Title: Quantum Field Theory (PHYS 601) - Lecture 14

Date: Oct 16, 2009 09:00 AM

URL: http://pirsa.org/09100122

Abstract:

Recall: $\partial_\mu A^\mu = 0$

$$A_\mu(\underset{\sim}{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2|\underset{\sim}{p}|}} \sum_{\lambda=0}^{3} \epsilon^\lambda_\mu(\underset{\sim}{p}) \left[a^\lambda_{\underset{\sim}{p}} e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}} + a^{\lambda\dagger}_{\underset{\sim}{p}} e^{-i\underset{\sim}{p}\cdot\underset{\sim}{x}} \right]$$

ϵ^0 timelike

ϵ^3 longitudinal

$\epsilon^{1,2}$ transverse $\epsilon^1 \cdot p = \epsilon^2 \cdot p = 0$

<u>Recall</u>: $\partial_\mu A^\mu = 0$

$$[A_\mu(\underline{x}), \pi^\nu(\underline{y})] = i\,\delta^{(3)}(\underline{x}-\underline{y})\,\delta_\mu^{\ \nu}$$

$$\Rightarrow \quad [a_{\underline{p}}^{\lambda}, a_{\underline{q}}^{\lambda'\dagger}] = -\eta^{\lambda\lambda'}(2\pi)^3\,\delta^{(3)}(\underline{p}-\underline{q})$$

$$\Rightarrow$$

Recall: $\partial_\mu A^\mu = 0$

$$[A_\mu(\underset{\sim}{x}), \pi^\nu(\underset{\sim}{y})] = i\,\delta^{(3)}(\underset{\sim}{x}-\underset{\sim}{y})\,\delta_\mu{}^\nu$$

$$\Rightarrow [a_k^\lambda, a_q^{\lambda' +}] = -\eta^{\lambda\lambda'}(2\pi)^3\,\delta^{(3)}(\underset{\sim}{k}-\underset{\sim}{q})$$

$\Rightarrow$ $a_k^{0+}|0\rangle$ has negative norm.

To remove these negative norm states, we'll try to impose $\partial_\mu A^\mu = 0$ in quantum theory. But how?

$$A_\mu(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2|p|}} \sum_{\lambda=0}^{3} \epsilon_\mu^\lambda(p) \left[a_p^\lambda e^{-ip\cdot x} + a_p^{\lambda\dagger} e^{+ip\cdot x} \right]$$

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<u>Option 1</u>: Could try to consider $\partial_\mu A^\mu = 0$ as a restriction on operators in Heisenberg picture. This is to strong — not consistent with commutation relations.

Option 2:

$\Rightarrow$ $a_k^{0\dagger}|0\rangle$ has negative norm.

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$$\partial_\mu A^\mu |\Psi\rangle = 0.$$

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But this doesn't work

To see this

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$$\sim a^\lambda_{\mathbf{k}} \qquad\qquad \sim a^{\lambda\dagger}_{\mathbf{k}}$$

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$$A_\mu(x) = A_\mu^+(x) + A_\mu^-(x).$$

$\sim a_p^\lambda$ $\qquad\qquad$ $\sim a_p^{\lambda\dagger}$

But if we empose this constraint on $|0\rangle$

$$\Rightarrow \partial_\lambda A^{\lambda+}|0\rangle = 0$$

but $\partial_\lambda A^{\lambda-}|0\rangle \neq 0 \Rightarrow$ this condition doesn't keep the vacuum

Option 3: We'll define physical states $|\Psi\rangle$ to obey

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$$\partial^\mu A^+_\mu |\Phi\rangle = 0$$

This ensures that, for any two physical states $|\Phi\rangle$ and $|\Phi'\rangle$

$$\langle \Phi' | \partial_\mu A^\mu | \Phi \rangle = 0$$

This is the Gupta-Bleuler condition.

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but $\partial_\mu A^{\mu -} |0\rangle \neq 0 \Rightarrow$ this condition doesn't keep the vacuum

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$$|\psi\rangle = |\psi_T\rangle \otimes |\phi\rangle$$

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$$A_\mu(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2|\vec{p}|}} \sum_{\lambda=0}^{3} \epsilon_\mu^\lambda(\vec{p}) \left[a_{\vec{p}}^\lambda e^{-ip\cdot x} + a_{\vec{p}}^{\lambda\dagger} e^{+ip\cdot x} \right]$$

$p = (1,0,0,1)$

$\epsilon^0 = (1,0,0,0)$

$\epsilon^3 = (0,0,0,1)$

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This condition does' eliminate negative norm states, but leaves us with zero norm states

