Title: Statistical Mechanics (PHYS 602) - Lecture 4

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Abstract:

More is the Same
Phase Transitions and Mean Field Theories

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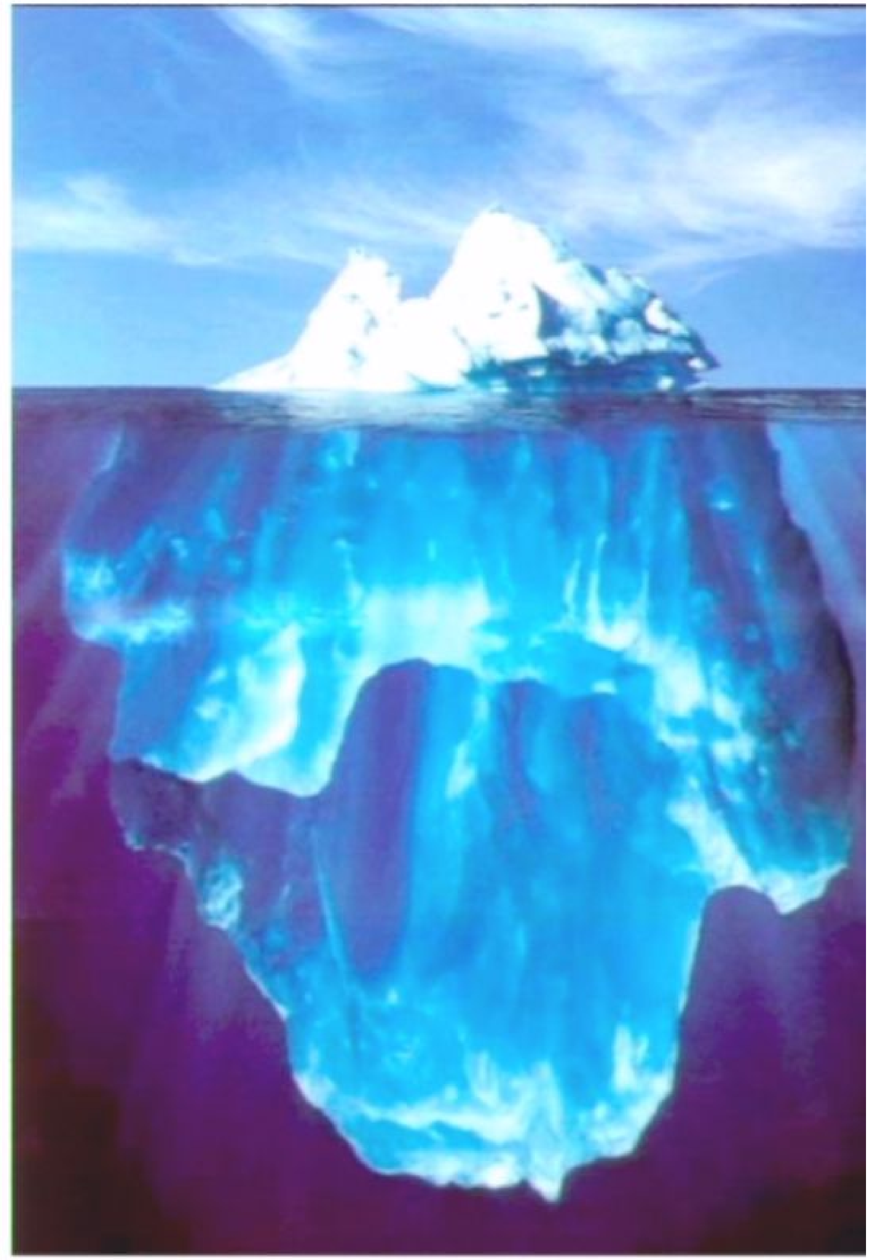
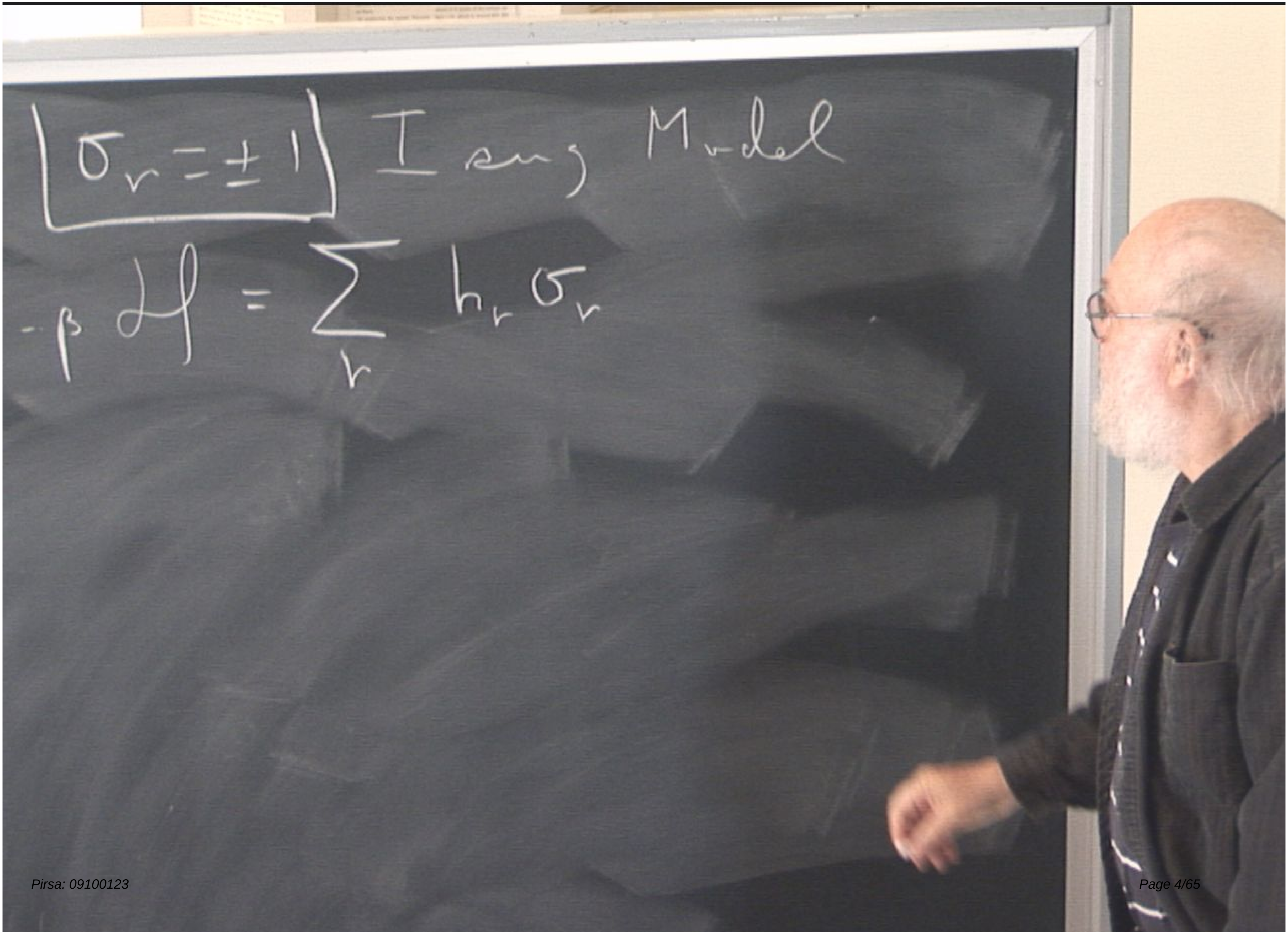
Abstract

This talk summarizes concepts derived from the study of phase transitions mostly within condensed matter physics. In its original form, the talk was aimed equally at condensed matter physicists and philosophers of science. The latter group are particularly interested in the logical structure of science. This talk bears some traces of its history. The key technical ideas go under the names of ``singularity'', ``order parameter'', ``mean field theory'', ``variational method'', and ``correlation length''. The key ideas here go under the names of "mean field theory", "phase transitions", "universality", "variational method", and "scaling".

Issues

Matter exists in different phases, different states of matter with qualitatively different properties: These phases are interesting in modern physics and provocative to modern philosophy. For example, no phase transition can ever occur in a finite system. Thus, in some sense phase transitions are not products of the finite world but of the human imagination.




