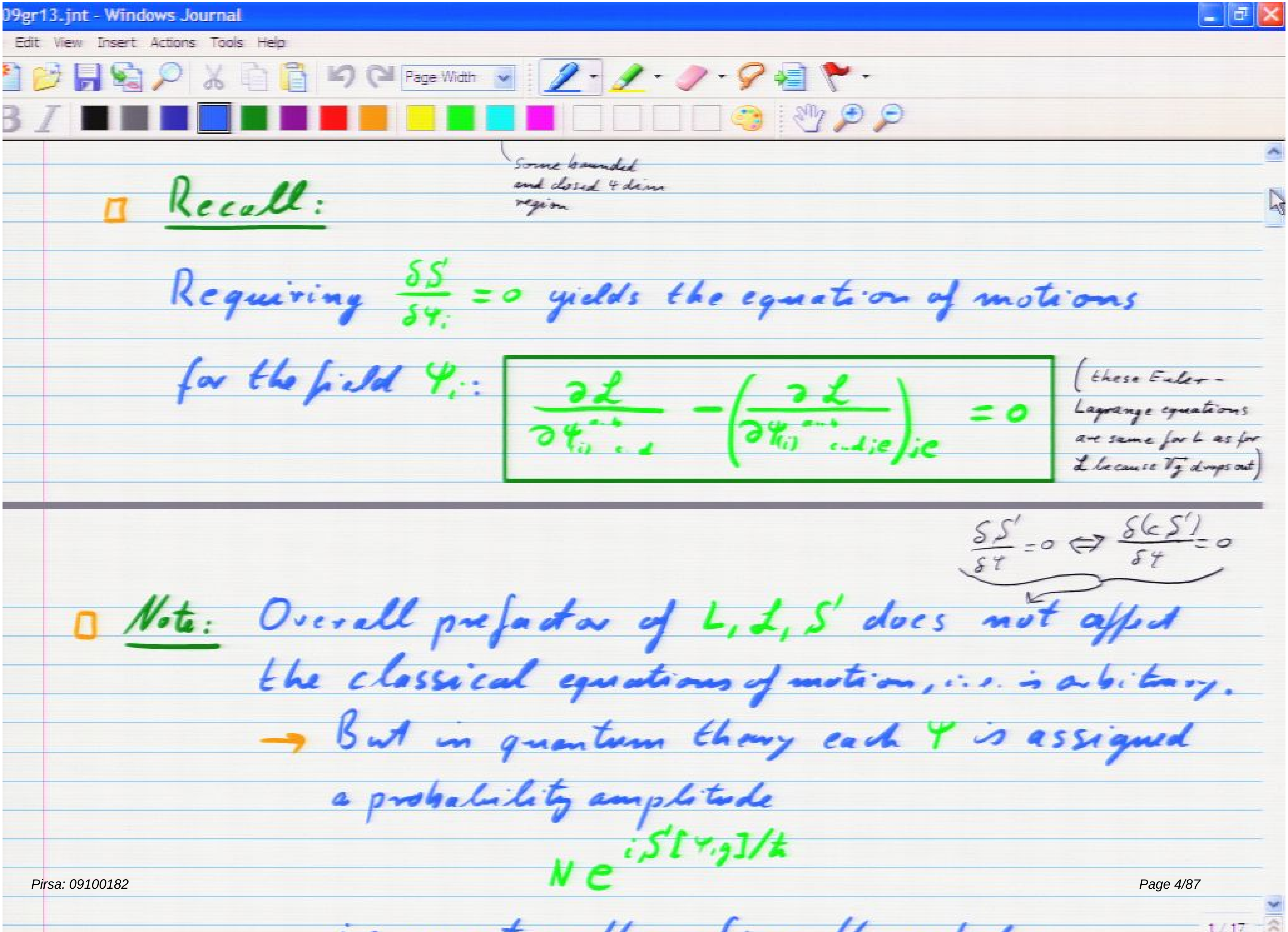
(these Euler-Lagrange equations are same for L as for $\mathcal{L}$ because $\sqrt{g}$ drops out)

$$\underbrace{\frac{\delta S'}{\delta \psi} = 0 \Leftrightarrow \frac{\delta (c S')}{\delta \psi} = 0}$$

☐ **Note:** Overall prefactor of $L, \mathcal{L}, S'$ does not affect the classical equations of motion, i.e. is arbitrary.

→ But in quantum theory each ψ is assigned



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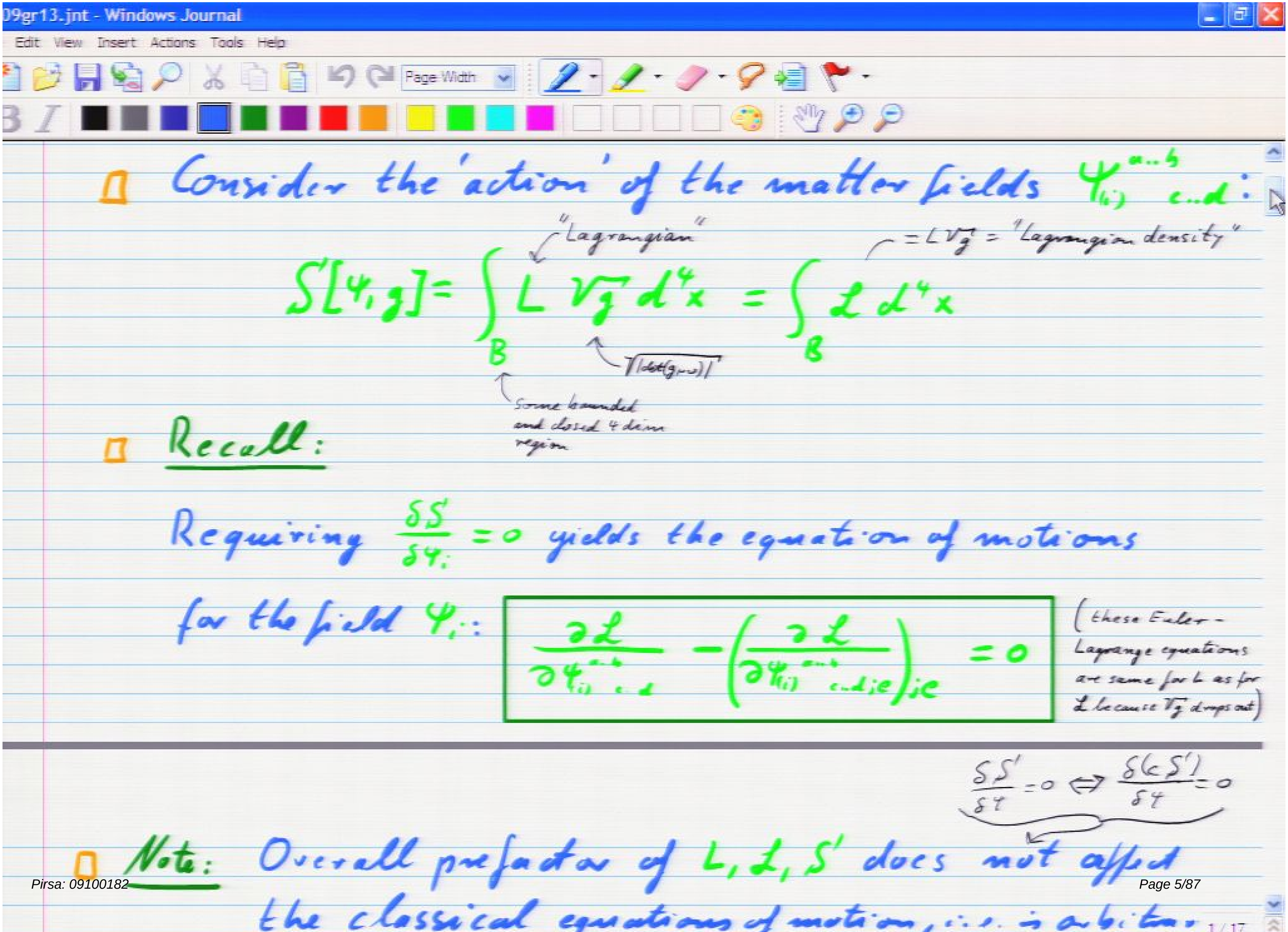

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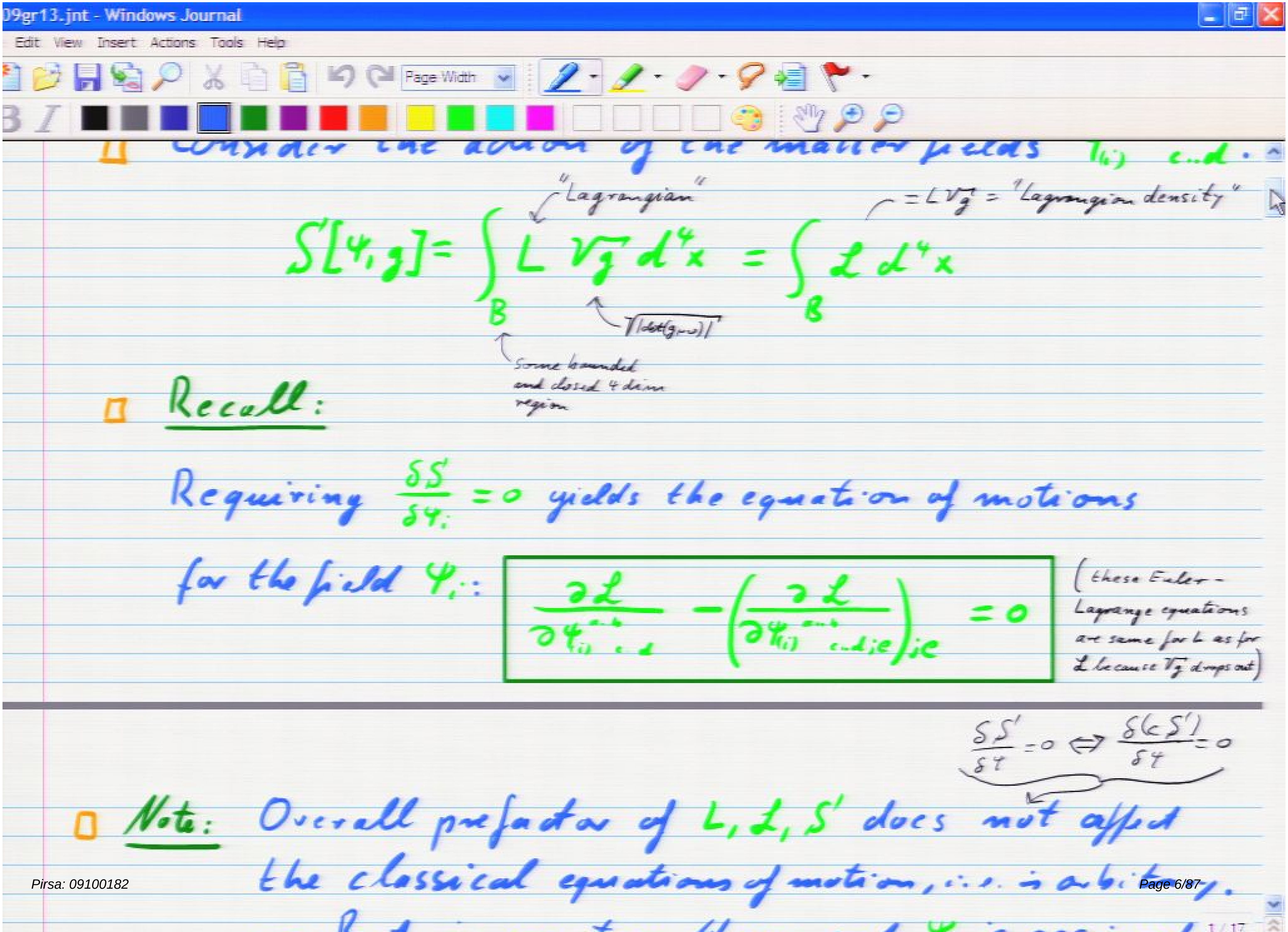

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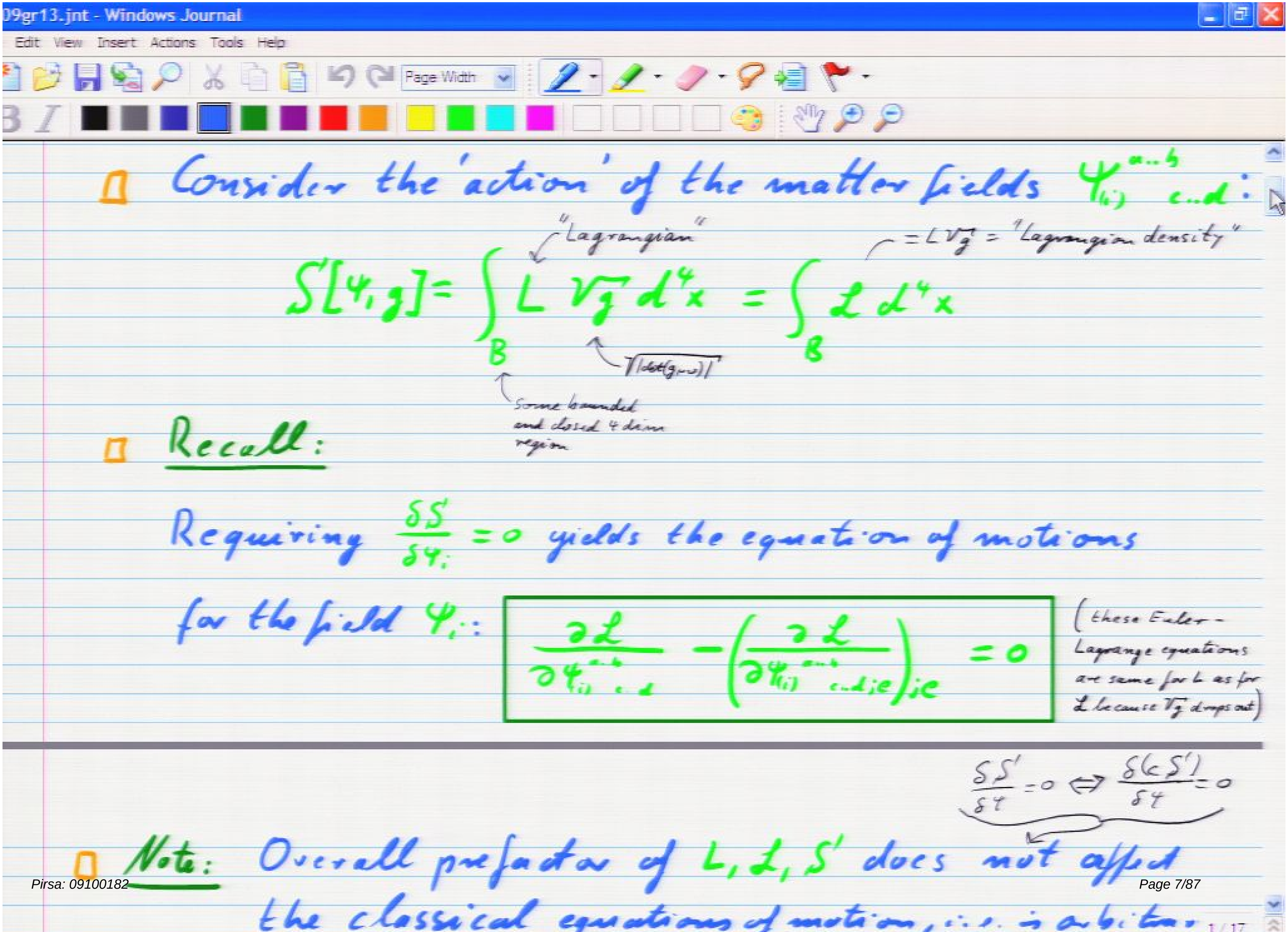

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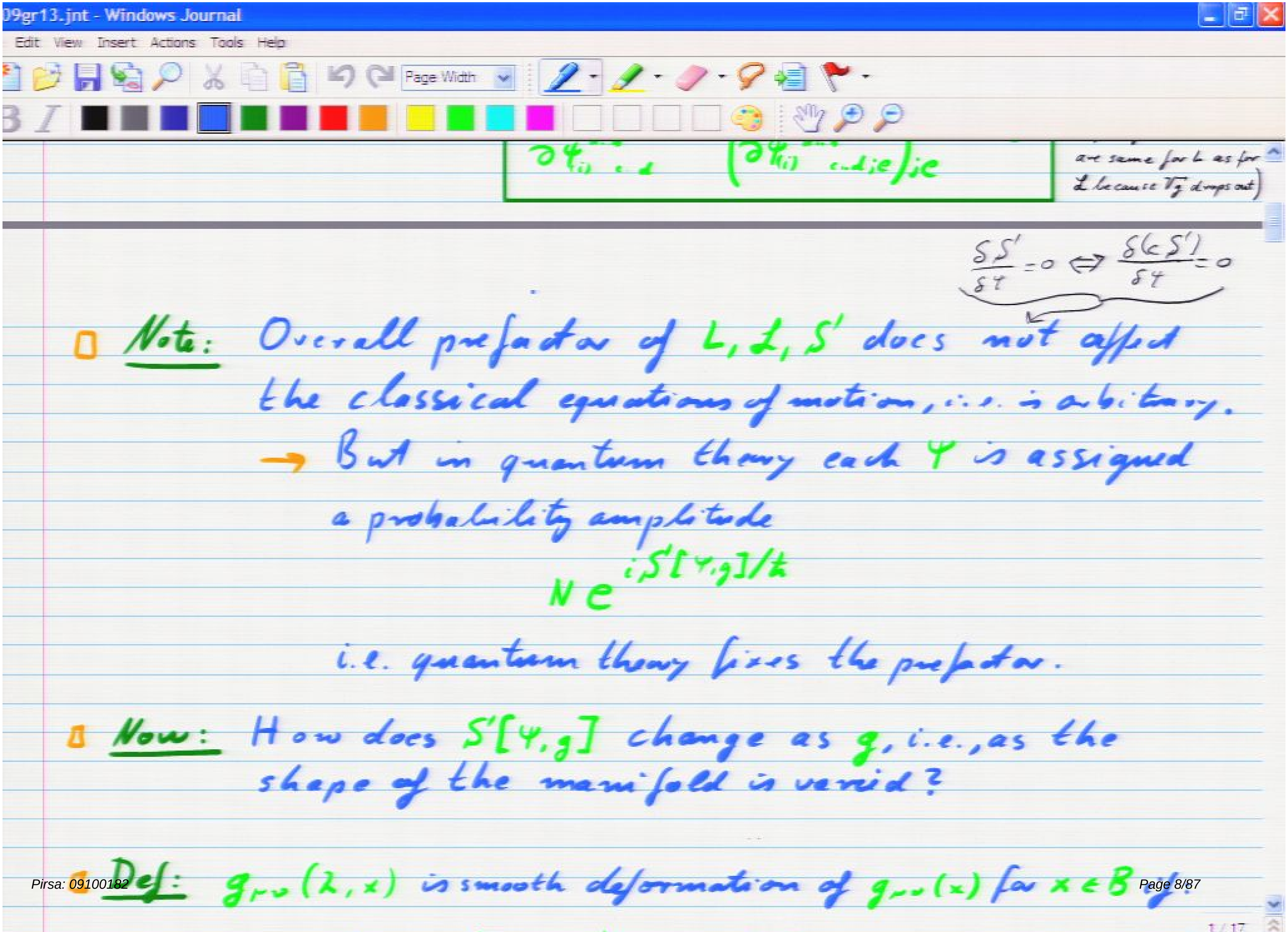

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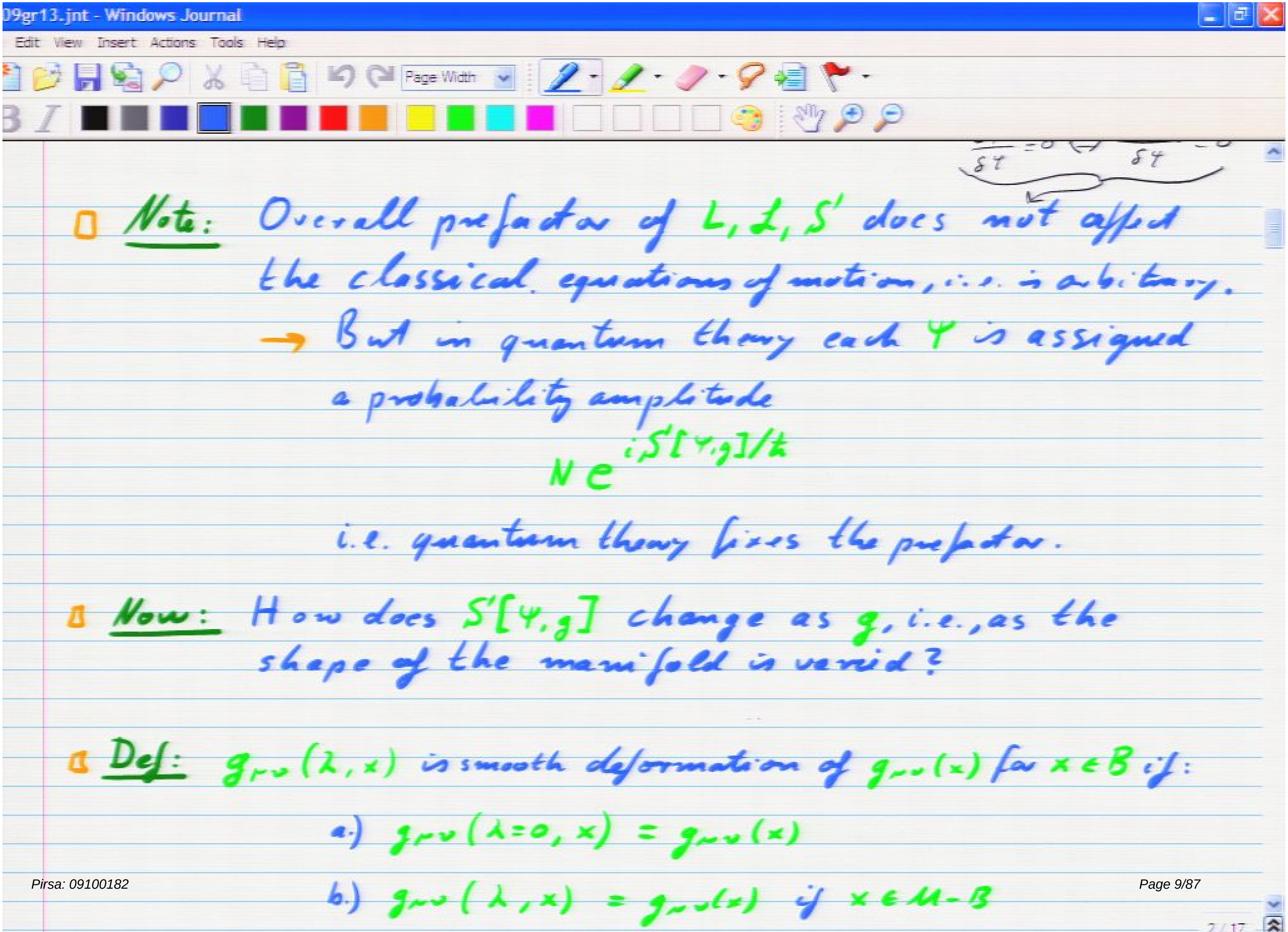
$$N e^{iS'[\varphi,g]/\hbar}$$

i.e. quantum theory fixes the prefactor.

☐ **Now:** How does $S'[\varphi,g]$ change as g, i.e., as the shape of the manifold is varied?

Def: $g_{\mu\nu}(\lambda,x)$ is smooth deformation of $g_{\mu\nu}(x)$ for $x \in B$



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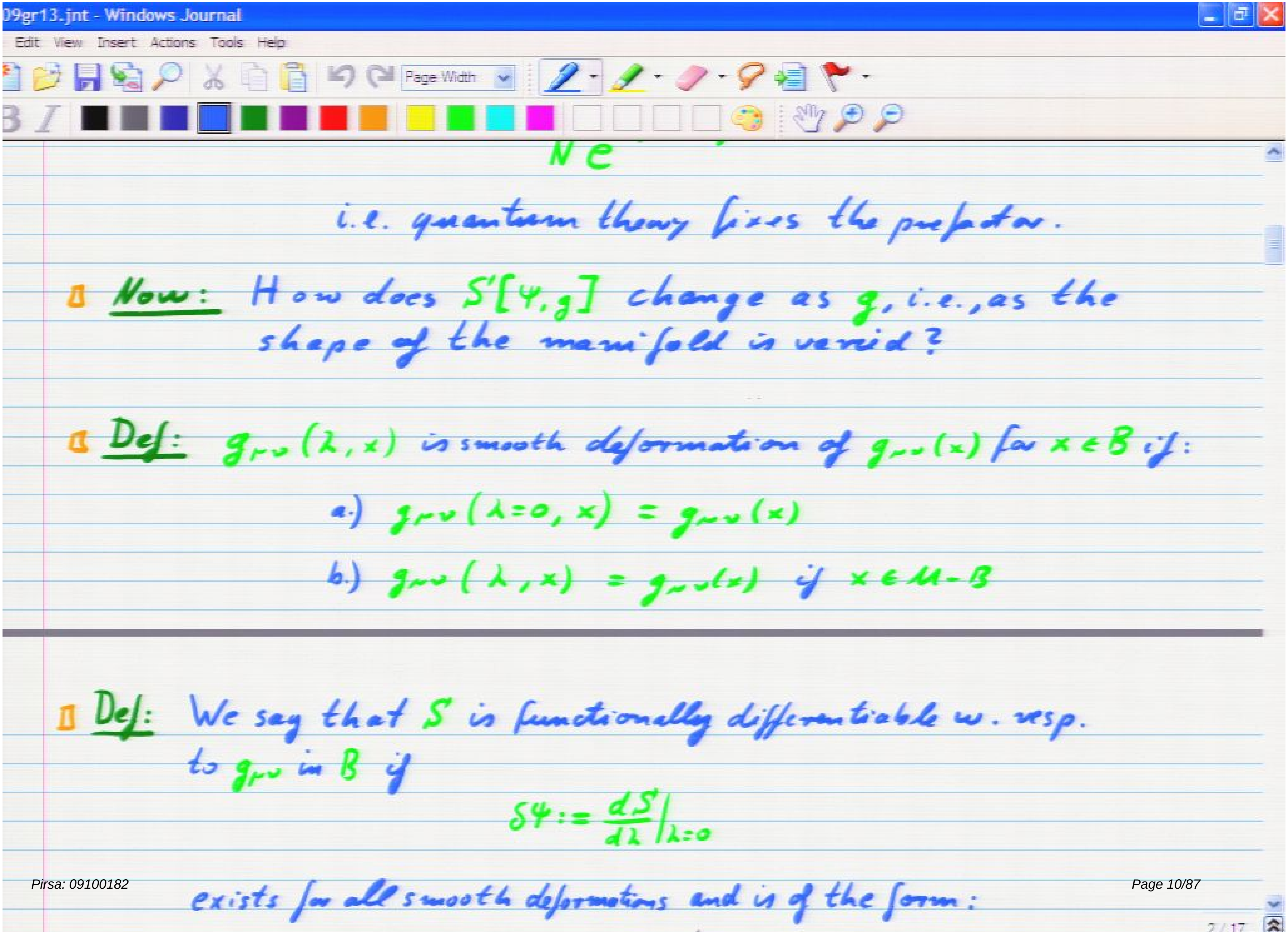

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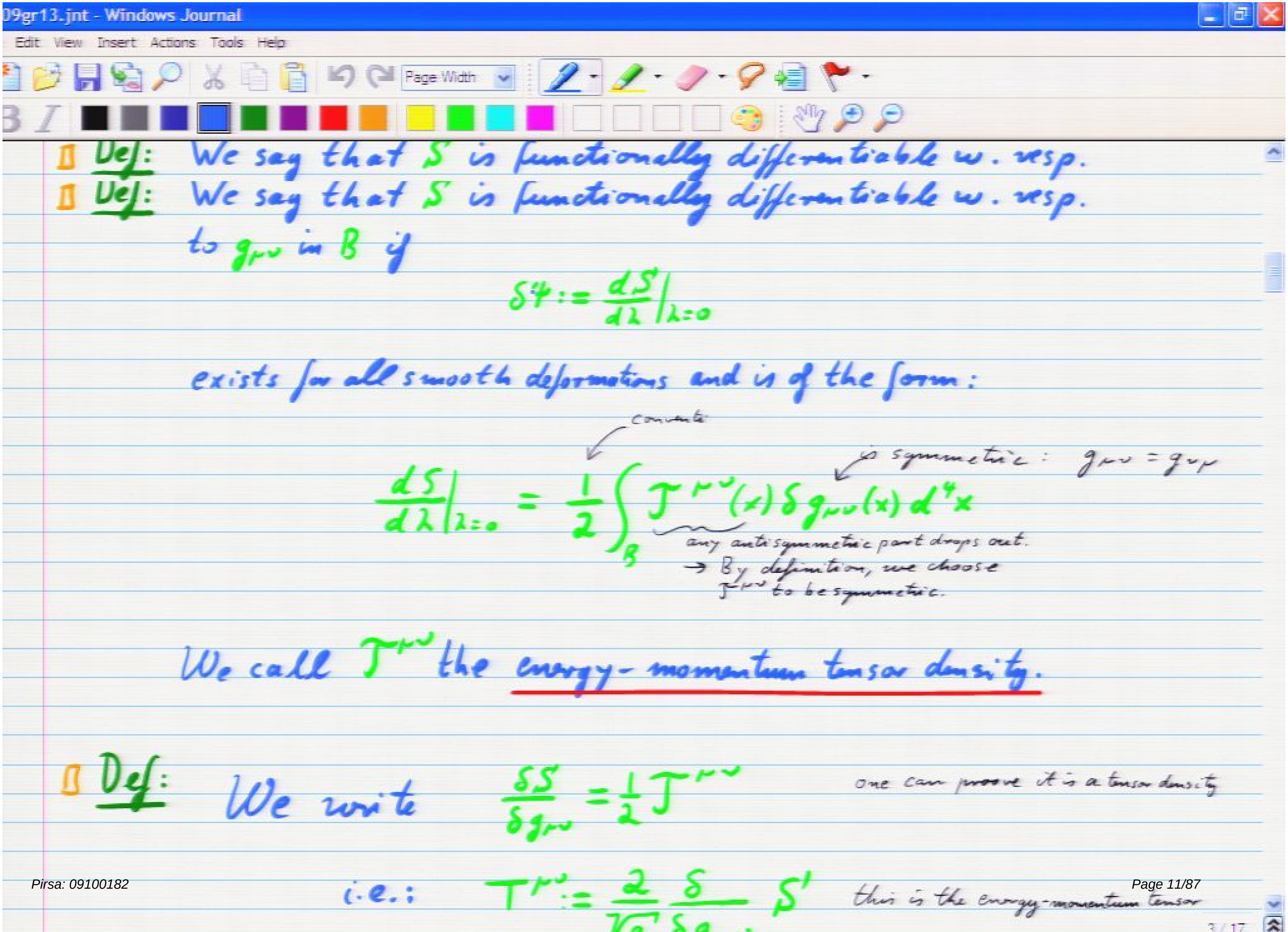
Def: We say that S is functionally differentiable w. resp. to $g_{\mu\nu}$ in B if

$$\delta\psi := \frac{dS'}{d\lambda}\Big|_{\lambda=0}$$

exists for all smooth deformations and is of the form:



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convenient

is symmetric: $g_{\mu\nu} = g_{\nu\mu}$

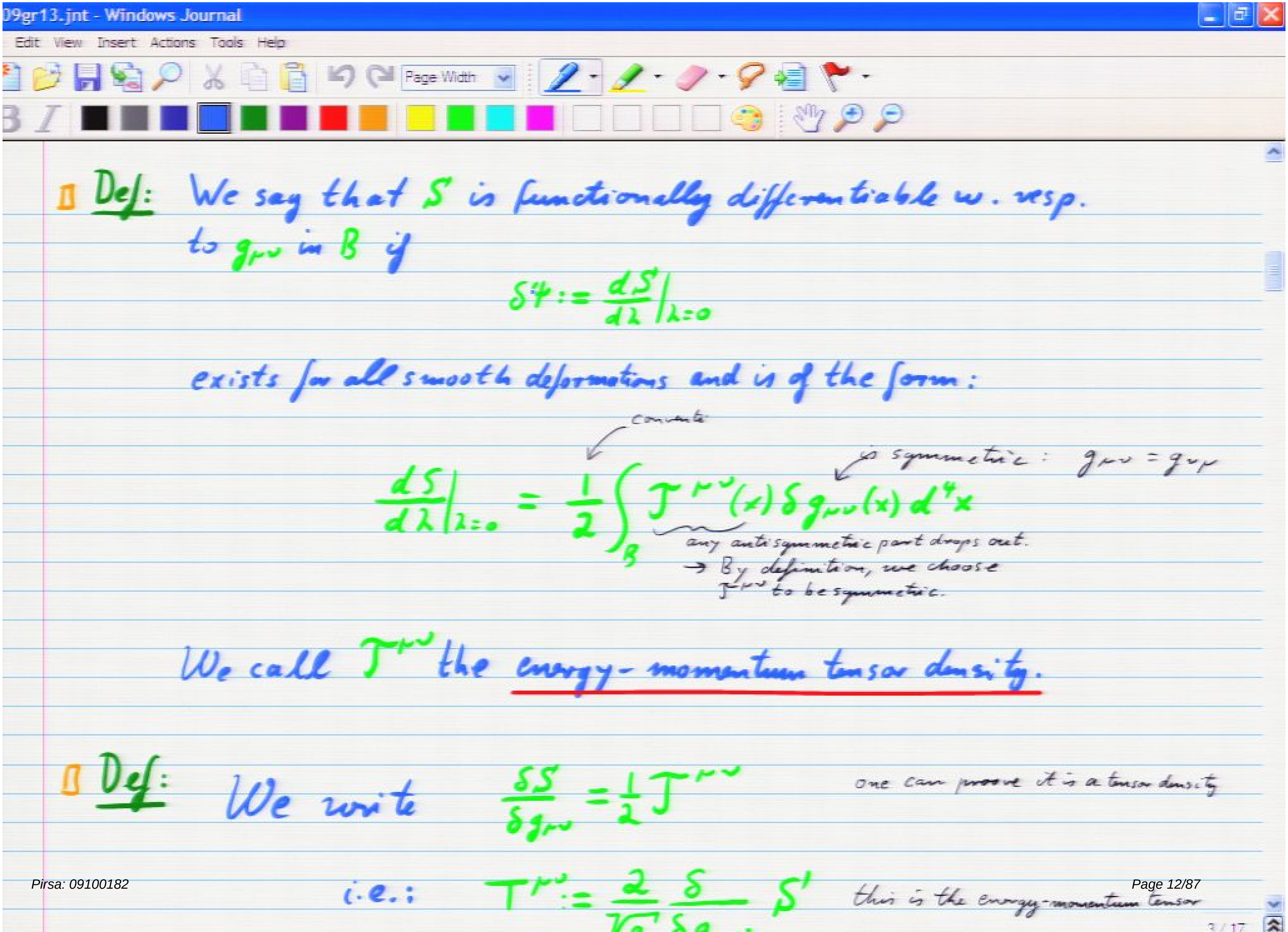
$$\frac{dS}{d\lambda}\Big|_{\lambda=0} = \frac{1}{2}\int_B \underbrace{J^{\mu\nu}(x)}\,\delta g_{\mu\nu}(x)\, d^4x$$

any antisymmetric part drops out.
→ By definition, we choose $J^{\mu\nu}$ to be symmetric.

We call $J^{\mu\nu}$ the <u>energy-momentum tensor density.</u>

Def: We write $\frac{\delta S}{\delta g_{\mu\nu}} = \frac{1}{2} J^{\mu\nu}$ one can prove it is a tensor density

i.e.: $T^{\mu\nu} := \frac{2}{\sqrt{g}}\frac{\delta}{\delta g} S'$ this is the energy-momentum tensor



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$$\frac{dS}{d\lambda}\Big|_{\lambda=0} = \overset{\text{covariant}}{\frac{1}{2}\int_B} \underbrace{J^{\mu\nu}(x)}_{\substack{\text{any antisymmetric part drops out.} \\ \to \text{By definition, we choose} \\ J^{\mu\nu} \text{ to be symmetric.}}} \overset{\text{is symmetric: } g_{\mu\nu} = g_{\nu\mu}}{\delta g_{\mu\nu}(x)} \, d^4x$$

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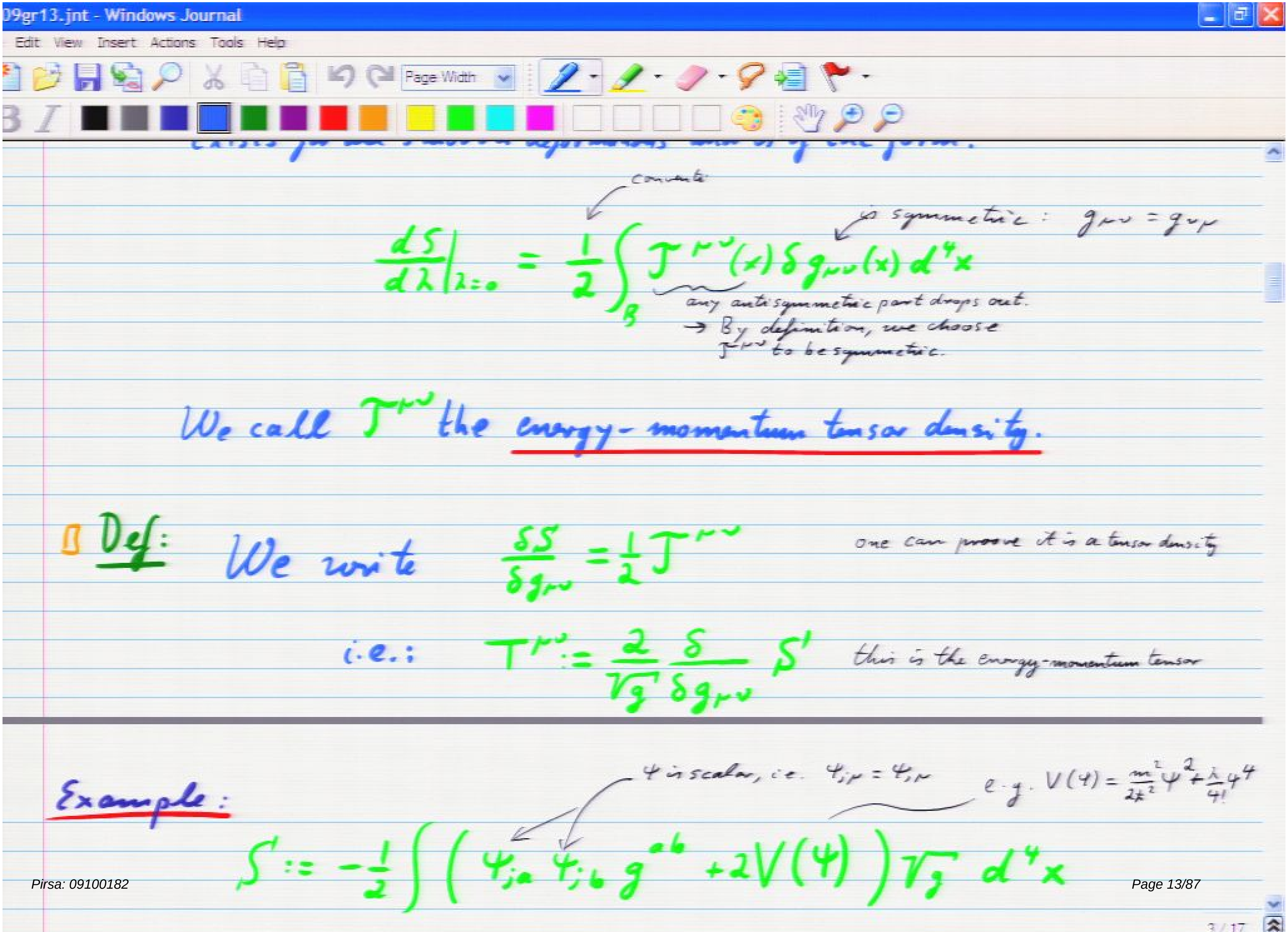