$$\langle\phi|\phi\rangle = 0$$

What does $\mathcal{H}_{phys}$ look like? Decompose

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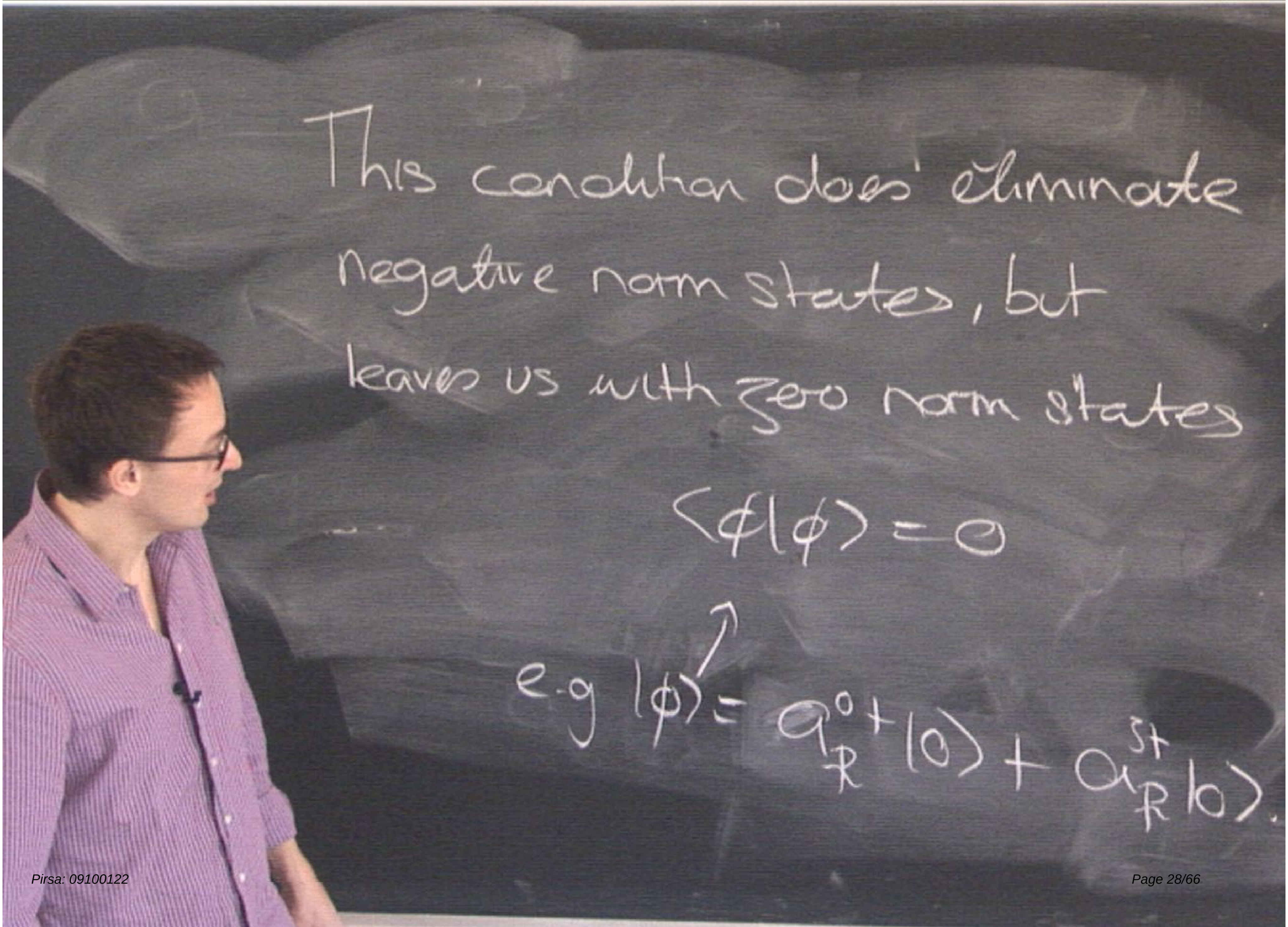
This condition does' eliminate
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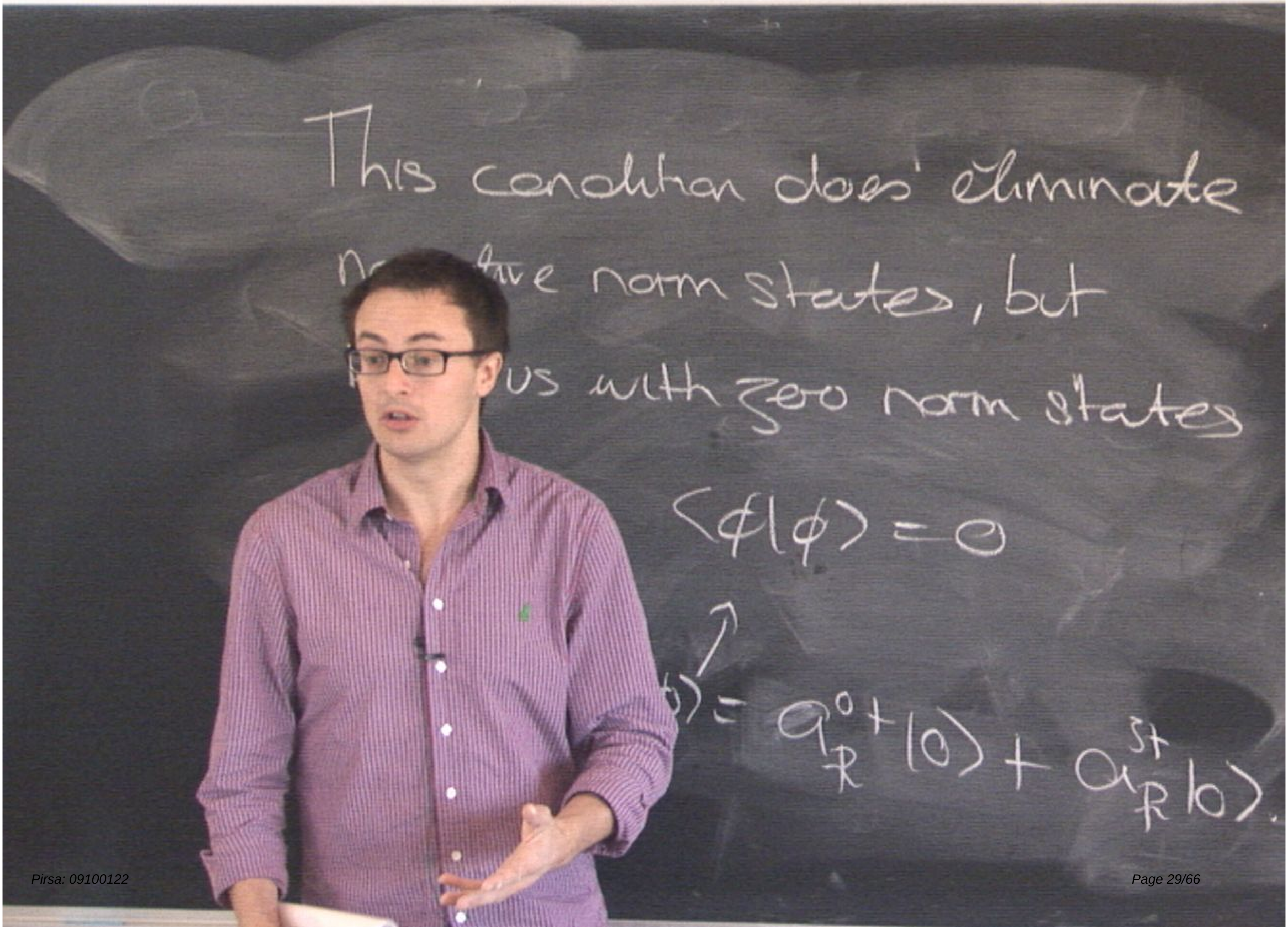
leaves us with zero norm states
$\langle \phi | \phi \rangle = 0$
e.g $|\phi\rangle = a_R^{0+}|0\rangle + a_R^{5+}|0\rangle$.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$$\langle \phi | \phi \rangle = 0$$

e.g. $|\phi\rangle = a_R^{0\dagger}|0\rangle + a_R^{3\dagger}|0\rangle$.

We treat these zero norm states as gauge equivalent to zero. Two

States which differ in their longitudinal + timelike polarization states are said to be physically equivalent.

