Title: Statistical Mechanics (PHYS 602) - Lecture 12

Date: Oct 14, 2009 10:30 AM

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Abstract:

Dynamics of bosons

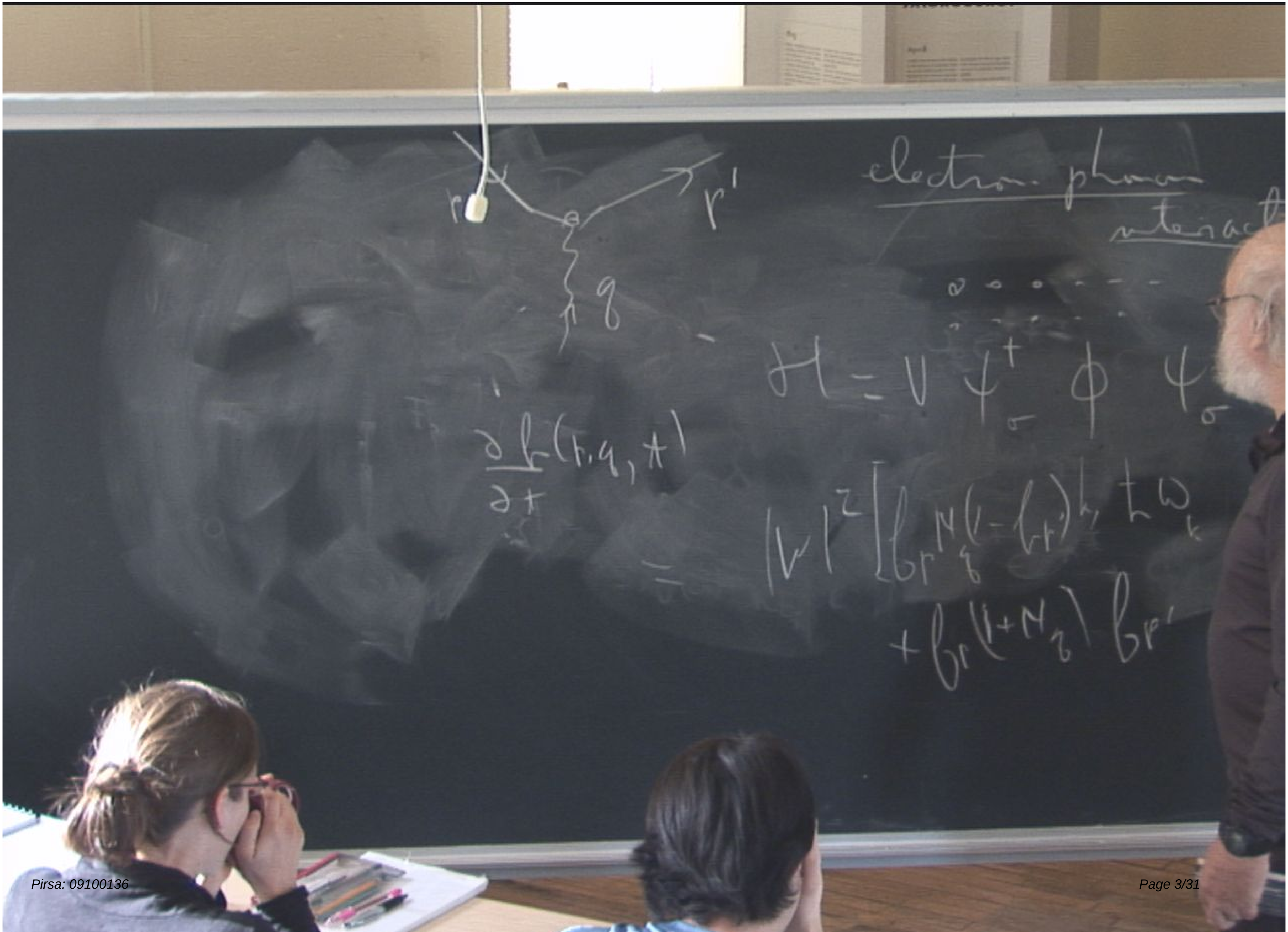
Some part of the story of bosons is much the same. A low temperature conserved boson system could be expected to obey the same sort of equation, under circumstances in which the bosons were conserved, and also the emission and absorption of phonons were not too significant.
Specifically, the equation would look like

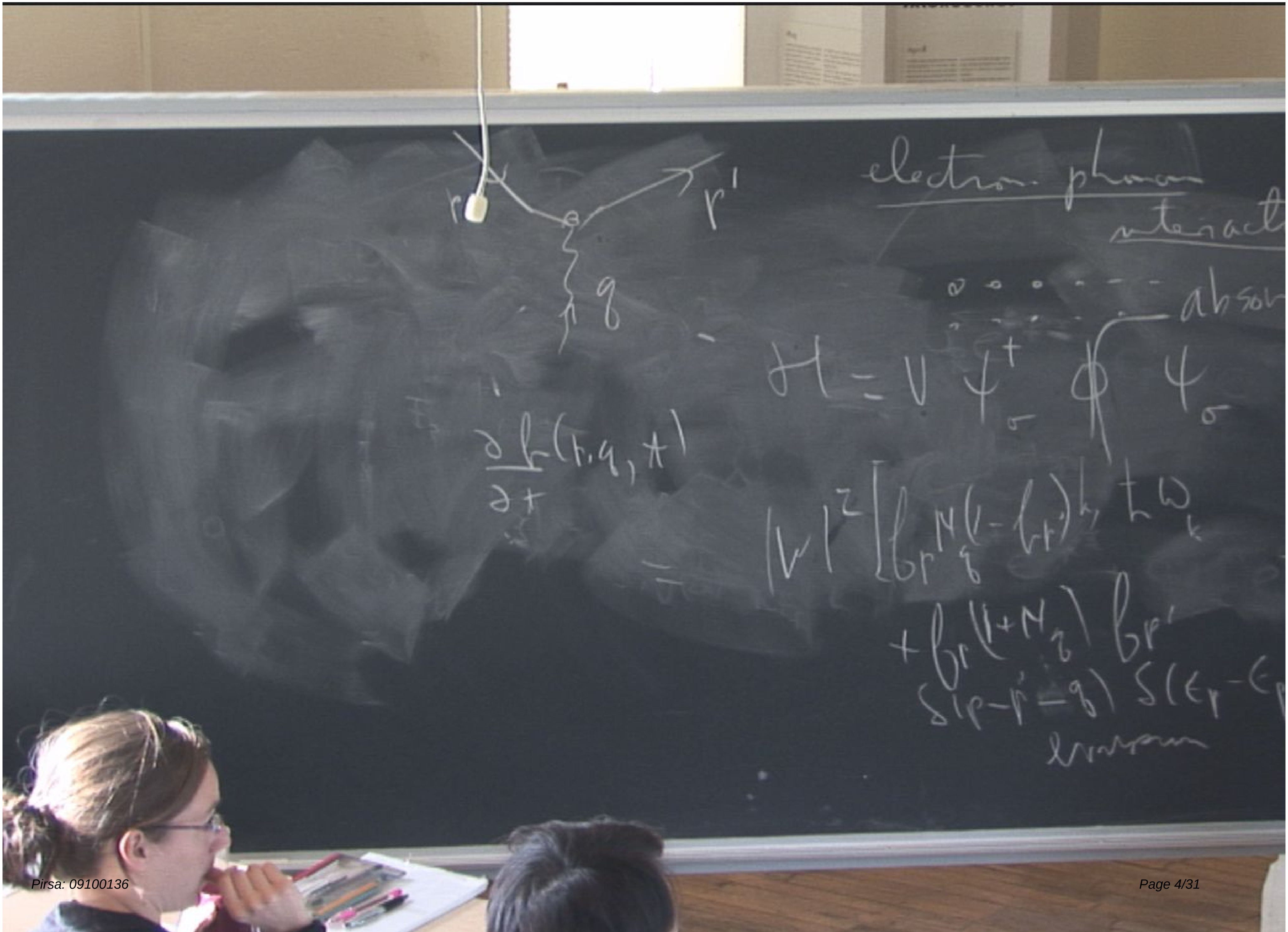
$$[\partial_t + (\nabla_{\mathbf{p}}\varepsilon)\cdot\nabla_{\mathbf{r}} - (\nabla_{\mathbf{r}}\,\varepsilon)\cdot\nabla_{\mathbf{p}}]\, f(\mathbf{p}) =$$

