Title: Quantum Field Theory (PHYS 601) - Lecture 13

Date: Oct 15, 2009 09:00 AM

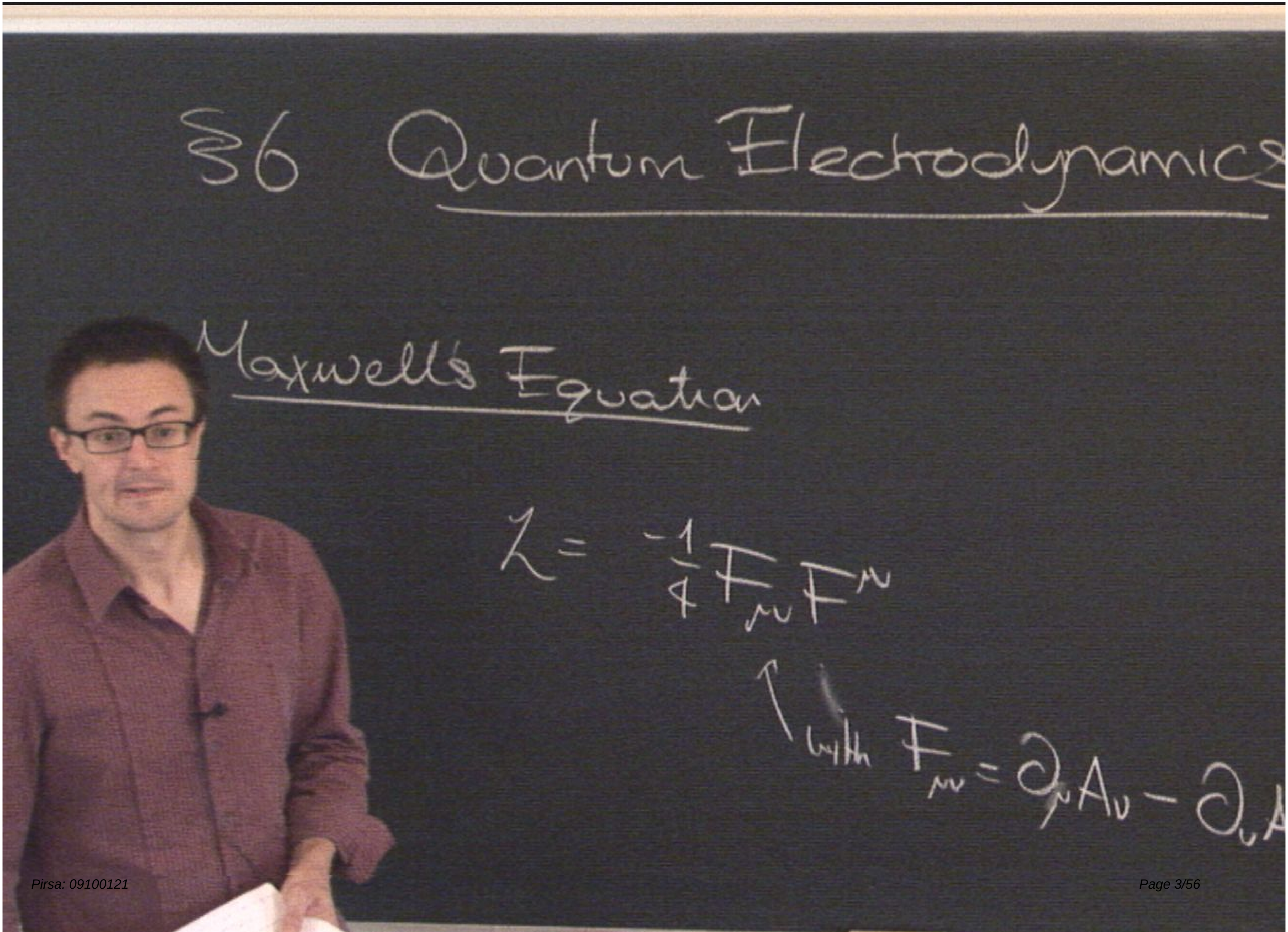
URL: http://pirsa.org/09100121

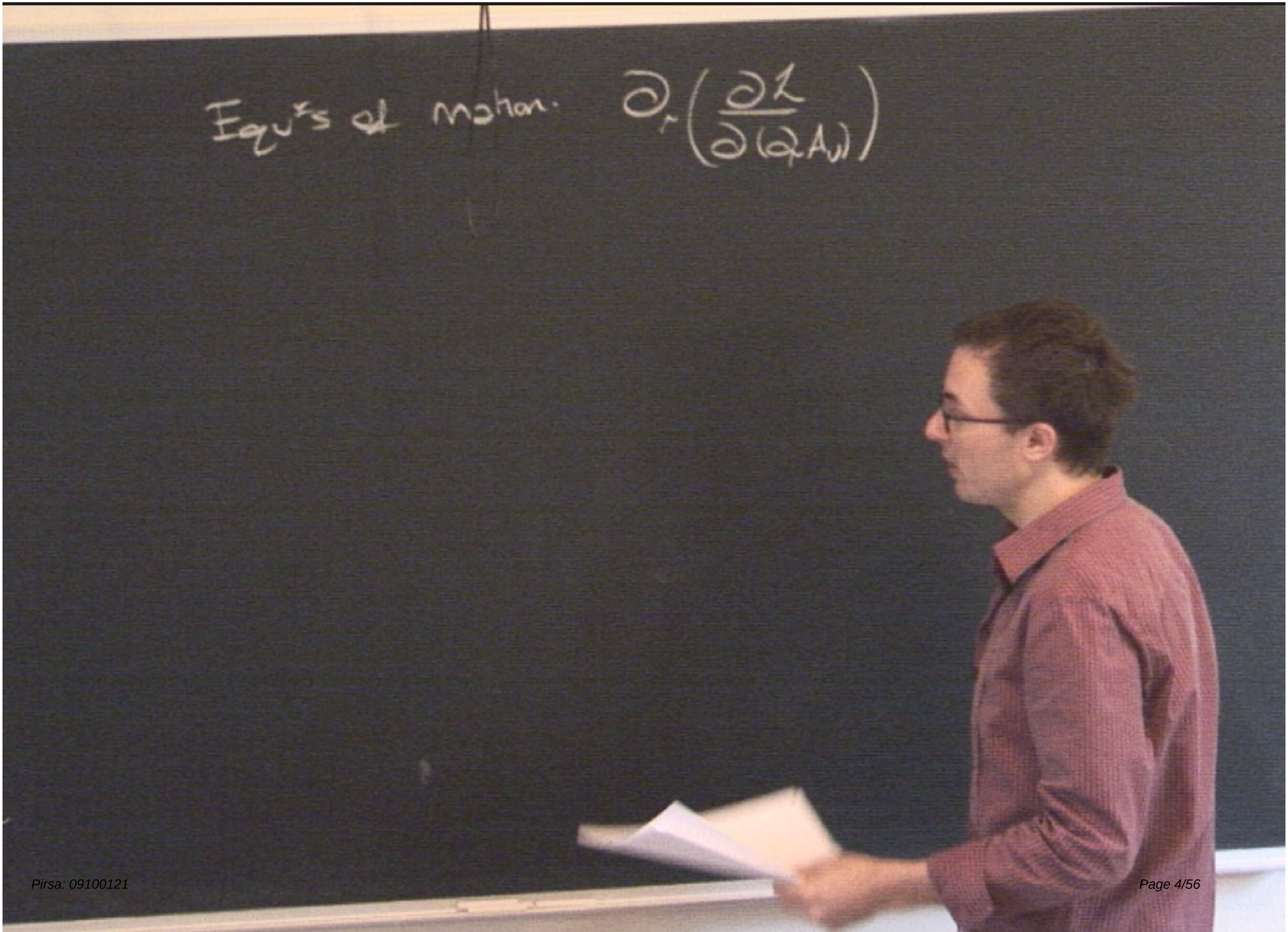
Abstract:

§6 Quantum Electrodynamics

Maxwell's Equation

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$


§6 Quantum Electrodynamics
Maxwell's Equation
$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$
with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A$


Equ^ns of motion. $\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)}\right)$