$\boxed{\sigma_r = \pm 1}$ Ising Model
$-\beta \mathcal{H} = \sum_r h_r \sigma_r$

$\boxed{\sigma_r = \pm 1}$ Ising Model

$$-\beta \mathcal{H} = \sum_r h_r \sigma_r + \sum_{\langle r s \rangle}$$

$\boxed{\sigma_r = \pm 1}$ Ising Model

$$-\beta \mathcal{H} = \sum_r h_r \sigma_r + \sum_{\langle rs \rangle} K \sigma_r \sigma_s$$

$\boxed{\sigma_r = \pm 1}$ Ising Model

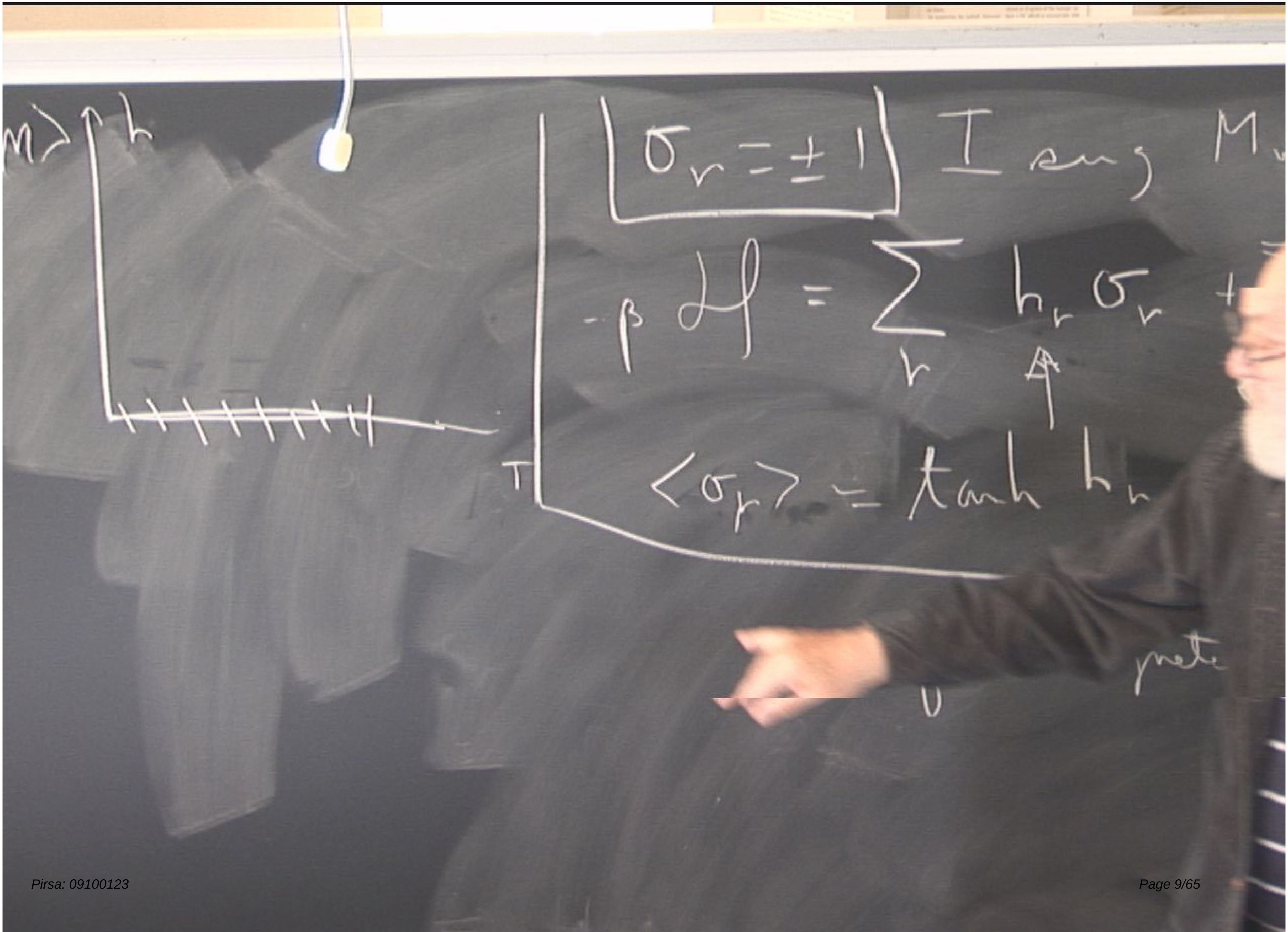
$$-\beta \mathcal{H} = \sum_r h_r \sigma_r + \sum_{\langle rs \rangle} K \sigma_r \sigma_s$$

$$\langle \sigma_r \rangle = \tanh h_r$$

$\boxed{\sigma_r = \pm 1}$ Ising Model

$$-\beta \mathcal{H} = \sum_r h_r \sigma_r + \sum_{\langle rs \rangle} K \sigma_r \sigma_s$$

$$\langle \sigma_r \rangle = \tanh h_r$$ "toy"


$\sigma_r = \pm 1$
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$-\beta \mathcal{H} = \sum_r h_r \sigma_r +$
$\langle \sigma_r \rangle = \tanh h_r$

$$\boxed{\sigma_r = \pm 1}$$ Ising Model

$$-\beta \mathcal{H} = \sum_r h_r \sigma_r + \sum K$$

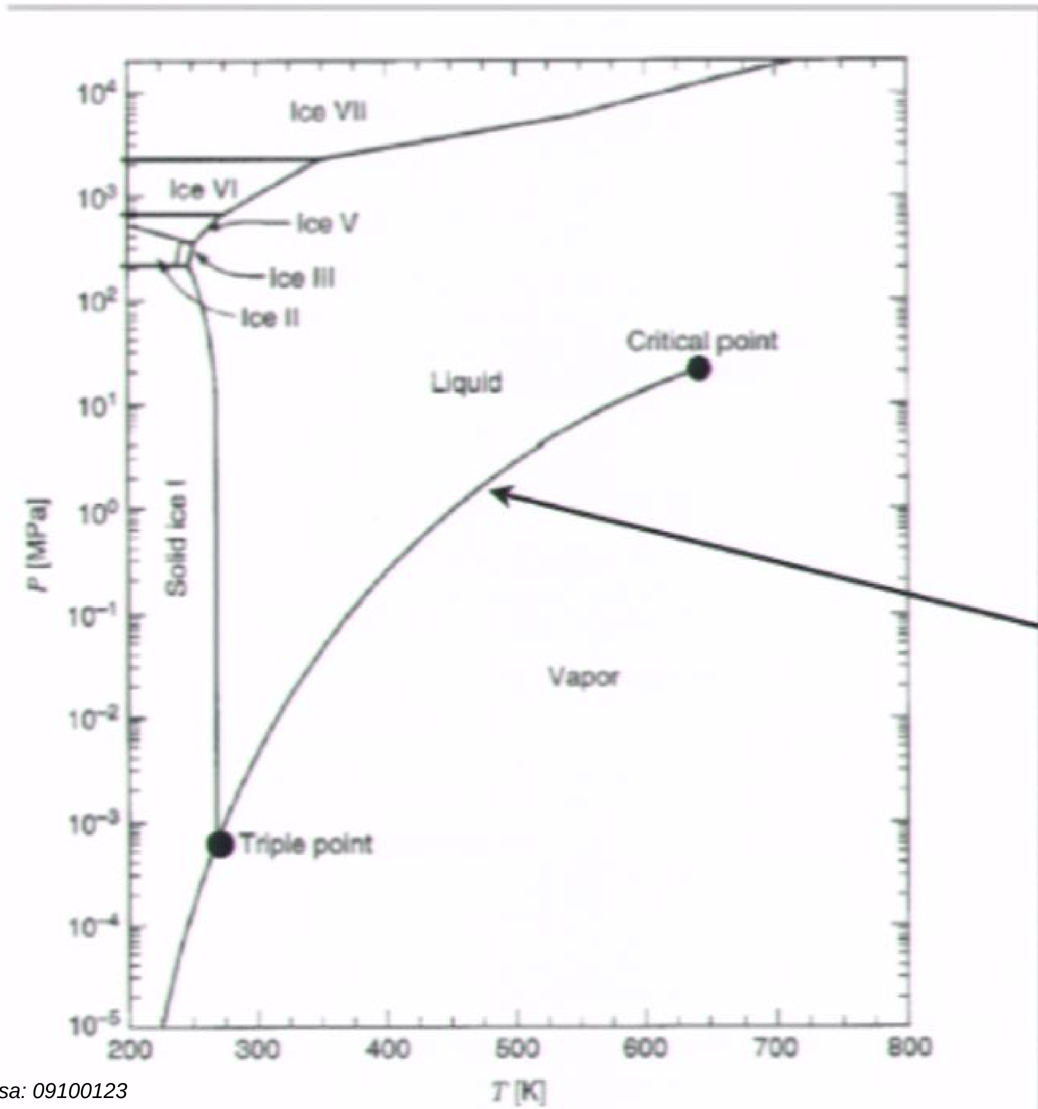
$$\langle \sigma_r \rangle = \tanh h_r$$

"tanh"

discontinuous jump in

ferromagnetism

Phase Diagram for Water



liquid-gas phase transition line

problem: why or how or what is a discontinuous jump?