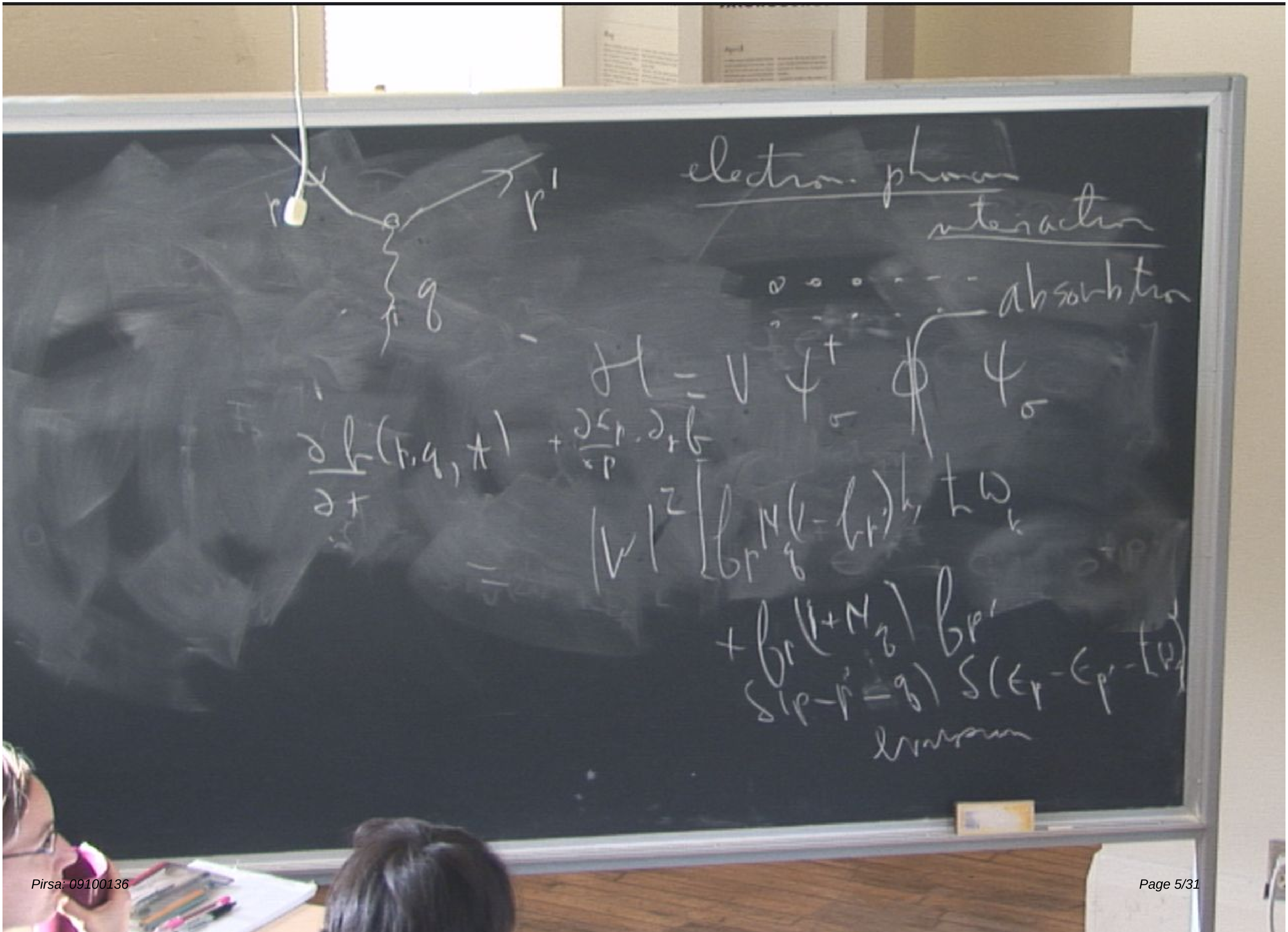
$$- \iiint d\mathbf{q}\ d\mathbf{p}'\ d\mathbf{q}'\ \delta(\mathbf{p}+\mathbf{q} - \mathbf{p}'-\mathbf{q}')\ \delta(\varepsilon(\mathbf{p})+ \varepsilon(\mathbf{q}) - \varepsilon(\mathbf{p}')-\varepsilon(\mathbf{q}'))$$

$$Q(\mathbf{p},\mathbf{q} \to \mathbf{p}',\mathbf{q}')\ [f(\mathbf{p})\ f(\mathbf{q})(1+ f(\mathbf{p}'))\ (1+ f(\mathbf{q}')) - f(\mathbf{p}')\ f(\mathbf{q}')\ (1+f(\mathbf{p}))\ (1+f(\mathbf{p}))\ ] \quad \text{vi.9}$$

Once again the new feature is shown in red. In the scattering events there are, for bosons, more scattering when the final single particle states are occupied than when they are empty. One says that fermions are unfriendly but bosons are gregarious (or at least attractive to their own tribe.). The two terms in the $1+f$ structure are described respectively as spontaneous and stimulated emission and brought in by Einstein, to make the bose dynamical equation have the right local equilibrium behavior. The logic used by Einstein includes the fact that for local equilibrium via equation vi.9, we must have $f/(1+f)$ be, as is $f/(1-f)$ in the fermion case, an exponential in conserved quantities and this result agrees with the known statistical mechanical result of equation vi.3.


electron-phonon


electron phonon
interact


electron. phonon
interaction
absorption

Landau's equation for low temperature fermion systems:

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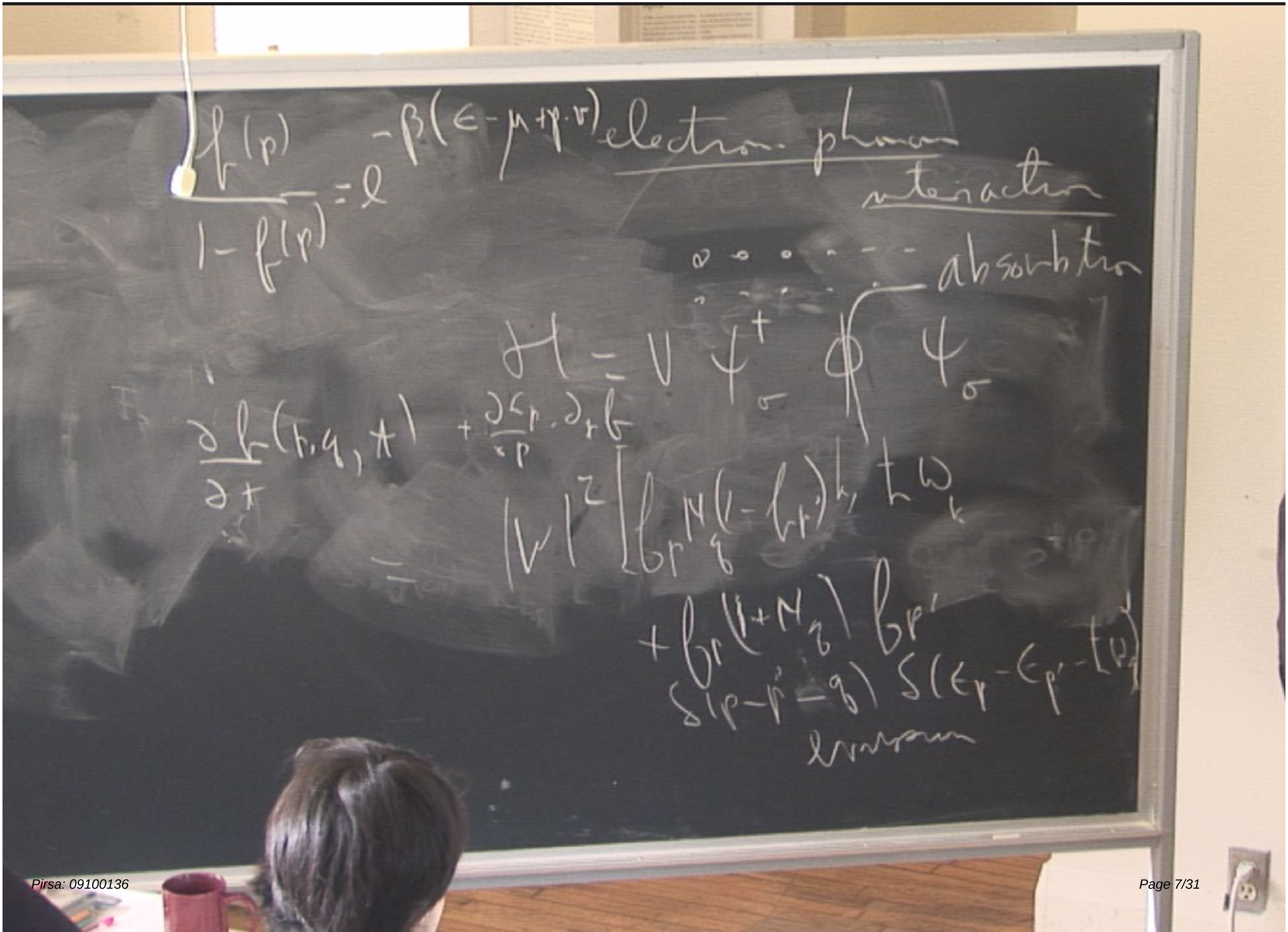
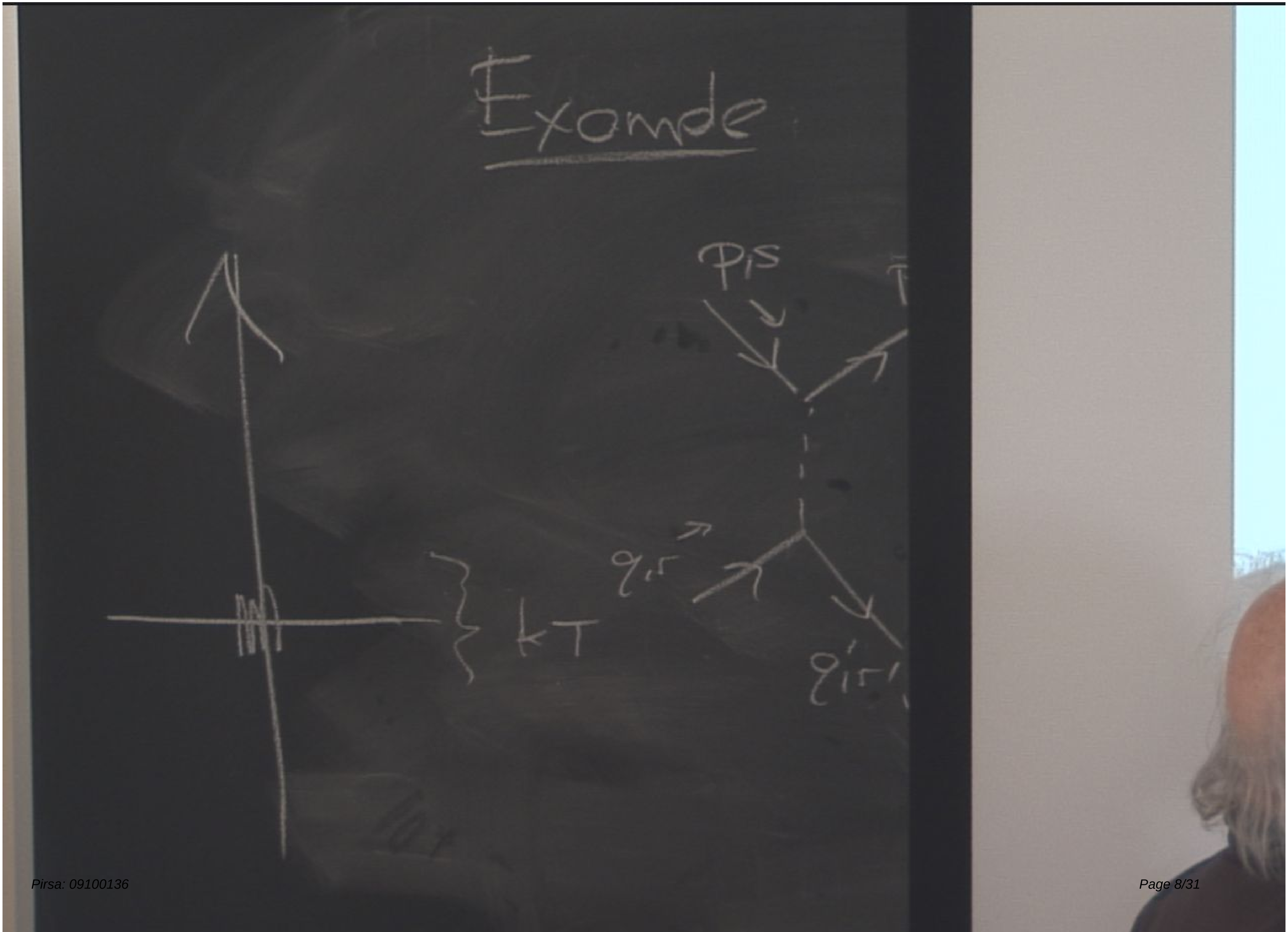
We can do just about everything with this equation that Boltzmann did with his more classical result. For example this equation also has an *H* theorem with *H* being an integral of $f \ln f + (1-f) \ln(1-f)$.

An important difference is that this equation gives us a particularly low scattering rate at low temperatures. Only modes with energies within kT of the fermi surface can participate in the scattering. As a result, the scattering rate ends up being proportional to T^2 at low temperatures.

Probably the most important result is that there is a local equilibrium solution of the right form, with $f/(1-f)$ equal to a linear combination of exponentials of conserved quantities, i.e. $\exp\{-\beta[\varepsilon(\mathbf{p})-\mu-\mathbf{p}\cdot\mathbf{v}- v^2/(2m)]\}$. this gives us
$f(\mathbf{p})=1/\ (1 + \exp\{\beta[\varepsilon(\mathbf{p})-\mu-\mathbf{p}\cdot\mathbf{v}- v^2/(2m)]\}.)$ as we knew it should be.

This approach gives us a piece of a theory of He^3, the fermion form of helium. To complete the theory one should also consider the emission and absorption of phonons, i.e. sound wave excitations.


f(p) / (1 - f(p)) = e^{-β(ε - μ + p·v)}
electron-phonon interaction
absorption
H = V ψ†_σ φ ψ_σ
∂f(r, q, t)/∂t + ∂ε_p/∂p · ∂_r f
|V|^2 [f_p N_q (1 - f_p') ħω_k
+ f_p (1 + N_q) f_p'
δ(p - p' - q) δ(ε_p - ε_p' - ħω_q)


Example
p, s
kT
q, r
q', r'

