Title: On Semi-classical States of Quantum Gravity and Noncommutative Geometry

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Abstract: The idea behind an intersection between loop quantum gravity and noncommutative geometry is to combine elements of unification with a setup of canonical quantum gravity. In my talk I will first review the construction of a semi-finite spectral triple build over an algebra of holonomy loops. Here, the loop algebra is a noncommutative algebra of functions over a configurations space of connections, and the interaction between the Dirac type operator and the loop algebra captures information of the kinematical part of canonical quantum gravity. Next, I will show how certain normalizable, semi-classical states are build which connects the spectral triple construction to the Dirac Hamiltonian in 3+1 dimensions. Thus, these states can be interpreted as one-particle fermion states in an ambient gravitational field. This analysis indicates that the spectral triple construction involves matter degrees of freedom.

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  - Noncommutative geometry - Connes' work on the standard model of particle physics.
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  - A spectral triple over a configuration space of connections.
  - A noncommutative algebra generated by holonomy loops.
  - The spectral triple is based on an ordered system of graphs.
- **Physical Interpretation**
  - The spectral triple encodes information of the Poisson bracket of general relativity - kinematical part of quantum gravity.
  - The spectral triple has semi-classical states which gives the Dirac Hamiltonian in 3+1 dimensions.

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  1. The resolvent of D, $(1 + D^2)^{-1}$, is compact.
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- First example: Riemannian geometry.

$$(B = C^\infty(M), H = L^2(M, S), D = \not{D})$$

- Connes 2008: reconstruction theorem, complete equivalence.

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- A noncommutative example from physics: *the standard model coupled to gravity* [Connes, Lott, Chamseddine, Marcolli, ...]
  - $B = C^\infty(M) \otimes B_F$, *"almost commutative algebra"*
    $B_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$,
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  - The classical action of the standard model coupled to gravity emerges from a heat kernel expansion of D.
- The fact that the Standard Model coupled to gravity fits into the framework of NCG is a non-trivial result.

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Formulation of the standard model coupled to general relativity as a single gravitational theory. The standard model emerges from a modification of space-time geometry:

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Our goal

To construct a framework which combines noncommutative geometry with elements of quantum gravity.

Loop quantum gravity

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- Poisson brackets

$$\{A^i_j(x), E^k_l(y)\} = \delta^i_l \delta^k_j \delta(x - y)$$

- Constraints related to the symmetries of GR (spatial diffeomorphism, Hamilton, Gauss)

- Shift focus from connections to holonomy and flux variables

$$h_L(A) = \mathrm{Hol}(L, A)$$

L loop on Σ

$$F_S^a(E) = \int_S \epsilon^i{}_{jk} E_i^a \, dx^j \, dx^k$$

S surface in Σ.

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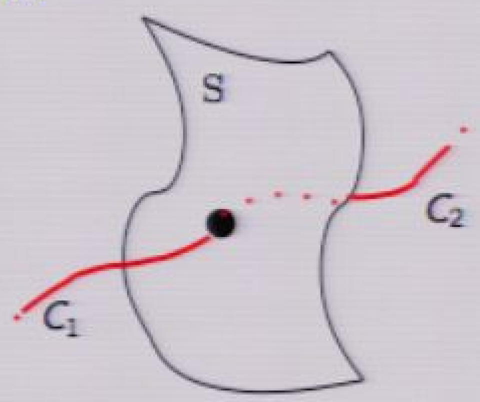

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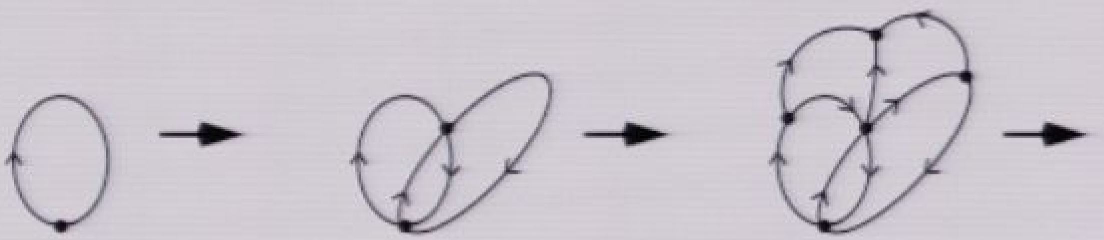
$$\{F^a_S(E), h_C(A)\} = \pm h_{C_1}(A) \tau^a h_{C_2}(A)$$

τ^a generator of $\mathfrak{su}(2)$, $C = C_1 C_2$ are curves in Σ.

Graphs

In LQG the algebra of holonomy loops is described via the inductive system of all finite, piece-wise analytic graphs



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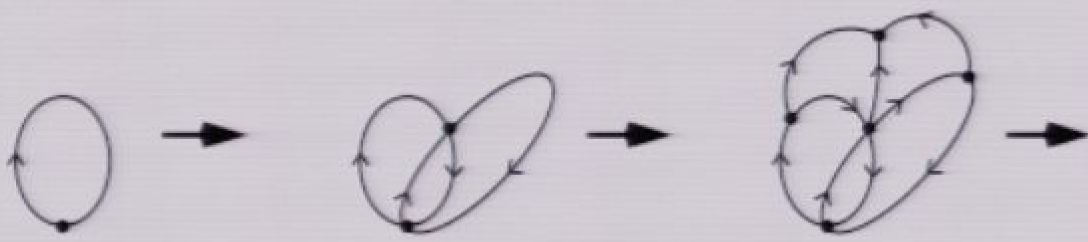

- Let $\mathcal{A}$ be the space of smooth connections with gauge group G. Denote by $\mathcal{A}_\Gamma$ the restriction of $\mathcal{A}$ to a finite graph Γ. Seen from Γ a connection $\nabla \in \mathcal{A}$ can be seen as a point in the space $G^{n(\Gamma)}$

$$\nabla = (g_1, \ldots, g_n) \in G^{n(\Gamma)} \simeq \mathcal{A}_\Gamma$$

where $n(\Gamma)$ is the number of edges ϵ_i in Γ and where $g_i = Hol(\nabla, \epsilon_i)$ is the holonomy of ∇ along ϵ_i.