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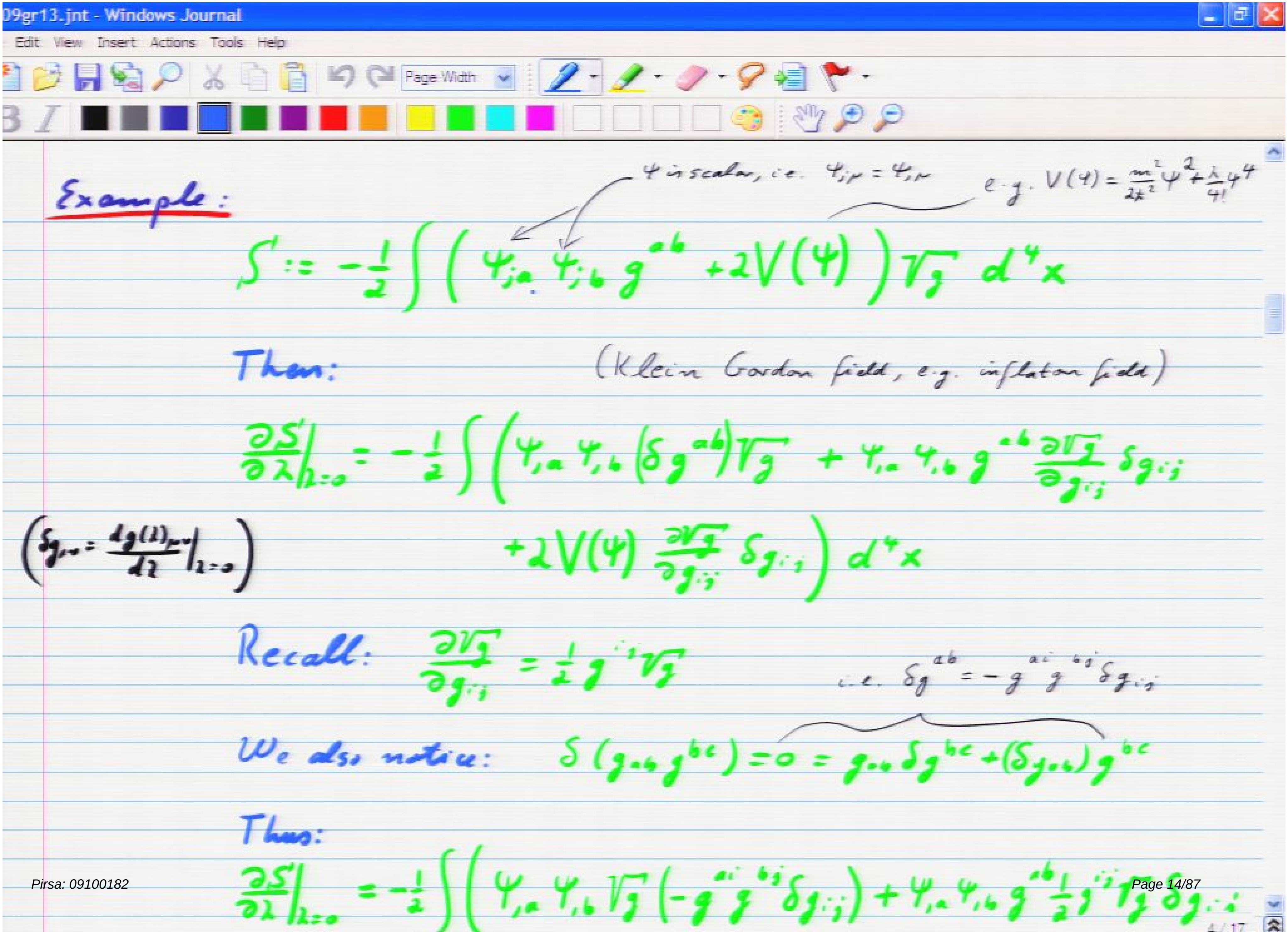
Example:

φ is scalar, i.e. $\varphi_{;\mu} = \varphi_{,\mu}$ e.g. $V(\varphi) = \frac{m^2}{2\hbar^2}\varphi^2 + \frac{\lambda}{4!}\varphi^4$

$$S' := -\frac{1}{2}\int \left(\varphi_{;a}\, \varphi_{;b}\, g^{ab} + 2V(\varphi) \right)\sqrt{g}\, d^4x$$



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Then: (Klein Gordon field, e.g. inflaton field)

$$\left.\frac{\partial S}{\partial\lambda}\right|_{\lambda=0} = -\frac{1}{2}\int\left(\psi_{,a}\,\psi_{,b}\left(\delta g^{ab}\right)\sqrt{g} + \psi_{,a}\,\psi_{,b}\,g^{ab}\frac{\partial\sqrt{g}}{\partial g_{ij}}\delta g_{ij}\right.$$

$$\left(\delta g_{\mu\nu} = \left.\frac{dg(\lambda)_{\mu\nu}}{d\lambda}\right|_{\lambda=0}\right) \qquad \left. + 2V(\psi)\frac{\partial\sqrt{g}}{\partial g_{ij}}\delta g_{ij}\right)d^4x$$

Recall: $\frac{\partial\sqrt{g}}{\partial g_{ij}} = \frac{1}{2}g^{ij}\sqrt{g}$

i.e. $\delta g^{ab} = -g^{ai}g^{bj}\delta g_{ij}$

We also notice: $\delta\left(g_{ab}g^{bc}\right) = 0 = g_{ab}\,\delta g^{bc} + \left(\delta g_{ab}\right)g^{bc}$

Thus:

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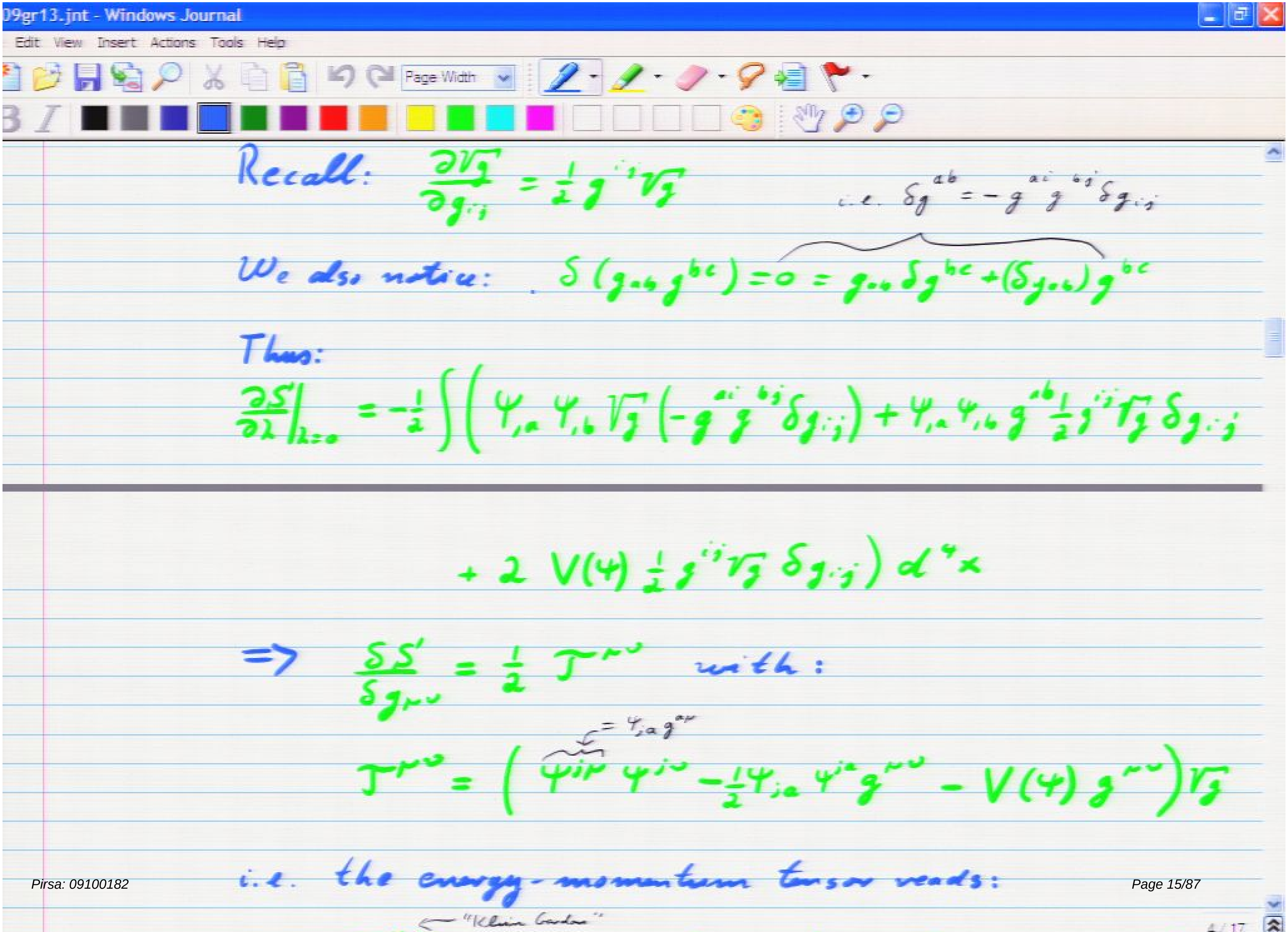

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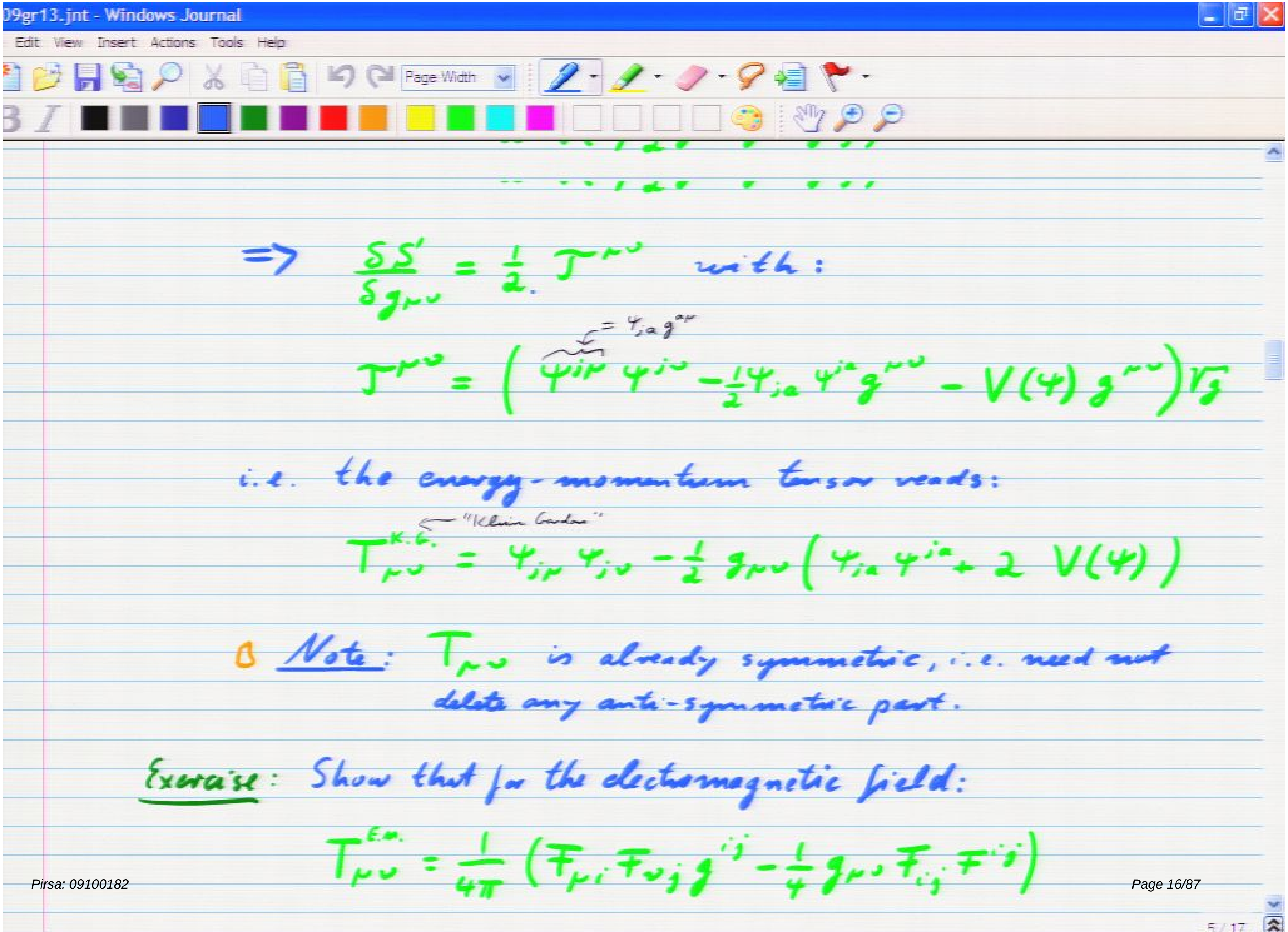
$\Rightarrow$ $\frac{\delta S'}{\delta g_{\mu\nu}} = \frac{1}{2}\,T^{\mu\nu}$ with:

$$T^{\mu\nu} = \Big(\overbrace{\psi^{;\mu}}^{=\psi_{;a}g^{a\mu}}\psi^{;\nu} - \frac{1}{2}\psi_{;a}\psi^{;a}g^{\mu\nu} - V(\psi)\,g^{\mu\nu}\Big)\sqrt{g}$$

i.e. the energy-momentum tensor reads:

← "Klein Gordon"



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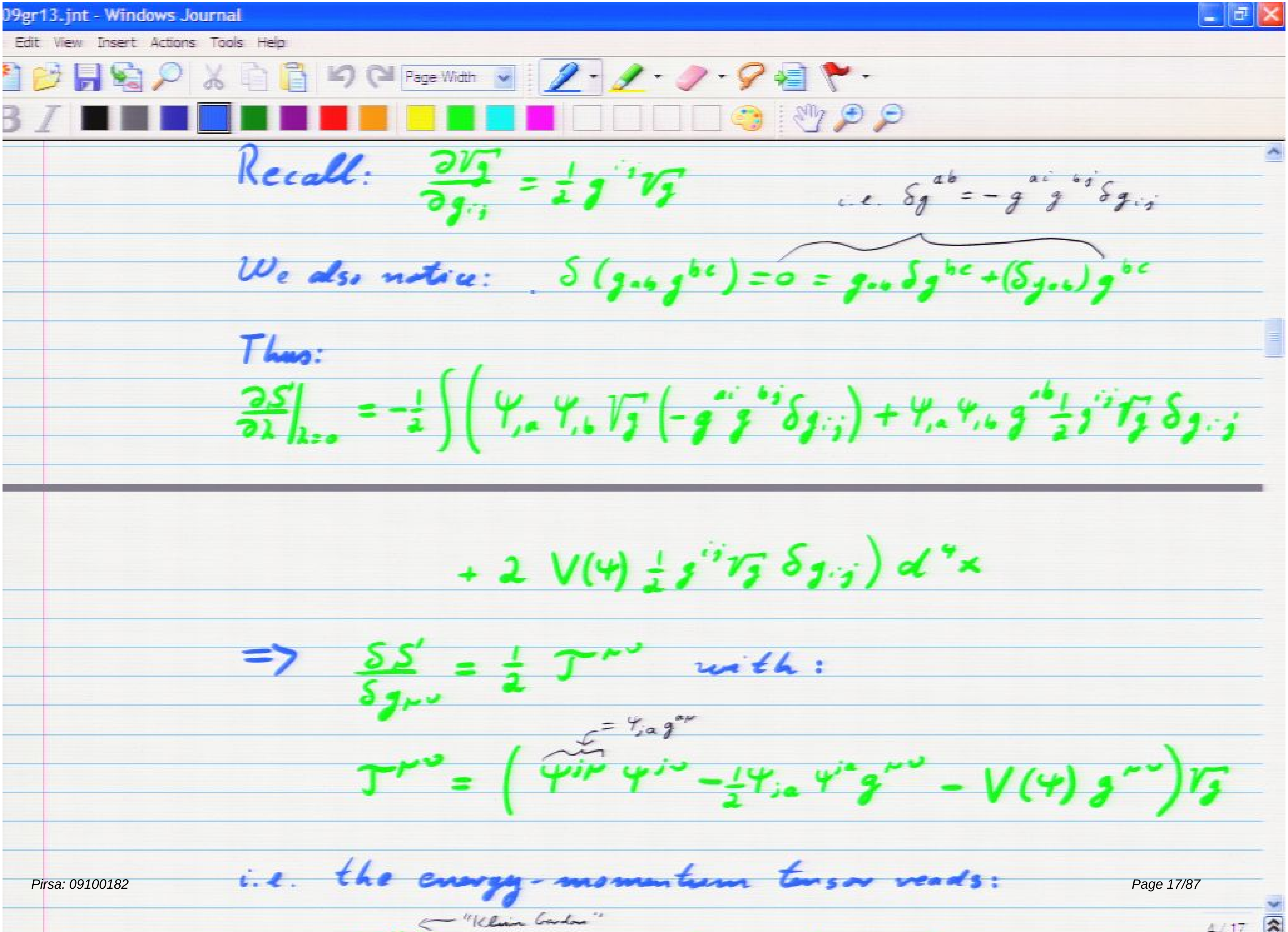
Note: $T_{\mu\nu}$ is already symmetric, i.e. need not delete any anti-symmetric part.

Exercise: Show that for the electromagnetic field:

$$T^{E.M.}_{\mu\nu} = \frac{1}{4\pi}\left(F_{\mu i}F_{\nu j}\, g^{ij} - \frac{1}{4} g_{\mu\nu} F_{ij}F^{ij}\right)$$



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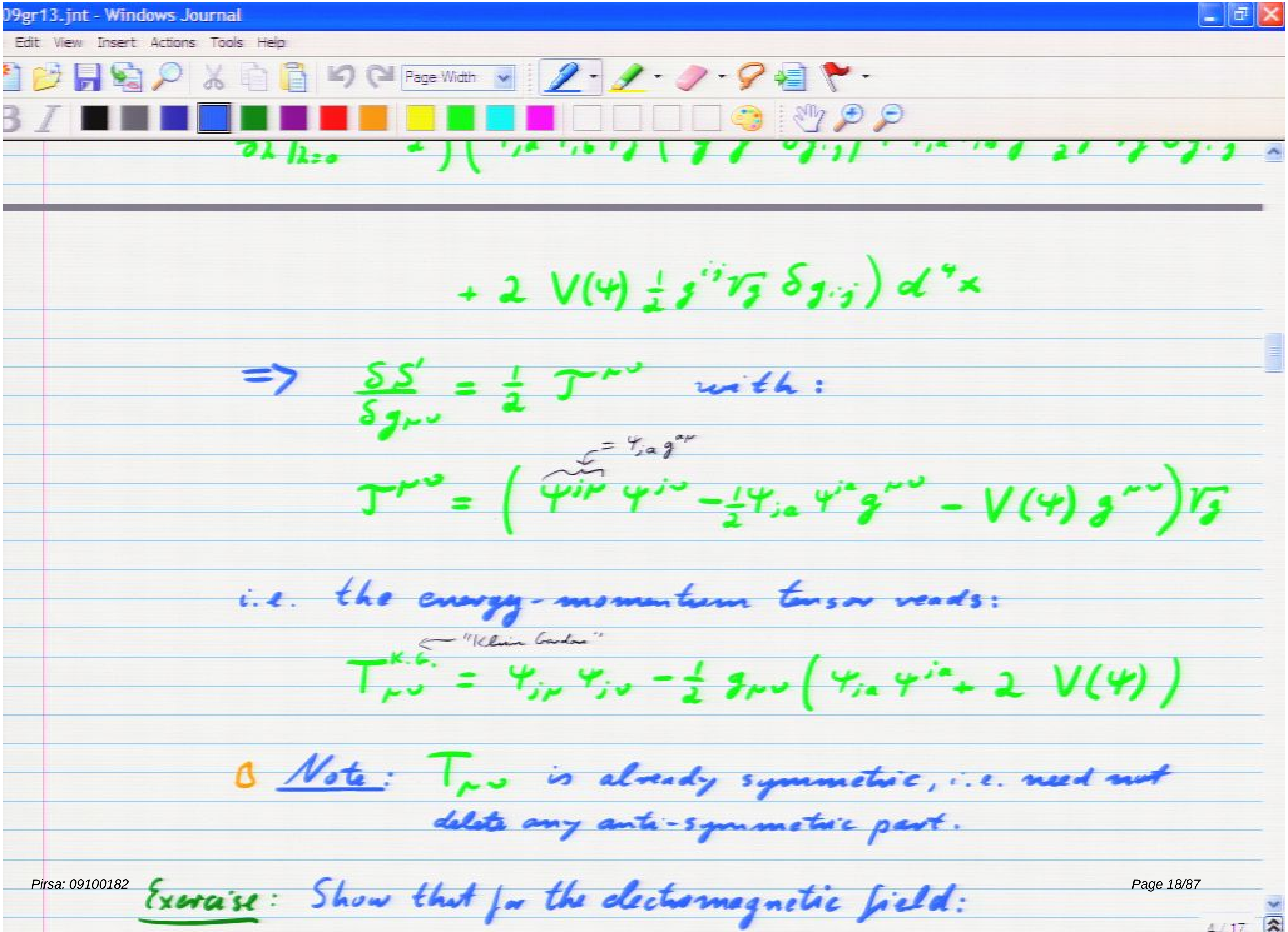

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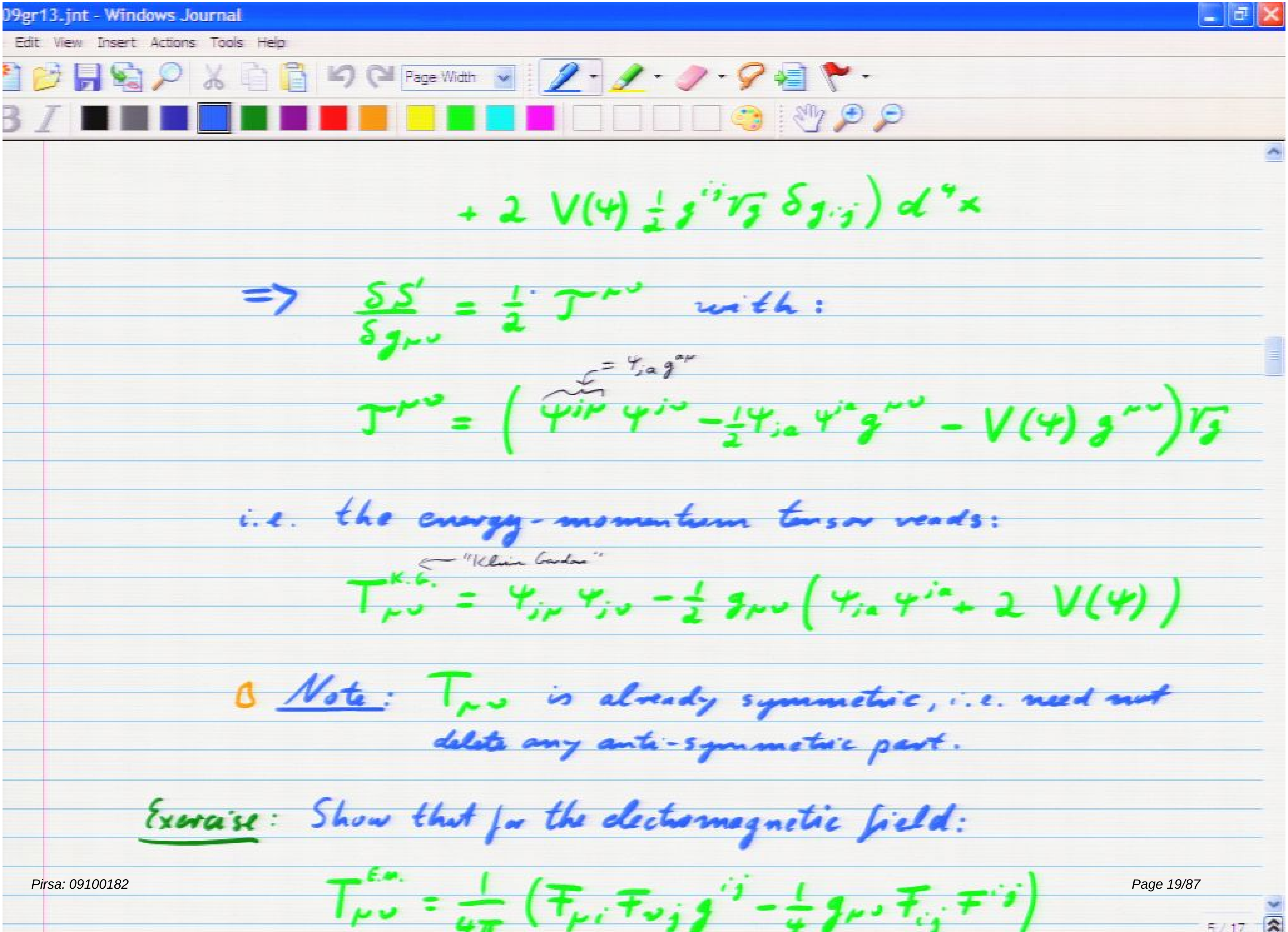

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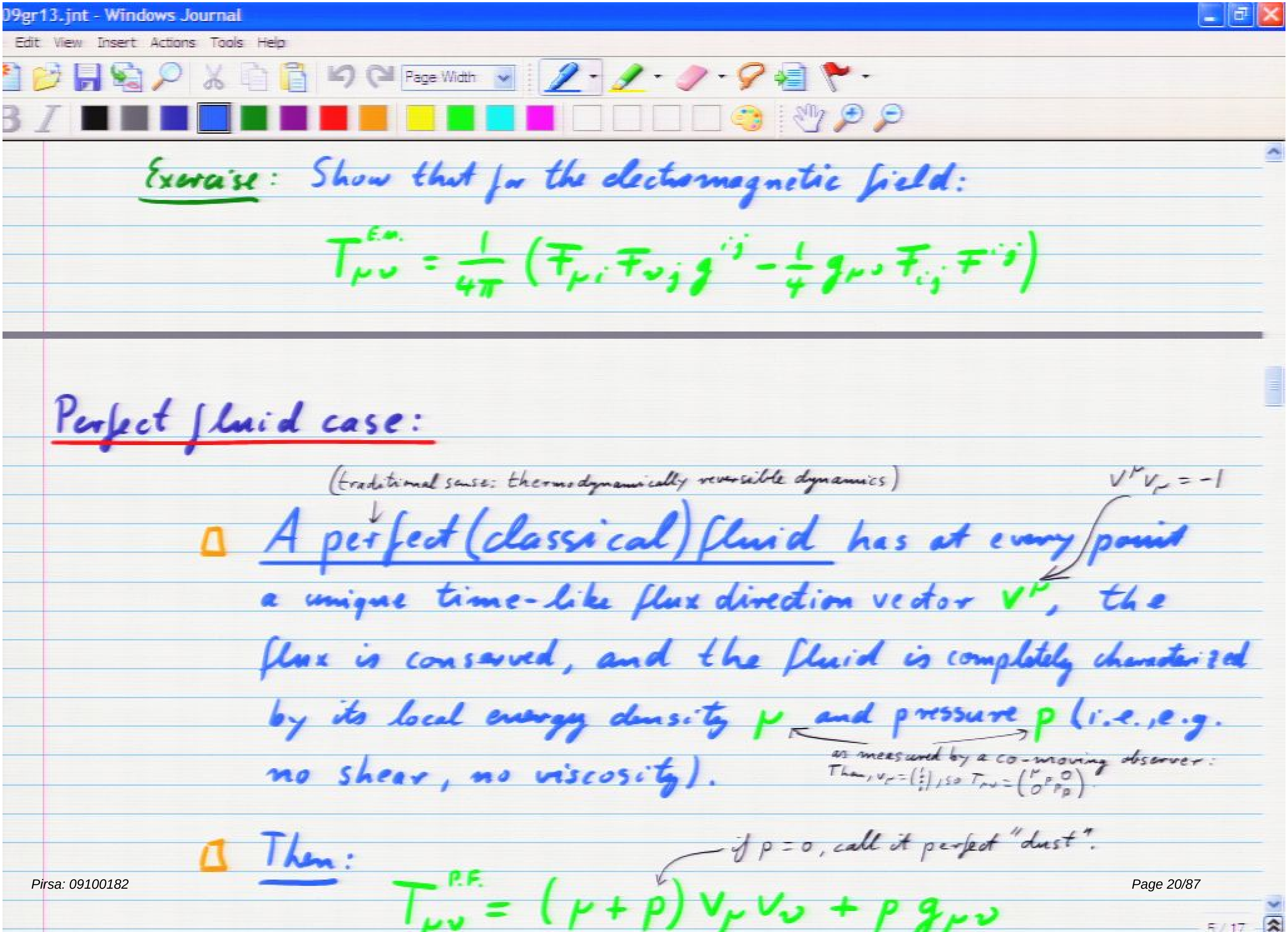

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Perfect fluid case:

(traditional sense: thermodynamically reversible dynamics)

$v^\mu v_\mu = -1$

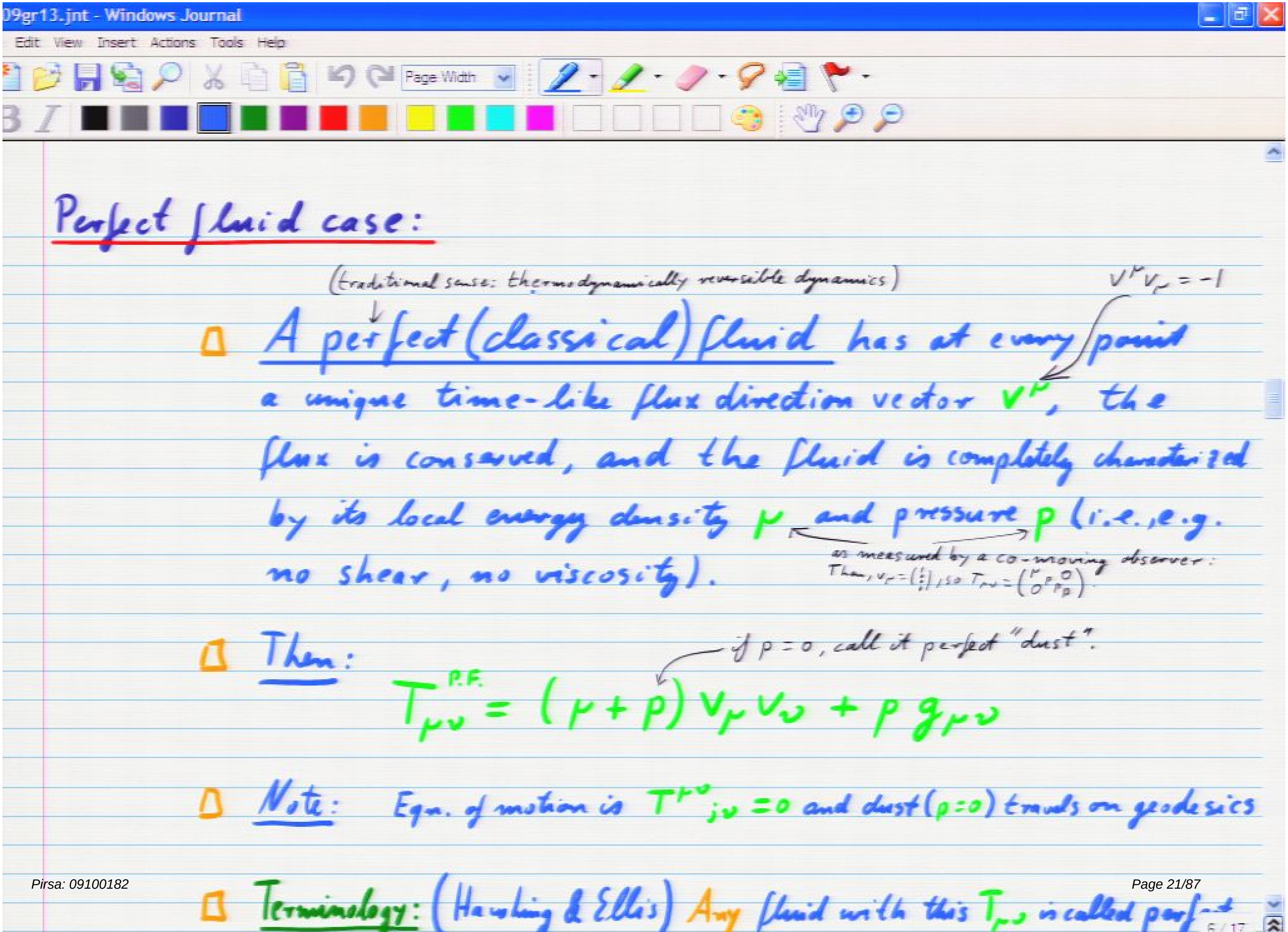
- A perfect (classical) fluid has at every point a unique time-like flux direction vector v^μ, the flux is conserved, and the fluid is completely characterized by its local energy density μ and pressure p (i.e., e.g. no shear, no viscosity).

as measured by a co-moving observer: Then, $v_\mu = \begin{pmatrix}1\\ \vdots\end{pmatrix}$, so $T_{\mu\nu} = \begin{pmatrix}\mu & p & 0\\ 0 & p & p\end{pmatrix}$.

- Then:

if $p=0$, call it perfect "dust".

$$T^{P.F.}_{\mu\nu} = (\mu + p)\, v_\mu v_\nu + p\, g_{\mu\nu}$$



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as measured by a co-moving observer: Then, $v_\mu = \begin{pmatrix}1\\ \vdots\end{pmatrix}$, so $T_{\mu\nu} = \begin{pmatrix}\mu & 0\\ 0 & p\delta\end{pmatrix}$.

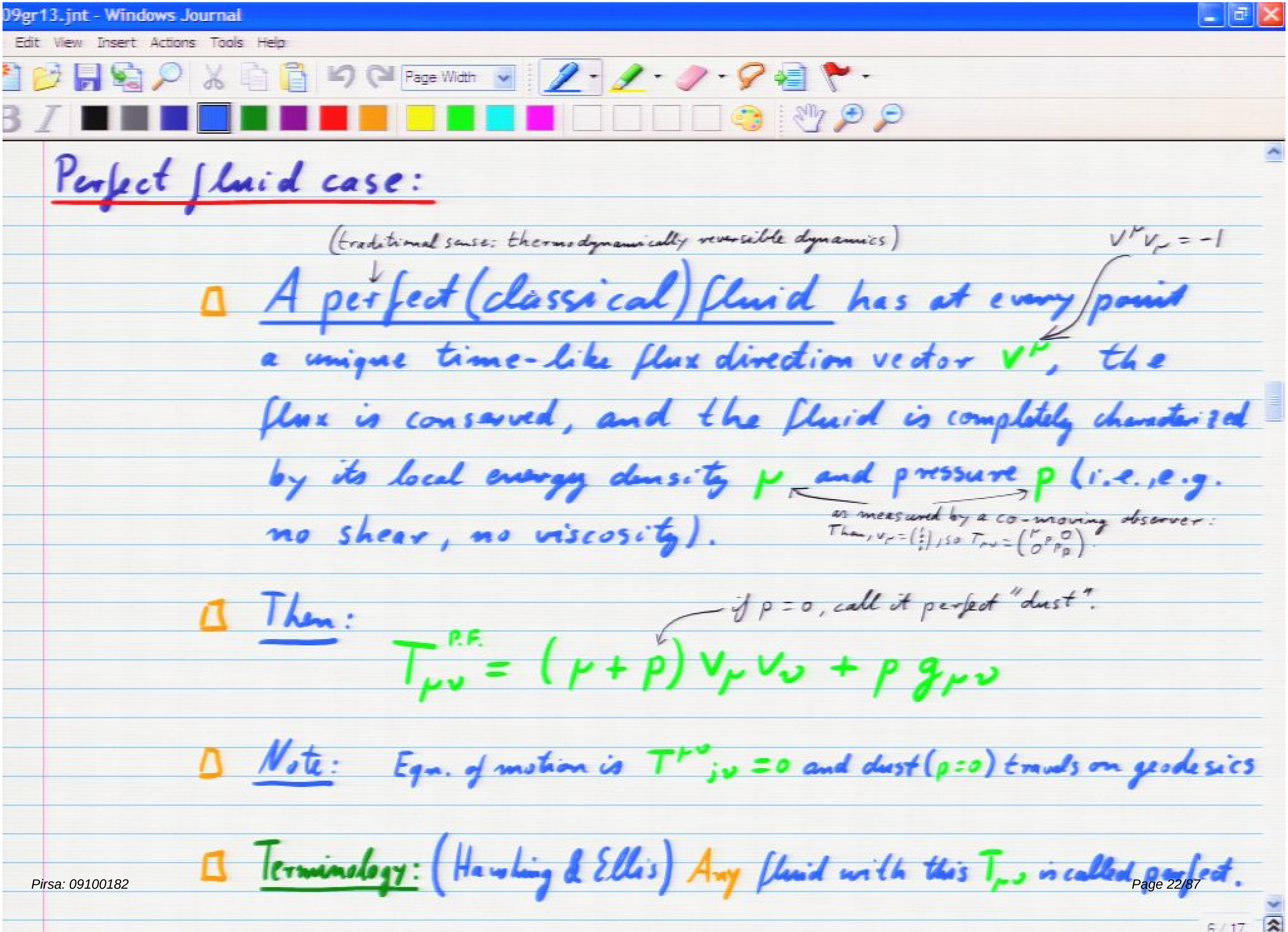
☐ Then:

if $p=0$, call it perfect "dust".

$$T^{P.F.}_{\mu\nu} = (\mu + p)\, v_\mu v_\nu + p\, g_{\mu\nu}$$

☐ Note: Eqn. of motion is ${T^{\mu\nu}}_{;\nu} = 0$ and dust $(p=0)$ travels on geodesics

☐ Terminology: (Hawking & Ellis) Any fluid with this $T_{\mu\nu}$ is called perfect



Perfect fluid case:

(traditional sense: thermodynamically reversible dynamics)

$V^\mu V_\mu = -1$

☐ A perfect (classical) fluid has at every point a unique time-like flux direction vector V^μ, the flux is conserved, and the fluid is completely characterized by its local energy density μ and pressure p (i.e., e.g. no shear, no viscosity).

as measured by a co-moving observer: Then, $v_\mu = \begin{pmatrix}1\\0\\0\end{pmatrix}$, so $T_{\mu\nu} = \begin{pmatrix}\mu & & 0\\ 0 & p & p\,p\end{pmatrix}$.