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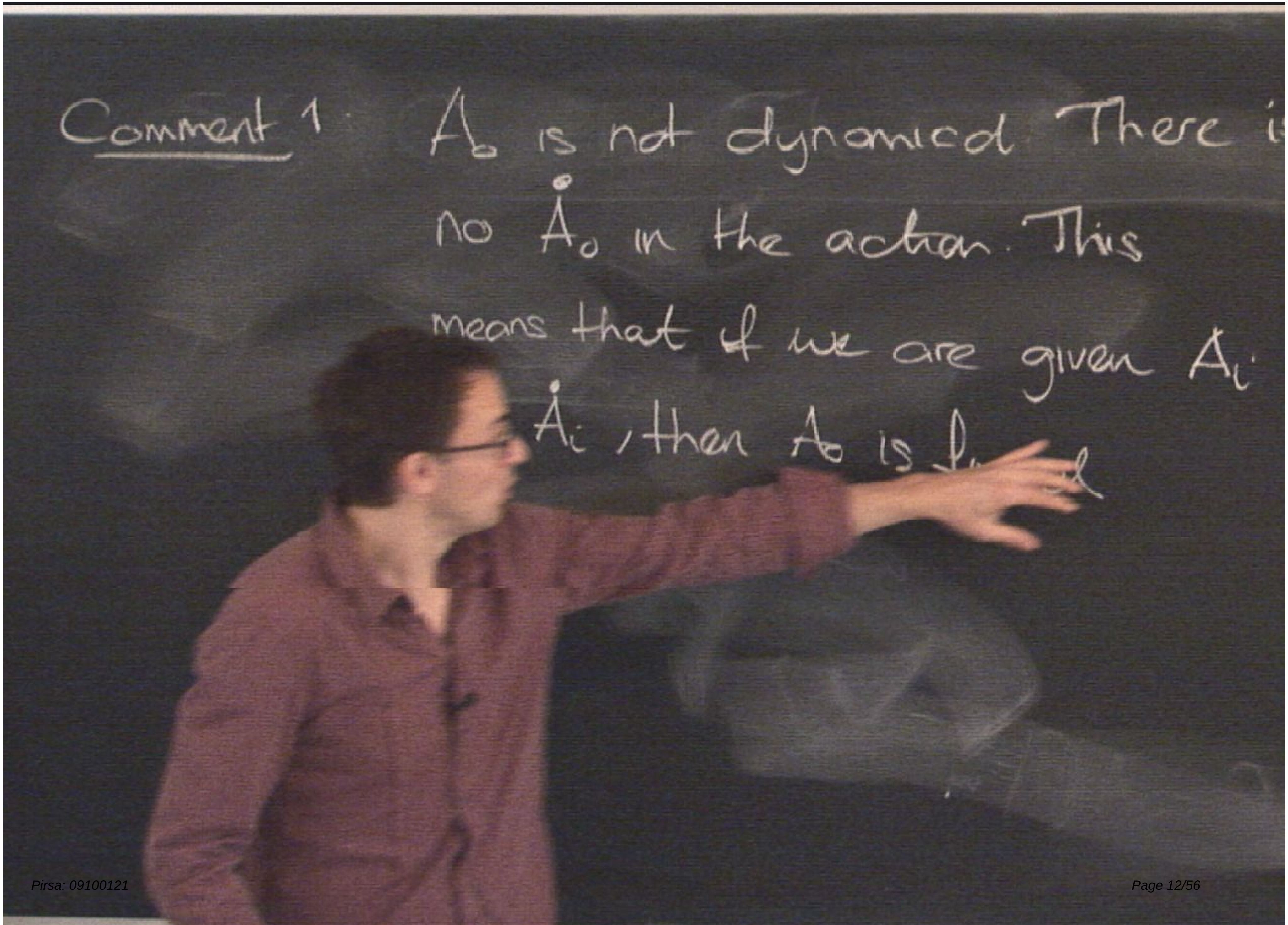
The massless vector field A_μ, $\mu = 0,1,2,3$ has 4 degrees of freedom — But the photon has 2 polarization states.