- Projective system of coarse grained spaces of connections:

$$
\begin{array}{ccccccc}
\ldots \leftarrow & \mathcal{A}_\Gamma & \leftarrow & \mathcal{A}_{\Gamma'} & \leftarrow & \mathcal{A}_{\Gamma''} & \leftarrow \ldots \\
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with structure maps

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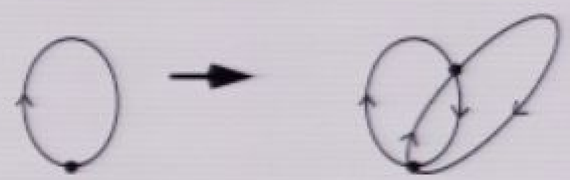
- **Example:**

$P : G^4 \to G$



$(g_1, g_2, g_3, g_4) \to g_1 \cdot g_3$

because $Hol(\nabla, \epsilon_1) \cdot Hol(\nabla, \epsilon_3) = Hol(\nabla, \epsilon_1 \cdot \epsilon_3)$

- **Result:**

$$\mathcal{A} \hookrightarrow \varprojlim \mathcal{A}_\Gamma =: \overline{\mathcal{A}}^a$$

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- **Important point:** This program is based on a *choice* of a loop algebra: piece-wise analytic. This choice is related to symmetry considerations.

Our Project

- **Aim:** to construct a spectral triple over an algebra of holonomy loops, i.e. functions on the configuration space $\mathcal{A}$:

$$L : \nabla \to \mathrm{Hol}(\nabla, L) \in M_n(\mathbb{C})$$

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- **Key point:** use holonomy loops instead of Wilson loops (LQG) $\rightarrow$ Noncommutative algebra $\rightarrow$ additional structure.
- **Hope/Idea:** to look for a (semi-) classical limit where the algebra of loops leads to an almost commutative algebra

$$\mathrm{NCG} + \mathrm{LQG} \stackrel{cl.limit}{\longrightarrow} C^\infty(M) \otimes B_F$$

to generate gauge + Higgs (requires noncommutativity)

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for all structures (Hilbert space, algebra, Dirac operator)
3. take the projective/inductive limit to obtain a spectral triple over the space of connections $\mathcal{A}$.

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  Problems:

  - *too many different embeddings between graphs to permit a Dirac type operator.*
  - *non-separability of the (kinematical) Hilbert space (!).*

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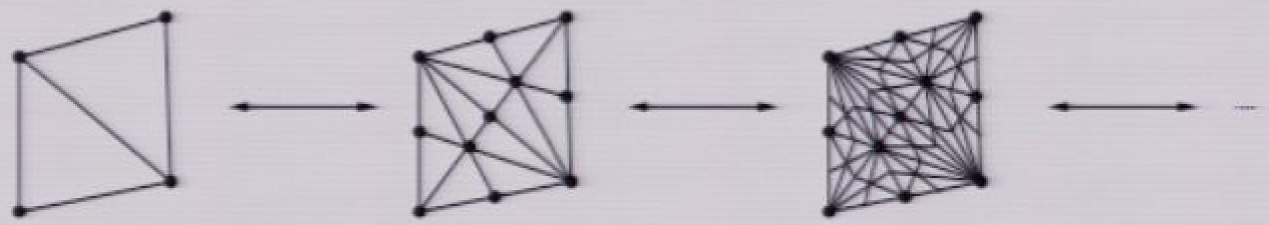

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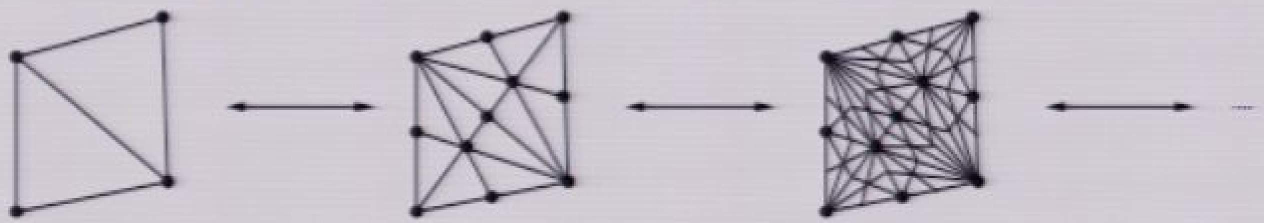