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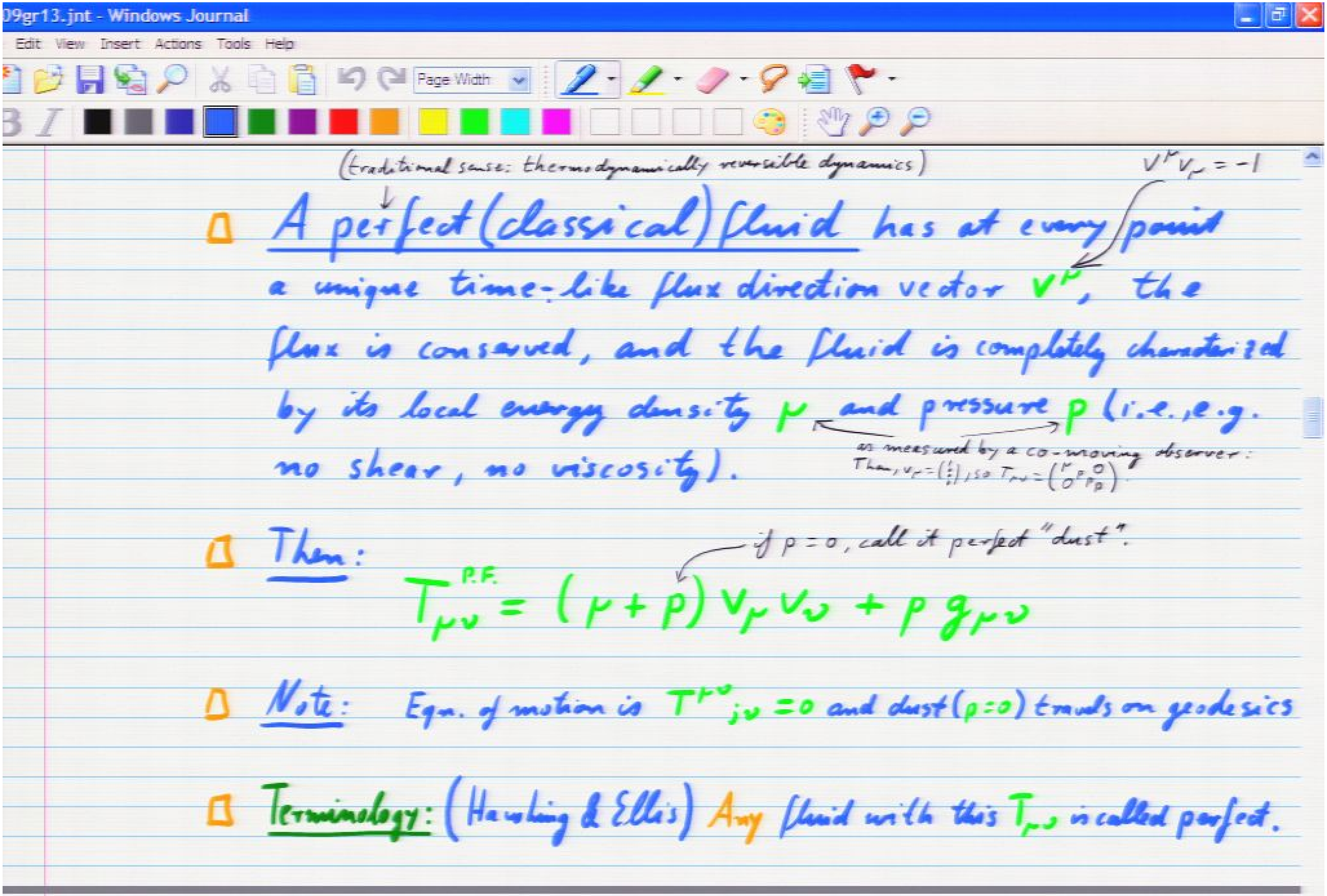

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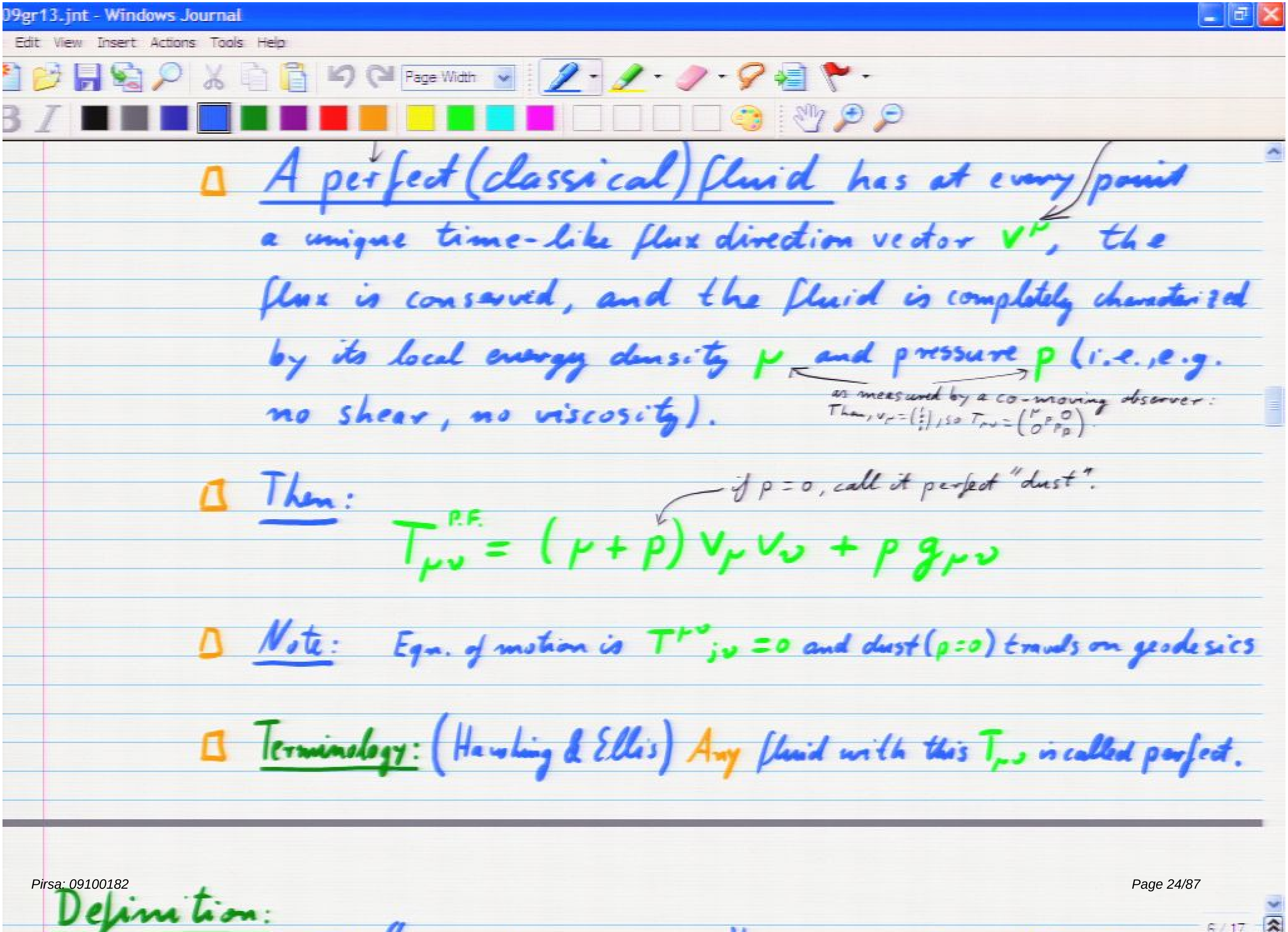

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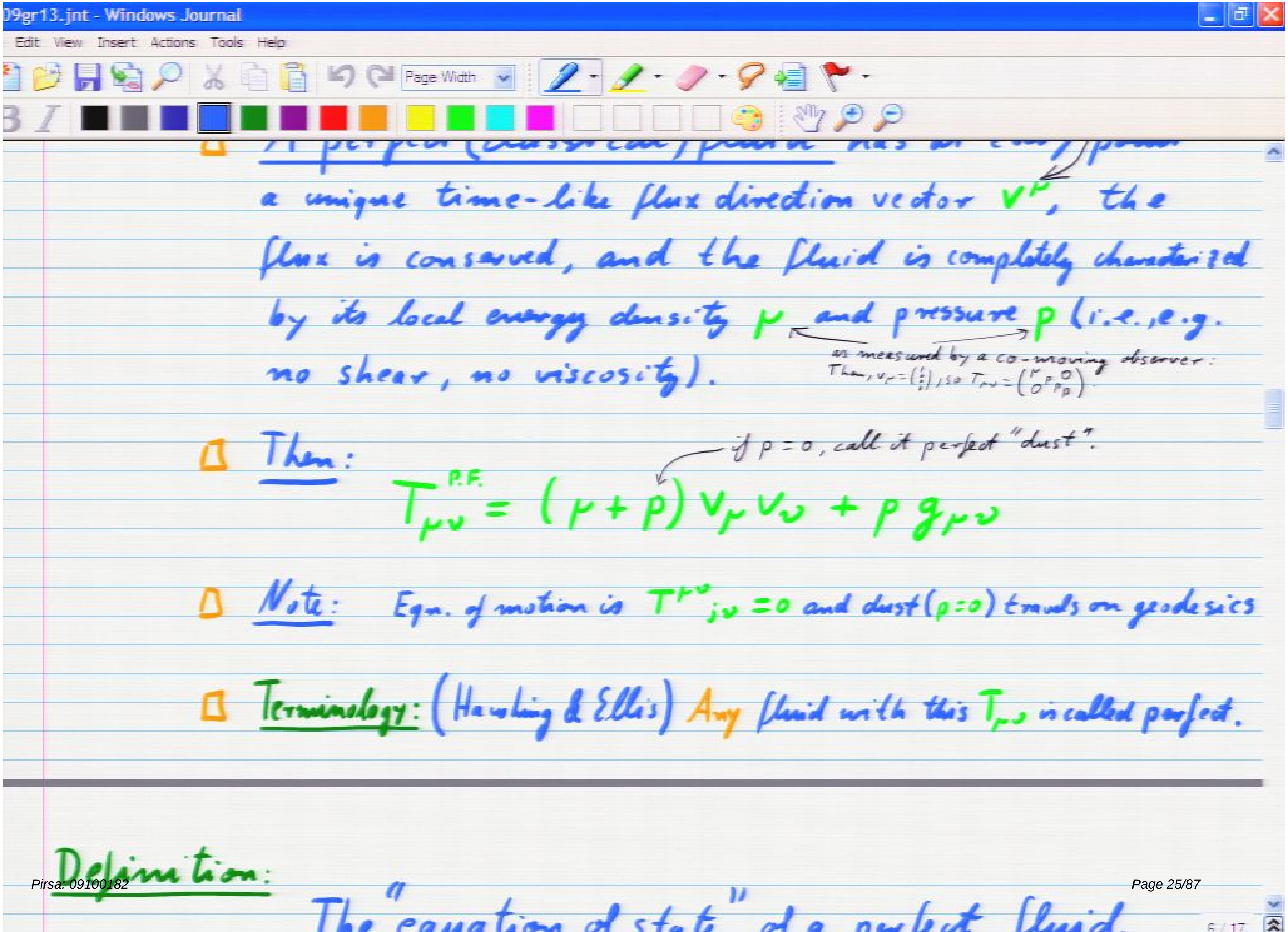
Definition:



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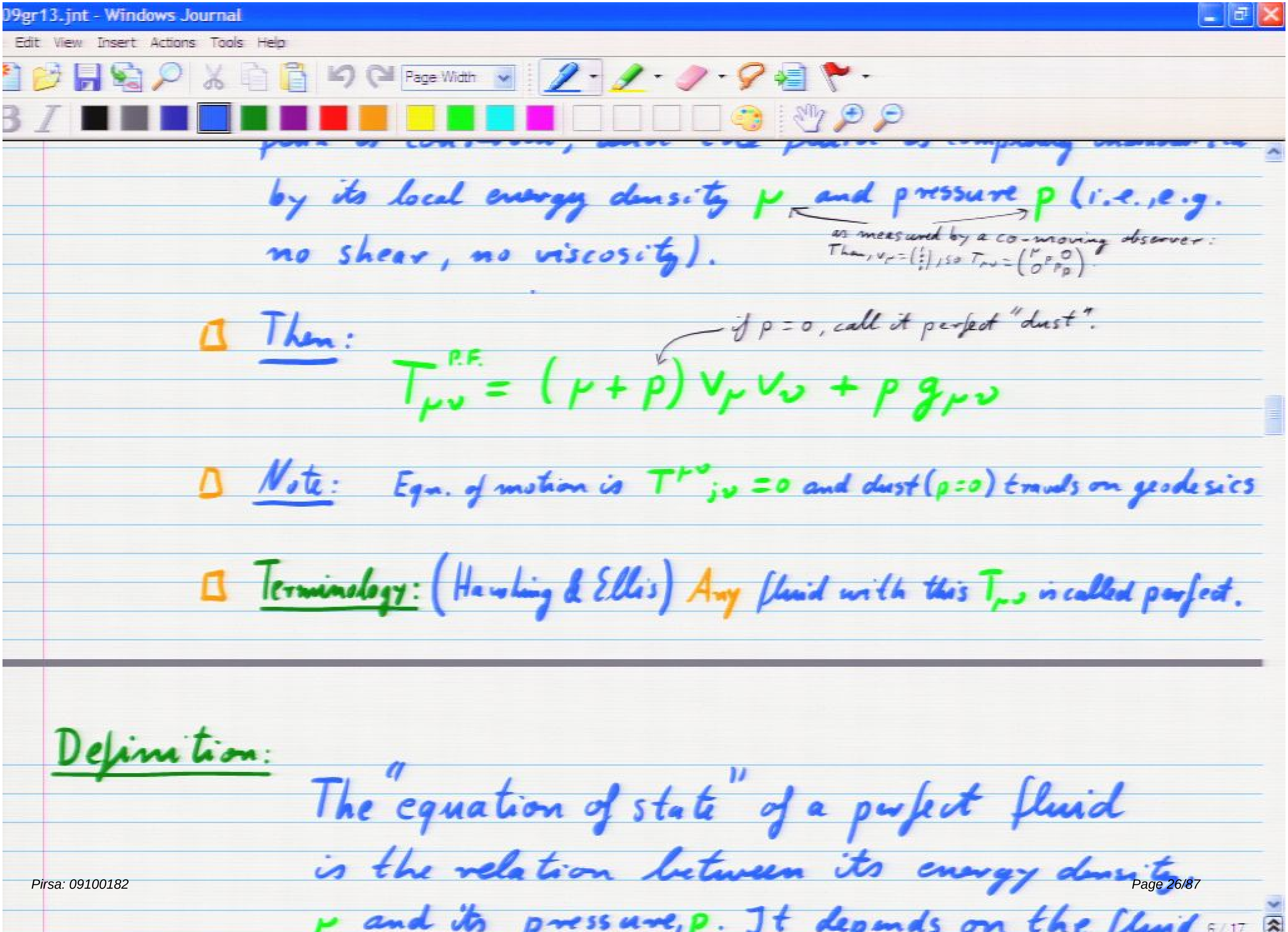
Definition:

The "equation of state" of a perfect fluid



by its local energy density μ and pressure p (i.e., e.g. no shear, no viscosity).

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$$T^{P.F.}_{\mu\nu} = (\mu + p) v_\mu v_\nu + p\, g_{\mu\nu}$$

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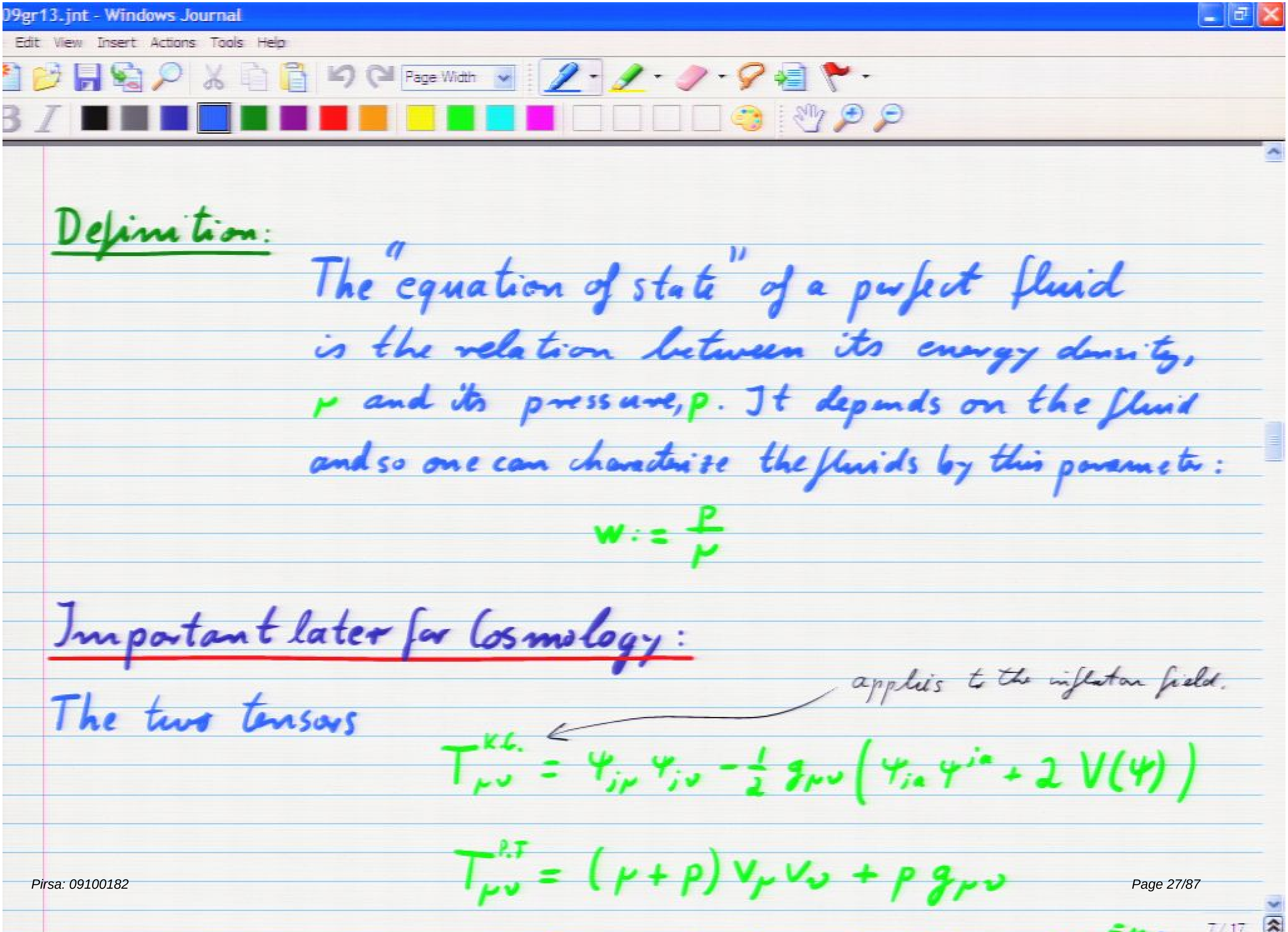
Definition:

The "equation of state" of a perfect fluid is the relation between its energy density μ and its pressure, p. It depends on the fluid



Definition:

The "equation of state" of a perfect fluid is the relation between its energy density, μ and its pressure, p. It depends on the fluid and so one can characterize the fluids by this parameter:

$$w := \frac{p}{\mu}$$

Important later for Cosmology:

The two tensors

applies to the inflaton field.

$$T^{K.G.}_{\mu\nu} = \varphi_{;\mu}\varphi_{;\nu} - \frac{1}{2} g_{\mu\nu}\left(\varphi_{;a}\varphi^{;a} + 2V(\varphi)\right)$$

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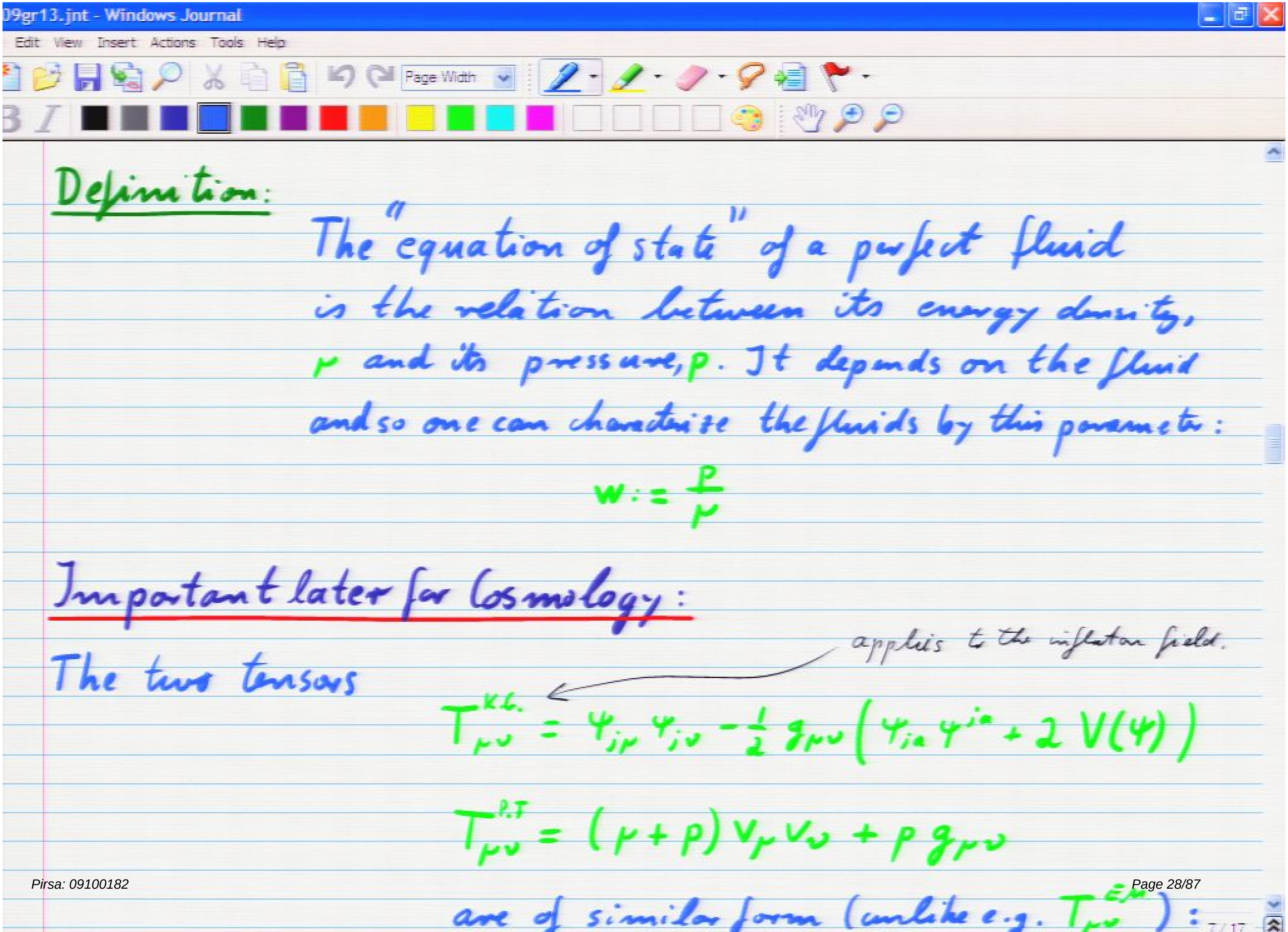

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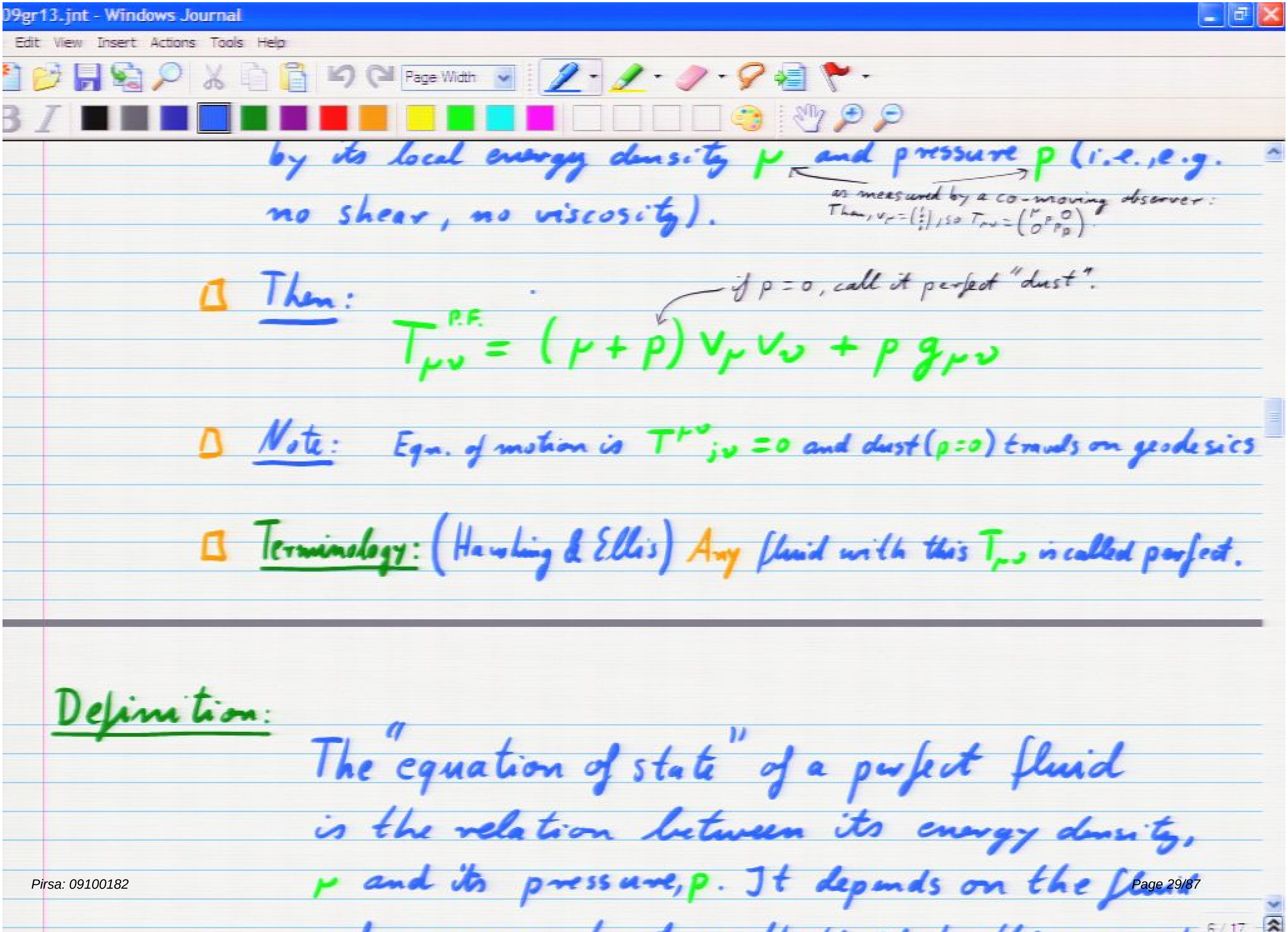

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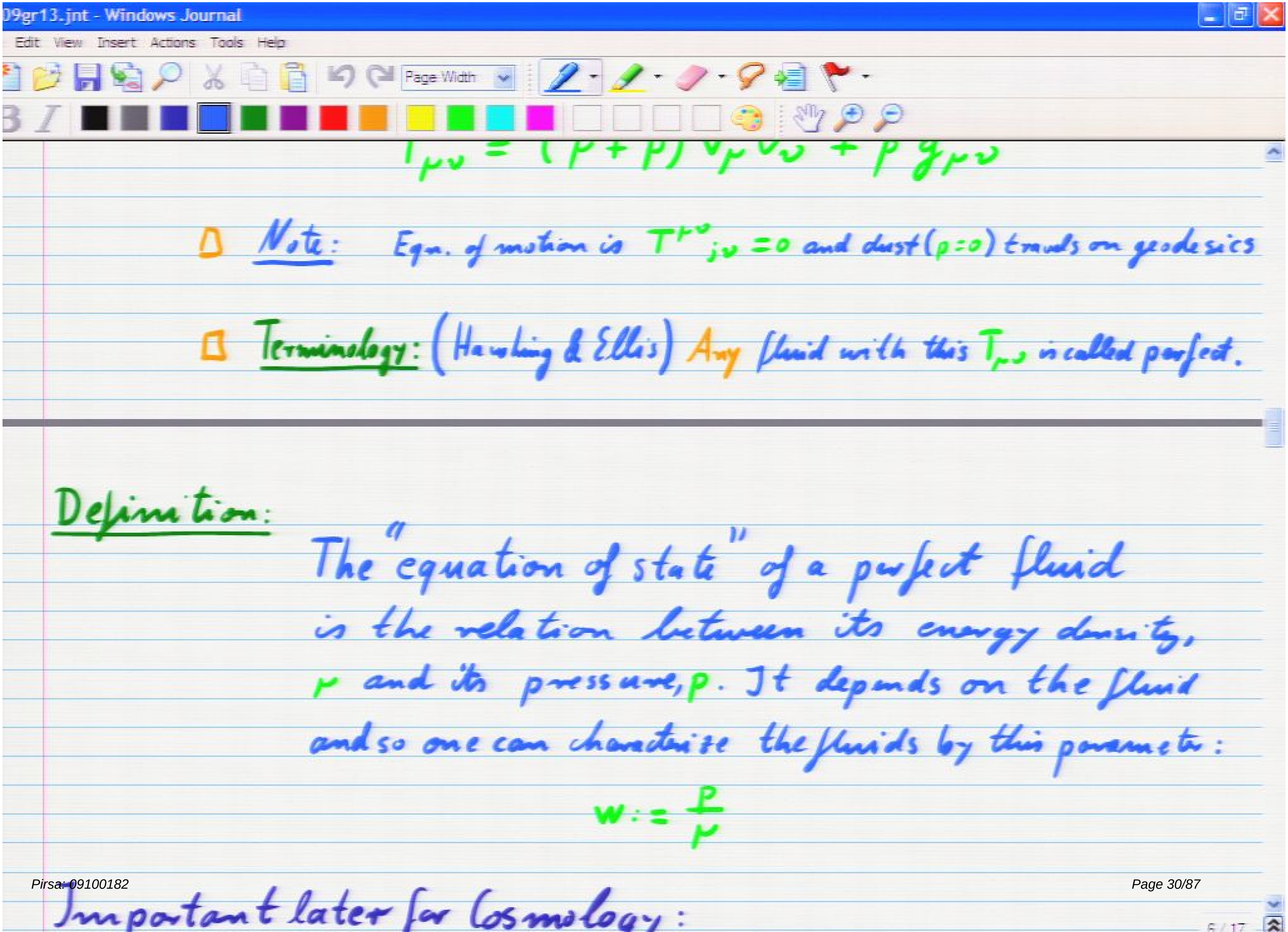

$$T_{\mu\nu} = (\rho + p) v_\mu v_\nu + p g_{\mu\nu}$$

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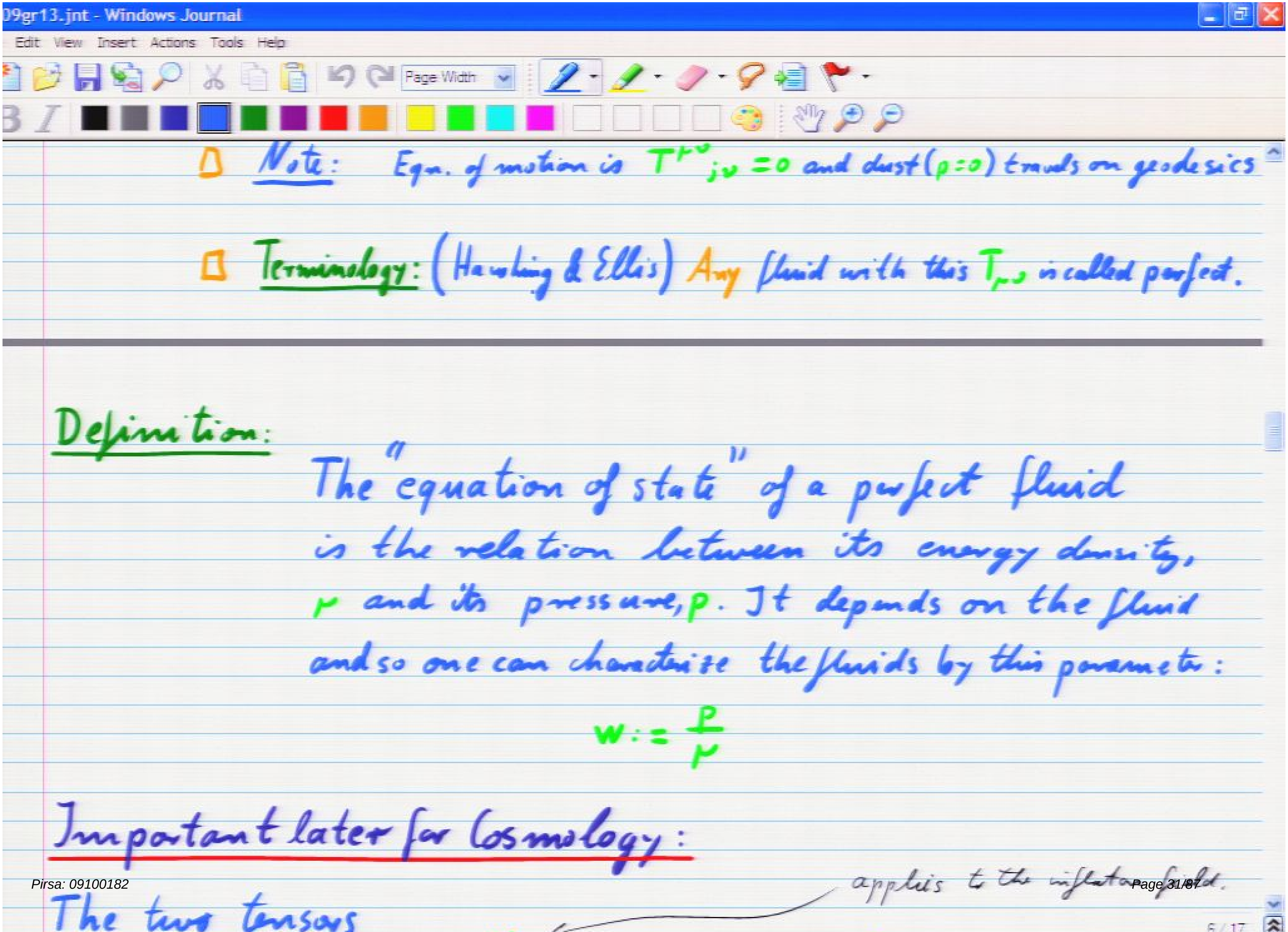

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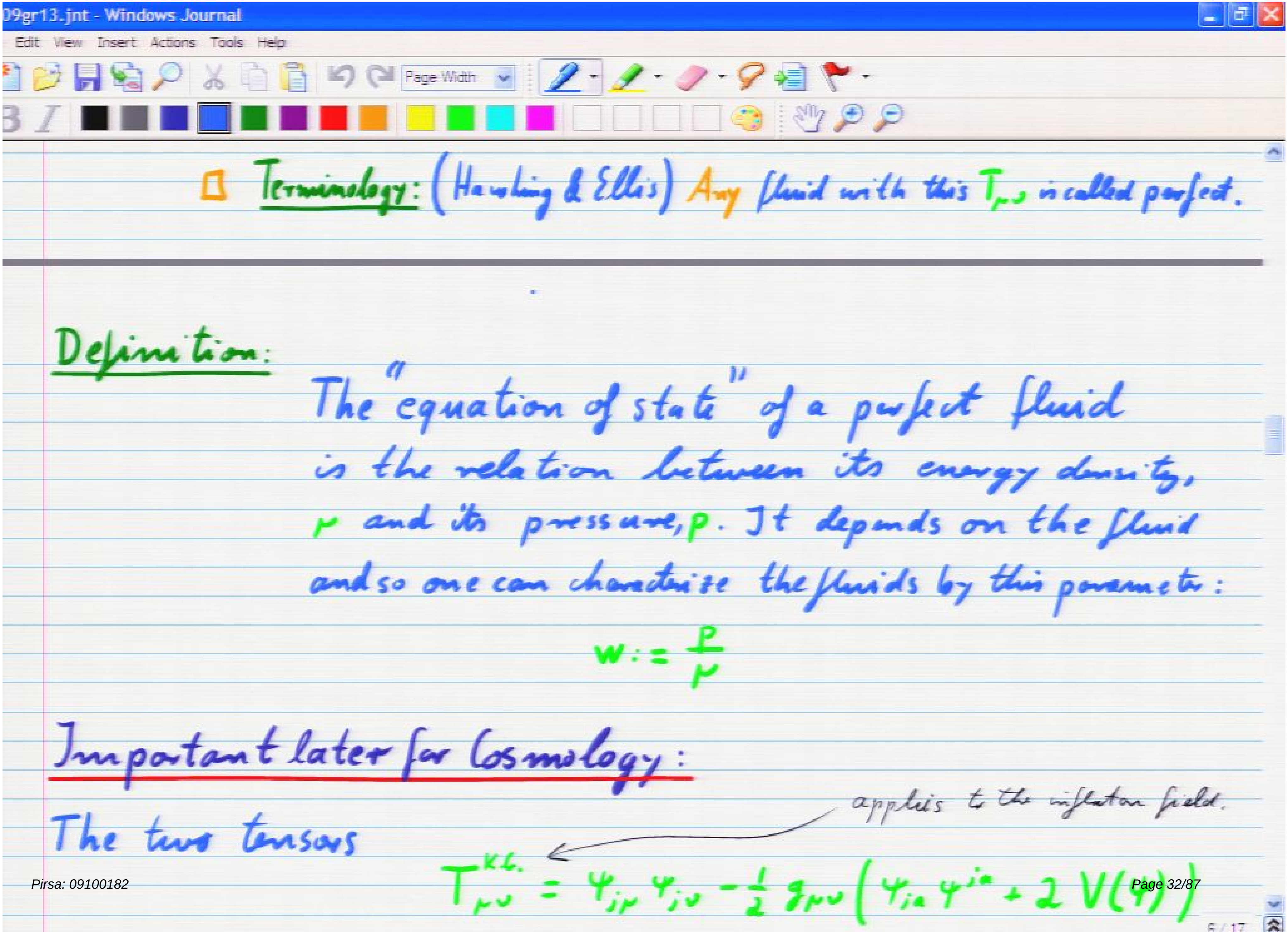
The two tensors



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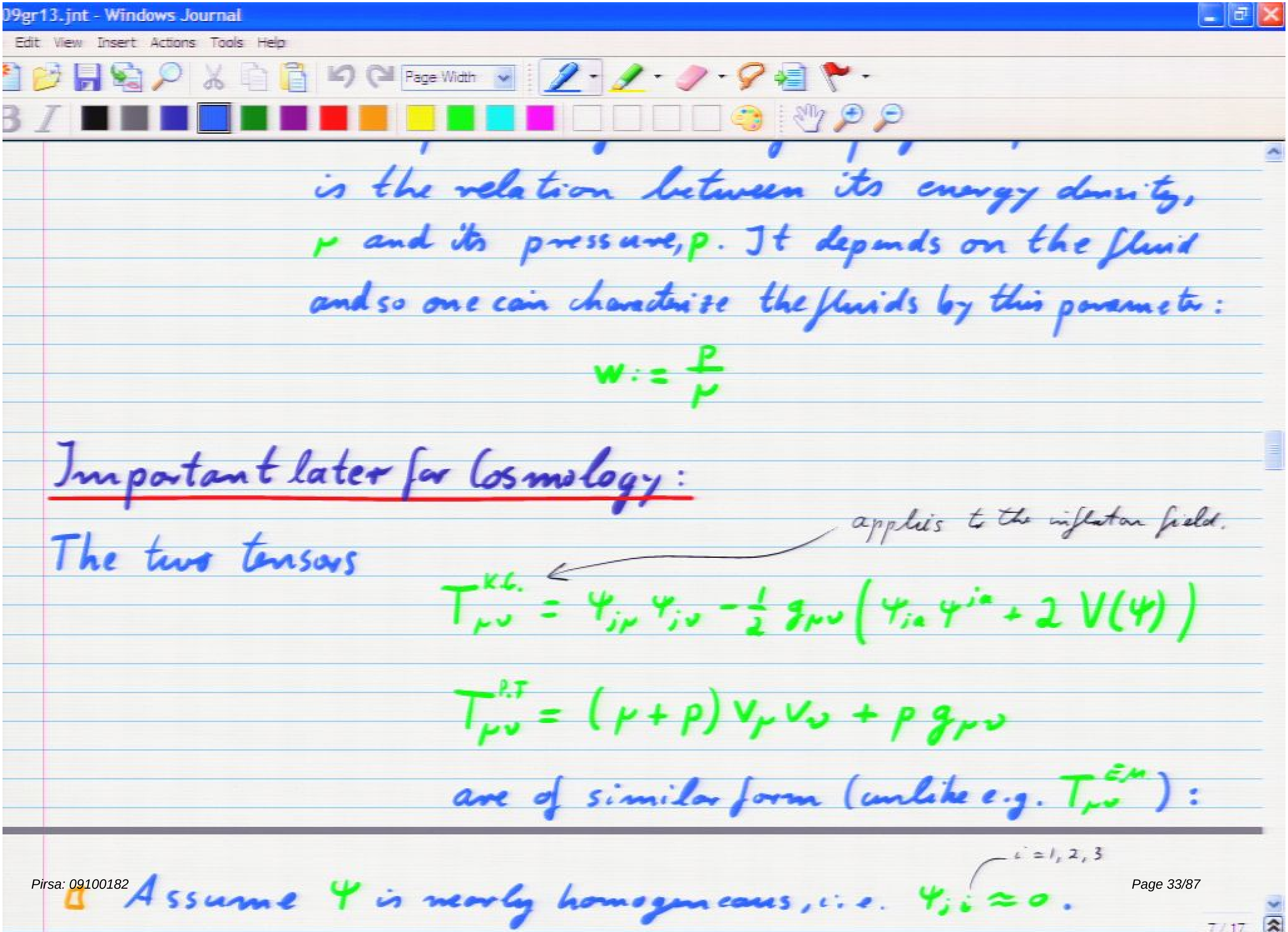

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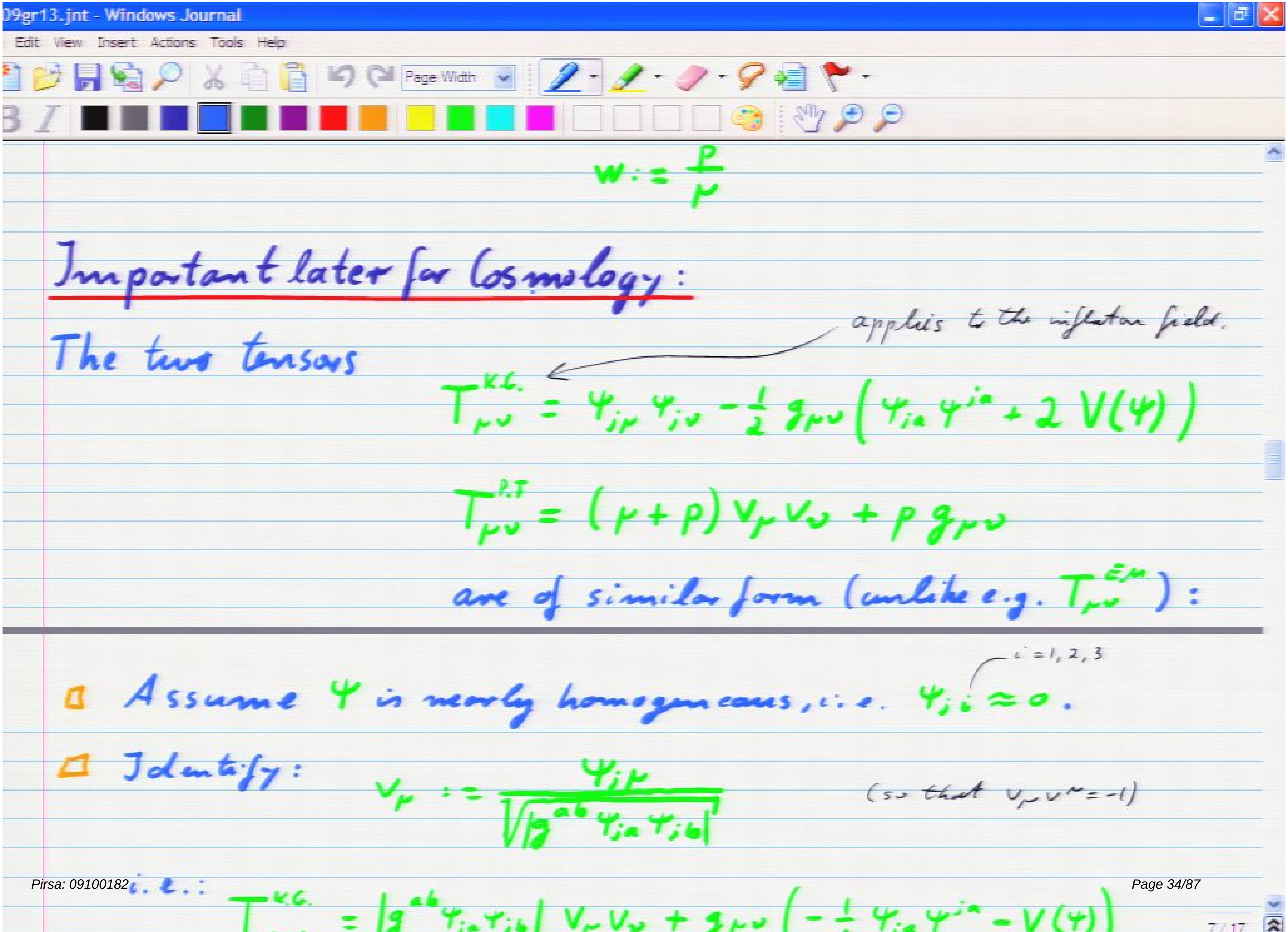
are of similar form (unlike e.g. $T^{EM}_{\mu\nu}$):

Assume φ is nearly homogeneous, i.e. $\underset{i=1,2,3}{\varphi_{;i}} \approx 0$.



$$w := \frac{p}{\rho}$$

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The two tensors

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i.e.:

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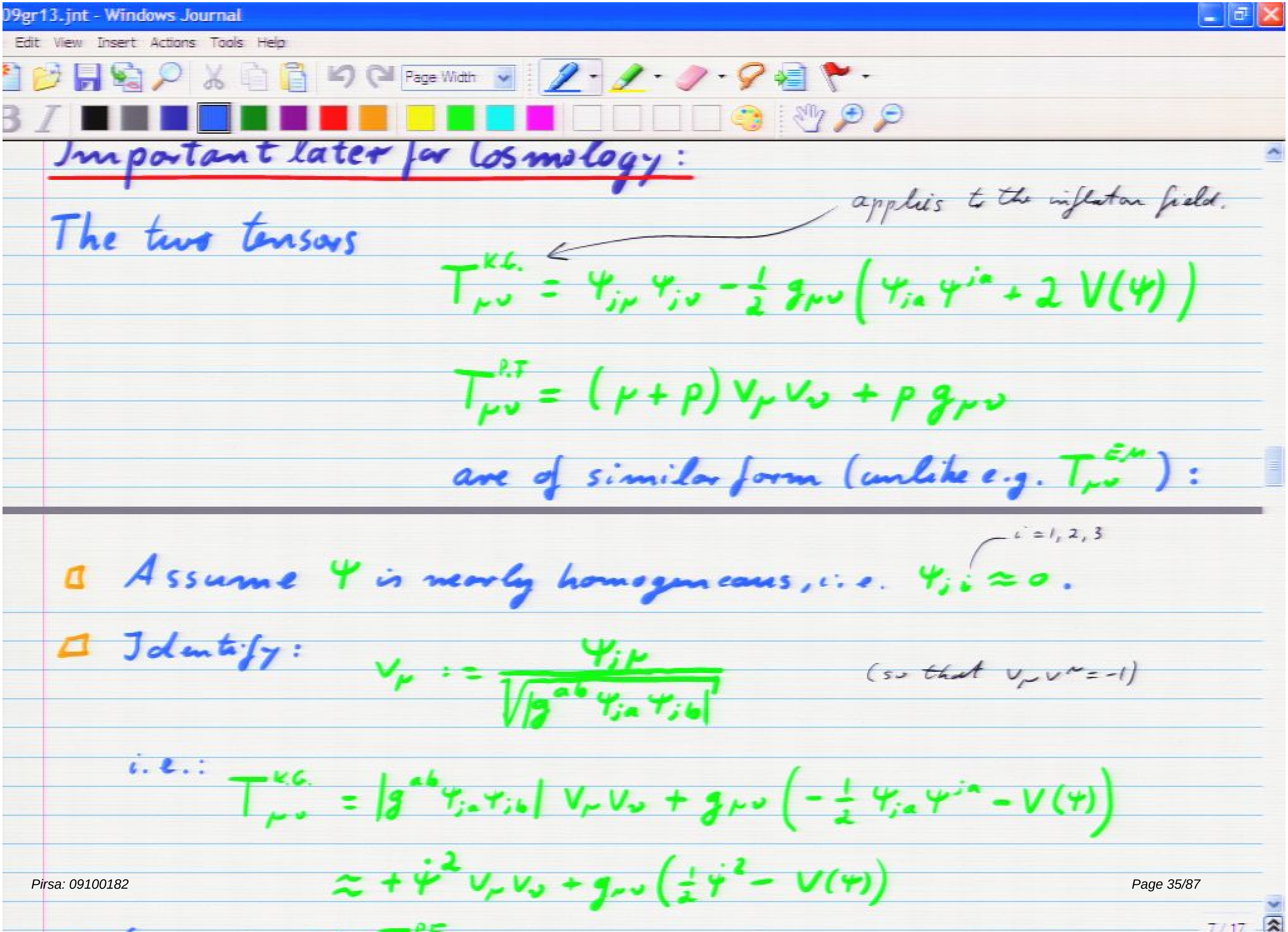

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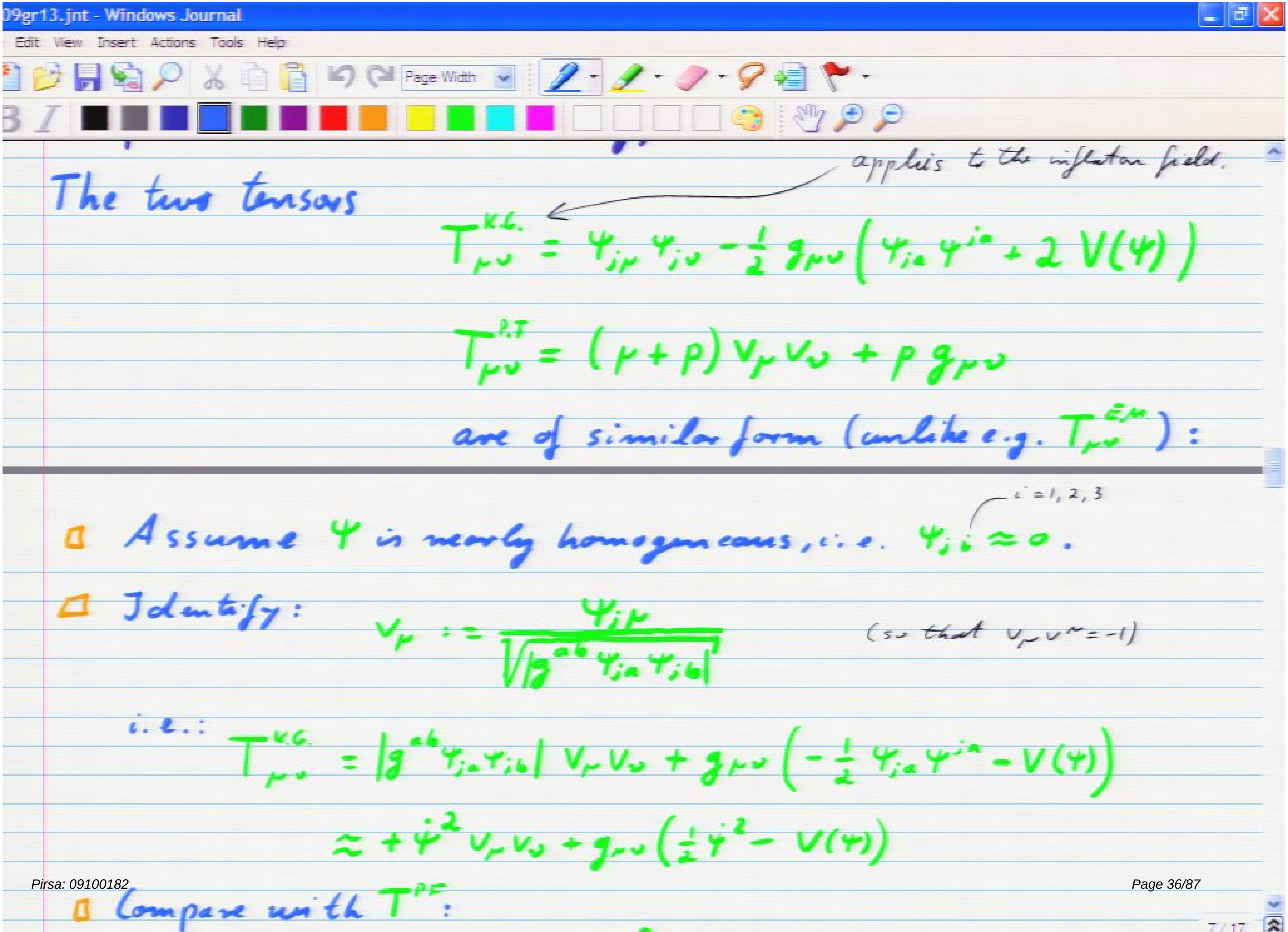
$$\approx +\dot{\varphi}^2 v_\mu v_\nu + g_{\mu\nu}\left(\frac{1}{2}\dot{\varphi}^2 - V(\varphi)\right)$$



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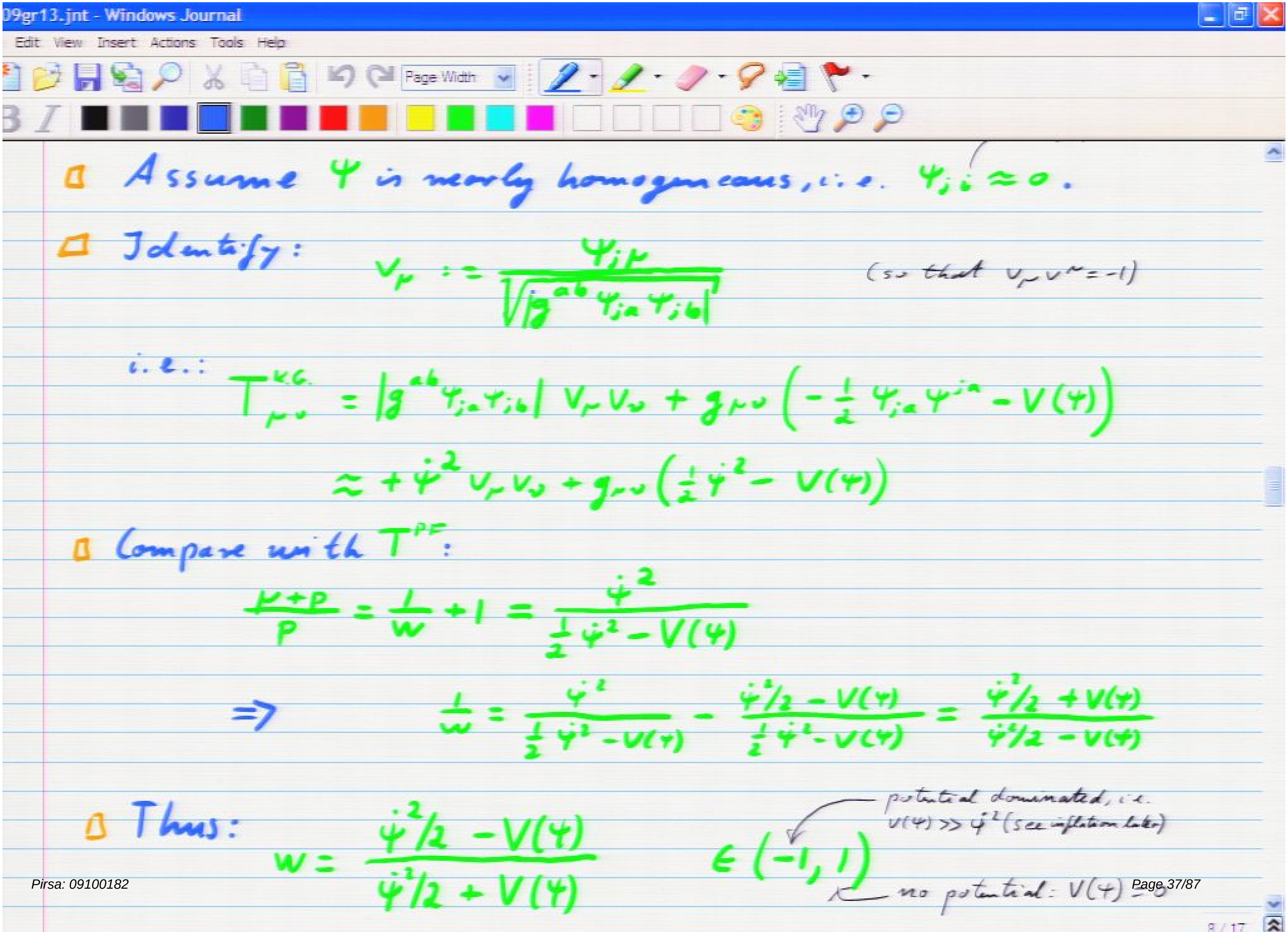
☐ Compare with T^{PF}:



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$$v_\mu := \frac{\varphi_{;\mu}}{\sqrt{|g^{ab}\varphi_{;a}\varphi_{;b}|}} \qquad (\text{so that } v_\mu v^\mu = -1)$$

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☐ Compare with T^{PF}:

$$\frac{\mu + p}{p} = \frac{1}{w} + 1 = \frac{\dot{\varphi}^2}{\frac{1}{2}\dot{\varphi}^2 - V(\varphi)}$$

$$\Rightarrow \qquad \frac{1}{w} = \frac{\dot{\varphi}^2}{\frac{1}{2}\dot{\varphi}^2 - V(\varphi)} - \frac{\dot{\varphi}^2/2 - V(\varphi)}{\frac{1}{2}\dot{\varphi}^2 - V(\varphi)} = \frac{\dot{\varphi}^2/2 + V(\varphi)}{\dot{\varphi}^2/2 - V(\varphi)}$$

☐ Thus:

$$w = \frac{\dot{\varphi}^2/2 - V(\varphi)}{\dot{\varphi}^2/2 + V(\varphi)} \qquad \in (-1, 1)$$

potential dominated, i.e. $V(\varphi) \gg \dot{\varphi}^2$ (see inflation later)

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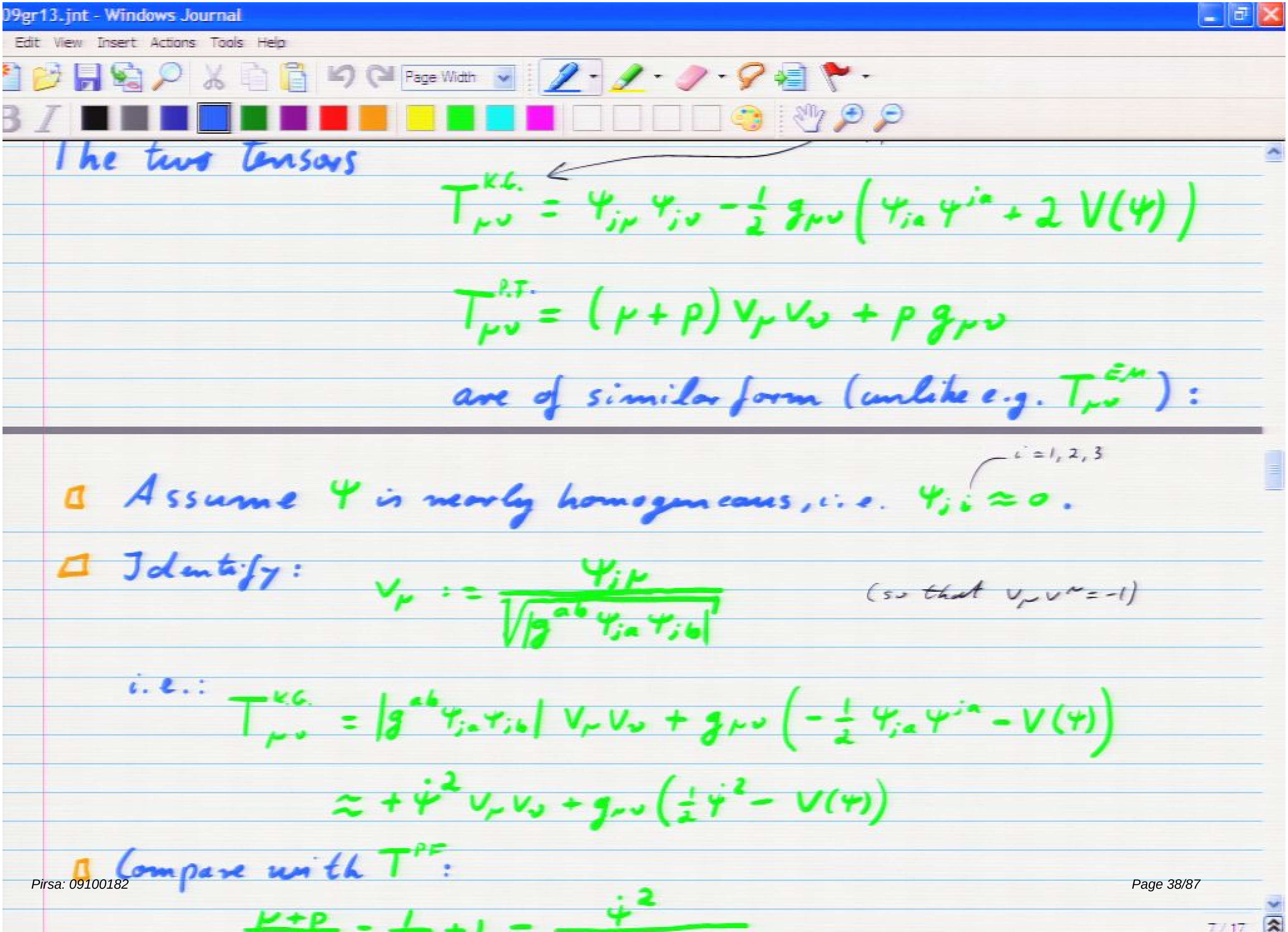

The two tensors

$$T^{K.G.}_{\mu\nu} = \psi_{;\mu}\psi_{;\nu} - \frac{1}{2} g_{\mu\nu}\left(\psi_{;a}\psi^{;a} + 2V(\psi)\right)$$

$$T^{P.F.}_{\mu\nu} = (\rho + p) V_\mu V_\nu + p\, g_{\mu\nu}$$

are of similar form (unlike e.g. $T^{EM}_{\mu\nu}$):

- Assume ψ is nearly homogeneous, i.e. $\overset{i=1,2,3}{\psi_{;i}} \approx 0$.
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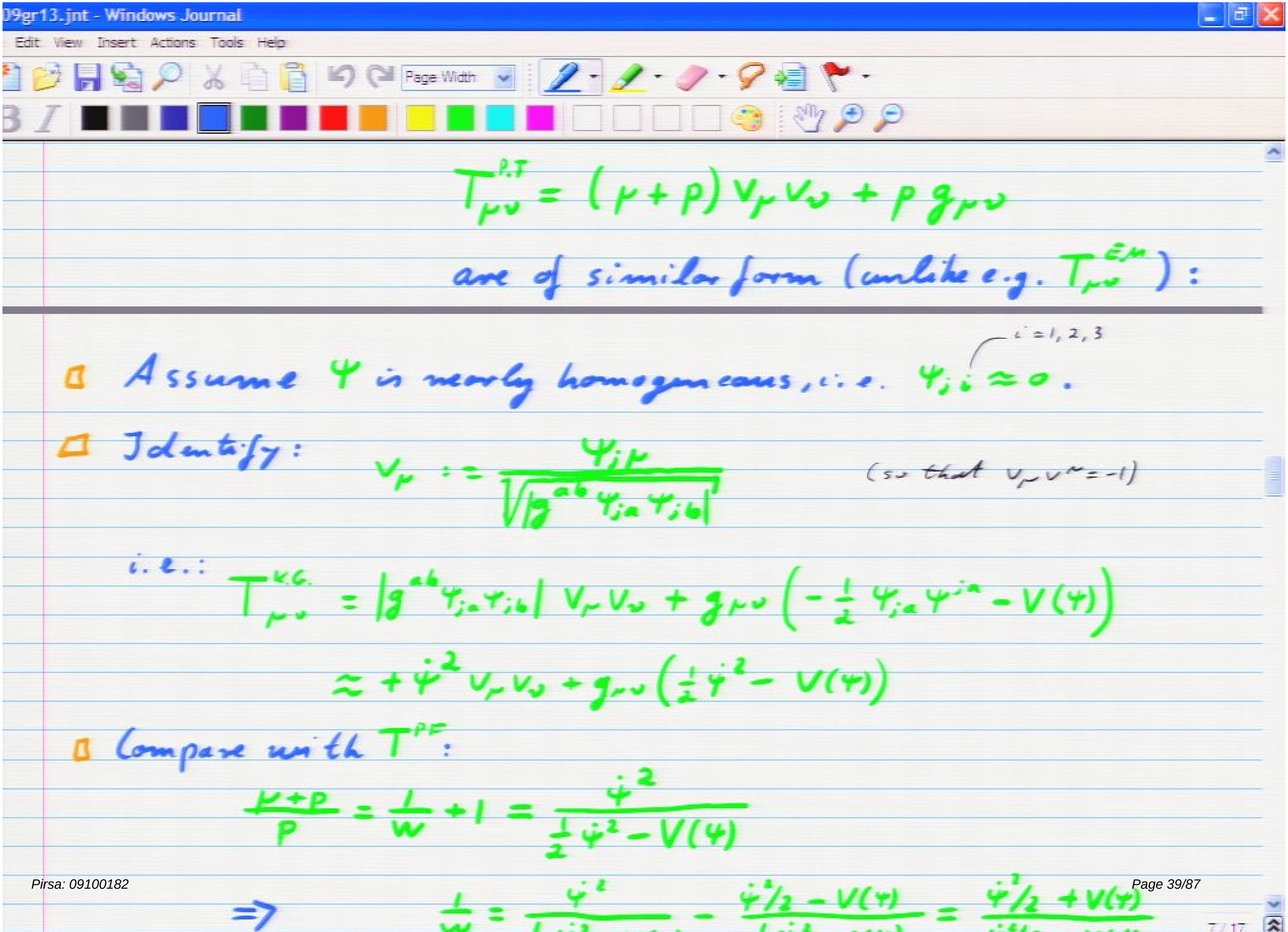

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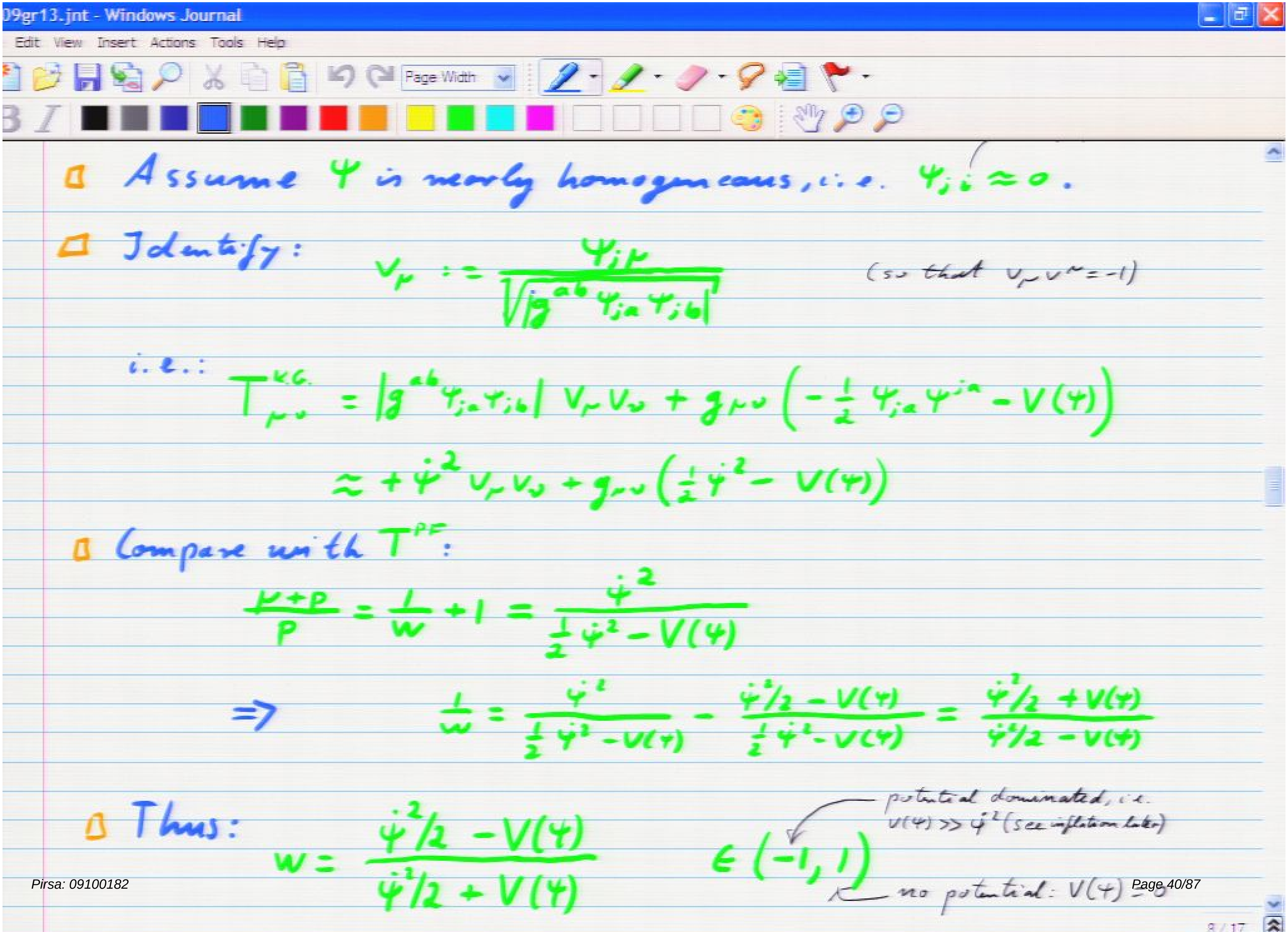

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potential dominated, i.e. $V(\psi) >> \dot{\psi}^2$ (see inflation later)

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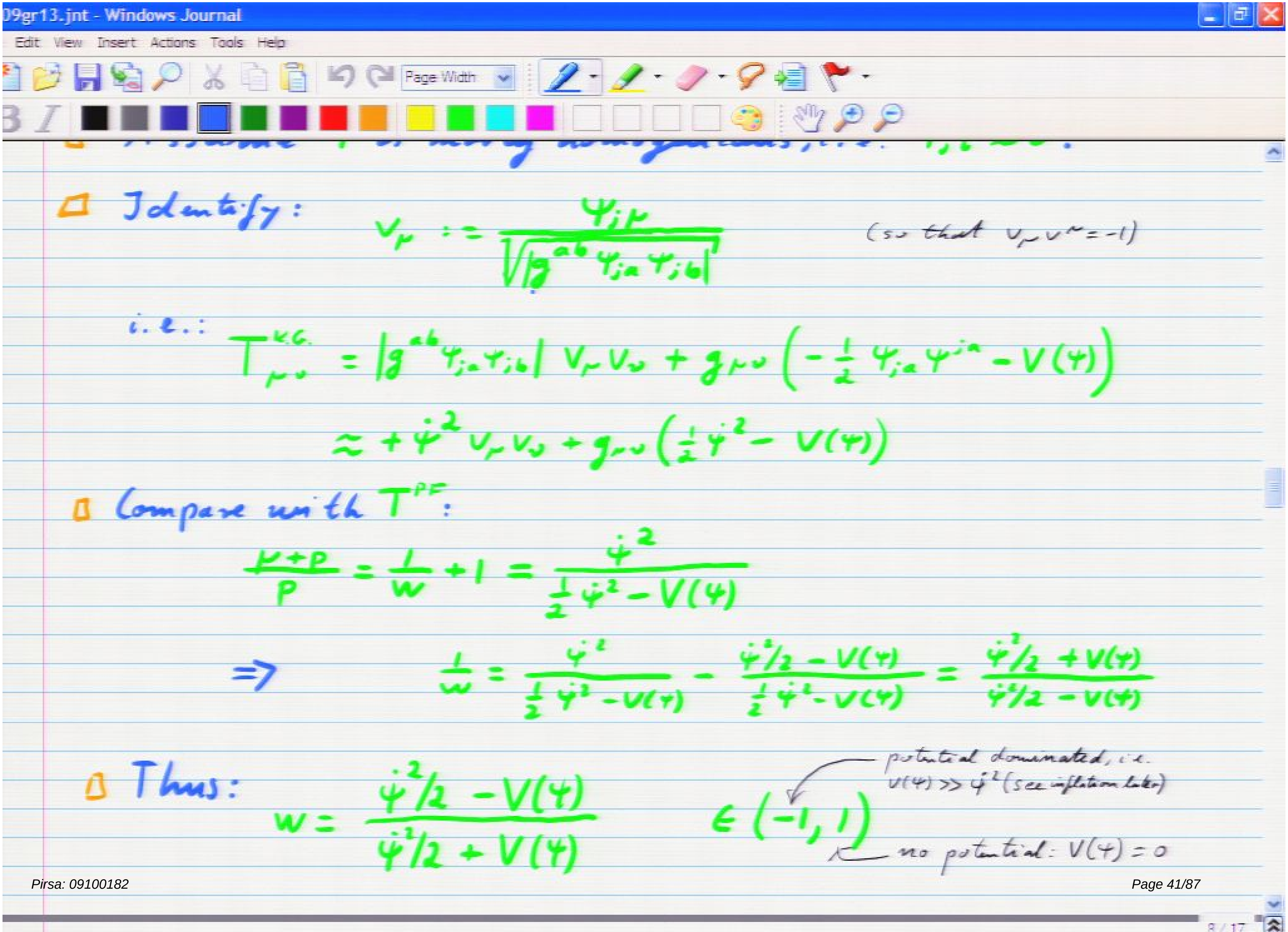

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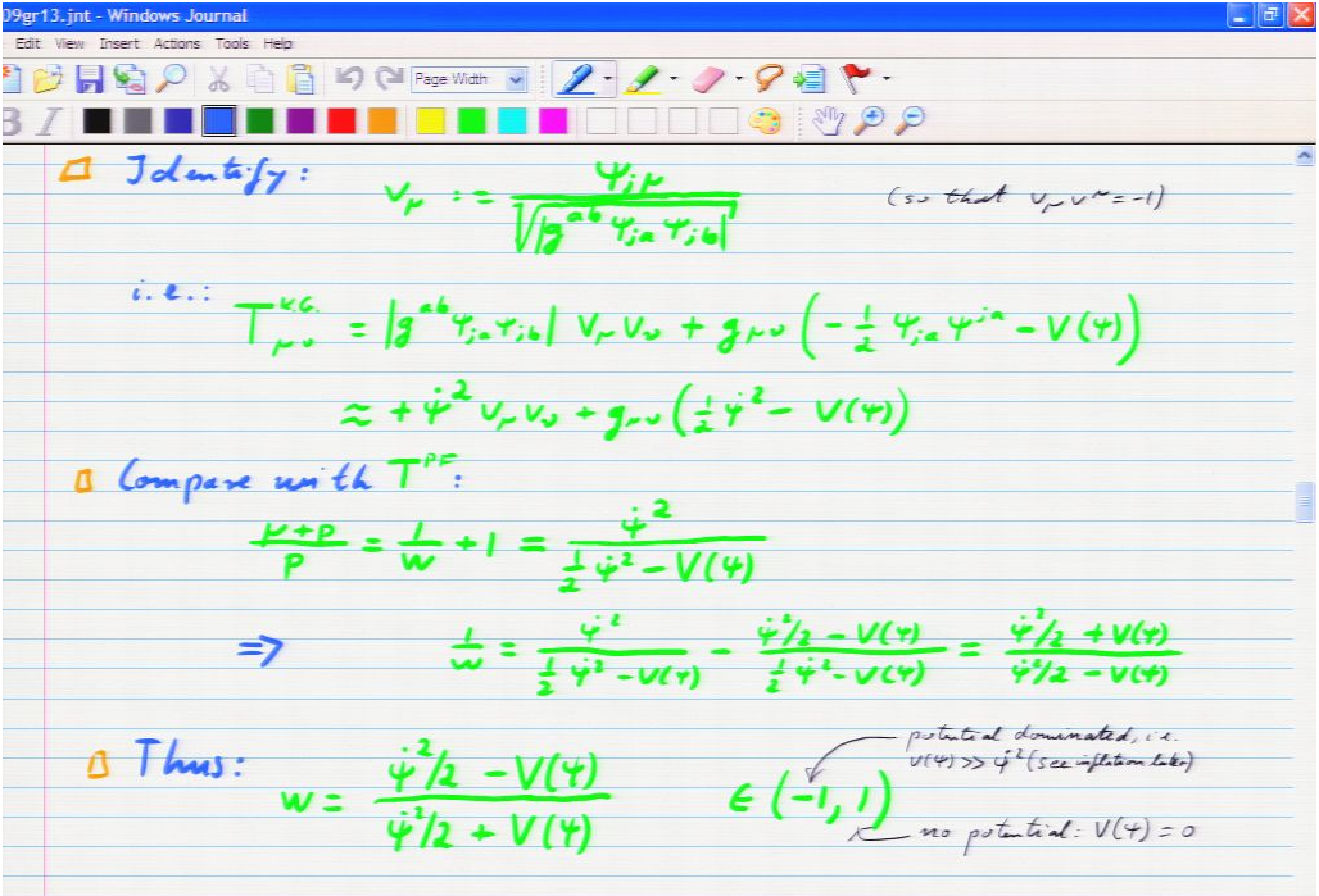

☐ Identify: $$v_\mu := \frac{\psi_{;\mu}}{\sqrt{|g^{ab}\psi_{;a}\psi_{;b}|}}$$ (so that $v_\mu v^\mu = -1$)

i.e.: $$T^{K.G.}_{\mu\nu} = |g^{ab}\psi_{;a}\psi_{;b}|\, v_\mu v_\nu + g_{\mu\nu}\left(-\frac{1}{2}\psi_{;a}\psi^{;a} - V(\psi)\right)$$

$$\approx +\dot{\psi}^2 v_\mu v_\nu + g_{\mu\nu}\left(\frac{1}{2}\dot{\psi}^2 - V(\psi)\right)$$

☐ Compare with T^{PF}:

$$\frac{\rho+p}{p} = \frac{1}{w} + 1 = \frac{\dot{\psi}^2}{\frac{1}{2}\dot{\psi}^2 - V(\psi)}$$

$$\Rightarrow \qquad \frac{1}{w} = \frac{\dot{\psi}^2}{\frac{1}{2}\dot{\psi}^2 - V(\psi)} - \frac{\dot{\psi}^2/2 - V(\psi)}{\frac{1}{2}\dot{\psi}^2 - V(\psi)} = \frac{\dot{\psi}^2/2 + V(\psi)}{\dot{\psi}^2/2 - V(\psi)}$$

☐ Thus: $$w = \frac{\dot{\psi}^2/2 - V(\psi)}{\dot{\psi}^2/2 + V(\psi)} \in (-1, 1)$$

potential dominated, i.e. $V(\psi) \gg \dot{\psi}^2$ (see inflation later)

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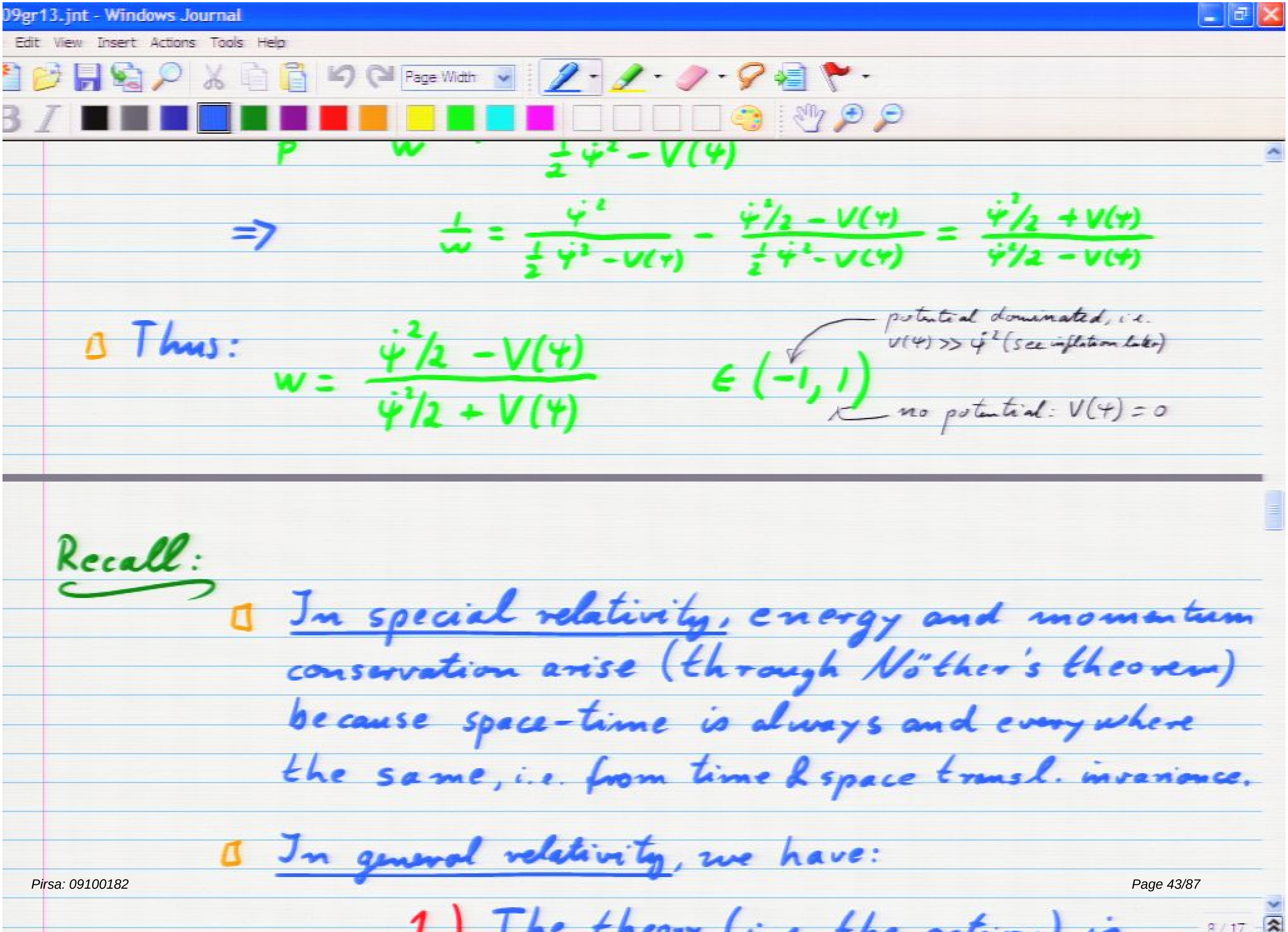

$$p \qquad w \qquad \frac{1}{2}\dot{\varphi}^2 - V(\varphi)$$

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Recall:

- In special relativity, energy and momentum conservation arise (through Nöther's theorem) because space-time is always and everywhere the same, i.e. from time & space transl. invariance.

- In general relativity, we have:

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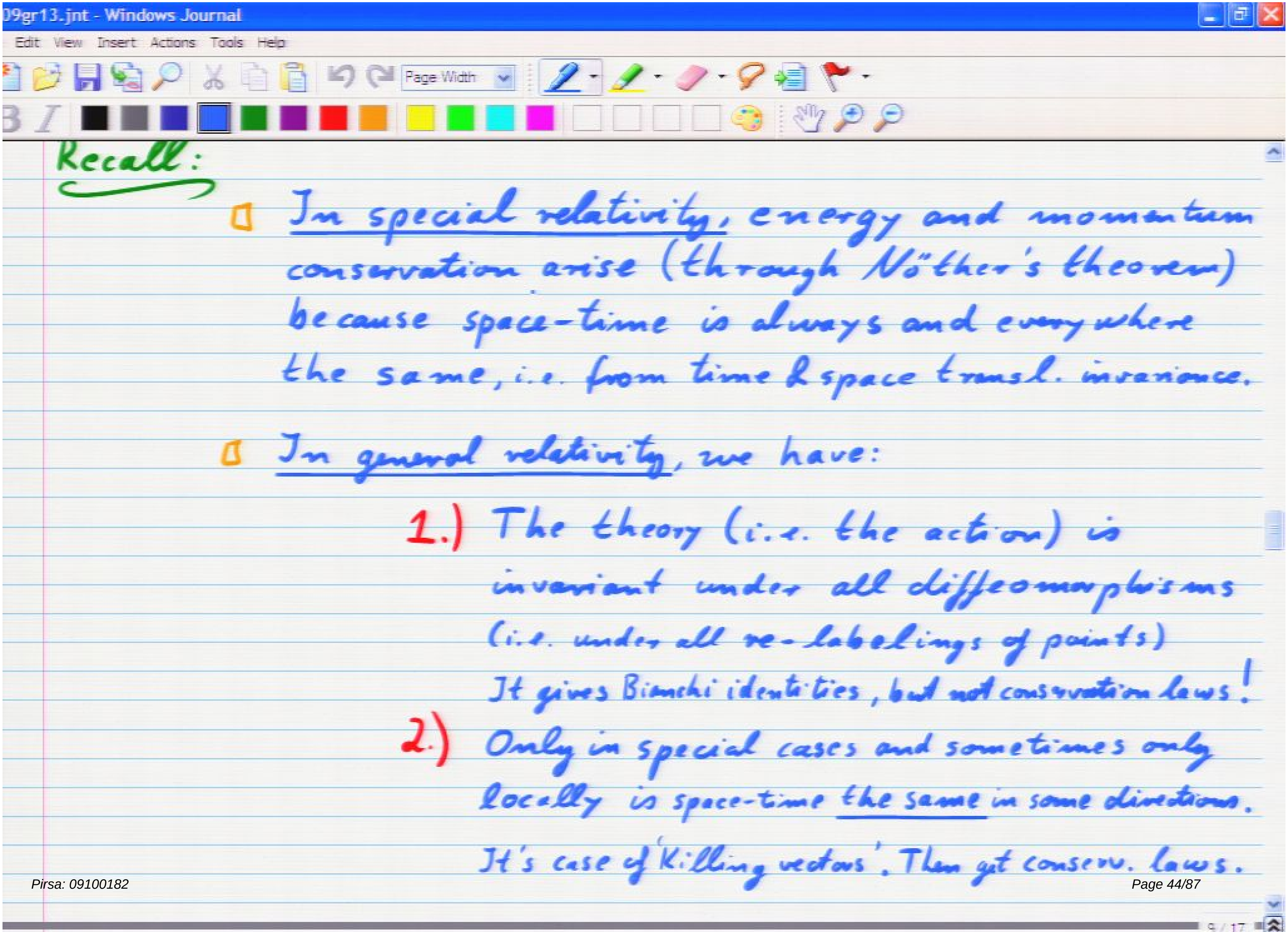

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  It gives Bianchi identities, but not conservation laws!

