Title: Statistical Mechanics (PHYS 602) - Lecture 6

Date: Oct 05, 2009 10:30 AM

URL: http://pirsa.org/09100127

Abstract:


rel
marginal
unrel.

rel.

marginal

unrel.

λ_2

rel.

marginal

irrel.

λ_2 λ_3 λ_4 λ_5

$\lambda_3 \uparrow$ rel

$\lambda_4 \uparrow$ mag

$\lambda_5 \uparrow$ irrel

rel.
marginal
irrel.

$\lambda_2 \quad \lambda_3 \quad \lambda_4 \quad \lambda_5 \ldots$

$\lambda_3 \uparrow$ rel $\quad \lambda_4 \uparrow$ mag $\quad \lambda_5 \uparrow$ irrel

$K_{nn} \quad K_{nnn}$

rel
marginal
irrel.

$K_4\ \sigma_1\ \sigma_2\ \sigma_3\ \sigma_4$

$\lambda_2 \quad \lambda_3 \quad \lambda_4 \quad \lambda_5$

$\uparrow \quad \uparrow \quad \uparrow$

rel mag irrel

$K_{nn} \quad K_{nnn} \quad K_4 + ! \ldots$

rel.

marginal

irrel.

$K_4\ \sigma_1\ \sigma_2\ \sigma_3\ \sigma_4$

$\lambda_2 \quad \lambda_3 \quad \lambda_4 \quad \lambda_5$

$\uparrow$ rel $\quad \uparrow$ mag $\quad \uparrow$ irrel

$K_{nn} \quad K_{nnn} \quad K_4 + !\ldots$

a $\quad 10^{-10}$ cm. $\qquad$ 1 cm

rel.
marginal
irrel.

$K_4 \ \sigma_1 \ \sigma_2 \ \sigma_3 \ \sigma_4$

$\lambda_2 \quad \lambda_3 \quad \lambda_4 \quad \lambda_5$

$\uparrow \quad \uparrow \quad \uparrow$

rel $\quad$ mag $\quad$ irrel

$K_{nn} \quad K_{nnn} \quad K_4 + \ldots$

a. 10^{-10} cm. $\quad$ 1 cm

neon. $\quad$ critical

linear combination

$(K_{nn} - K_{nnn} + K_4 \cdots)$

rel.

marginal

irrel.

$K_4\ \sigma_1\ \sigma_2\ \sigma_3\ \sigma_4$

$\lambda_2 \quad \lambda_3 \quad \lambda_4 \quad \lambda_5 \dots$

$\uparrow \quad \uparrow \quad \uparrow$

rel mag irrel

$K_{nn} \quad K_{nnn} \quad K_4 + ! \dots$

$| \ |$ ⟶ 1 cm

a. 10^{-10} cm.

critical

linear combinatn near.

$(K - K_c)\ (K_{nn} - K_{nnn} + K_4 \cdots)$ relevant

other linear comb ⟶ 0 irrelevant

rel.

marginal

irrel.

$K_4 \quad \sigma_1 \quad \sigma_2 \quad \sigma_3 \quad \sigma_4$

$\lambda_2 \quad \lambda_3 \quad \lambda_4 \quad \lambda_5 \ldots$

$\uparrow$ rel $\quad \uparrow$ marg $\quad \uparrow$ irrel

$K_{nn} \quad K_{nnn} \quad K_4 + \ldots$

$| \; | \longrightarrow 1$ cm

$a \quad 10^{-10}$ cm. $\quad$ near. $\quad$ critical

linear combination

$O = (K - K_c)\,(K_{nn} - K_{nnn} + K_4 \ldots)$ relevant

other linear comb $\longrightarrow$ O irrelevant rel

marginal

irrel.

$K_4 \quad \sigma_1 \quad \sigma_2 \quad \sigma_3 \quad \sigma_4$

$\lambda_2 \quad \lambda_3 \quad \lambda_4 \quad \lambda_5$

$\uparrow \quad \uparrow \quad \uparrow$

rel marg irrel

$K_{nn} \quad K_{nnn} \quad K_4 + \ldots$

a 10^{-10} cm. $\longrightarrow$ 1 cm

near.   critical

linear combination

$O = (K - K_c)\,(K_{nn} - K_{nnn} + K_4 \cdots)$ relevant

other linear comb $\longrightarrow$ 0 irrelevant

"Universal"

$\lambda_2 \quad \lambda_3 \quad \lambda_4 \quad \lambda_5 \ldots$

$\uparrow$ rel $\quad \uparrow$ mag $\quad \uparrow$ irrel

$K_{nn} \quad K_{nnn} \quad K_4 + \ldots$

rel.

marginal $\Leftarrow$

irrel.

$K_4\ \sigma_1\ \sigma_2\ \sigma_3\ \sigma_4$

~~redundant~~

$1 : 1 \longrightarrow 1$ cm

a. 10^{-10} cm.

near. critical

linear combinatn

$(\ldots t_c)\,(K_{nn} - K_{nnn} + K_4 \cdots)$ relevant

other linear comb $\longrightarrow 0$ irrelevant

" Universal "

de

rel

marginal $\Leftarrow$

unrel.

$\lambda_2 \quad \lambda_3 \quad \lambda_4 \quad \lambda_5$

$\uparrow \quad \uparrow \quad \uparrow$

rel mag

$K_{nn} \quad K_{nnn} \quad K_4 \; + \ldots$

$K_4 \; \sigma_1 \sigma_2 \sigma_3 \sigma_4$

~~redundant~~

$1 \;\; 1 \longrightarrow 1 \text{ cm}$

$a \quad 10^{-10}$ cm. critical

near.

linear combinatn

$= (K - K_c)(K_{nn} - K_{nnn} + K_4 \cdots)$ relevant

other linear comb^b $\longrightarrow 0$ irrelev

" Universal "