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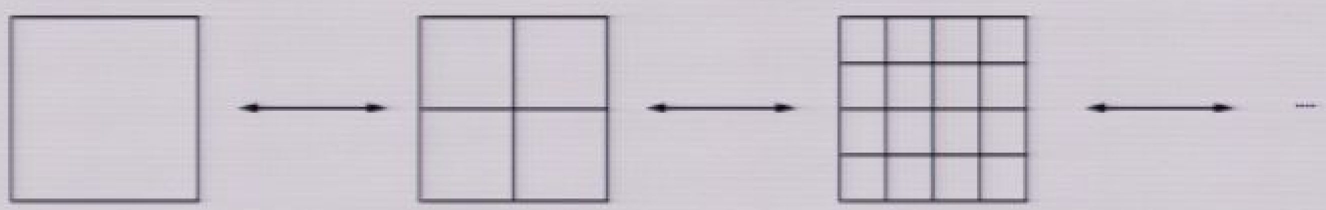

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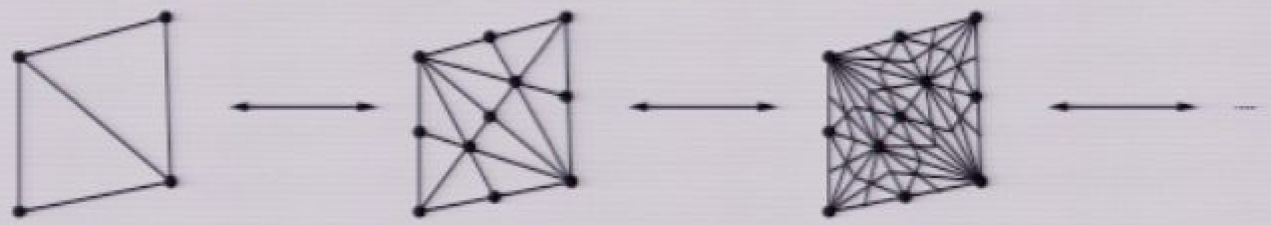

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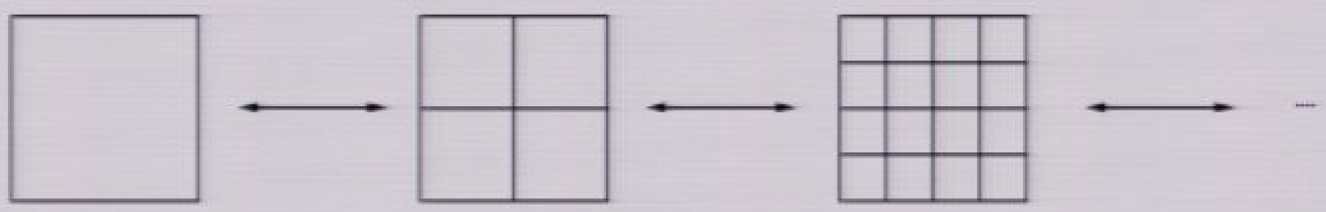

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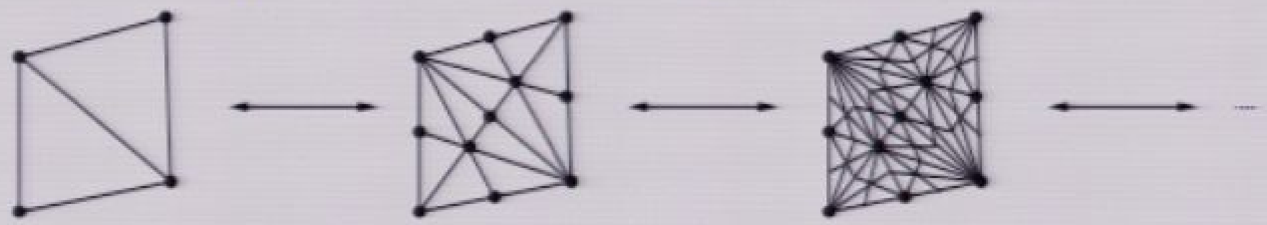

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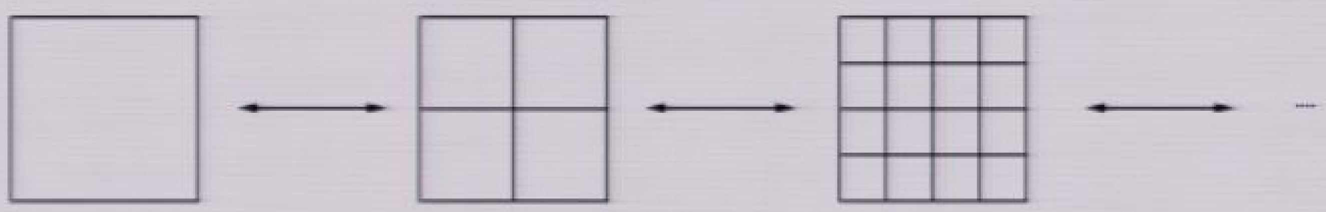
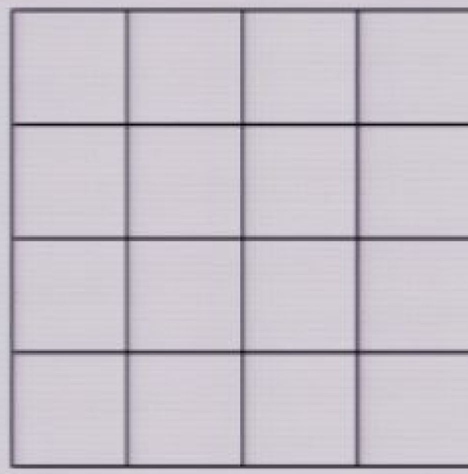


- Both systems of graphs permit a spectral triple construction.
- We now believe that cubic lattices are natural (end of talk).

