Title: Holographic calculation of entanglement entropy in conformal field theories

Date: Oct 09, 2009 12:00 PM

URL: http://pirsa.org/09100142

Abstract: Universal scaling behavior of the entanglement entropy in conformal field theories uncovered by a holographic calculation.

entanglement and entropy of entanglement

entanglement entropy (von-Neumann entropy)

= a measure of entanglement in a given quantum state $|\Psi\rangle$

(i) bipartition the Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$

(ii) take partial trace $\rho_{\text{tot}} = |\Psi\rangle\langle\Psi|$ ← pure state

$$\rho_A = \text{tr}_B |\Psi\rangle\langle\Psi| = \sum_j p_j |\psi_j\rangle_A \langle\psi_j|_A \qquad \left(\sum_j p_j = 1\right)$$

↑ mixed state

(iii) entanglement entropy

$$S_A = -\text{tr}_A \left[\rho_A \ln \rho_A\right] = -\sum_j p_j \ln p_j$$

(iv) entanglment entropy spectrum

$$\{\xi_i\}_i \quad \text{where} \quad p_i =: \exp(-\xi_i)/Z$$

entanglement and entropy of entanglement

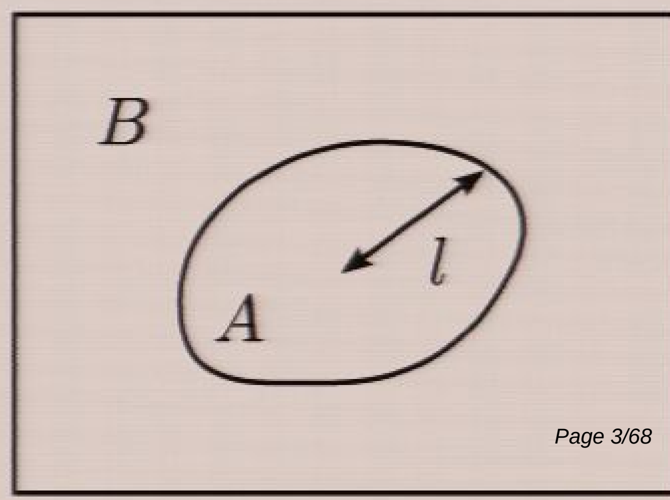
e.g. two qubit system

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle_A|\downarrow\rangle_B - |\downarrow\rangle_A|\uparrow\rangle_B) \longrightarrow S_A = \ln 2$$

$$|\Psi\rangle = |\uparrow\rangle_A|\downarrow\rangle_B \longrightarrow S_A = 0$$

application to many-body systms and field theories:

A, B : submanifold of the total system



motivation for entanglement entropy

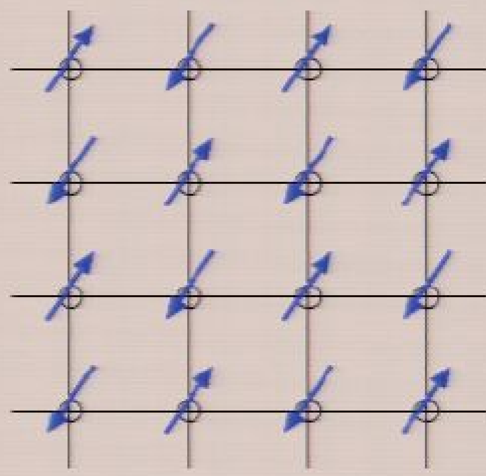
(1) EE can be a good "order parameter" for quantum systems (?)

- quantum liquid phases: no LRO for _any_ local order parameter

  -fractional quantum Hall effect
  -gappless/gapped quantum spin liquid
  -quantum critical points, non-Landau-Ginzburg transition

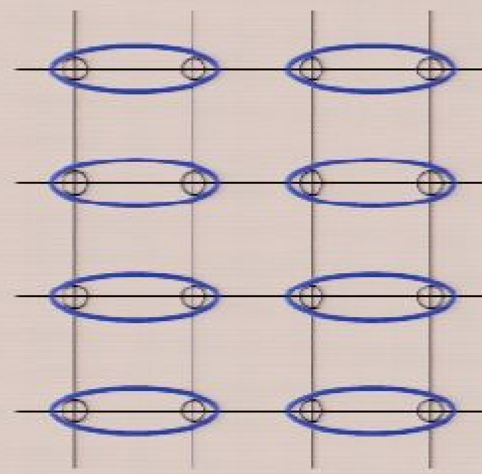
Neel state

$|\Psi_{\rm Neel}\rangle =$



Valence bond solid (VBS) state

$|\Psi_{\rm VBS}\rangle =$



$= \frac{1}{\sqrt{2}}[|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle]$

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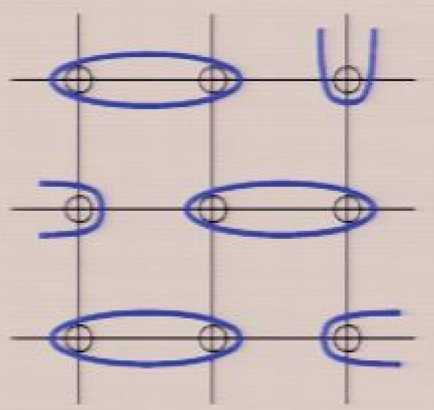
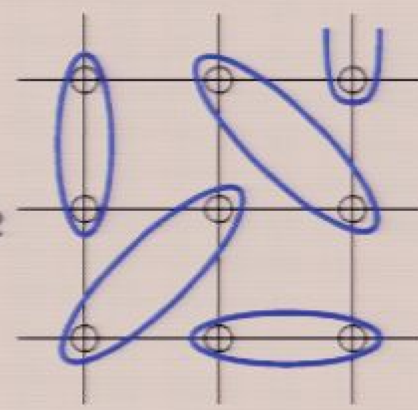
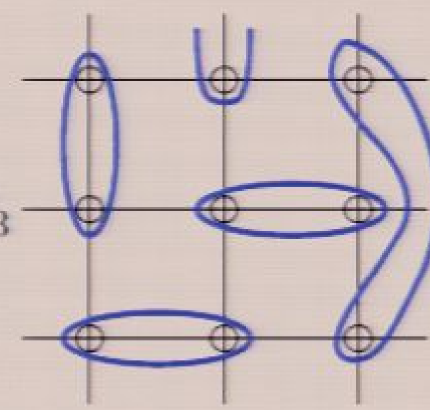
$|\Psi_{\rm RVB}\rangle = C_1$  $+C_2$  $+C_3$  $+\cdots$

Resonating valence bond (RVB) state

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- defined purely in terms of wavefunctions

- EE measures a response to external gravity

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density matrix renomalization group (DMRG)

use computational complexity to classify quantum states ?

EE in pure 4D SU(2) Yang-Mills theory

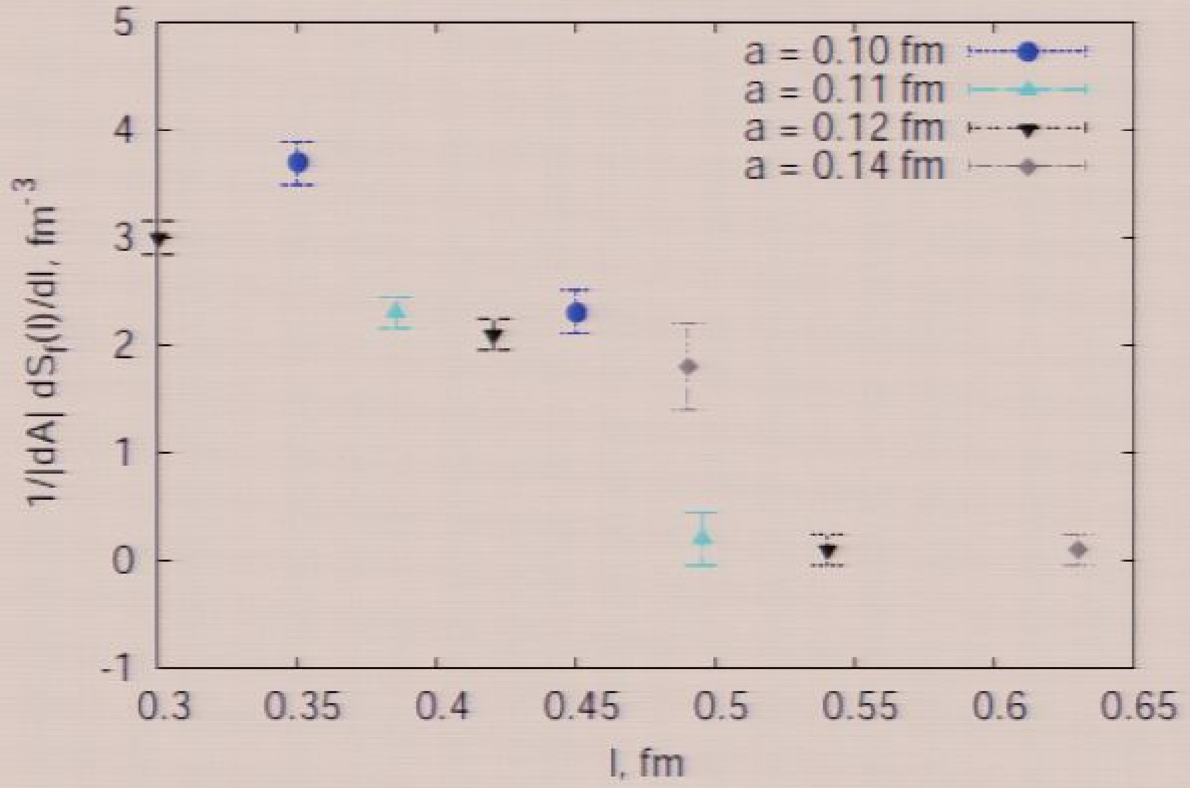


FIG. 5: The discontinuity of the derivative of the entanglement entropy over l near $l_c \approx 0.5\, fm$.

Buividovich, Polikarpov (NPB802, pp458, 2008)

holographic calculations: Nishioka, Takayanagi (2006,2007), Klebanov, Kutasov, Murugan (2007)

scaling of entanglement entropy

von-Neumann entropy is defined for a region (geometric entropy)
natural object to look at is how EE depends on the size and shape of the region for a given quantum system.

Area law (gapped system, CFT in (d+1)D with d>1, etc.)

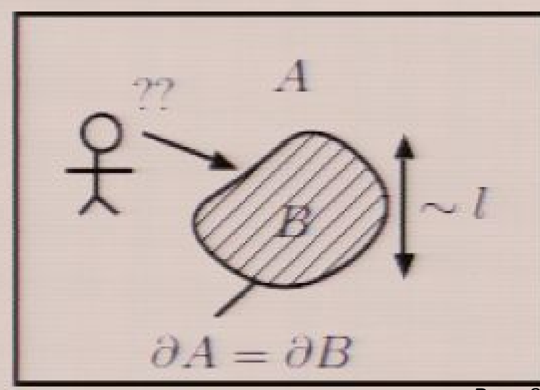
$$S_A = \text{const.} \left(\frac{l}{a}\right)^{d-1} + \cdots$$

Srednicki (93)

↕

Black Hole Entropy (Beckenstein-Hawking)

$$S_{BH} = \frac{\text{Area of horizon}}{4G_N}$$



scaling of entanglement entropy

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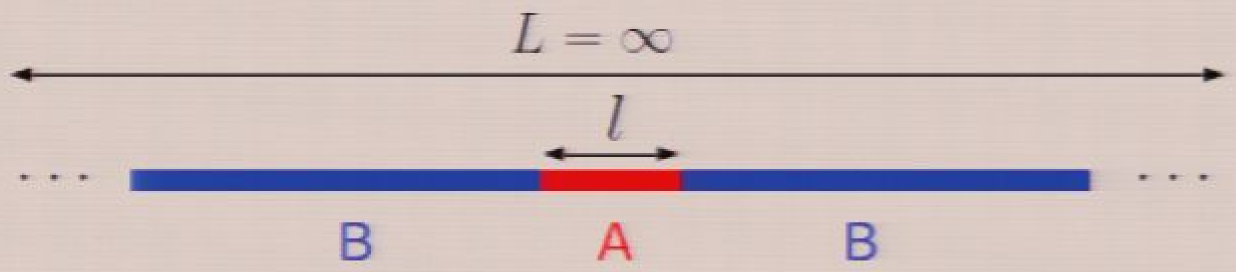

boundary entropy (Zhou, Barthel, Fjaerestad, Schollwock)

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$\log(g)$
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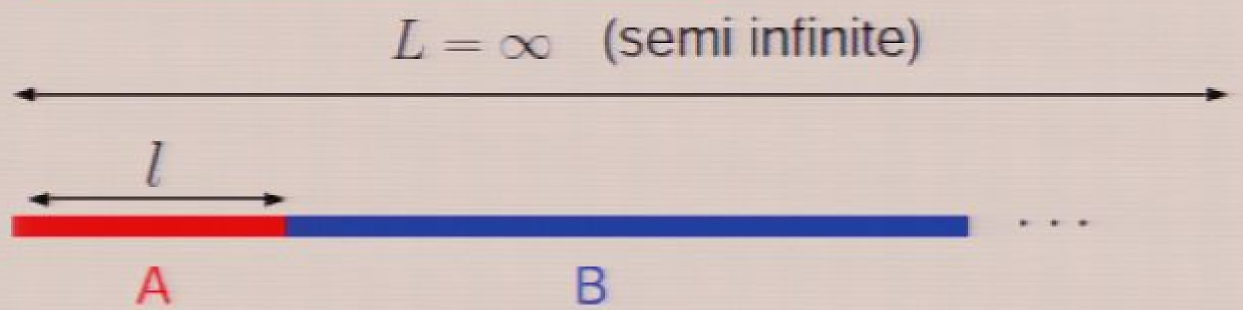

scaling of entanglement entropy

- detecting topological order in (2+1)D Kitaev & Preskill Levin & Wen (2006)

$$S_A = \gamma \frac{l}{a} - \log(D)$$

$$D = \sqrt{\sum_a d_a^2}$$

← quantum dimension (d_a)

↖ quasi-particle type (a)

$\log D = \log \sqrt{q}$ FQHE at nu = 1/q (Chern-Simons theory)

$\log D = \log 2$ Z2 lattice gauge theory

- z=2 Lifsitz critical point in (2+1)D Fradkin & Moore (2006)

$$S_A = \gamma \frac{l}{a} + \alpha c \log(l/a) + \cdots$$

- free fermions with Fermi surface Gioev & Klich, Wolf (2006)

$$S_A = C l^{d-1} \log(l/a)$$

$$C \propto \int_{\partial A} \int_{\mathrm{FS}} |\boldsymbol{n}_r \cdot \boldsymbol{n}_k| dS_r dS_k$$

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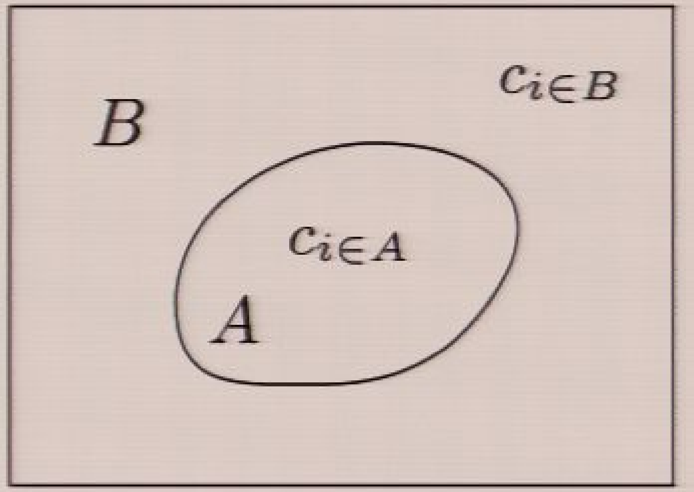

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$$K_{ij} = \sum_\alpha \phi_\alpha(i)\phi_\alpha^*(j)\epsilon_\alpha$$

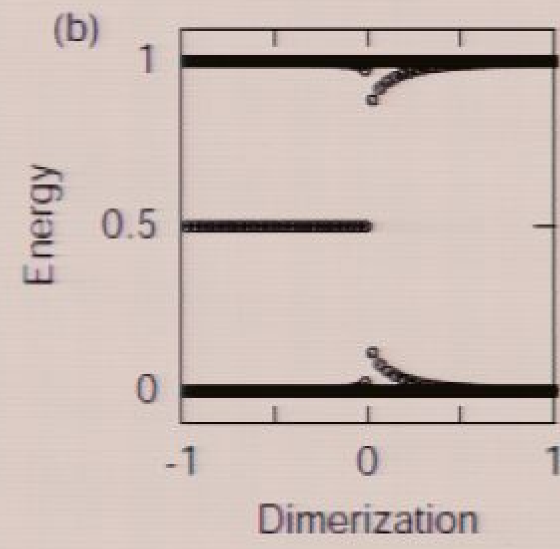
entanglement entropy is given by

$$S_A = -\sum_\alpha \left[\zeta_\alpha \ln \zeta_\alpha + (1-\zeta_\alpha)\ln(1-\zeta_\alpha)\right] \qquad \zeta_\alpha = \frac{1}{1+e^{\epsilon_\alpha}}$$