Ising model

~~results depend on marginal couplings~~

rel

margin

unrel

$K_4\ \sigma_1\ \sigma_2$

~~redundant~~

linear

$0 = (K - K_c)\ (K$

other linear

" Unmm

Ising model

~~results depend on marginal coupling~~

rel

margin

unrel

$K_4\ \sigma_1 \sigma_x$

~~redun~~

linear

$O = (K - K_c)\ (K$

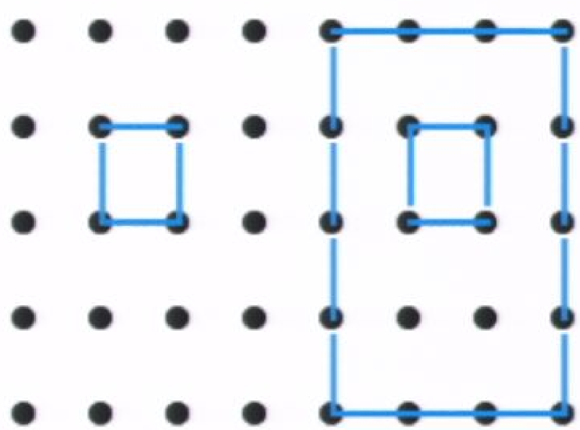
other linear

" Unnn

High Temperature Expansion

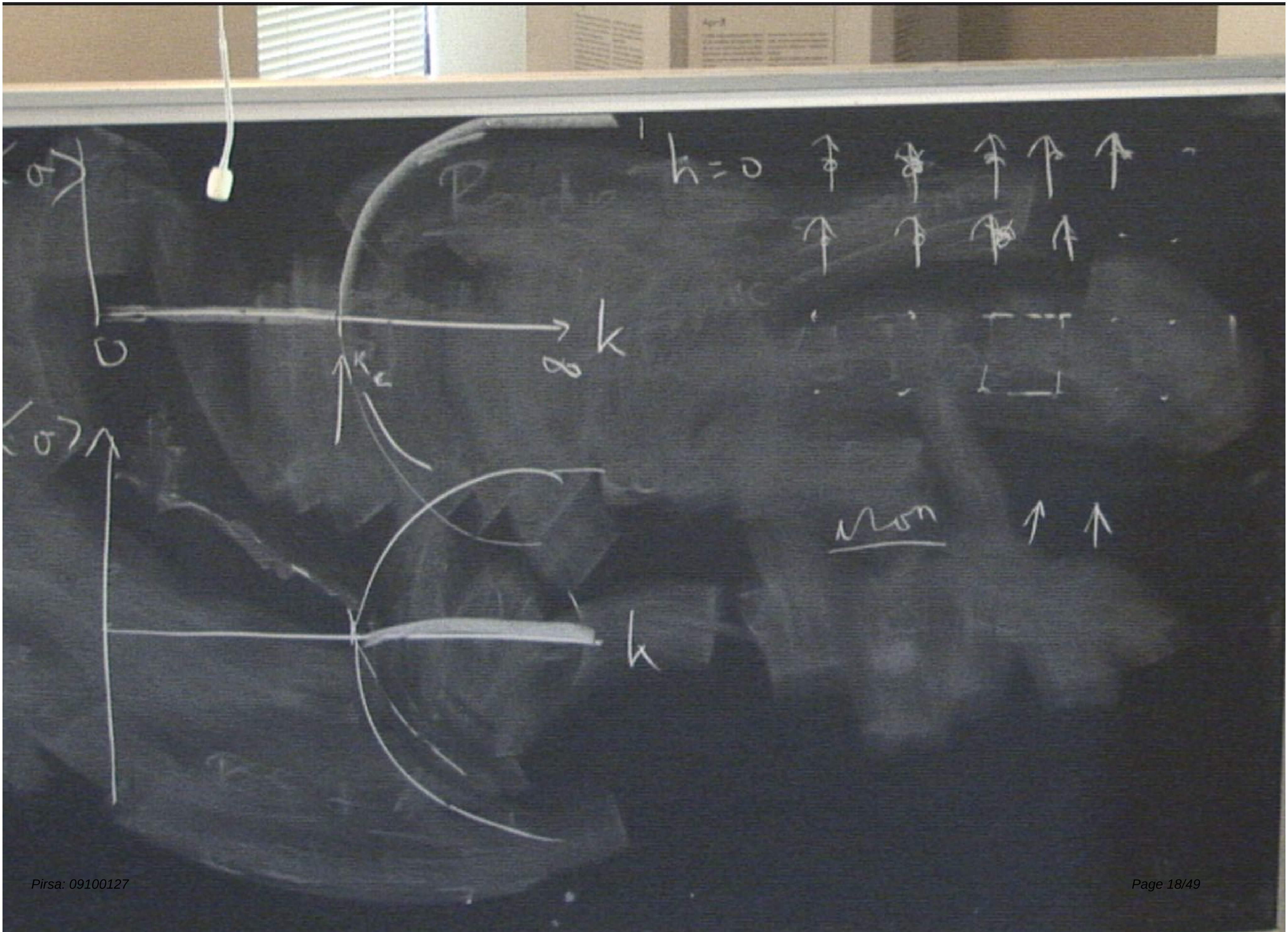
Nearest neighbor structure
Bonds=exp($K\sigma\sigma'$) connect nearest neighbors
Bond=cosh K +$\sigma\sigma'$ sinh K =cosh K[1+ $\sigma\sigma'$ tanh K]

Z=(2 cosh K cosh K)^N < products of [1+ $\sigma\sigma'$ tanh K] >
= (2 cosh K cosh K)^N sum < products of (tanh K)^M >
for nonzero terms, when there are N sites



To get a non-zero value each spin must appear on a even number of bonds. You then get the lattice covered by closed polygons.

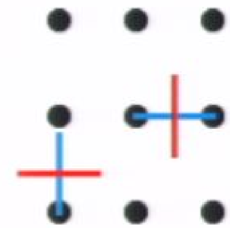
With a lot of hard work one can calculate a series up to ten or even twenty terms long and estimate behavior of thermodynamic functions from these seres



⟨σ⟩
0
k_c
∞
k
h = 0
⟨σ⟩
k

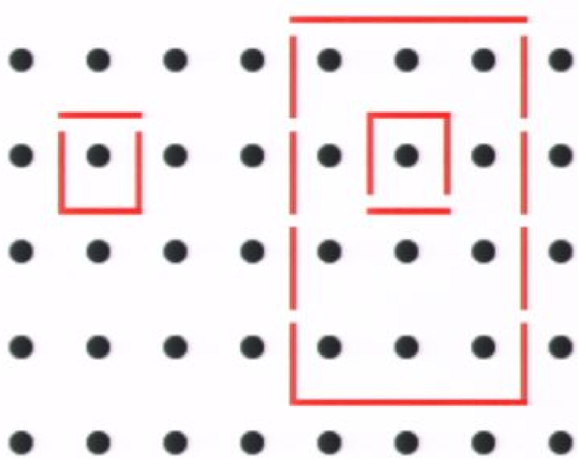
Low Temperature Expansion

Nearest neighbor structure
Bonds=exp($K\sigma\sigma'$) $= e^{K}\delta_{\sigma,\sigma'} + e^{-K}\delta_{\sigma,-\sigma'}$
Bond $= e^{K}[\delta_{\sigma,\sigma'} + e^{-2K}\delta_{\sigma,-\sigma'}]$
We draw these bonds differently from the high T bonds. We draw them rotated 90 degrees in comparison to the other bonds.



note e^{-2K}
$= \tanh \widetilde{K}$

$Z=2(e^{K})^{N}$ < products of $[\delta_{\sigma,\sigma'} + e^{-2K}\delta_{\sigma,-\sigma'}]$ >
$= 2e^{2NK}$ sum < products of $(e^{-2K})^{M}$ >
for nonzero terms



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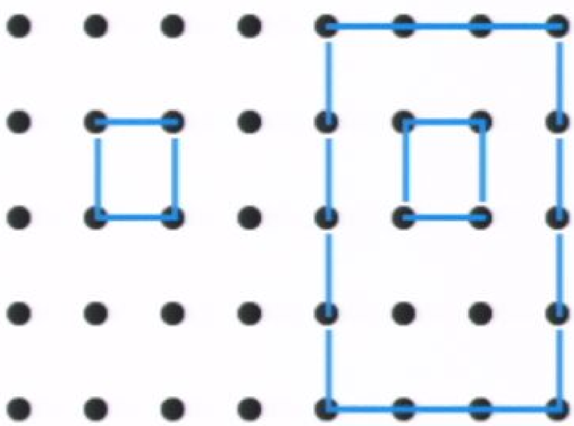

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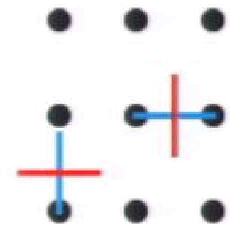

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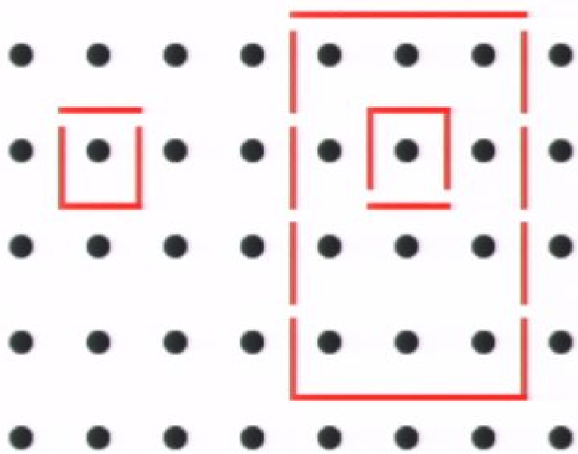