We can check that no physical observable depends on $|\phi\rangle$.

e.g. $$H = \int \frac{d^3p}{(2\pi)^3} |p| \left(\sum_{i=1}^{3} a_p^{i\dagger} a_p^i - a_p^{0\dagger} a_p^0 \right)$$

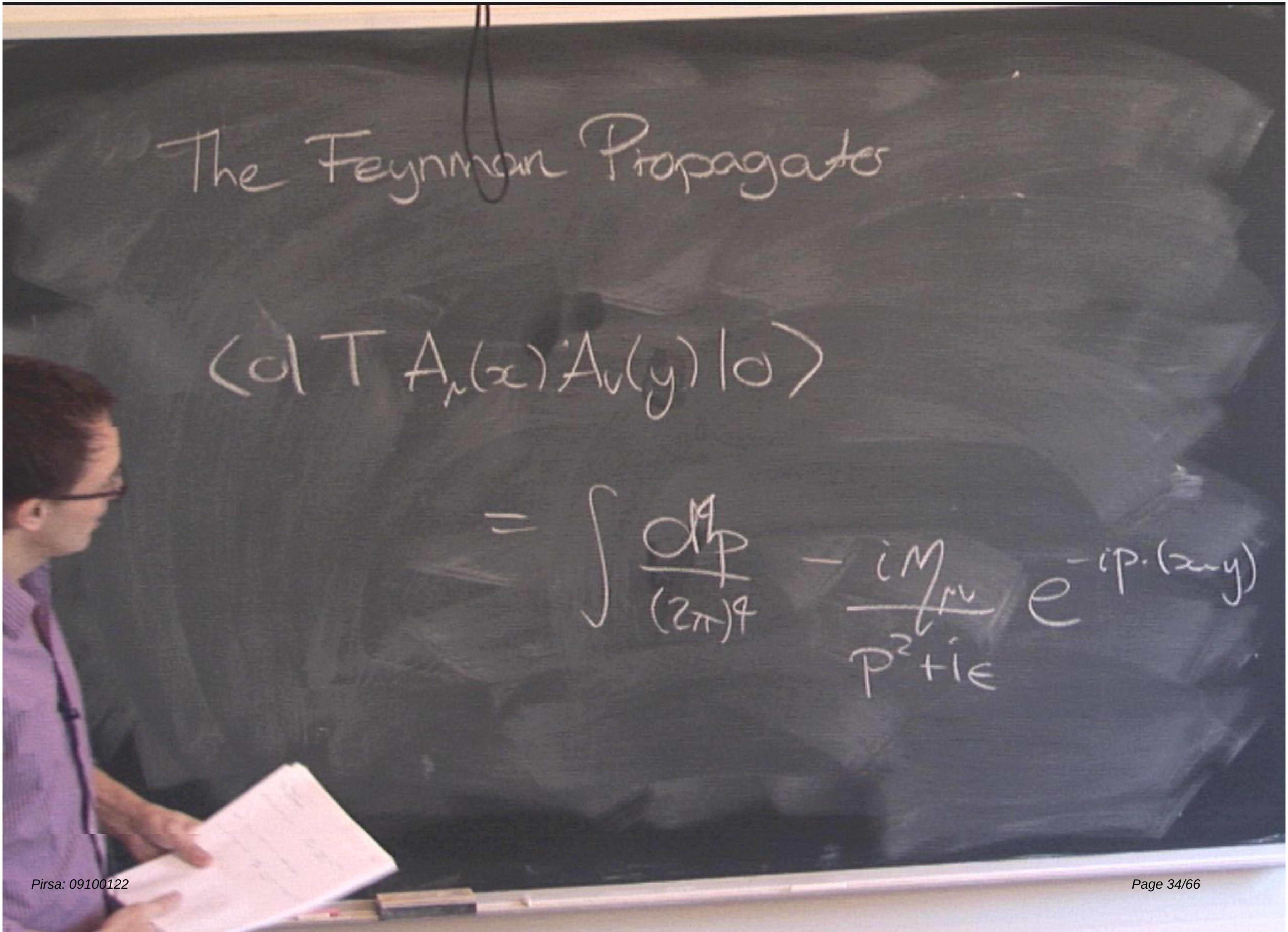
But $$(a_p^3 - a_p^0)|\phi\rangle = 0 \Rightarrow \langle\phi| a_p^{3\dagger} a_p^3 |\phi\rangle$$

$$= \langle\phi| a_p^{0\dagger} a_p |\phi\rangle$$

The Feynman Propagator

$$\langle 0 | T\, A_\mu(x) A_\nu(x) | 0 \rangle$$

$$= \int$$


The Feynman Propagator
$\langle 0| T A_\mu(x) A_\nu(y) |0\rangle$
$= \int \frac{d^4p}{(2\pi)^4} \frac{-i\eta_{\mu\nu}}{p^2+i\epsilon} e^{-ip\cdot(x-y)}$

Coupling to Matter

We want an interacting Lagrgangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - A_\mu j^\mu$$

$$\Rightarrow \partial_\mu F^{\mu\nu} = j^\nu$$

Consistency $\Rightarrow$ $0 = \partial_\mu \partial_\nu F^{\mu\nu}$

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Coupling to Fermions

The Dirac Lagrangian $\mathcal{L} = \bar{\Psi}(i\not{\partial} - m)\psi$

has a symmetry $\psi \to e^{i\alpha}\psi$

$$\Rightarrow \quad j^{\mu} = \bar{\psi}\gamma^{\mu}\psi$$

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$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\not{\partial} - m)\psi - \underbrace{e}_{\text{coupling constant.}}\bar{\psi}\gamma^\mu A_\mu$$

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coupling constant.

To see that $\mathcal{L}$ is gauge invariant, write

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\not{D} - m)\psi.$$

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$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}\,(i\not{D} - m)\,\psi$$

$$\mathcal{D}_\mu \psi = \partial_\mu \psi + ieA_\mu \psi$$

is the <u>covariant derivative</u>

Claim This Lagrangian is gauge invariant, under

$$A_\mu \to A_\mu + \partial_\mu \lambda(x)$$

$$\psi \to e^{-ie\lambda(x)}\,\psi .$$

Proof: Look at $D_\mu \psi = \partial_\mu \psi + ieA_\mu \psi$

$$\rightarrow \partial_\mu (e^{-ie\lambda}\psi) + ieA_\mu (e^{-ie\lambda}\psi)$$

$$+ ie\partial_\mu \lambda (e^{-ie\lambda}\psi).$$

$$\to \partial_\mu\left(e^{-ie\lambda}\psi\right) + ieA_\mu\left(e^{-ie\lambda}\psi\right)$$

$$+ ie\partial_\mu\lambda\left(e^{-ie\lambda}\psi\right).$$

$$-ie\partial_\mu\lambda\cdot\left(e^{-ie\lambda}\psi\right) + e^{-ie\lambda}\partial_\mu\psi$$

$$+ ieA_\mu\left(e^{-ie\lambda}\psi\right) + ie\partial_\mu\lambda\left(e^{-ie\lambda}\psi\right)$$

