Title: Entanglement entropy in topological insulators

Date: Oct 08, 2009 04:00 PM

URL: http://pirsa.org/09100141

Abstract: Entanglement entropy and entanglement entropy spectrum in topological insulators and related systems.

classification of topological insulators and superconductors

result:

AZ$\backslash d$	0	1	2	3	4	5	6	7	8	9
A	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	$\cdots$
AIII	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\cdots$
AI	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\cdots$
BDI	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\cdots$
D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\cdots$
DIII	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\cdots$
AII	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\cdots$
CII	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\cdots$
C	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	$\cdots$
CI	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	$\cdots$

SR, Schnyder, Furusaki, Ludwig (for d=1,2,3, 2008)

Kitaev (all d and periodicity, 2009)

Qi, Hughes, Zhang (cases with one discrete symmetry and field theory description, 2008)

classification of topological insulators and superconductors

spatial dimensions

presence/absence
of topological band structure

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CI	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	$\cdots$

symmetry classes of quadratic fermionic
Hamiltonians (Altland-Zirnbauer)

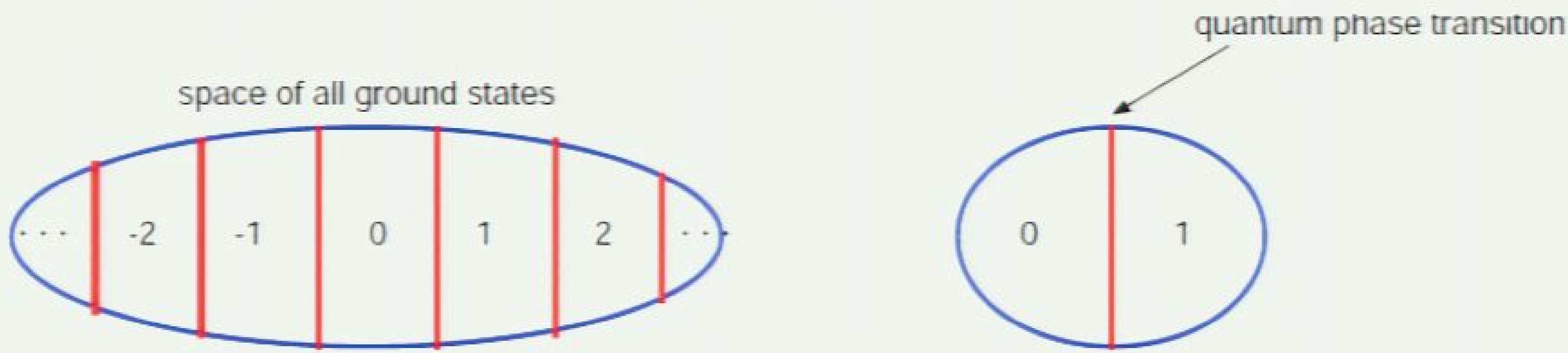
$\mathbb{Z}$ integer classification

$\mathbb{Z}_2$ Z2 classification

0 no top. ins./SC

underlying strategies for classification

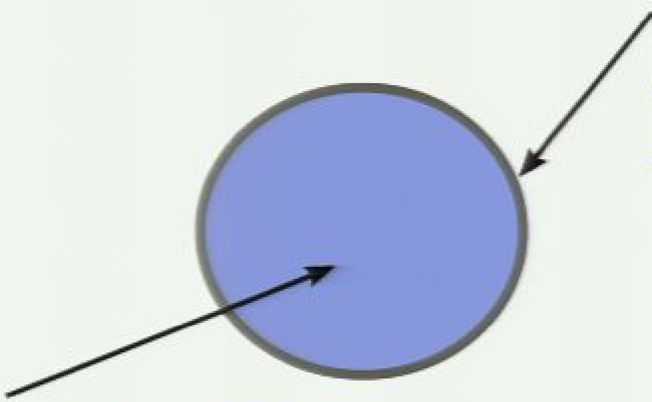
- discover a topological invariant



- bulk-boundary correspondence

Anderson delocalization

non-linear sigma model on G/H
+ (discrete) topological term

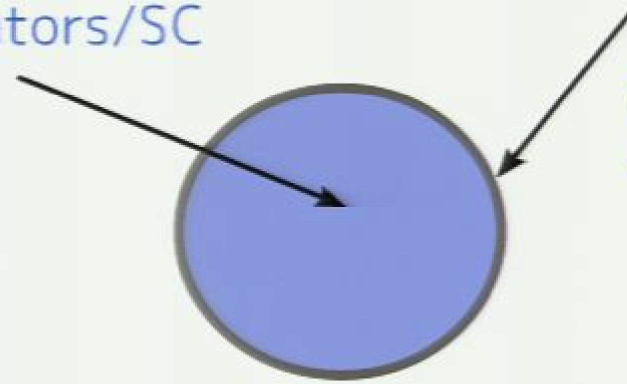


topological insulators/SC

bulk-boundary correspondence

topological insulators/SC

fully gapped,
no excitations



Anderson delocalization

non-linear sigma model on G/H
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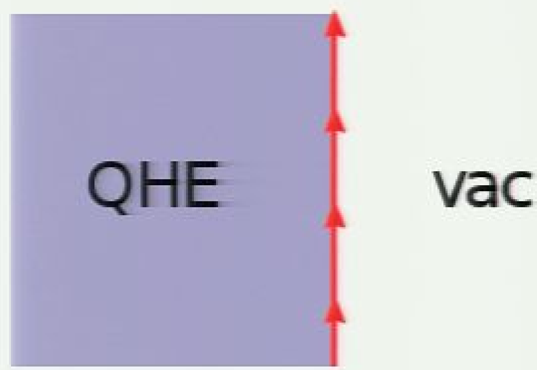
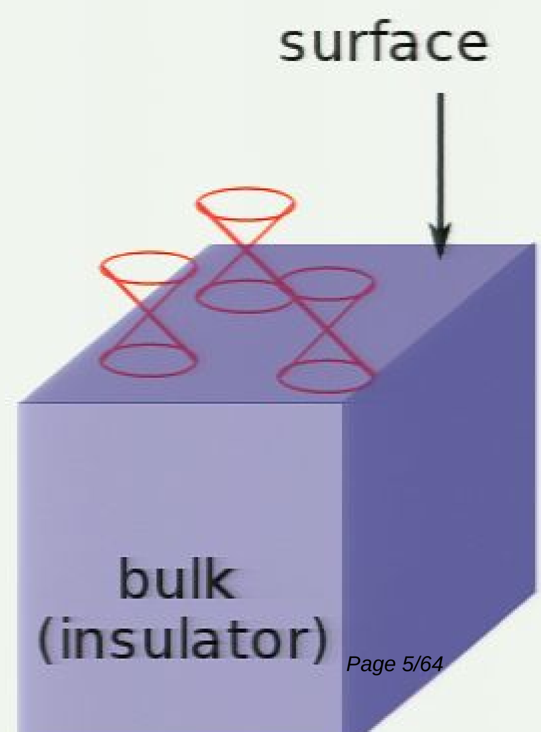
IQHE

QSHE

chiral p+ip wave SC

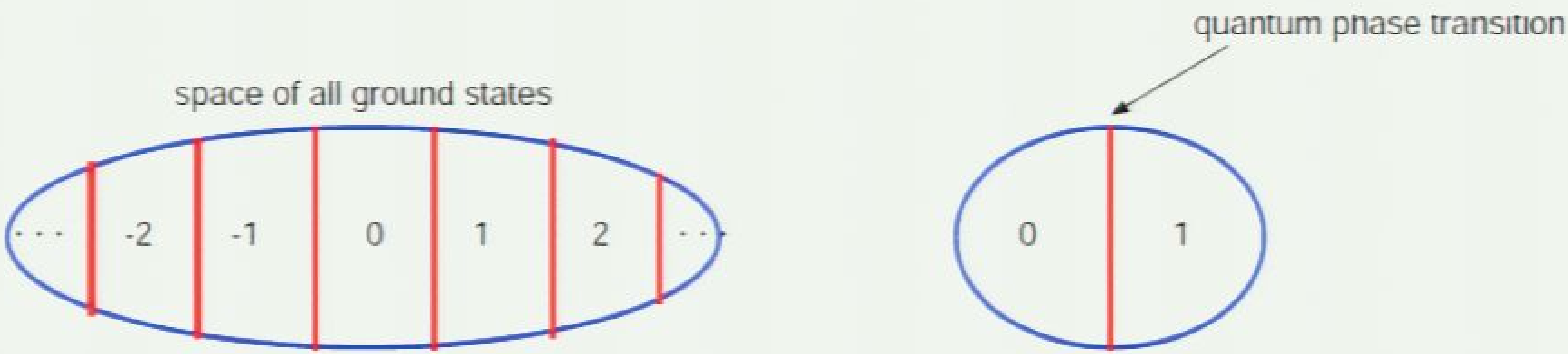




case study: Z2 topological insulator in d=3

underlying strategies for classification

- discover a topological invariant



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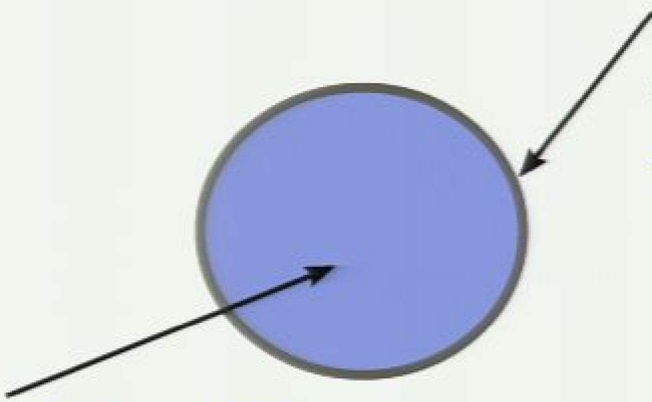