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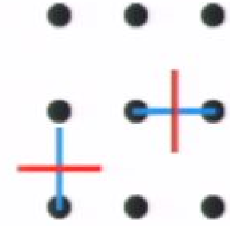

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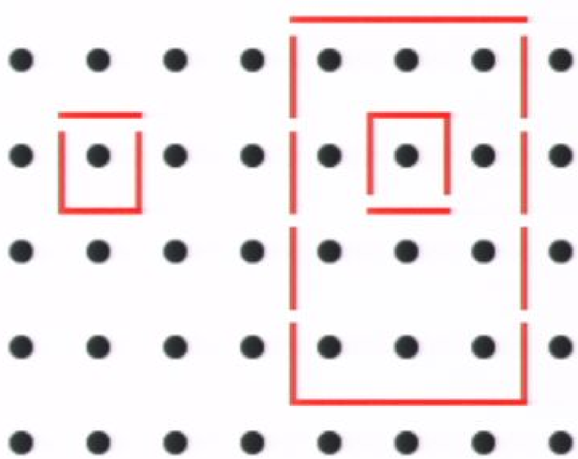

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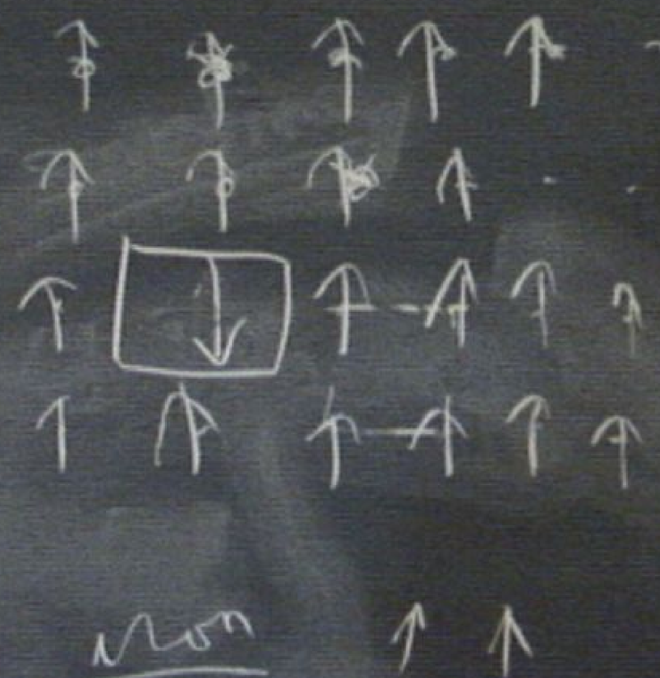
$$Z = \left[2\cosh^2 k\right]^N f(\tanh k)$$

$$= Z\,(e^{2k})^N f(e^{-2k})$$

$h = 0$



non

e^{-8k}

k

Duality Hendrik Kramers and Gregory Wannier

Since the two expressions both give Z we get a relationship beteen a high temperature theory of Z and a low temperature one. We write our sum of products as exp[Nf(.)] where the . can be either $\exp(-2K)$ or $\tanh K$ depending on which expansion we are going to use. We then have

$$\ln Z = N[K] + N f[\exp(-2K)] = N \ln [\, 2 \cosh K \cosh K] + N f[\tanh K]$$

Let us assume that there is only one singularity in $\ln Z$ as K goes through the interval between zero and infinity. Since $\tanh K$ is an increasing function of K and $\exp(-2K)$ is a decreasing function of K, the singularity must be at the point where the two things are equal

$\tanh K_c = \exp(-2K_c)$.

After a little algebra we get $\sinh 2K_c = 1$

which is the criticality condition for two-dimensional Ising model. This criticality condition was later verified by Onsager's exact solution of the 2d ising model.

Further we might notice that $\ln Z$ must have a form of singularity in which the singular part of the partition function is even about this point.

Specific Heat = $d^2 \ln Z / dT^2$

Further we might notice that in two dimensions ln Z must have a form of singularity which is even about the critical value of the coupling.

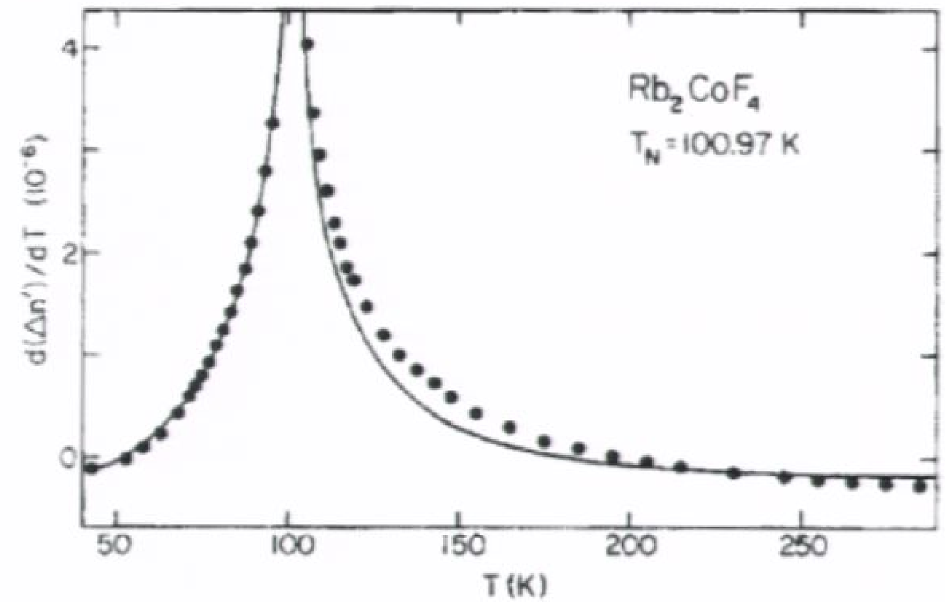


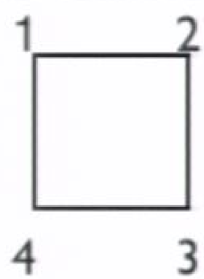
Figure 7. Variation of the magnetic specific heat, as a function of temperature for Rb_2CoF_4. The solid points (·) are experimental results of optical birefringence measurements shown previously to be proportional to the magnetic specific heat. The solid line is the exact Onsager solution for the two-dimensional Ising model with amplitude and critical temperature adjusted to fit the data, and a small constant background term subtracted. After Ref. 22.

22. P. Nordblad, D.P. Belanger, A.R. King, V. Jaccarino and H. Ikeda., Phys. Rev. B 28. 278 (1983).

Duality Behavior

The structure of duality behavior depends upon both the lattice structure and th symmetry group of the interactions on that lattice. Many of the deepest results c string theory, gauge theories, and modern mathematics similarly deal with the simultaneous effect of internal symmetries, and the symmetries of space, or of space-time. Once again the condensed matter gives us a chance to work out things which show up in a more complicated form in other situations.