The construction

A single cubic lattice

- Let Γ be a finite 3-dim finite cubic lattice with edges $\{\epsilon_i\}$ and vertices $\{v_i\}$
$$\epsilon_j : \{0,1\} \rightarrow \{v_i\}$$
- Assign to each edge ϵ_i a group element $g_i \in G$.
$$\nabla : \epsilon_i \longrightarrow g_i$$



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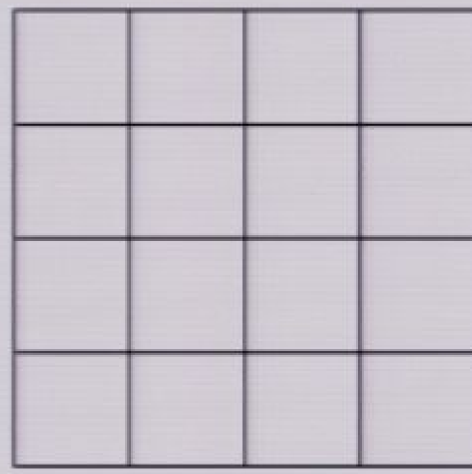

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G is a compact Lie-group. The space of such maps is denoted $\mathcal{A}_\Gamma$. Notice:

$$\mathcal{A}_\Gamma \simeq G^n \quad \text{because} \quad \mathcal{A}_\Gamma \ni \nabla \to (\nabla(\epsilon_1), \ldots, \nabla(\epsilon_n)) \in G^n$$

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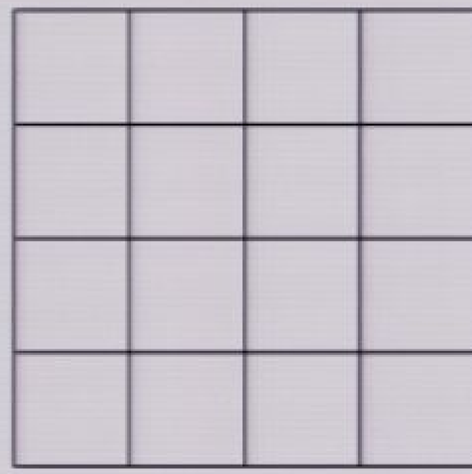


- Assign to each edge ϵ_i a group element $g_i \in G$.

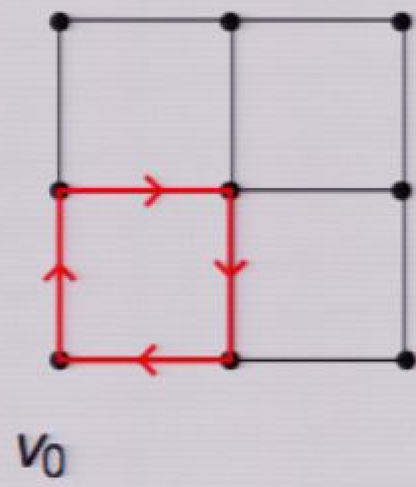
$$\nabla : \epsilon_i \to g_i$$

G is a compact Lie-group. The space of such maps is denoted $\mathcal{A}_\Gamma$. Notice:

$$\mathcal{A}_\Gamma \simeq G^n \quad \text{because} \quad \mathcal{A}_\Gamma \ni \nabla \to (\nabla(\epsilon_1), \ldots, \nabla(\epsilon_n)) \in G^n$$

- The space $\mathcal{A}_\Gamma$ is a coarse-grained approximation of $\mathcal{A}$.

- **Algebra:** A loop L is a finite sequence of edges $L = \{\epsilon_{i_1}, \epsilon_{i_2}, \ldots, \epsilon_{i_n}\}$ running in Γ (choose basepoint v_0). Discard trivial backtracking.



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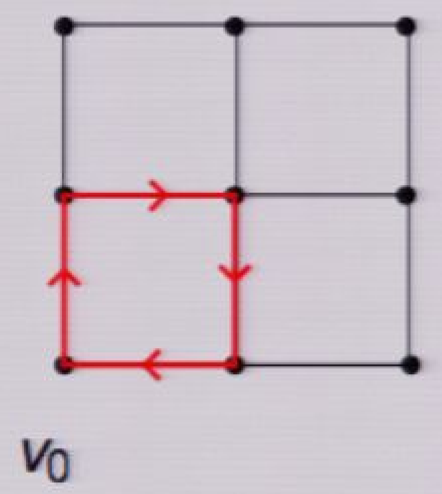

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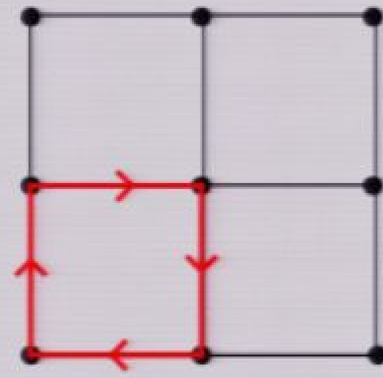

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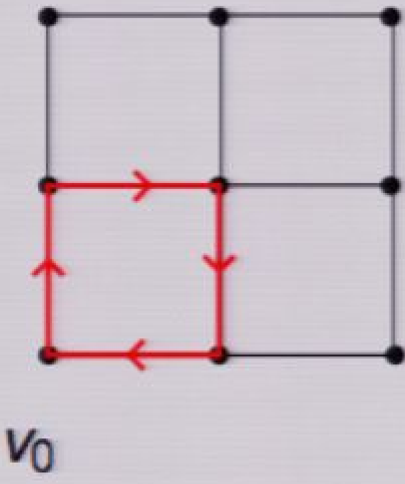

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  - Define

$$\nabla(L) = \nabla(\epsilon_{i_1}) \cdot \nabla(\epsilon_{i_2}) \cdot \ldots \cdot \nabla(\epsilon_{i_n})$$

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- These elements have a natural norm

$$\|a\| = \sup_{\nabla \in \mathcal{A}_\Gamma} \| \sum a_i \nabla(L_i) \|_G$$

where the norm on the rhs is the matrix norm in G. The closure of the $\star$-algebra of loops with respect to this norm is a C^*-algebra. We denote this loop algebra by $\mathcal{B}$.