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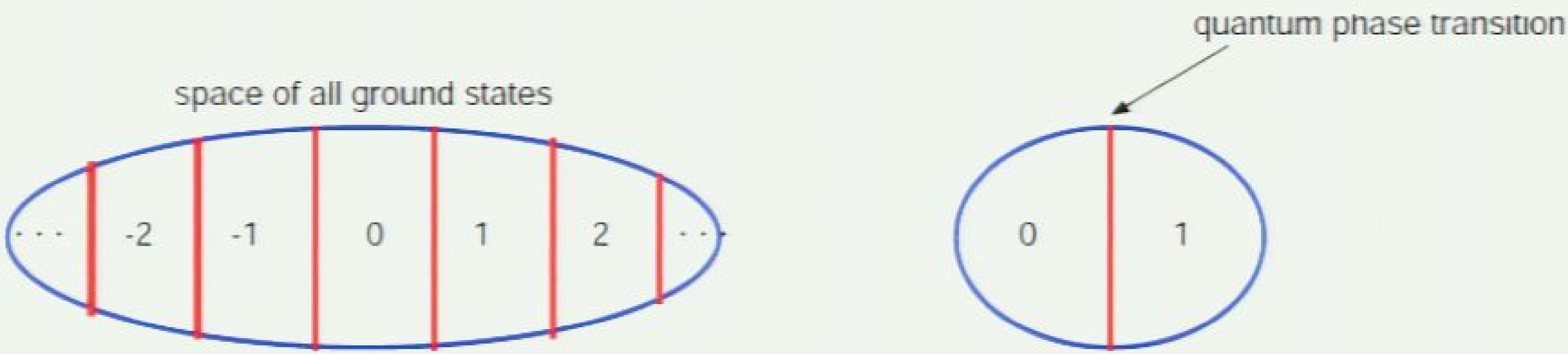
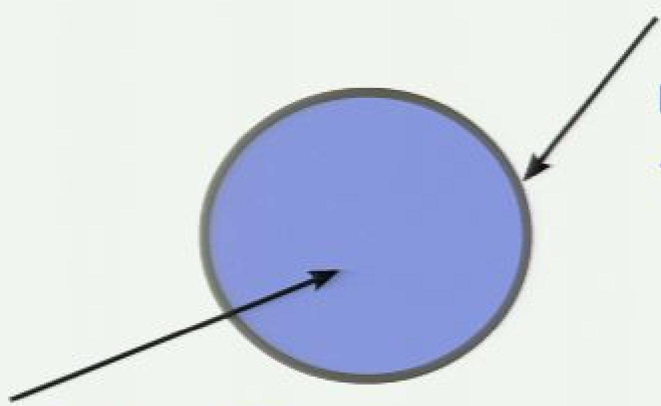


- bulk-boundary correspondence



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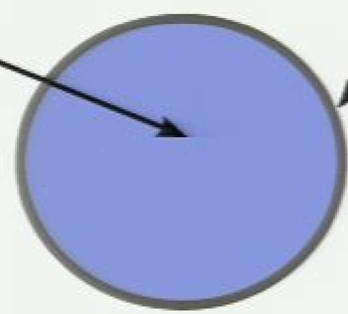

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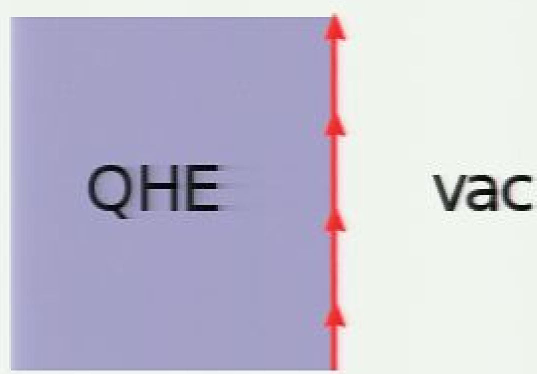
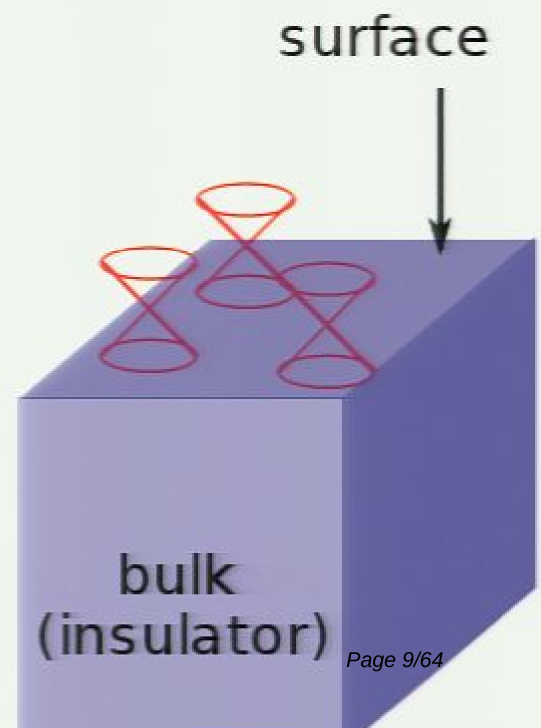
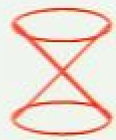




case study: Z2 topological insulator in d=3

Anderson localization with time-reversal symmetry

$$\mathcal{H} = v_F\,(\sigma_x p_x + \sigma_y p_y) + V(\mathbf{r})$$



$$V(\mathbf{r}) = \sum_i U(\mathbf{r} - \mathbf{R}_i)$$

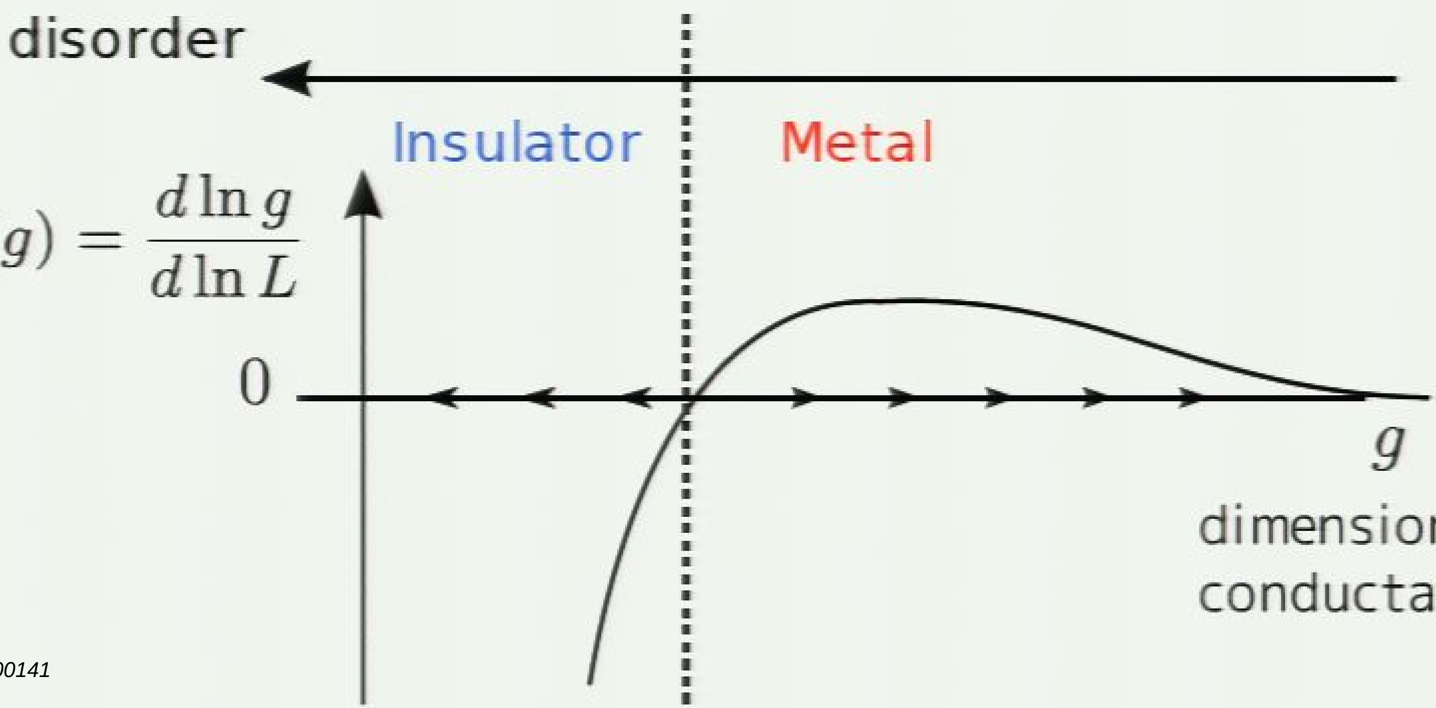
random potential (impurities)

$$i\sigma_y \mathcal{H}^* (-i\sigma_y) = \mathcal{H}$$

time-reversal symmetry
('symplectic' symmetry class)

conventional theory:

$$\beta(g) = \frac{d\ln g}{d\ln L}$$



"gang of four" (79)   Hikami Larkin Nagaoka (80)

numerical beta function

Nomura, Koshino, SR (07)

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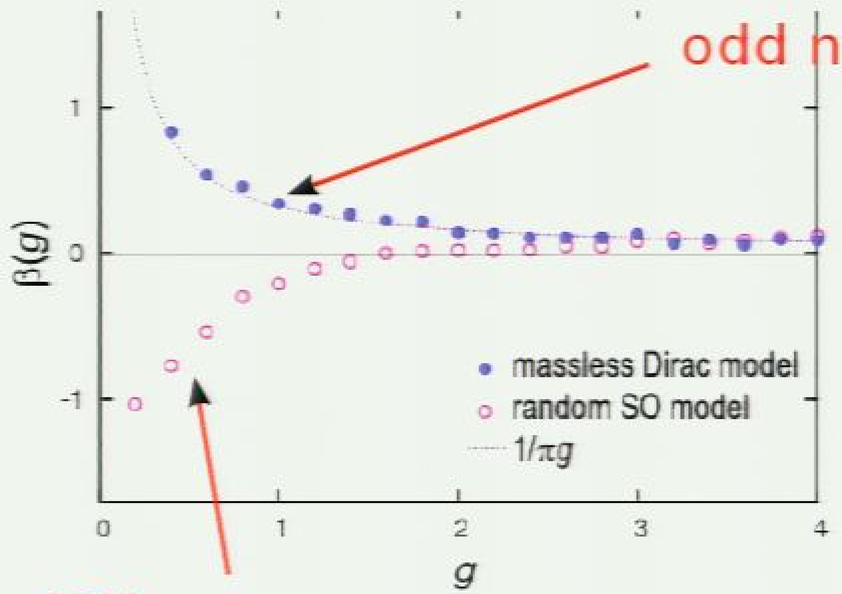
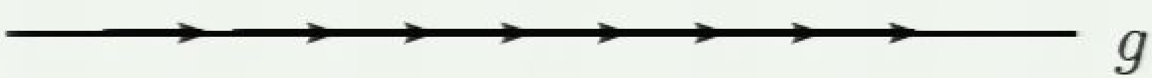


odd number of Dirac cones

even number of Dirac cones
(~ non-relativistic case)



surface of 3D Z2 top. insulator = perfect metal !
"topological metal"