<u>Comment</u> 1: A_0 is not dynamical. There is no $\dot{A}_0$ in the action.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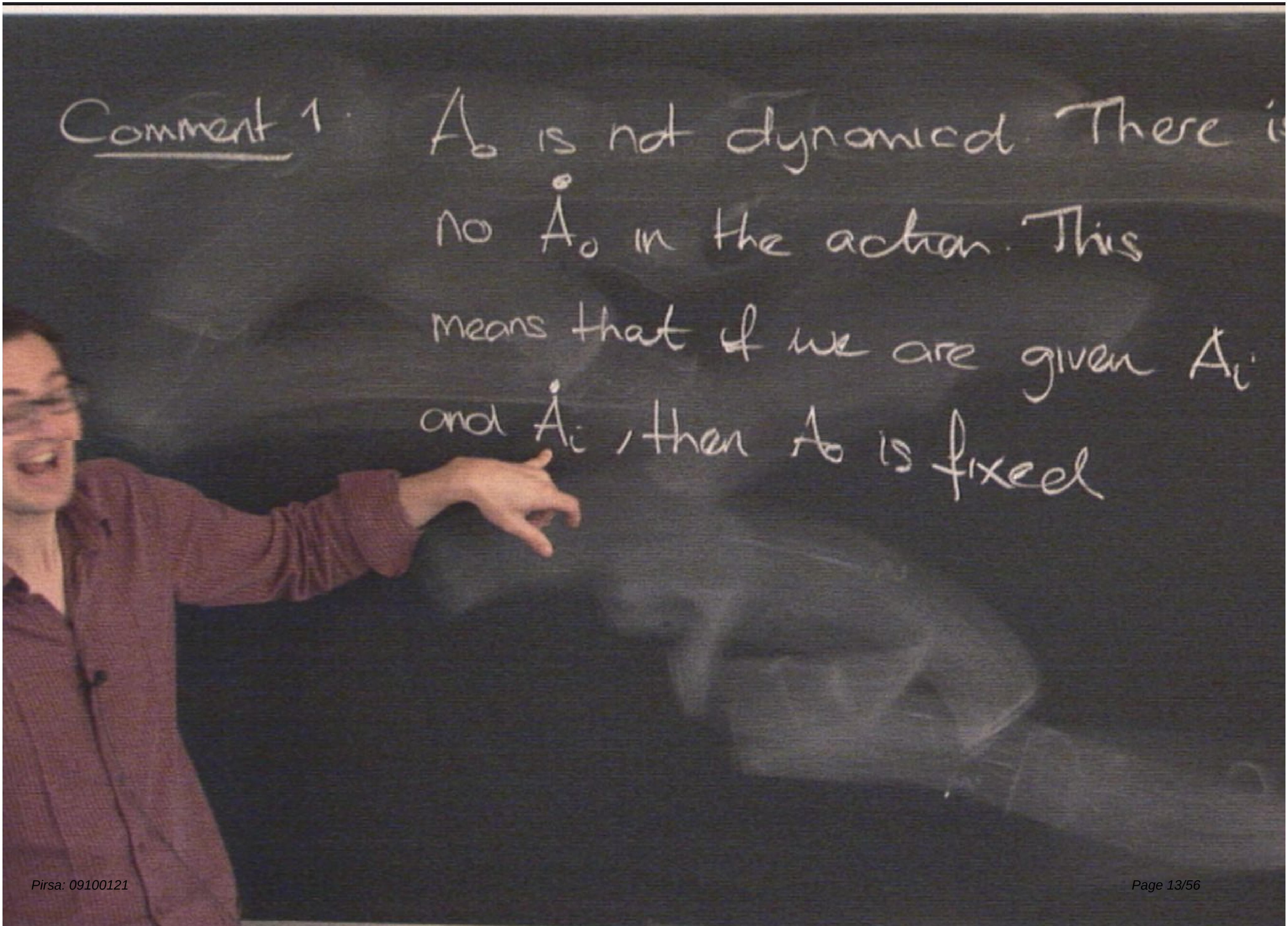
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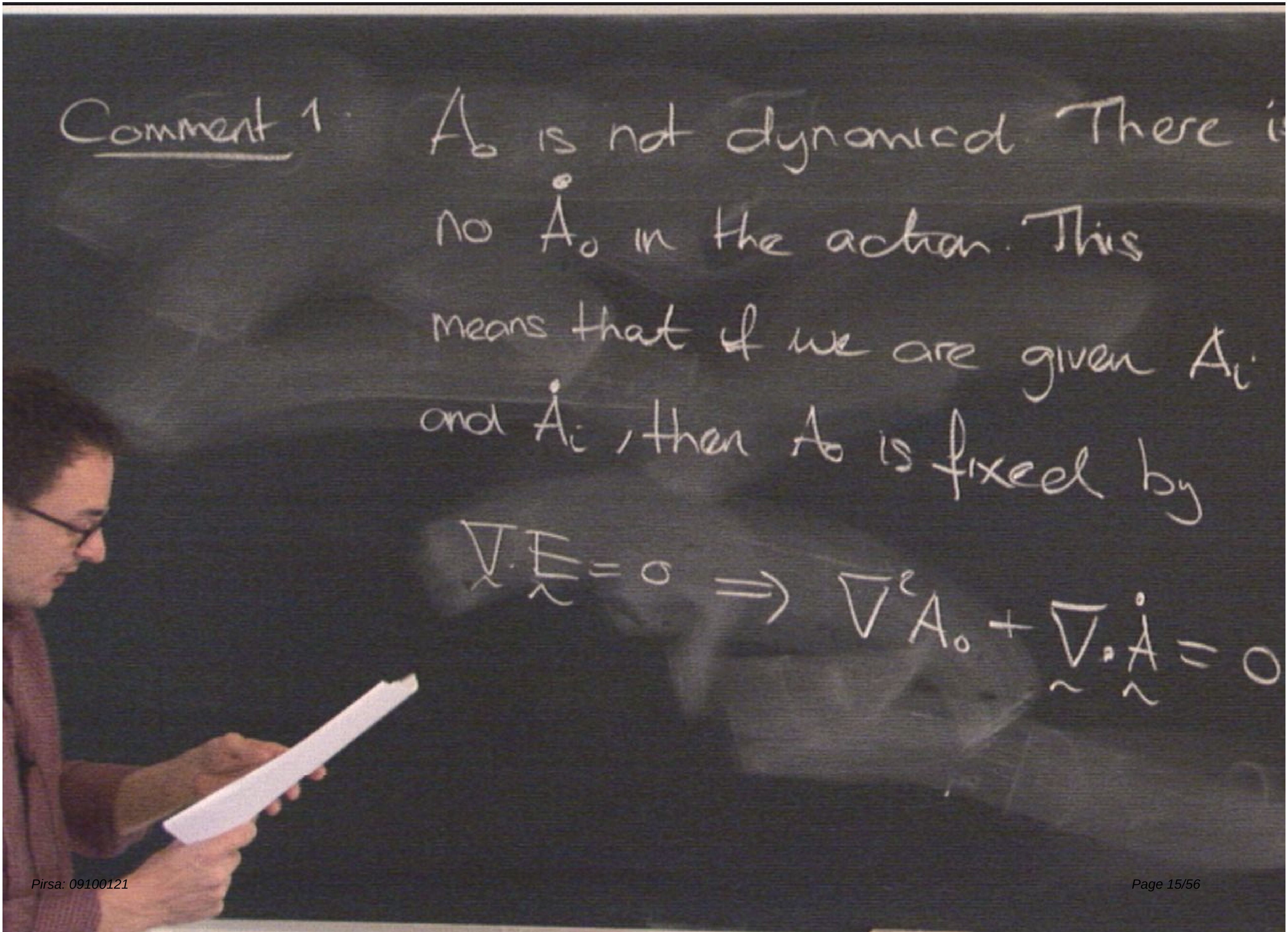
$$\underset{\sim}{\nabla}\cdot\underset{\sim}{E}=0 \implies$$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$$\underset{\sim}{\nabla}\cdot\underset{\sim}{E} = 0 \implies \nabla^2 A_0 + \underset{\sim}{\nabla}\cdot\dot{\underset{\sim}{A}} = 0$$