$\boxed{\sigma_r = \pm 1}$ Ising Model

$$-\beta \mathcal{H} = \sum_r h_r \sigma_r + \sum_{\langle rs \rangle} K \sigma_r \sigma_s$$

T

$$\langle \sigma_r \rangle \simeq \tanh h_r \qquad \text{"tiny"}$$

discontinuous jump

ferromagnetism

problem: why or how or what is a distrib

$$p(\{\sigma\}) = \frac{e^{-\beta \mathcal{H}\{\sigma\}}}{Z}$$

problem: why or how or what is a discontinuous p

$$\rho(\{\sigma\}) = \frac{e^{-\beta H\{\sigma\}}}{Z}$$

consequences

su

problem: why or how or what is a discontinuous jump?

$$P(\{\sigma\}) = \frac{e^{-\beta H\{\sigma\}}}{Z}$$

consequences → simulate

→ solve analytically

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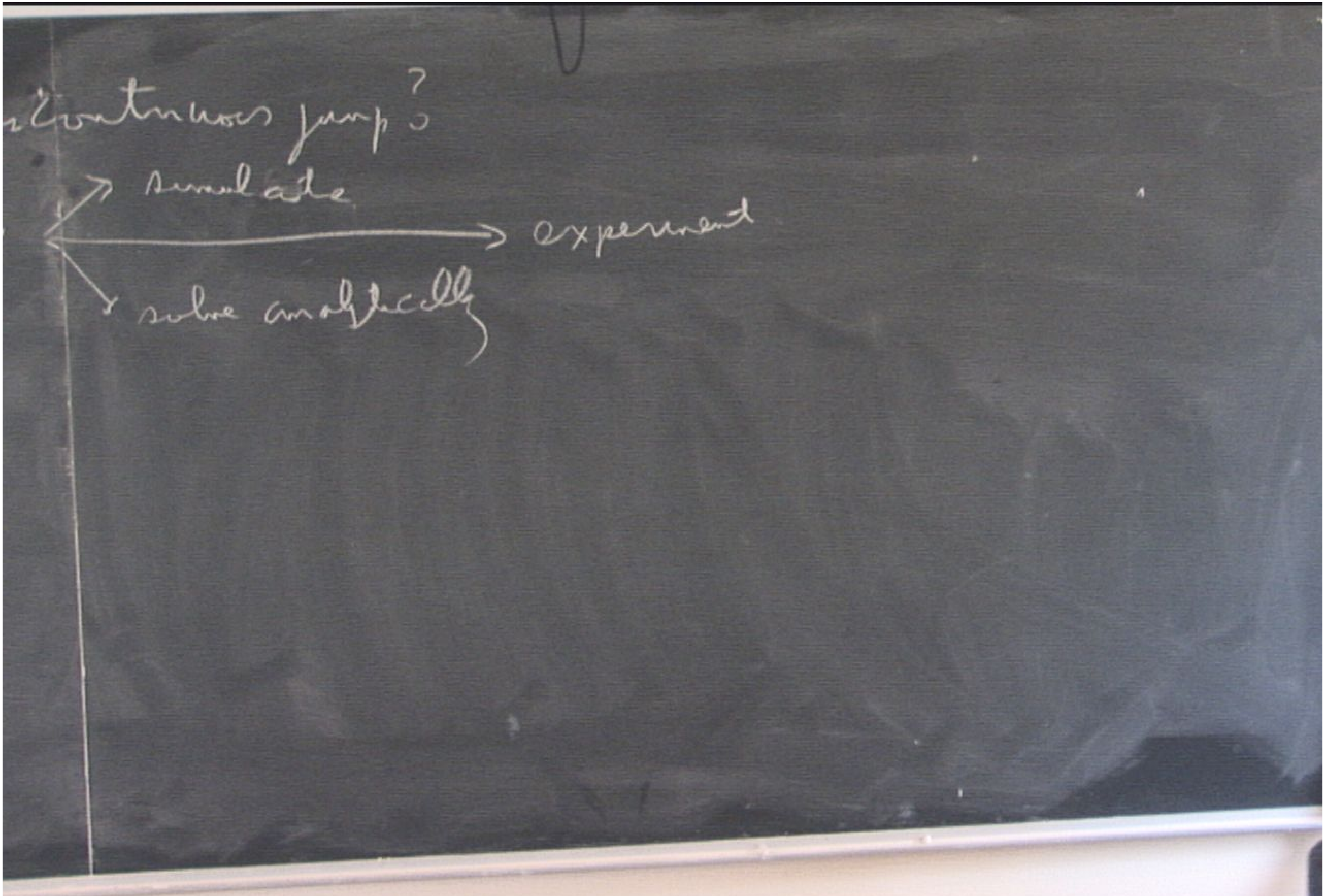

continuous jump?
simulate
experiment
solve analytically

discontinuous jump?

simulate

experiment

solve analytically

$Z(k)$

1d

continuous jump?

simulate

experiment

solve analytically

$Z(K)$

1 d

$$K = -\frac{J}{k_B T}$$

discontinuous jump?

simulate

experiment

solve analytically

$Z(K)$

1 d

$$K = -\frac{J}{k_B T}$$

$$\frac{\partial^2}{\partial T^2} \ln Z(K) \longrightarrow \infty$$

$K = K_c$

$\langle m \rangle$ h

critical pt.

T

discontinuous jump

$\boxed{\sigma_r = \pm 1}$ Ising Model:

$$-\beta \mathcal{H} = \sum_r h_r \sigma_r + \sum_{\langle r a \rangle} K \sigma_r$$

$$\langle \sigma_r \rangle = \tanh h_r$$

"tiny"

ferromagnetism

problem: why or how or what is a discontinuous jump?