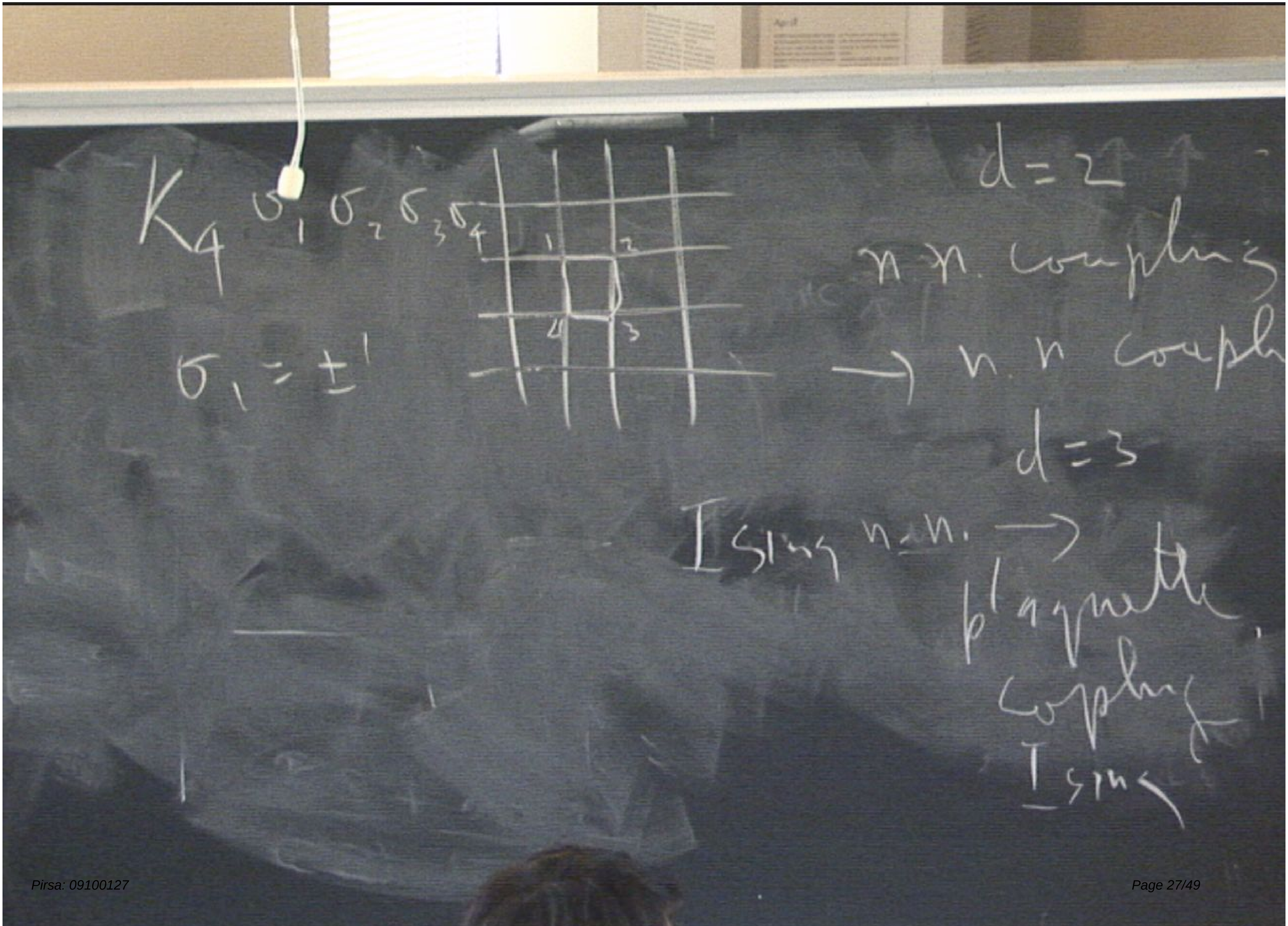
For example one can consider interactions on plaquettes like



Here the basic variables live on the bonds connecting nearest neighbor lattice sites, as for example, Ising variables σ_{23} σ_{34} and the basic interaction is on a look which goes around a unit square $K\,\sigma_{12}\,\sigma_{23}\,\sigma_{34}\,\sigma_{41}$. On a three dimensional lattice a set of interactions like this

a. has a fundamental gauge symmetry (symmetry operation at every point) of the form σ_{34} goes into $\mu_3\,\sigma_{34}\,\mu_4$. This is a symmetry operation since the μ's cancel in each plaquette interaction.

b. has a phase transition at sufficiently high value of K in three dimensions


K4 σ1 σ2 σ3 σ4
σ1 = ±1
1 2
4 3
d = 2
n.n. coupling
→ n.n. coupl
d = 3
Ising n.n. →
plaquette coupling
Ising

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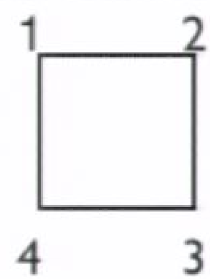

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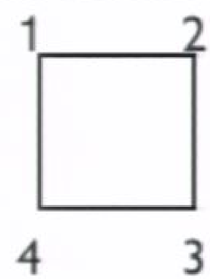

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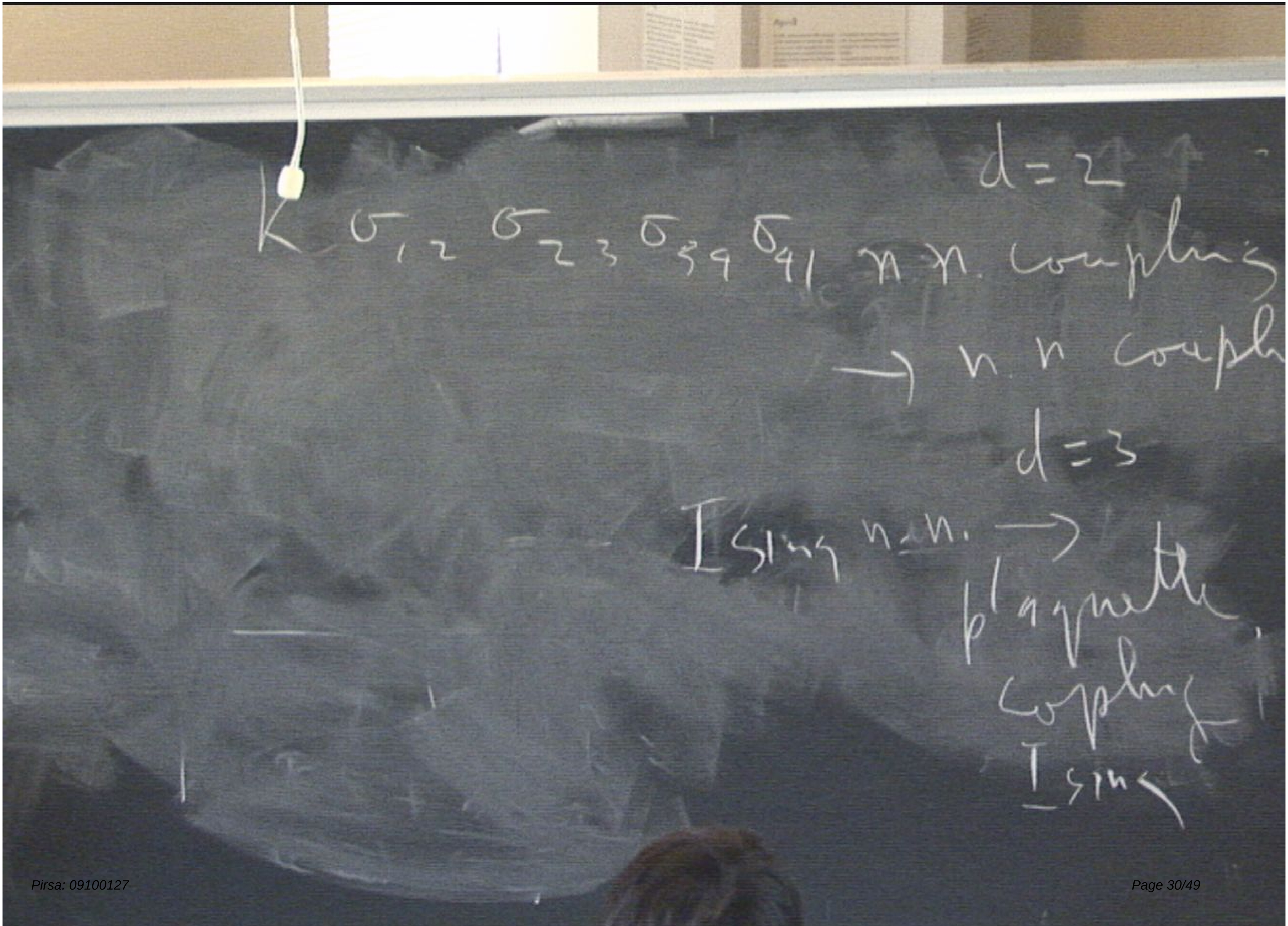