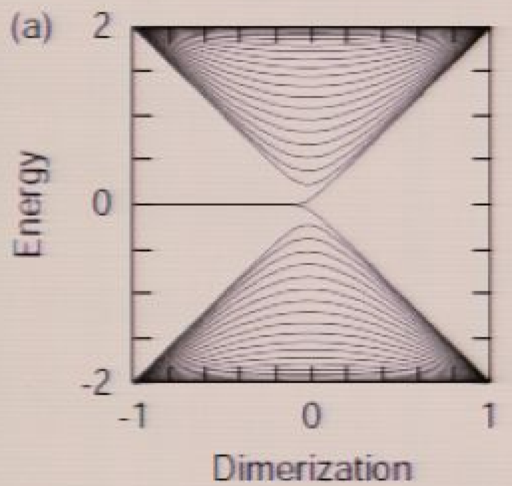
entanglement entropy in 1D topological insulator

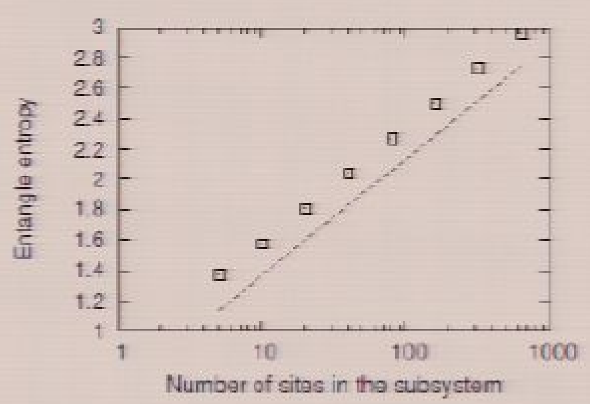
SR and Hatsugai (2006)

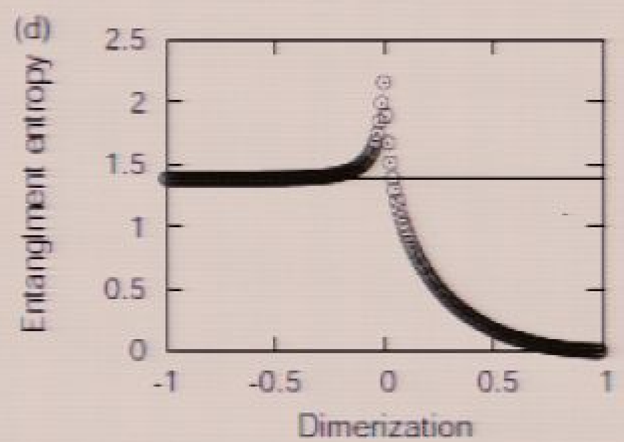
$$H = \sum_i \left[(1+\delta t) c^{\dagger}_{i\bullet} c_{i\circ} + c^{\dagger}_{i\circ} c_{i+1\bullet} + h.c. \right]$$



$$S_A = \frac{c}{3} \ln \frac{l}{a}$$







zero mode of EE spetrum when Berry phase = pi

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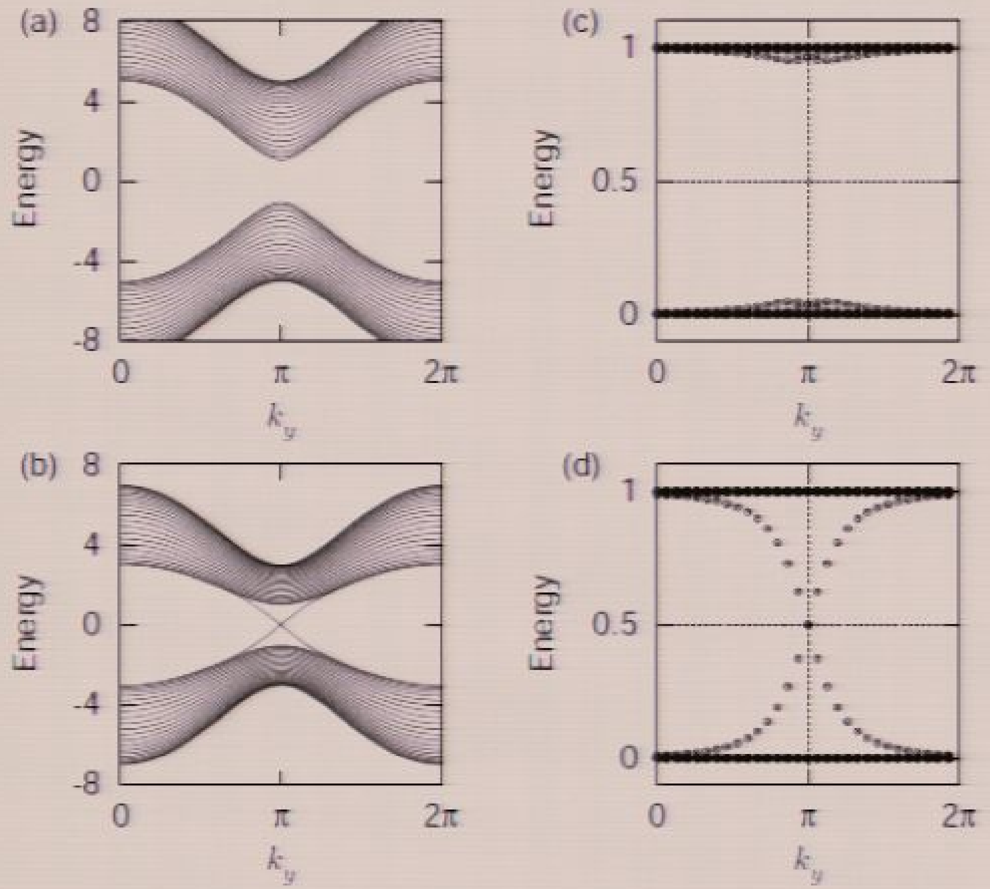
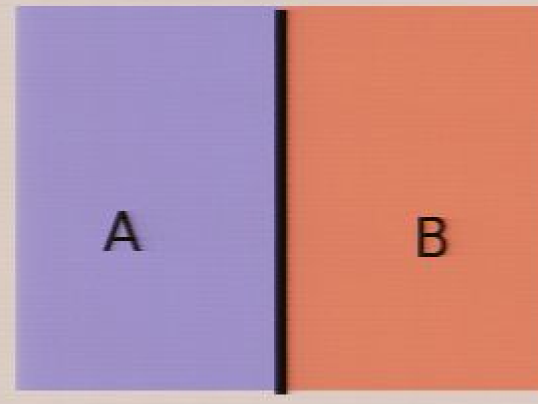
energy spectrum with edges

EE spectrum





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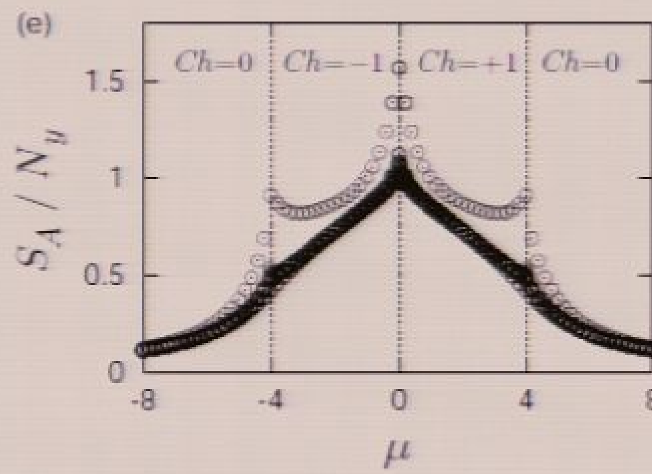

EE spectrum shows "edge states".

(although there is no real edge)

c.f. for FQHE: Li, Haldane (2008)

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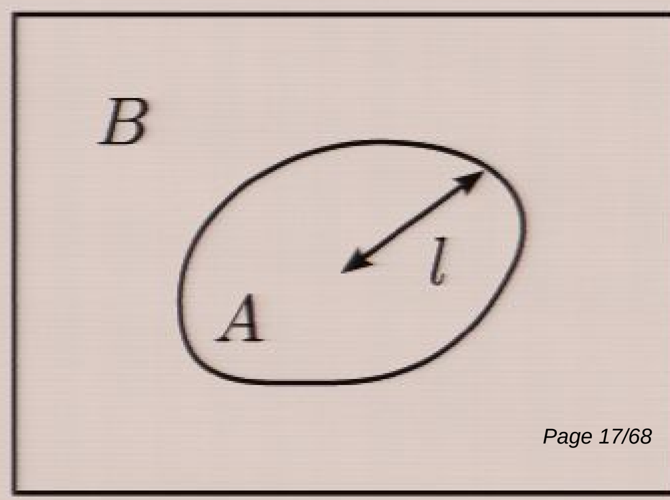

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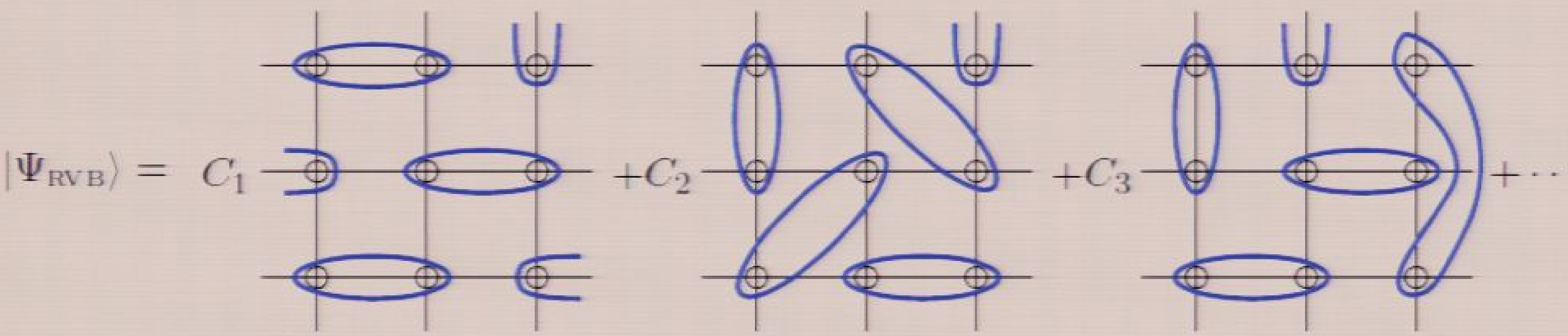
$$|\Psi_{\rm RVB}\rangle = C_1 \,[\cdot]\, + C_2 \,[\cdot]\, + C_3 \,[\cdot]\, + \cdots$$



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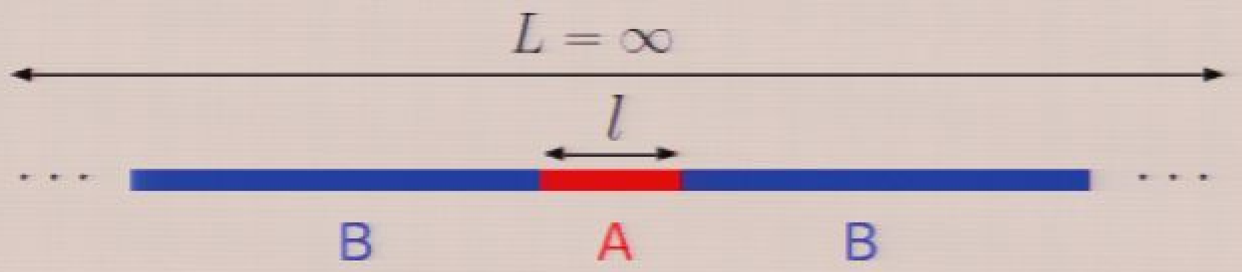

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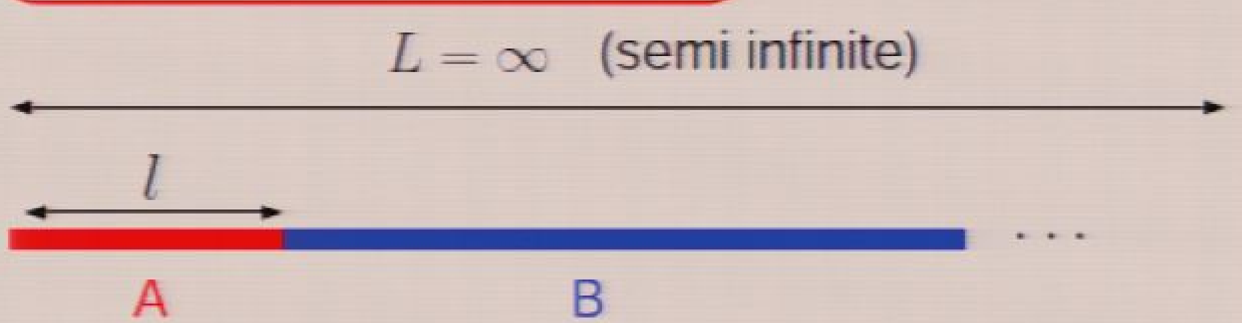

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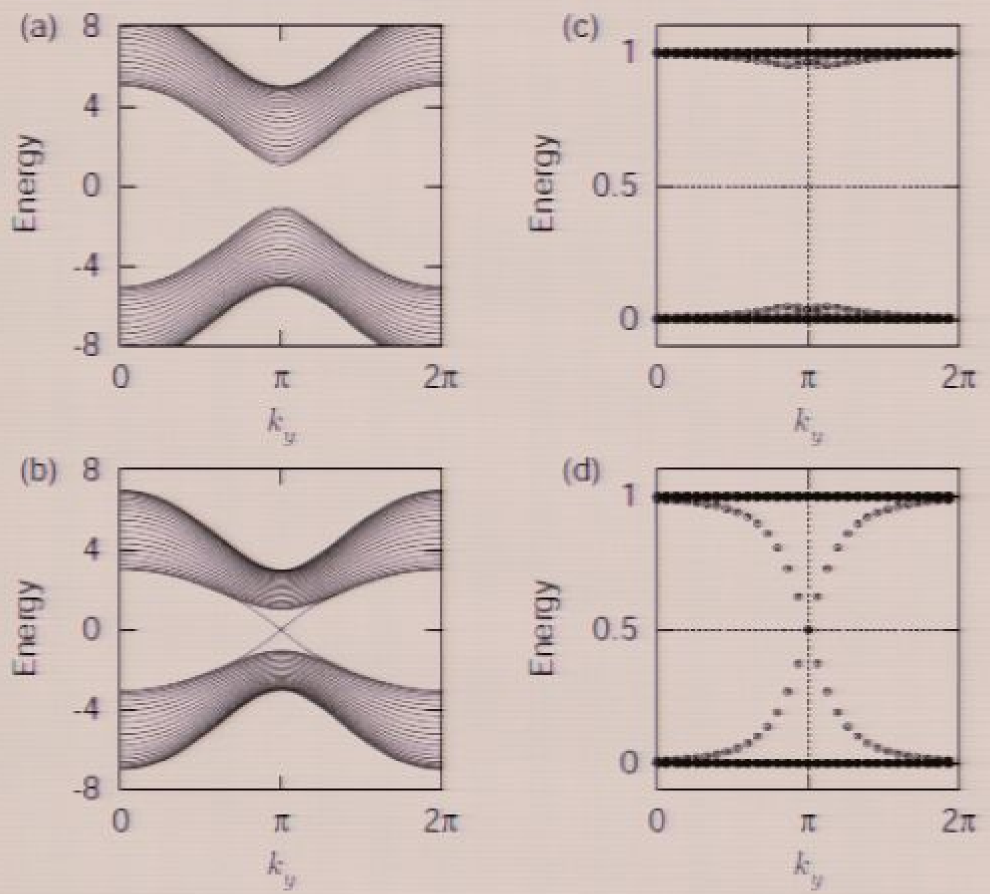

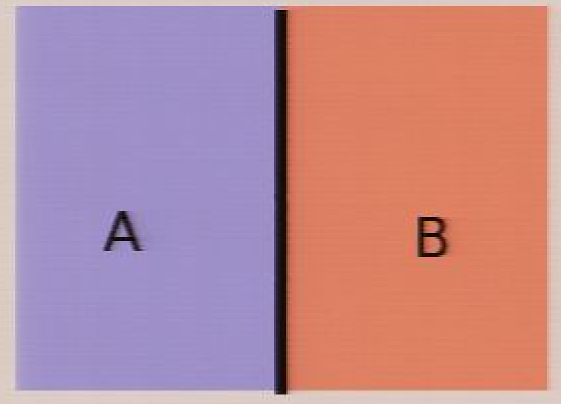

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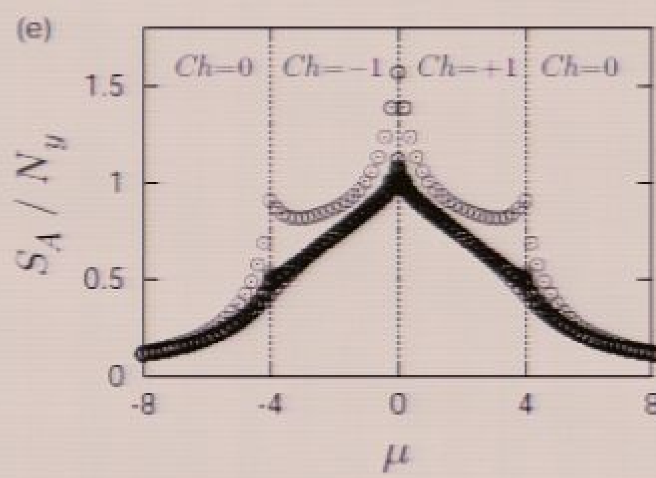