$$\ldots\{\sigma\}) = \frac{e^{-\beta H\{\sigma\}}}{Z}$$

consequences → simulate

consequences → solve analytically

problem → model → solution → understanding

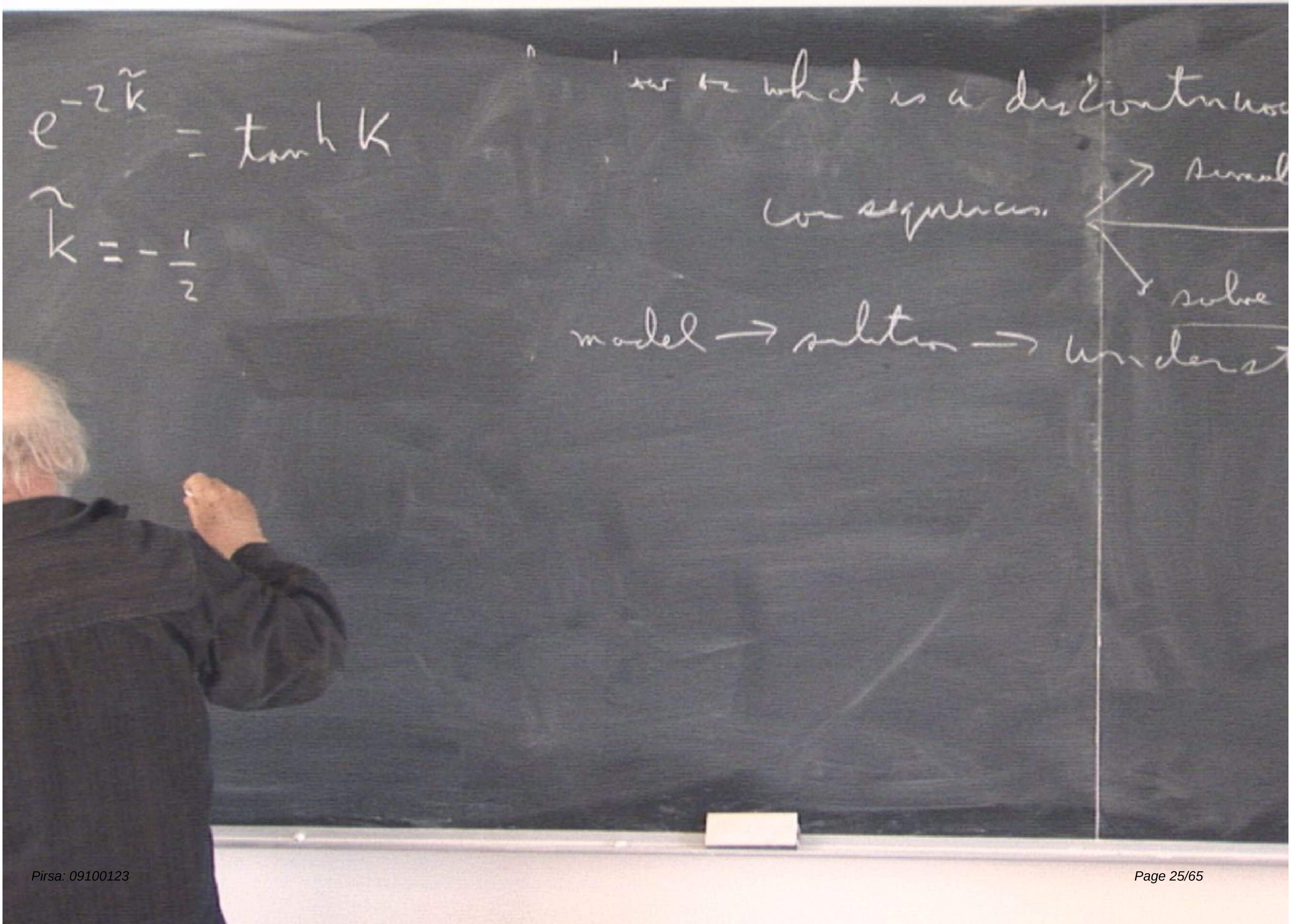
$= \tanh K$

as to what is a discontinuous jum

consequences → simulate

consequences → solve analyt

model → solution → understandin


$e^{-2\tilde{k}} = \tanh K$
$\tilde{k} = -\frac{1}{2}$
what is a
consequences
solve
model → solution →

$$e^{-2\tilde{k}} = \tanh K$$

$$\tilde{k} = -\frac{1}{2} \ln \tanh K$$

answer to what is a discontinuo

consequences → simul

consequences → solve

model → solution → underst

$$e^{-2\tilde{k}} = \tanh K$$

$$\tilde{k} = -\frac{1}{2} \ln \tanh k$$

$$= D(k)$$

... to what is a discontinuo...

Consequences → surrou...

→ solve

model → solution → underst...

... as to what is a discontinuo...

$$e^{-2\tilde{k}} = \tanh k$$

$$\tilde{k} = -\frac{1}{2} \ln \tanh k$$

$$= D(k) \,;\; D(\tilde{k}) = k$$

$$D(D(k)) = k$$

consequences ... sumal ...

... solve

model $\rightarrow$ solution $\rightarrow$ underst...

... as to what is a discontinu...

$$e^{-2\tilde{k}} = \tanh K$$

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$$= D(k) ; \; D(\tilde{k}) = k$$

model → solution → understa...

$$D(D(k)) = k$$

Con sequences → Answe... / solve

$$e^{-2\tilde{k}} = \frac{e^{k} - e^{-k}}{e^{k} + e^{-k}} = \frac{1 - e^{-2k}}{1 + e^{-2k}}$$

$$e^{-2\tilde{k}} \left(1 + x^{-2k}\right) = 1 - e^{-2k}$$

$$e^{-2\tilde{k}} e^{-2k} = 1 - e^{-2k} - e^{-2\tilde{k}}$$

... how to what is a discontinuous ...

$$e^{-2\tilde{k}} = \tanh K$$

consequences.

$$\tilde{k} = -\frac{1}{2} \ln \tanh k$$

$$= D(k); \; D(\tilde{k}) = k$$

model → solution → understand

solve

$$D(D(k)) = k$$

$$e^{-2\tilde{k}} = \frac{e^{k} - e^{-k}}{e^{k} + e^{-k}} = \frac{1 - e^{-2k}}{1 + e^{-2k}}$$

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...nuous jump?

simulate the

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solve analytically

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d dim stat mech

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...ntinuous jump?

simulate → experiment

solve analytically → understanding

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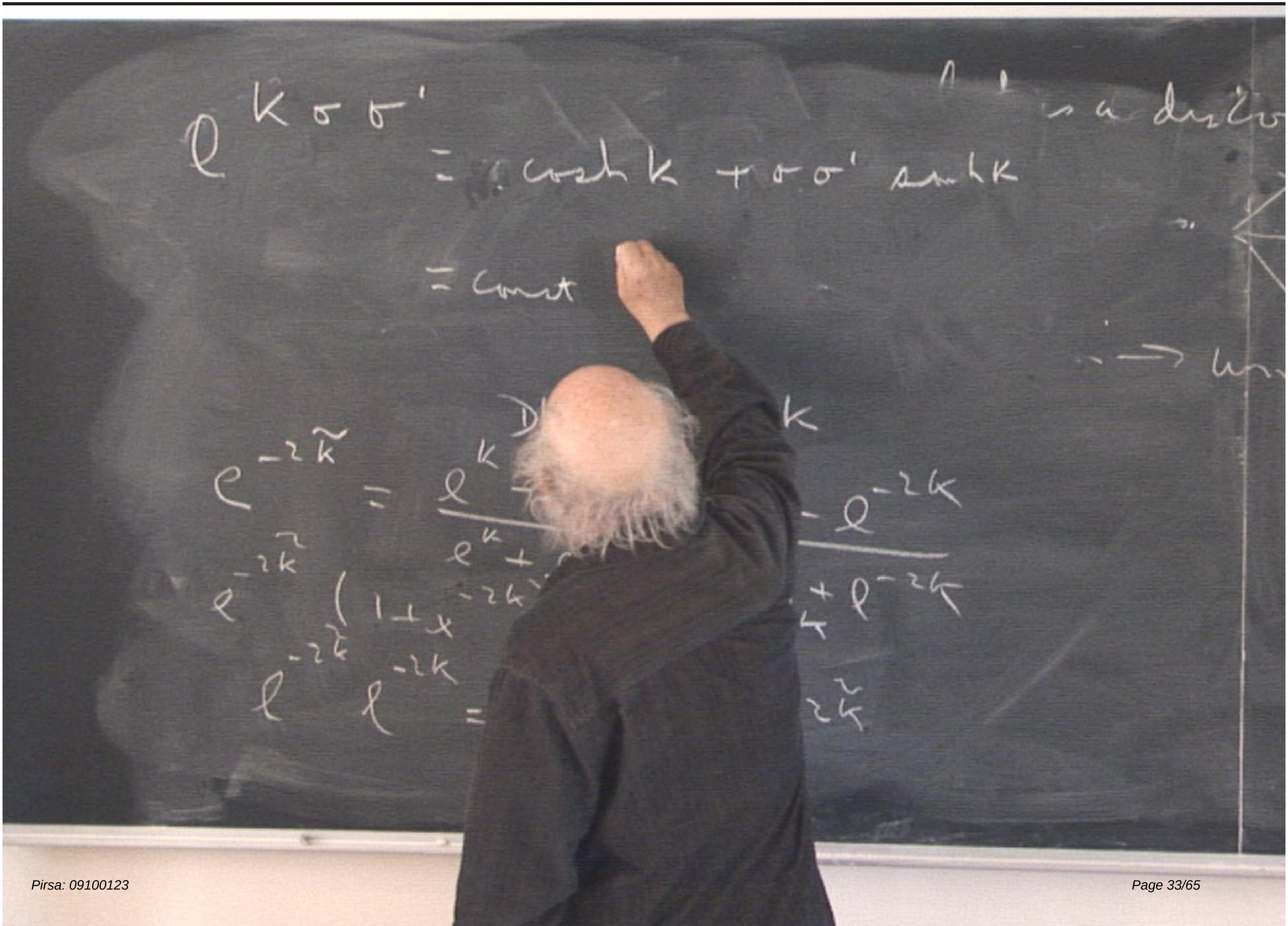
d-1 " Q.M.