d = 2
K σ12 σ23 σ34 σ41 n.n. coupling
→ n.n coupl
d = 3
Ising n.n. →
plaquette
coupling
Ising

$K\,\mu_1\sigma_{12}\mu_2\mu_2\sigma_{23}\mu_3\mu_3\sigma_{34}\mu_4\mu_4\sigma_{41}\mu_1$ d = 2

n.n. coupling

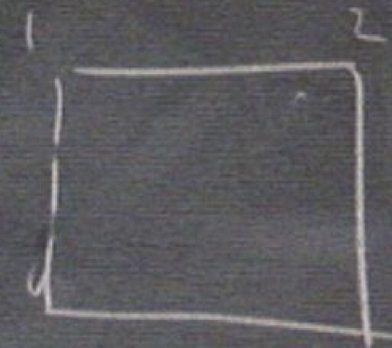
$\sigma_{12} \to \mu_1 \sigma_{12} \mu_2 \longrightarrow$ n.n. coupl

$\mu_1^2 = 1$

d = 3

Ising n.n. $\longrightarrow$ plaquette coupling Ising

$\Psi(r)$



$$\exp\left[ie\int_1^2 dx^\mu A_\mu(x)\right]$$

plaquette

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \Lambda(x)$$

$$\exp\left[ie\int_1^2 dx^\mu A_\mu(x)\right] \to e^{ie\Lambda(1)}\left[\quad\right] e^{-ie\Lambda(2)}$$

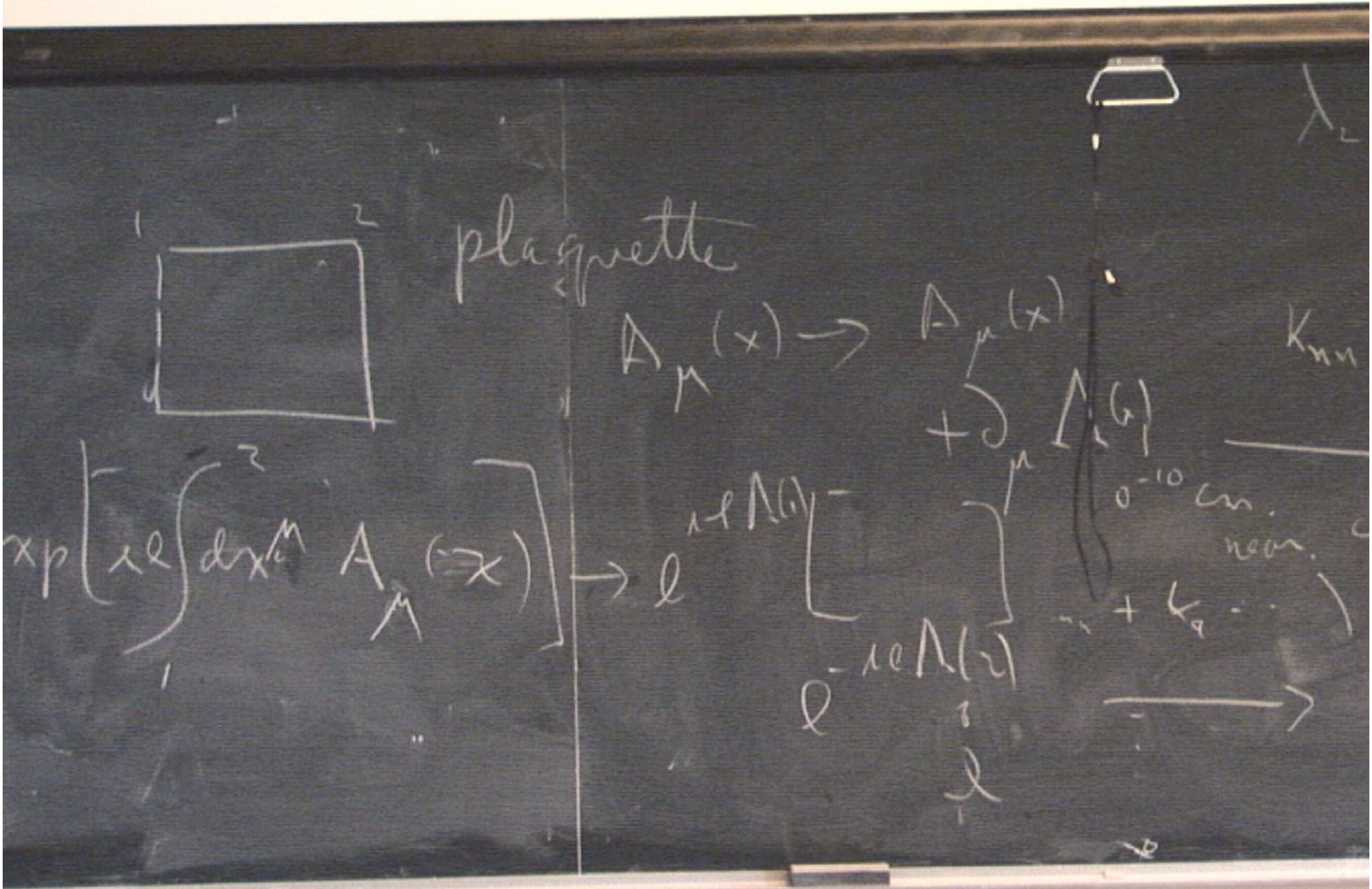
λ_2

K_{nn}

10^{-10} cm.

neon.

$+ k_4 \cdots)$


plaquette
neon.

How does one know?

Because this plaquette model is dual to the three dimensional Ising model an has a critical coupling, $K_c = D(K_c^{Ising})$.
The plaquette construction and gauge symmetry is due to K. Wilson.
The duality argument is due to F. Wegner

electromagnetic argument: basic variable is link =exp [ie $\int dx_\mu A_\mu$] where the integral goes from one lattice site to its neighbor. Then, a gauge transform
gives A_μ goes into $A_\mu + \partial_\mu \Lambda$ and link goes into exp[-ieΛ]link exp[ieΛ] where the two Λ's are evaluated at the two ends of the t=link. As in the case described above, A product of four link varaibles has a gauge symmetry, in this case the gauge symmetry of electromagnetism.

But let's get back to the Ising model.