Anderson localization with time-reversal symmetry

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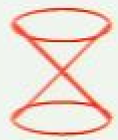

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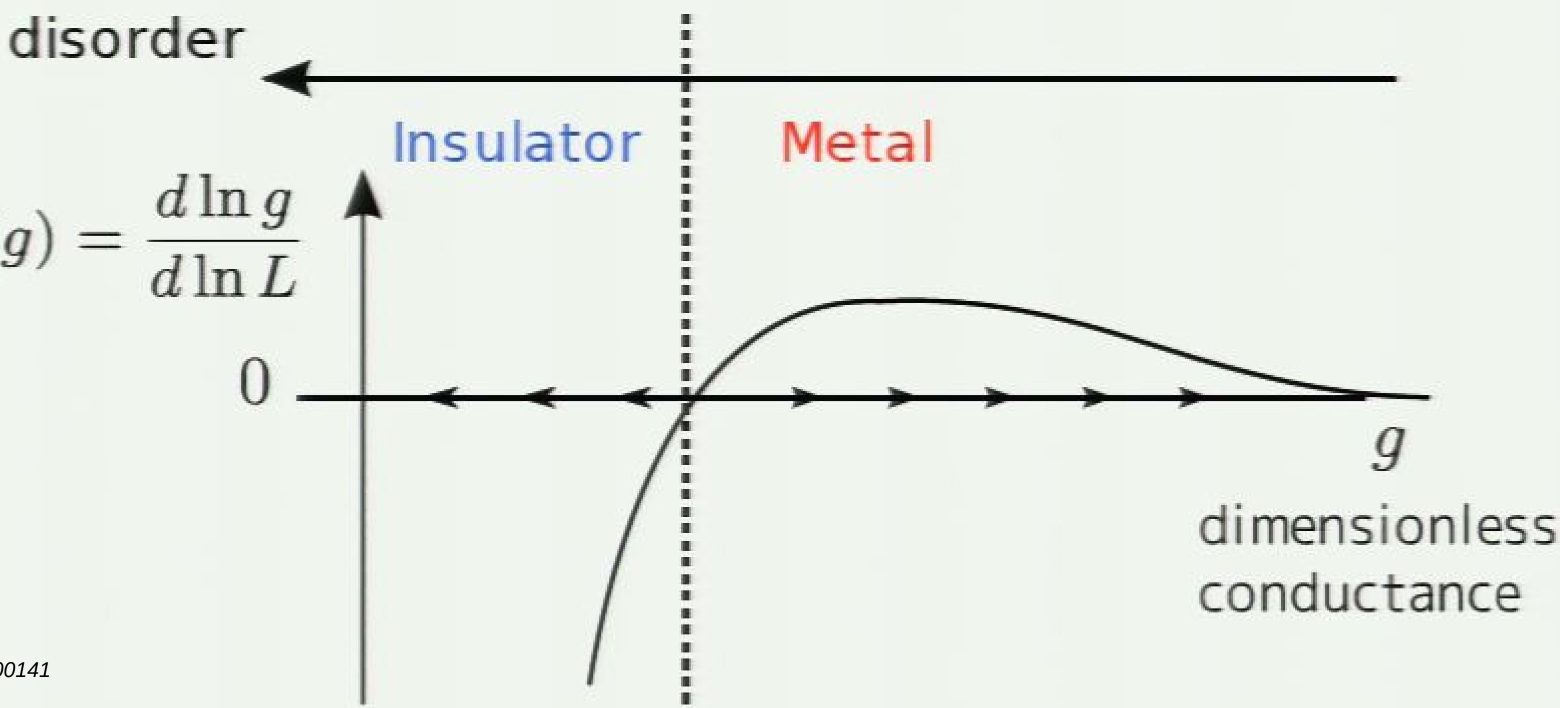

Z2 topological term in symplectic symmetry class

SR, Mudry, Obuse, Furusaki (07)

microscopic model:

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↓

effective field theory: non-linear sigma model

(diffusive motion of electrons

$$Q(\mathbf{r}) \in \mathrm{O}(4N)/[\mathrm{O}(2N) \times \mathrm{O}(2N)]$$

$$S = \sigma_{xx} \int d^2 r \, \mathrm{tr}\left[\partial_\mu Q \partial_\mu Q\right] \qquad \pi_2(\mathrm{O}(4N)/\mathrm{O}(2N) \times \mathrm{O}(2N)) = \mathbb{Z}_2$$

even number of Dirac

$$\boxed{Z = \int \mathcal{D}[Q] e^{-S}}$$

odd number of Dirac
-> Z2 topological term

$$\boxed{Z = \int \mathcal{D}[Q] (-1)^{n[Q]} e^{-S} \quad n[Q] = 0, 1}$$

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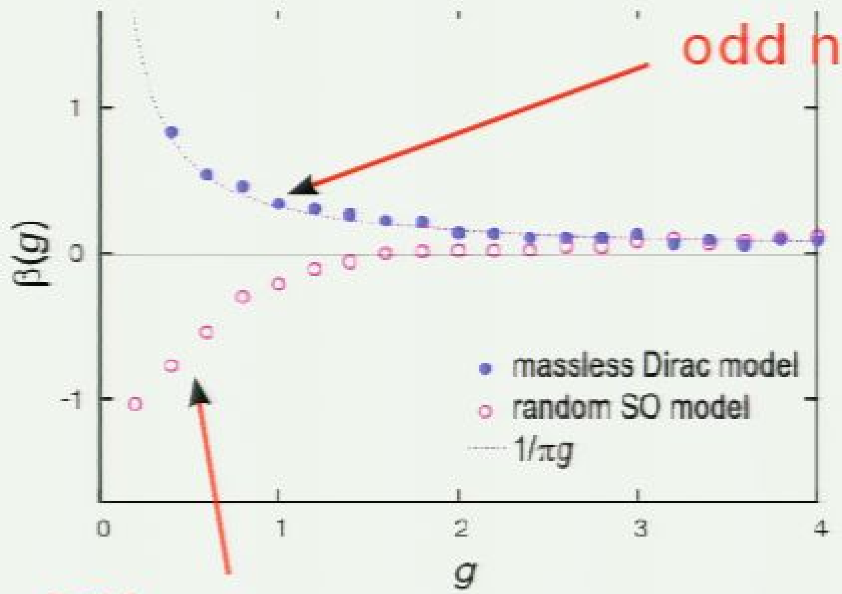

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g

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$\downarrow$

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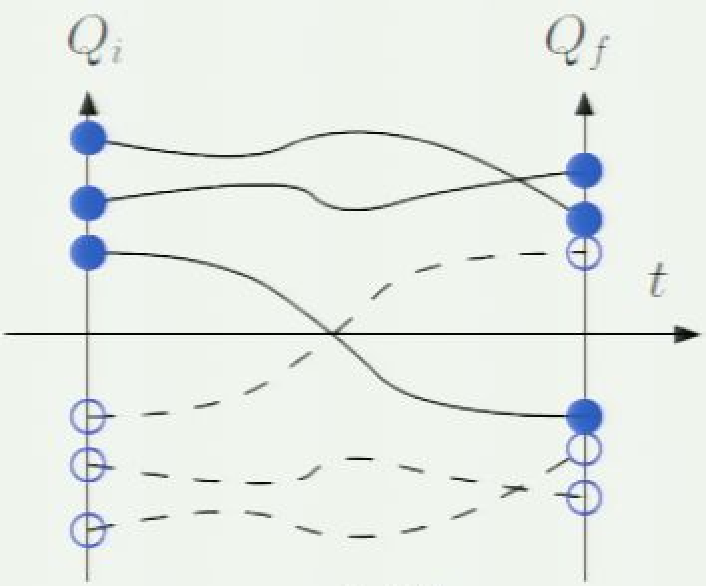
Z2 topological term in symplectic symmetry class

spectral flow

SR,Mudry,Obuse, Furusaki, PRL (07)

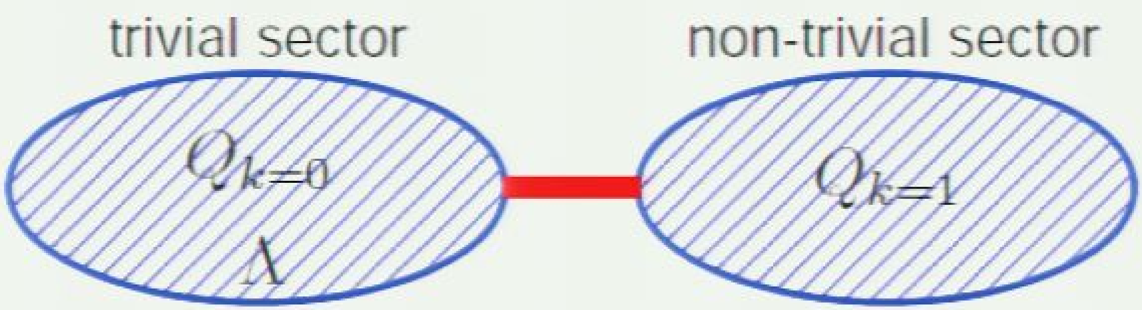
sign of Pfaffian $\longleftrightarrow$ spectral flow