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entanglement entropy in QFTs

ground state wavefunctional:

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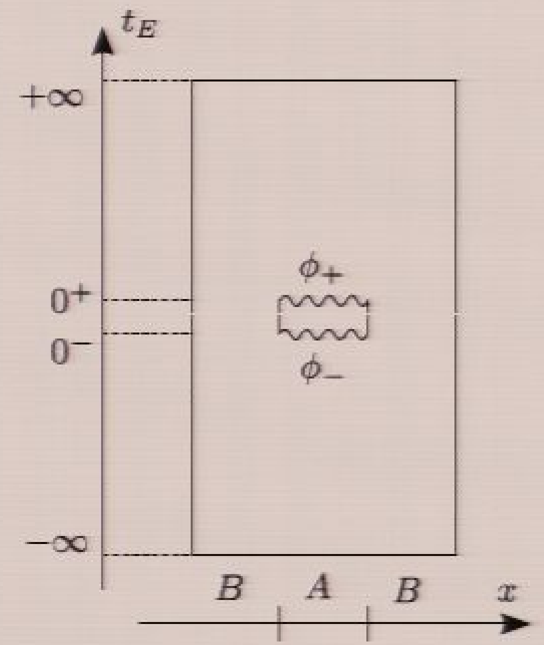

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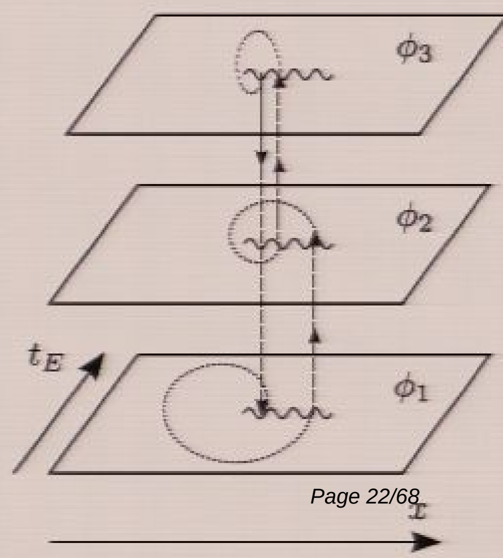

QFT on a singular curved space

replica trick --> entanglement entropy

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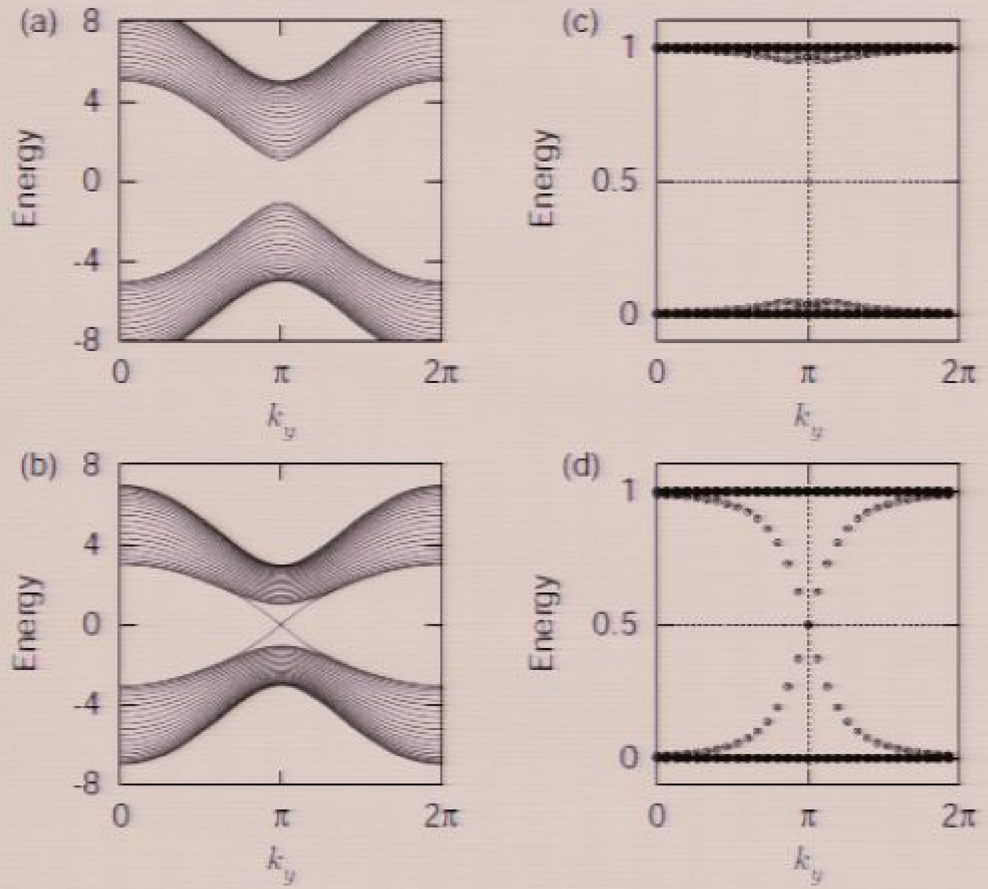

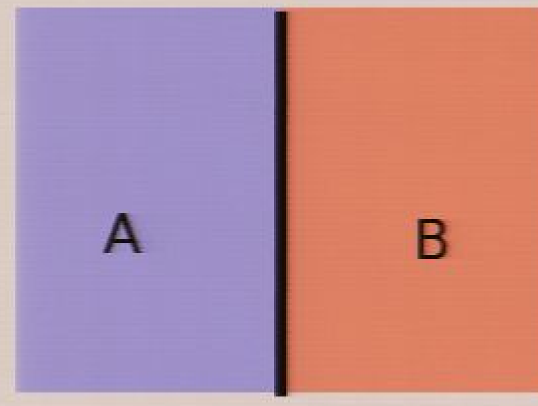

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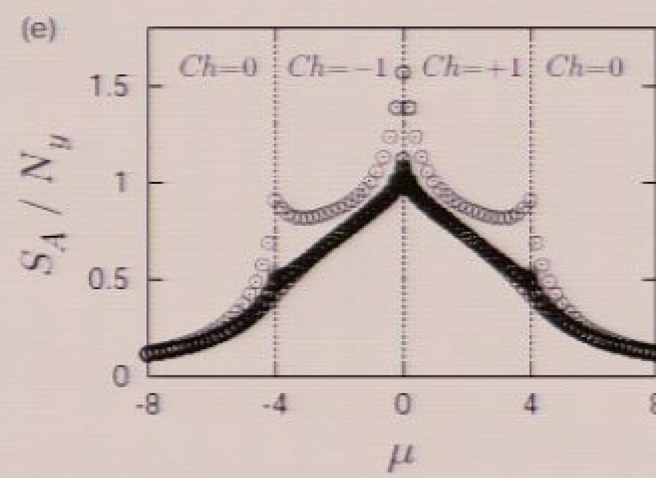

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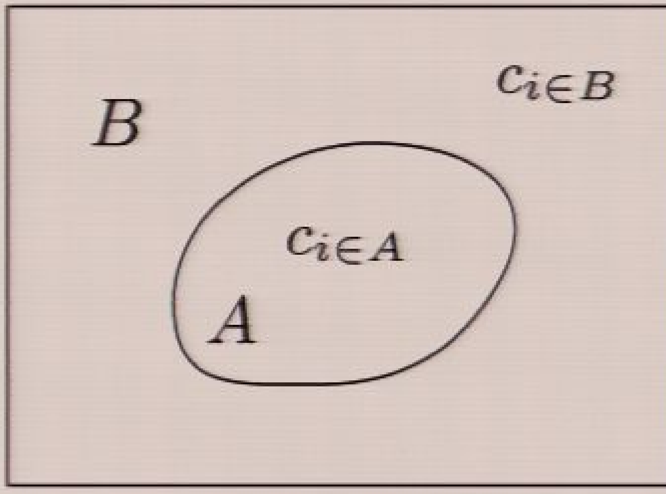

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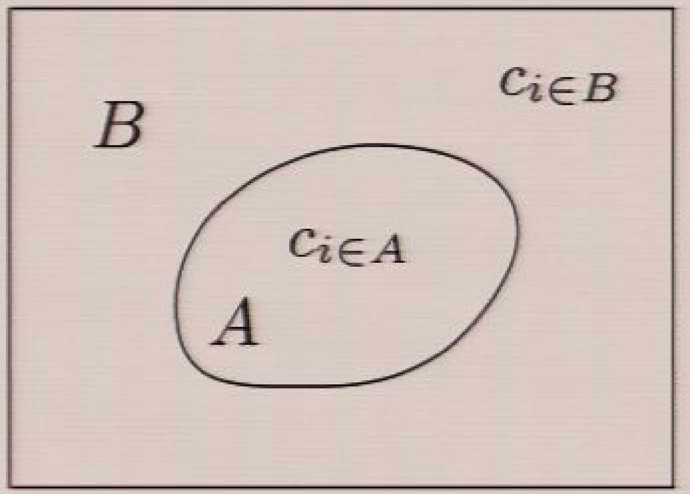

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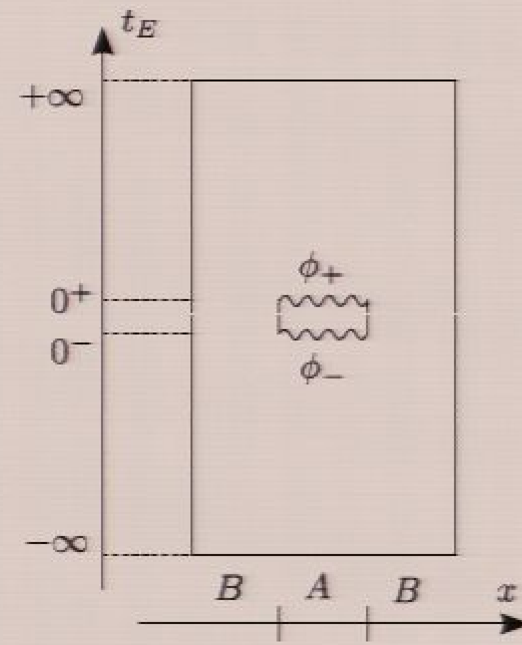

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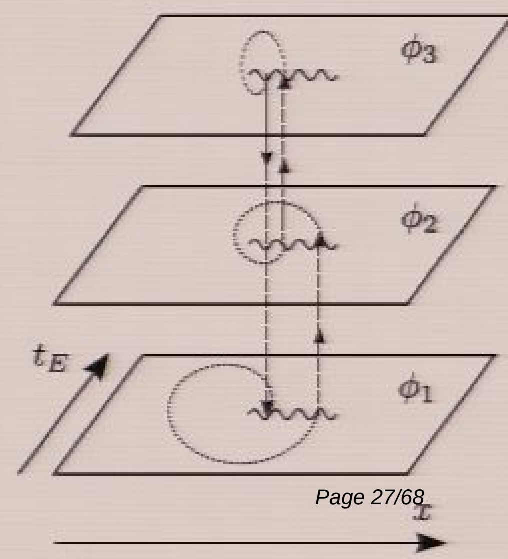

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entanglement entropy in CFTs

Weyl rescaling: $g_{\mu\nu} = \delta_{\mu\nu} e^{2\rho} \quad l \sim e^{2\rho}$

$$l\frac{d}{dl}\ln \mathrm{tr}_A \rho_A^n = 2\int d^{d+1}x\, g_{\mu\nu}(x)\frac{\delta}{\delta g_{\mu\nu}(x)}(\ln Z_n - n\ln Z_1)$$

$$= -\frac{1}{2\pi}\left\langle \int d^{d+1}x\sqrt{g}T^\mu_\mu \right\rangle_{M_n} + \frac{n}{2\pi}\left\langle \int d^{d+1}x\sqrt{g}T^\mu_\mu \right\rangle_{T_1}$$

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2D CFT $\quad \left\langle T^\mu_\mu \right\rangle = -\frac{c}{12}R \longrightarrow$ $\boxed{S_A = \frac{c}{3}\ln\frac{l}{a}}$

Holzhey,Larsen, Wilczek (94) Calabrese, Cardy (04)

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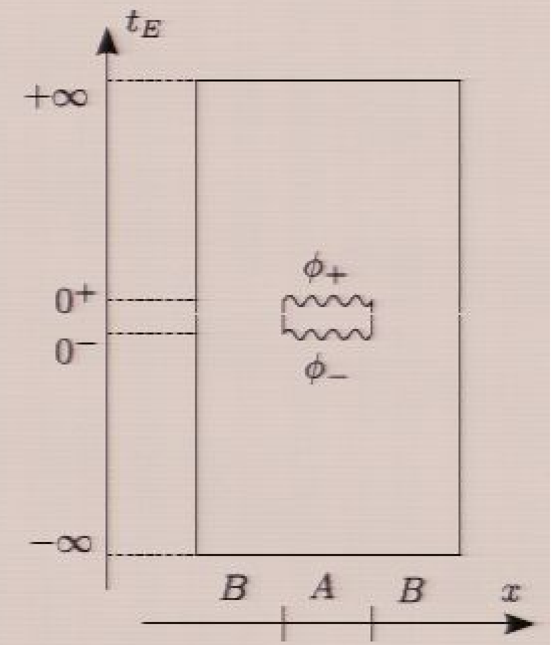
SR, Takayanagi (06)

odd dimentional CFTs: no central charges !?

entanglement entropy in QFTs

ground state wavefunctional:

$$\Psi[\phi_0(x)] = \mathcal{N}^{-1} \int_{-\infty}^{t=0} \mathcal{D}\phi e^{-S} \prod_{x \in A} \delta(\phi(0,x) - \phi_0(x))$$



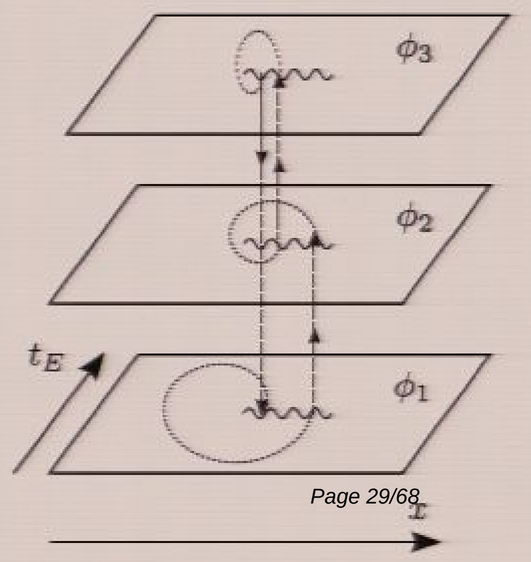
reduced density matrix:

$$[\rho_A]_{\phi_+,\phi_-} = Z^{-1} \int_{-\infty}^{+\infty} \mathcal{D}\phi e^{-S}$$

$$\times \prod_{x \in A} \delta(\phi(+0,x) - \phi_+(x)) \delta(\phi(-0,x) - \phi_-(x))$$

$$\mathrm{tr}_A \rho_A^n = (Z_1)^{-n} \int_{\mathcal{R}_n} \mathcal{D}\phi e^{-S} = \frac{Z_n}{(Z_1)^n}$$

QFT on a singular curved space



replica trick --> entanglement entropy

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entanglement entropy in CFTs

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SR, Takayanagi (06)

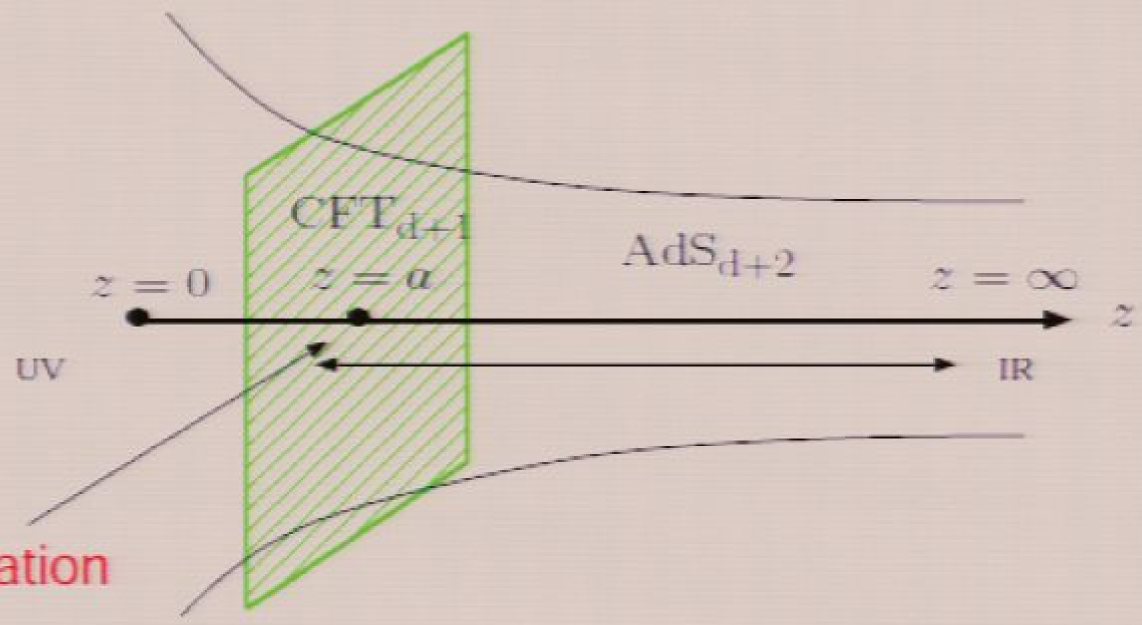
odd dimentional CFTs: no central charges !?