- **Hilbert space:** There is a 'natural' Hilbert space

$$\mathcal{H} = L^2(G^n, Cl(T^*G^n) \otimes M_l(\mathbb{C}))$$

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$$f_L \cdot \psi(\nabla) = (1 \otimes \nabla(L)) \cdot \psi(\nabla), \quad \psi \in \mathcal{H}$$

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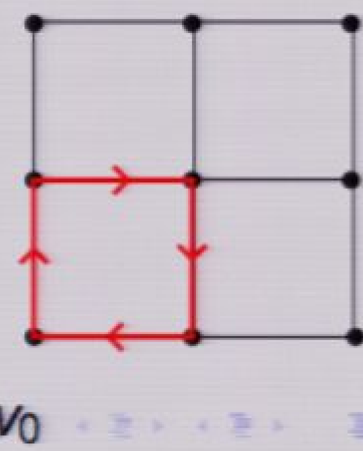
- Example:

$$L = \{\epsilon_1, \epsilon_4, \epsilon_6^*, \epsilon_3^*\}$$

$$f_L \sim g_1 \cdot g_4 \cdot (g_6)^{-1} \cdot (g_3)^{-1}$$



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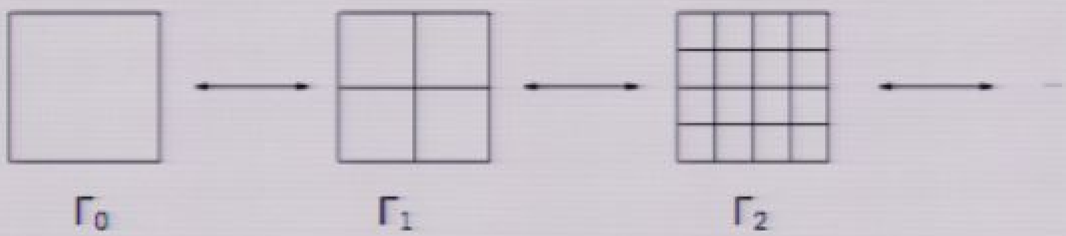

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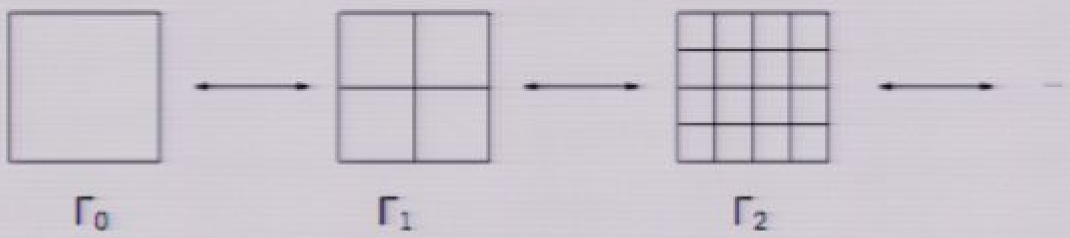

On the level of the associated manifolds $\mathcal{A}_{\Gamma_i}$ this gives rise to projections

$$G^{n_0} \xleftarrow{P_{10}} G^{n_1} \xleftarrow{P_{21}} G^{n_2} \xleftarrow{P_{32}} G^{n_3} \xleftarrow{P_{43}} \dots$$

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- For the Hilbert space compatibility is easily obtained and compatibility for the algebra is clear.
- For the Dirac operator care must be taken.

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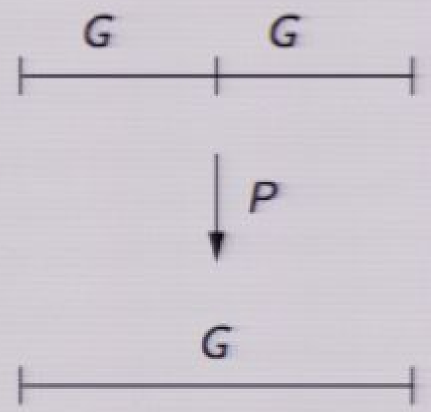
$$P : G^2 \rightarrow G\,, \quad (g_1, g_2) \rightarrow g_1 \cdot g_2$$



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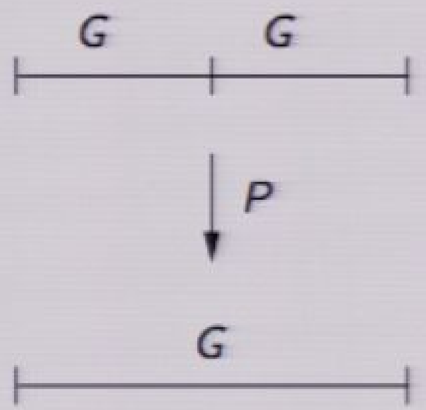