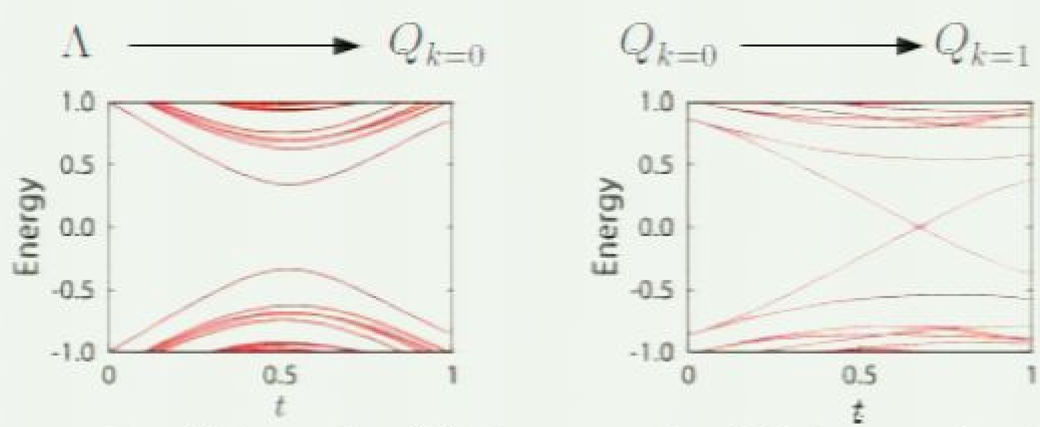
$Q_t := (1-t)Q_i + tQ_f$



$$\mathrm{Pf}\, D[Q] \equiv \prod_i^{\lambda_i>0} \lambda_i$$

$$D[Q] := \sigma \cdot p - \Delta \sigma_z Q$$





$\mathrm{sign}(\Lambda) = \mathrm{sign}(Q_0)$

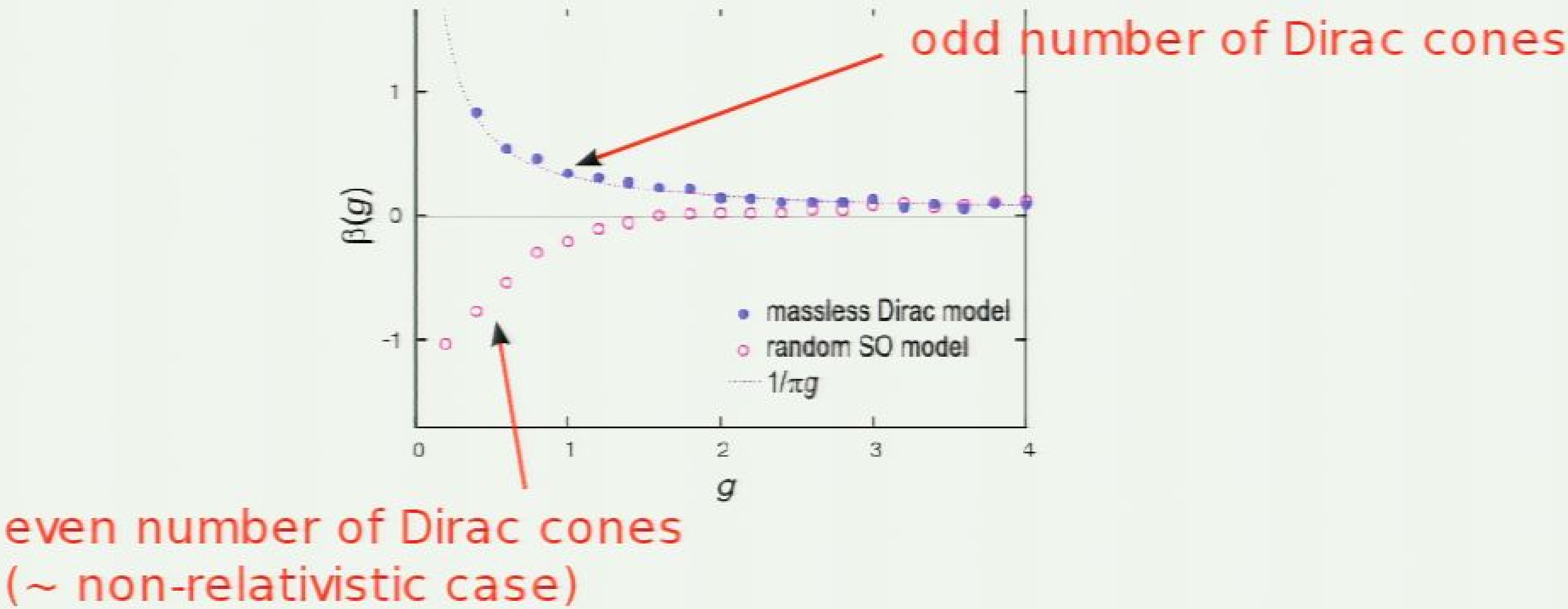
$\mathrm{sign}(Q_0) = -\mathrm{sign}(Q_1)$

Witten's su(2) global anomaly

numerical beta function

Nomura, Koshino, SR (07)

$$\mathcal{H} = v_F\left(\sigma_x p_x + \sigma_y p_y\right) + V(\mathbf{r})$$



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$$\boxed{Z = \int \mathcal{D}[Q](-1)^{n[Q]}e^{-S} \quad n[Q] = 0, 1}$$

Anderson delocalization in d=2 spatial dimensions

		TRS	PHS	SLS	fermionic replica NLsM	
Wigner-Dyson (standard)	A	0	0	0	$U(2N)/U(N)\times U(N)$	Pruisken
	AI	+1	0	0	$Sp(4N)/Sp(2N)\times Sp(2N)$	
	AII	-1	0	0	$O(2N)/O(2N)\times O(2N)$	$\mathbf{Z}_2$
chiral (sublattice)	AIII	0	0	1	$U(N)$	WZW
	BDI	+1	+1	1	$U(2N)/Sp(N)$	
	CII	-1	-1	1	$U(N)/O(N)$	$\mathbf{Z}_2$
BdG	D	0	+1	0	$O(2N)/U(N)$	Pruisken
	C	0	-1	0	$Sp(N)/U(N)$	Pruisken
	DIII	-1	+1	1	$O(N)$	WZW
	CI	+1	-1	1	$Sp(N)$	WZW

newly derived !

- Bernard-Le Clair: 13-fold symmetry classifcation of 2d Dirac fermions
- AIII, CI, DIII; exact results
- "abnormal terms" in NLsM

WZW type $\quad Z=\int \mathcal{D}[g]e^{2\pi i\nu\Gamma_{\mathrm{WZW}}}e^{-S[g]} \qquad \Gamma_{\mathrm{WZW}}=\frac{1}{24\pi^2}\int_{\mathcal{M}^3}\mathrm{tr}\left[(g^{-1}dg)^3\right]$

Z2 type $\quad Z=\int \mathcal{D}[Q](-1)^{N[Q]}e^{-S[Q]}$ $\qquad$ SR, Mudry, Obuse Furusaki (07)

abnormal terms in non-linear sigma models

classification at boundary

complex case:

	$G/H \setminus d$	$d=0$	$d=1$	$d=2$	$d=3$
A	$\mathrm{U}(N+M)/\mathrm{U}(N)\times\mathrm{U}(M)$	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AIII	$\mathrm{U}(N)$	0	$\mathbb{Z}$	0	$\mathbb{Z}$

real case:

	$G/H \setminus d$	$d=0$	$d=1$	$d=2$	$d=3$
AI	$\mathrm{Sp}(N+M)/\mathrm{Sp}(N)\times\mathrm{Sp}(M)$	$\mathbb{Z}$	0	0	0
BDI	$\mathrm{U}(2N)/\mathrm{Sp}(N)$	0	$\mathbb{Z}$	0	0
D	$\mathrm{O}(2N)/\mathrm{U}(N)$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0
DIII	$\mathrm{O}(N)$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$
AII	$\mathrm{O}(N+M)/\mathrm{O}(N)\times\mathrm{O}(M)$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0
CII	$\mathrm{U}(N)/\mathrm{O}(N)$	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
C	$\mathrm{Sp}(N)/\mathrm{U}(N)$	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$
CI	$\mathrm{Sp}(N)$	0	0	0	$\mathbb{Z}$

classifcation in bulk

AZ$\setminus d$	0	1	2	3	4	5	6	7	8	9
A	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	$\cdots$
AIII	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\cdots$
AI	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\cdots$
BDI	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\cdots$
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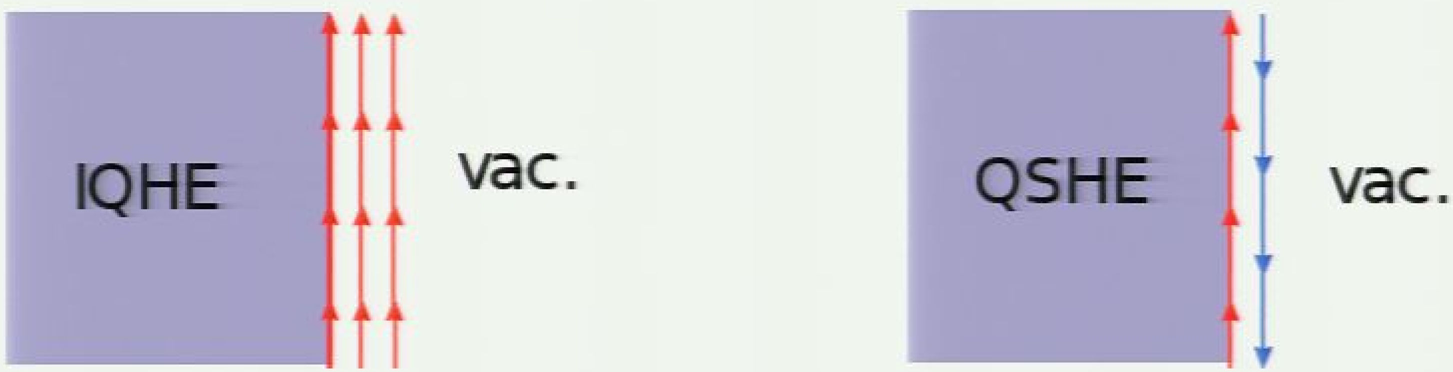
CII	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\cdots$
C	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	$\cdots$
CI	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	$\cdots$