$$-\beta H = \sum_j K \sigma_j \sigma_{j+1}$$

$$-\beta \mathcal{H}_Q = \tilde{K}_0 + \tilde{K}_1 T_1$$

$$\mathrm{trace}\left(e^{-\beta \mathcal{H}_Q}\right)^N = Z =$$


= cosh k + σσ' sinh k
= const

$$e^{K\sigma\sigma'}$$

$$= \cosh K + \sigma\sigma' \sinh K$$

$$\doteq \text{const}\left(1\, e^{\tilde{K}} + \sigma\sigma'\, e^{-\tilde{K}}\right)$$

$$\tanh K = e^{-2\tilde{K}}$$

$$e^{-2\tilde{K}} = \frac{\qquad K}{e^{K} + e^{-K}}$$

$$e^{-2\tilde{K}}\left(1 + x^{-2K}\right) = 1 - e^{-2K}$$

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...nuous jump?

simulate

$\longrightarrow$ experiment

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understanding

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$$-\beta \mathcal{H} = \sum_j K \sigma_j \sigma_{j+1}$$

$$-\beta \mathcal{H}_0 = \tilde{K}_0 + \tilde{K}_1 T_1$$

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$$e^{K\sigma\sigma'}$$

$$= \cosh K + \sigma\sigma' \sinh K$$

$$\stackrel{\circ}{=} \text{const}\left(1\, e^{\tilde{K}} + \sigma\sigma'\, e^{-\tilde{K}}\right)$$

$$\tanh K = e^{-2\tilde{K}}$$

$$e^{-\beta\, 2 h} = T$$

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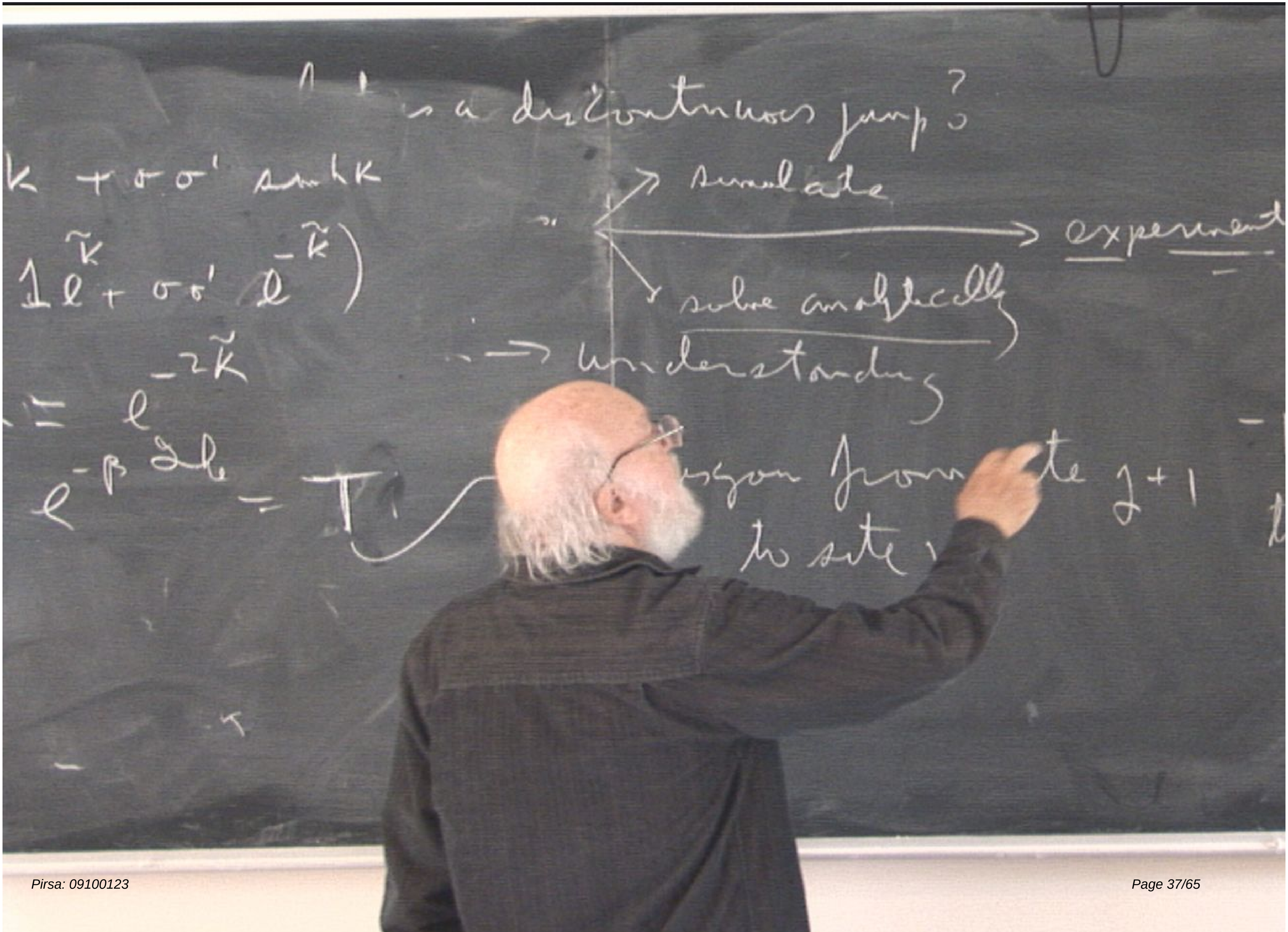
is a discontinuous jump?
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solve analytically
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to site

Dual Couplings

$$-\mathcal{H} = \widetilde{K}_0 \mathbf{1} + \widetilde{K} \tau_1 \qquad (4.22)$$

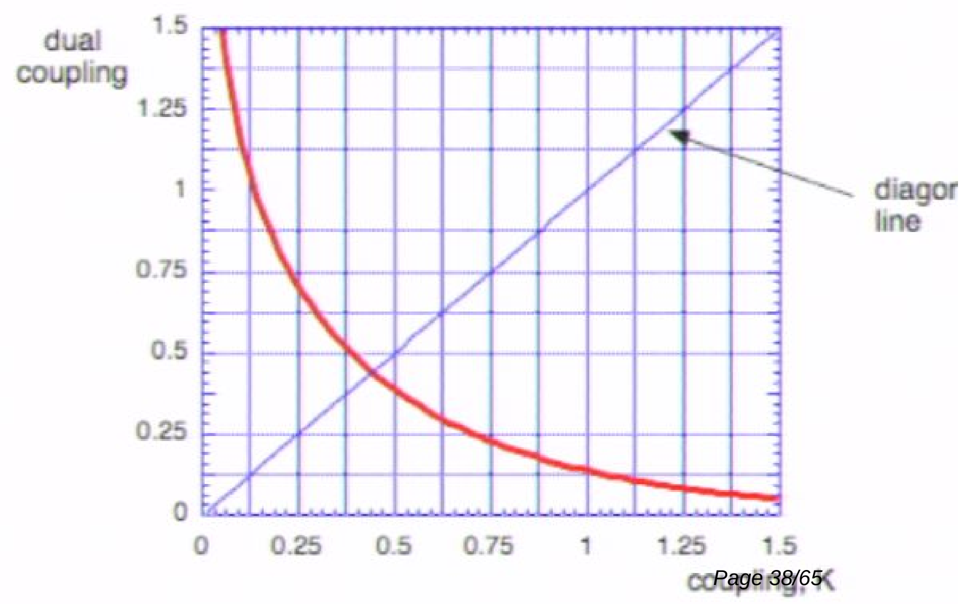
The quantity $\widetilde{K}$ is said to be the dual of K. For a simpler notation, we call this function by another name so that the dual of K is $D(K)$ This name implies in part that the function $D(K)$ has the property that if it is applied twice that you get precisely the same thing once more:

$D(D(K))=K$ or $D^{-1}(K)=D(K)$

How would I find the function D(K)?

$\widetilde{K}= D(K)=-[\ln (\tanh K)]/2$ $\widetilde{K}_0= [\ln(\sinh 2K)]/2$

This function has the property that when K is strong its dual is weak and *vice versa*. This property has proven to be very important in both statistical physics and particle physics. Often we know both a basic model and its dual. Often models are hard to solve in strong coupling. But the dual models have weak coupling when the basic model has strong coupling. So then we get an indirect solution of the basic model.



Solution of the one-dimensional Ising model

From equation 4.20, we find that the partition function of the one-dimensional Ising model is

$$Z = \text{ trace } (e^K \mathbf{1} + e^{-K}\tau_1)^N$$

But the trace is a sum over eigenvalues and the eigenvalues of τ_1 are plus or minus one. Thus, the answer is:

$$Z = (2\cosh K)^N + (2\sinh K)^N \qquad (4.25)$$

If N is very large, the first term is much larger than the second and thus in this limit of large system size:

$$-\beta F = \ln Z = N \ln(2\cosh K) \qquad (4.26)$$

What quantum mechanics problem have we solved?