$+$

$$\to \partial_\mu(e^{-ie\lambda}\psi) + ieA_\mu(e^{-ie\lambda}\psi)$$

$$+ ie\partial_\mu\lambda(e^{-ie\lambda}\psi).$$

$$= \cancel{-ie\partial_\mu\lambda\cdot(e^{-ie\lambda}\psi)} + e^{-ie\lambda}\partial_\mu\psi$$

$$+ ieA_\mu(e^{-ie\lambda}\psi) + \cancel{ie\partial_\mu\lambda(e^{-ie\lambda}\psi)}$$

$$=$$

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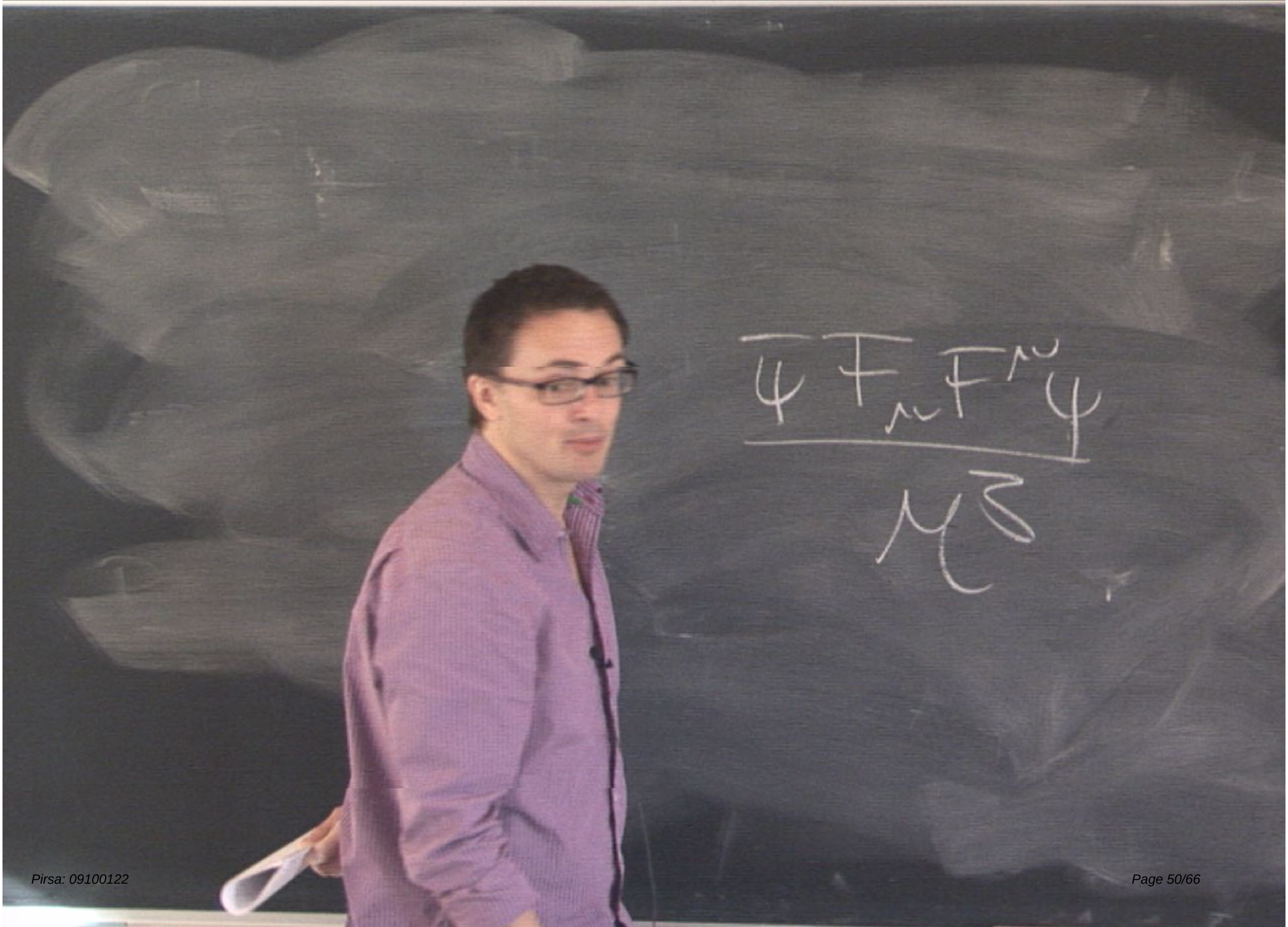
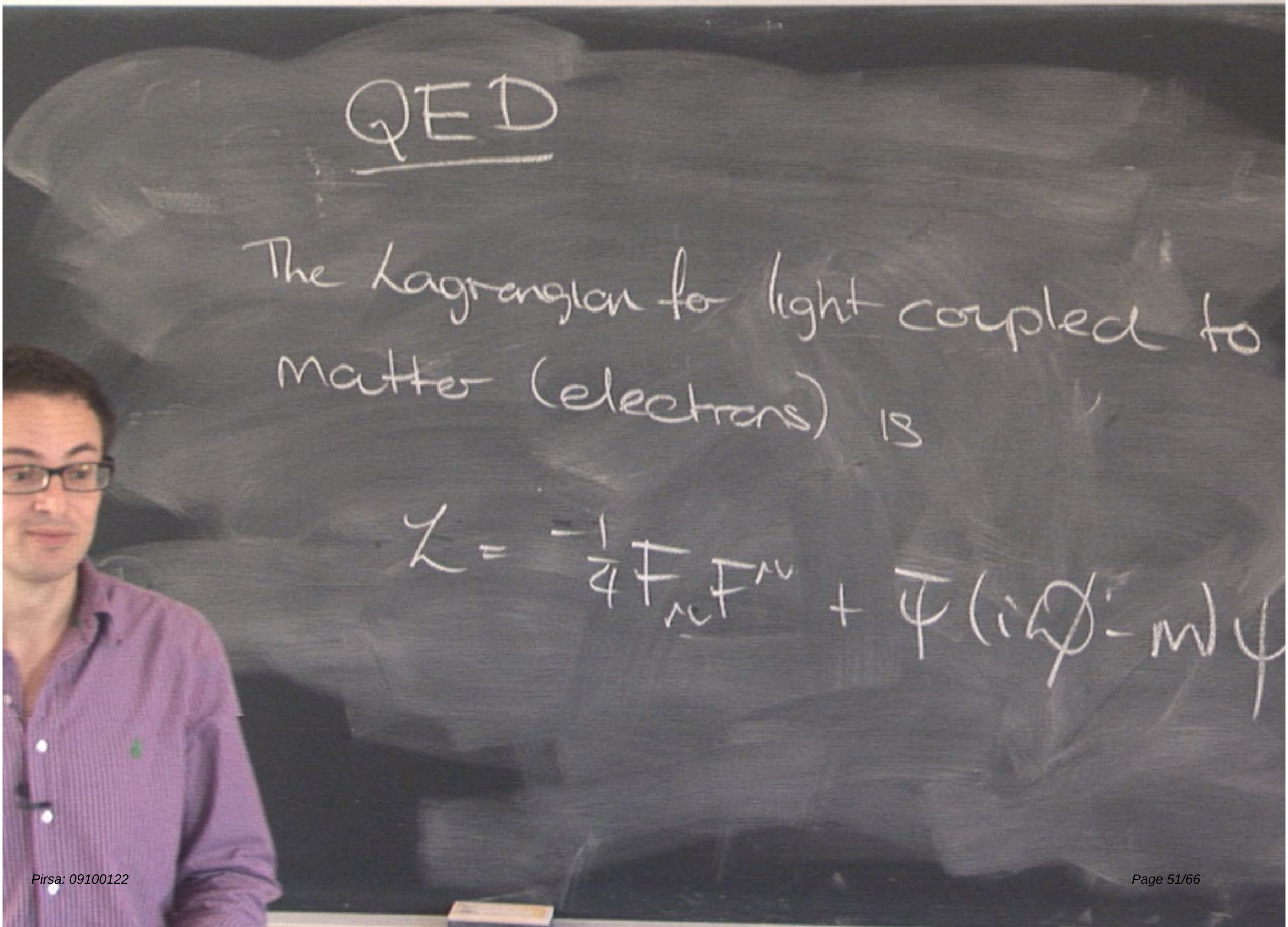
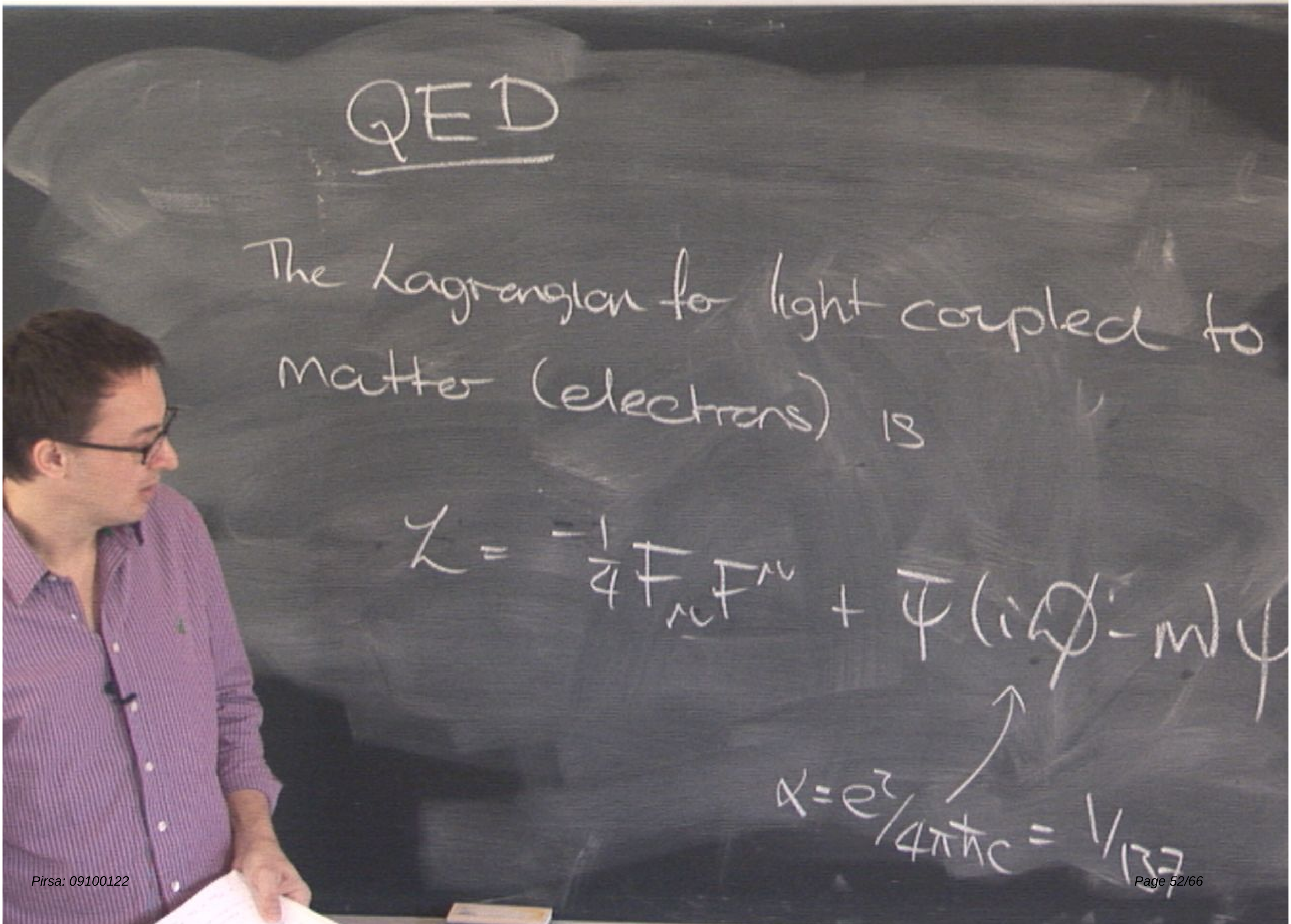
$$= e^{-ie\lambda} D_\mu\psi$$