NLSM target space = G/H

Z2 = existence of Z2 topological term
in d dimensions

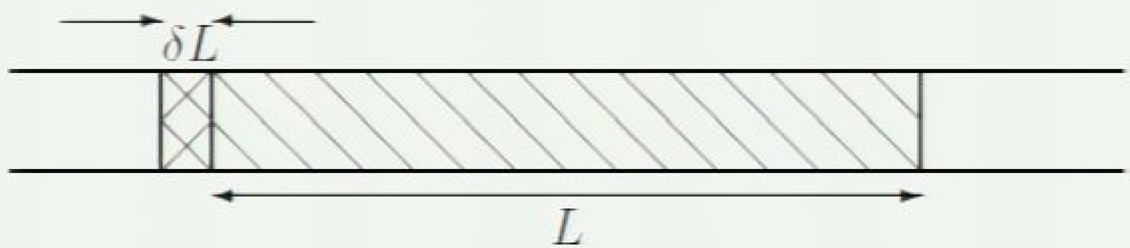
Z = existence of WZW term
in (d-1) dimensions

classification in (2+1)-dimensions

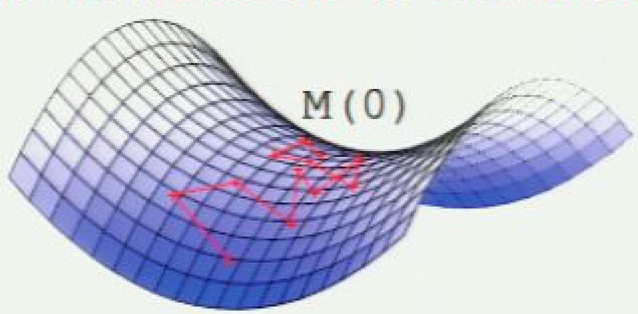


classification of (1+1)-dimensional Anderson delocalization

$$\mathcal{M}_E(L+\delta L) = \mathcal{M}_E(\delta L)\mathcal{M}_E(L)$$



$\Rightarrow$ "Brownian motion" of the transfer matrix

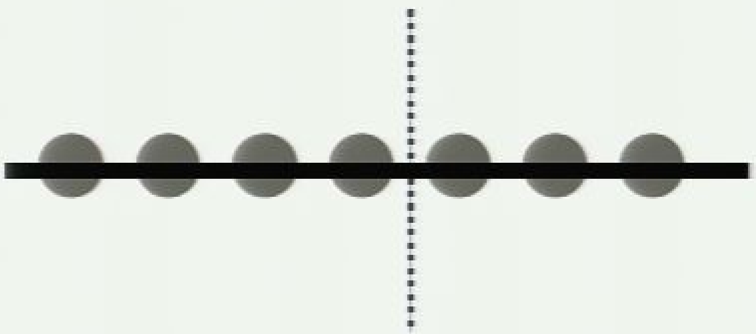


transfer matrix ensembles

12 transfer matrix ensembles
(not 10)

	T	C	$S = CT$	Hamiltonian $\mathcal{H}$	Transfer matrix element of (matrices)	Top.
A (unitary)	0	0	0	$U(N)$	$U(p,q)/U(p)\times U(q)$	$\mathbb{Z}$
AI (orthogonal)	+1	0	0	$U(N)/O(N)$	$Sp(2n,\mathbb{R})/U(2n)$	—
AII (symplectic)	−1	0	0	$U(2N)/Sp(2N)$	$\frac{SO^*(4n)}{U(2n)}$ (even), $\frac{SO^*(4n+2)}{U(2n+1)}$ (odd)	$\mathbb{Z}_2$
AIII (chiral unitary)	0	0	1	$\frac{U(N+M)}{U(N)\times U(M)}$	$\frac{Gl(n,\mathbb{C})}{U(n)}$	—
BDI (chiral orthogonal)	+1	+1	1	$\frac{O(N+M)}{O(N)\times O(M)}$	$\frac{Gl(n,\mathbb{R})}{O(n)}$	—
CII (chiral symplectic)	−1	−1	1	$\frac{Sp(2N+2M)}{Sp(2N)\times Sp(2M)}$	$\frac{U^*(2n)}{Sp(2n)}$	—
D	0	+1	0	$O(N)$	$\frac{SO(p,q)}{SO(p)\times SO(q)}$	$\mathbb{Z}$
C	0	−1	0	$Sp(2N)$	$\frac{Sp(2p,2q)}{Sp(2p)\times Sp(2q)}$	$\mathbb{Z}$
DIII	−1	+1	1	$SO(2N)/U(N)$	$\frac{SO(2n,\mathbb{C})}{SO(2n)}$ (even), $\frac{SO(2n+1,\mathbb{C})}{SO(2n+1)}$ (odd)	$\mathbb{Z}_2$
CI	+1	−1	1	$Sp(2N)/$ [illegible]	$Sp(2n,\mathbb{C})/$ [illegible]	—

classification in (1+1)-dimensions



presence/absence of end states

classification of (1+0)-dim. systems

13 random matrix ensembles (not 10)

(Ivanov, zero modes in RMT, 2001)

Name	Hamiltonian $\mathcal{H}$ element of	d=1 top. ins.
A	$U(N)$	
AI	$U(N)/O(N)$	
AII	$U(2N)/Sp(2N)$	
AIII	$U(p+q)/U(p)\times U(q)$	$\mathbb{Z}$
BDI	$SO(p+q)/SO(p)\times SO(q)$	$\mathbb{Z}$
CII	$Sp(2p+2q)/Sp(2p)\times Sp(2q)$	$\mathbb{Z}$
D(even)	$SO(2N)$	
D(odd) $\hat{=}$ B	$SO(2N+1)$	$\mathbb{Z}_2$
C	$Sp(2N)$	
DIII (even)	$SO(2N)/U(N)$	
DIII (odd)	$SO(4N+2)/U(2N+1)$	$\mathbb{Z}_2$
CI	$Sp(2N)/U(N)$	

classification of boundaries

-classification of 2D Dirac Hamiltonians

Bernard-LeClair (2001)

$$\mathcal{H} = \begin{pmatrix} V_+ + V_- & -i\bar{\partial} + A_+ \\ +i\partial + A_- & V_+ - V_- \end{pmatrix}$$

13 classes (not 10 !)

AIII, CI, DIII has an extra class.

		TR	SU(2)	description
Wigner-Dyson (standard)	A	×	○ ×	unitary
	AI	○	○	orthogonal
	AII	○	×	symplectic (spin-orbit)
chiral (sublattice)	AIII	×	○ ×	chiral unitary
	AIII	×	○ ×	chiral unitary extra
	BDI	○	○	chiral orthogonal
	CII	○	×	chiral symplectic
BdG	C	×	○	singlet SC
	D	×	×	singlet/triplet SC
	CI	○	○	singlet SC
	CI	○	○	singlet SC extra
	DIII	○	×	singlet/triplet SC
	DIII	○	×	singlet/triplet SC extra

even/odd effect

always gapless

AZ$\backslash d$	0	1	2	3	4	5	6	7	8	9
A	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	$\cdots$
AIII	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\cdots$
AI	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\cdots$
BDI	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\cdots$
D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\cdots$
DIII	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\cdots$
AII	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\cdots$
CII	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\cdots$
C	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	$\cdots$
CI	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	$\cdots$

SR, Schnyder, Furusaki, Ludwig (for d=1,2,3, 2008)

Kitaev (all d and periodicity, 2009)

Qi, Hughes, Zhang (cases with one discrete symmetry, 2008)

IQHE

p+ip wave SC

polyacetylene

3He B

TMTSF

Z2 topological insulator

QSHE

d+id wave SC

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AI	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\cdots$
BDI	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\cdots$
D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\cdots$
DIII	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\cdots$
AII	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\cdots$
CII	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\cdots$
C	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	$\cdots$
CI	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	$\cdots$

some outcomes of classification:

- 3He B is newly identified as a topological SC (superfluid) in d=3.
- topological singlet SC in d=3 is predicted.

spatial dimensions

presence/absence
of topological band structure

AZ\\d	0	1	2	3	4	5	6	7	8	9
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AI	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\cdots$
BDI	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\cdots$
D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\cdots$
DIII	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\cdots$
AII	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\cdots$
CII	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\cdots$
C	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	$\cdots$
CI	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	$\cdots$

symmetry classes of quadratic fermionic
Hamiltonians (Altland-Zirnbauer)

$\mathbb{Z}$ integer classification

$\mathbb{Z}_2$ Z2 classification

0 no top. ins./SC

IQHE

p+ip wave SC

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TMTSF

AZ\d	0	1	2	3	4	5	6	7	8	9
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AIII	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\cdots$
AI	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\cdots$
BDI	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\cdots$
D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\cdots$
DIII	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\cdots$
AII	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\cdots$
CII	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\cdots$
C	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	$\cdots$
CI	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	$\cdots$

Z2 topological insulator

QSHE

d+id wave SC

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A	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	$\cdots$
AIII	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\cdots$
AI	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\cdots$
BDI	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\cdots$
D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\cdots$
DIII	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\cdots$
AII	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\cdots$
CII	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\cdots$
C	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	$\cdots$
CI	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	$\cdots$

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0 no top. ins./SC

momentum space topology

Berry gauge field (k-space gauge field)

$$\mathcal{A}^{\hat{a}\hat{b}}(k) = A_{\mu}^{\hat{a}\hat{b}}(k)\mathrm{d}k_{\mu} = \langle u_{\hat{a}}^{-}(k)|\mathrm{d}u_{\hat{b}}^{-}(k)\rangle, \quad \mu = 1, \cdots, d, \quad \hat{a}, \hat{b} = 1, \cdots, N_{-},$$

$$\mathcal{F}^{\hat{a}\hat{b}}(k) = \mathrm{d}\mathcal{A}^{\hat{a}\hat{b}} + \left(\mathcal{A}^{2}\right)^{\hat{a}\hat{b}} = \frac{1}{2}F_{\mu\nu}^{\hat{a}\hat{b}}(k)\mathrm{d}k_{\mu}\wedge\mathrm{d}k_{\nu}.$$

topological invariant in d=2

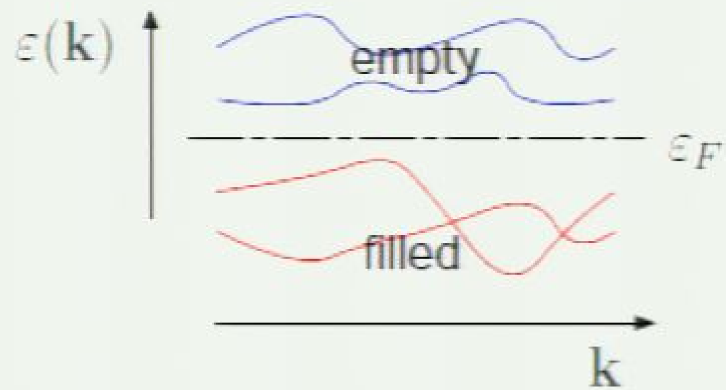
$$\mathrm{Ch}_1[\mathcal{F}] = \frac{\mathrm{i}}{2\pi}\int_{\mathrm{BZ}^{d=2}} \mathrm{tr}\,(\mathcal{F}) = \frac{\mathrm{i}}{2\pi}\int \mathrm{d}^2k\,\mathrm{tr}\,(F_{12})$$

$$\boxed{\sigma_{xy} = \frac{e^2}{h}\times \mathrm{Ch}_1[\mathcal{F}]}$$

Thouless, Kohmoto, Naitingale, den Nijs (1982)

momentum space topology

projector: $$Q(k) = 2 \sum_{a \in \text{filled}} |u_a(k)\rangle\langle u_a(k)| - 1$$