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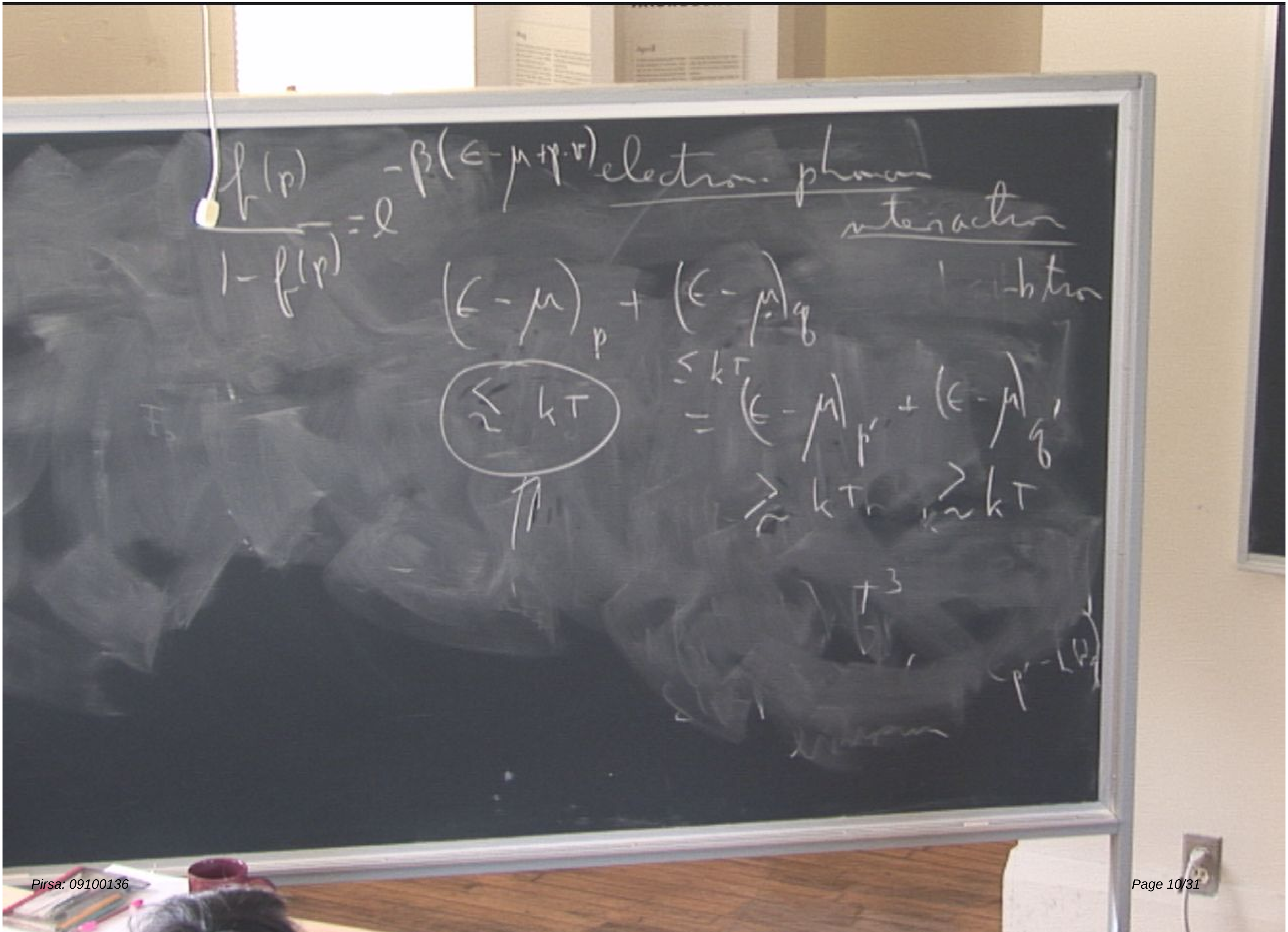
electron-phonon
interaction