More about quantum from the Long Chain

We should be able to say more about quantum problems based upon the analysis of the long chain. For example let us imagine that we wish to calculate the average of some quantum operator, $X(q)$, which happens to be diagonal in the q-representation. The text book goes through a long song and dance to prove a rather obvious result. You have seen that the trace in equation 4.10 pushes us into a sum over energy states, and if N is very large that sum reduces to a projection onto the ground state of the system. Specifically,

$$Z = \text{trace}_{q_1} \text{trace}_{q_2} \ldots \text{trace}_{q_n} \prod_{j=1}^{N} \exp(w(q_j, q_{j+1})) \qquad (4.10)$$

becomes $Z=\exp(-\tau\epsilon_0)$

So if we insert an X, for any any operator X, in that sum the result should give what happen: to that X in the ground state, specifically

$(1/Z)\,\text{Trace}_{\{q\}}\ \exp[W\{q\}]\ X = \langle 0|\ X(q)\ |0\rangle$

In this way, we can use statistical mechanics to calculate the average of any operator in the ground state. If we do not take N to infinity, we can do the corresponding calculation to calculate the average of any operator at a inverse temperature ($\beta-$ value) equal to N τ.

By playing with the times in an appropriate fashion, we can even calculate time-dependent correlation functions in the ground state or in a finite-temperature state.

Statistical Correlations in a Long Chain

We should be able lots about the statistical mechanics of a the long chain with Ising style interactions. For example, let us calculate the average of the jth spin on a long chain or the correlations among the spins in the chain. Start from

$$Z = \mathrm{Tr}\,\exp[\textstyle\sum_{j=1}^{N} K\sigma_{j+1}\sigma_j]$$

$$<\sigma_k> = (1/Z)\mathrm{Tr}\,\sigma_k \exp[\textstyle\sum_{j=1}^{N} K\sigma_{j+1}\sigma_j]$$

$$<\sigma_k\sigma_{k+r}> = (1/Z)\mathrm{Tr}\,\sigma_k\sigma_{k+r} \exp[\textstyle\sum_{j=1}^{N} K\sigma_{j+1}\sigma_j]$$

Here Tr means "sum over all the N spin-values". We use periodic boundary conditions. In this equation all the σ's are numbers, and they commute with each other.

We can make the calculation easier by replacing all the couplings by their expressions in terms of Pauli spins matrices giving these three calculations as, first,

$$Z=\mathrm{trace}_\tau \prod_{j=1}^{N} \exp(\tilde{K}_0 + \tilde{K}\tau_1) = \mathrm{trace}_\tau \exp[N(\tilde{K}_0 + \tilde{K}\tau_1)]$$

$$= (2\cosh K)^N + (2\sinh K)^N \qquad \approx \quad (2\cosh K)^N$$

The $\approx$ is an approximate equality which holds for large N. Note that in this limit the term with eigenvalue of $\tau_1 = 1$ dominates because the dual coupling is positive.

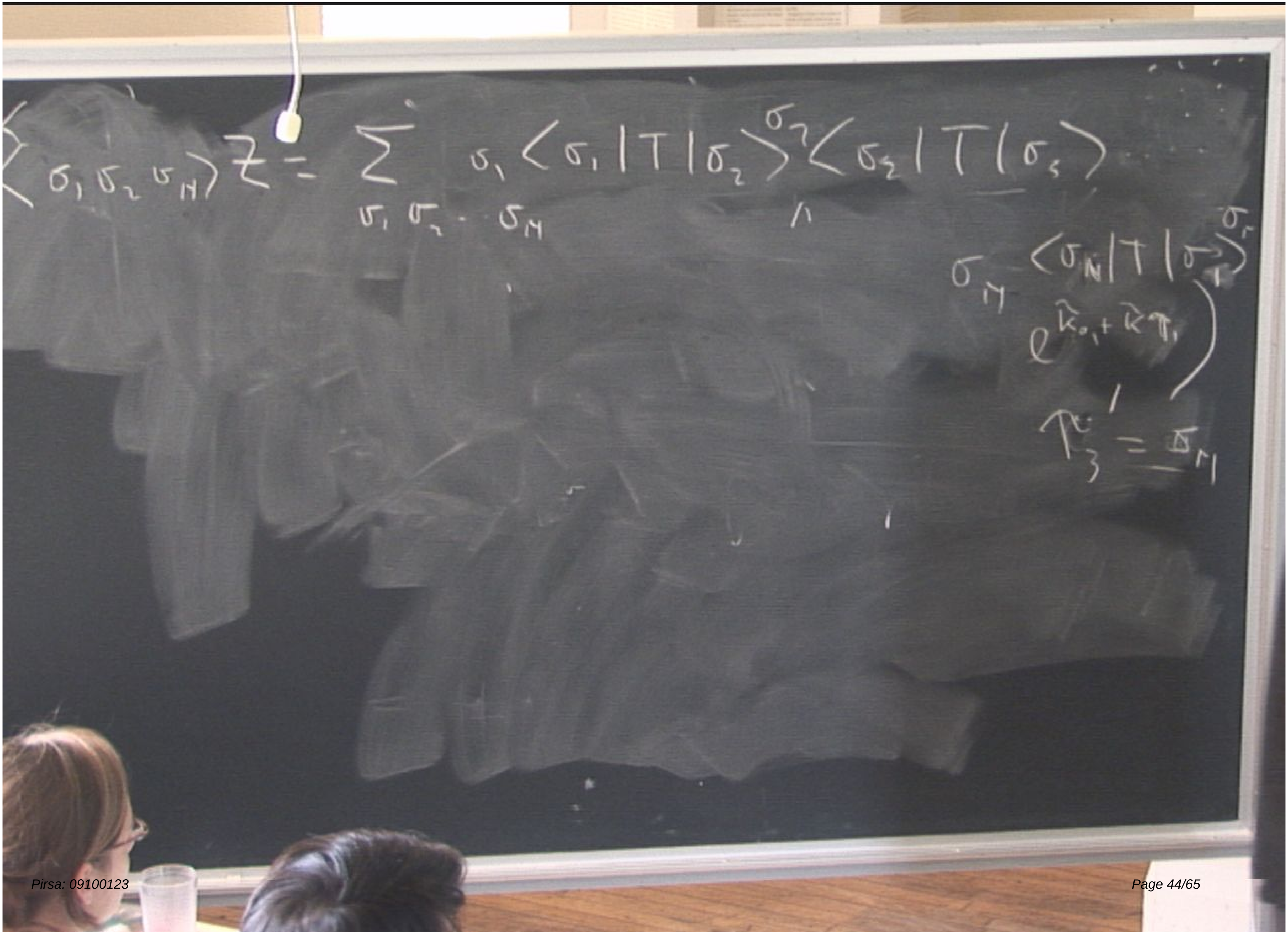
$$Z = \sum_{\sigma_1, \sigma_2 \cdots \sigma_M} \langle \sigma_1 | T | \sigma_2 \rangle \langle \sigma_2 | T | \sigma_3 \rangle \cdots$$

$$Z = \sum_{\sigma_1 \sigma_2 \cdots \sigma_M} \langle \sigma_1 | T | \sigma_2 \rangle \langle \sigma_2 | T | \sigma_3 \rangle$$

$$\langle \sigma_1 | T | \sigma_N \rangle$$

$$e^{\tilde{K}\sigma_1 + \tilde{K}\tau_1}$$

$$\tau^z_3 = \sigma_M$$


⟨σ1 σ2 σN⟩ Z = Σ σ1 ⟨σ1 | T | σ2⟩ σ2 ⟨σ2 | T | σ3⟩
σ1 σ2 · σN
σN ⟨σN | T | σ1⟩ σ1
e
= σN

$$\langle \sigma_1 \sigma_2 \sigma_M \rangle Z = \sum_{\sigma_1 \sigma_2 \cdots \sigma_M} \sigma_1 \langle \sigma_1 | T | \sigma_2 \rangle \sigma_2 \langle \sigma_2 | T | \sigma_3 \rangle \cdots$$

$$\sigma_M \langle \sigma_N | T | \sigma_1 \rangle \sigma_1$$

$$\langle \sigma_1 \rangle Z$$

$$e^{\tilde{K}\sigma_1 + \tilde{K}\tau_1}$$

$$\tau_3 = \sigma_{r1}$$

$$\langle \sigma_1 \sigma_2 \sigma_N \rangle Z = \sum_{\sigma_1 \sigma_2 \cdots \sigma_N} \sigma_1 \langle \sigma_1 | T | \sigma_2 \rangle \sigma_2 \langle \sigma_2 | T | \sigma_3 \rangle$$