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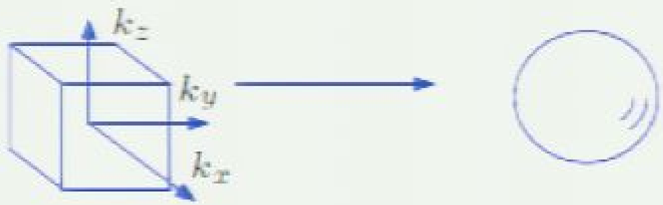
Q : BZ $\longrightarrow$ $U(m+n)/U(m)\times U(n)$



quantum ground state = map from Bz onto Grassmannian

$\pi_2[U(m+n)/U(m)\times U(n)] = \mathbf{Z}$ $\longrightarrow$ IQHE in 2D

$\pi_3[U(m+n)/U(m)\times U(n)] = 0$

$\longrightarrow$ no top. insulator in 3D without constraint (Class A)

(for large enough m,n)

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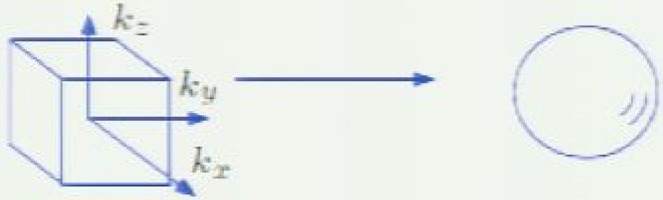
$$Q^2 = 1, \ Q^\dagger = Q, \ \mathrm{tr}\, Q = m - n$$

filled empty

Q : BZ $\longrightarrow$ $U(m+n)/U(m) \times U(n)$



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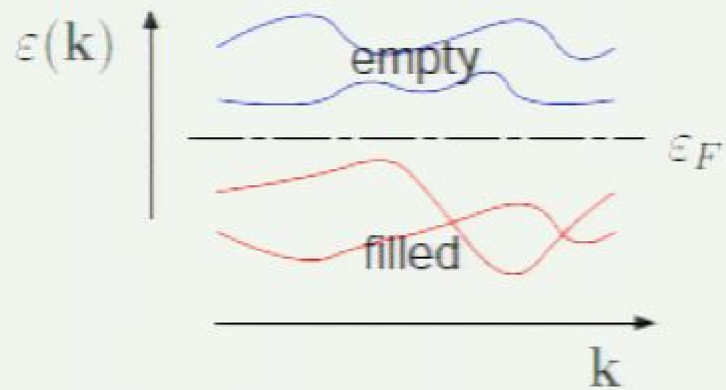

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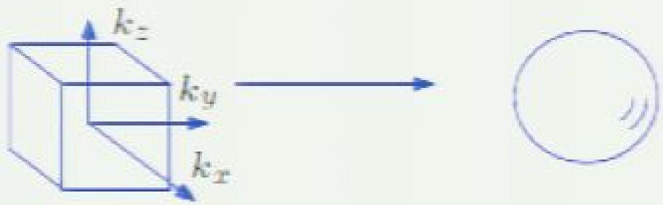

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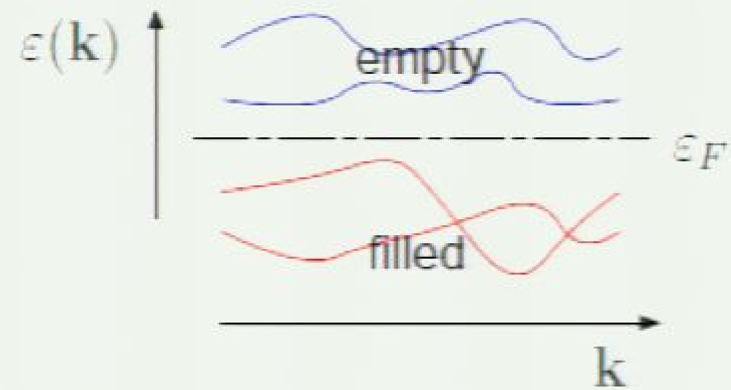

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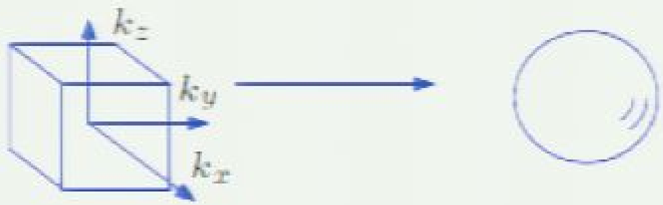

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AI	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\cdots$
BDI	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\cdots$
D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\cdots$
DIII	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\cdots$
AII	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\cdots$
CII	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\cdots$
C	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	$\cdots$
CI	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	$\cdots$

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AIII	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\cdots$
AI	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\cdots$
BDI	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\cdots$
D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\cdots$
DIII	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\cdots$
AII	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\cdots$
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C	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	$\cdots$
CI	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	$\cdots$

symmetry classes of quadratic fermionic
Hamiltonians (Altland-Zirnbauer)

$\mathbb{Z}$ integer classification

$\mathbb{Z}_2$ Z2 classification

0 no top. ins./SC

momentum space topology

-projectors in classes AIII

chiral symmetry $\Gamma\mathcal{H}\Gamma = -\mathcal{H}$ $\longrightarrow$ $Q(k) = \begin{pmatrix} 0 & q(k) \\ q^\dagger(k) & 0 \end{pmatrix}$

q : BZ $\longrightarrow$ $U(m)$

$\pi_3[U(m)] = \mathbf{Z}$ $\longrightarrow$ topological insulators labeled by an integer

$$\boxed{\nu = \int_{\mathrm{Bz}} \frac{1}{24\pi^2} \mathrm{tr}\left[(q^{-1}dq)^3\right]}$$

-discrete symmetries limit possible values of nu

$q^T(-k) = -q(k)$	DIII	AIII & DIII	$\nu \in \mathbf{Z}$
$q^T(-k) = q(k)$	CI	CI	$\nu \in 2\mathbf{Z}$
$q^*(-k) = q(k)$	BDI	CII & BDI	$\nu = 0$
$i\sigma_y q^*(-k)(-i\sigma_y) = -q(k)$	CII	Z2 insulators in CII (later)	

AZ\d	0	1	2	3	4	5	6	7	8	9
A	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	$\cdots$
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C	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	$\cdots$
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SR, Schnyder, Furusaki, Ludwig (for d=1,2,3, 2008)

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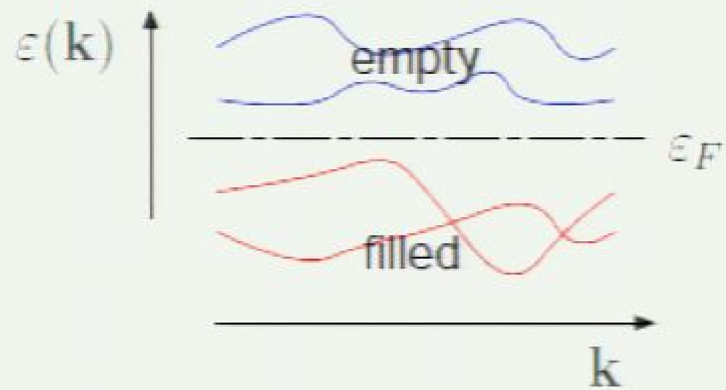
momentum space topology

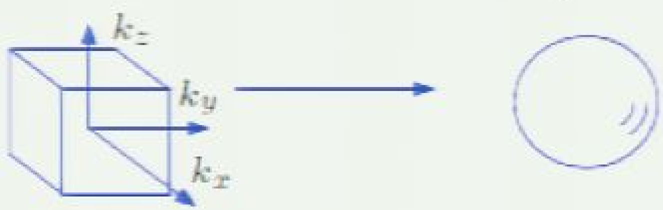
projector: $$Q(k) = 2 \sum_{a\in\text{filled}} |u_a(k)\rangle\langle u_a(k)| - 1$$



$$Q^2 = 1,\ Q^\dagger = Q,\ \mathrm{tr}\, Q = m - n$$

filled empty

Q : BZ $\longrightarrow$ $U(m+n)/U(m)\times U(n)$



$\boxed{\text{quantum ground state = map from Bz onto Grassmannian}}$

$\pi_2[U(m+n)/U(m)\times U(n)] = \mathbf{Z}$ $\longrightarrow$ IQHE in 2D

$\pi_3[U(m+n)/U(m)\times U(n)] = 0$

$\longrightarrow$ no top. insulator in 3D without constraint (Class A)

(for large enough m,n)

momentum space topology

Berry gauge field (k-space gauge field)

$$\mathcal{A}^{\hat{a}\hat{b}}(k) = A^{\hat{a}\hat{b}}_{\mu}(k)\mathrm{d}k_{\mu} = \langle u^{-}_{\hat{a}}(k)|\mathrm{d}u^{-}_{\hat{b}}(k)\rangle, \quad \mu = 1, \cdots, d, \quad \hat{a}, \hat{b} = 1, \cdots, N_{-},$$

$$\mathcal{F}^{\hat{a}\hat{b}}(k) = \mathrm{d}\mathcal{A}^{\hat{a}\hat{b}} + \left(\mathcal{A}^{2}\right)^{\hat{a}\hat{b}} = \frac{1}{2}F^{\hat{a}\hat{b}}_{\mu\nu}(k)\mathrm{d}k_{\mu}\wedge\mathrm{d}k_{\nu}.$$

topological invariant in d=2

$$\mathrm{Ch}_1[\mathcal{F}] = \frac{\mathrm{i}}{2\pi}\int_{\mathrm{BZ}^{d=2}} \mathrm{tr}\,(\mathcal{F}) = \frac{\mathrm{i}}{2\pi}\int \mathrm{d}^2k\,\mathrm{tr}\,(F_{12})$$

$$\boxed{\sigma_{xy} = \frac{e^2}{h} \times \mathrm{Ch}_1[\mathcal{F}]}$$

Thouless, Kohmoto, Naitingale, den Nijs (1982)

momentum space topology

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spatial dimensions

presence/absence
of topological band structure

AZ\d	0	1	2	3	4	5	6	7	8	9
A	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	$\cdots$
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symmetry classes of quadratic fermionic
Hamiltonians (Altland-Zirnbauer)