AdS/CFT

Maldacena (97)

Poincare coordinate

$$ds^2 = R^2 \cdot \frac{(dz)^2 - (dx_0)^2 + (dx_1)^2 + \cdots + (dx_d)^2}{z^2}$$



$z > a$

UV regularization

entanglement entropy in CFTs

Weyl rescaling: $g_{\mu\nu} = \delta_{\mu\nu} e^{2\rho} \quad l \sim e^{2\rho}$

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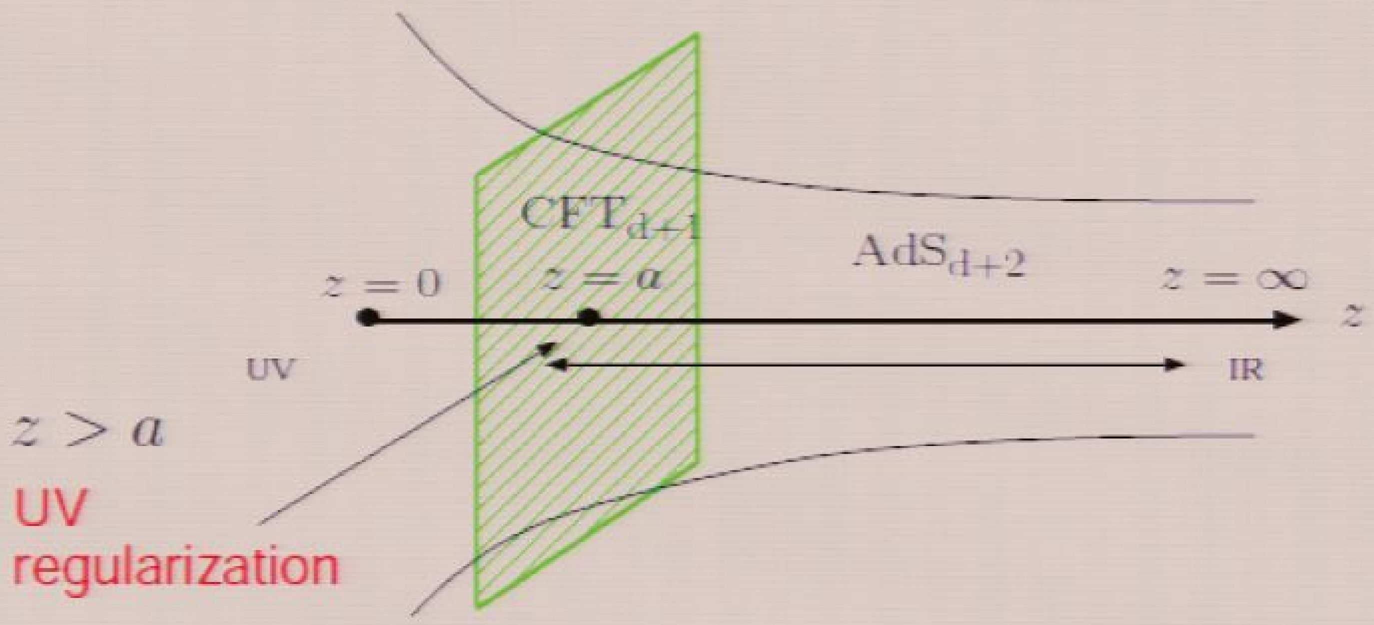

entanglement entropy in CFTs

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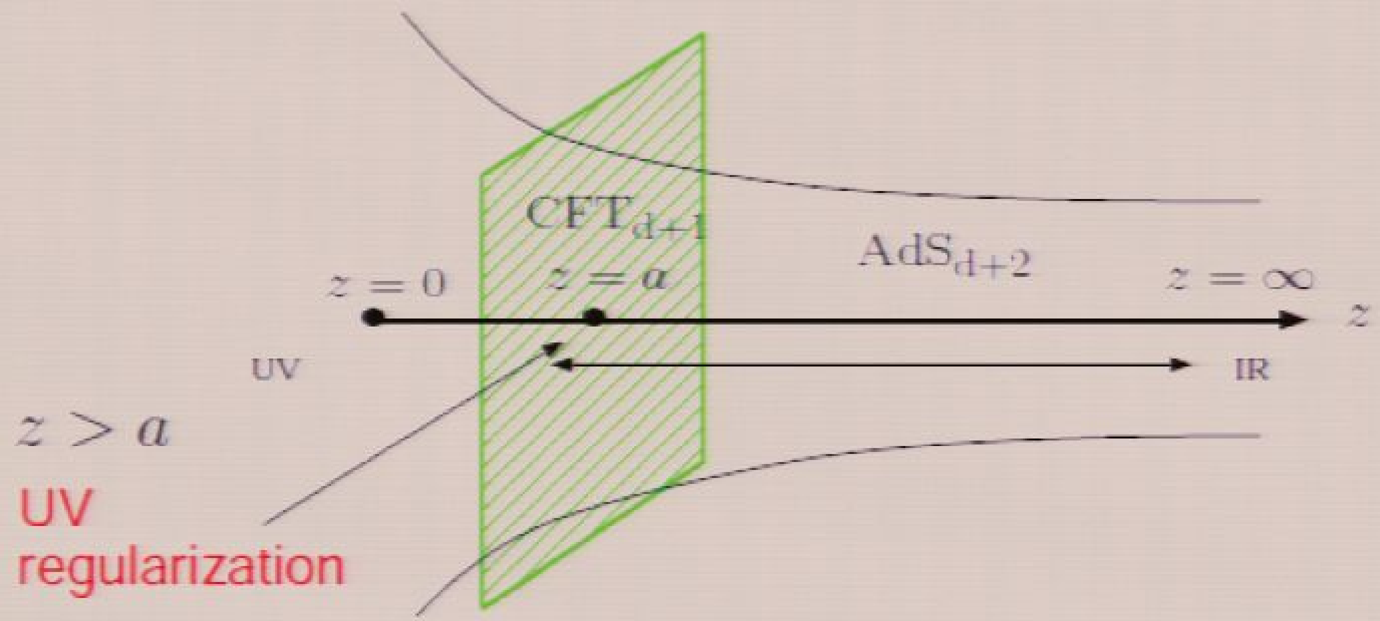
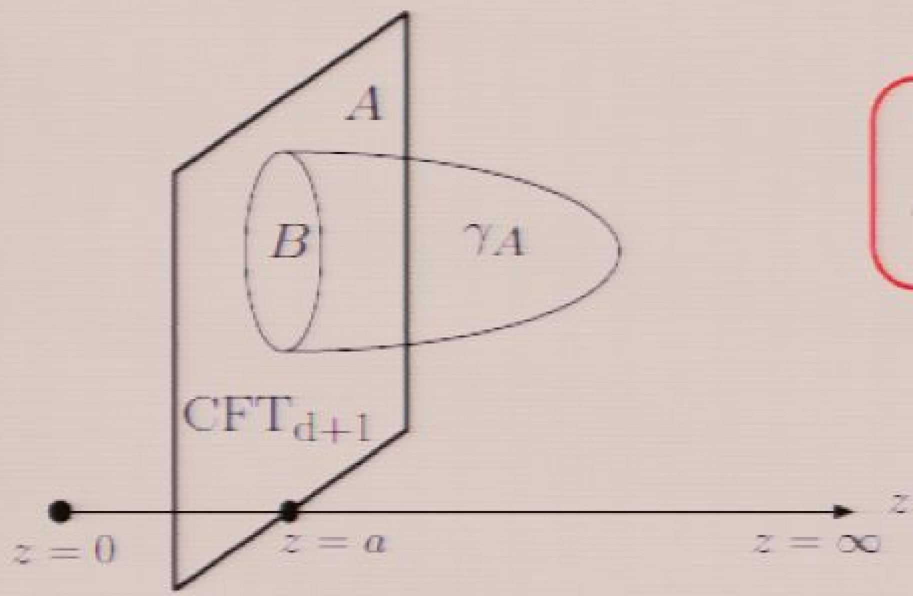
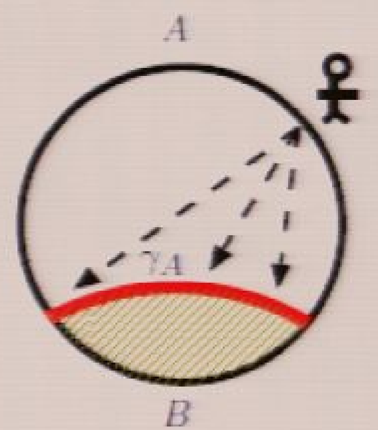
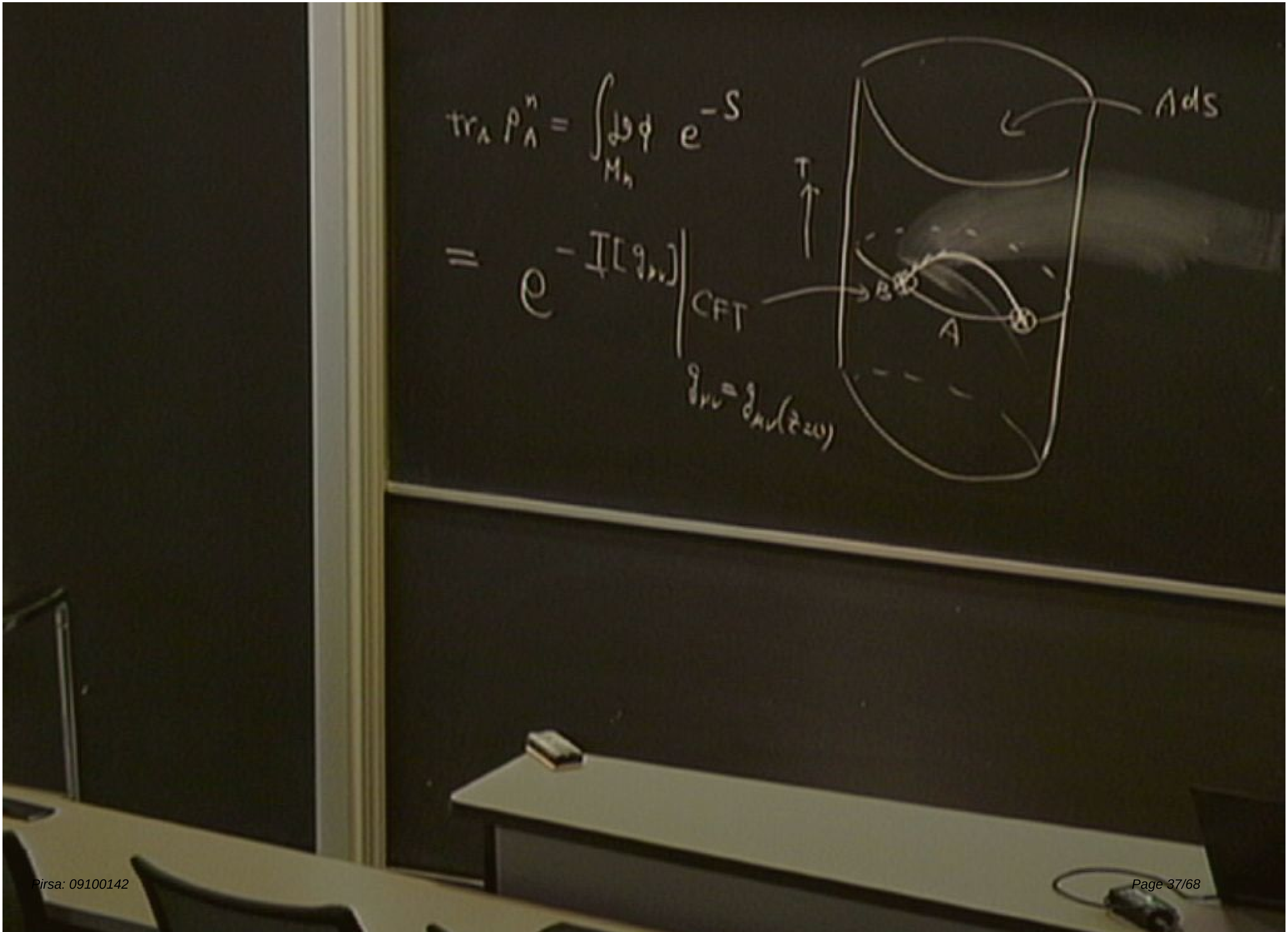