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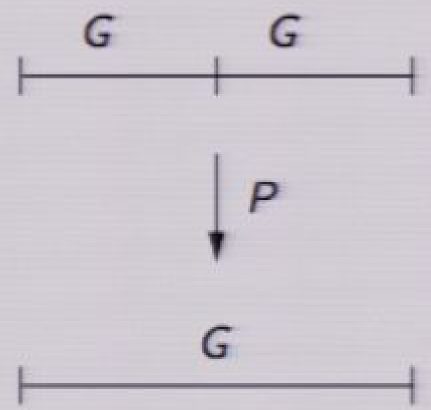

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- We have found several ways to solve this consistency problem. One solution is to consider the change of variables:

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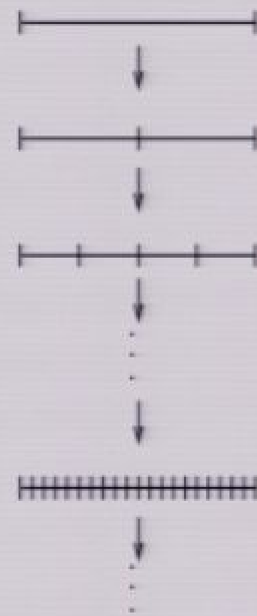

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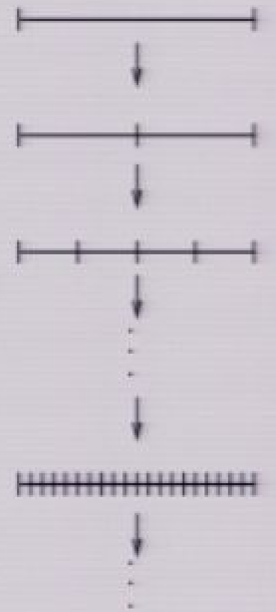

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- Result: For a compact Lie-group G the triple $(\mathcal{B}, D, \mathcal{H})_{\overline{\mathcal{A}}}$ is a semi-finite* spectral triple:
  - D's resolvent $(1 + D^2)^{-1}$ is compact (wrt. trace) and
  - the commutator $[D, b]$ is bounded

  provided the sequence $\{a_i\}$ approaches ∞.

**semi-finite: everything works up to a symmetry group with a trace (CAR algebra)* [Carey, Phillips, Sukochev].

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$$\overline{\mathcal{A}} := \lim_{\substack{\longleftarrow \\ \Gamma}} \mathcal{A}_\Gamma$$

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- $\overline{\mathcal{A}}$ is a space of generalized connections. To see this map the graphs $\{\Gamma_i\}$ into a manifold Σ

$$h : \Gamma_i \to \Gamma_i \in \Sigma$$

Spaces of connections

- Denote by $\mathcal{A}$ the space of smooth G-connections. There is a natural map

$$\chi : \mathcal{A} \to \overline{\mathcal{A}}, \quad \chi(\nabla)(\epsilon_i) = Hol(\nabla, \epsilon_i)$$

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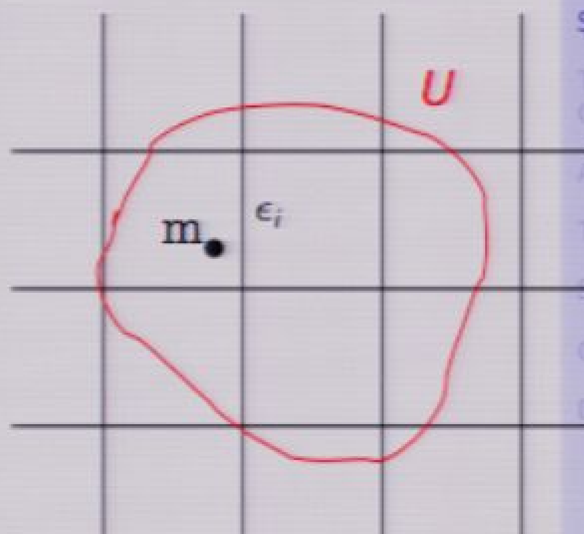

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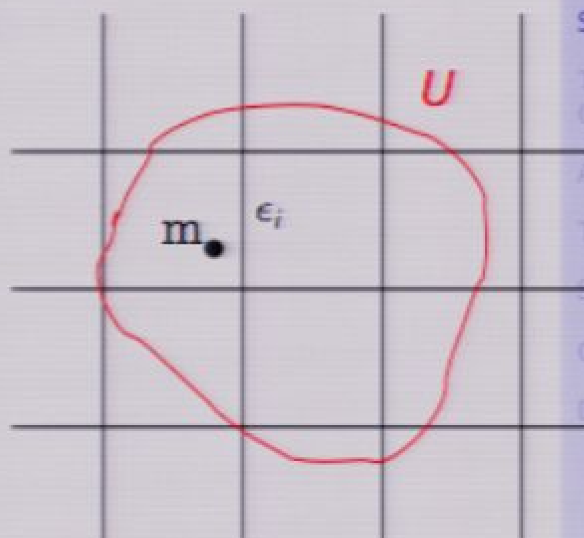

- This result mirrors the similar result from LQG, based on a system of piece-wise analytic graphs. Here: it is possible to capture information of $\mathcal{A}$ with a countable system of graphs.

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  ... somewhere between **lattice gauge theory** and LQG.

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- Recall the Poisson bracket between loop and flux variables.