$\mathbb{Z}$ integer classification

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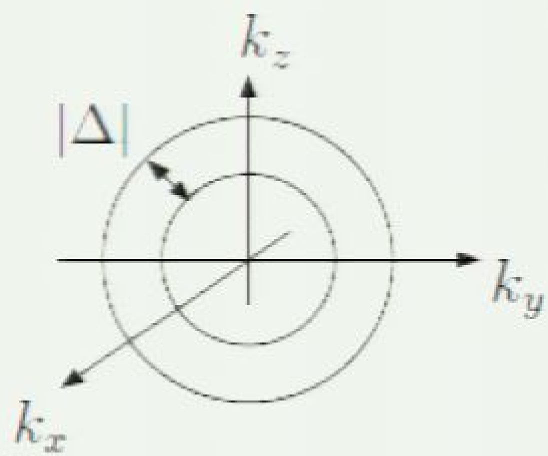
3He B is a topological "superconductor" in class DIII

$$H = \frac{1}{2}\int d^3 r \Psi^\dagger \mathcal{H} \Psi \quad \mathcal{H} = \begin{pmatrix} \xi & \Delta \\ \Delta^\dagger & -\xi \end{pmatrix}$$

$$\xi_{\boldsymbol{k}} = \frac{\boldsymbol{k}^2}{2m} - \mu \qquad \Delta_{\boldsymbol{k}} = |\Delta| i\sigma_y \boldsymbol{k}\cdot\boldsymbol{\sigma}$$



isotropic gap

strong pairing | weak pairing

μ

$\nu = 0$ 0 $\nu = 1$

stable surface Majorana fermion

3d analogue of Moore-Read state

$$\Psi(\{\boldsymbol{r}_i\},\{\sigma_i\}) \sim \mathrm{Pf}\left(\frac{[(\boldsymbol{r}_i - \boldsymbol{r}_j)\cdot i\boldsymbol{\sigma}\sigma_y]_{\sigma_i\sigma_j}}{|\boldsymbol{r}_i - \boldsymbol{r}_i|^3}\right)$$

summary of results and future issues

complete classification of topological phases in free fermion systems in all dimensions and symmetry classes

some predictions:

- surface of 3d Z2 topological insulator: perfect metal

- 3He B is identified as a topological SC:

  stable gapless Majorana surface mode

- there are topological singlet SC with good T and in d=3 spatial dimensions

a big open issue: interactions

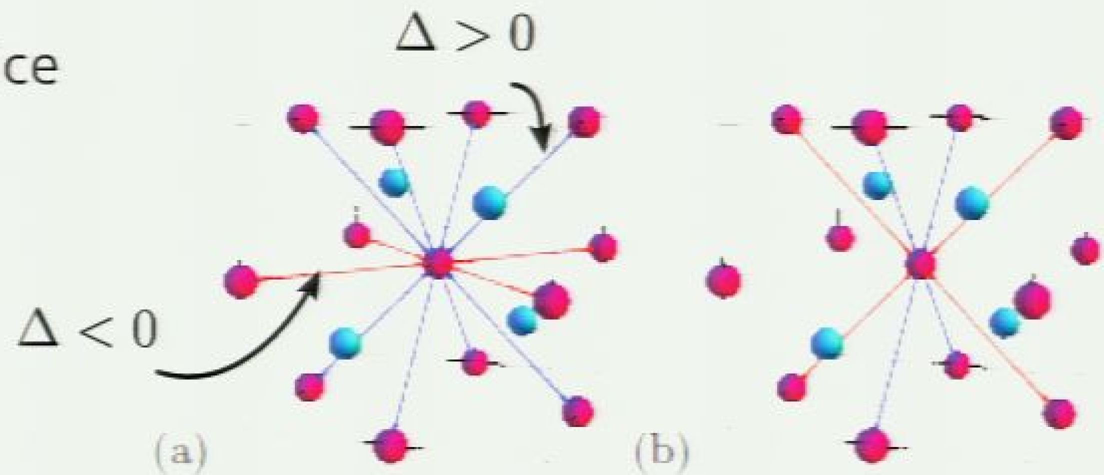
- do non-interacting topological phases survive interactions ?
- can topological phases arise solely due to interactions ?
- is there "fractional" topological insulators/superconductors ?
- is there a topological classification for bosonic systems (e.g., spin systems)

topological singlet superconductor in 3 dimensions

d-wave SC on the diamond lattice

(nn, nnn hopping + pairing)

$$H = \sum_{ij}\sum_{s=\uparrow,\downarrow} t_{ij} c_{is}^{\dagger} c_{js} + \sum_{ij} \Delta_{ij} c_{i\uparrow}^{\dagger} c_{j\downarrow}^{\dagger} + \mathrm{h.c.}$$

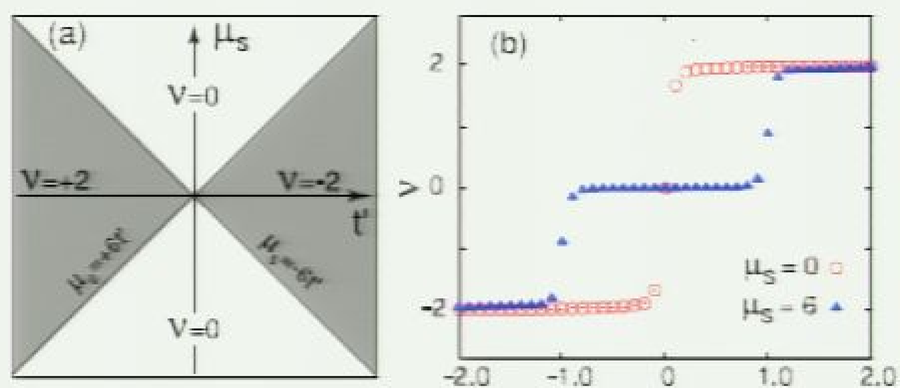


pairing

nnn hopping

phase diagram

topological invariant



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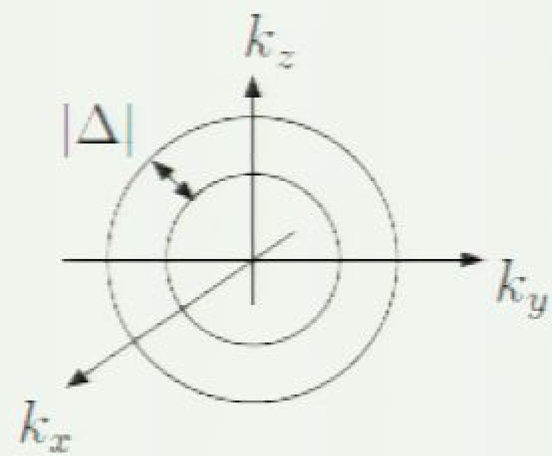

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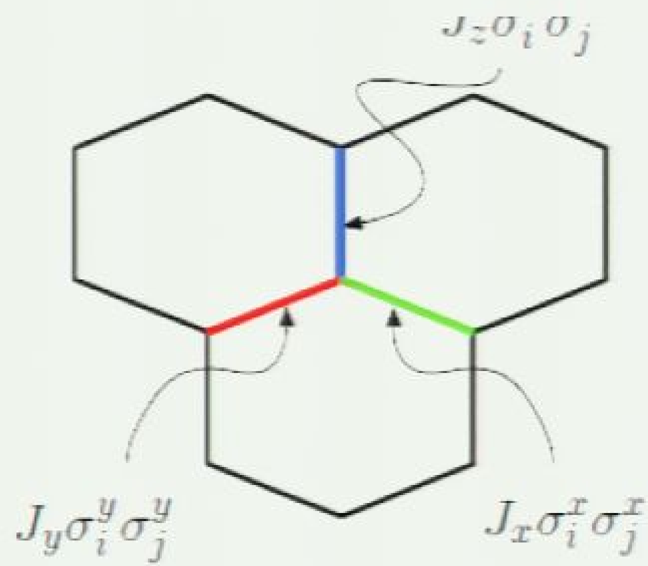
A. Kitaev, Ann. Phys. (2005)

$$H = \sum_{\mu=1}^{3} J_\mu \sum_{i,j} \sigma_i^\mu \sigma_j^\mu$$



- exact solvable in terms of emergent fermions

- has both Abelian and non-Abelian phases

  ~ "trivial insulator" ~ "topological insulator"

- produces many exotic behaviors

honeycomb lattice Kitaev model in 2 dimensions

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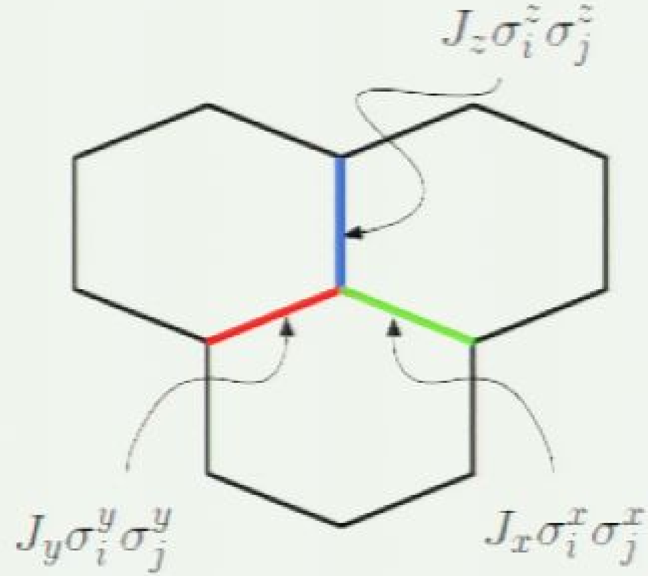