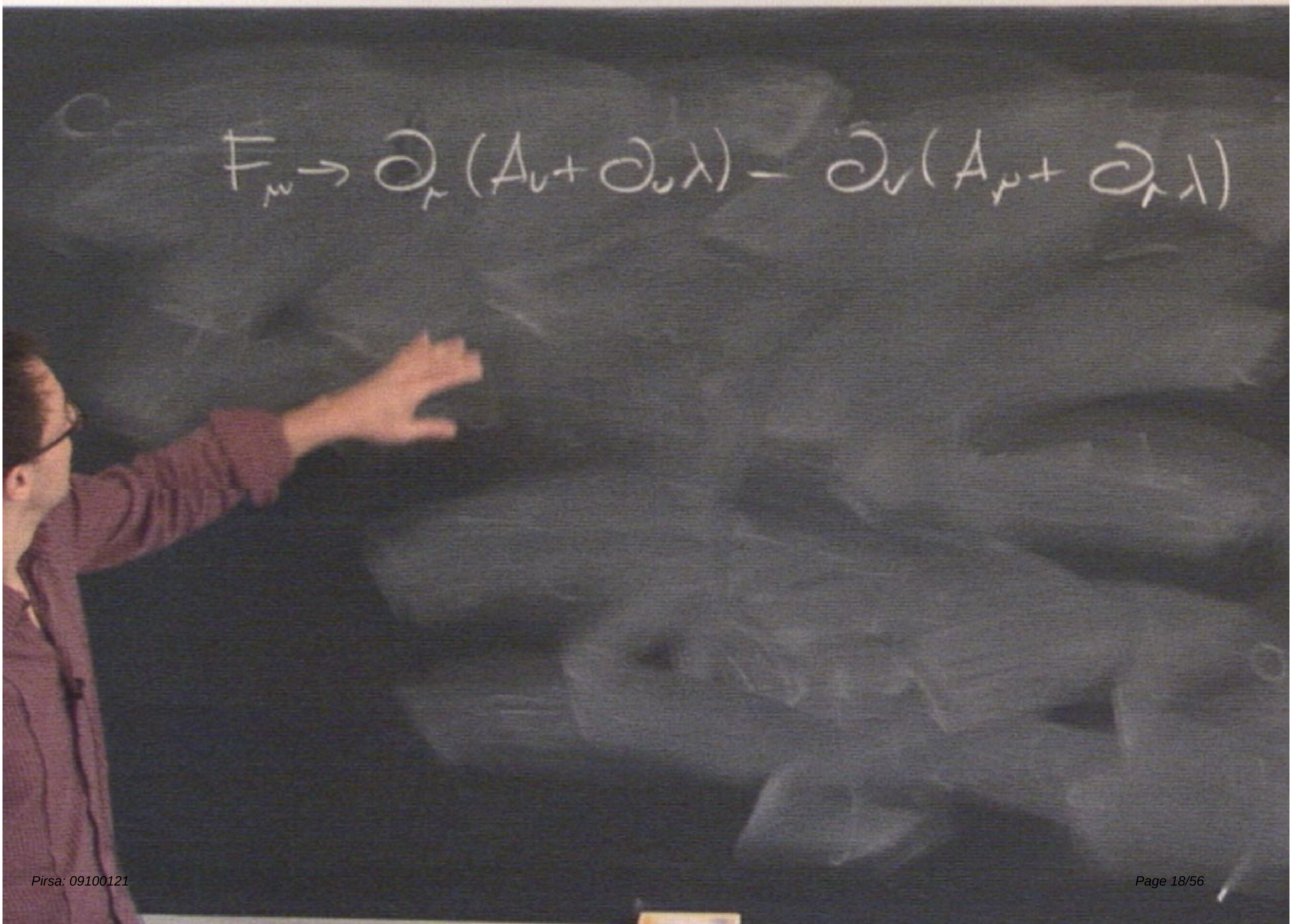
$$\implies A_0(\underset{\sim}{x}) = \int d^3x' \, (\nabla\cdot\dot{\underset{\sim}{A}})(x') /$$