Renormalization for d-2 Ising model

A. Pokrovskii & A. Patashinskii, Ben Widom, myself, Kenneth Wilson.

$Z = \text{Trace}_{\{\sigma\}} \exp(W_K\{\sigma\})$

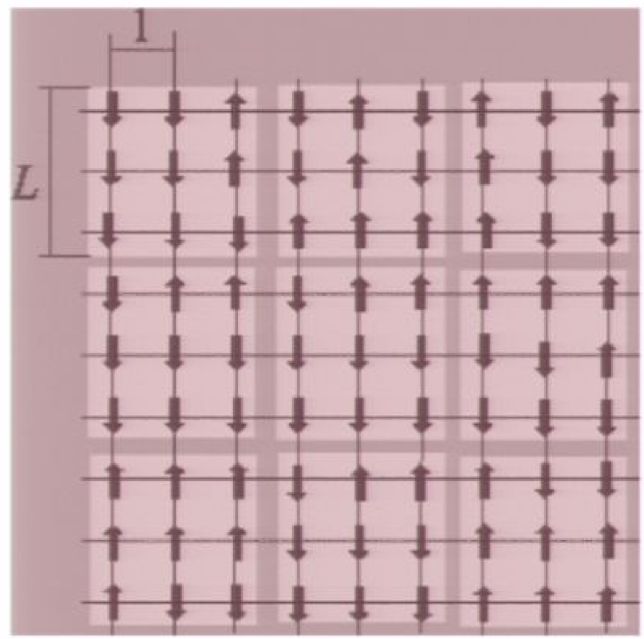
nagine that each box in the picture has in it a ariable called $\mu_\mathbf{R}$, where the **R**'s are a set of new ttice sites with nearest neighbor separation 3*a*. Each ew variable is tied to an old ones via a enormalization matrix $G\{\mu, \sigma\} = \prod_\mathbf{R} g(\mu_\mathbf{R}, \{\sigma\})$ here g couples the $\mu_\mathbf{R}$ to the

r's in the corresponding box. We take each $\mu_\mathbf{R}$ to e ±1 and define g so that,

$\sum_\mu g(\mu, \{s\}) = 1$. For example, μ might be lefined to be an Ising variable with the ame sign as the sum of σ's in its box. Now we are ready. Define

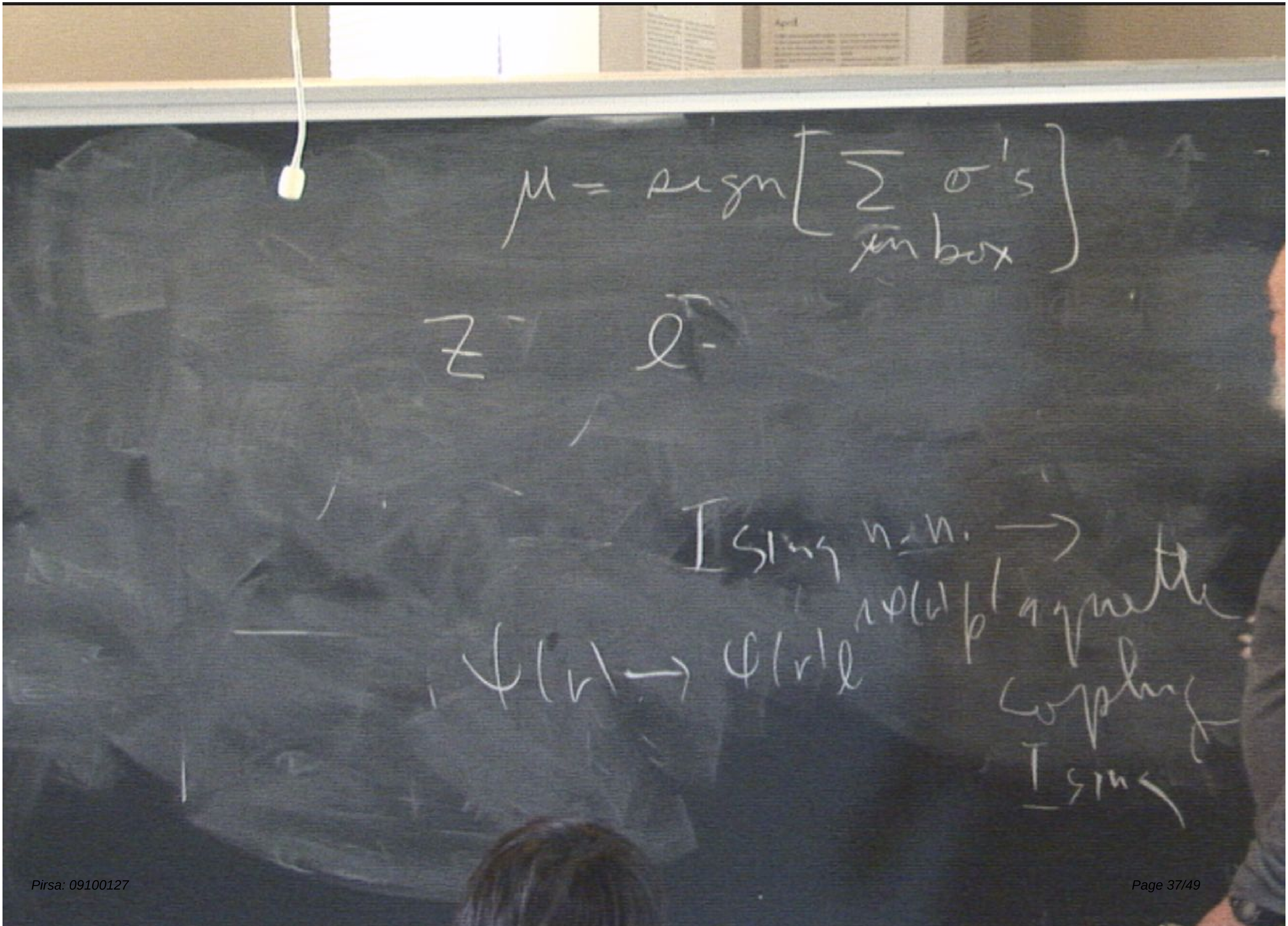
$\exp(W'\{\mu\}) = \text{Trace}_{\{\sigma\}} G\{\mu, \sigma\} \exp(W_K\{\sigma\})$