holographic derivation of entanglement entropy

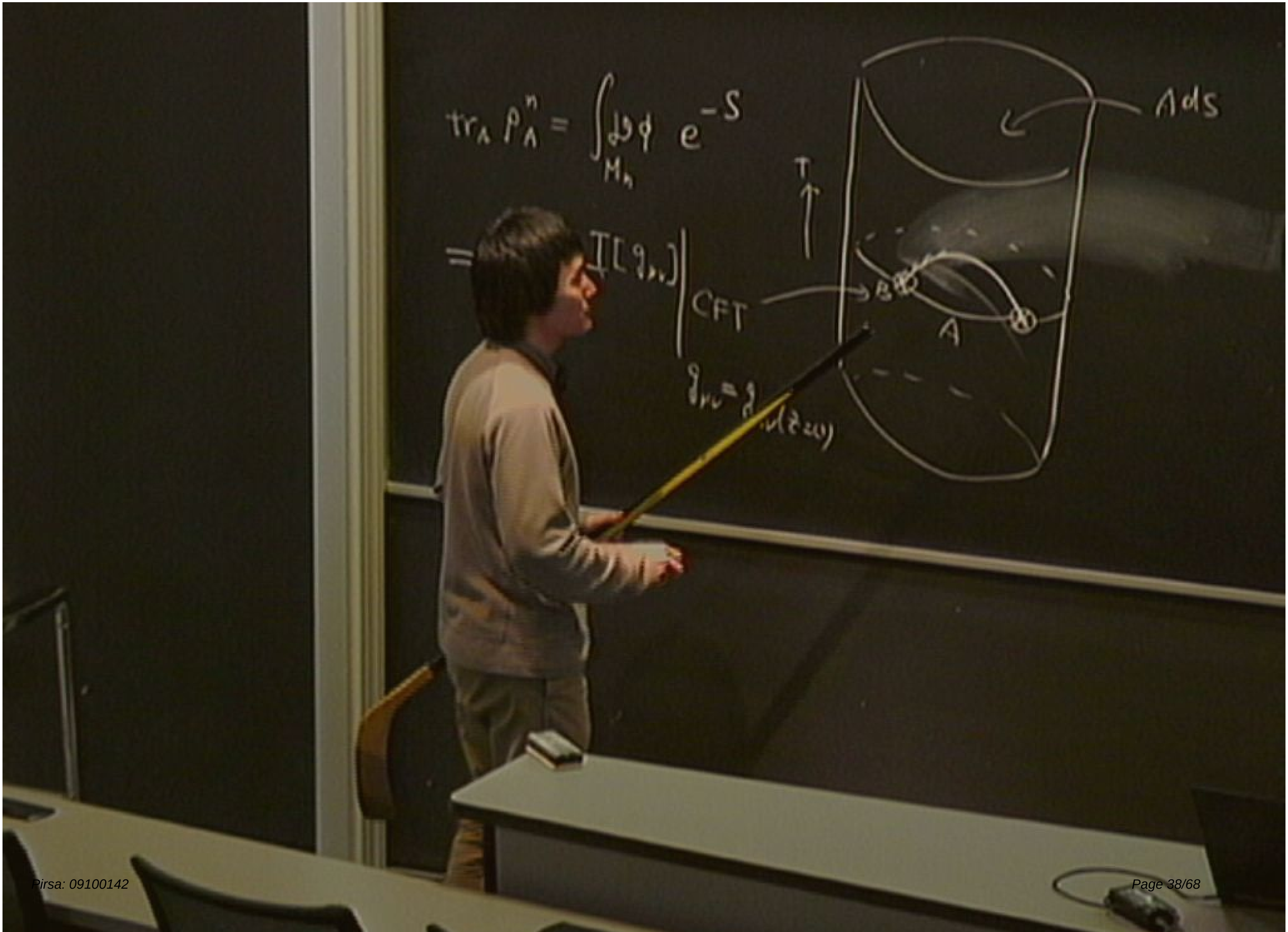


$$S_A = \frac{\text{Area of minimal surface } \gamma_A}{4G_N}$$

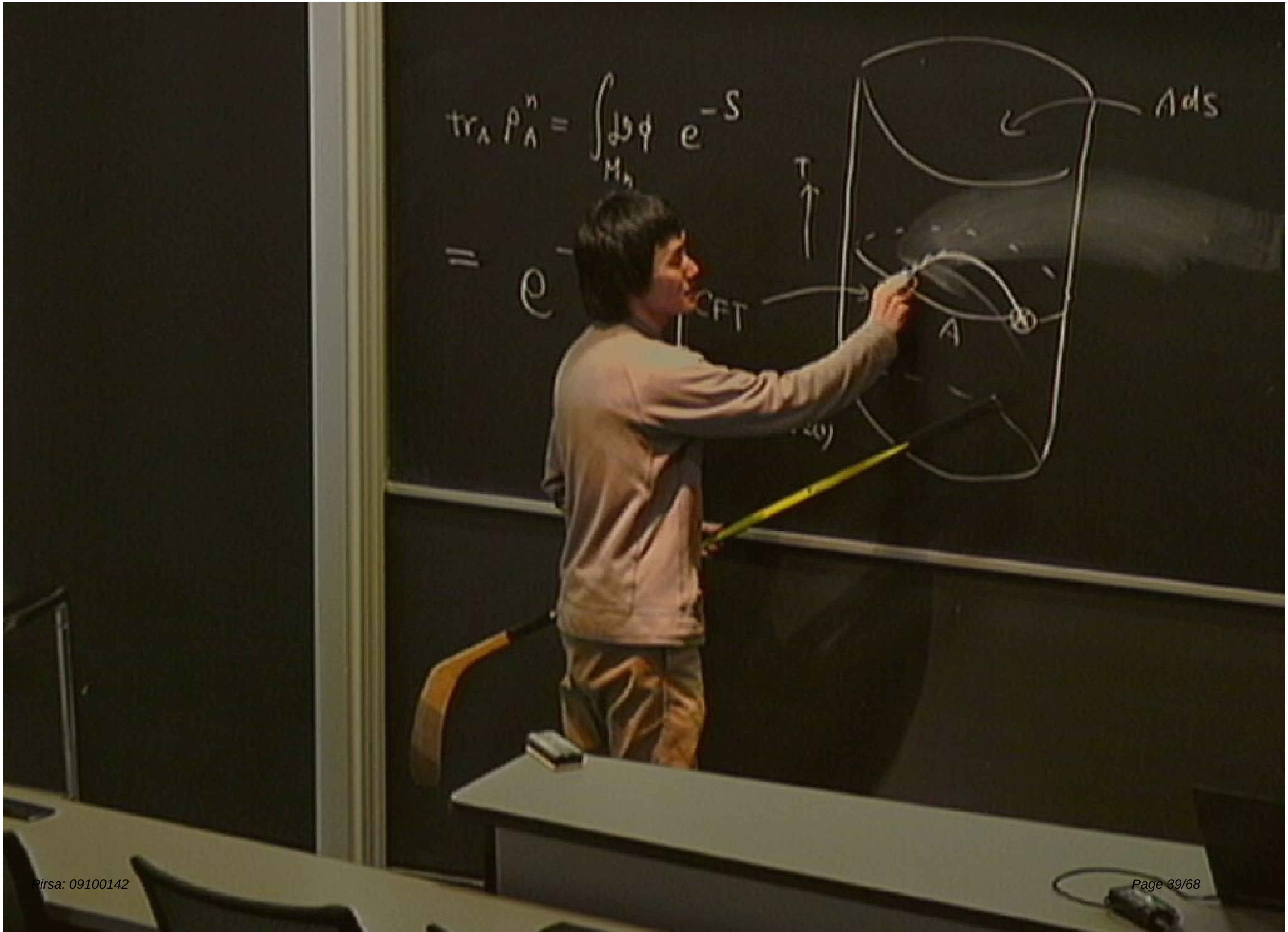


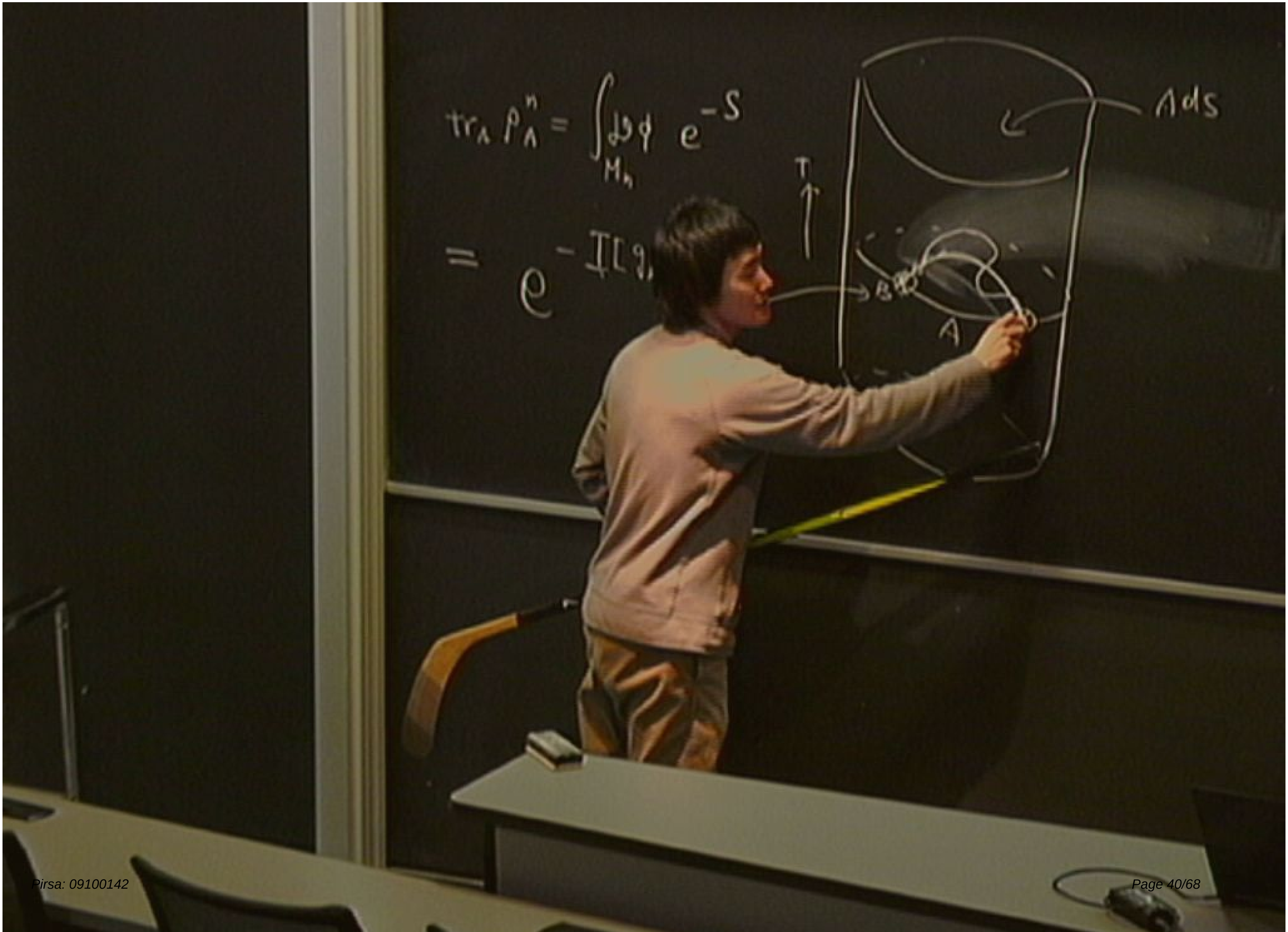


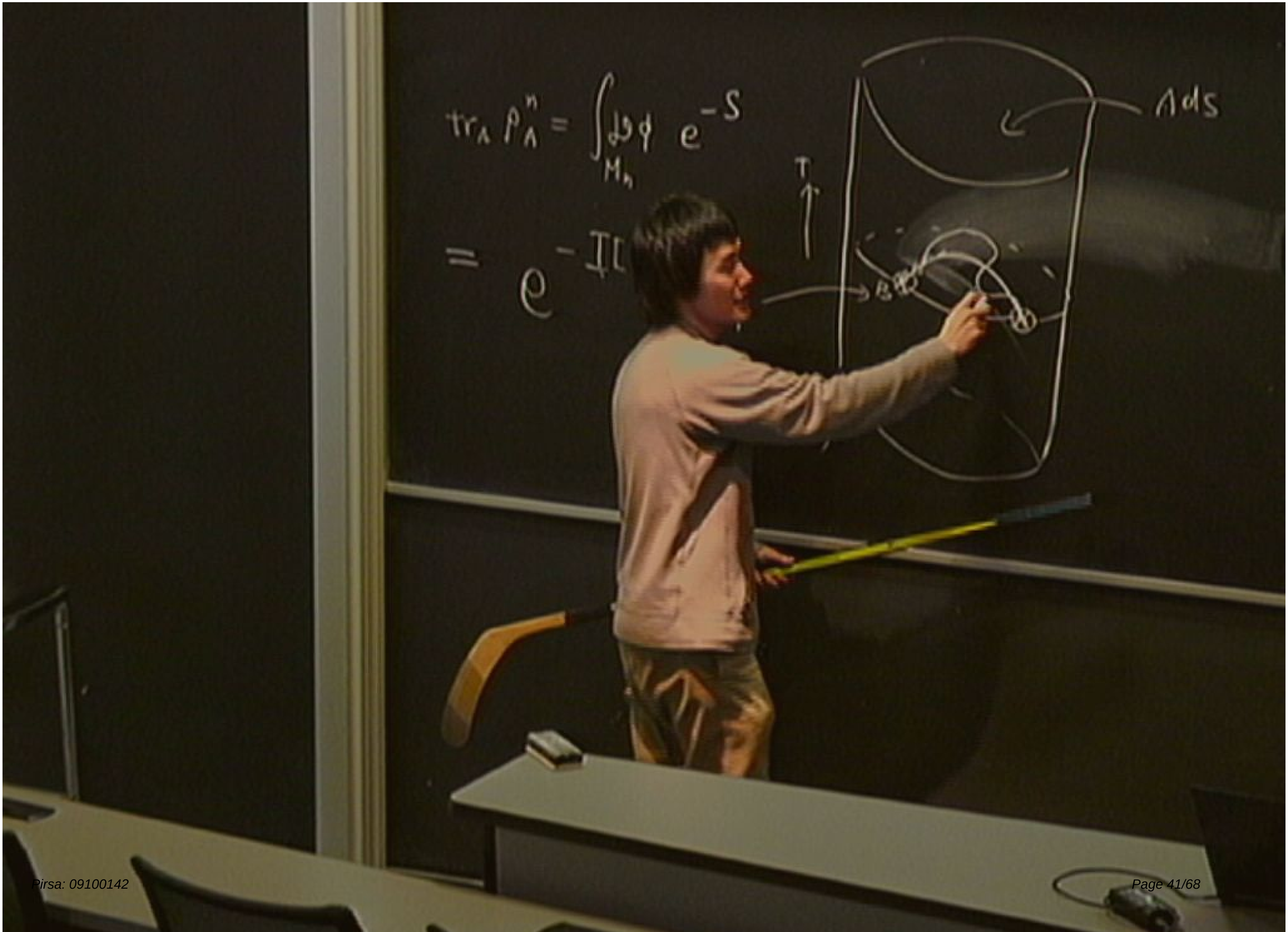
$\mathrm{tr}_A \rho_A^n = \int_{M_n} \mathcal{D}\phi \, e^{-S}$
$= e^{-I[g_{\mu\nu}]} \Big|_{CFT}$
$g_{\mu\nu} = g_{\mu\nu}(z=0)$
CFT
AdS
T
B
A



AdS
CFT
A
B
T


$\mathrm{tr}_A \rho_A^n = \int_{M_n} D\phi \; e^{-S}$
AdS
CFT
A


AdS
A
B
T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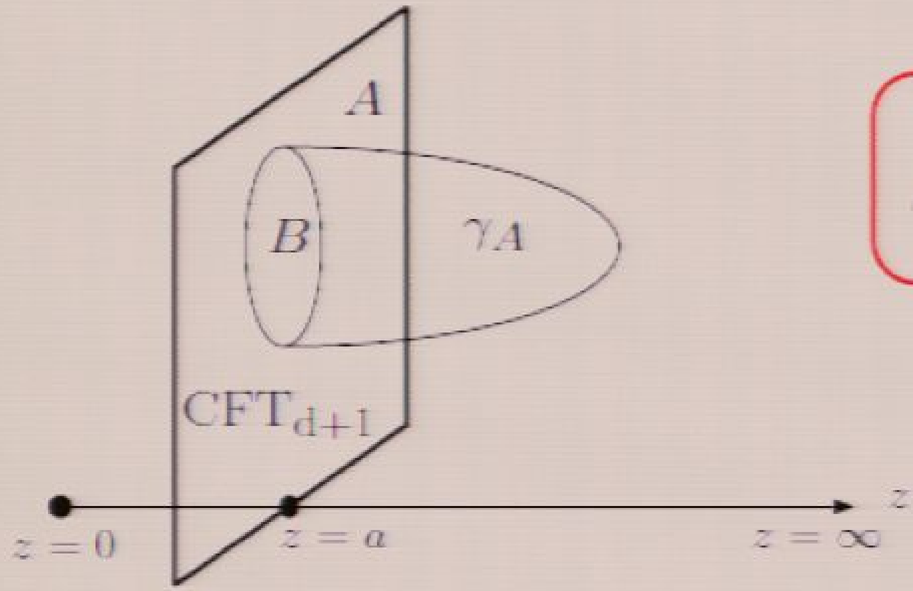
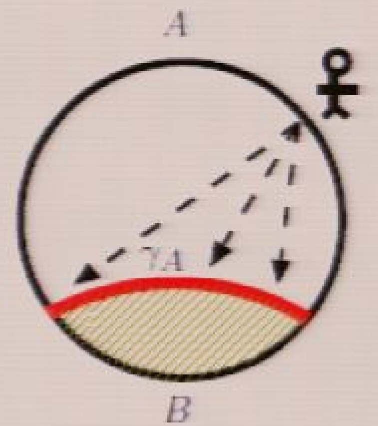
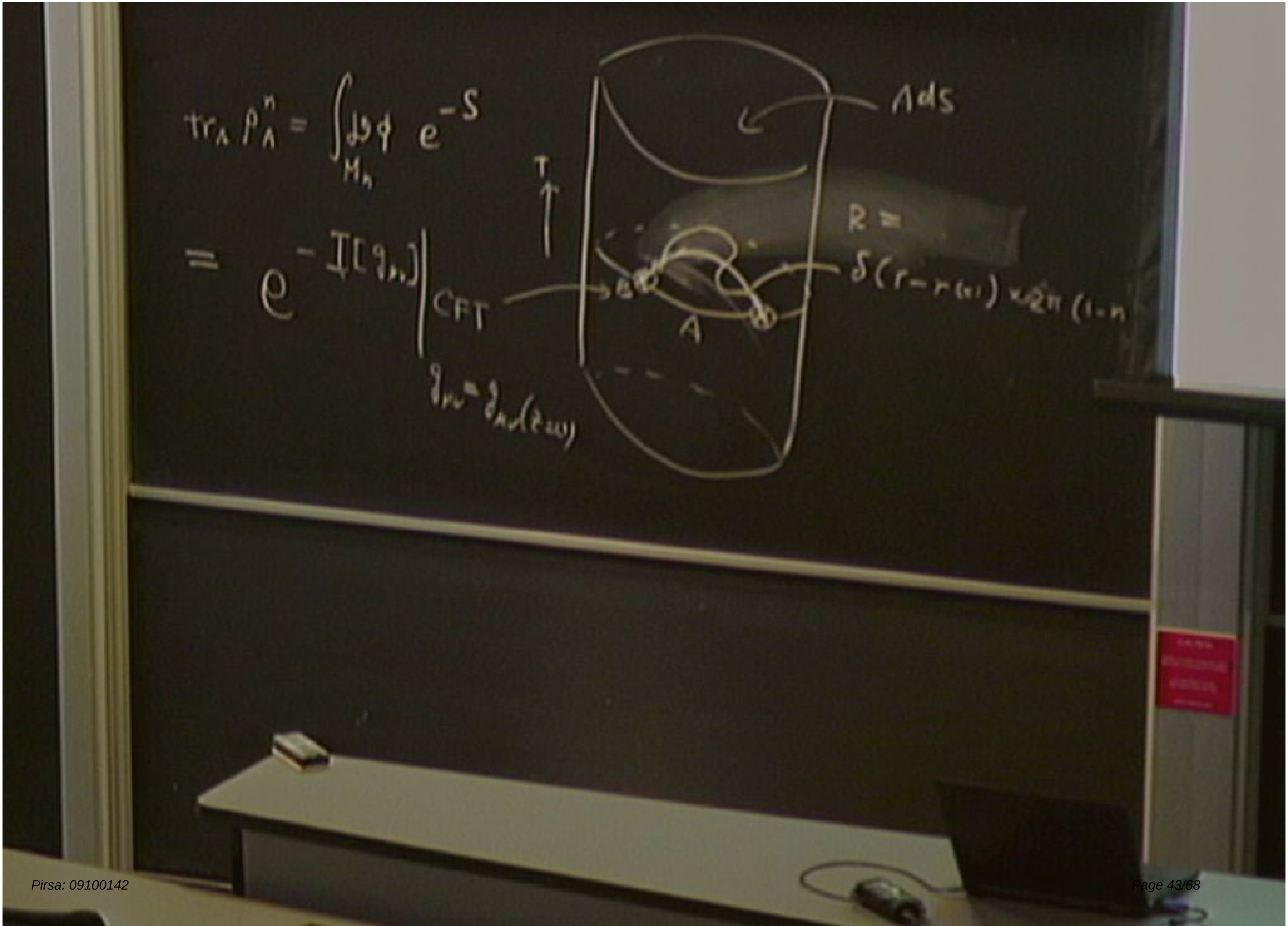
$= e^{-I[\ldots}$
AdS
T
B