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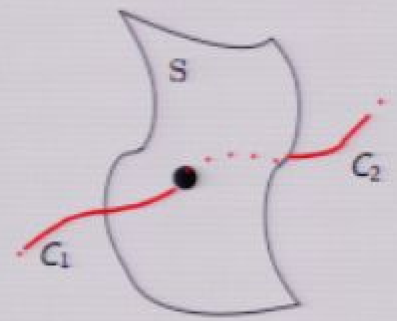

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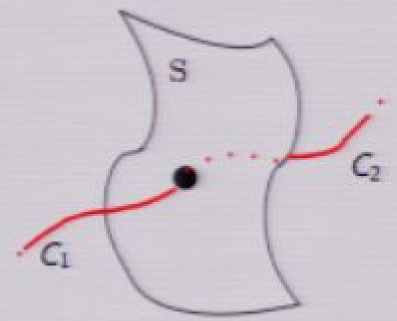

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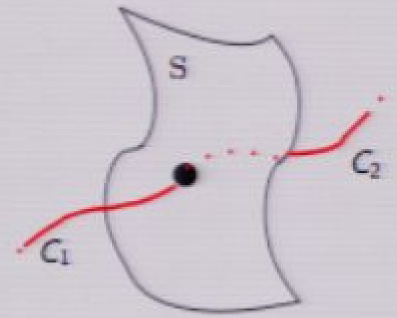

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- Next, the commutator between D and the loop L is

$$[D, f_L] = [D, g_{i_1}]g_{i_2} \ldots g_{i_k} + g_{i_1}[D, g_{i_2}] \ldots g_{i_k} + \ldots$$

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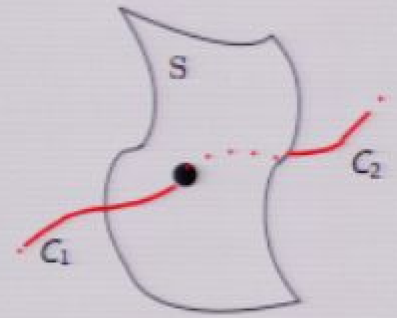

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$$[D, g] = \sum \left(\pm g\sigma^a\right) \cdot e_i^a \qquad (a_i \equiv 1)$$

where $e_i^a \in Cl(T^*G^n)$ and σ^a are generators of the Lie algebra $\mathfrak{g}$.

- Next, the commutator between D and the loop L is

$$[D, f_L] = [D, g_{i_1}]g_{i_2} \dots g_{i_k} + g_{i_1}[D, g_{i_2}] \dots g_{i_k} + \dots$$

- **In short**: the action of D is to insert Lie algebra generators at each vertex in the loop.

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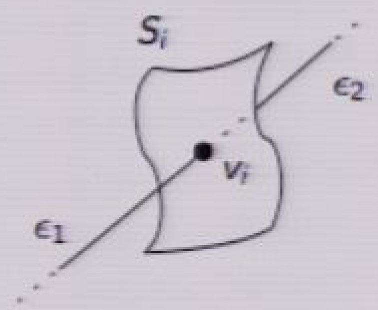

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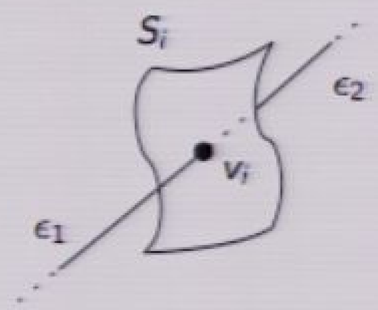

- The corresponding surfaces are 'dummy' in the sense that only the intersection points play any role in the following.

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- **Point:** the spectral triple construction captures information about the *kinematical* part of GR.

Semi-classical states

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\begin{aligned}
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where E and A are classical fields ($t \sim l_P^2$).

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- **Remark I:** these are the same states which Thomas Thiemann and collaborators have used to construct semi-classical states in a LQG-setup.

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gives, to lowest order, the expectation value of D

$$\begin{aligned}\lim_{t\to 0}\langle\bar{\Psi}(g)|D|\Psi(g)\rangle \;\; = \;\; & a_n 2^{-2n}\big(-\bar{\psi}(v_1)\sigma^a E_a^1(\psi(v_2)-\psi(v_1)) \\ & +(\bar{\psi}(v_2)-\bar{\psi}(v_1))\sigma^a E_a^1\psi(v_1) \\ & +\bar{\psi}(v_1)\{\epsilon A_1, \sigma^a E_a^1\}\psi(v_1)\big)\end{aligned}$$

where we used $g \sim 1 + \epsilon A_1$, with $\epsilon = 2^{-n}$. Here "1" denotes the direction of ϵ_1.

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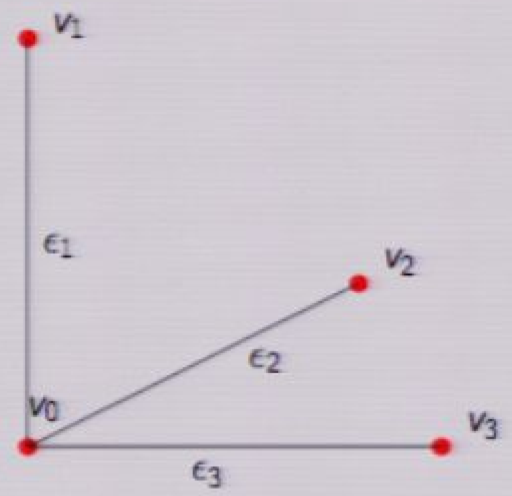