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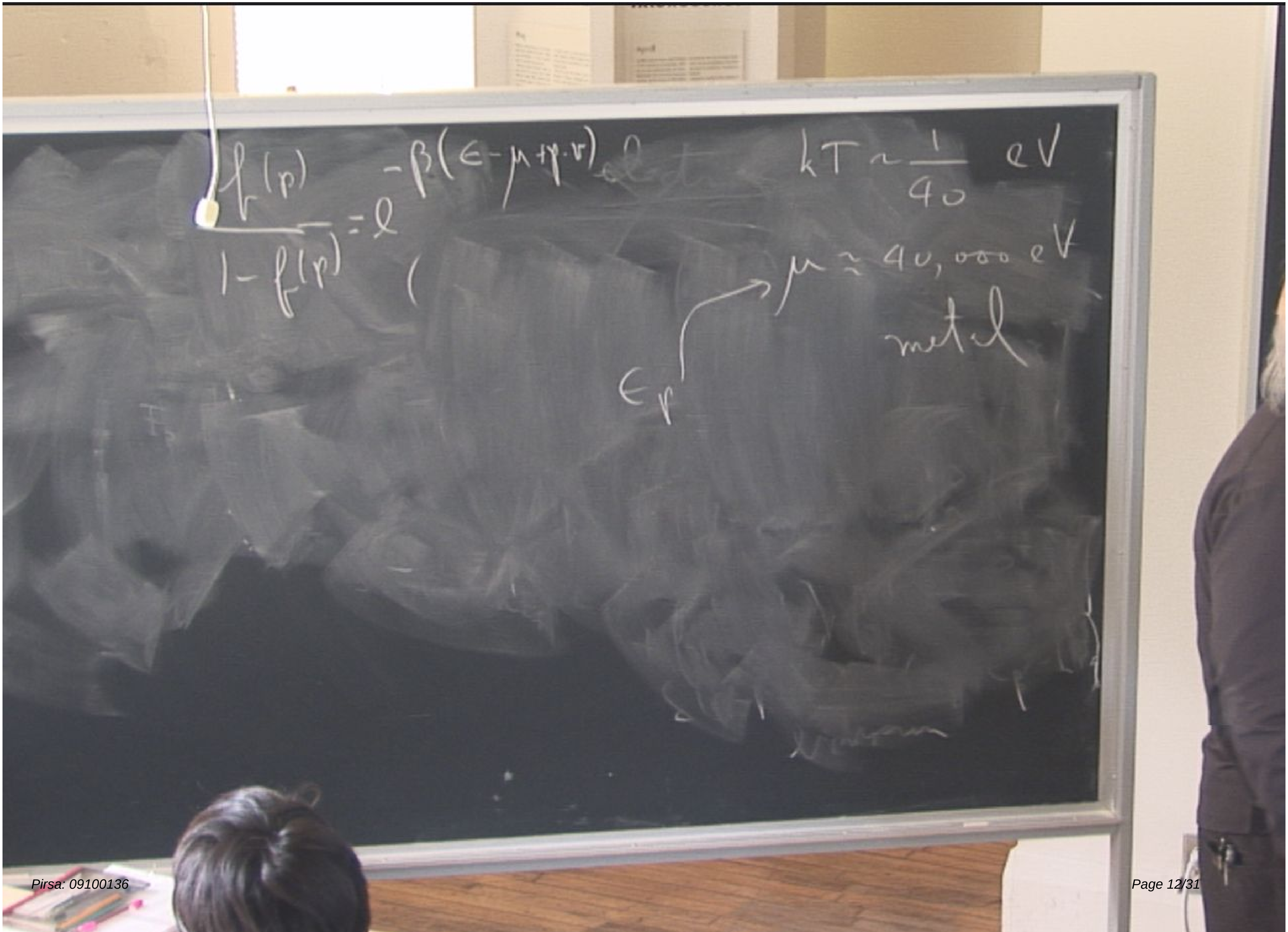
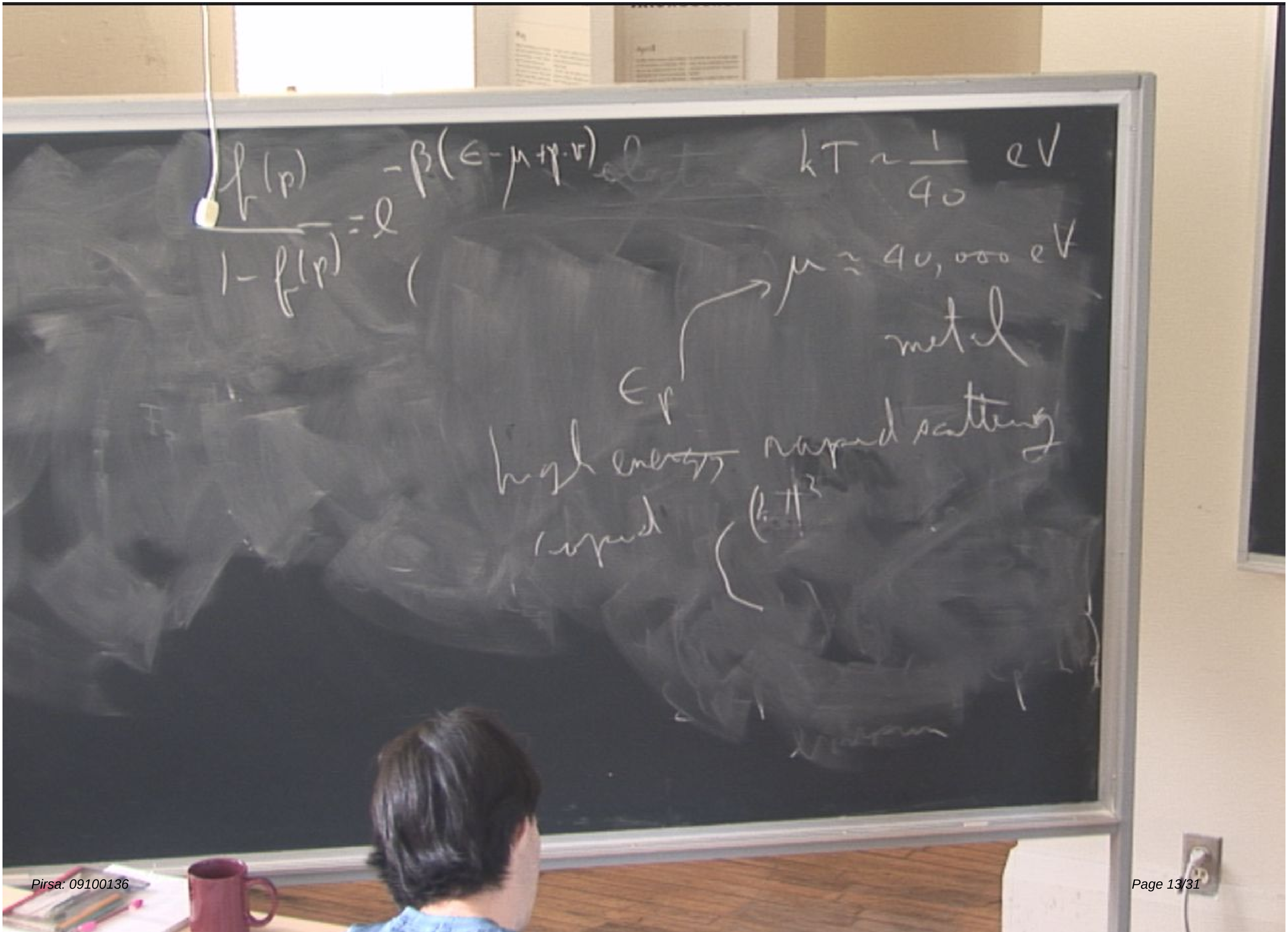
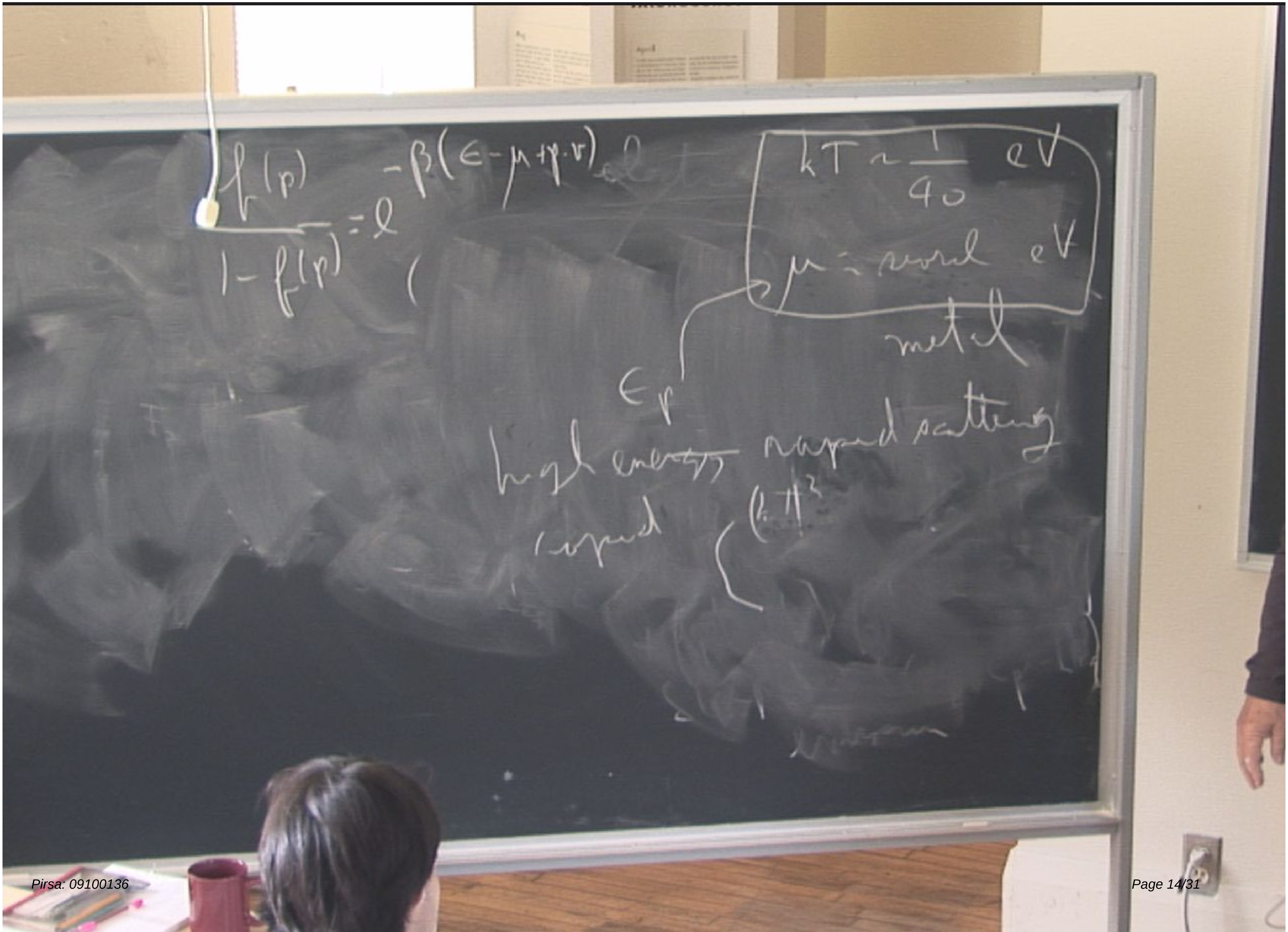