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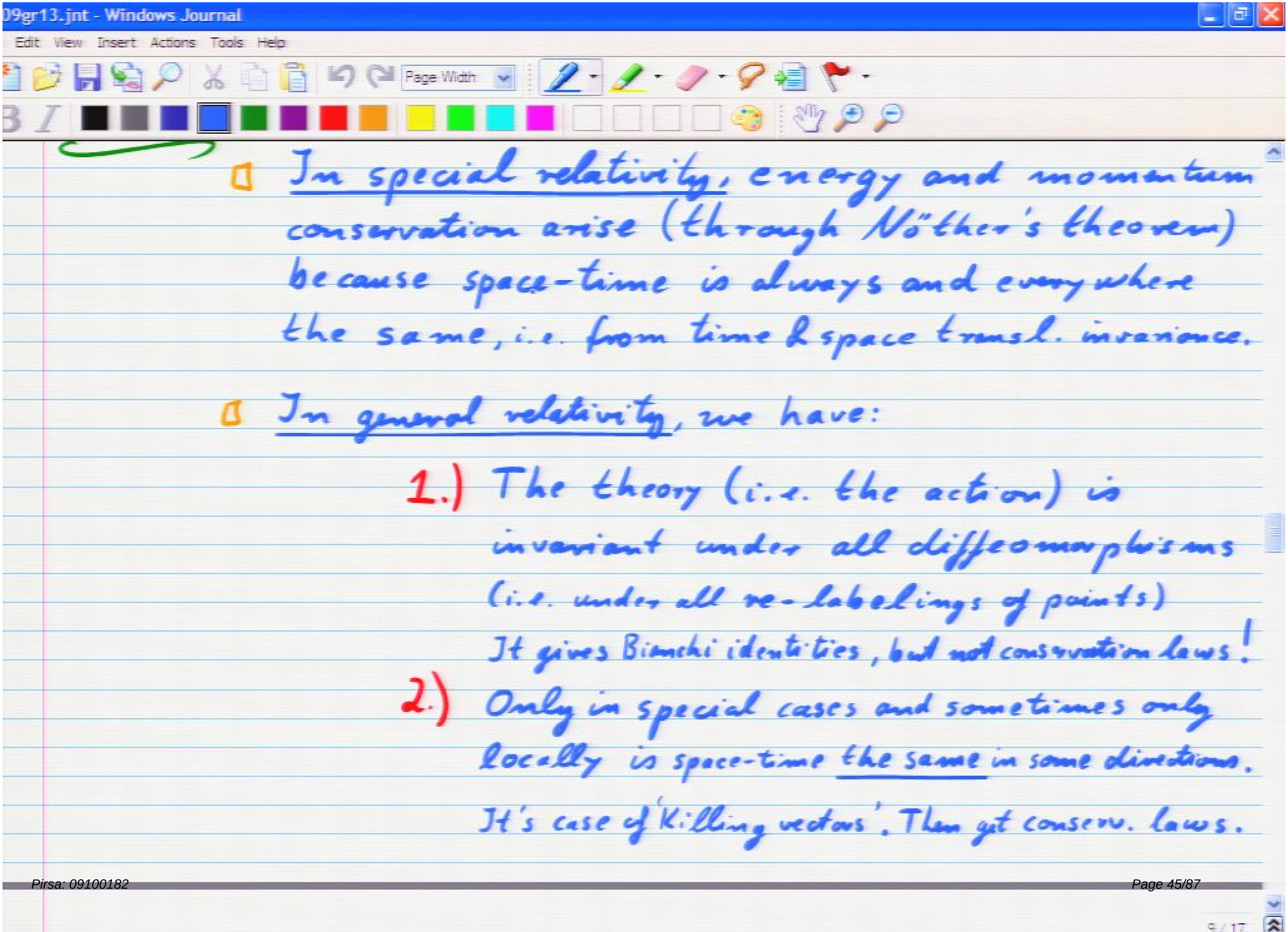

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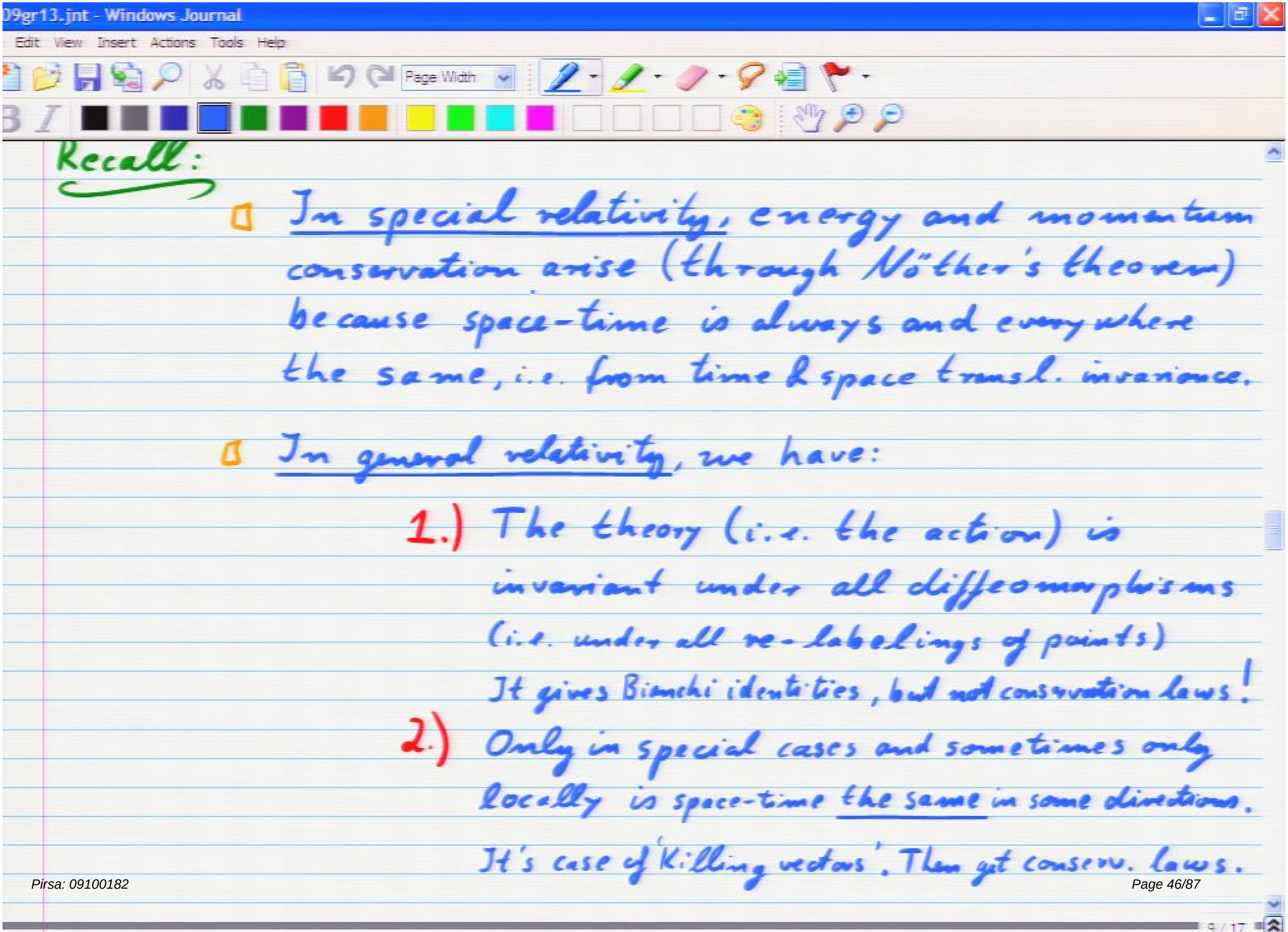

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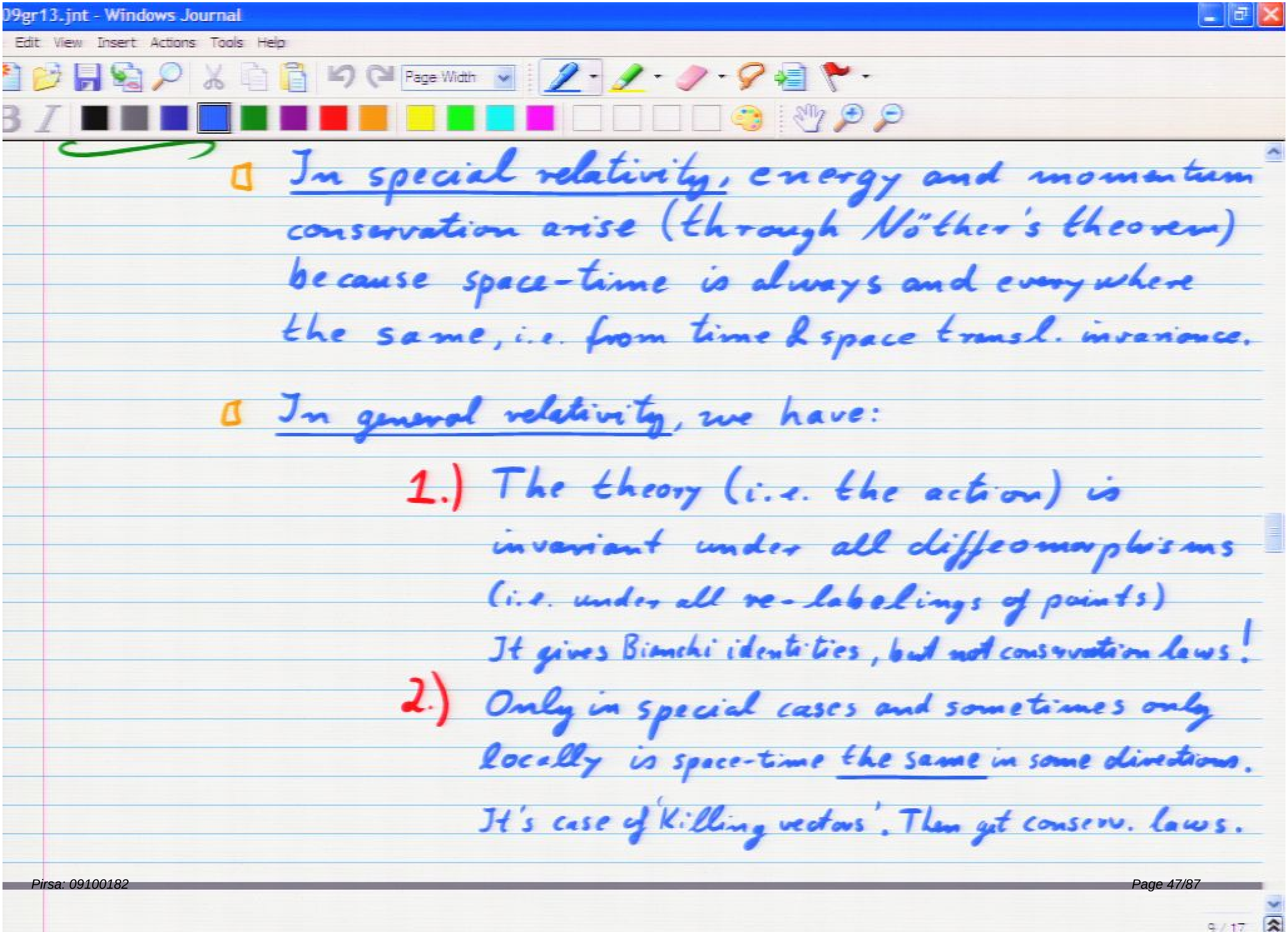

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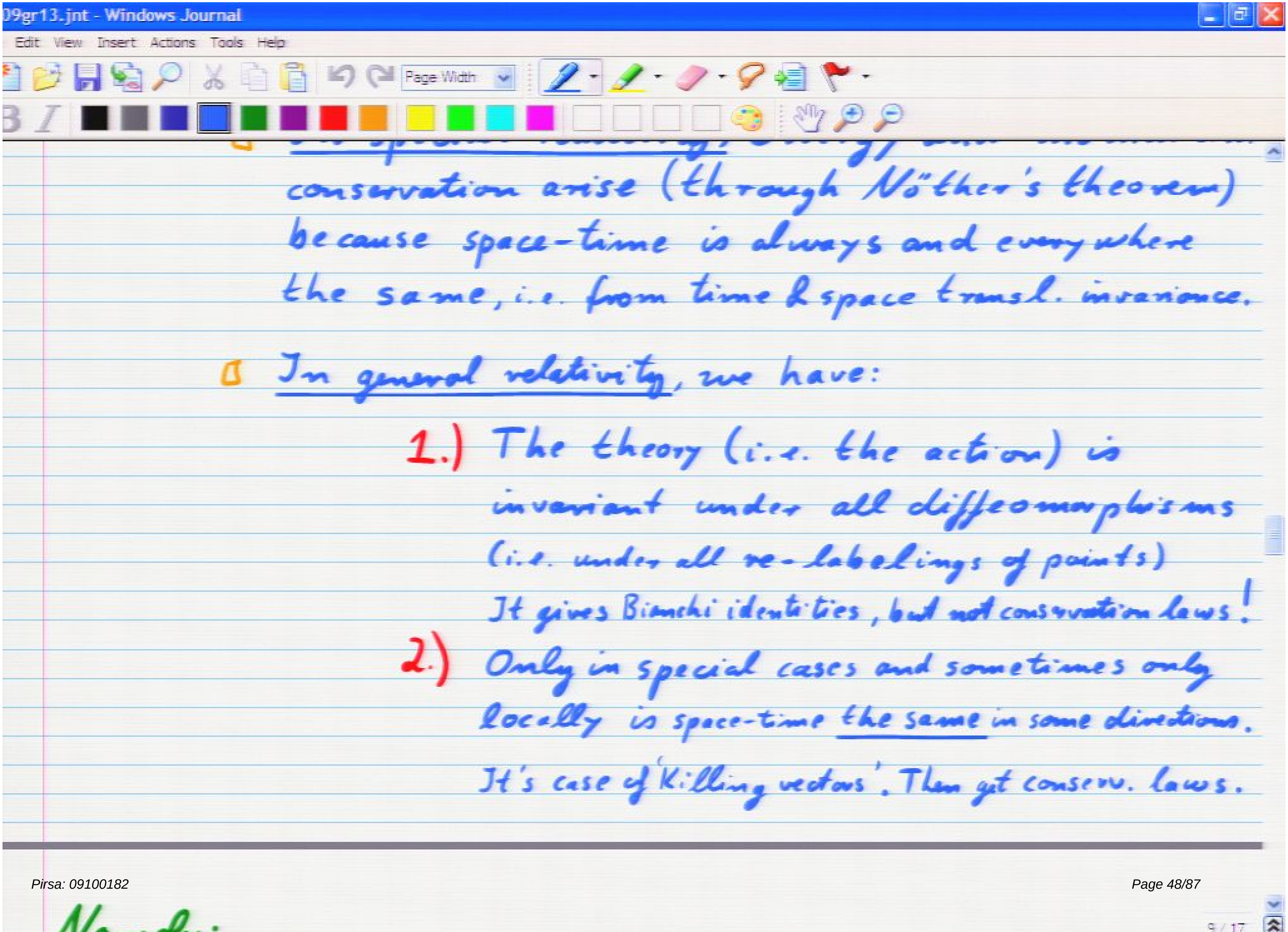

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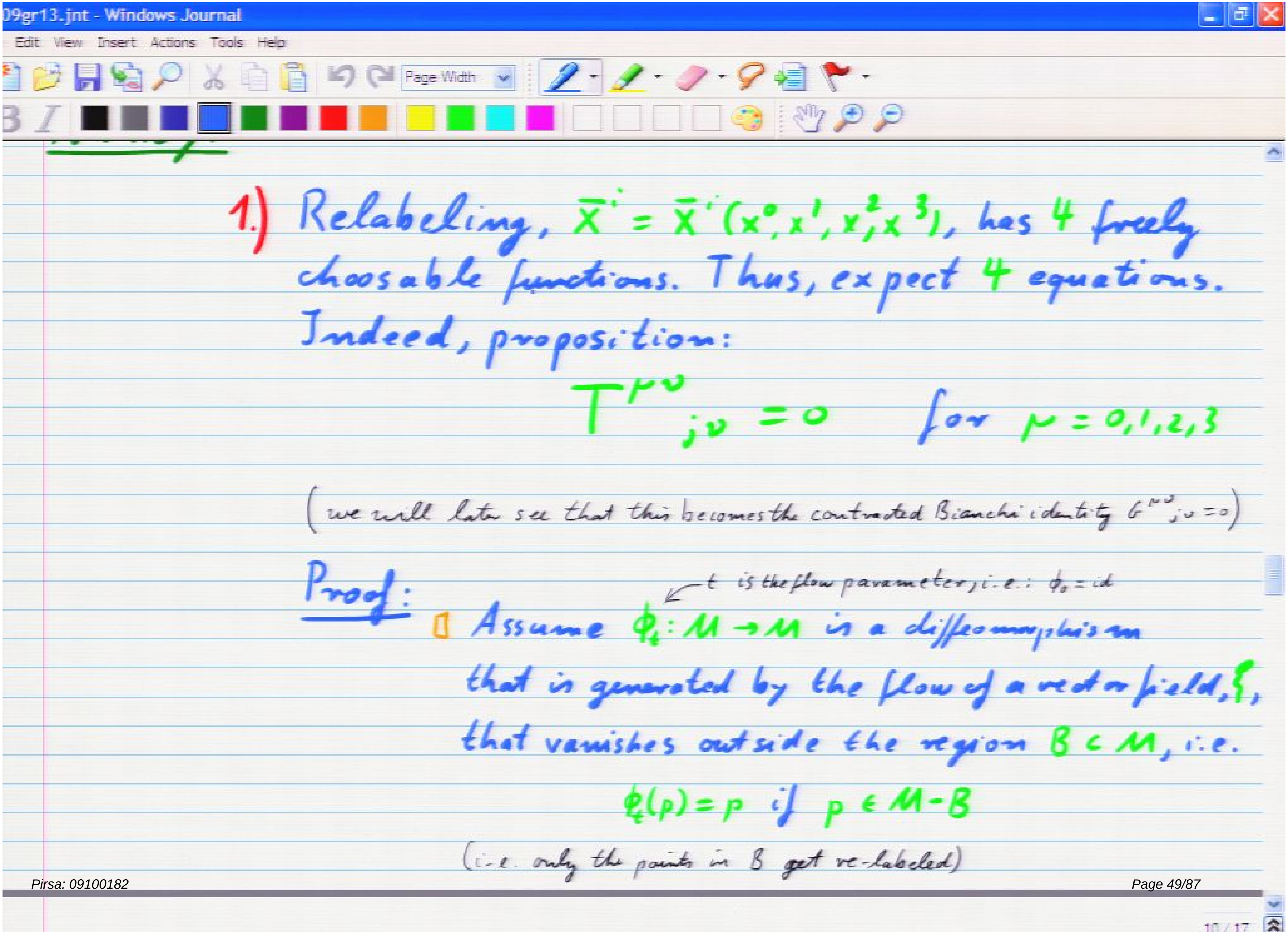

1.) Relabeling, $\bar{x}^i = \bar{x}^i(x^0, x^1, x^2, x^3)$, has 4 freely choosable functions. Thus, expect 4 equations. Indeed, proposition:

$$T^{\mu\nu}{}_{;\nu} = 0 \qquad \text{for } \mu = 0,1,2,3$$

(we will later see that this becomes the contracted Bianchi identity $G^{\mu\nu}{}_{;\nu} = 0$)

Proof:

$\swarrow$ t is the flow parameter, i.e.: $\phi_0 = id$

▯ Assume $\phi_t : M \to M$ is a diffeomorphism that is generated by the flow of a vector field, ξ, that vanishes outside the region $B \subset M$, i.e.

$$\phi_t(p) = p \quad \text{if} \quad p \in M - B$$

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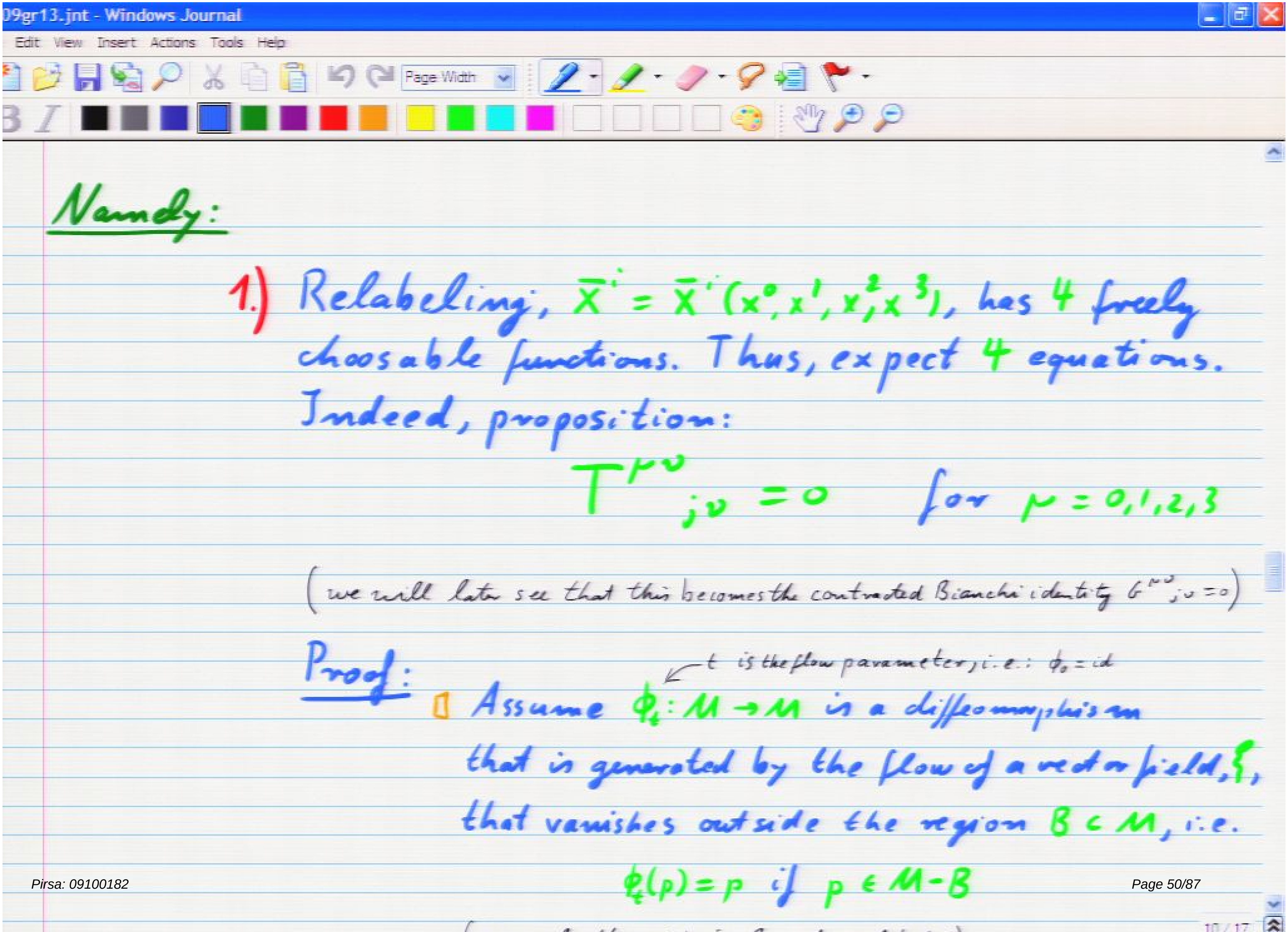

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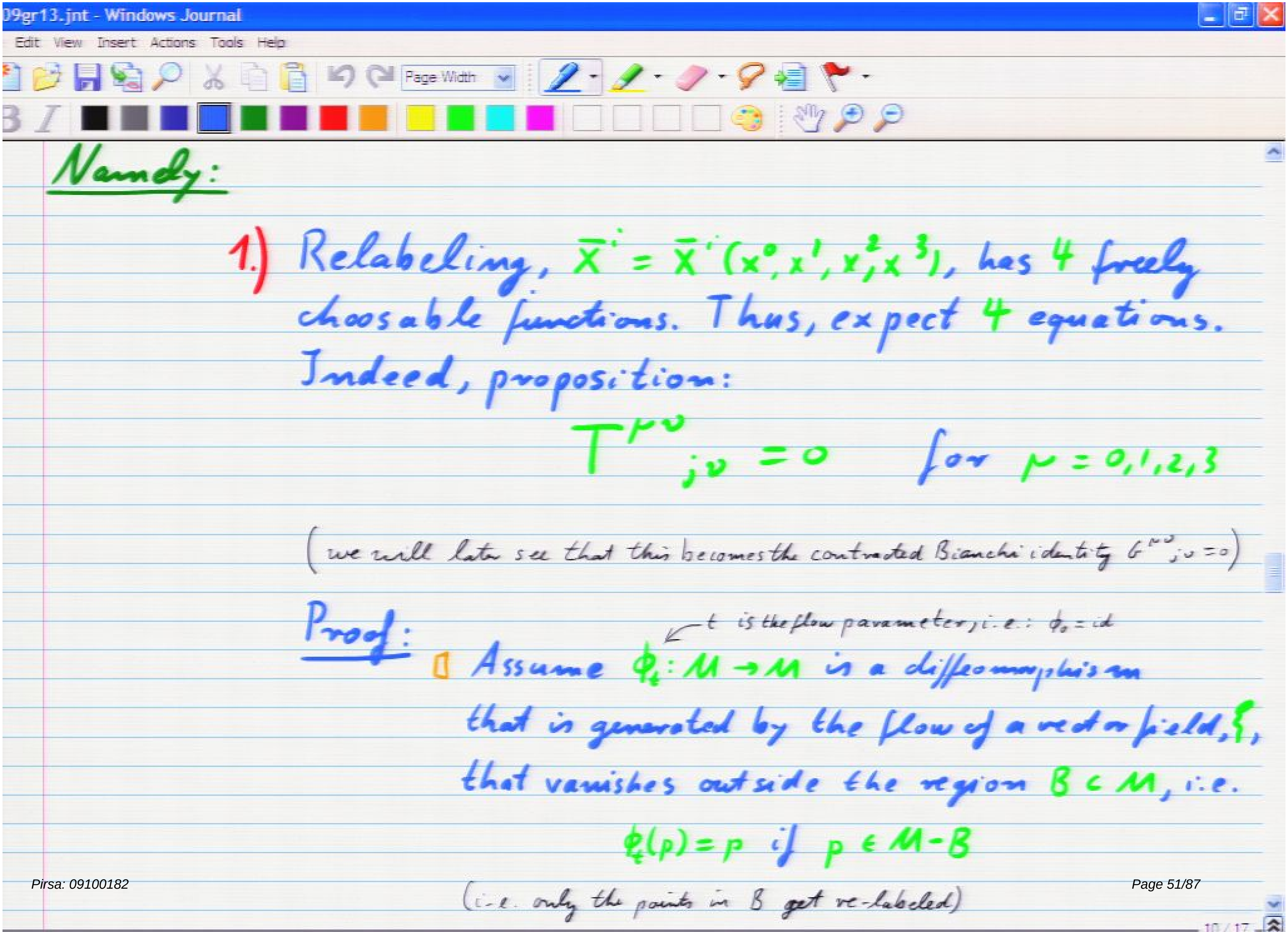

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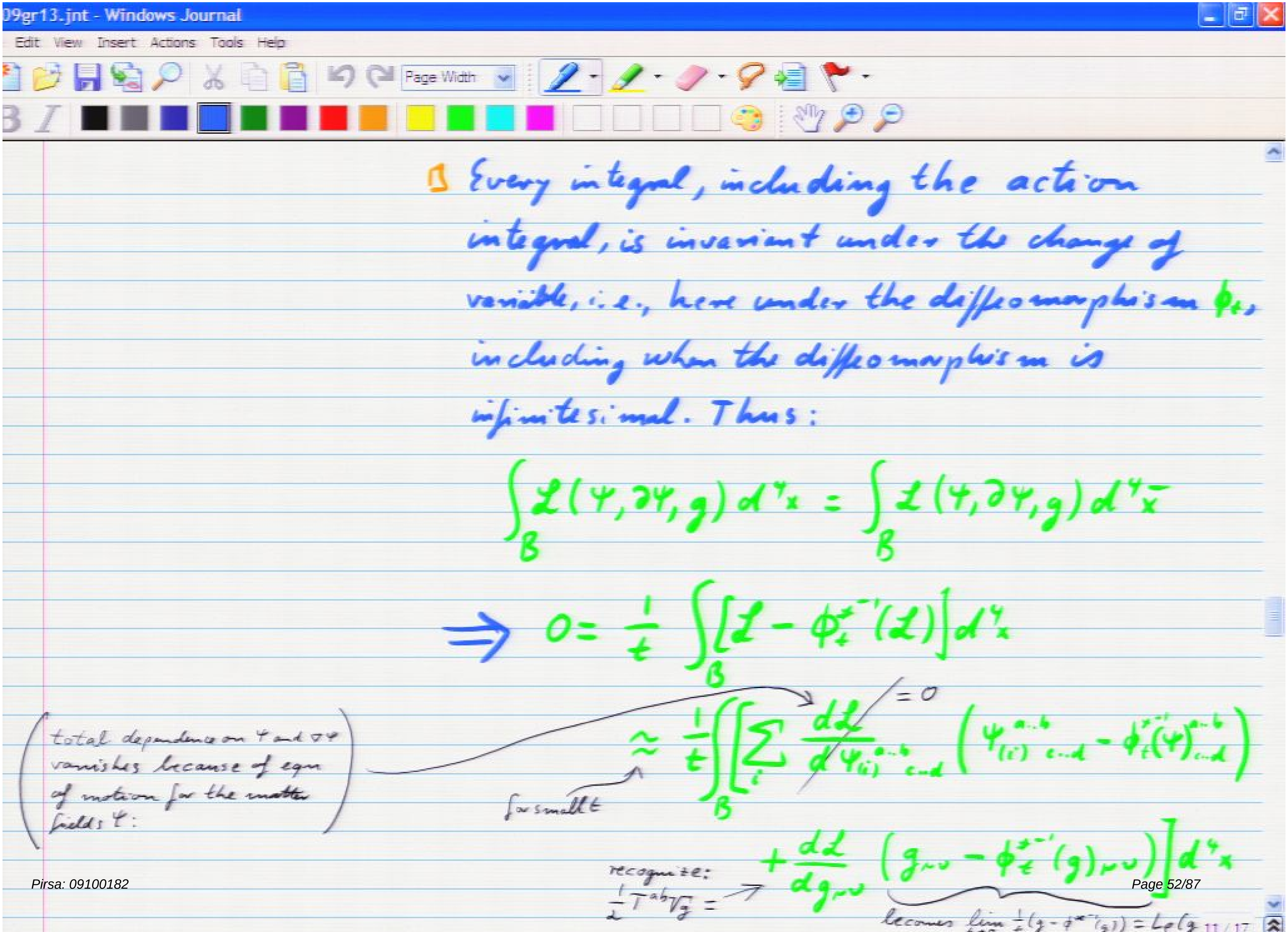

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$$\int_B \mathcal{L}(\psi,\partial\psi,g)\,d^4x = \int_B \mathcal{L}(\psi,\partial\psi,g)\,d^4\bar{x}$$

$$\Rightarrow\ 0 = \frac{1}{t}\int_B \left[\mathcal{L} - \phi_t^{*-1}(\mathcal{L})\right] d^4x$$

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$$\underset{\text{for small } t}{\approx} \frac{1}{t}\int_B \Bigg[\sum_i \underbrace{\cancel{\frac{d\mathcal{L}}{d\psi_{(i)}{}^{a..b}{}_{c..d}}}}_{=0}\left(\psi_{(i)}{}^{a..b}{}_{c..d} - \phi_t^{*-1}(\psi)^{a..b}{}_{c..d}\right)$$

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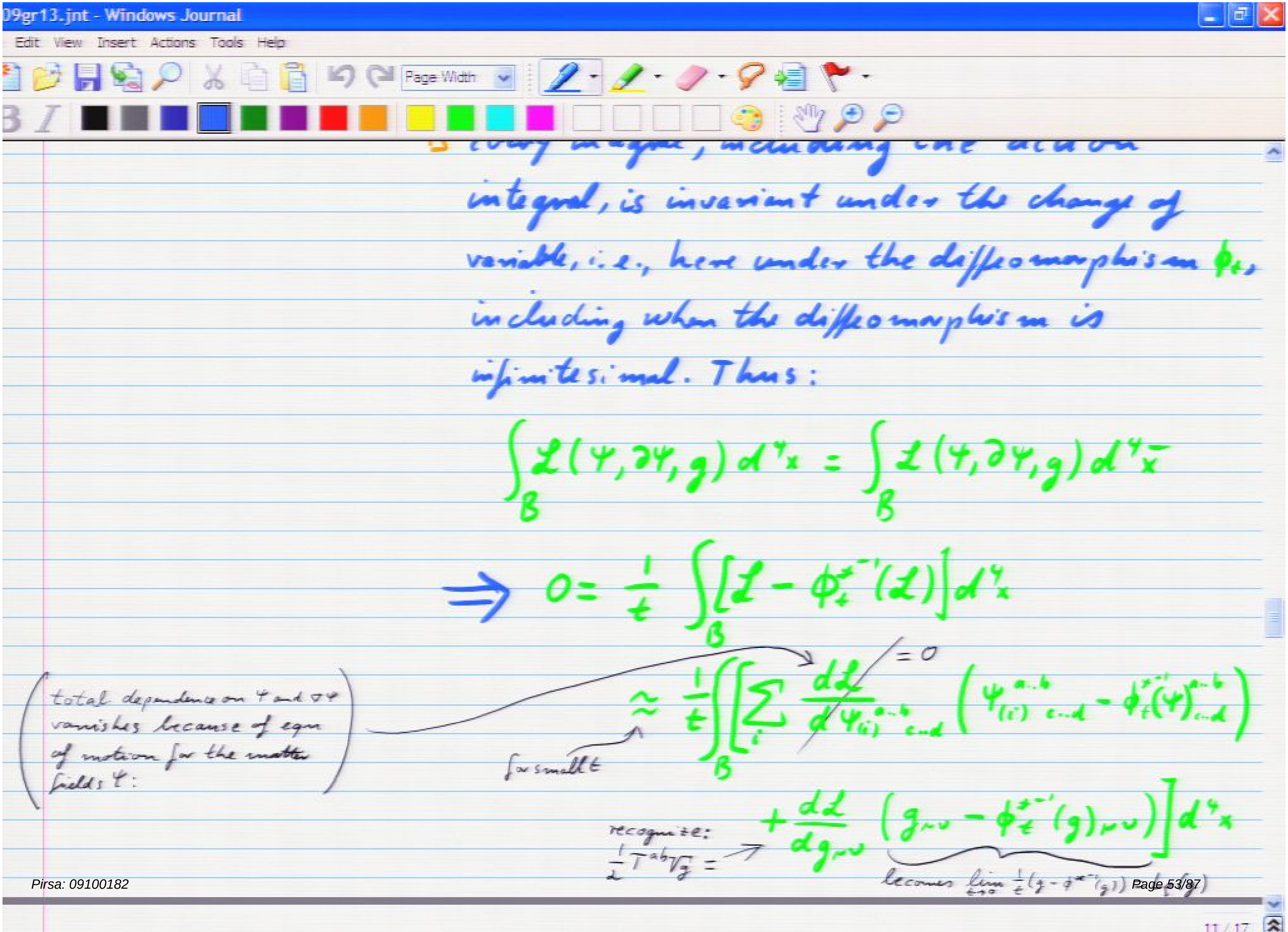

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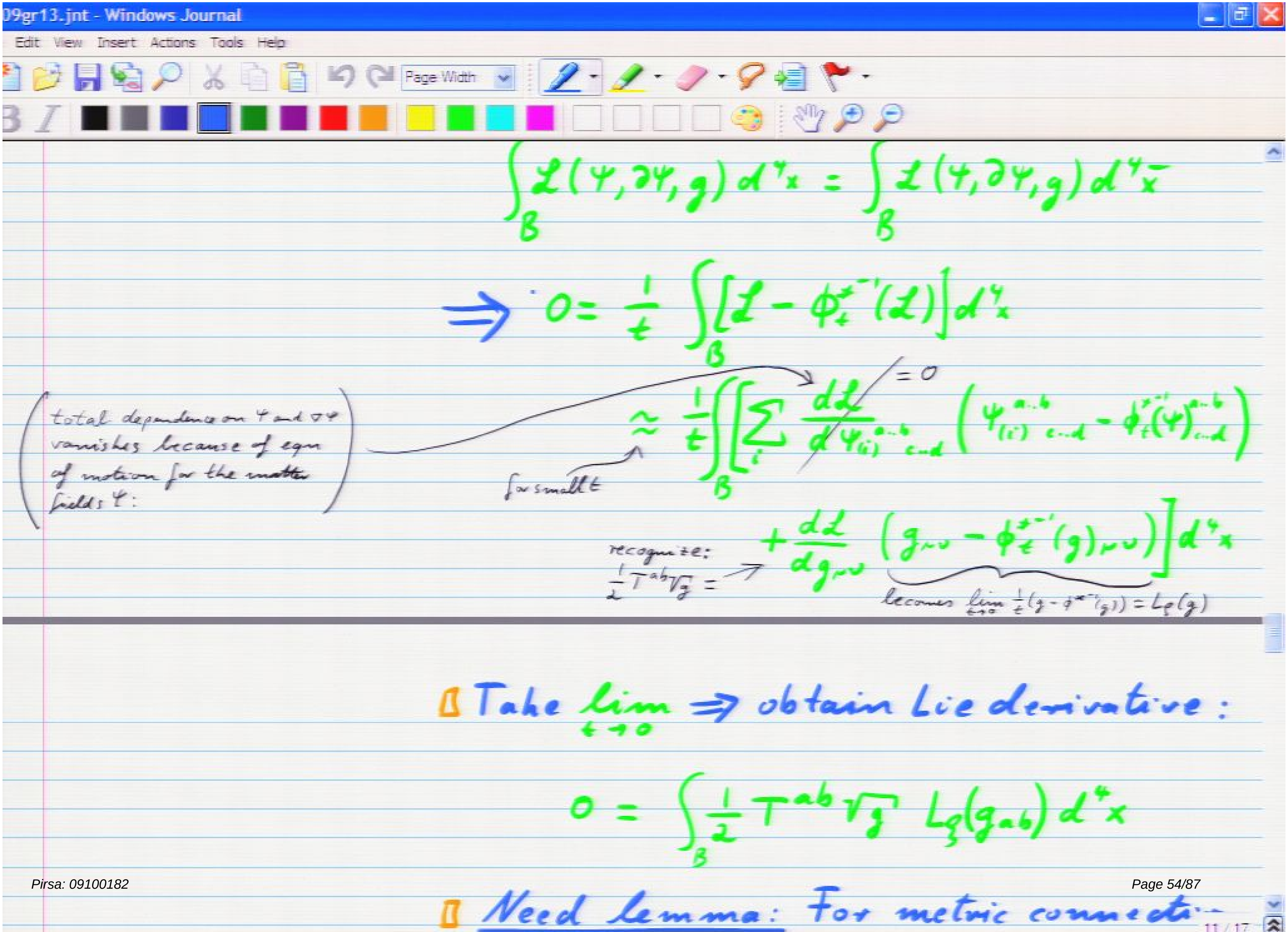

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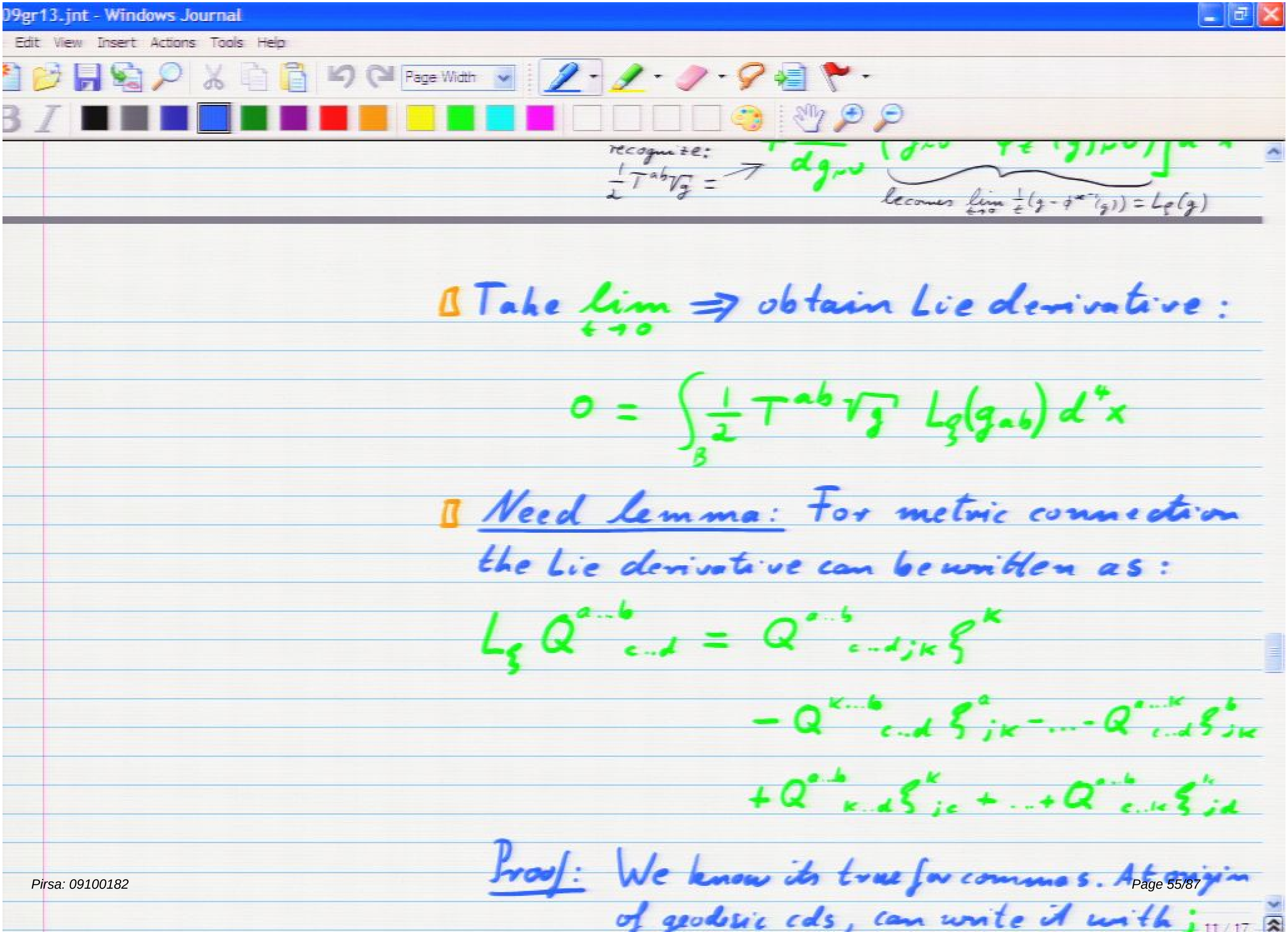