holographic derivation of entanglement entropy



$$S_A = \frac{\text{Area of minimal surface } \gamma_A}{4G_N}$$

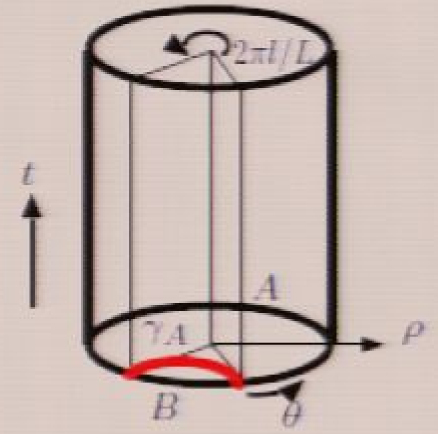



AdS
CFT
A
B
R =

holographic derivation of entanglement entropy

d=1 ⟶ AdS_3/CFT_2 minimal surf = geodesic

$$S_A = \frac{c}{3}\log(l/a) + O(1)$$



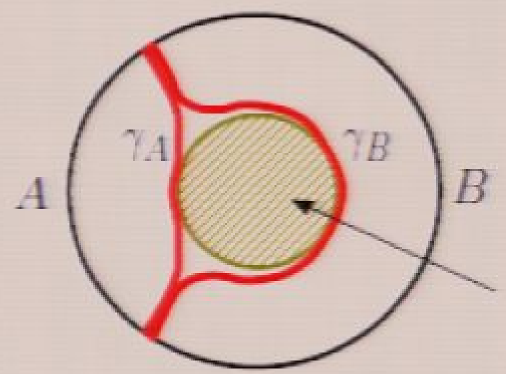
$$c = \frac{3R}{2G_N^{(3)}}$$

finite system

$$S_A = \frac{c}{3}\log\left(\frac{L}{\pi a}\sin\frac{\pi l}{L}\right) + O(1)$$

finite temperature

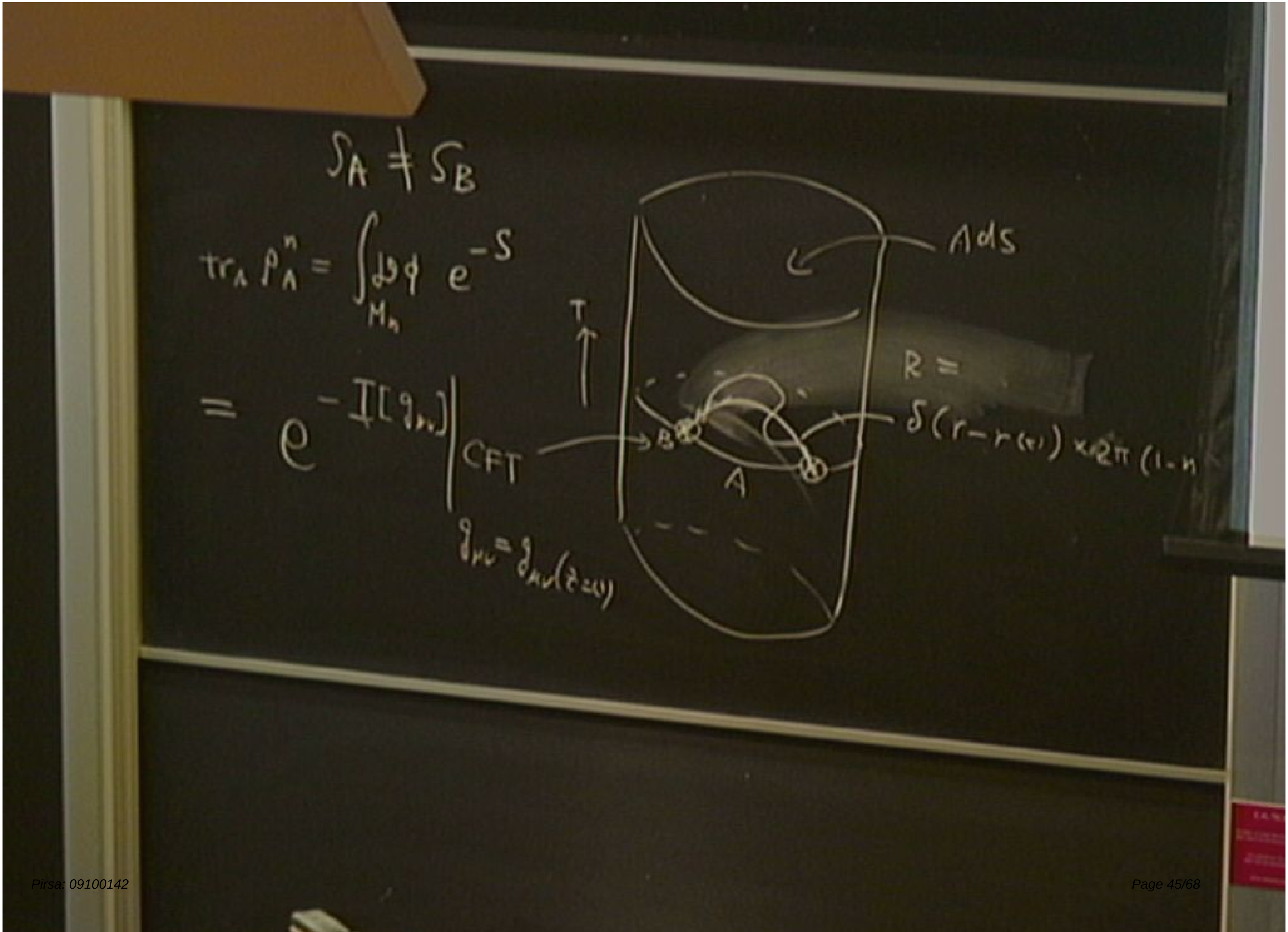
$$S_A = \frac{c}{3}\log\left(\frac{\beta}{\pi a}\sinh\frac{\pi l}{\beta}\right) + O(1)$$



c.f.

Exact agreement between the log term of EE in 4D CFT (two central charges)

A proof from the bulk-boundary relation (GKP-Witten relation) (Furusaev)


S_A ≠ S_B
tr_A ρ_A^n = ∫_{M_n} Dφ e^{-S}
= e^{-I[g_μν]} | CFT
g_μν = g_μν(z=0)
AdS
R =
δ(r − r(z)) × 2π (1 − n
T
B
A

holographic derivation of entanglement entropy

d=1 ⟶ AdS_3/CFT_2   minimal surf = geodesic

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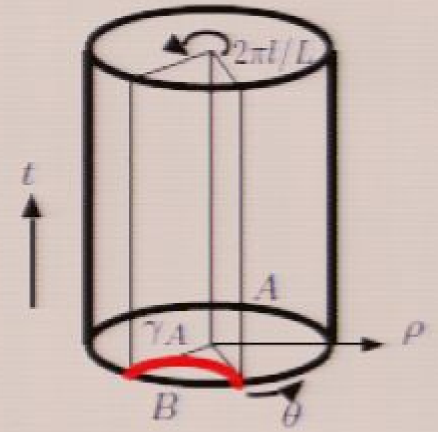

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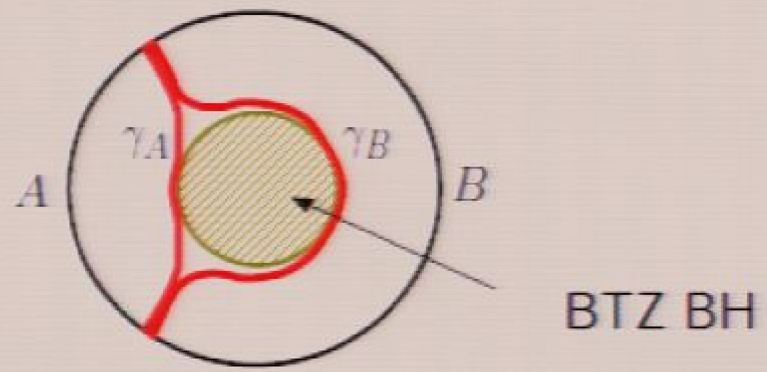

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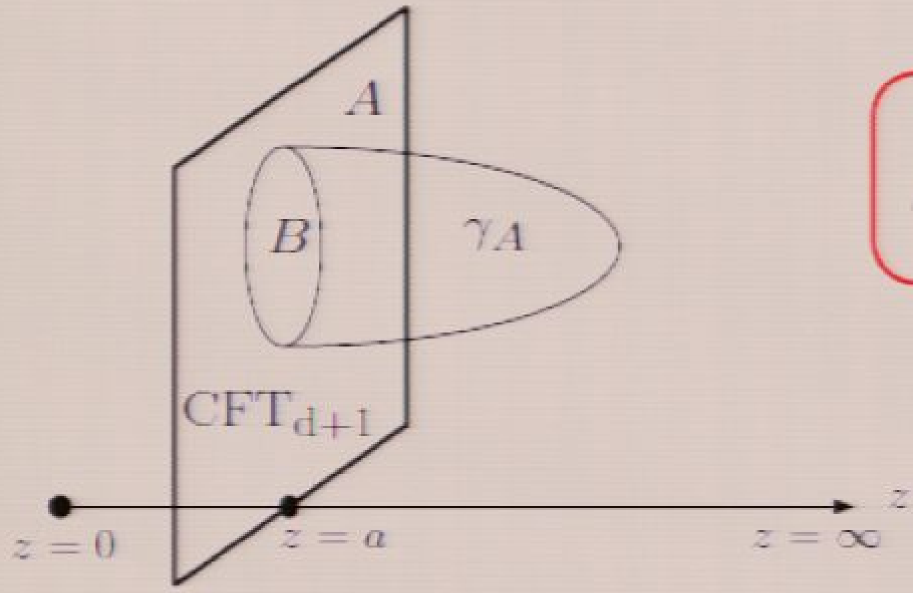

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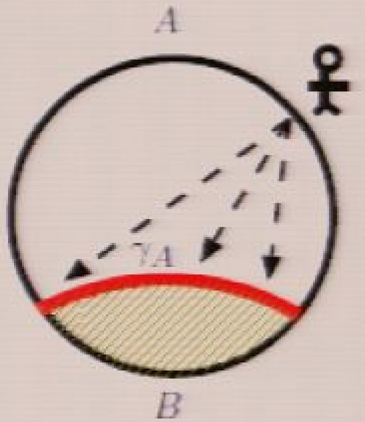

entanglement entropy in CFTs

Weyl rescaling: $g_{\mu\nu} = \delta_{\mu\nu} e^{2\rho} \quad l \sim e^{2\rho}$

$$
\begin{aligned}
l\frac{d}{dl}\ln \mathrm{tr}_A \rho_A^n &= 2\int d^{d+1}x\, g_{\mu\nu}(x)\frac{\delta}{\delta g_{\mu\nu}(x)}\left(\ln Z_n - n\ln Z_1\right) \\
&= -\frac{1}{2\pi}\left\langle \int d^{d+1}x\sqrt{g}T^\mu_\mu \right\rangle_{M_n} + \frac{n}{2\pi}\left\langle \int d^{d+1}x\sqrt{g}T^\mu_\mu \right\rangle_{T_1}
\end{aligned}
$$

$$
l\frac{d}{dl}S_A = -l\frac{d}{dl}\frac{\partial}{\partial n}\ln \mathrm{tr}_A \rho_A^n = -\frac{1}{2\pi}\frac{\partial}{\partial n}\left\langle \int d^{d+1}x\sqrt{g}T^\mu_\mu \right\rangle_{M_n} \qquad (n \to 1)
$$

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SR, Takayanagi (06)

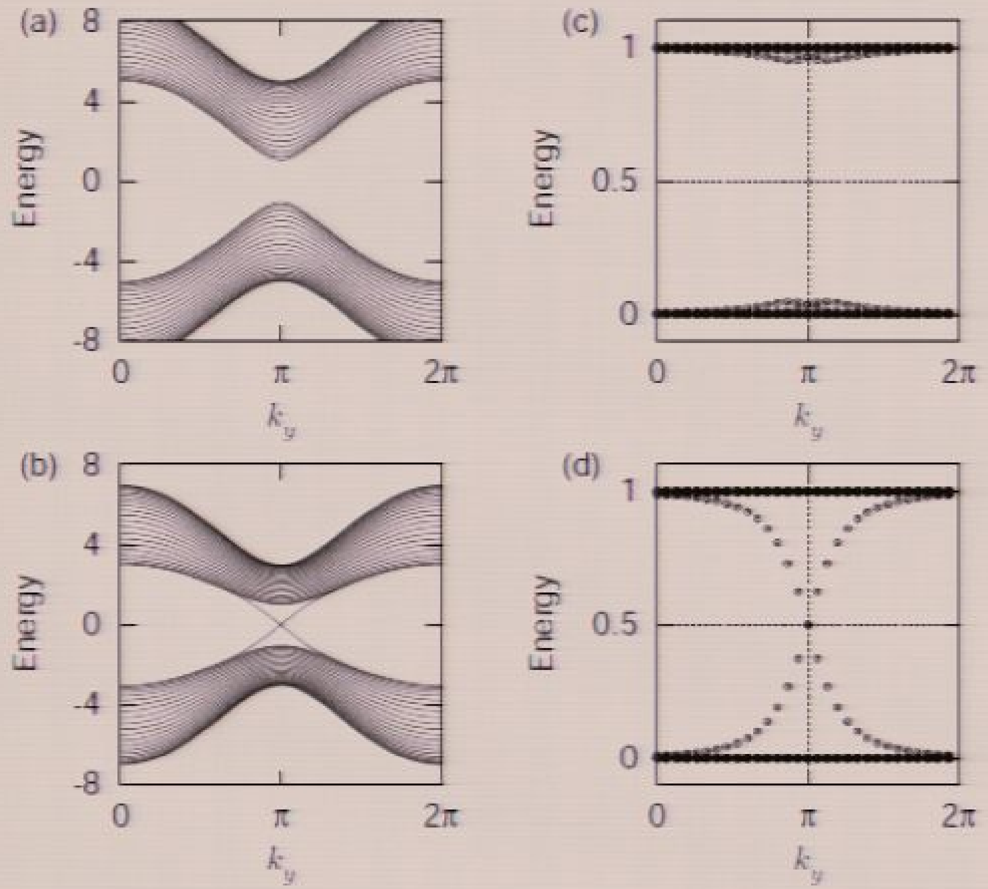
odd dimentional CFTs: no central charges !?

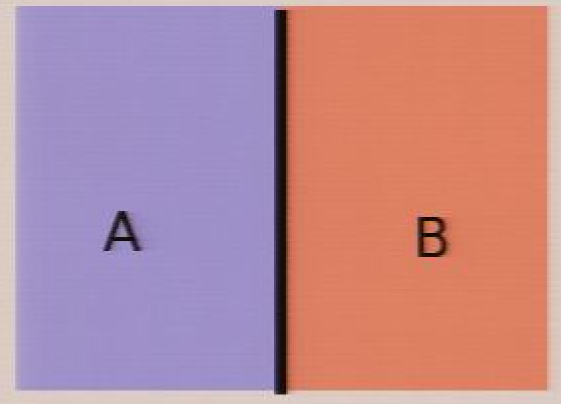
entanglement entropy in topological superconductor

SR, Hatsugai (2006)

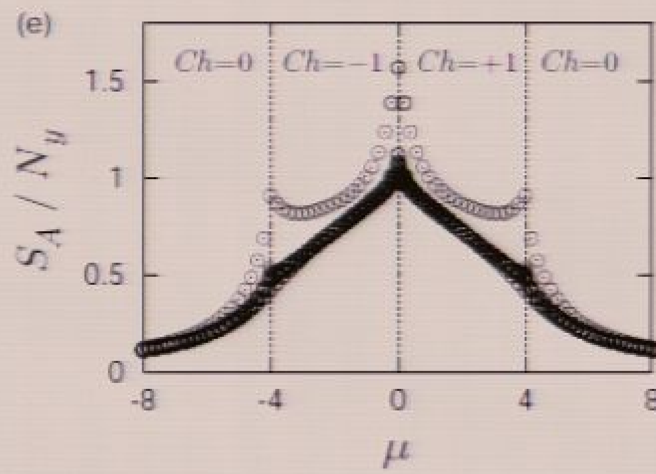
energy spectrum with edges

EE spectrum





EE



EE spectrum shows "edge states".