f(p) / (1 − f(p)) = e^{−β(ε − μ + p·v)}
kT ~ 1/40 eV
μ ≈ 40,000 eV
metal
ε_F


kT ~ 1/40 eV
μ ≈ 40,000 eV
metal
high energy
rapid scattering


kT ~ 1/40 eV
eV
metal
high energy
scattering

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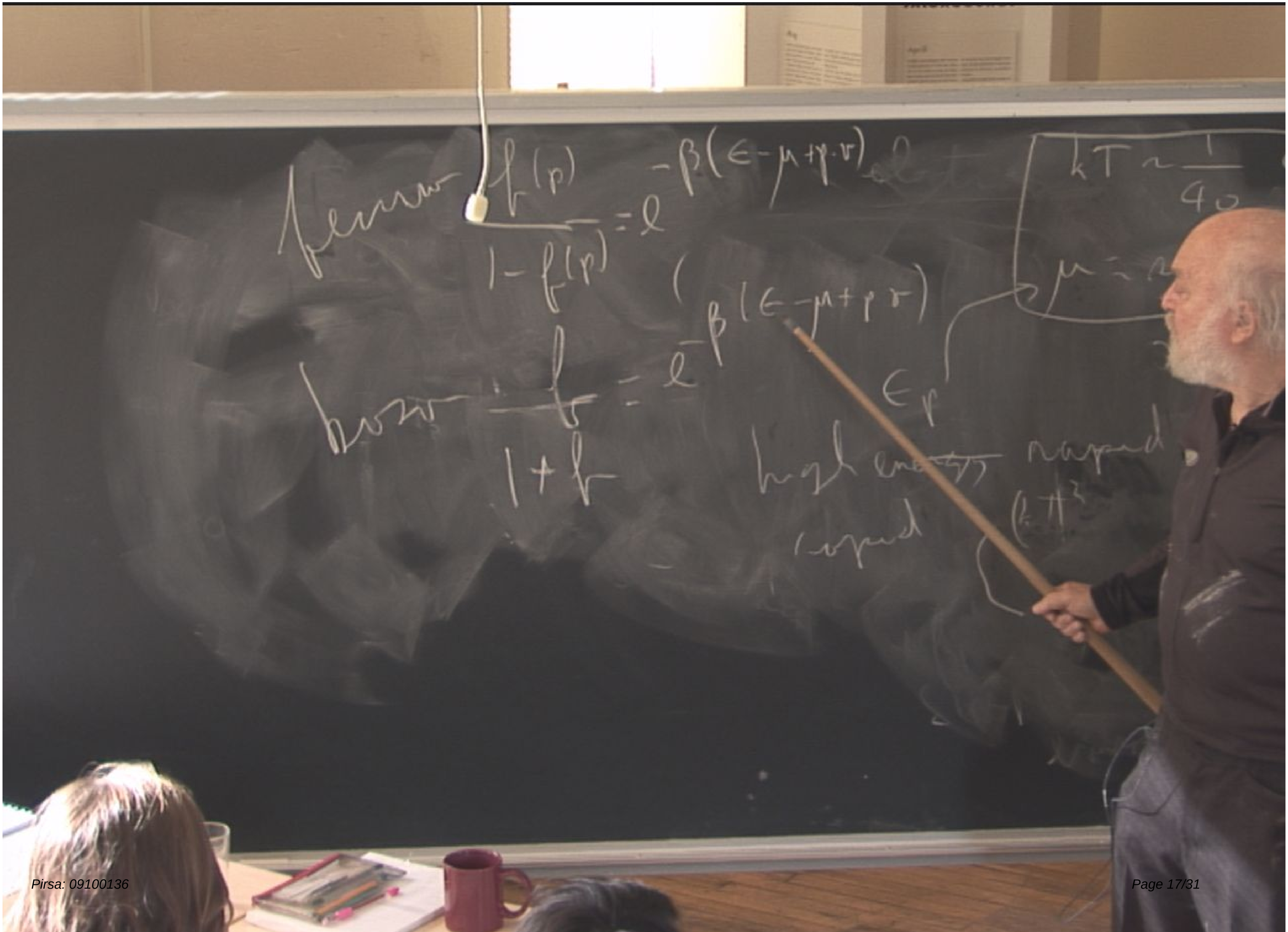

References

Daniel Kleppner., "Rereading Einstein on Radiation", *Physics Today,* (February 2005).

A Einstein, *Phys. Z.* **18** 121 (1917). English translation, D. ter Haar, *The Old Quantum Theory,* Pergamon Press, New York, p. 167 (1967).

More is the Same
Phase Transitions and Mean Field Theories

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Abstract

This talk summarizes concepts derived from the study of phase transitions mostly within condensed matter physics. In its original form, the talk was aimed equally at condensed matter physicists and philosophers of science. The latter group are particularly interested in the logical structure of science. This talk bears some traces of its history. The key technical ideas go under the names of ``singularity'', ``order parameter'', ``mean field theory'', ``variational method'', and ``correlation length''. The key ideas here go under the names of "mean field theory", "phase transitions", "universality", "variational method", and "scaling".

Phase Transitions Describe Basic Physics

Ideas derived from phase transitions and other intellectual products of condensed matter physics are crucial for many branches of modern science and for modern theoretical physics. The basic ideas include:

a. Phase transitions always require an infinite number of degrees of freedom. This infinity may come either from something that happens over an infinite period of time, over an infinite amount of space, or via the development of some sort of infinite complexity within the system.

b. The structure of space and time are important determinants of what we see happen. The dimensions of space matter as do whether the system in question repeats itself infinitely often. The topology of the surrounding space and of created structure are quite important.

c. Condensed matter systems provide a good area for study because they provide an observable platform for amazing diverse and rich phenomena, well beyond the untutored imagination of scientists.

Gibbs: A phase transition is a singularity in thermodynamic behavior.

This occurs only in an infinite system

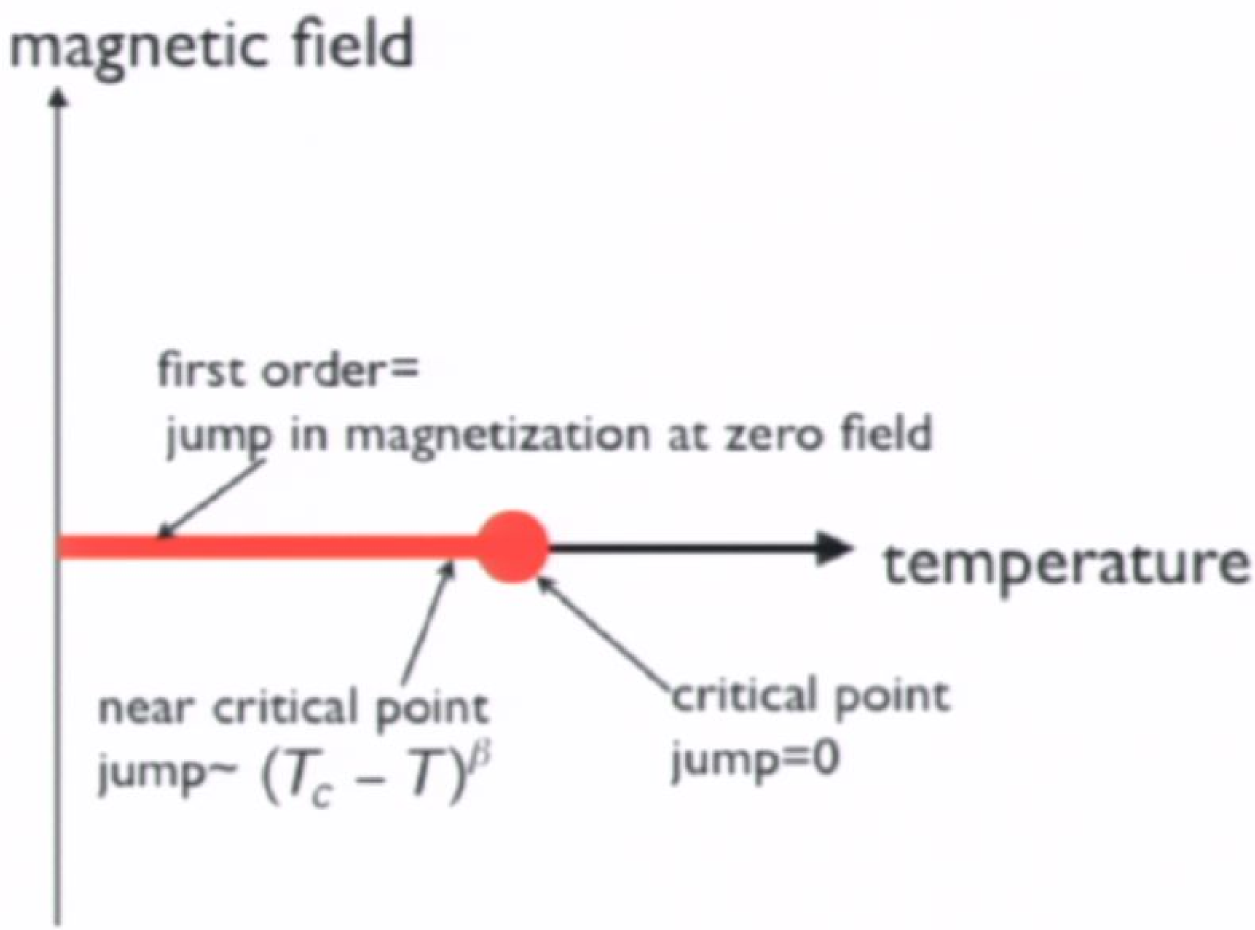
Ehrenfest:

- First order= discontinuous jump in thermodynamic quantities.