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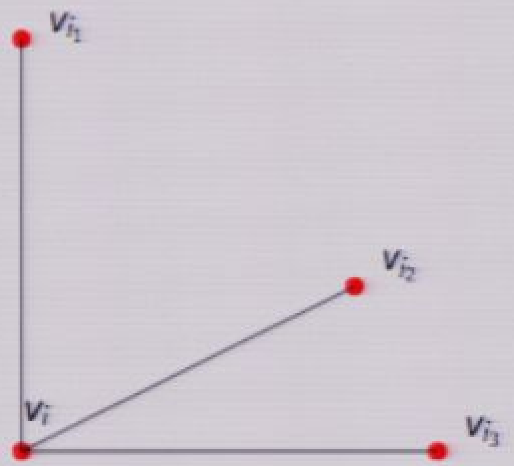
$$\Psi(g_1, g_2, g_3) = \Big(e_2^a e_3^a g_1 \psi(v_1) - e_1^a e_3^a g_2 \psi(v_2) + e_1^a e_2^a g_3 \psi(v_3) + \frac{i}{5} e_1^a e_2^b e_3^c (\delta^{ab}\sigma^c + \delta^{ac}\sigma^b + \delta^{bc}\sigma^a)\psi(v_0) \Big) \times \phi^t_{\epsilon_1}\phi^t_{\epsilon_2}\phi^t_{\epsilon_3}$$

Semiclassical states on $\overline{\mathcal{A}}$



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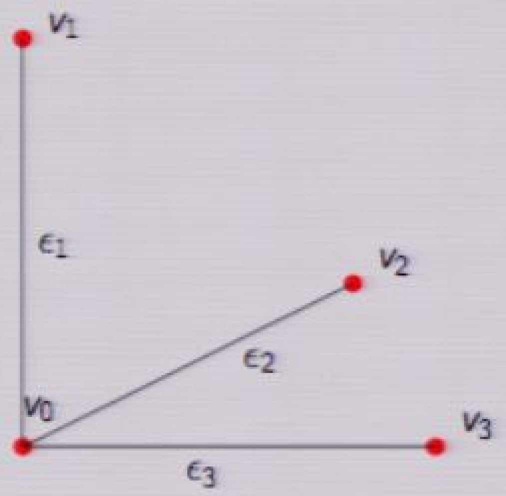

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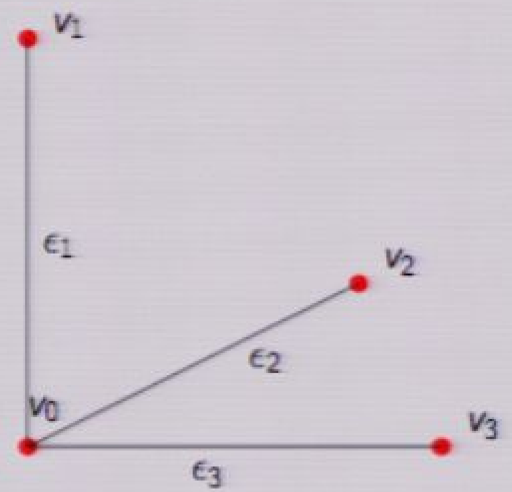

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On Semi-Classical States in Quantum Gravity & Noncommutative Geometry

Jesper Møller Grimstrup

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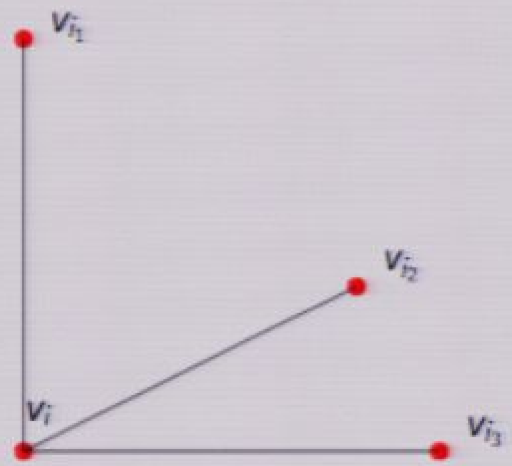

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which is the Dirac Hamiltonian in 3+1 dim. (principal part)

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- The cross terms in an expression like

$$M(\mathbf{e}^a_m d_{\mathbf{e}^a_m})(\mathbf{e}^b_n d_{\mathbf{e}^b_n})L \stackrel{quant.}{\longleftrightarrow} ME^m_a\sigma^a E^n_b\sigma^b F^c_{mn}\sigma^c$$

could come from terms like $D_M D_N$.

Spectral action functional

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- The construction is well defined in any dimensions.

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- **Connes distance formula:** Given a spectral triple $(\mathcal{A}, D, \mathcal{H})$ over a manifold $\mathcal{M}$ the distance formula reads

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- If they differ only on short scales, then the distance will be 'small' (difference weighted with large a's - small distance).

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  - does this point towards a natural formulation of a Wheeler-deWitt equation?

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- The general framework which we present appear to be quite rich - we have not analyzed all the structure (real structure, left-right invariant vector fields, the Hilbert space ...).
- Alternative approach: forget about NCG - formulate LQG with a countable system of graphs
  - different completion of the space of smooth connections;
  - different approach to the diffeomorphism group.

"A striking aspect of this approach to geometry of $\bar{\mathcal{A}}/\mathcal{G}$ is that its general spirit is the same as that of non-commutative geometry and quantum groups: even though there is no underlying differential manifold, geometrical notions can be developed by exploiting the properties of the *algebra* of functions."

- Ashtekar, Lewandowski, 1996

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