$$\sigma_N \quad \langle \sigma_N | T |$$

$$\langle \sigma_1 \rangle Z = \sum_{\sigma_1 \cdots \cancel{\sigma_N}} (\sigma_1) \langle \sigma_1 | T^N | \sigma_1 \rangle$$

$$e^{\tilde{K}_0 + \tilde{K} \tau_1}$$

$$= \text{trace} (\quad) \; T^N$$

$$\tau_3 \quad (\tilde{K}_0 + \tilde{K} \tau_1)^N$$

$$\tau_3' = $$

$$= \text{trace} \; \hat{\tau}_3 \, e$$

$$\langle \sigma_1 \sigma_2 \sigma_N \rangle Z = \sum_{\sigma_1 \sigma_2 \cdots \sigma_N} \sigma_1 \langle \sigma_1 | T | \sigma_2 \rangle \sigma_2 \langle \sigma_2 | T | \sigma_3 \rangle$$

$$\sigma_N \langle \sigma_N | T | \sigma$$

$$\langle \sigma_1 \rangle Z = \sum_{\sigma_1 \cdots \sigma_N} (\sigma_1) \langle \sigma_1 | T^N | \sigma_1 \rangle$$

$$e^{\tilde{K}_{0,} + \tilde{K} \tau_1}$$

$$\tau_3' = \sigma$$

$$= \mathrm{trace}\,(\;)\, T^N$$

$$\tau_3 \quad (\tilde{K}_0 + \tilde{K} \tau_1) N$$

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$$\sigma_M \langle \sigma_N | T | \sigma_1 \rangle \sigma_1$$

$$Z = \sum_{\sigma_1 \cdots \sigma_M} (\sigma_1) \langle \sigma_1 | T^M | \sigma_1 \rangle$$

$$e^{\tilde{K}_0 + \tilde{K}\tau_1}$$

$$= \text{trace} (\;) \; T^M$$

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$$\tau_3 = \sigma_M$$

$$= \text{trace} \left[\hat{\tau}_3 \, e \right.$$

$\mathrm{trace}\ \tau_3\, e^{(K_0 + k\tau_1)N}$

$$\text{trace}\ \tau_3\, e^{(K_0 + k\tau_1)N}$$

$$\langle \tau_3 = +1 |\, 1\ e^{(\tilde{K}_0 + \tilde{k}\tau_1)N}\, | \tau_3 = +1 \rangle$$

$$+ \langle \tau_3 = -1 | \,(-)\, | \tau_3 = -1 \rangle \longrightarrow$$

$$= 0$$

$$\langle \sigma_1 \sigma_2 \sigma_M \rangle Z = \sum_{\sigma_1 \sigma_2 \cdots \sigma_M} \sigma_1 \langle \sigma_1 | T | \sigma_2 \rangle \sigma_2 \langle \sigma_2 | T | \sigma_3 \rangle \cdots \sigma_M \langle \sigma_M | T |$$

$$\langle \sigma_1 \rangle Z = \sum_{\sigma_1 \cdots \sigma_M} (\sigma_1) \langle \sigma_1 | T^M | \sigma_1 \rangle$$

$$= \text{trace}\,(\hat{\tau}_3)\, T^M$$

$$= \text{trace}\left[\hat{\tau}_3\, e^{(\tilde{K}_0 + \tilde{K}\hat{\tau}_1) M}\right]$$

$$e^{\tilde{K}_0 + \tilde{K}\tau_1}$$

$$\tau_3' =$$

Statistical Correlations in a Long Chain

We should be able lots about the statistical mechanics of a the long chain with Ising style interactions. For example, let us calculate the average of the jth spin on a long chain or the correlations among the spins in the chain. Start from

$$Z = \mathrm{Tr}\,\exp[\,\textstyle\sum_{j=1}^{N} K\sigma_{j+1}\sigma_j\,]$$

$$<\sigma_k> = (1/Z)\mathrm{Tr}\,\sigma_k\,\exp[\,\textstyle\sum_{j=1}^{N} K\sigma_{j+1}\sigma_j\,]$$

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$$= (2\cosh K)^N + (2\sinh K)^N \qquad \approx \quad (2\cosh K)^N$$

The ≈ is an approximate equality which holds for large N. Note that in this limit the term with eigenvalue of $\tau_1 = 1$ dominates because the dual coupling is positive.

Average magnetization in a Long Chain

We know the answer: the system has full symmetry between spin up and spin down so that the average magnetization must be zero. Nonetheless, let's calculate

$Z\langle\sigma_j\rangle = \text{trace}_\tau \{\exp[-H\,\tau(j-1)]\ \tau_3\ \exp[-H\,\tau(N-j+1)]\}$

Since we can rearrange terms under a trace, as trace (ab)=trace (ba), this expression simplifies to

$$= \text{trace}_\tau \{\tau_3\ \exp[-H\,\tau N]\} = \text{trace}_\tau \{\tau_3\ \exp[-(K_0+K\,\tau_1 N]\}$$

To evaluate the last expression we must take diagonal matrix elements of τ_3 between eigenstates of τ_1. Both such matrix elements are zero. why? Because τ_3 acts to change the value of τ_1 so that $\tau_3\,|\tau_1=1\rangle = |\tau_1=-1\rangle$ so that $\langle\tau_1=1|\,\tau_3\,|\tau_1=1\rangle = \langle\tau_1=1|\,|\tau_1=-1\rangle=0$. Therefore the entire result is zero and the average has the value zero, as expected.

$$\langle\sigma_j\rangle = 0$$

At zero magnetic field, the magnetization of the one-dimensional Ising model is zero. Thus, this Ising model has no ordered state. In fact no one-dimensional system with finite interactions has one. This model is always in the disordered phase at all finite temperatures.

Correlations in Large N limit

Let N be large. Z simplifies to $Z = \exp(N\tilde{K}_0 + N\tilde{K})$ since the τ_1=1 term dominates the trace

We start from*

$$Z <\sigma_j\sigma_{j+r}>= \text{trace}_\tau \{e^{-(j-1)H}\ \tau_3\ e^{-rH}\ \tau_3\ e^{-(N-j-r+1)H}\}, \quad \text{for large } N$$

* Note how the ordering in space converts into an ordering in time.

Since we can rearrange terms under a trace, as trace (ab)=trace (ba), this expression simplifies to

$$(\text{trace}_\tau\ e^{-NH}) <\sigma_j\sigma_{j+r}>= \text{trace}_\tau \{e^{-(N-r)H}\ \tau_3\ e^{-rH}\ \tau_3\}, \quad \text{so that}$$

The K_0 term is the same on both sides of the equation. It cancels.
For large N, the τ_1 =1 term dominates both traces. Since the effect of τ_3 is to change the eigenvalue of τ_1 this result is

$$\exp[\tilde{K}N] <\sigma_j\sigma_{j+r}>\approx \{\exp[\tilde{K}N]\ \exp[-2\tilde{K}r]\}, \quad \text{for large } N \qquad \text{Consequently}$$

$$<\sigma_j\sigma_{j+r}>\approx \exp[-2\tilde{K}r], \quad \text{for large } N \qquad \text{iii. 7}$$

The result is that correlations fall off exponentially with distance, with the typical falloff distance, denoted as ξ , being the distance between lattice points (usually called a) times $1/(2D(K))=1/(2\tilde{K})$.

$$\langle \sigma_1 \sigma_2 \sigma_M \rangle Z = \sum_{\sigma_1 \sigma_2 \cdots \sigma_M} \sigma_1$$

$$|\tau_1 = -1\rangle = \hat{\tau}_3 \, |\hat{\tau}_1 = +1\rangle$$

$$\langle \sigma_1 \rangle Z = \sum_{\sigma_1 \cdots \sigma_M} (\sigma_1) \langle \sigma_1 | T^M | \sigma_1 \rangle$$

$$\sigma_M \quad \langle \sigma_M | T |$$

$$e^{\tilde{K}_0 + \tilde{K} \tau_1}$$

$$= \mathrm{trace} \left(\hat{\tau}_3 \right) T^M$$

$$\hat{\tau}_3' =$$

$$= \mathrm{trace} \left[\hat{\tau}_3 \, e^{(\tilde{K}_0 + \tilde{K} \hat{\tau}_1) M} \right]$$

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* Note how the ordering in space converts into an ordering in time.