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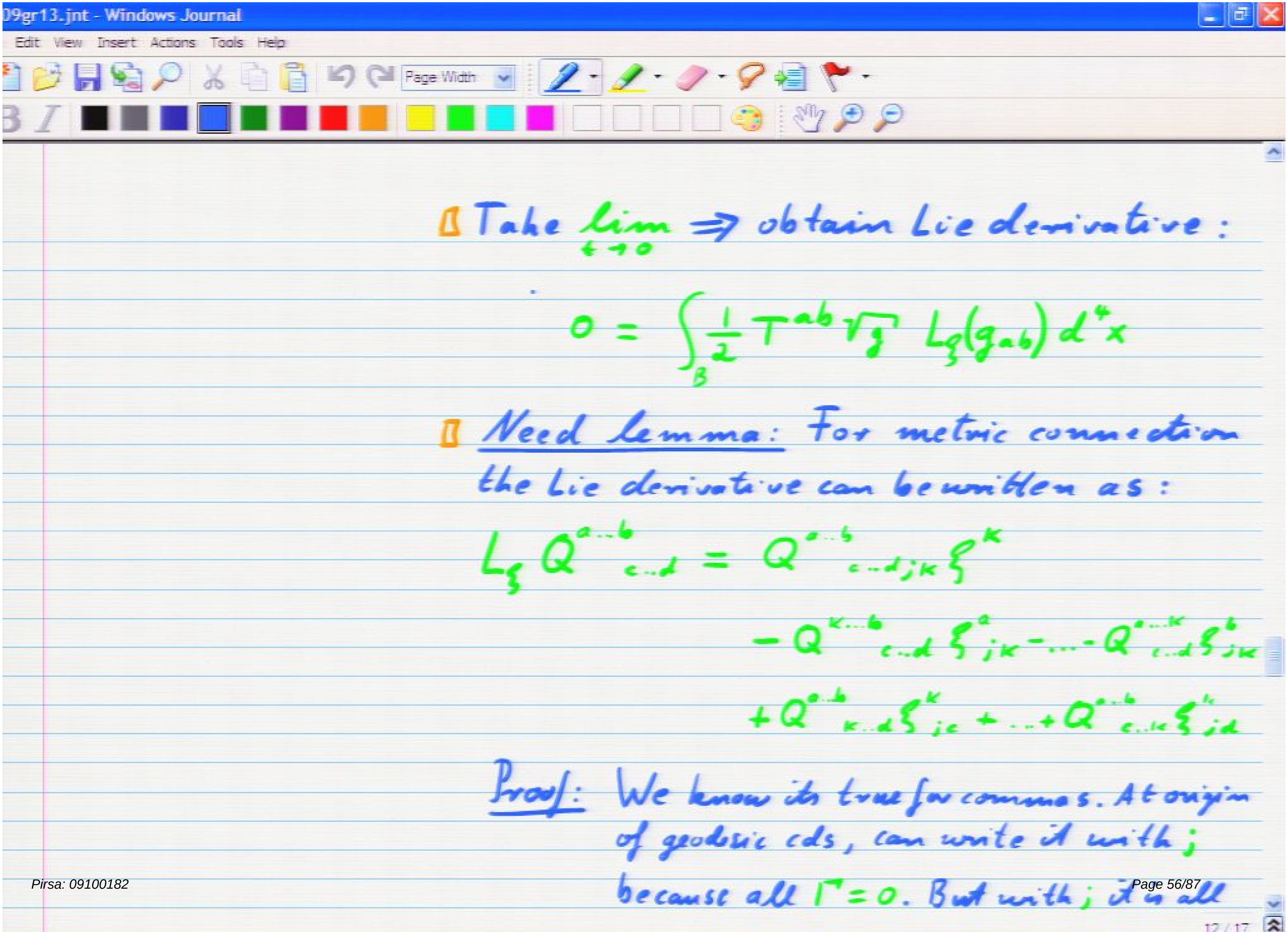

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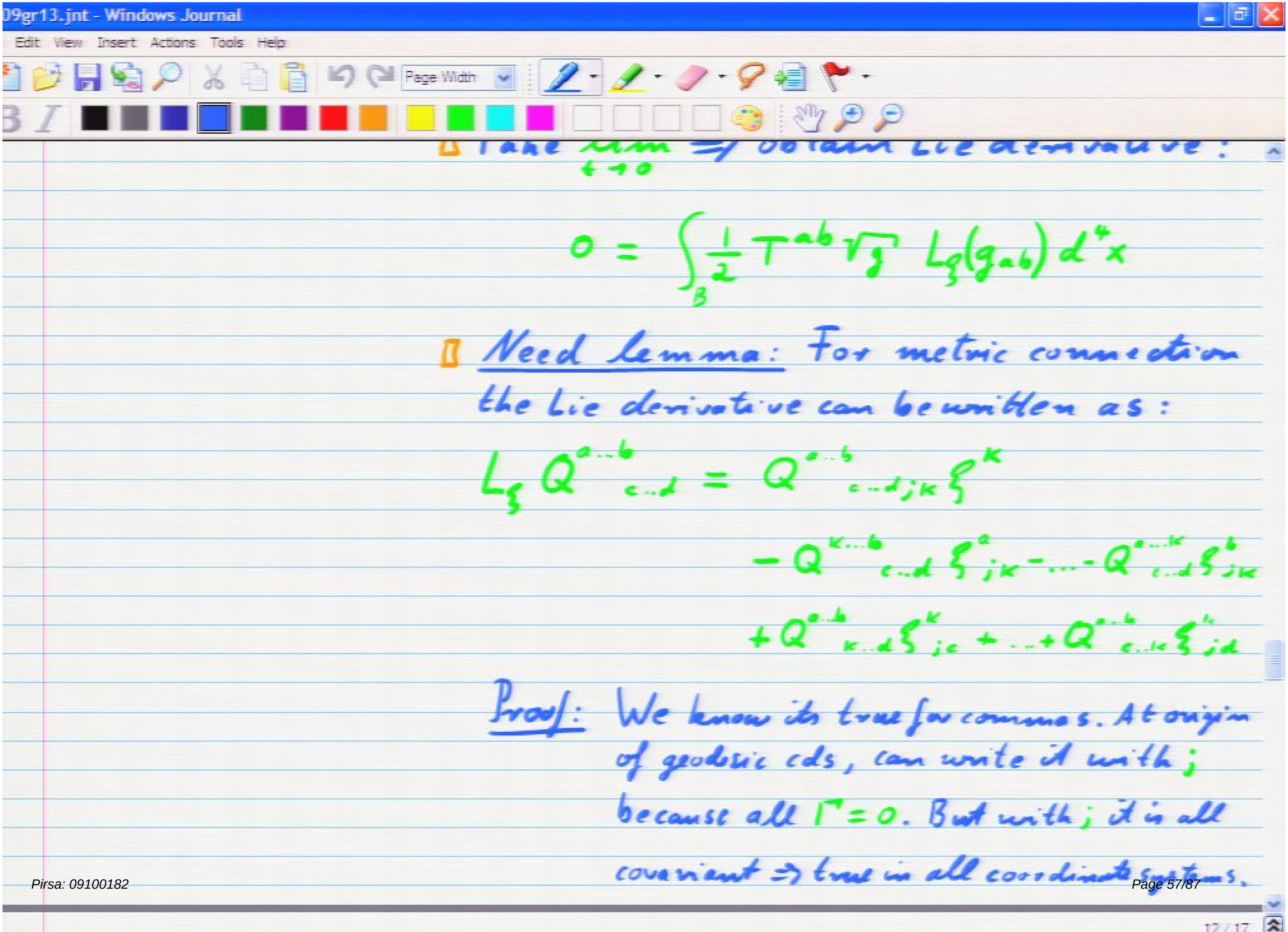

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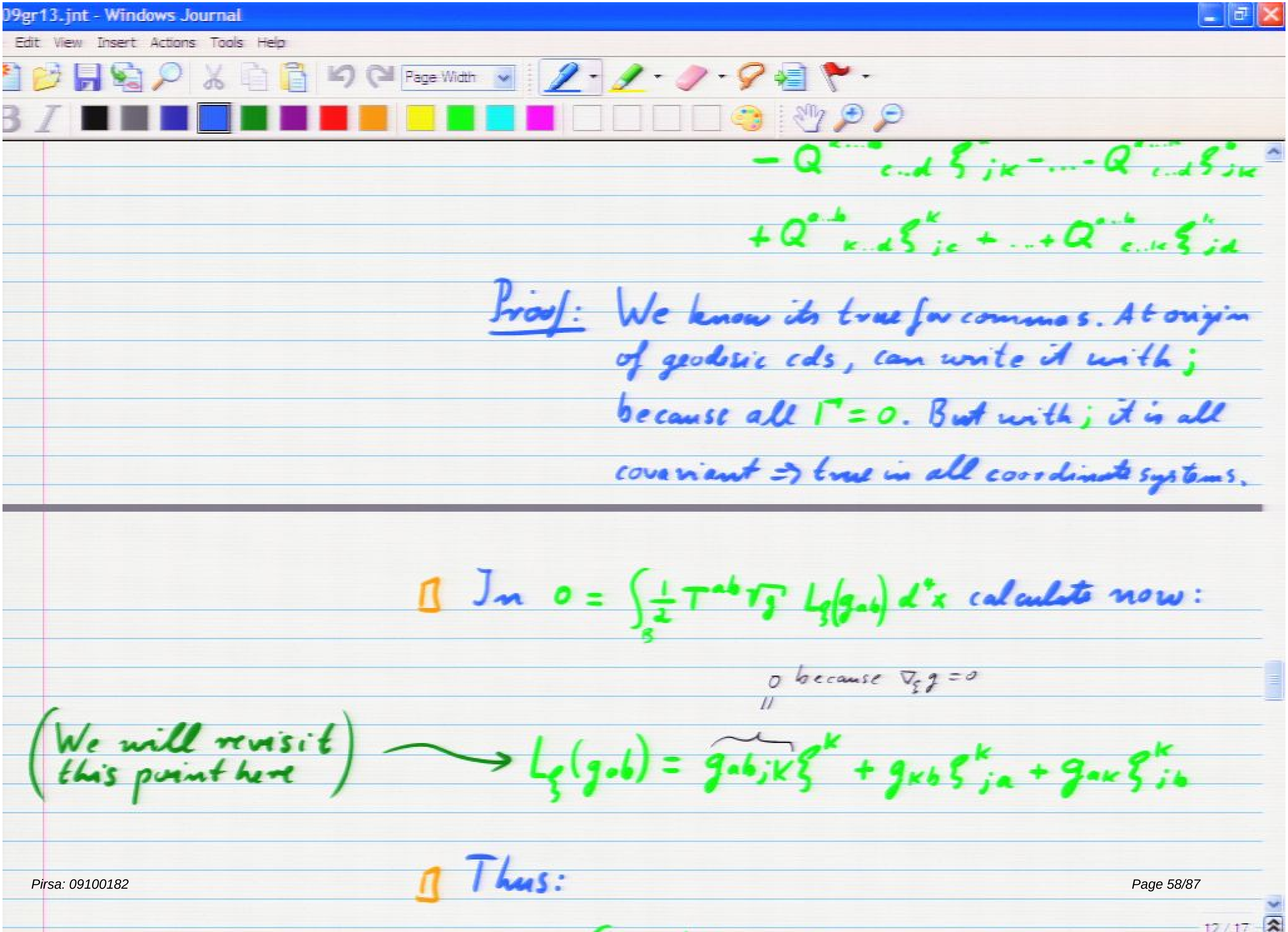

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▯ In $0=\int_{\mathcal{B}}\frac{1}{2}T^{ab}\sqrt{g}\,L_\xi(g_{ab})\,d^4x$ calculate now:

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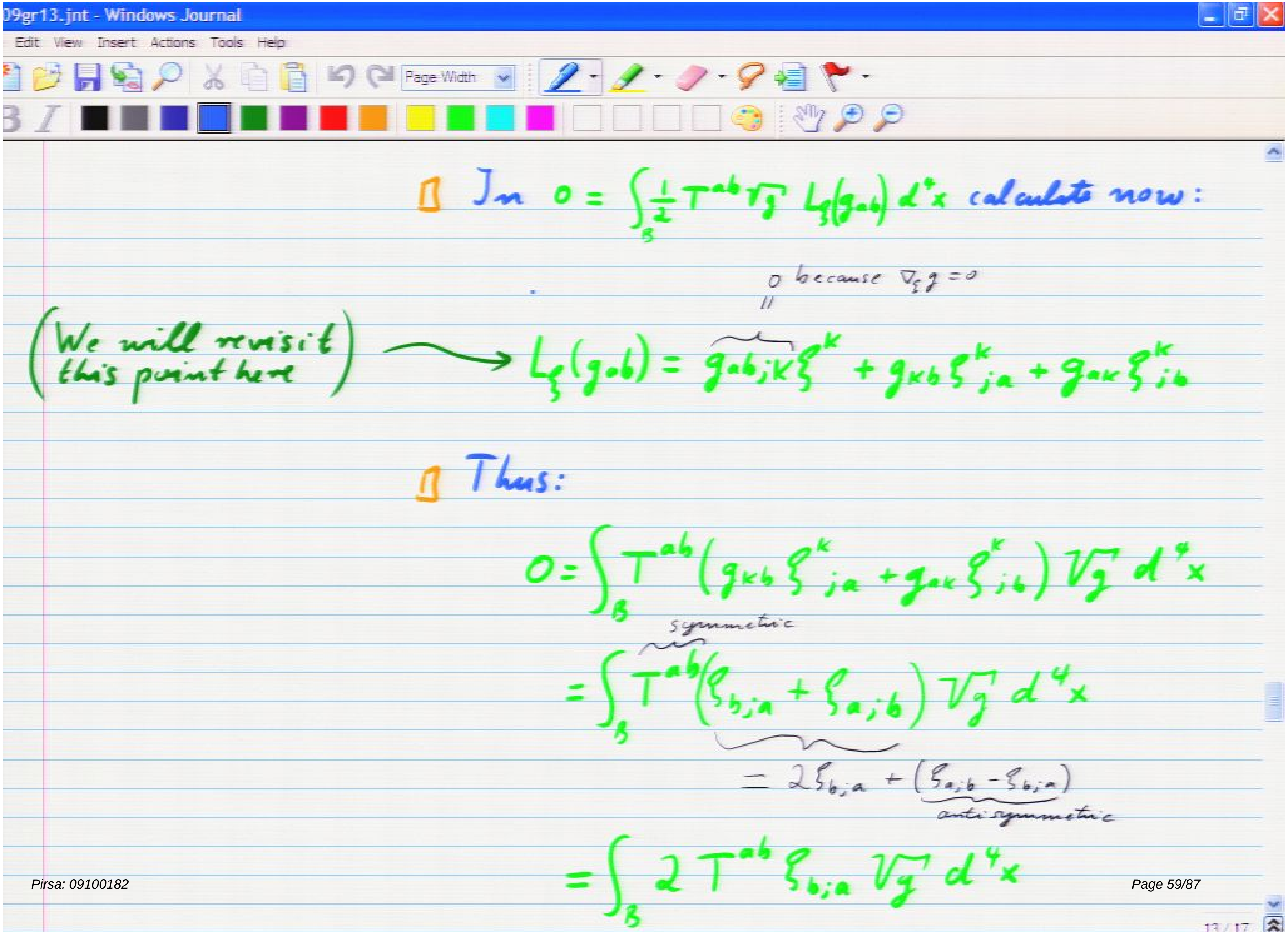

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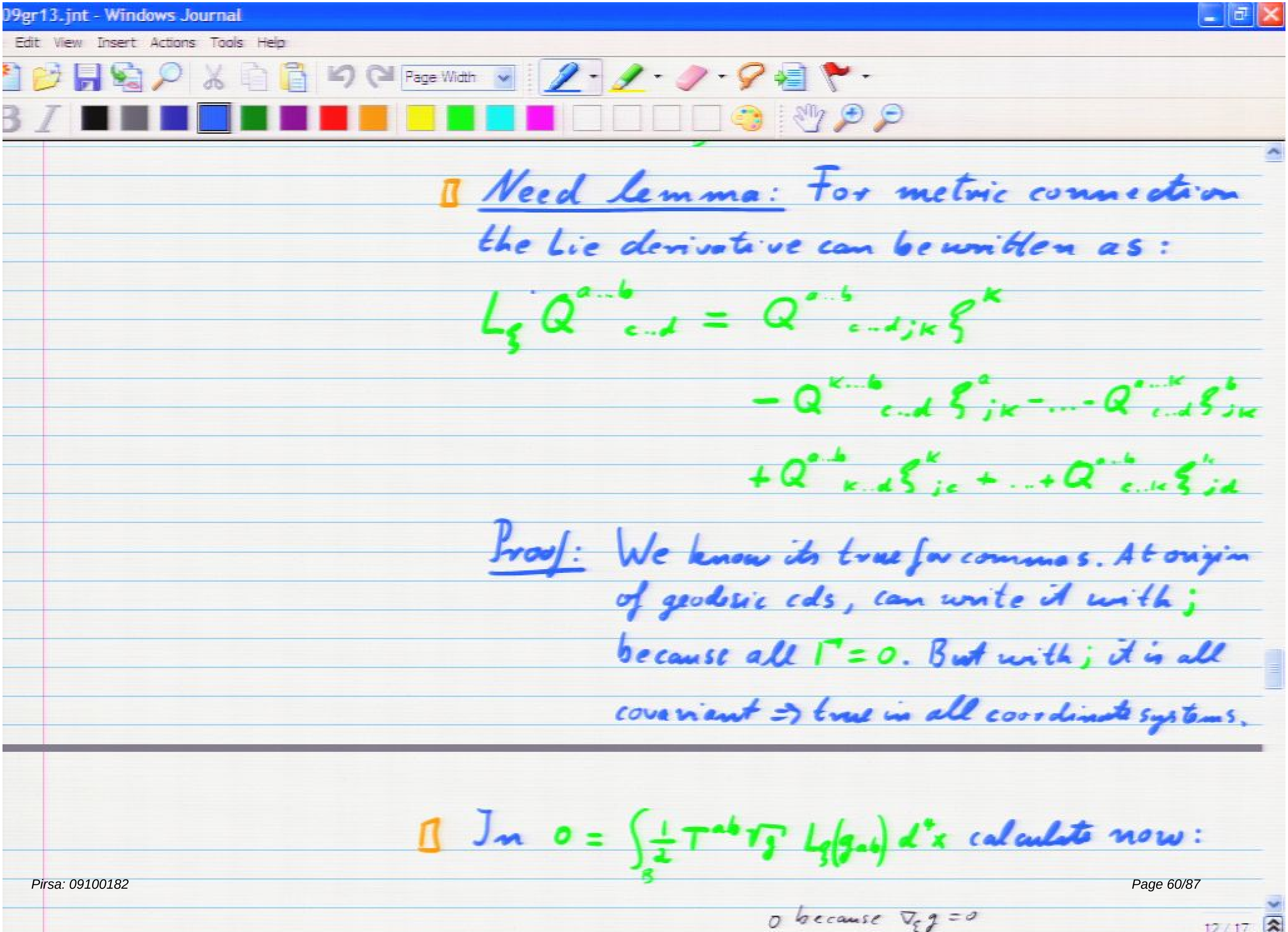

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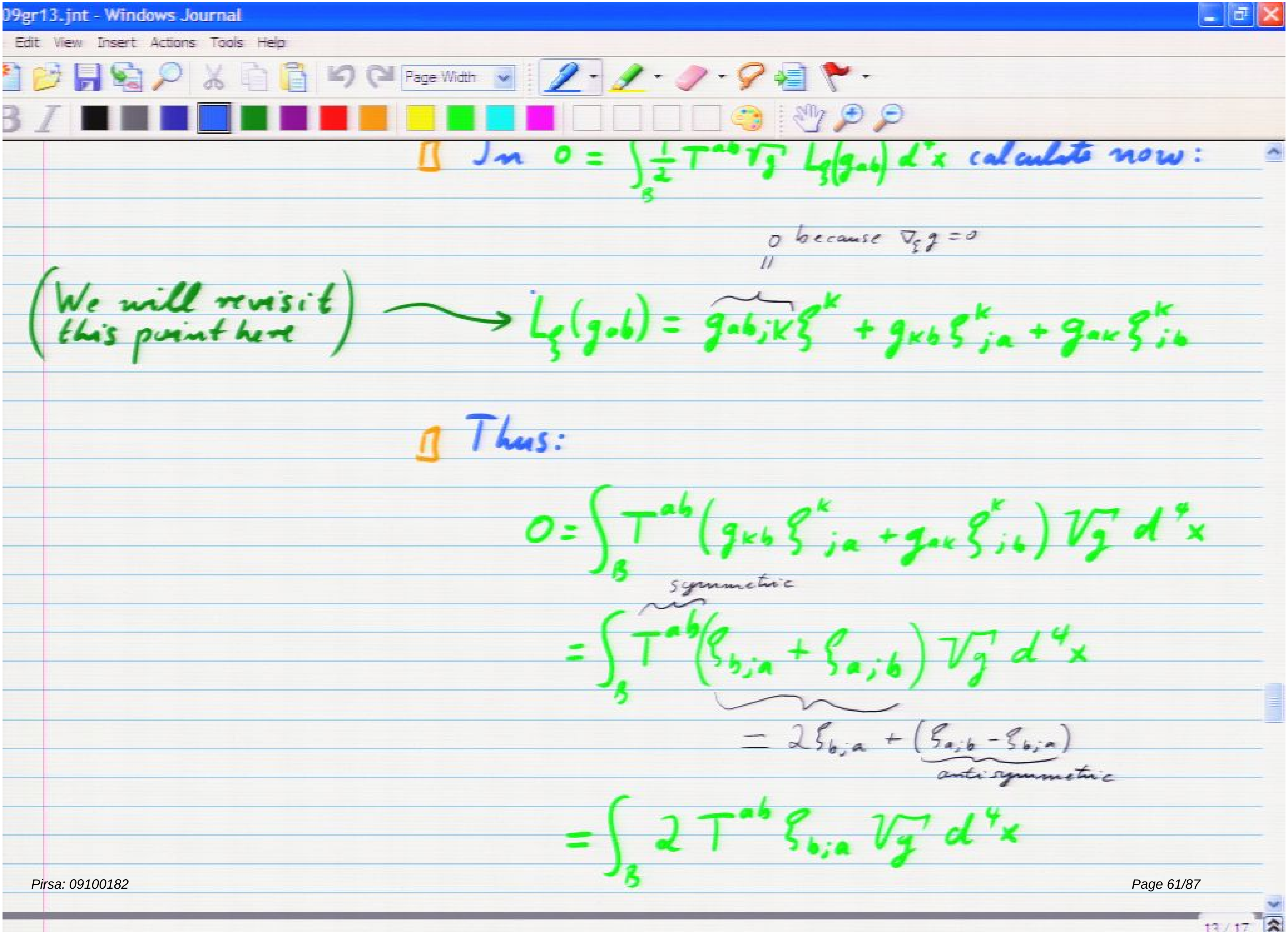

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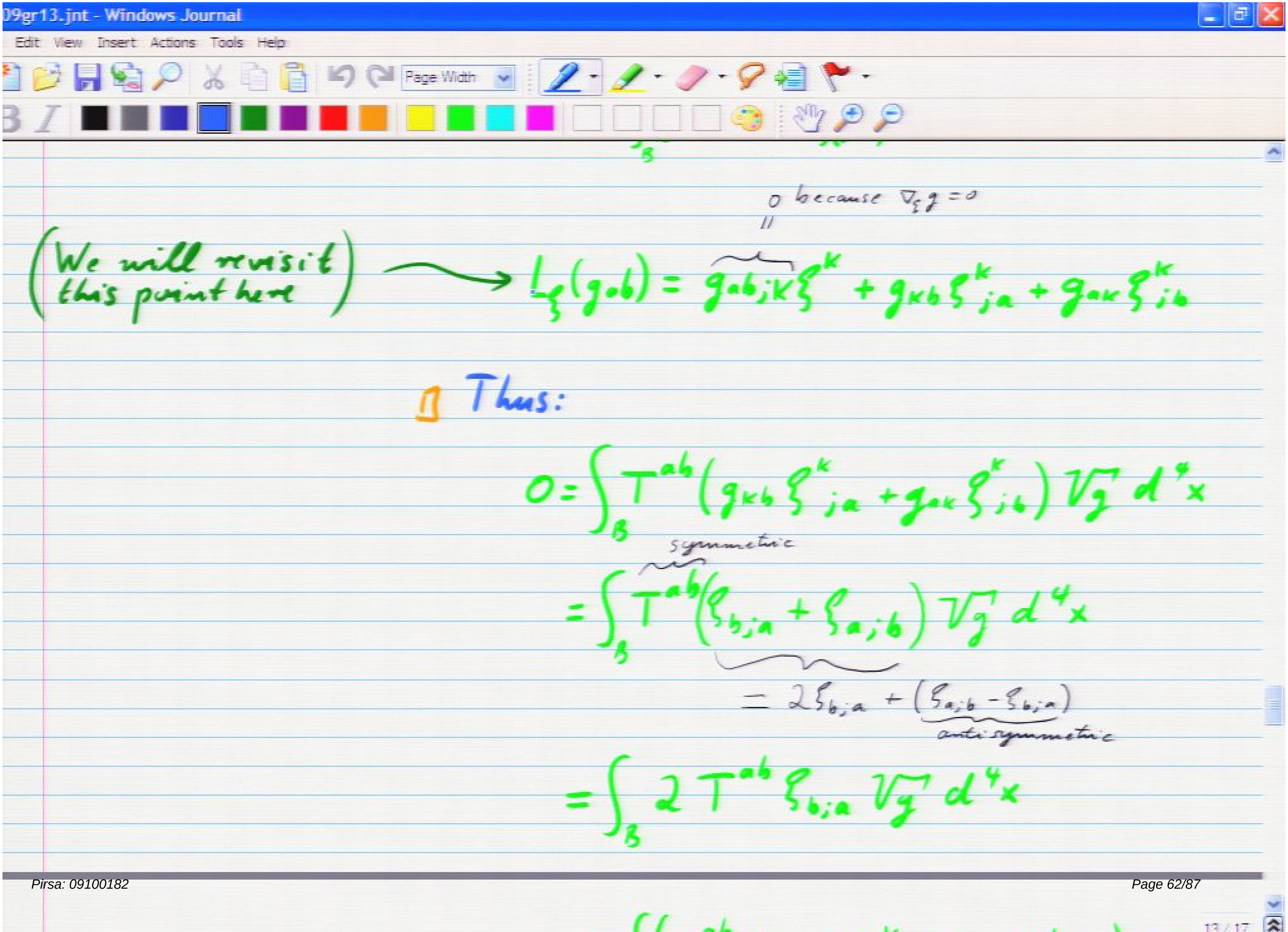

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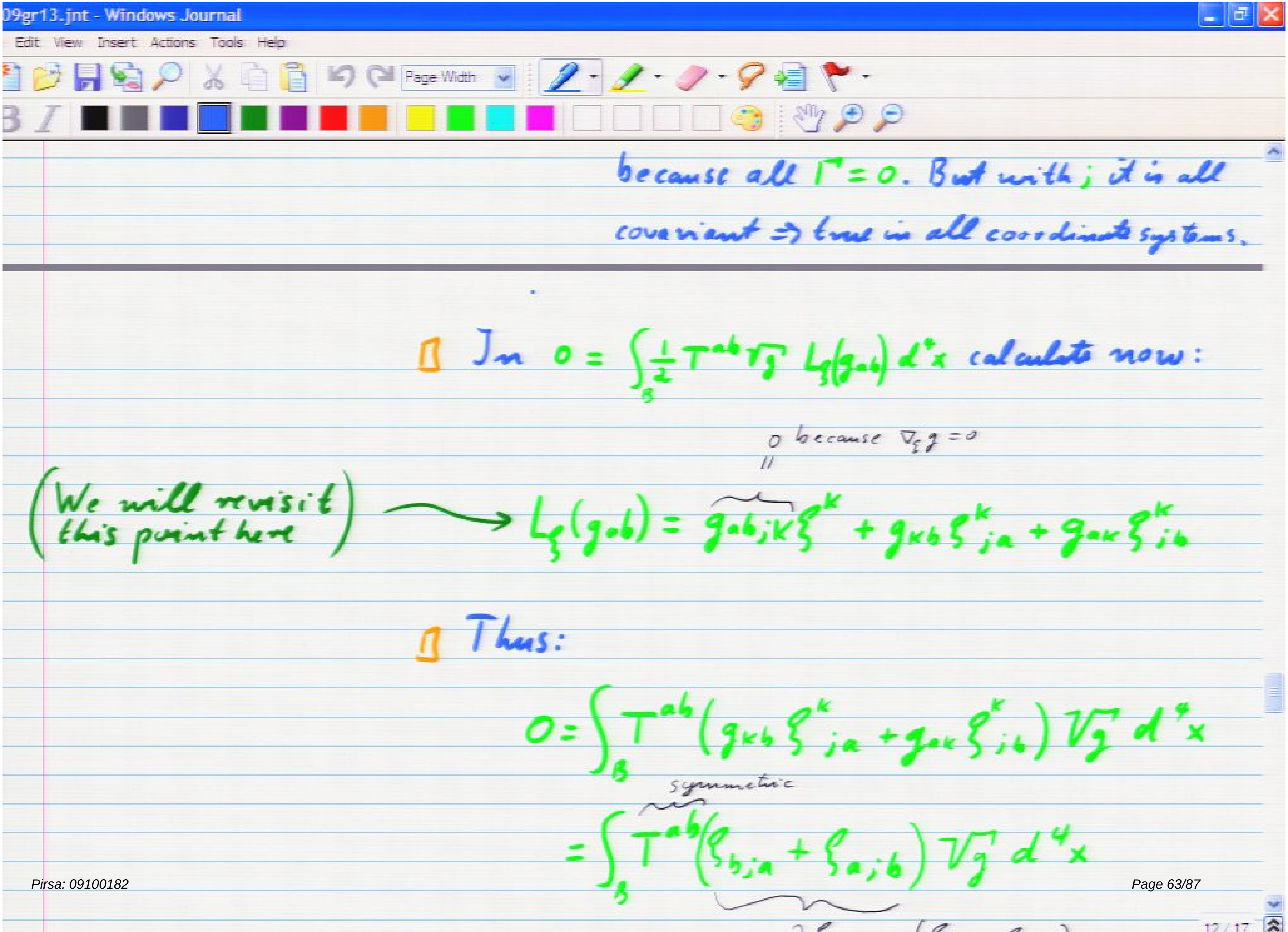

because all $\Gamma = 0$. But with ; it is all covariant ⇒ true in all coordinate systems.

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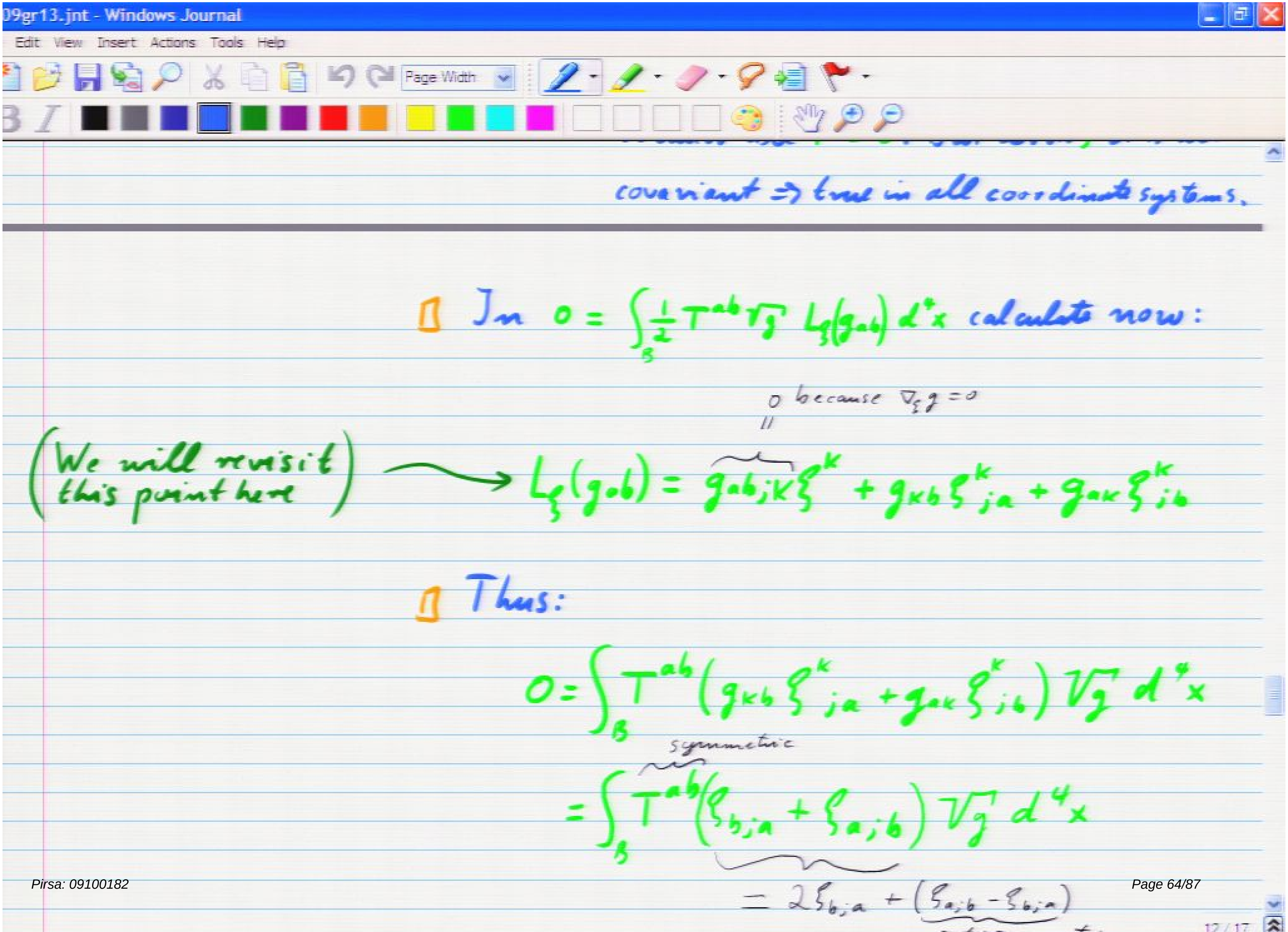

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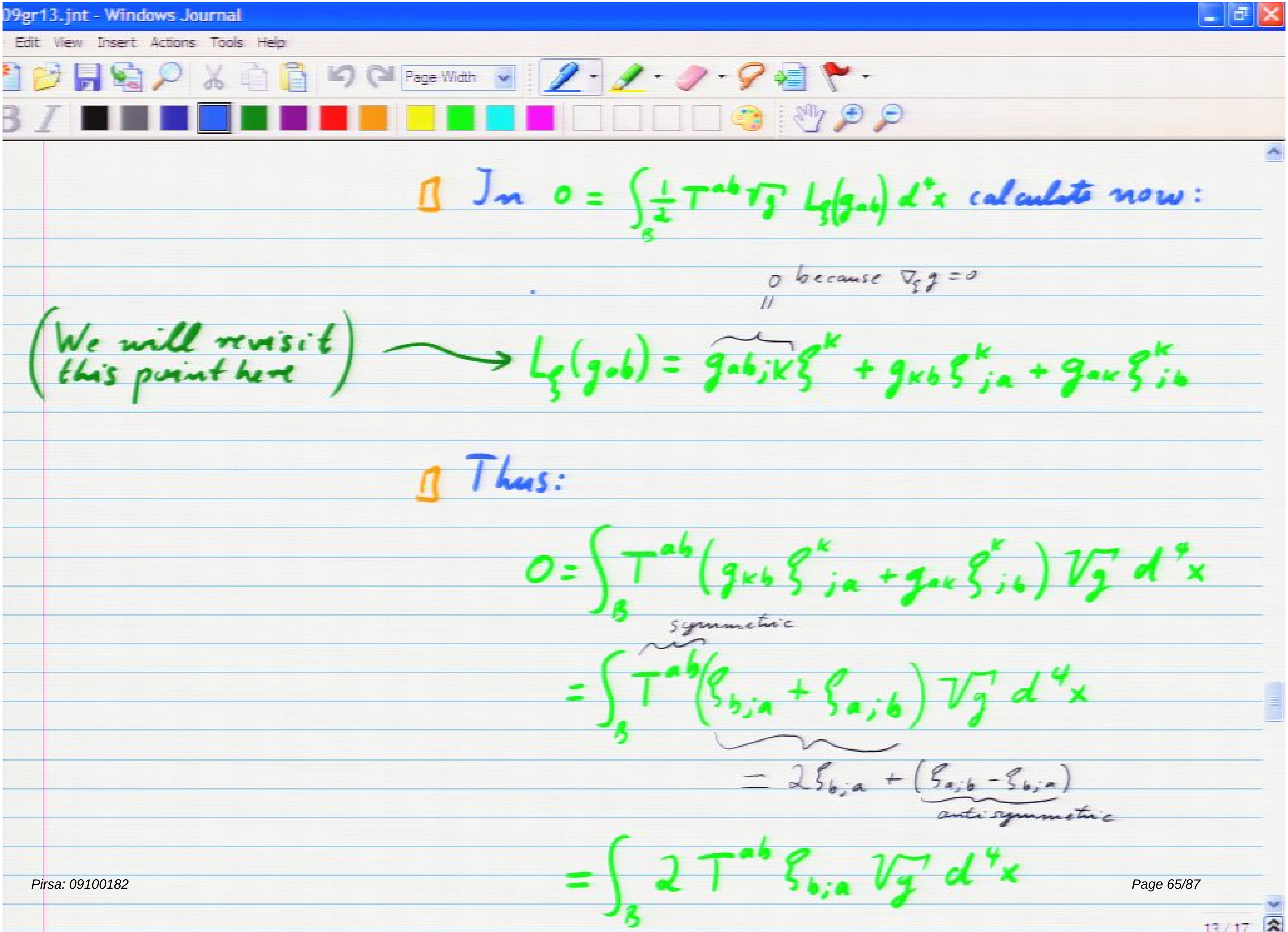
$$0 = \int_B T^{ab}\left(g_{kb}\xi^k_{;a} + g_{ak}\xi^k_{;b}\right)\sqrt{g}\, d^4x$$

$$= \int_B \overbrace{T^{ab}}^{\text{symmetric}}\underbrace{\left(\xi_{b;a} + \xi_{a;b}\right)}_{= 2\xi_{b;a} + (\xi_{a;b} - \xi_{b;a})}\sqrt{g}\, d^4x$$



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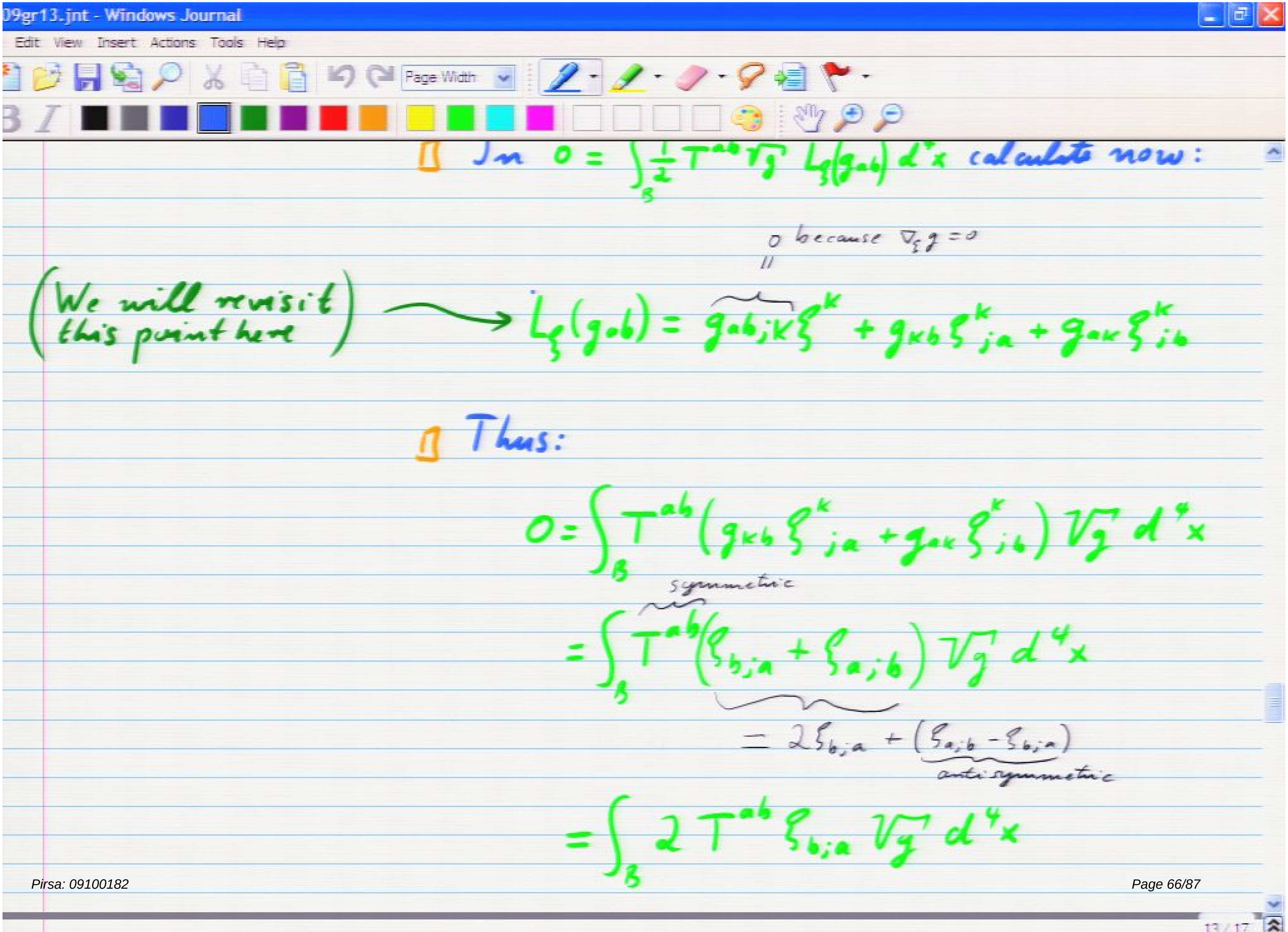

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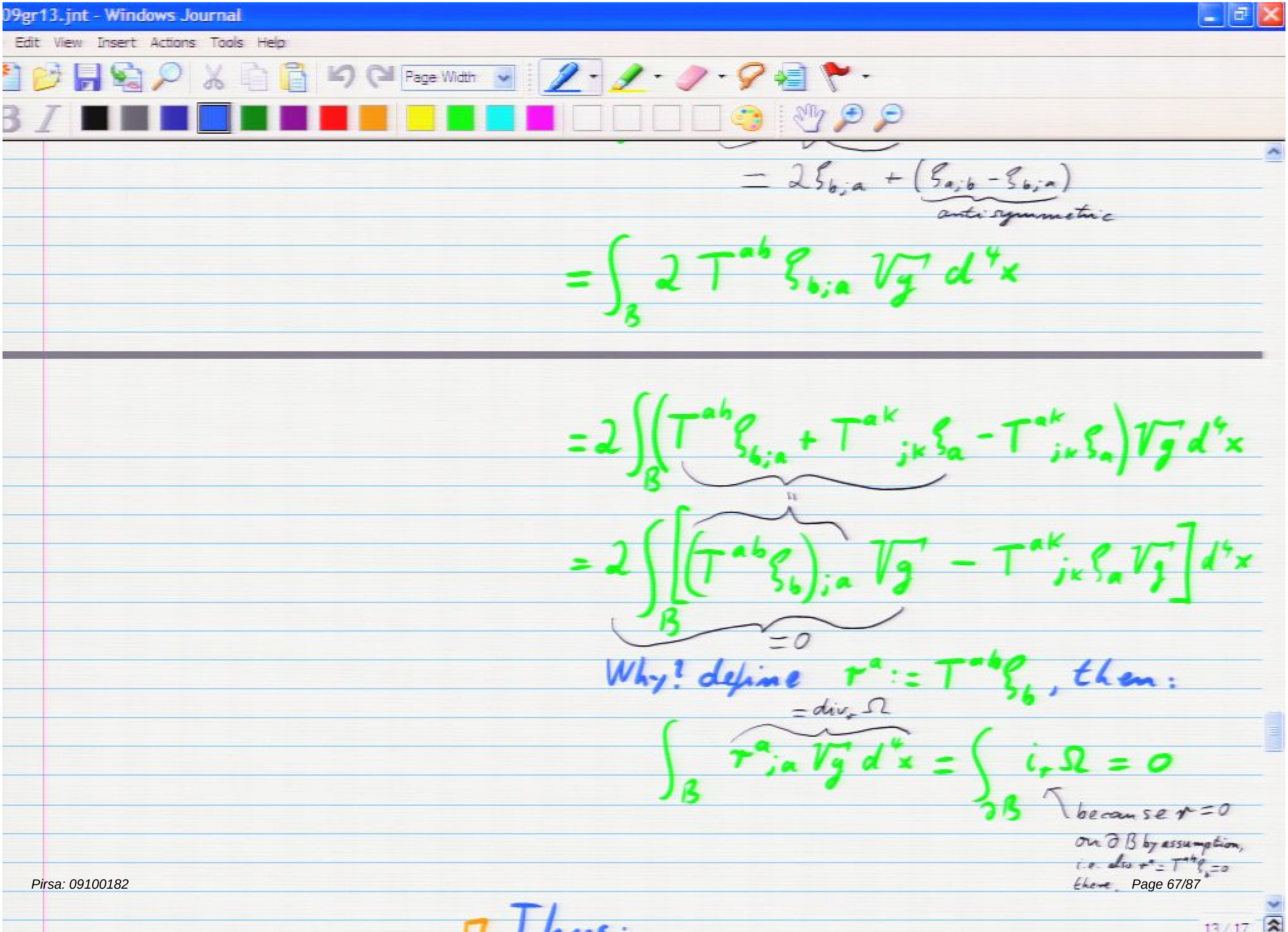

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Why? define $r^a := T^{ab}\xi_b$, then:

$$\int_B \overbrace{r^a{}_{;a}\sqrt{g}\,d^4x}^{=\mathrm{div}_r\Omega} = \int_{\partial B} i_r\Omega = 0$$

$\nwarrow$ because $r = 0$ on ∂B by assumption, i.e. also $r^a = T^{ab}\xi_b = 0$ there.