Comment 2:

The theory has a **very** large symmetry group.

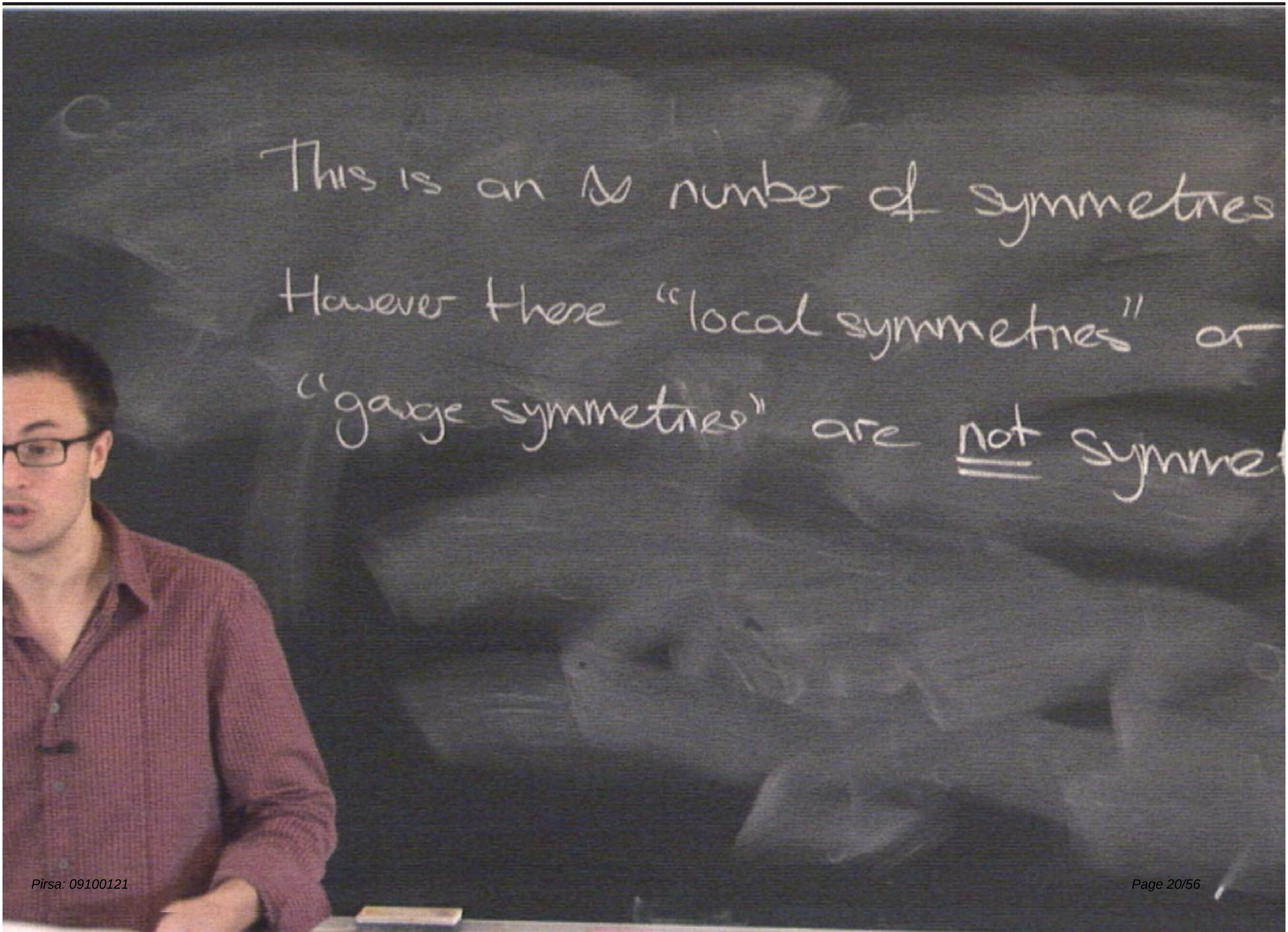
$$A_\mu(x) \longrightarrow A_\mu(x) + \partial_\mu \lambda(x).$$

require that it dies off as $x \to \infty$.