Since we can rearrange terms under a trace, as trace (ab)=trace (ba), this expression simplifies to

$$(\text{trace}_\tau\ e^{-NH}) <\sigma_j \sigma_{j+r}> = \text{trace}_\tau \{e^{-(N-r)H}\ \tau_3\ e^{-rH}\ \tau_3\}, \qquad \text{so that}$$

The K_0 term is the same on both sides of the equation. It cancels.
For large N, the τ_1 =1 term dominates both traces. Since the effect of τ_3 is to change the eigenvalue of τ_1 this result is

$$\exp[\tilde{K}N] <\sigma_j \sigma_{j+r}> \approx \{\exp[\tilde{K}N] \exp[-2\tilde{K}r]\}, \qquad \text{for large } N \qquad \text{Consequently}$$

$$<\sigma_j \sigma_{j+r}> \approx \exp[-2\tilde{K}r], \qquad \text{for large } N \qquad \text{iii. 7}$$

The result is that correlations fall off exponentially with distance, with the typical falloff distance, denoted as ξ , being the distance between lattice points (usually called *a*) times 1/(2D(K))=1/(2$\tilde{K}$).

Correlation Length

$$<\sigma_j \sigma_{j+r}> = exp(-2r\tilde{K}) \qquad = \exp(-ar/\xi)$$

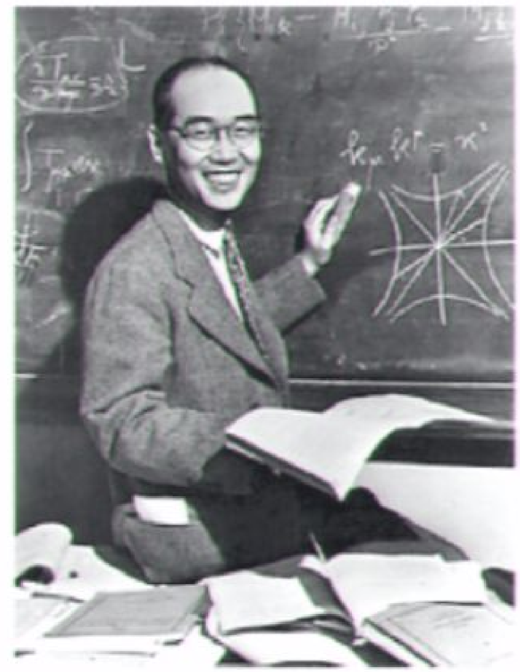
Here *ar* is distance between the sites of the two spins.

The result is that correlations fall off exponentially with distance, with the typical falloff distance, denoted as ξ, being the distance between lattice points (usually called *a*) times $1/(2D(K))=1/(2\tilde{K})$. This falloff distance is very important in field theory, particle physics, and phase transition theory. In the latter context it is called the coherence length. It is also called the Yukawa distance because it first came up in Hideki Yukjawa's description of mesons. Here, in the one dimensional Ising model, we have a very large coherence length for large K. Specifically

$$1/(2\tilde{K}) = \xi/a \to \exp(2K)/2 \text{ as } K \to \infty$$

while is very small in the opposite limit of small K.

$$\xi/a \to 1/(-\ln(2K)) \text{ as } K \to 0$$



Hideki Yukawa

Large correlation lengths, or equivalently small masses, play an important role in statistical and particle physics since they indicate a near-by phase transition or change in behavior.

ξ = correlation length= a/[2D(K)] =a/($2\tilde{K}$)

$$\sigma_j \sigma_{j+r} \rangle = (\tanh K)^r = e^{-(\ln r)/\xi}$$

$N \to \infty \quad r \ll N$

$$\sigma_1\rangle \quad Z = \sum_{\sigma_1 \ldots \sigma_N} (\sigma_1) \langle \sigma_1 | T^N | \sigma_1 \rangle \qquad \sigma_N \langle \sigma_N | T | \sigma_1 \rangle$$

$$e^{\tilde{K}_0 + \tilde{K} \tau_1}$$

$$= \mathrm{trace}\,(\;)\; T^N \qquad \tau_3 = \sigma_N$$

$$\tau_3 \quad \left[(\tilde{K}_0 + \tilde{K}\tau_1)N\right]$$

$$= \mathrm{trace}\, \tau_3\, e$$

$$\langle \sigma_j \, \sigma_{j+r} \rangle = (\tanh K)^r = e^{-(a r)/\xi}$$

$N \to \infty \quad r <$

Correlation Length

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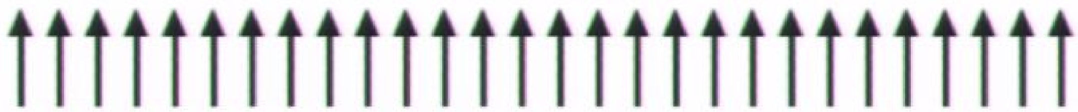
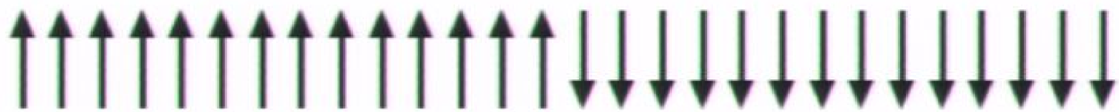
Bloch Walls in 1 d

In the Ising model at large values of the coupling, K, the spins tend to line up.



However, with a cost in probability exp(-2K) a whole region might flip its spins, producing a defect called a Bloch wall



This kind of defect produces the decay of correlations in the Ising model at low temperatures. In any long Ising chain, many such defects will be randomly placed and ruin any possibility of correlations over infinitely long distances.

This is the simplest example of what is called a topological excitation, a defect which breaks the ordering in the system by separating two regions with different kinds of order. Since ordering is crucial in many situations, so are topological excitations.

Notice that, at low temperatures, this kind of excitation is much more likely than a simple flip of a single spin. The wall costs a factor of exp(-K); the flip costs exp(-2K).

Renormalization for 1D ising,

following ideas of Kenneth Wilson, this calculation is due to David Nelson and myself

$$Z = \sum_{\sigma_1, \sigma_2, \cdots} \exp(W_K\{\sigma\}) = \sum_{\sigma_1, \sigma_2, \cdots} \exp(K\sigma_1\sigma_2 + K\sigma_2\sigma_3 + \cdots)$$

Rearrange calculation: Rename spins separated by two lattice sites: let $\mu_1=\sigma_1$; $\mu_2=\sigma_3$, $\mu_3=\sigma_5$, ...; and sum over every other spin, σ_2, σ_4 .....

$$Z = \sum_{\mu_1, \mu_2, \cdots} \sum_{\sigma_2, \sigma_4, \cdots} \exp(K\mu_1\sigma_2 + K\sigma_2\mu_2 + \cdots) = \sum_{\mu_1, \mu_2, \cdots} \exp(w'\{\mu\})$$

Note that sum over σ_2, σ_4 ..... generates only nearest neighbor interactions for the μ's

$w'\{\mu\}$= const + K' μ_1 μ_2 + K' μ_2 μ_3 + ...

K' describes same system as before, with a new separation between lattice sites, which is twice as big as the old separation. Since the physical system is the same, physical quantities like the correlation length and the entropy are unchanged, but their description in terms of couplings and lattice constants has changed. In particular, the new lattice spacing is a' =2a, but the correlation length is exactly the same $\xi' = \xi$. Since we know that the correlation length is given by

ξ = a/[2D(K)], we know that the new coupling obeys a/[2D(K)] = a'/[2D(K')] we find that the new coupling obeys D(K') = 2 D(K) before we do any detailed renormalization calculations. Since D is a decreasing function of K we know that the new, renormalized, coupling is smaller than the old one.

renormalization

$$\sum_{\sigma_2} e^{K\sigma_1\sigma_2}\, e^{K\sigma_2\sigma_3} = \text{const}\; e^{K'\sigma_1\sigma_3}$$

renormalization

$\xleftrightarrow{2a}$

$$\sum_{\sigma_2} e^{K\sigma_1\sigma_2}\, e^{K\sigma_2\sigma_3} = \text{const}\; e^{K'\sigma_1\sigma_3}$$

$$K' = R(K)$$

Ising Model in d=2

$$-H/(kT) = K \sum_{nn} \sigma_r \sigma_s + h \sum_r \sigma_r$$

$\sigma_r = \pm 1$

nn indicates a sum over nearest neighboirs

square lattice

Onsager calculated partition function and phase transition for this situation

Nearest neighbor structure
s's are nearest neighbors to r
Bonds=exp($K\sigma\sigma'$) connect nearest neighbors