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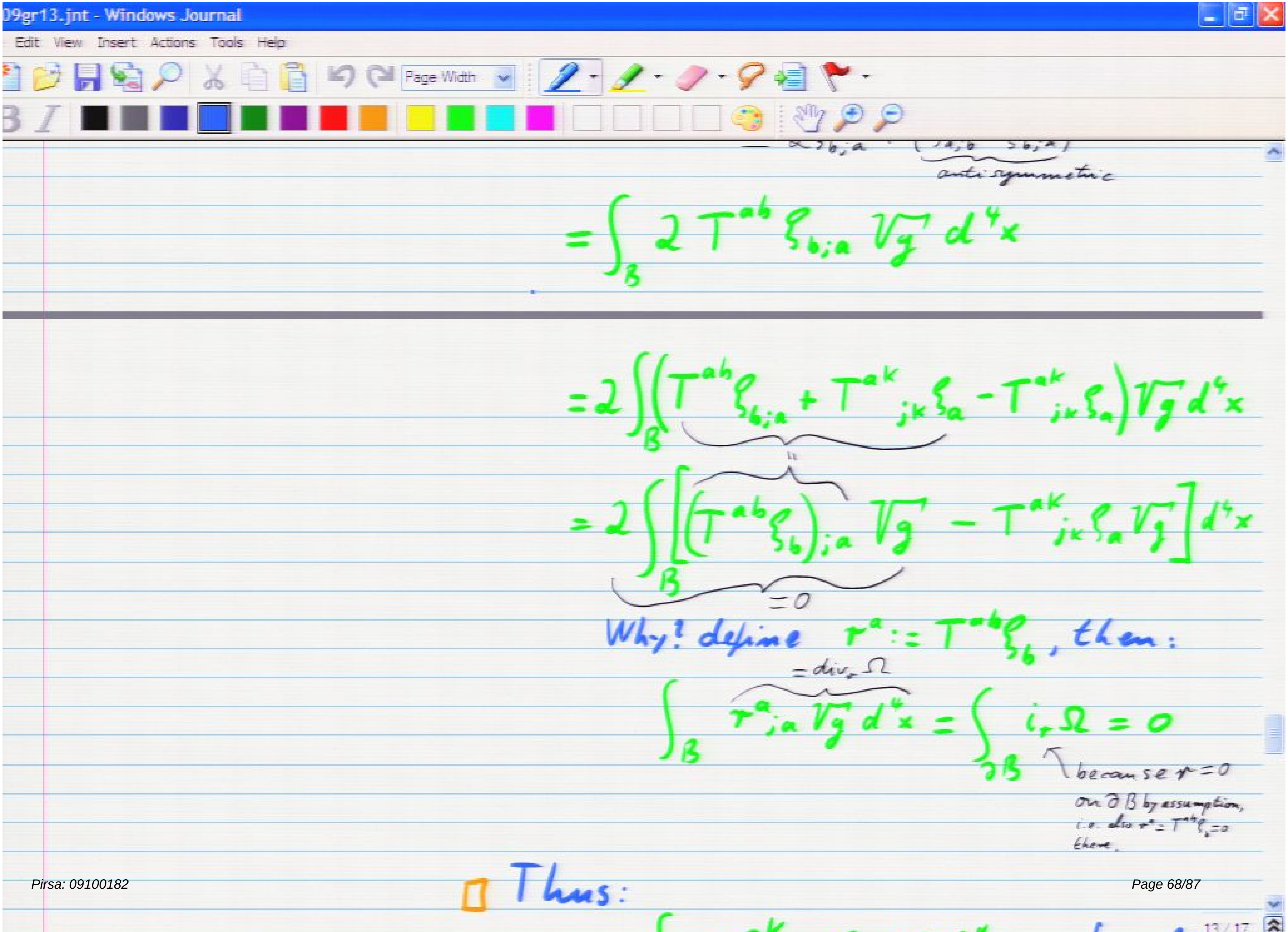

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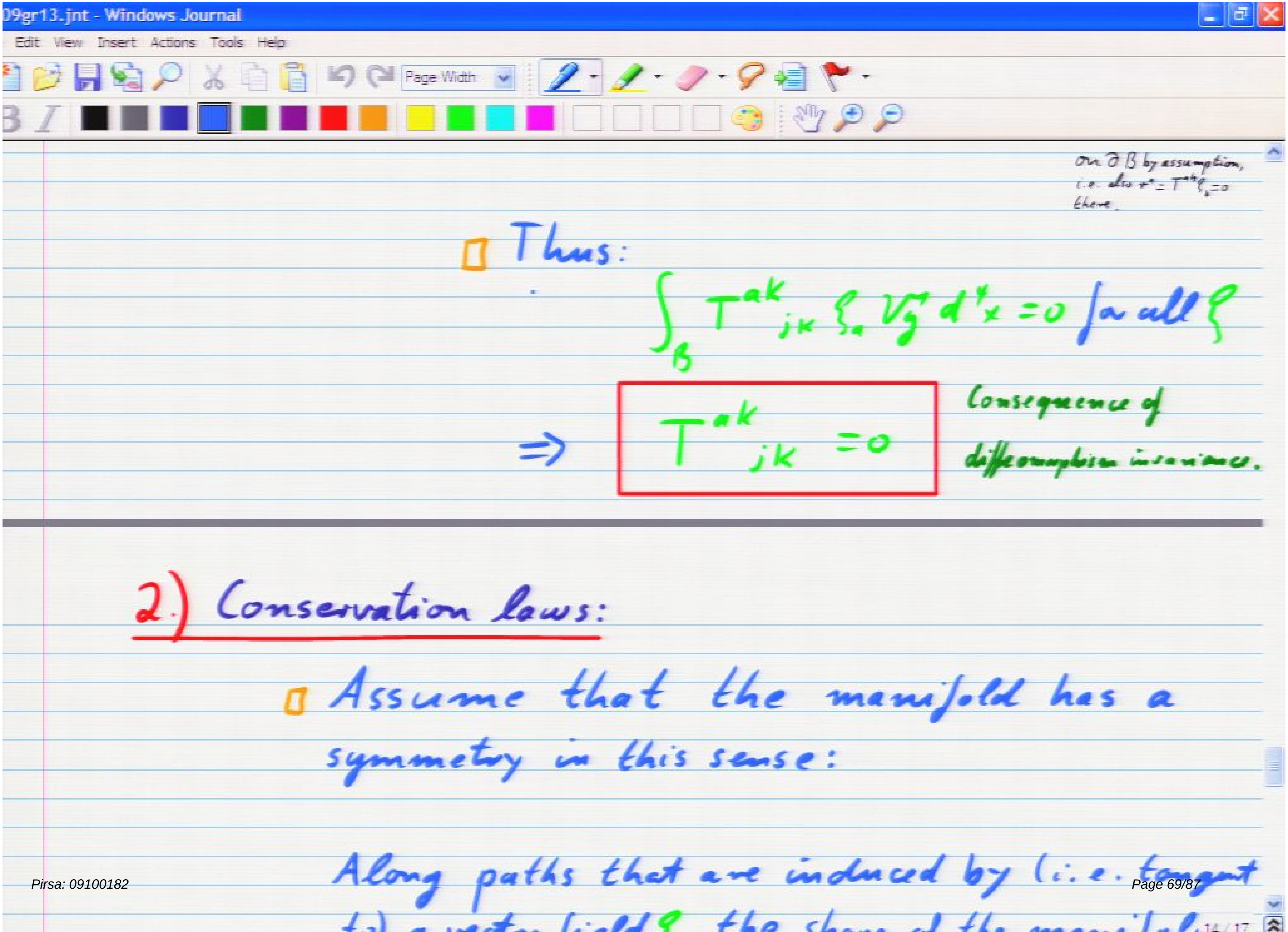

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Thus:

$$\int_B T^{ak}{}_{;k}\,\xi_a \sqrt{g}\, d^4x = 0 \text{ for all } \xi$$

$$\Rightarrow \boxed{T^{ak}{}_{;k} = 0}$$

Consequence of diffeomorphism invariance.

2) Conservation laws:

Assume that the manifold has a symmetry in this sense:

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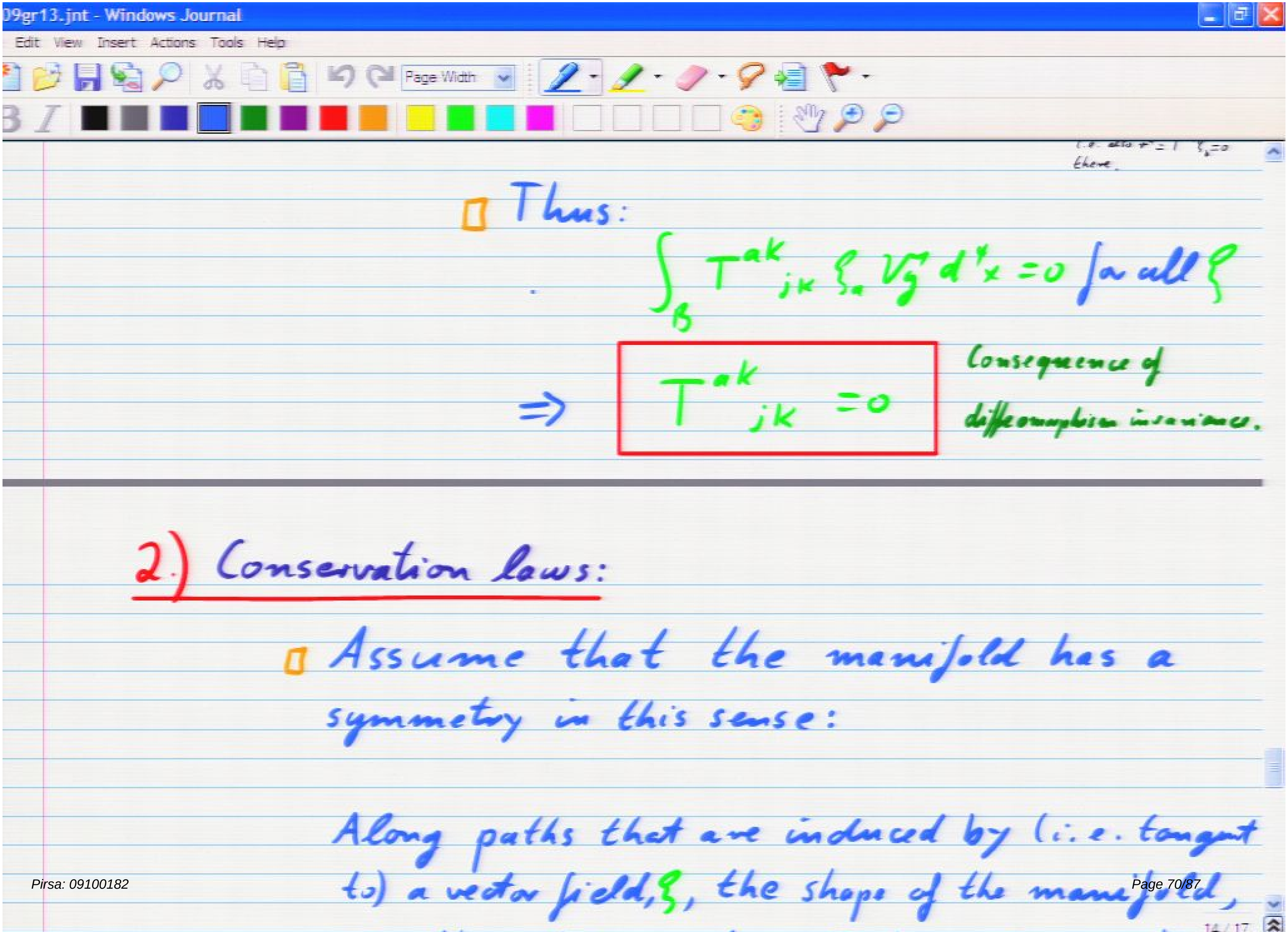

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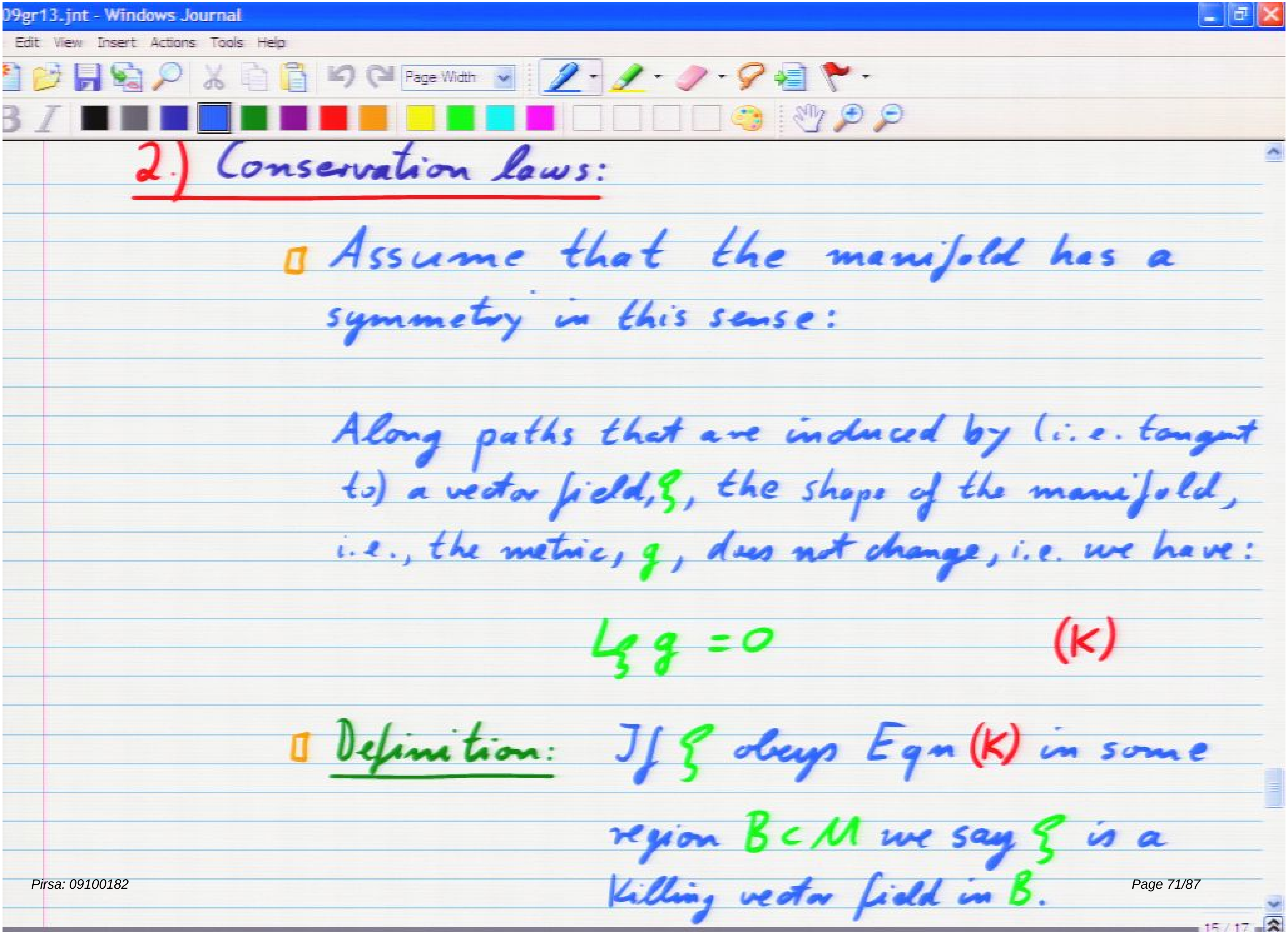

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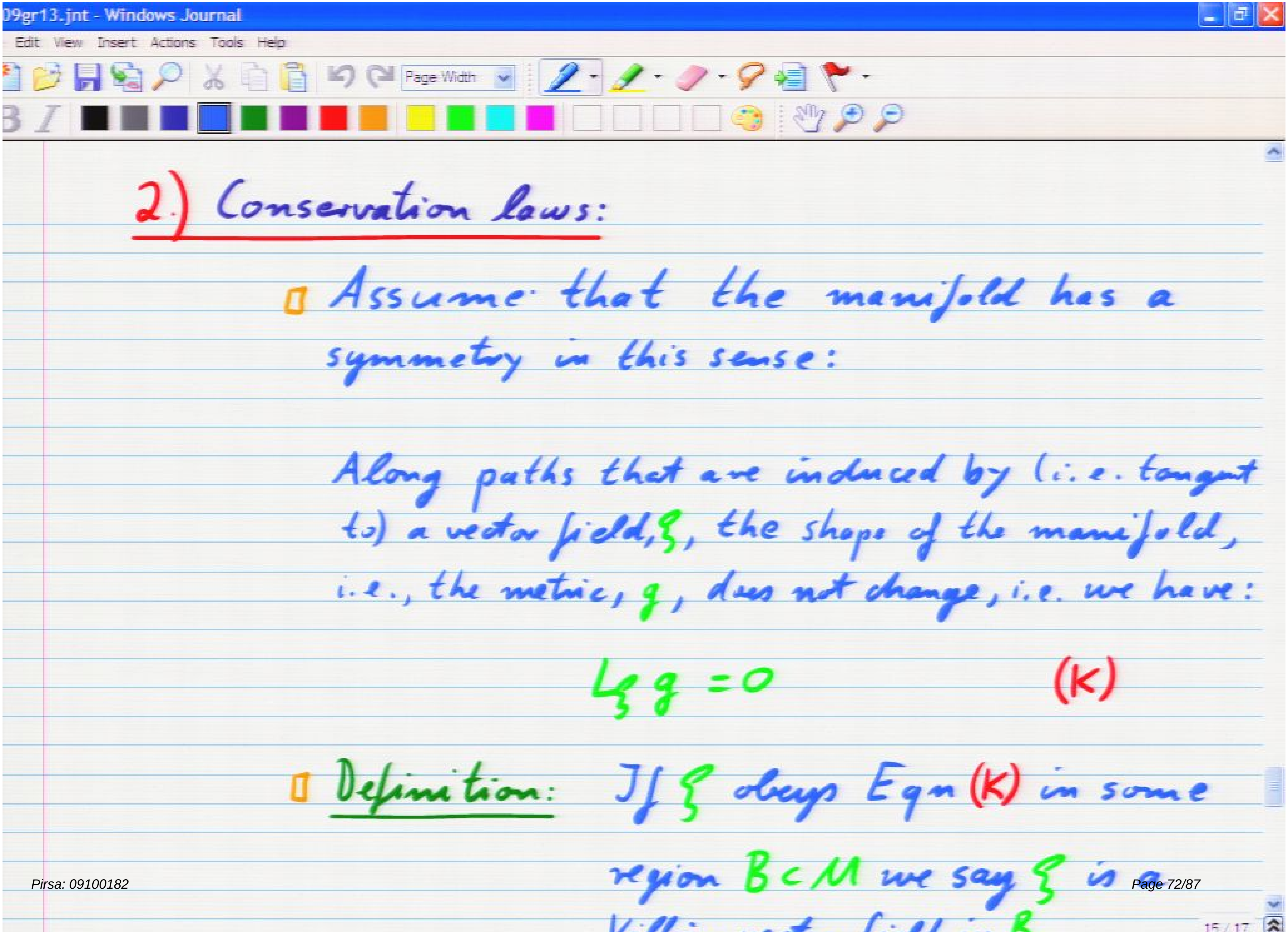
$$L_\xi g = 0 \qquad (K)$$

- **Definition:** If ξ obeys Eqn (K) in some region $B \subset M$ we say ξ is a Killing vector field in B.



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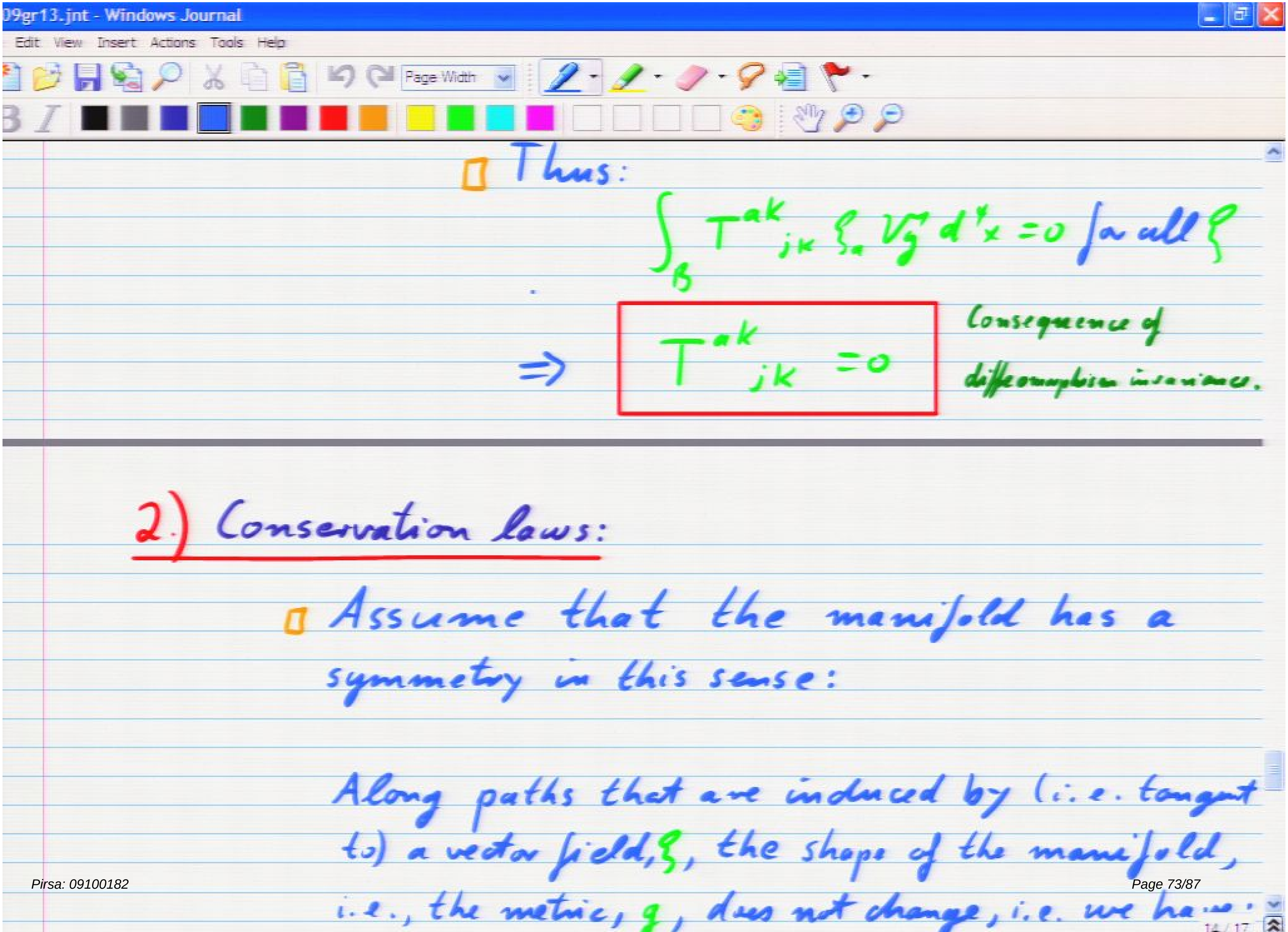

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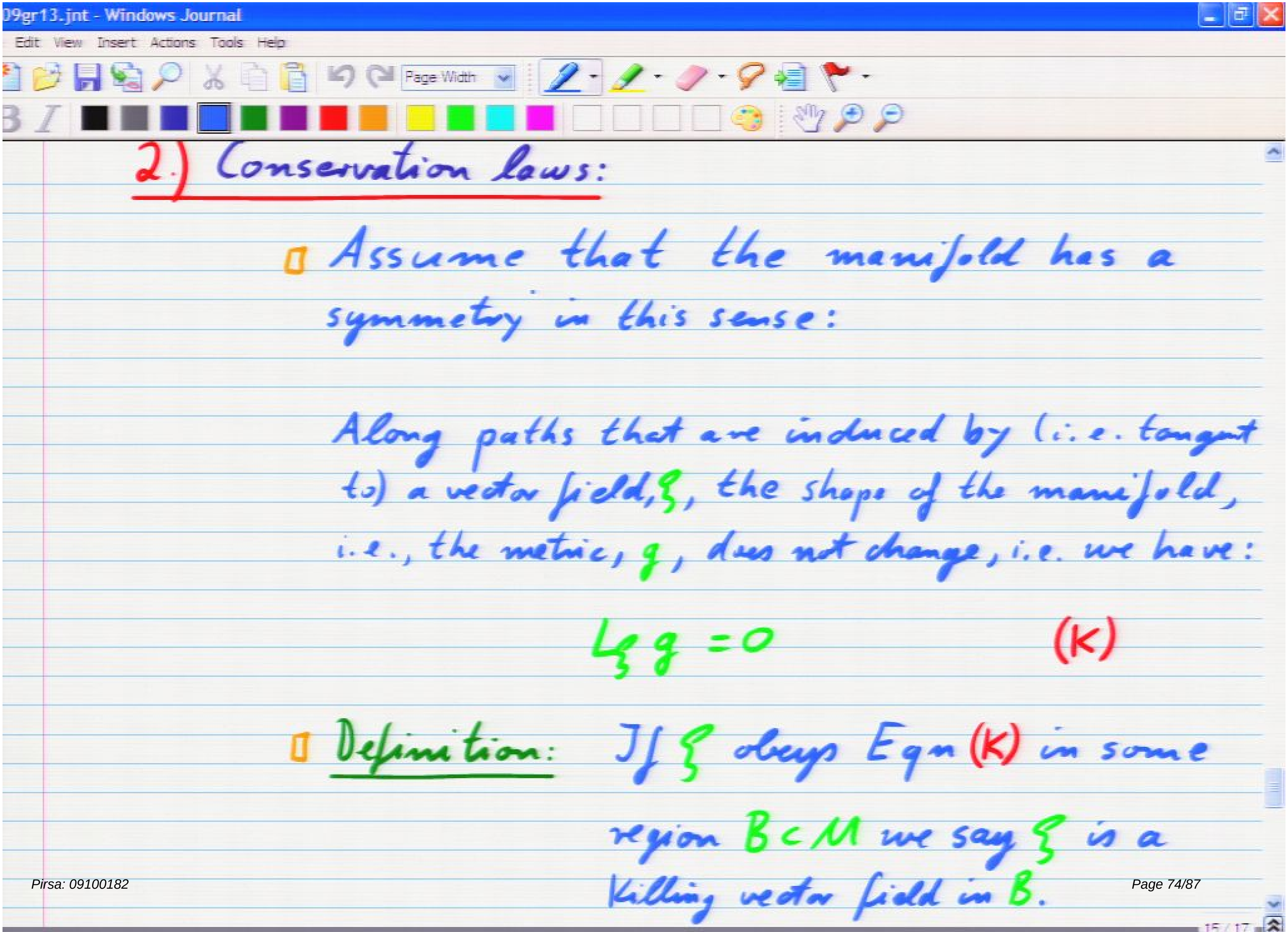

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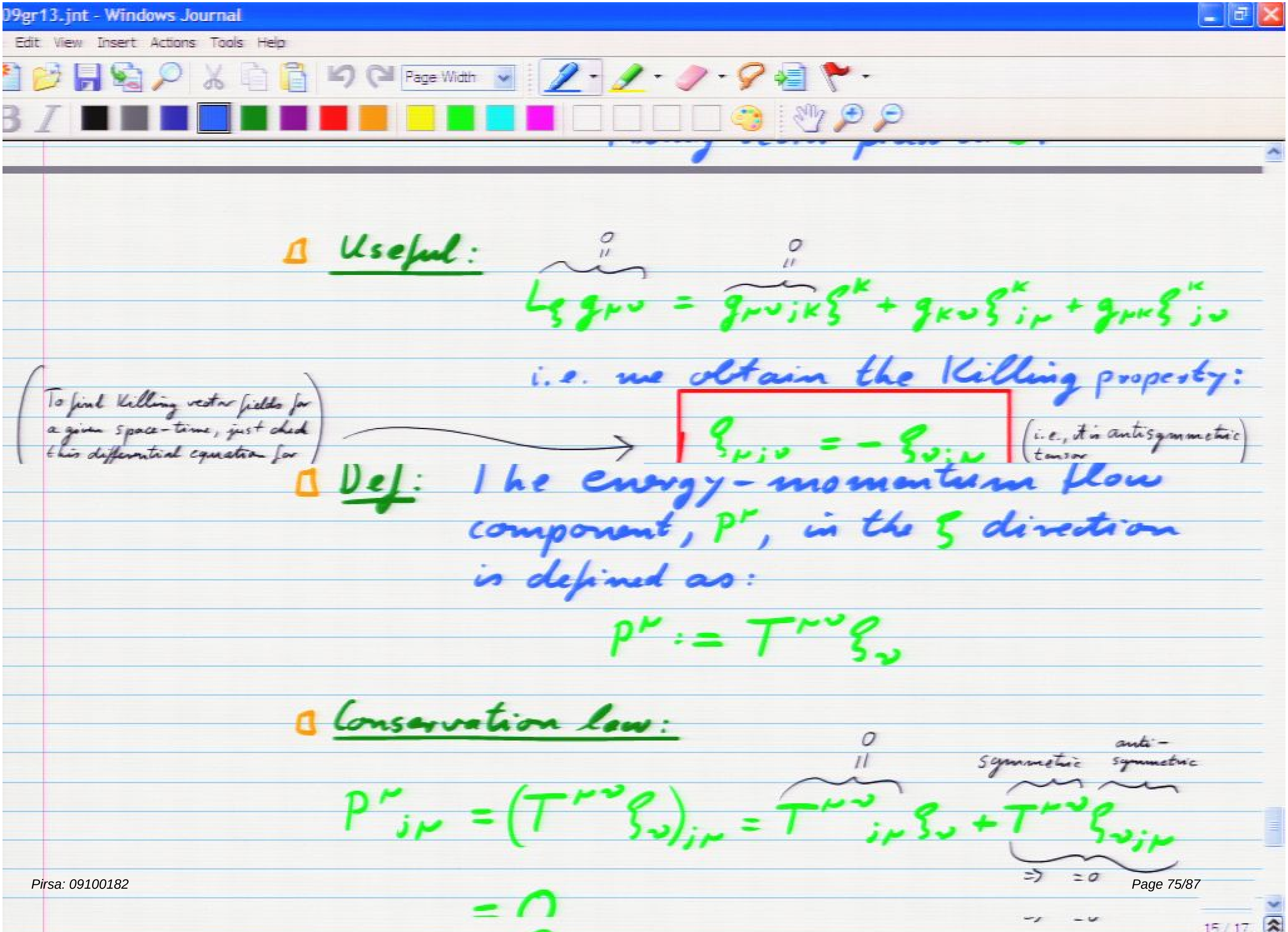

Useful:

$$L_\xi g_{\mu\nu} = \overbrace{g_{\mu\nu;\kappa}}^{\overset{0}{\shortparallel}}\xi^\kappa + \overbrace{g_{\kappa\nu}\xi^\kappa{}_{;\mu} + g_{\mu\kappa}\xi^\kappa{}_{;\nu}}^{\overset{0}{\shortparallel}}$$

i.e. we obtain the Killing property:

(To find Killing vector fields for a given space-time, just check this differential equation for) $\longrightarrow$

$$\boxed{\xi_{\mu;\nu} = -\xi_{\nu;\mu}}$$

(i.e., it is antisymmetric tensor)

Def: The energy-momentum flow component, P^μ, in the ξ direction is defined as:

$$P^\mu := T^{\mu\nu}\xi_\nu$$

Conservation law:

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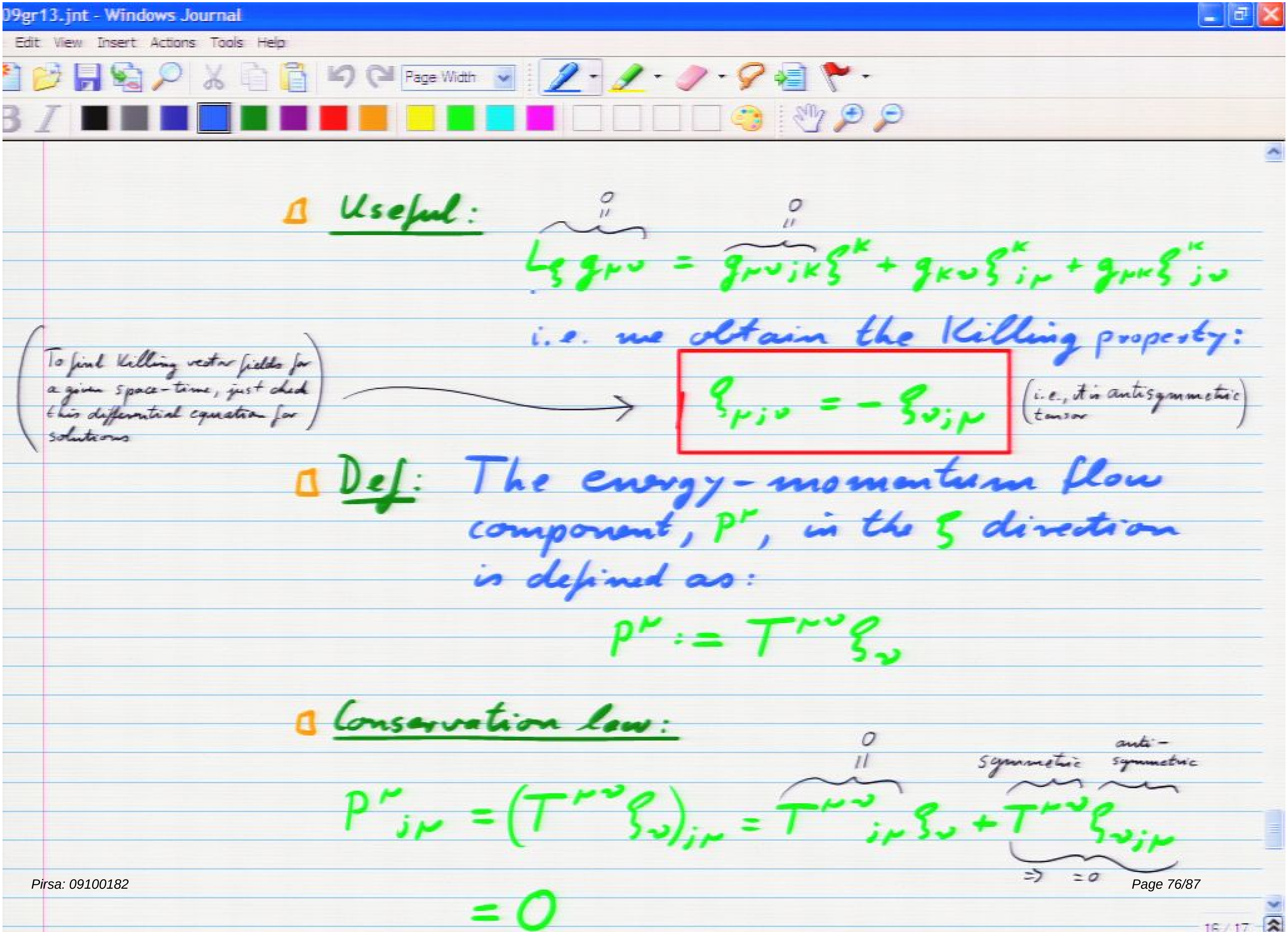
$$= 0$$