- Second order has continuous thermodynamic quantities, but infinity in derivative of thermodynamic quantities.

- ....

Magnetic Phase Diagram



Phase Transition is a change from one behavior to another

A first order phase transition involves a
liscontinuous jump in some statistical variable.
'he discontinuous property is called the order
arameter. Each phase transition has its own
rder parameter. The possible order
arameters range over a tremendous variety of
hysical properties. These properties include
he density of a liquid-gas transition, the
nagnetization in a ferromagnet, the size of a
onnected cluster in a percolation transition,
nd a condensate wave function in a superfluid
r superconductor. A continuous transition
ccurs when the discontinuity in the jump
pproaches zero. This section is about the
levelopment of mean field theory as a basis for
partial understanding of phase transition
henomena

http://blogs.trb.com/news/local/longisland/politics/blog/2008/04/

http://azahar.files.wordpress.com/2008/12/

theory starts from: phase transitions are singularities in free energy

J. Willard Gibbs:

gets to: phase transitions are a property of infinite systems

proof: .... consider Ising model for example

problem defined by

$$-H/(kT) = K \sum_{nn} \sigma_r \sigma_s + h \sum_r \sigma_r$$

free energy defined by

$$-F/(kT) = \ln \sum_{\{\sigma_r = \pm 1\}} \exp[-H\{\sigma_r\}/(kT)]$$

H is a smooth function of K and h.

Since a finite sum of exponentials of smooth functions

s a positive smooth function, it follows that

the free energy is smooth too

Mean Field Theory: more is the same

one spin

Ising model, spin, simplified atom $\sigma = \pm 1$

one spin in a magnetic field $H = -\sigma\mu B = -kT\sigma h$

statistical average: $<\sigma>= \tanh(h)$

many spins

spin in a magnetic field, dimension d $-H / kT = K\sum_{nn}\sigma_r\sigma_s + h\sum_r \sigma_r$

focus on one spin, at ***r***: that spin feels h and K $\sum_{\mathbf{s}\text{ nn to }\mathbf{r}} \sigma_{\mathbf{s}}$