$\frac{\bar{\psi} F_{\mu\nu} F^{\mu\nu} \psi}{M^3}$


QED
The Lagrangian for light coupled to matter (electrons) is
$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\not{\partial} - m)\psi$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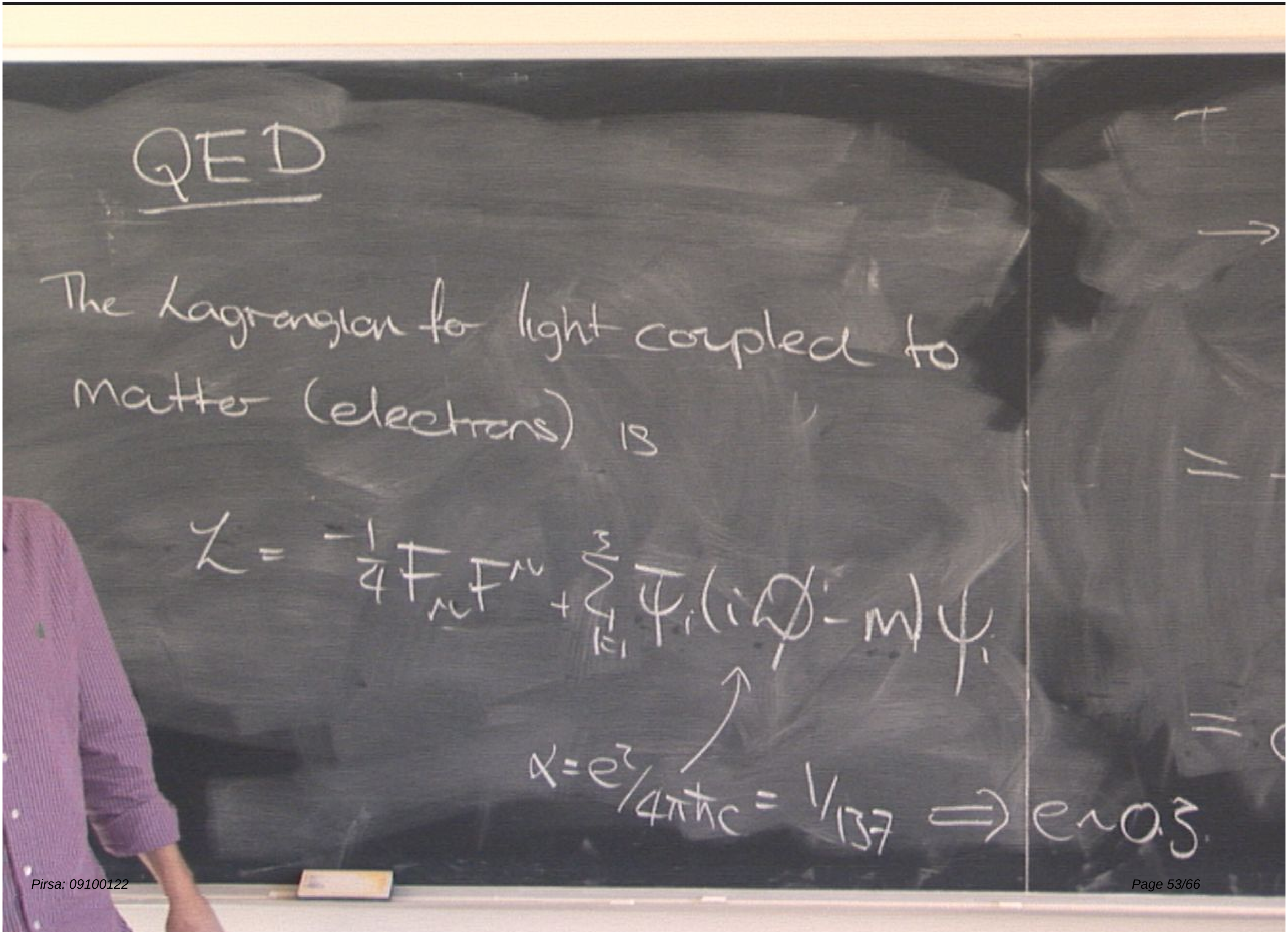
$\alpha = e^2/4\pi\hbar c = 