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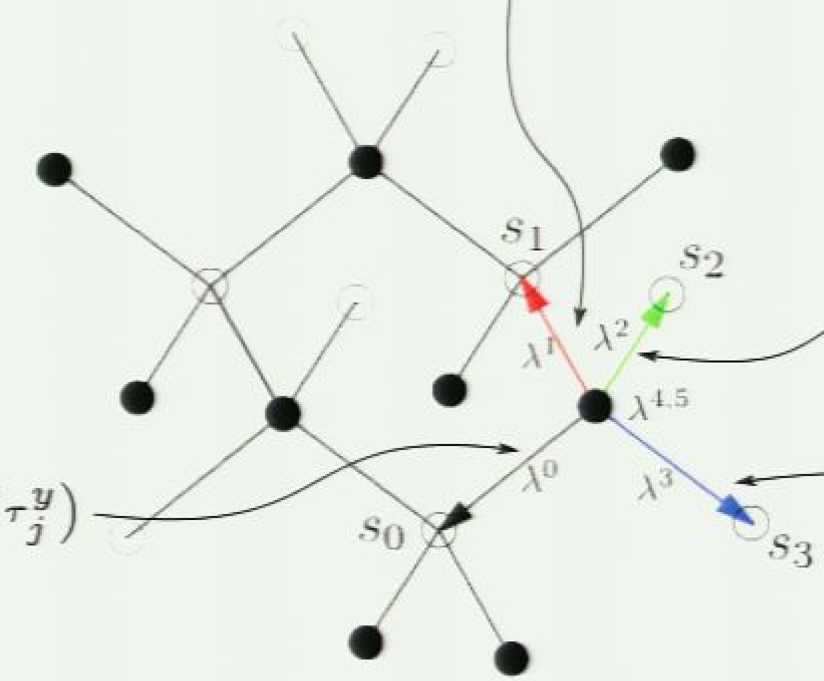
Kitaev type model on the diamond lattice

"spin-orbit" Kitaev model ("gamma matrix" Kitaev model)

$$H = -\sum_{\mu=0}^{3} J_\mu \sum_{\mu-\mathrm{links}} \sigma_i^\mu \sigma_j^\mu \left(\tau_i^x \tau_j^x + \tau_i^y \tau_j^y\right)$$

$$-J_1 \sigma_i^1 \sigma_j^1 \left(\tau_i^x \tau_j^x + \tau_i^y \tau_j^y\right)$$

$$-J_2 \sigma_i^2 \sigma_j^2 \left(\tau_i^x \tau_j^x + \tau_i^y \tau_j^y\right)$$



$$-J_2 \sigma_i^0 \sigma_j^0 \left(\tau_i^x \tau_j^x + \tau_i^y \tau_j^y\right)$$

$$-J_3 \sigma_i^3 \sigma_j^3 \left(\tau_i^x \tau_j^x + \tau_i^y \tau_j^y\right)$$

gamma matrices

$$R: \quad R \begin{pmatrix} \tau^x \\ \tau^y \\ \tau^z \end{pmatrix} R^{-1} = \begin{pmatrix} \tau^z \\ \tau^y \\ -\tau^x \end{pmatrix}$$

pi/2 rotation around tau^y

$$T: \quad T\vec{\tau}T^{-1} = -\vec{\tau}, \quad T\vec{\sigma}T^{-1} = -\vec{\sigma} \qquad TiT^{-1} = -i$$

"time-reversal"

Dirac rep. gamma matrices

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} = \vec{\sigma} \otimes \tau^x$$

$$\alpha^0 = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix} = 1_2 \otimes \tau^z$$

Weyl rep. gamma matrices

$$\vec{\zeta} = \begin{pmatrix} -\vec{\sigma} & 0 \\ 0 & +\vec{\sigma} \end{pmatrix} = -\vec{\sigma} \otimes \tau^z$$

$$\zeta^0 = \begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix} = 1_2 \otimes \tau^x$$

$$\{\alpha^\mu, \alpha^\nu\} = \{\zeta^\mu, \zeta^\nu\} = 2\delta^{\mu\nu}$$

$$H = -\sum_{\mu=0}^{3} J_\mu \sum_{i,j} \left(\alpha_i^\mu \alpha_j^\mu + \zeta_i^\mu \zeta_j^\mu\right)$$

solution through emergent Majorana fermions

introduce six Majorana fermions $\lambda^{0,1,2,3,4,5}$ $\lambda^{a\dagger} = \lambda^a$ $\lambda^{a2} = 1$

4 dim. Hilbert space

constraint: $D := i\lambda^0\lambda^1\lambda^2\lambda^3\lambda^4\lambda^5 = 1$ 8 dim. Hilbert space

$$\alpha^\mu = i\lambda^\mu\lambda^4 \quad \zeta^\mu = i\lambda^\mu\lambda^5 \quad u_{ij} = i\lambda_i^{\mu_{ij}}\lambda_j^{\mu_{ij}}, \quad \mu_{ij} = 0,1,2,3$$

$$H = i\sum_{\mu=0}^{3} J_\mu \sum_{i,j} u_{i,j}\left(\lambda_i^4\lambda_j^4 + \lambda_i^5\lambda_j^5\right)$$

$$\boxed{[H, u_{jk}] = 0 \quad u_{jk}^2 = 0 \Rightarrow u_{jk} = \pm 1}$$

(i) pick up a configuration for u (by Lieb theorem)

(ii) solve auxiliary Majorana hopping problem

(iii) projection

$$|\Psi\rangle = \prod_j \left(\frac{1+D_j}{2}\right)|\Psi\rangle$$

symmetry and topology of Majorana hopping problem

$H = \lambda^T X \lambda$

symmetry class DIII

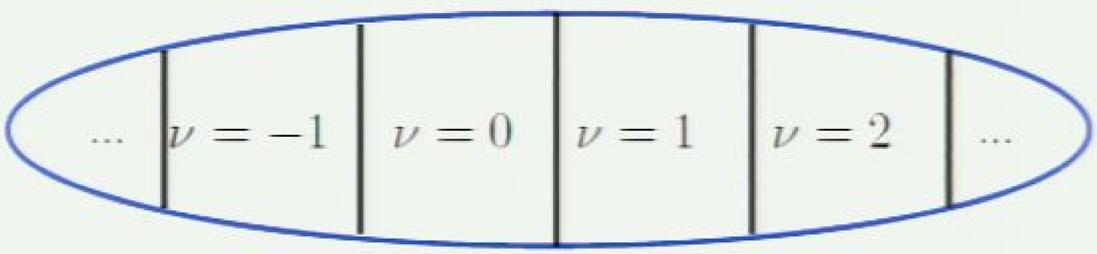
(i) $X^T = -X$ Majorana condition

$\lambda^{4,5}$: pseudo spin

(ii) $is_y X^*(-is_y) = X$ "time-reversal"

$(TR)H(TR)^{-1} = H$

space of all quantum ground state of Majorana hopping problem
with "time-reversal symmetry" is partitioned into different topological classes



topological invariant

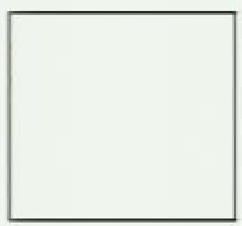
$$\boxed{\nu = \int_{\mathrm{Bz}} \frac{1}{24\pi^2}\epsilon^{\mu\nu\rho}\mathrm{tr}\left[(q^{-1}\partial_\mu q)(q^{-1}\partial_\nu q)(q^{-1}\partial_\rho q)\right]}$$

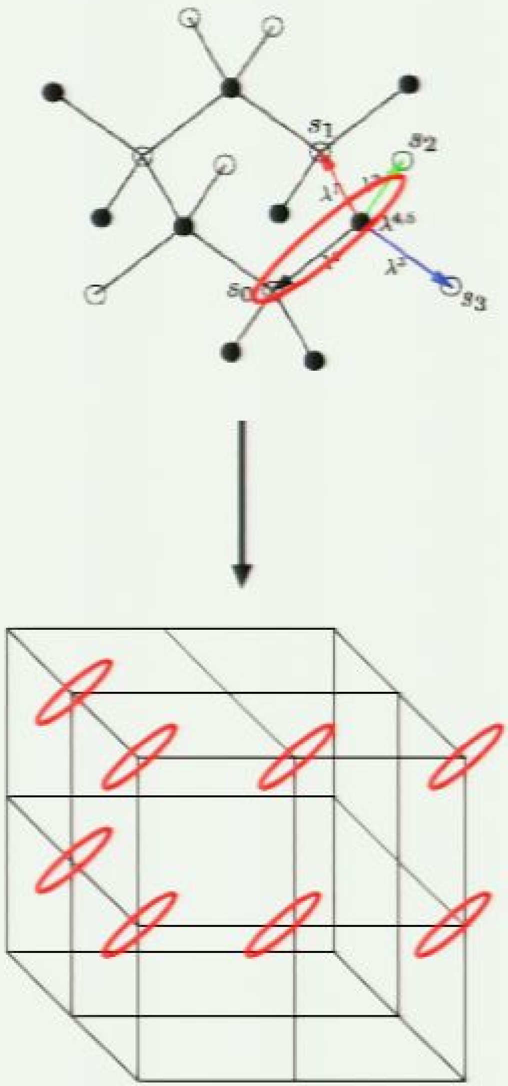
$q(k)$: spectral projector

strong pairing phase (trivial phase) $J_0 \gg J_{1,2,3}$

$$H_{\text{eff}} = g \sum_{p}^{\text{plaquette}} F_p$$

$$F_p = (\sigma^x \tau^z)_r (\sigma^y \tau^x)_{r+x} (\sigma^z \tau^y)_{r+x+y} (\sigma^0 \tau^0)_{r+y}$$





and cyclic permutations

Z2 lattice gauge theory like model
on cubic lattice

similar models: Levin-Wen (03) Hamma-Zanardi-Wen (05)

summary and outlook

- constructed a new vacuum of bosonic systems together with a Hamiltonian which happens to have it as an exact ground state.

- lessons to be learned from the model:

  fermions can emerge from purely bosonic model

  wavefunction for bosons can be obtained from fermionic wfn through projection

  c.f. variational approach

- experimental realization ?

- excitations ?