more is the same

one spin

statistical average: $<\sigma>= \tanh h$

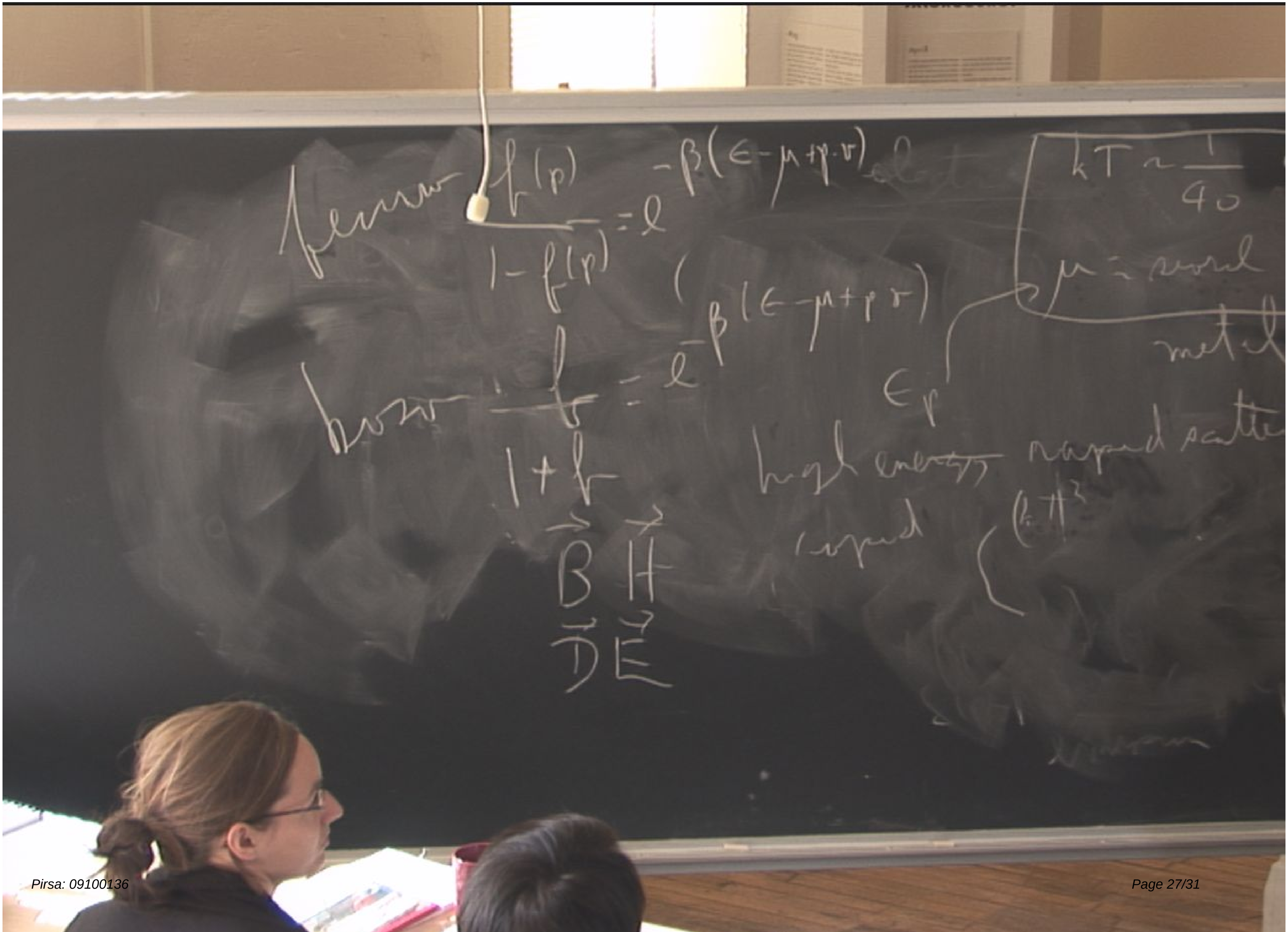
many spins

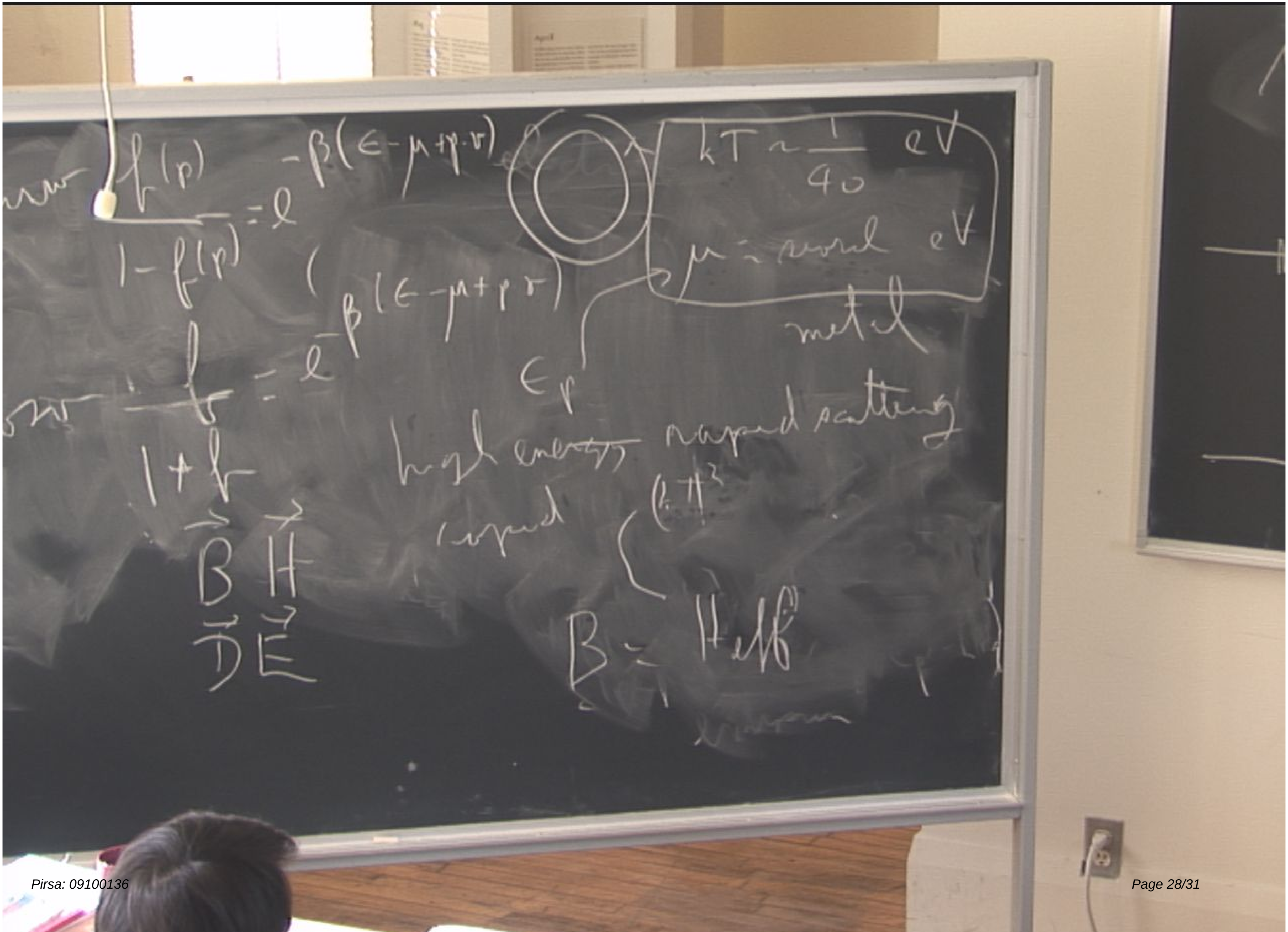
focus on one spin $\quad -H_{eff} / (kT) = \sigma_r[h_r + K\sum_s <\sigma_s>]$

statistical average: $\quad h_{eff} = [h + Kz < \sigma >]$ z=number of nn

$$< \sigma >= \tanh(h_{eff})$$

or, if there is space variation, $h_{eff} = h_{\mathbf{r}} + K \sum_{\mathbf{s}\text{ nn to }\mathbf{r}} <\sigma_{\mathbf{s}}>$


fermion
boson
high energy
metal


kT ~ 1/40 eV
metal
high energy
B H
D E

Mean Field Theory is Only Partially Right

Mean field theory says that spin moves in the average field produced by all other spins. But actual value is often larger in magnitude than mean value and fluctuates in sign. Net result is error, with unknown sign. The same ideas can be applied to lots of problems. (In particle physics mean field theory often goes with the words "one loop approximation" or "tadpole diagram".)

As we shall discuss in detail, mean field theory gives an interesting and instructive theory of phase transitions, but one which is only partially right. Near the critical point, for lower dimensional systems, including three dimensions, fluctuations dominate the system behavior and mean field theory gives the wrong answer, badly wrong. Very near first order phase transitions, fluctuations also count, but in a less obvious manner.

However in high dimensions, usually above four, mean field theory gives a good picture of phase transitions. It also has features which point the way toward the right theory. It is also simple to use

Calculate results of mean field theory

Numerical method: I calculate $\langle\sigma\rangle$ at high temperatures, small K, by using Newton's method starting from $\langle\sigma\rangle=h$. I then increase K step by step at fixed h, and find $\langle\sigma\rangle$ in each step by using the last step as the starting point for Newton's method.

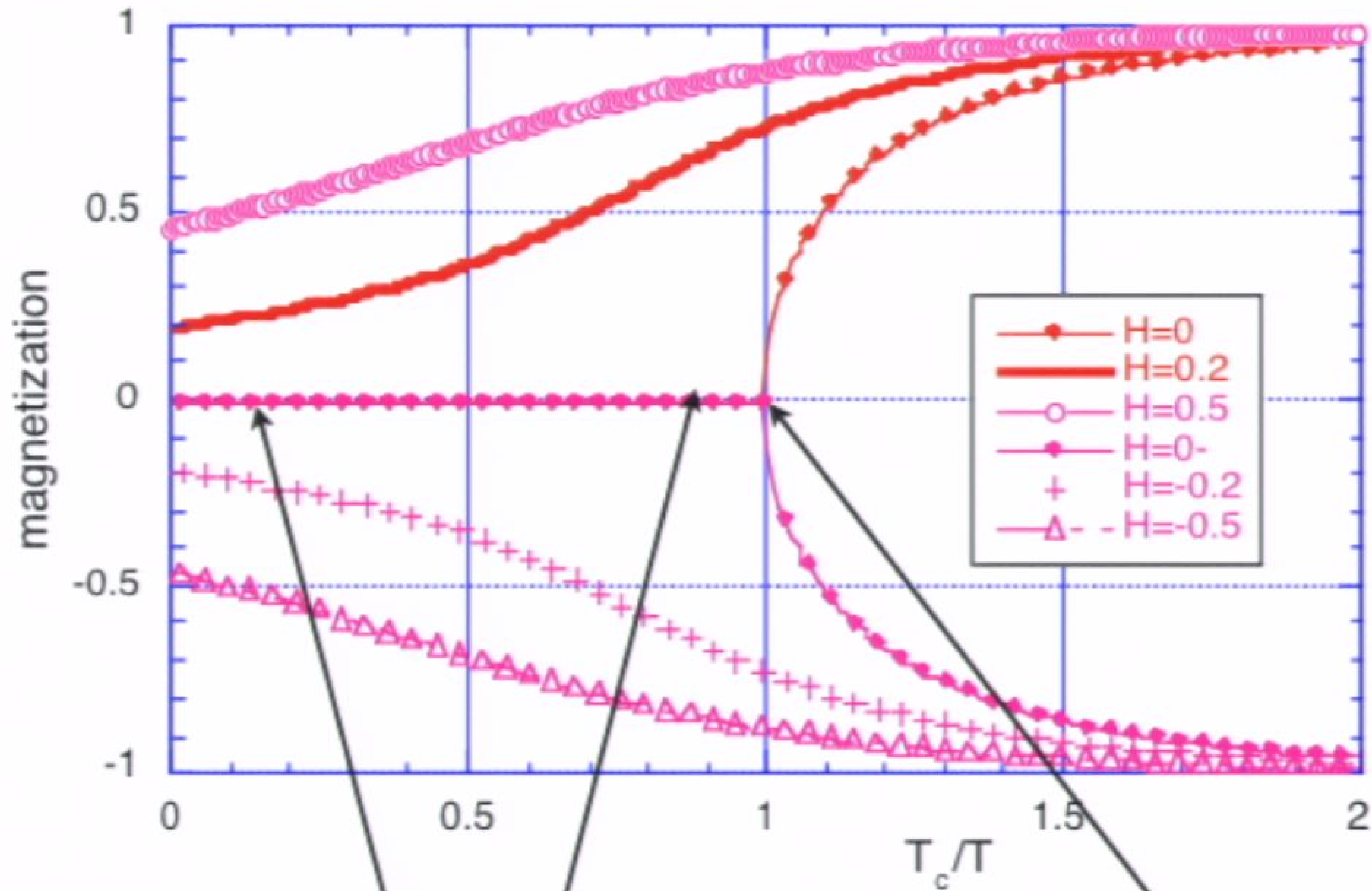
I then calculate $-\beta F$ using the fact that $\partial_h(-\beta F) = \langle\sigma\rangle$ and the known value of the free energy at high temperatures and small fields.

One of the useful results that emerges from this numerical calculation is the value of the coupling *K* which produces the first splitting in the two h=0 magnetization curves. This bifurcation occurs at Kz=1. Consequently we identify this value of K as the critical one. Since K is a physical coupling strength divided by temperature, we can write Kx in terms of the temperature and the critical temperature as

$K_z = T_c/T$

order parameter in mean field transition

legend "H" should be h



First order phase transition

critical point