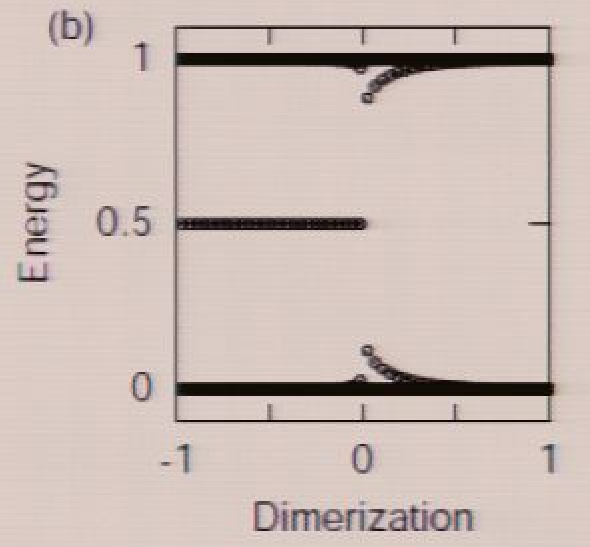
(although there is no real edge)

c.f. for FQHE: Li, Haldane (2008)

entanglement entropy in 1D topological insulator

SR and Hatsugai (2006)

$$H = \sum_i \left[(1+\delta t) c^\dagger_{i\bullet} c_{i\circ} + c^\dagger_{i\circ} c_{i+1\bullet} + h.c. \right]$$



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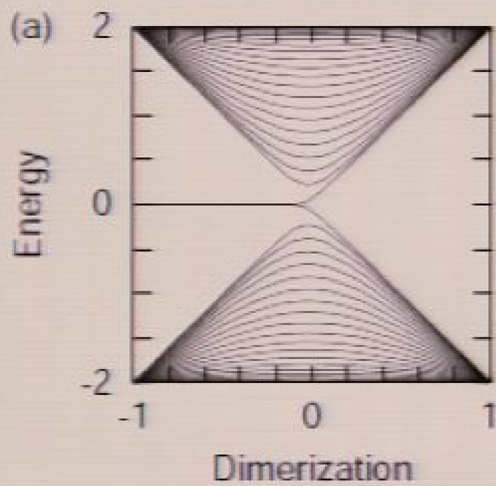
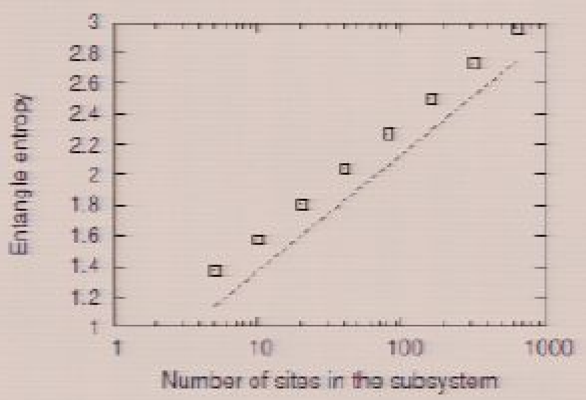
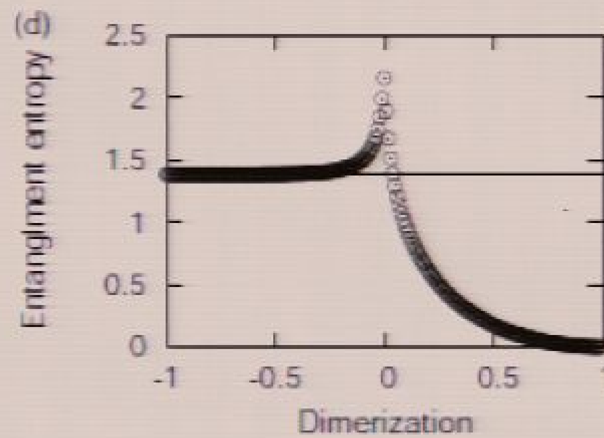




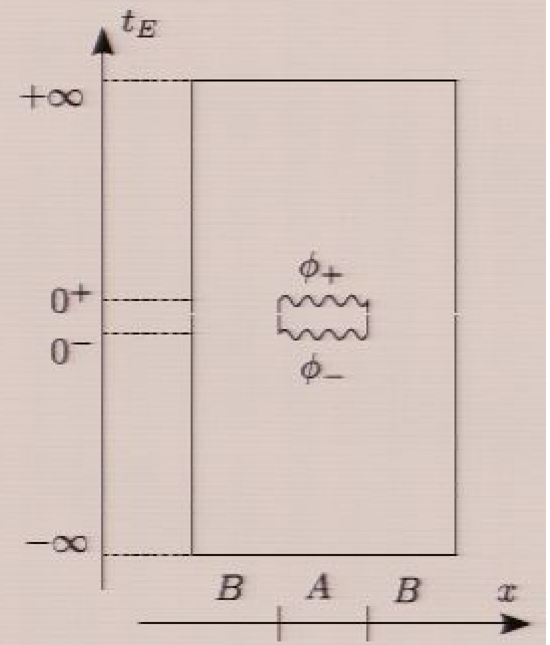


zero mode of EE spetrum when Berry phase = pi

entanglement entropy in QFTs

ground state wavefunctional:

$$\Psi[\phi_0(x)] = \mathcal{N}^{-1} \int_{-\infty}^{t=0} \mathcal{D}\phi e^{-S} \prod_{x\in A} \delta(\phi(0,x) - \phi_0(x))$$

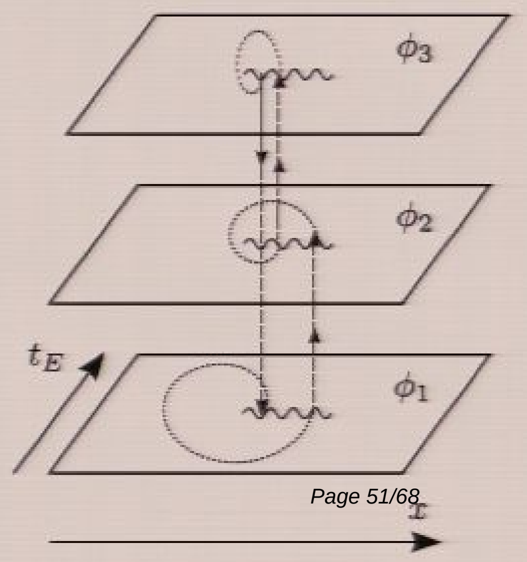


reduced density matrix:

$$[\rho_A]_{\phi_+,\phi_-} = Z^{-1} \int_{-\infty}^{+\infty} \mathcal{D}\phi e^{-S}$$

$$\times \prod_{x\in A} \delta(\phi(+0,x) - \phi_+(x))\delta(\phi(-0,x) - \phi_-(x))$$

$$\mathrm{tr}_A \rho_A^n = (Z_1)^{-n} \int_{\mathcal{R}_n} \mathcal{D}\phi e^{-S} = \frac{Z_n}{(Z_1)^n}$$



QFT on a singular curved space

replica trick --> entanglement entropy

$$S_A = -\frac{\partial}{\partial n} \mathrm{tr}_A \rho_A^n\Big|_{n=1} = -\frac{\partial}{\partial n} \ln \mathrm{tr}_A \rho_A^n\Big|_{n=1}$$

holographic derivation of entanglement entropy

d=1 ⟶ AdS_3/CFT_2   minimal surf = geodesic

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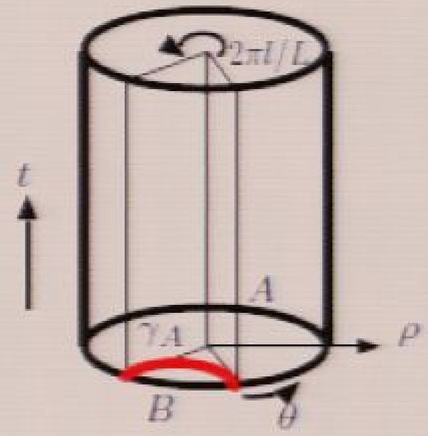

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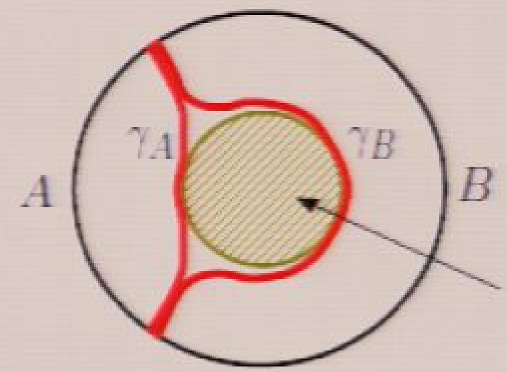
finite temperature

$$S_A = \frac{c}{3}\log\left(\frac{\beta}{\pi a}\sinh\frac{\pi l}{\beta}\right) + O(1)$$



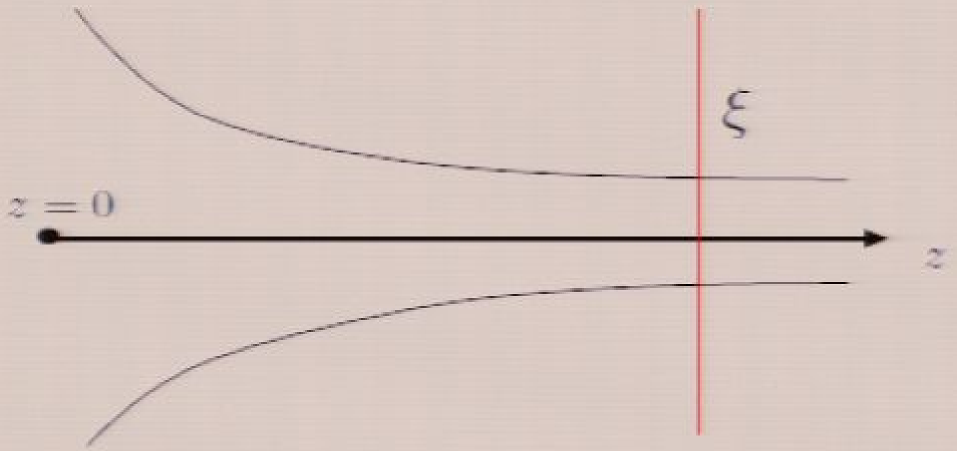
BTZ BH

c.f.

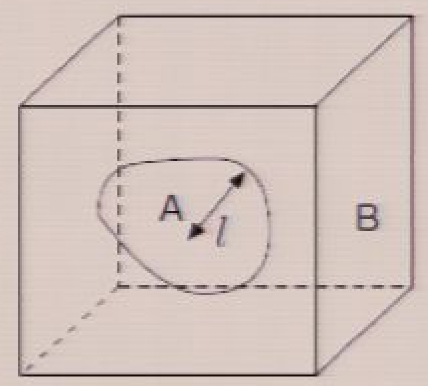
Exact agreement between the log term of EE in 4D CFT (two central charges)

A proof from the bulk-boundary relation (GKP-Witten relation) (Furusaev)

massive deformation



ξ : correlation length $(\ll l)$



a : cut off

$$S_A = p_1\left(\frac{l}{a}\right)^{d-1} + p_3\left(\frac{l}{a}\right)^{d-3} + \cdots + p_{d-2}\left(\frac{l}{a}\right)^{2} +$$
$$+p_1'\left(\frac{\xi}{a}\right)^{d-1} + p_3'\left(\frac{\xi}{a}\right)^{d-3} + \cdots + p_{d-2}'\left(\frac{\xi}{a}\right)^{2} + q\log \xi/a + O(1)$$

d: odd

$$S_A = p_1\left(\frac{l}{a}\right)^{d-1} + p_3\left(\frac{l}{a}\right)^{d-3} + \cdots + p_{d-2}\left(\frac{l}{a}\right)^{2} +$$
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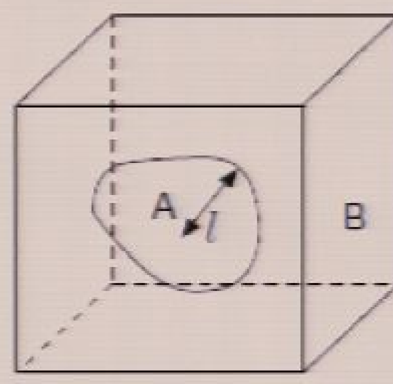
d: even

conformal field theory in (d+1) dimensions

$$S_A = p_1\left(\frac{l}{a}\right)^{d-1} + p_3\left(\frac{l}{a}\right)^{d-3} + \cdots + p_{d-2}\left(\frac{l}{a}\right)^{2} + q\log l/a + O(1) \qquad \text{d: odd}$$

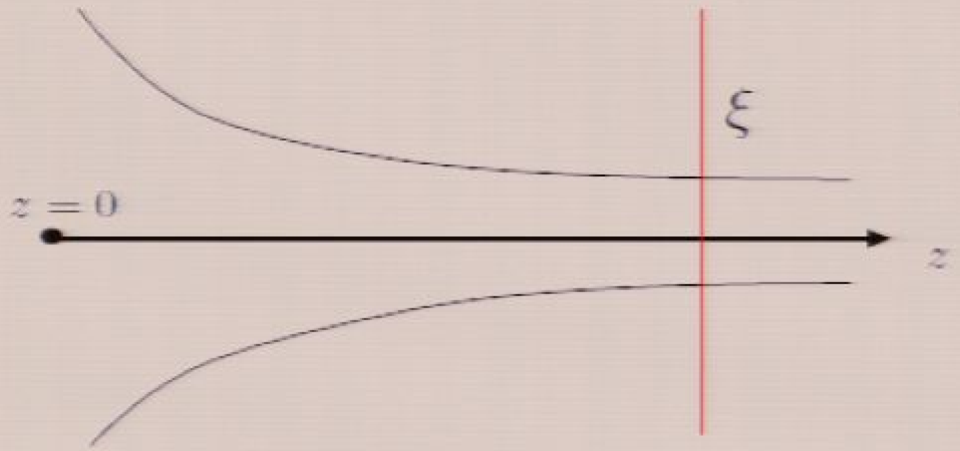
$$S_A = p_1\left(\frac{l}{a}\right)^{d-1} + p_3\left(\frac{l}{a}\right)^{d-3} + \cdots + p_{d-1}\left(\frac{l}{a}\right)^{1} + p_d + O(a/l) \qquad \text{d: even}$$

q and p_d : universal and conformal invariant



a : cut off

massive deformation



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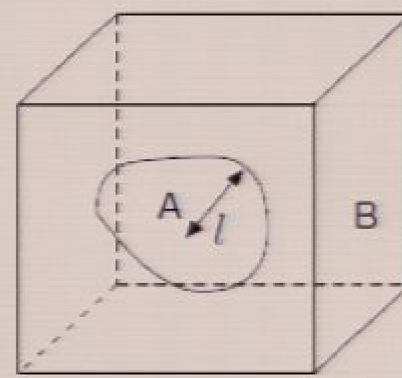

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AdS5/CFT4

type II B on AdS_5 x S_5 ⟷ 4D N=4 SU(N) SYM

$$G^{(10)} = 8\pi^6 \alpha'^4 g_s^2 \qquad G_N^{(5)} = \frac{G_N^{(10)}}{\pi^3 R^5} \qquad R = (4\pi g_s \alpha'^2 N)^{1/4}$$

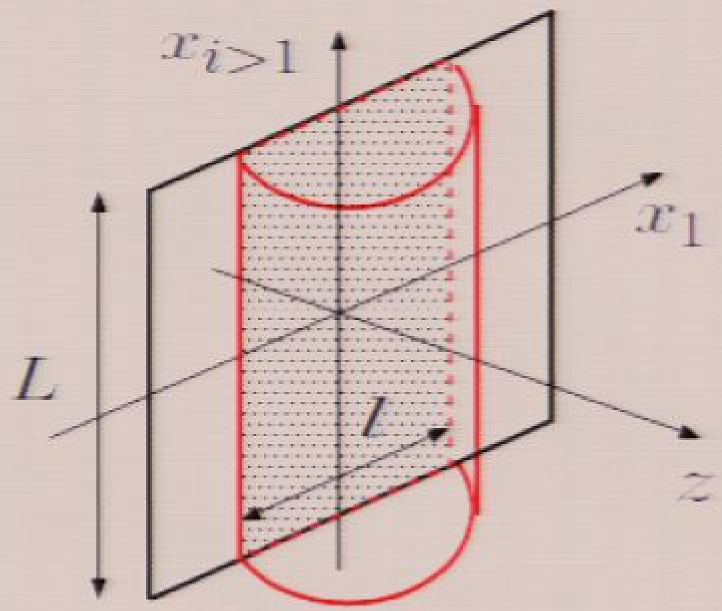
gravity calculation (strongly coupled SYM)

$$S_A = Const'.\frac{N^2L^2}{a^2} - 0.051\frac{N^2L^2}{l^2}$$

↕

free field calculation

$$S_A = Const.\frac{N^2L^2}{a^2} - 0.078\frac{N^2L^2}{l^2}$$



c.f. 4/3 problem in thermal entropy, Gubser-Klevanov-Peet (96)

conformal field theory in (d+1) dimensions

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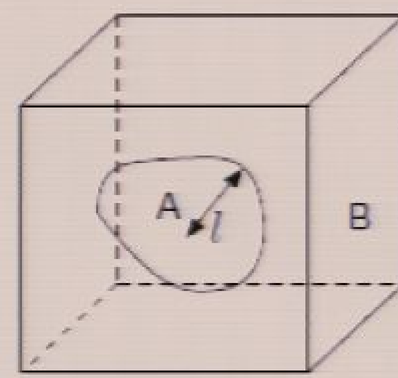