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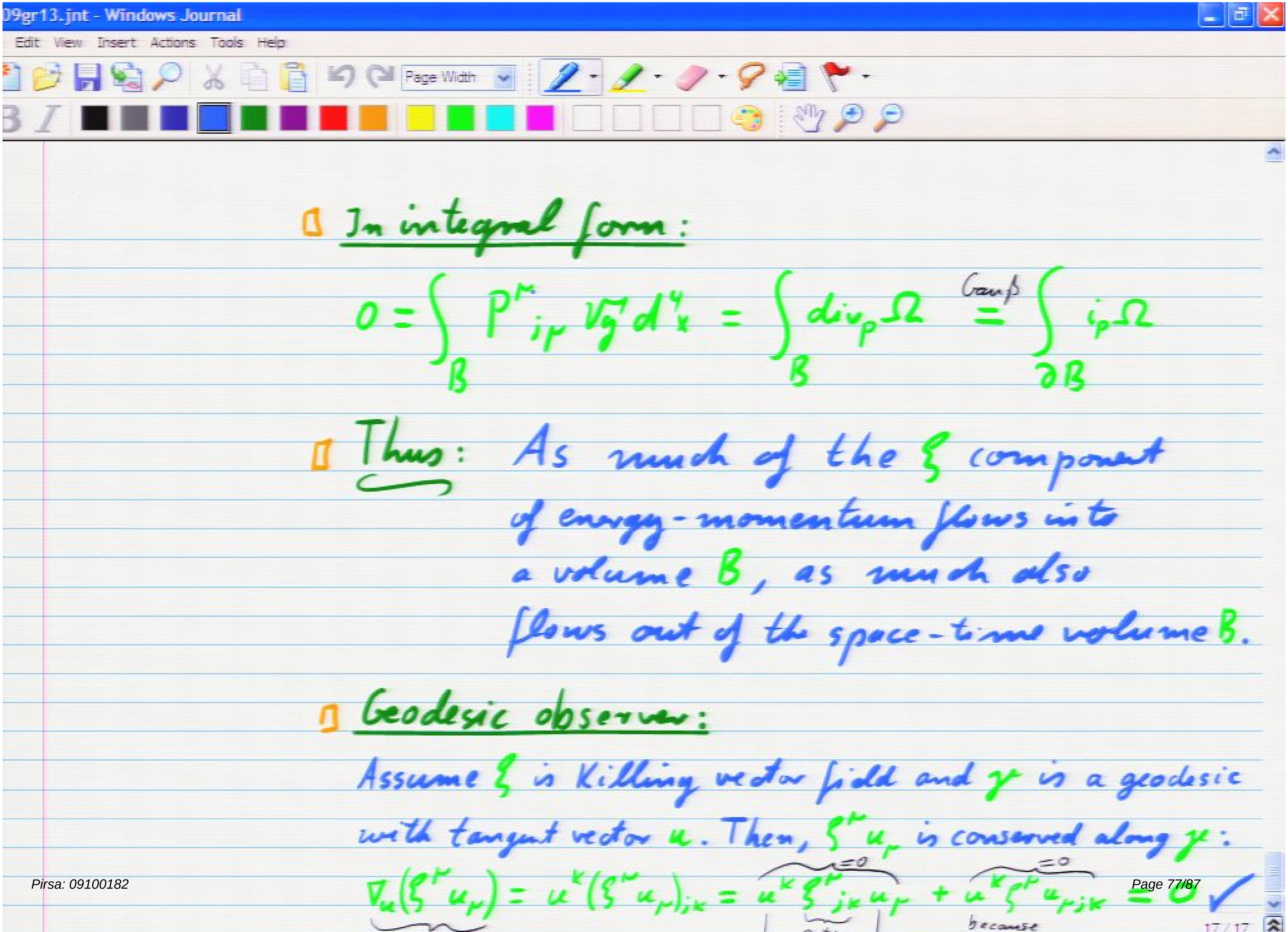


In integral form:

$$0 = \int_B P^\mu{}_{;\mu} \sqrt{g}\, d^4x = \int_B \operatorname{div}_P \Omega \overset{\text{Gauß}}{=} \int_{\partial B} i_P \Omega$$

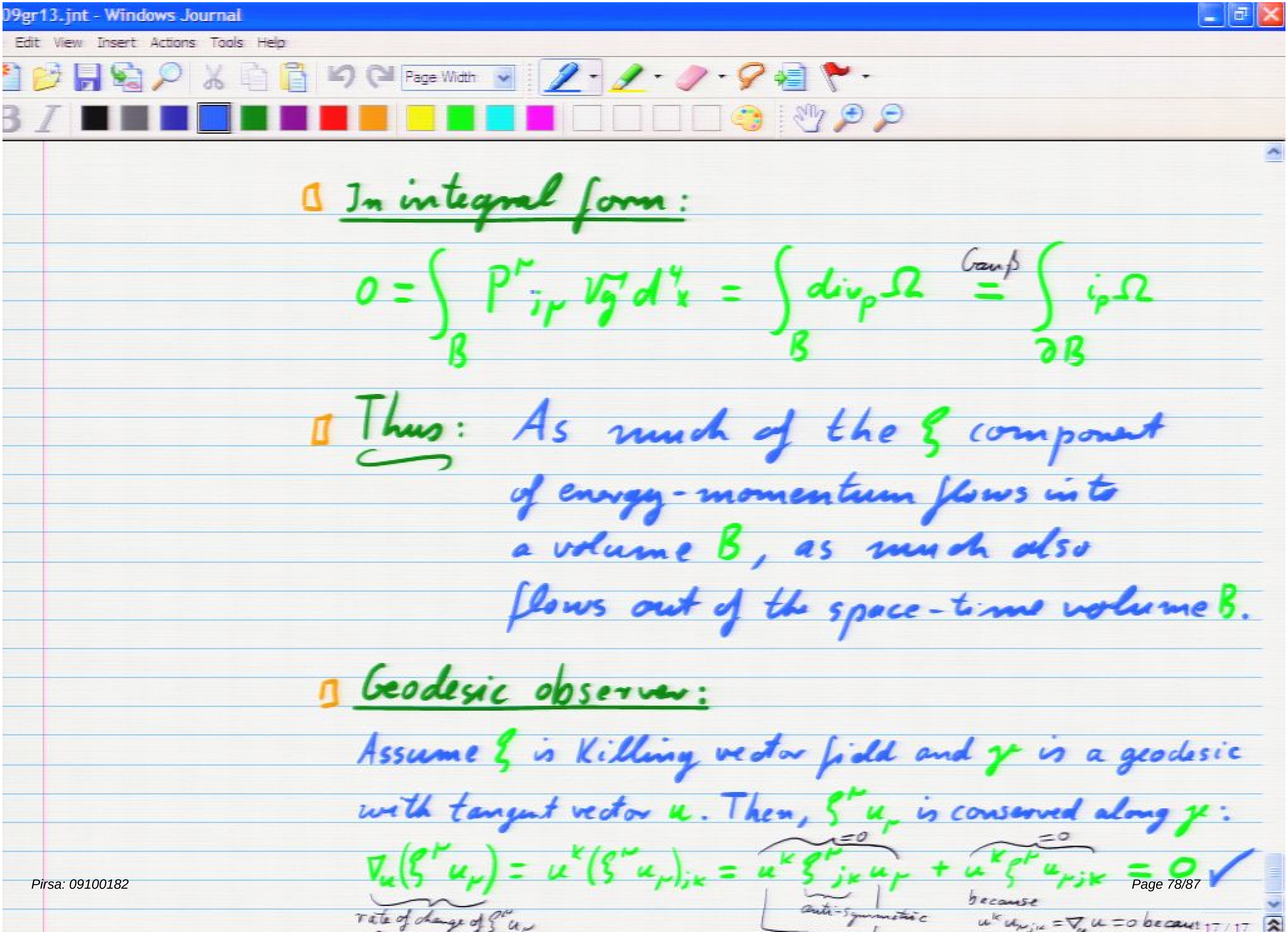
Thus: As much of the ξ component of energy-momentum flows into a volume B, as much also flows out of the space-time volume B.

Geodesic observer:

Assume ξ is Killing vector field and γ is a geodesic with tangent vector u. Then, $\xi^\mu u_\mu$ is conserved along γ:

$$\nabla_u(\xi^\mu u_\mu) = u^\kappa(\xi^\mu u_\mu)_{;\kappa} = \overbrace{u^\kappa \xi^\mu{}_{;\kappa} u_\mu}^{=0} + \overbrace{u^\kappa \xi^\mu u_{\mu;\kappa}}^{=0} = 0 \checkmark$$

because



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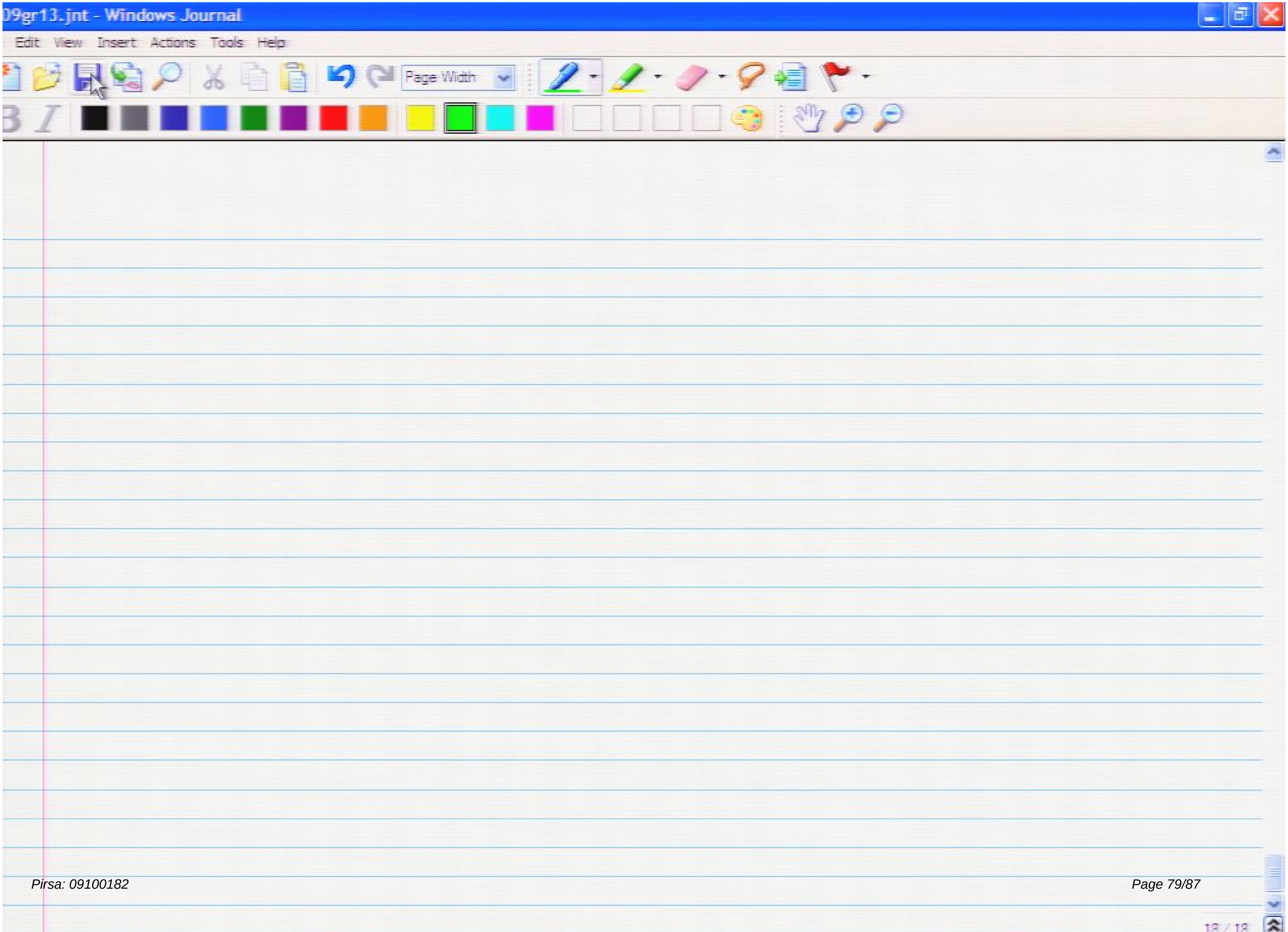
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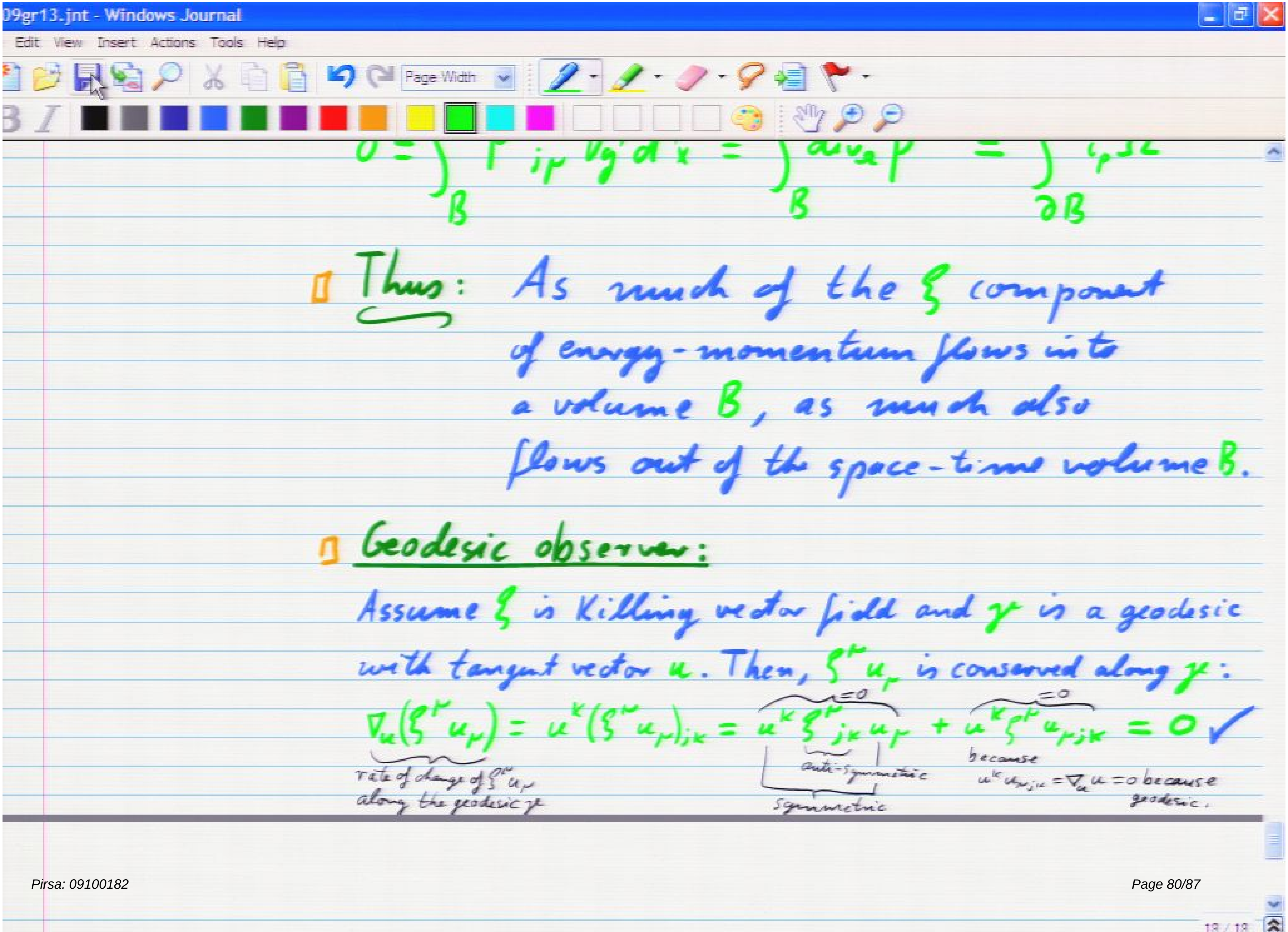

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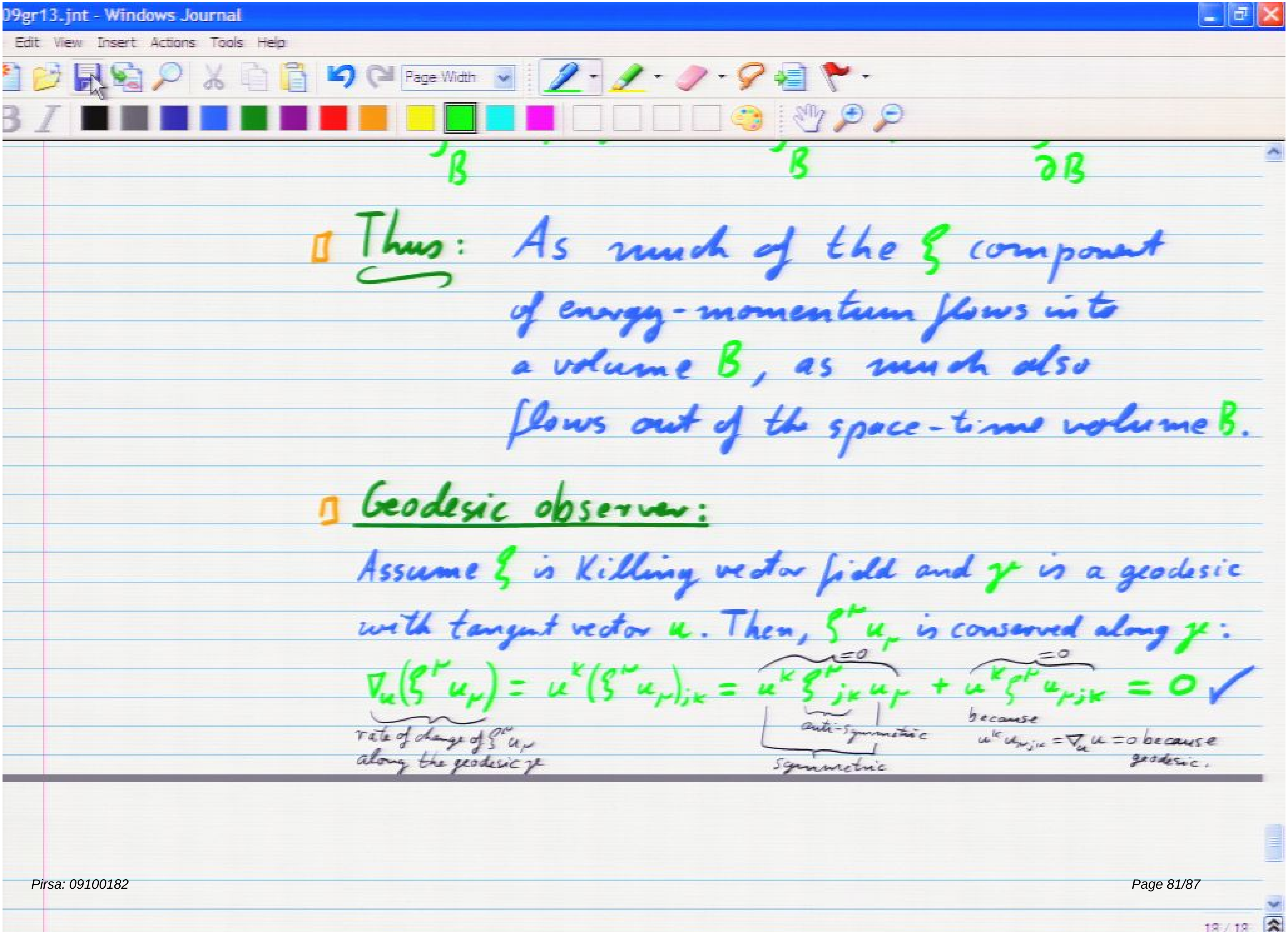

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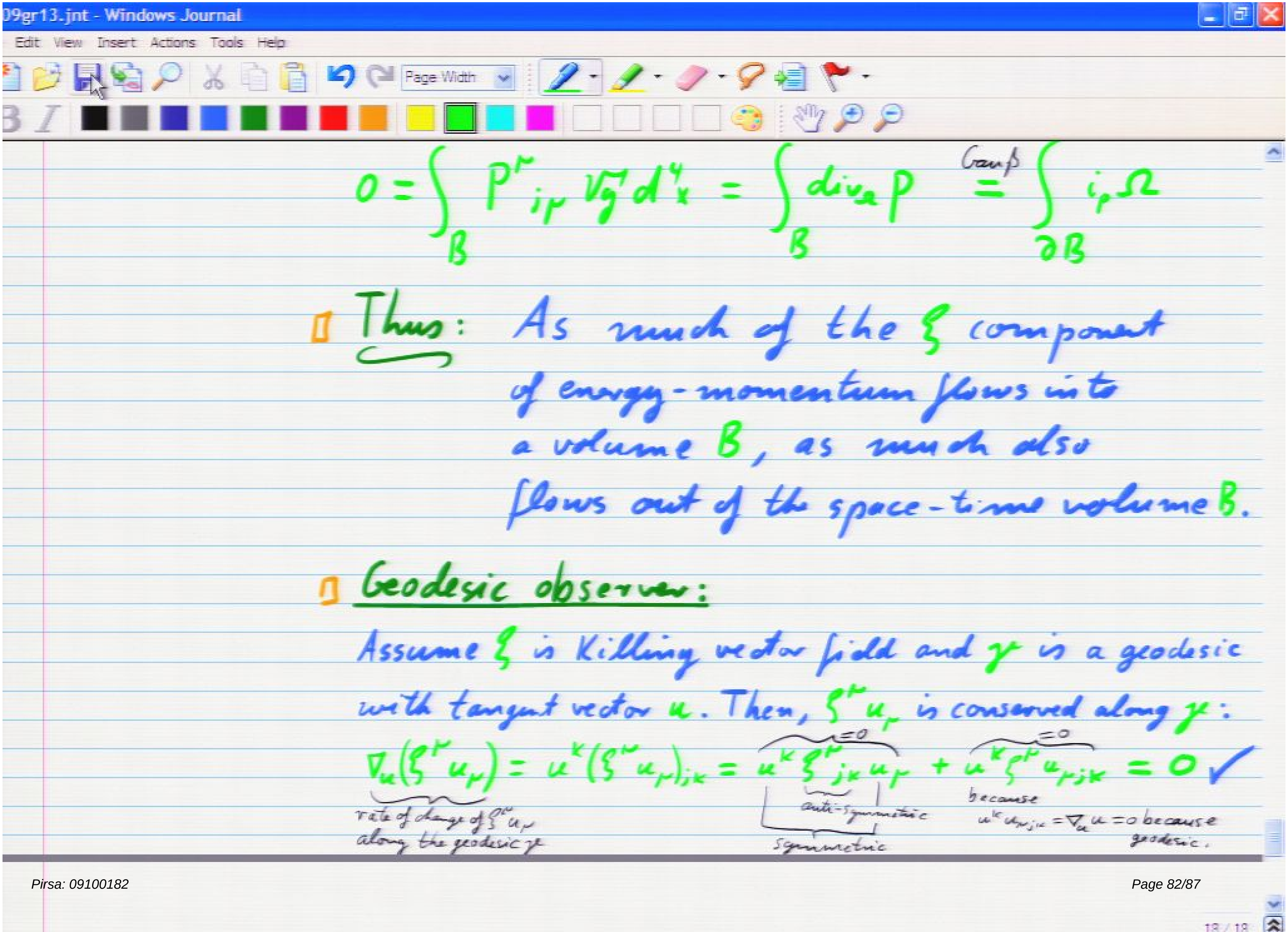

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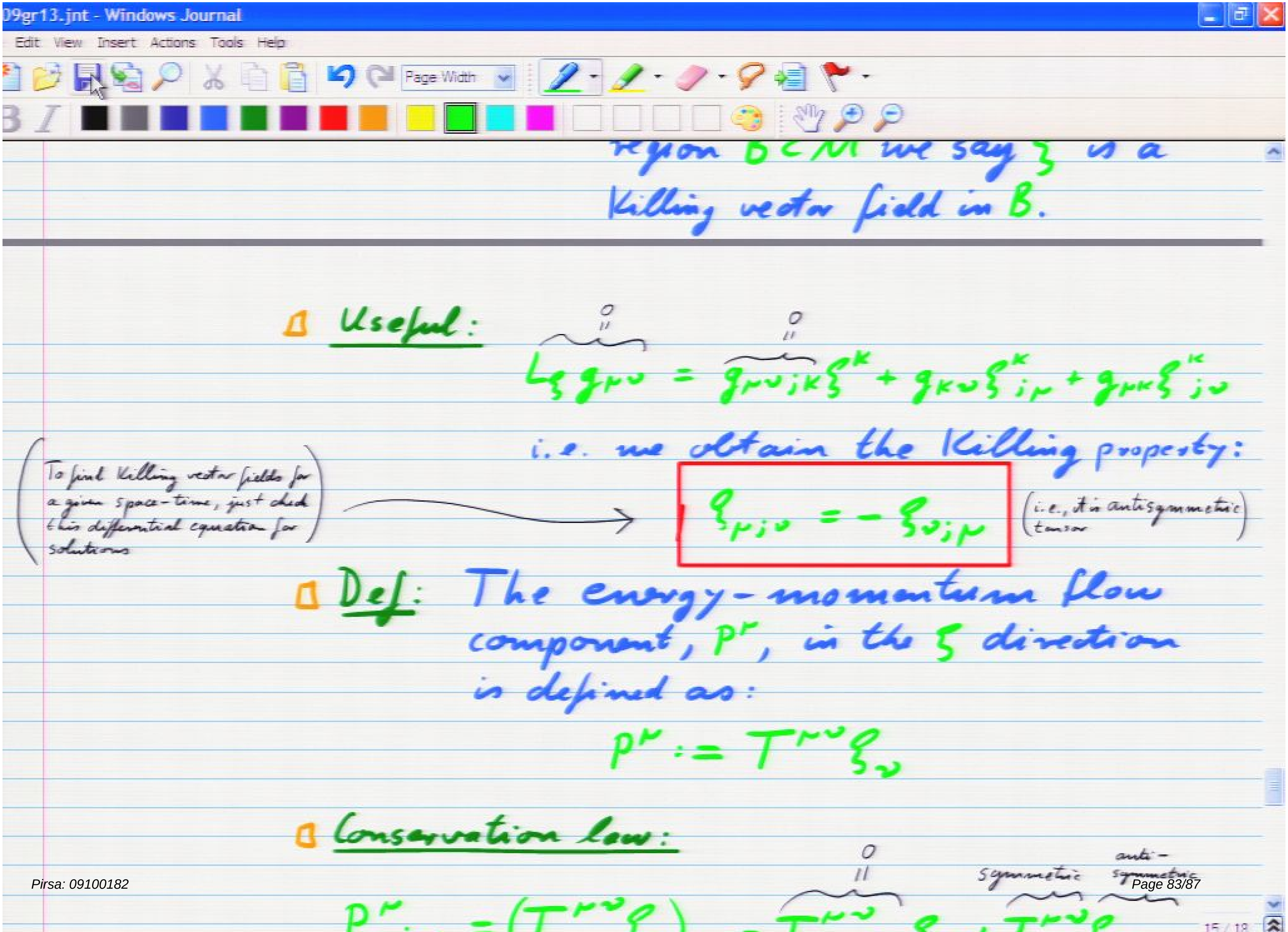

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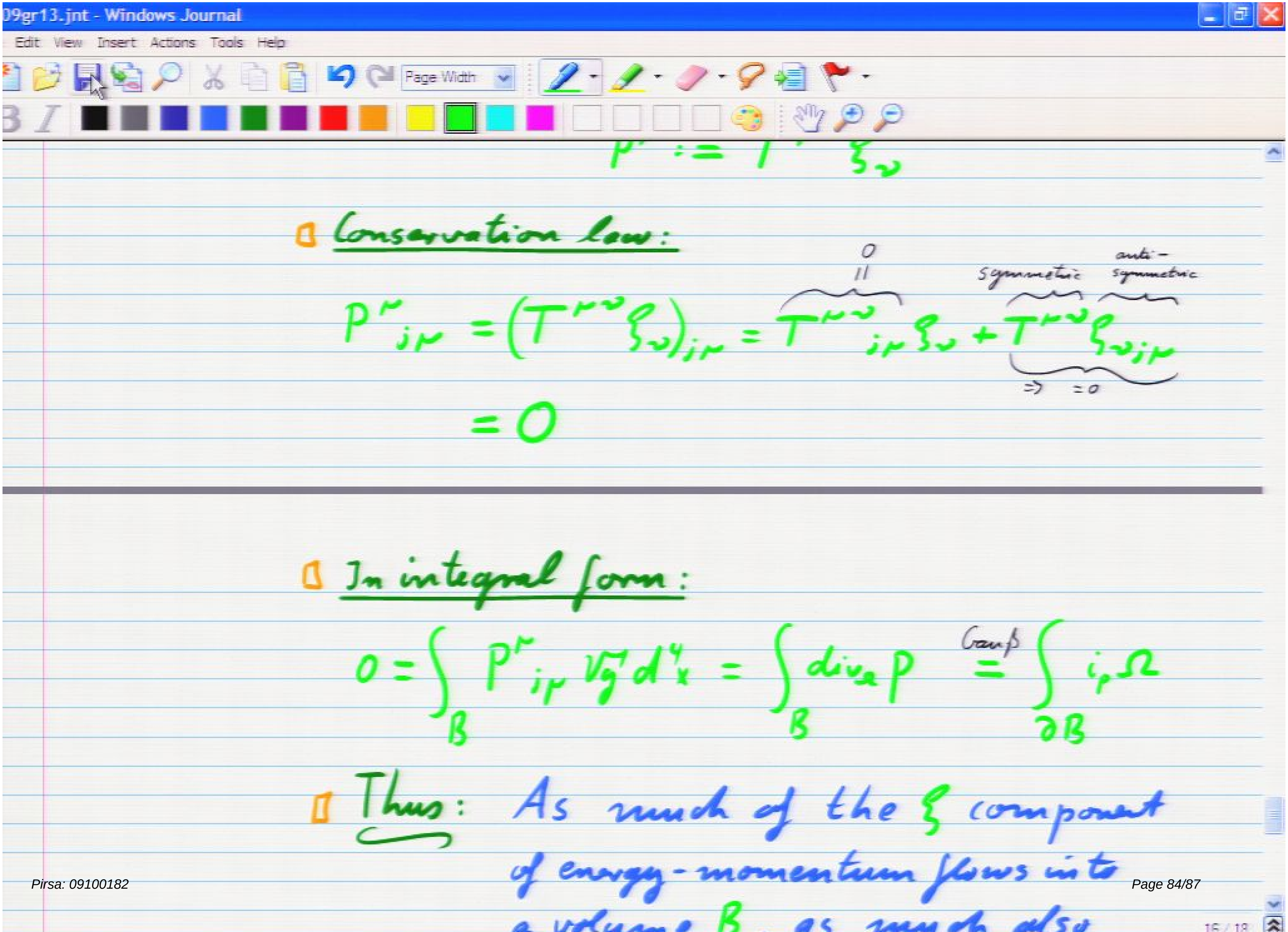

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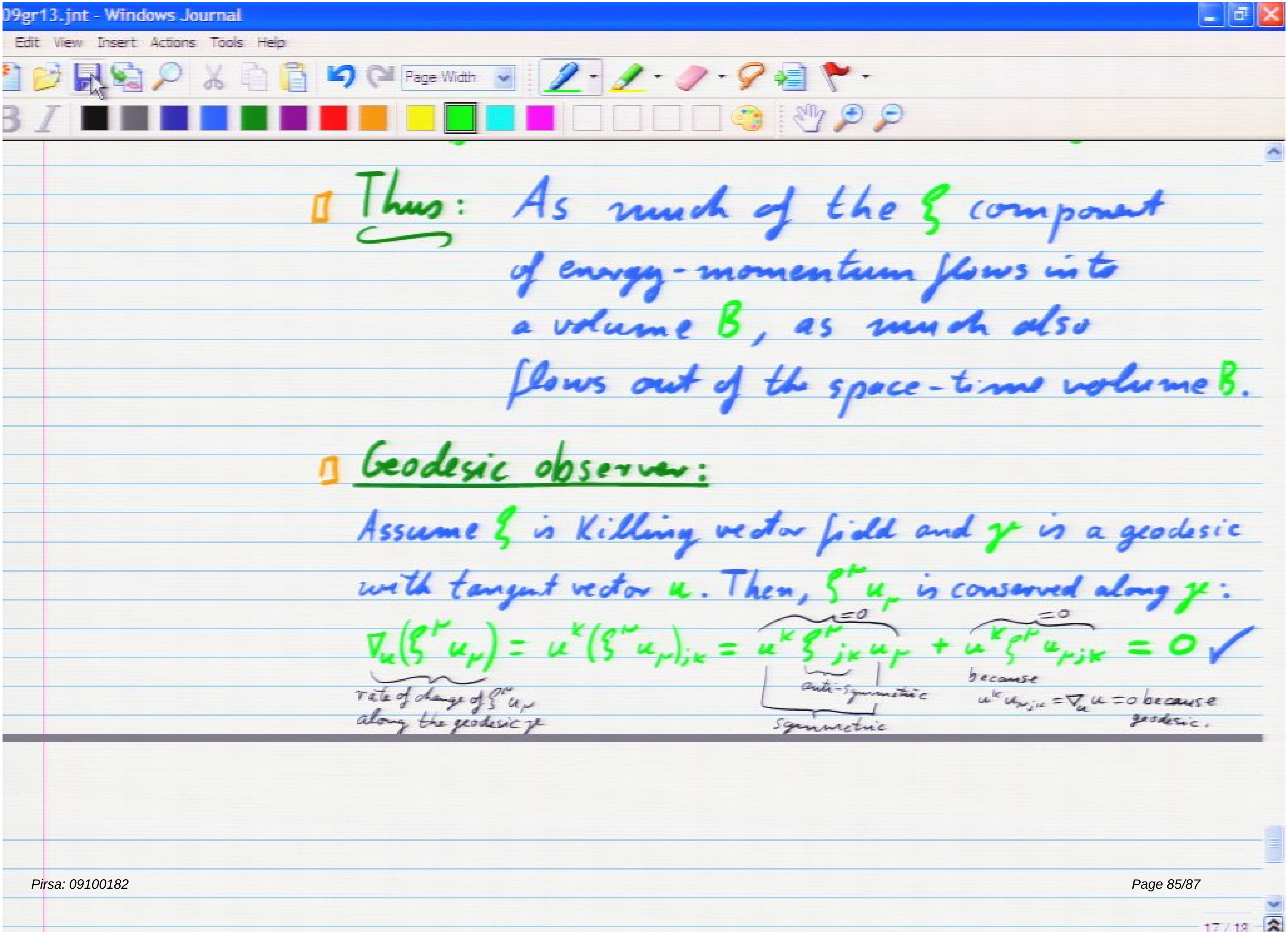

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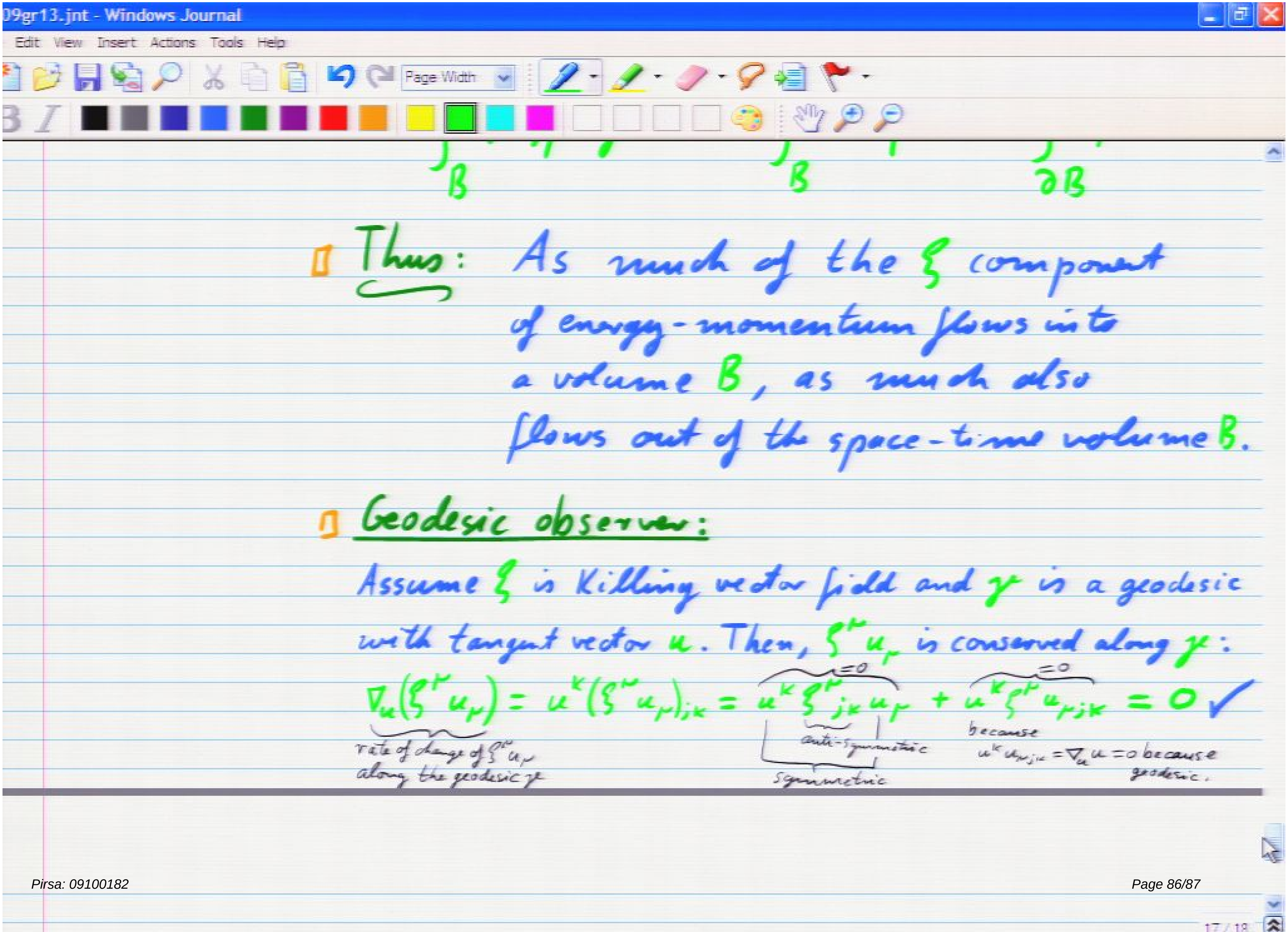

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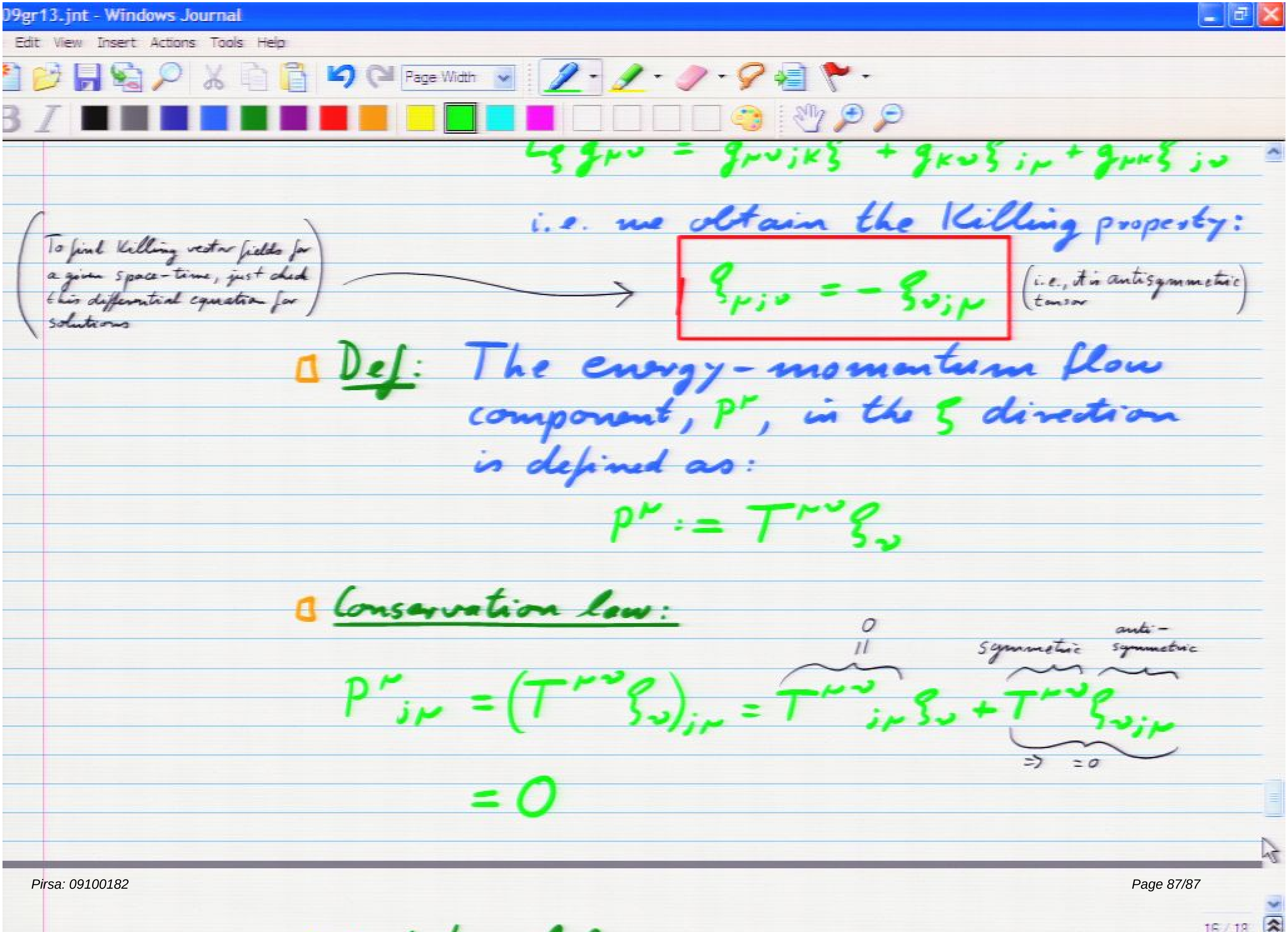

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