1/137$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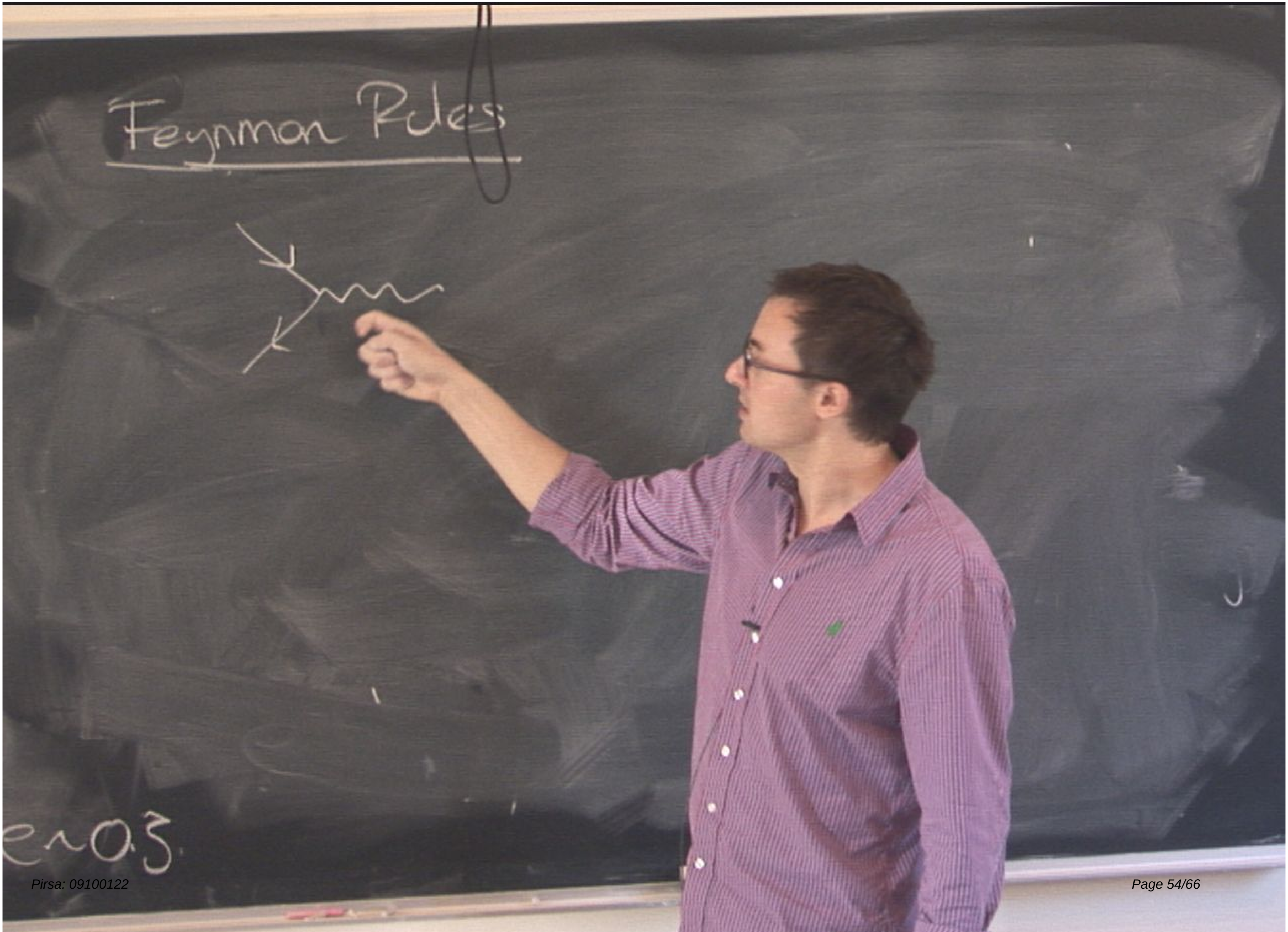
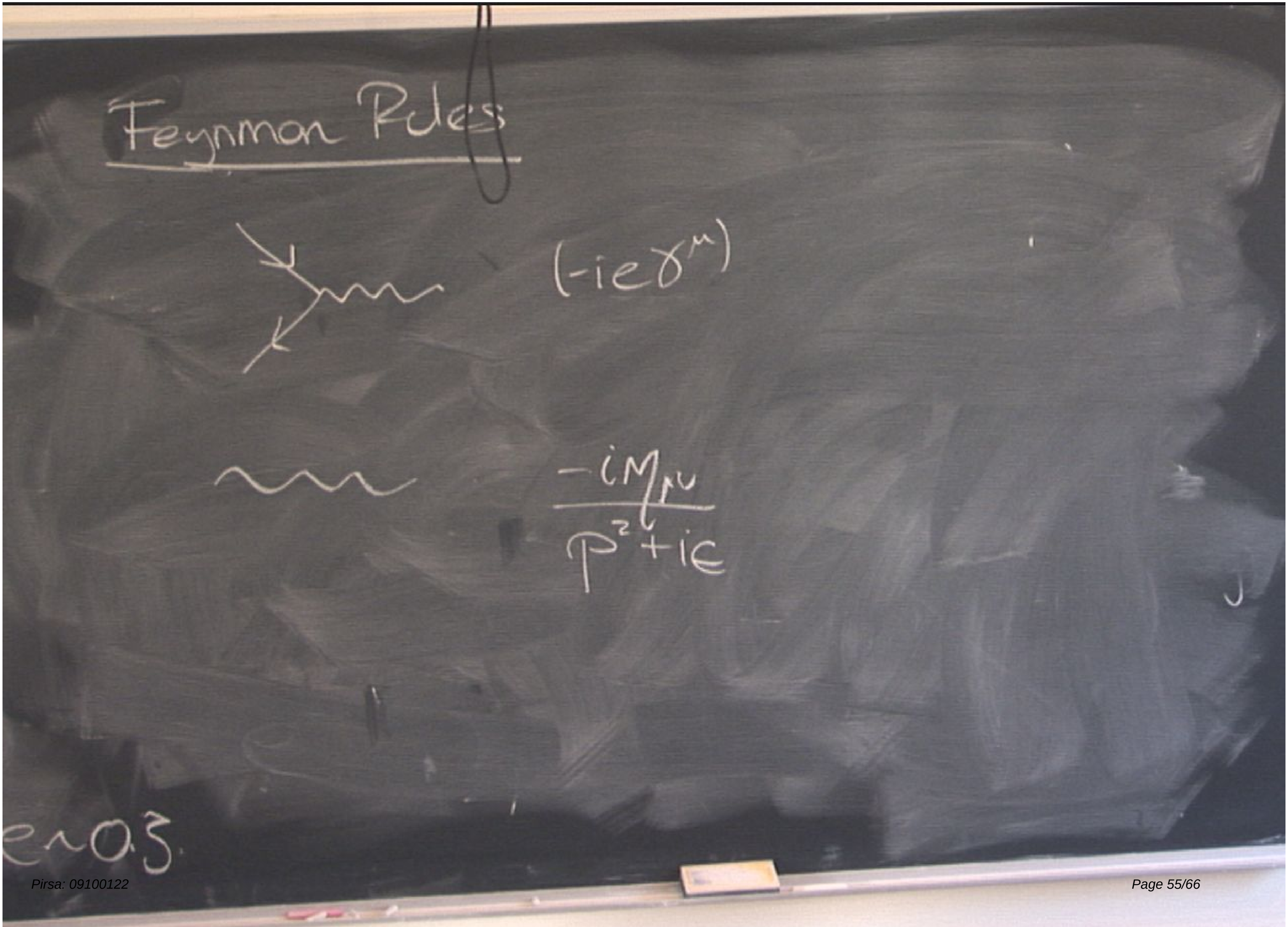
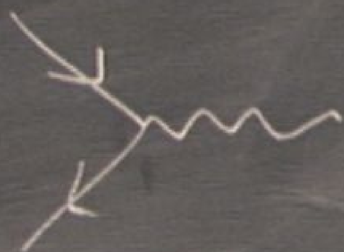
The Lagrangian for light coupled to matter (electrons) is
$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \sum_{i=1}^{3}\bar{\psi}_i(i\not{D} - m)\psi_i$
$\alpha = e^2/4\pi\hbar c = 1/137 \Rightarrow e \sim 0.3$