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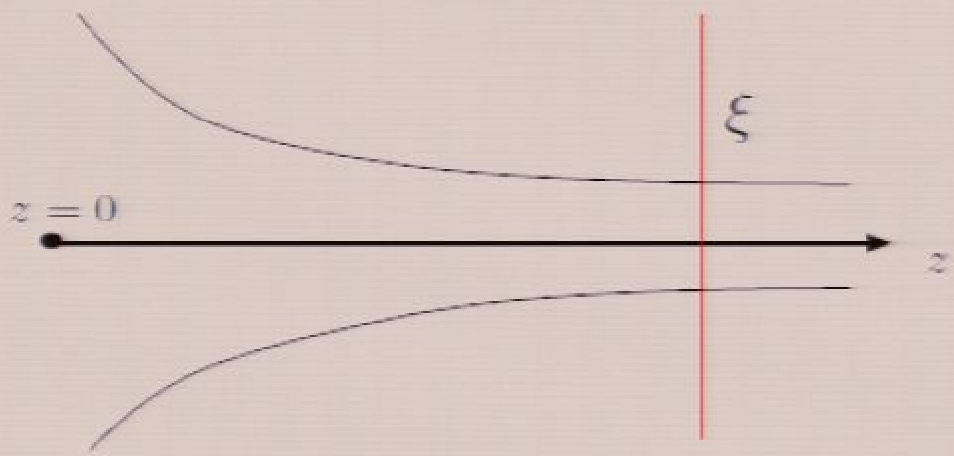

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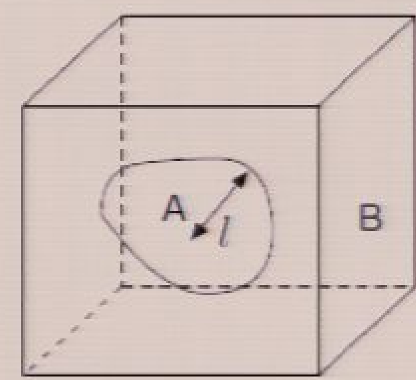

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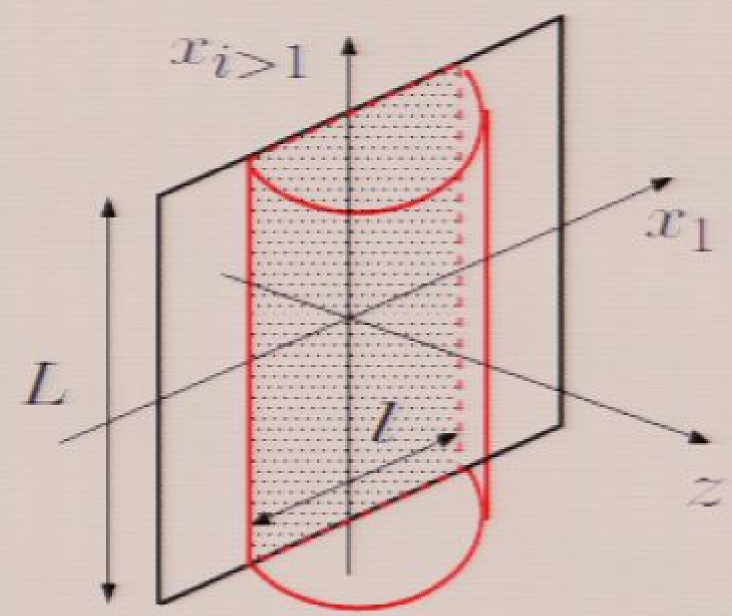

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summary

- entanglement entropy and spectrum in topological insulators
- entanglement entropy in 4D CFTs from Weyl anomaly
- holographic calculation of entanglement entropy in CFTs
- holographic calculation of topological entanglement entropy ?
- EE may be a good test for a gravity dual of a CM system ?

conformal field theory in (d+1) dimensions

$$S_A = p_1 \left(\frac{l}{a}\right)^{d-1} + p_3 \left(\frac{l}{a}\right)^{d-3} + \cdots + p_{d-2} \left(\frac{l}{a}\right)^{2} + q \log l/a + O(1)$$ d: odd

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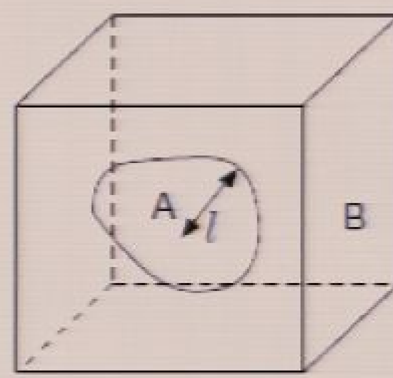
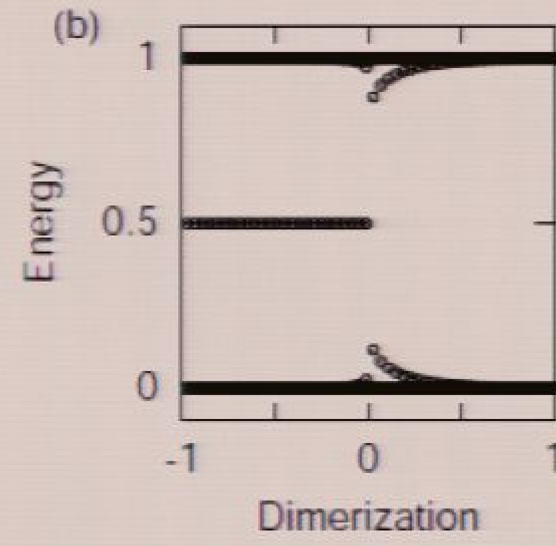


a : cut off

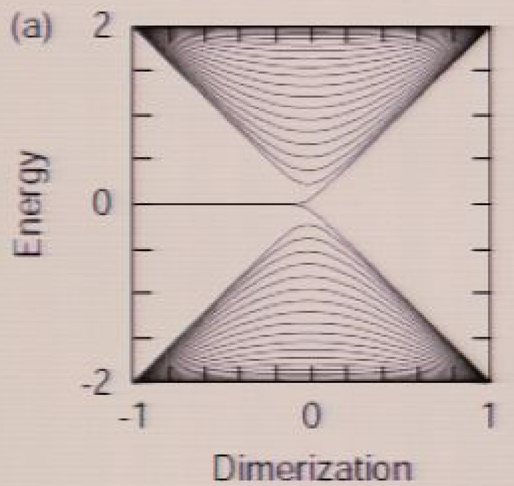
entanglement entropy in 1D topological insulator

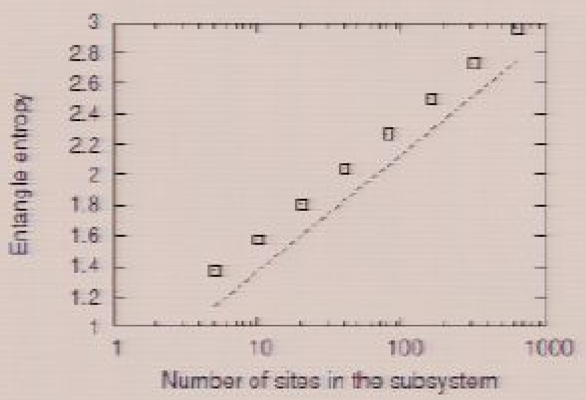
SR and Hatsugai (2006)

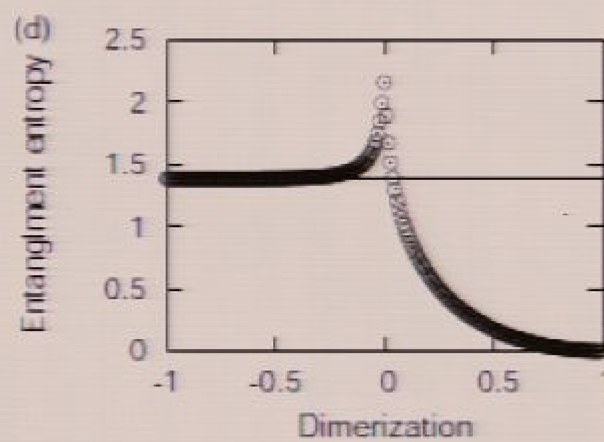
$$H = \sum_i \left[(1+\delta t) c^\dagger_{i\bullet} c_{i\circ} + c^\dagger_{i\circ} c_{i+1\bullet} + h.c. \right]$$



$$S_A = \frac{c}{3} \ln \frac{l}{a}$$







zero mode of EE spetrum when Berry phase = pi

scaling of entanglement entropy

von-Neumann entropy is defined for a region (geometric entropy)
natural object to look at is how EE depends on the size and shape of the region for a given quantum system.

Area law (gapped system, CFT in (d+1)D with d>1, etc.)

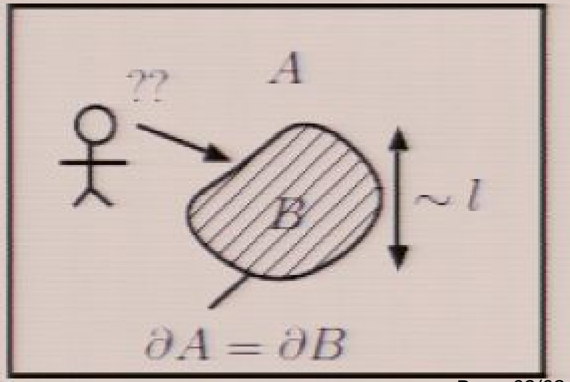
$$S_A = \text{const.} \left(\frac{l}{a}\right)^{d-1} + \cdots$$

Srednicki (93)

↕

Black Hole Entropy (Beckenstein-Hawking)

$$S_{BH} = \frac{\text{Area of horizon}}{4G_N}$$



motivation for entanglement entropy

(1) EE can be a good "order parameter" for quantum systems (?)

- quantum liquid phases: no LRO for _any_ local order parameter

  -fractional quantum Hall effect
  -gappless/gapped quantum spin liquid
  -quantum critical points, non-Landau-Ginzburg transition

- defined purely in terms of wavefunctions

- EE measures a response to external gravity

(2) useful for inventing efficient algorightms for simulating quantum many-body systems

density matrix renomalization group (DMRG)

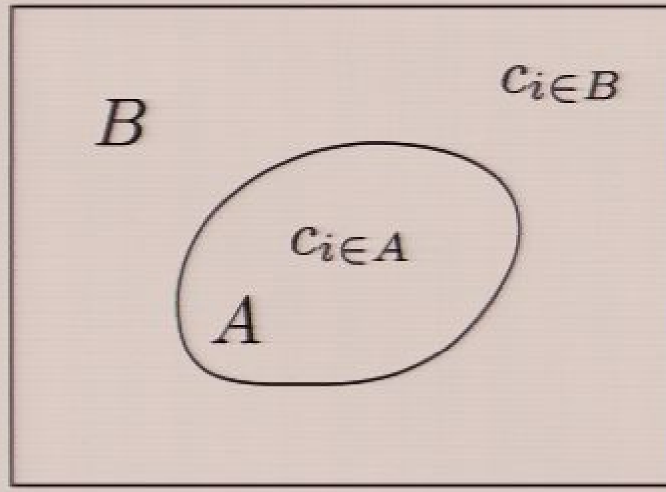
use computational complexity to classify quantum states ?

entanglement entropy for free fermions

Peschel (03)

reduced density matrix for free fermions

$$\rho_A = \mathcal{N} \exp\left(- \sum_{ij \in A} c_i^\dagger K_{ij} c_j \right)$$



K_{ij} is obtained by diagonalizing the correlation matrix:

$$\langle c_i^\dagger c_j \rangle_{i,j\in A} = \sum_\alpha \phi_\alpha^*(i)\phi_\alpha(j) \frac{1}{1+e^{\epsilon_\alpha}}$$

$$K_{ij} = \sum_\alpha \phi_\alpha(i)\phi_\alpha^*(j)\epsilon_\alpha$$

entanglement entropy is given by

$$S_A = -\sum_\alpha [\zeta_\alpha \ln \zeta_\alpha + (1-\zeta_\alpha)\ln(1-\zeta_\alpha)] \qquad \zeta_\alpha = \frac{1}{1+e^{\epsilon_\alpha}}$$

scaling of entanglement entropy

- detecting topological order in (2+1)D Kitaev & Preskill Levin & Wen (2006)

$$S_A = \gamma \tfrac{l}{a} - \log(D) \qquad D = \sqrt{\textstyle\sum_a d_a^2}$$

quantum dimension (d_a); quasi-particle type (a)

$\log D = \log \sqrt{q}$ FQHE at nu = 1/q (Chern-Simons theory)

$\log D = \log 2$ Z2 lattice gauge theory

- z=2 Lifsitz critical point in (2+1)D   Fradkin & Moore (2006)

$$S_A = \gamma \tfrac{l}{a} + \alpha c \log(l/a) + \cdots$$

- free fermions with Fermi surface Gioev & Klich, Wolf (2006)

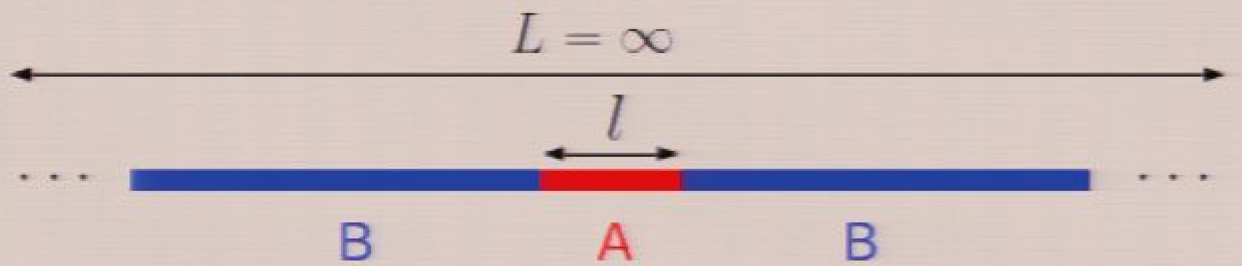
$$S_A = C l^{d-1} \log(l/a) \qquad C \propto \int_{\partial A} \int_{\mathrm{FS}} |\boldsymbol{n}_r \cdot \boldsymbol{n}_k| dS_r dS_k$$

scaling of entanglement entropy

detecting CFT QCP in 1D (Holzhey,Larsen & Wilczek)

$$S_A = \frac{c}{3}\log l/a + c'$$

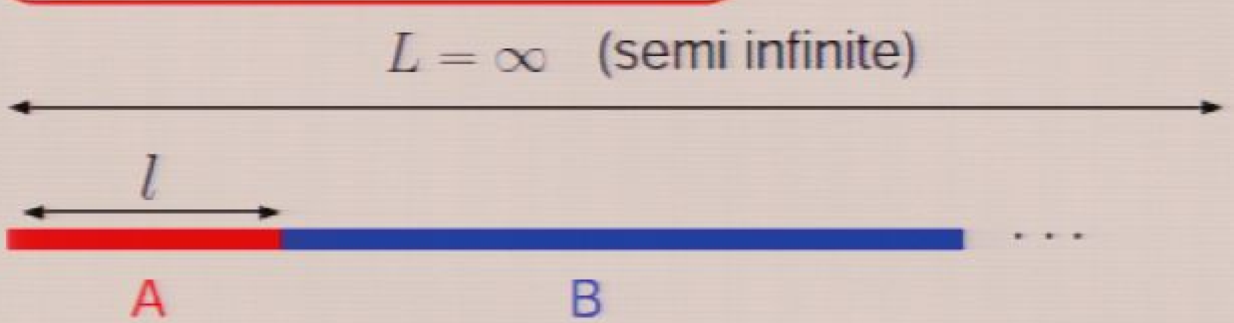
c : central charge
a : cut off



boundary entropy (Zhou, Barthel, Fjaerestad, Schollwock)

$$S_A = \frac{c}{6}\log 2l/a + c'/2 + \log(g)$$

$\log(g)$
:Affleck-Ludwig's boundary entropy



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