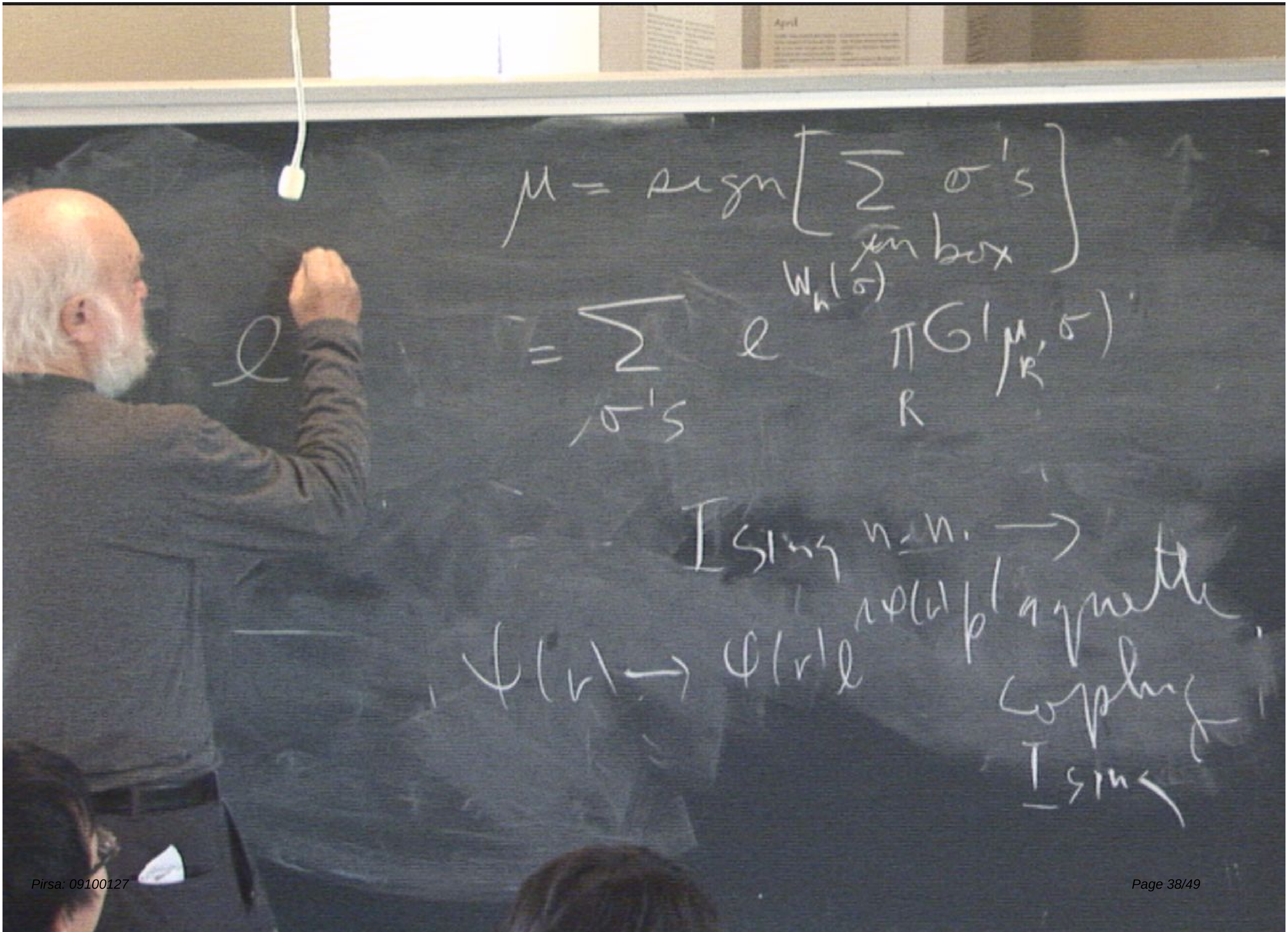
$Z = \text{Trace}_{\{\mu\}} \exp(W'\{\mu\})$



fewer degrees of freedom produces "block renormalization"

f we could ask our fairy god-mother what we wished for now it would be that we came back to the same problem as we had at the beginning: $W'\{\mu\} = W_{K'}\{\mu\}$

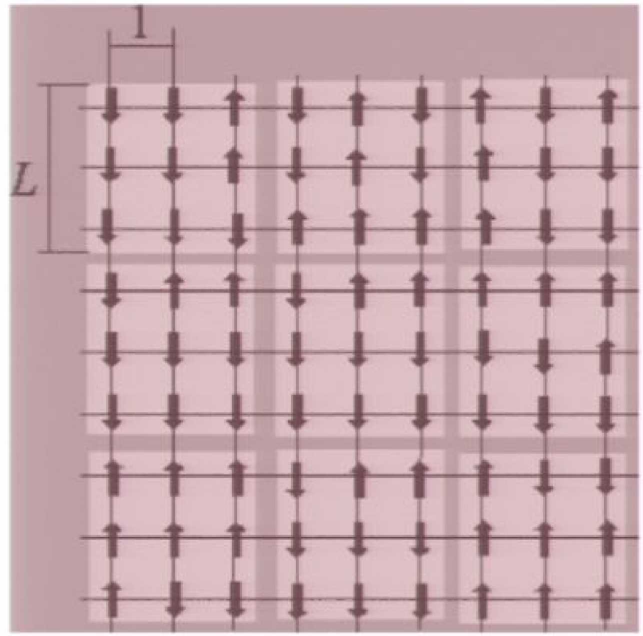

μ = sign[Σ σ's in box]
Z
Ising n.n. →
plaquette
coupling
Ising


μ = sign[Σ σ's in box]
= Σ σ's e^{W_n(σ)} Π_R G(μ_R, σ)
Ising n.n. → plaquette coupling Ising
ψ(r) → ψ(r) e^{iφ(r)}

Renormalization for d-2 Ising model

A. Pokrovskii & A. Patashinskii, Ben Widom, myself, Kenneth Wilson.

$Z=\mathrm{Trace}_{\{\sigma\}}\exp(W_K\{\sigma\})$



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Renormalization: $a \dashrightarrow 3a = a'$ $\quad W_K\{\sigma\} \dashrightarrow W_{K'}\{\mu\}$ $\quad Z' = Z$ $\quad K' = R(K)$

Scale Invariance at the critical point: --> $K_c = R(K_c)$

Temperature Deviation: $K=K_c+t$ $\quad K'=K_c+t'$

if $t=0$ then $t'=0$

ordered region ($t>0$) goes into ordered region ($t'>0$)

disordered region goes into disordered region

if t is small, $t'=bt$. $\quad b=(a'/a)^x$ defines x. b can be found through a numerical calculation.

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One can do many more roughly analogous calculations and compare with experiment and numerical simulation. Everything works!

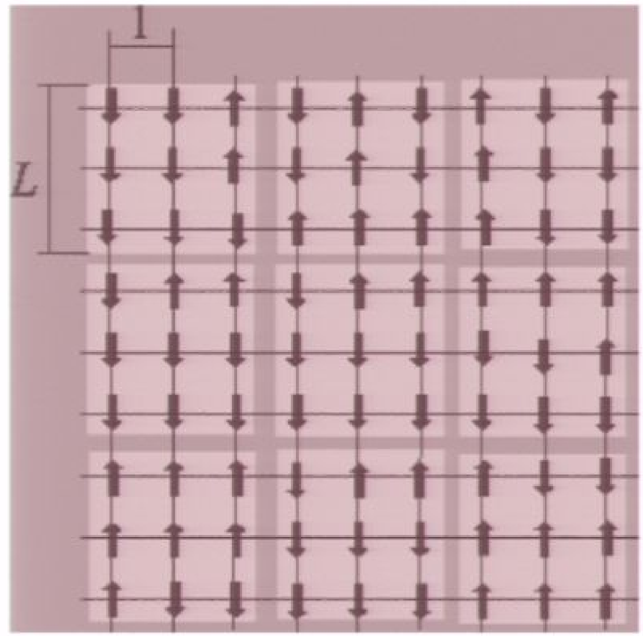
However notice that this is not a complete theory. It is a *phenomenological* theory. We have no way to find x from theory

Renormalization for d-2 Ising model

A. Pokrovskii & A. Patashinskii, Ben Widom, myself, Kenneth Wilson.

$Z = \text{Trace}_{\{\sigma\}} \exp(W_K\{\sigma\})$

nagine that each box in the picture has in it a ariable called $\mu_{\mathbf{R}}$, where the **R**'s are a set of new ttice sites with nearest neighbor separation $3a$. Each ew variable is tied to an old ones via a enormalization matrix $G\{\mu, \sigma\} = \prod_{\mathbf{R}} g(\mu_{\mathbf{R}}, \{\sigma\})$ here g couples the $\mu_{\mathbf{R}}$ to the



fewer degrees of freedom produces "block renormalization"

r's in the corresponding box. We take each $\mu_{\mathbf{R}}$ to e ± 1 and define g so that,

$\sum_\mu g(\mu, \{s\}) = 1$. For example, μ might be lefined to be an Ising variable with the ame sign as the sum of σ's in its box. Now we are ready. Define

$\exp(W'\{\mu\}) = \text{Trace}_{\{\sigma\}} G\{\mu, \sigma\} \exp(W_K\{\sigma\})$