Feynman Rules
e~0.3


Feynman Rules
$(-ie\gamma^{\mu})$
$\frac{-i\eta_{\mu\nu}}{p^2+i\epsilon}$
$e \sim 0.3$

Feynman Rules



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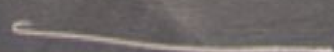
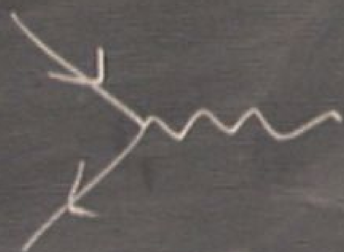


$$\frac{i(\not{p}+M)}{p^2-M+i\epsilon}$$

$e \sim 0.3$

Feynman Rules



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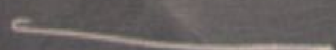

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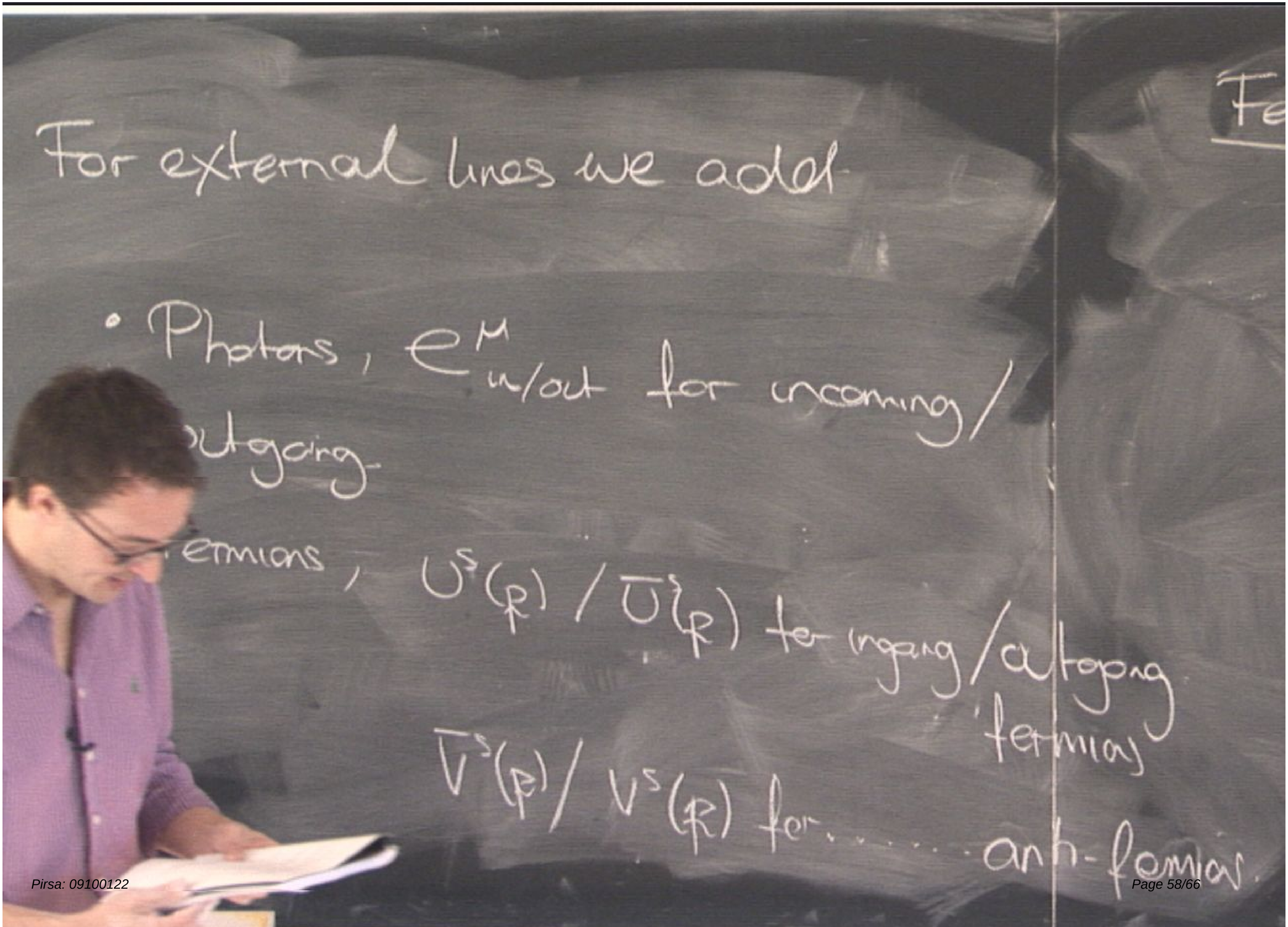
Fe
For external lines we add
• Photons, $\epsilon^{\mu}_{in/out}$ for incoming/ outgoing
ermions, $u^s(p)$ / $\bar{u}^s(p)$ for ingoing/outgoing fermions
$\bar{v}^s(p)$ / $v^s(p)$ for...... anti-fermions.