$Z = \text{Trace}_{\{\mu\}} \exp(W'\{\mu\})$

f we could ask our fairy god-mother what we wished for now it would be that we came back to the same problem as we had at the beginning: $W'\{\mu\} = W_{K'}\{\mu\}$

$$Z = \sum_{\{\sigma\}} e^{W_k\{\sigma\}}$$

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$$\sum_{\{\mu\}} G\{\mu,\sigma\} = 1$$

$$e^{W'\{\sigma\}} = \sum_{\{\sigma\}} e^{W_k(\sigma)} G\{\mu,\sigma\}$$

$$Z = \sum_{\{\sigma\}} e^{W_k\{\sigma\}}$$

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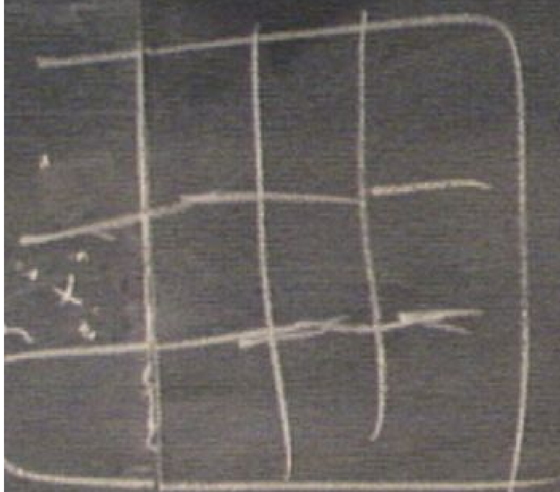
$$e^{W'\{\mu\}} = \sum_{\{\sigma\}} e^{W_k(\sigma)}$$

$$Z = \sum_{\{\mu\}} e^{W'(\mu)}$$

$\mu_R = \mathrm{sign}(\sum \sigma's)$

$G\{\mu, \sigma\} = \prod_R \delta_{\mu_R, \mathrm{sign}(\sum \sigma)}$



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$$e^{W'\{\mu\}} = \sum_{\{\sigma\}} e^{W_k(\sigma)}$$

$$Z = \sum_{\{\mu\}} e^{W'(\mu)}$$

Renormalization: $a \dashrightarrow 3a = a'$ $W_K\{\sigma\} \dashrightarrow W_{K'}\{\mu\}$ $Z' = Z$ $K' = R(K)$

Scale Invariance at the critical point: --> $K_c = R(K_c)$
Temperature Deviation: $K = K_c + t$ $K' = K_c + t'$
if $t=0$ then $t'=0$
ordered region ($t>0$) goes into ordered region ($t'>0$)
disordered region goes into disordered region
if t is small, $t' = bt$. $b = (a'/a)^x$ defines x. b can be found through a numerical calculation.

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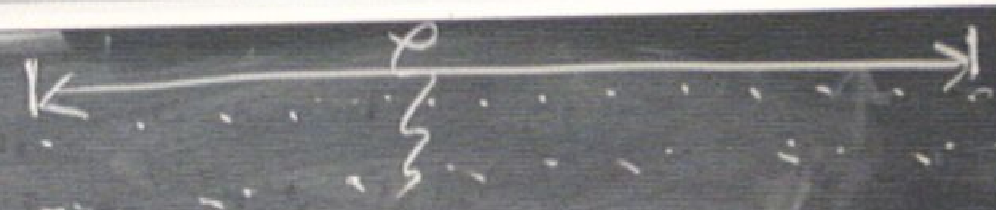
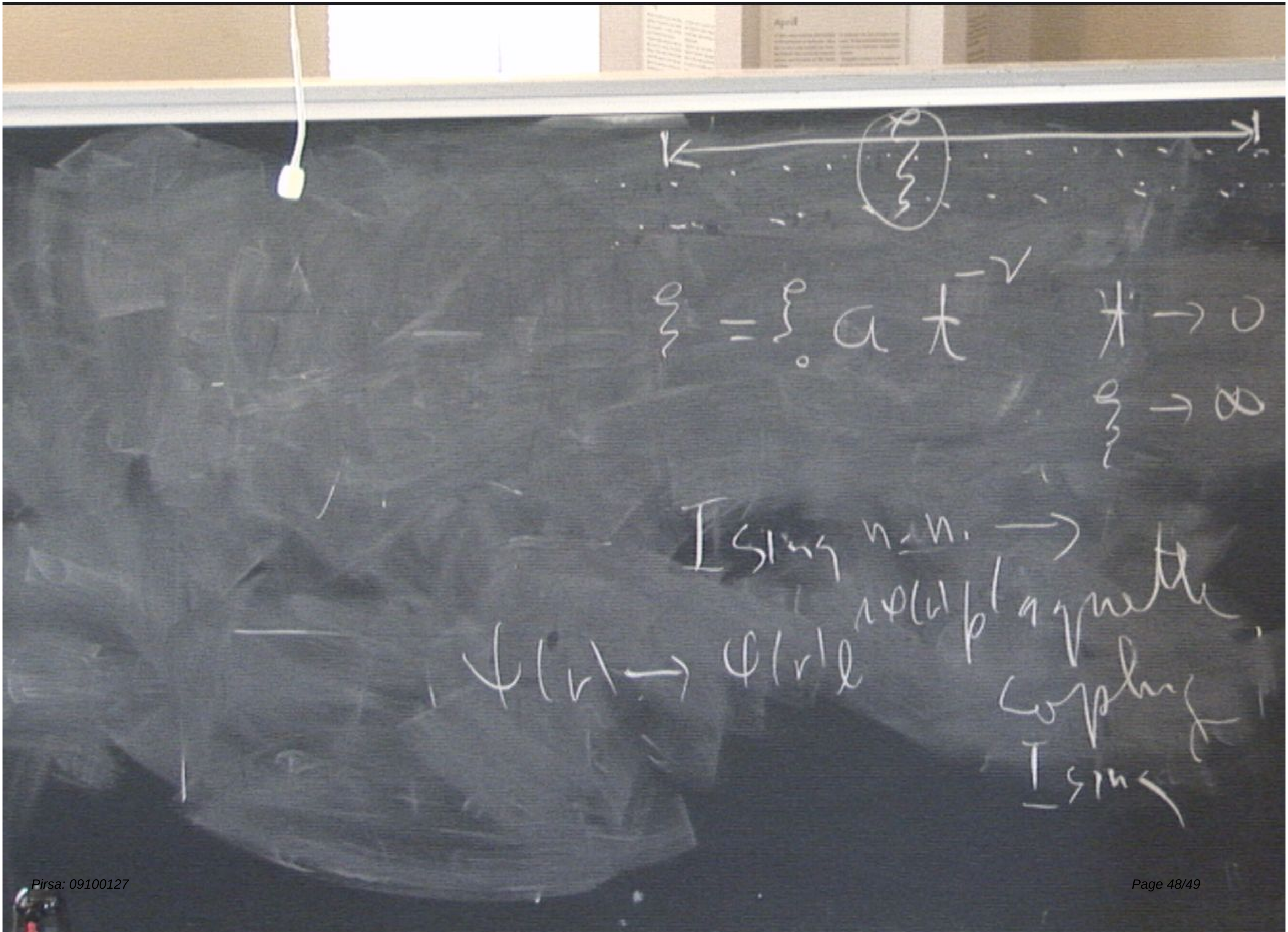


$$\xi = a\, t^{-\nu} \qquad t \to 0$$

$$\xi \to \infty$$

Ising n.n. $\longrightarrow$ plaquette coupling Ising

$$\varphi(r) \to \varphi(r)\, e^{i\varphi(r)}$$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