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Some Amplitudes.

Electron scattering $e^-e^- \to e^-e^-$.

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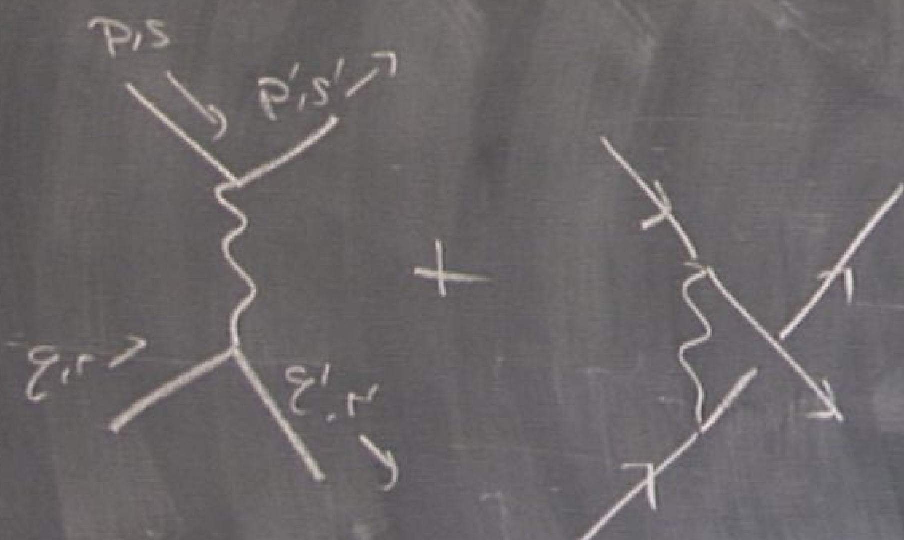
Electron scattering $e^-e^- \to e^-e^-$:



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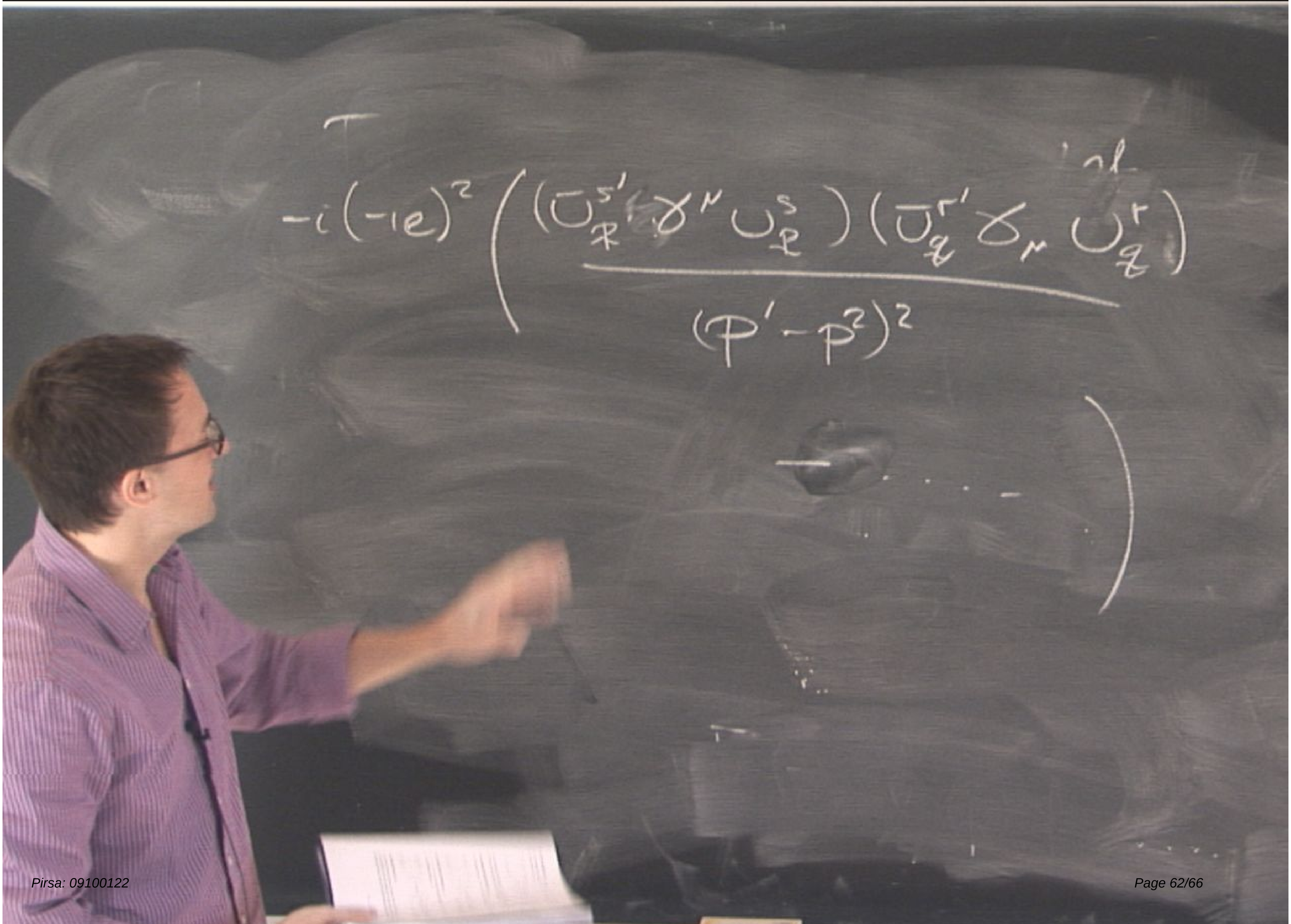
$-i(-ie)^2 \left(\frac{(\bar{u}^{s'}_{p'}\gamma^\mu u^s_p)(\bar{u}^{r'}_{q'}\gamma_\mu u^r_q)}{(p'-p)^2} \right.$
$- \ldots \left. \right)$

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$$\left. - \cdots \right)$$

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