$F_{\mu\nu} \rightarrow \partial_\mu (A_\nu + \partial_\nu \lambda) - \partial_\nu (A_\mu + \partial_\mu \lambda)$

$$F_{\mu\nu} \to \partial_\mu (A_\nu + \partial_\nu \lambda) - \partial_\nu (A_\mu + \partial_\mu \lambda)$$

$= F_{\mu\nu}$ is invariant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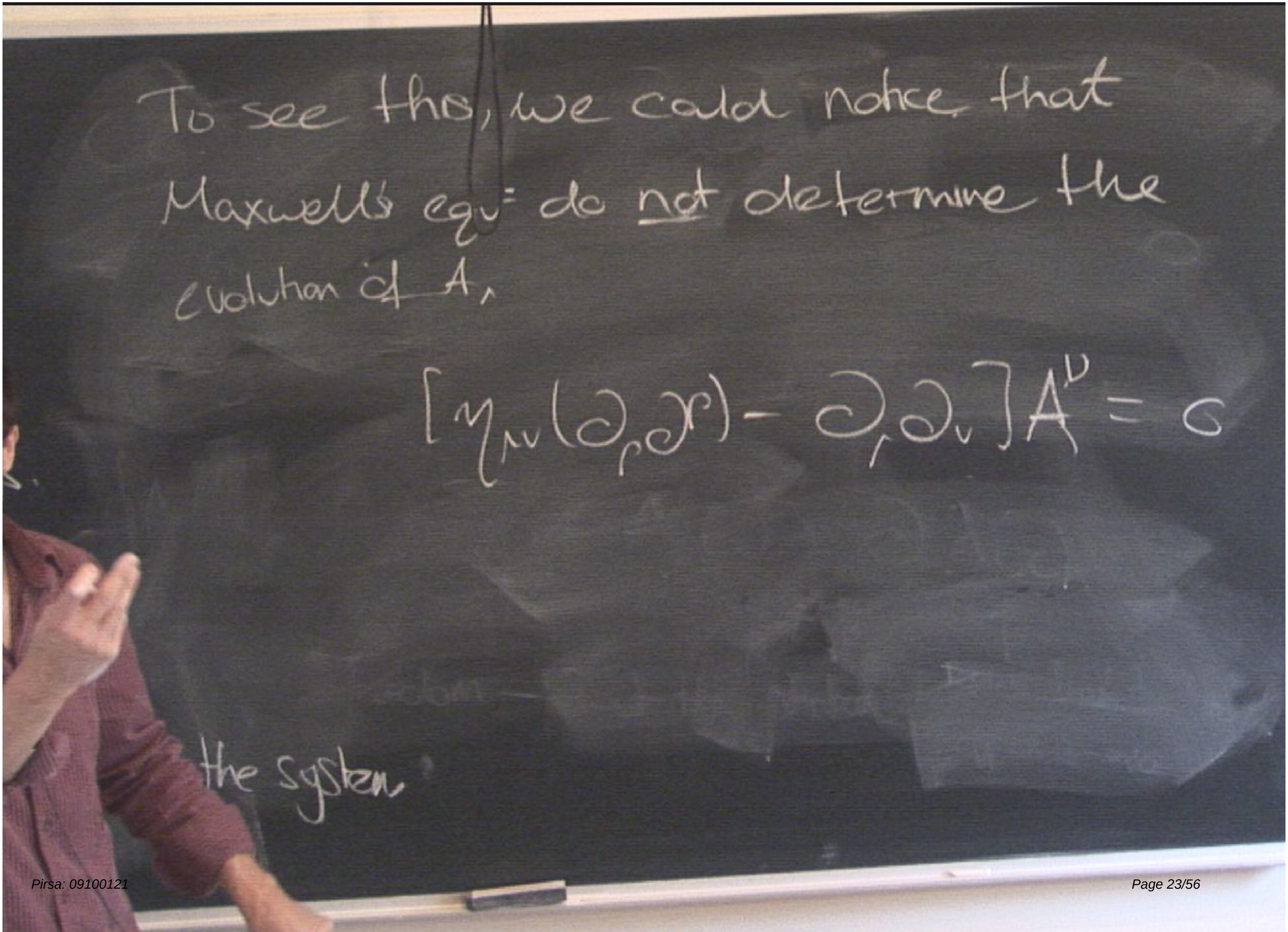
To see this, we could notice that
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$[\eta_{\mu\nu}(\partial_\rho\partial^\rho) - \partial_\mu\partial_\nu]A^\nu = 0$
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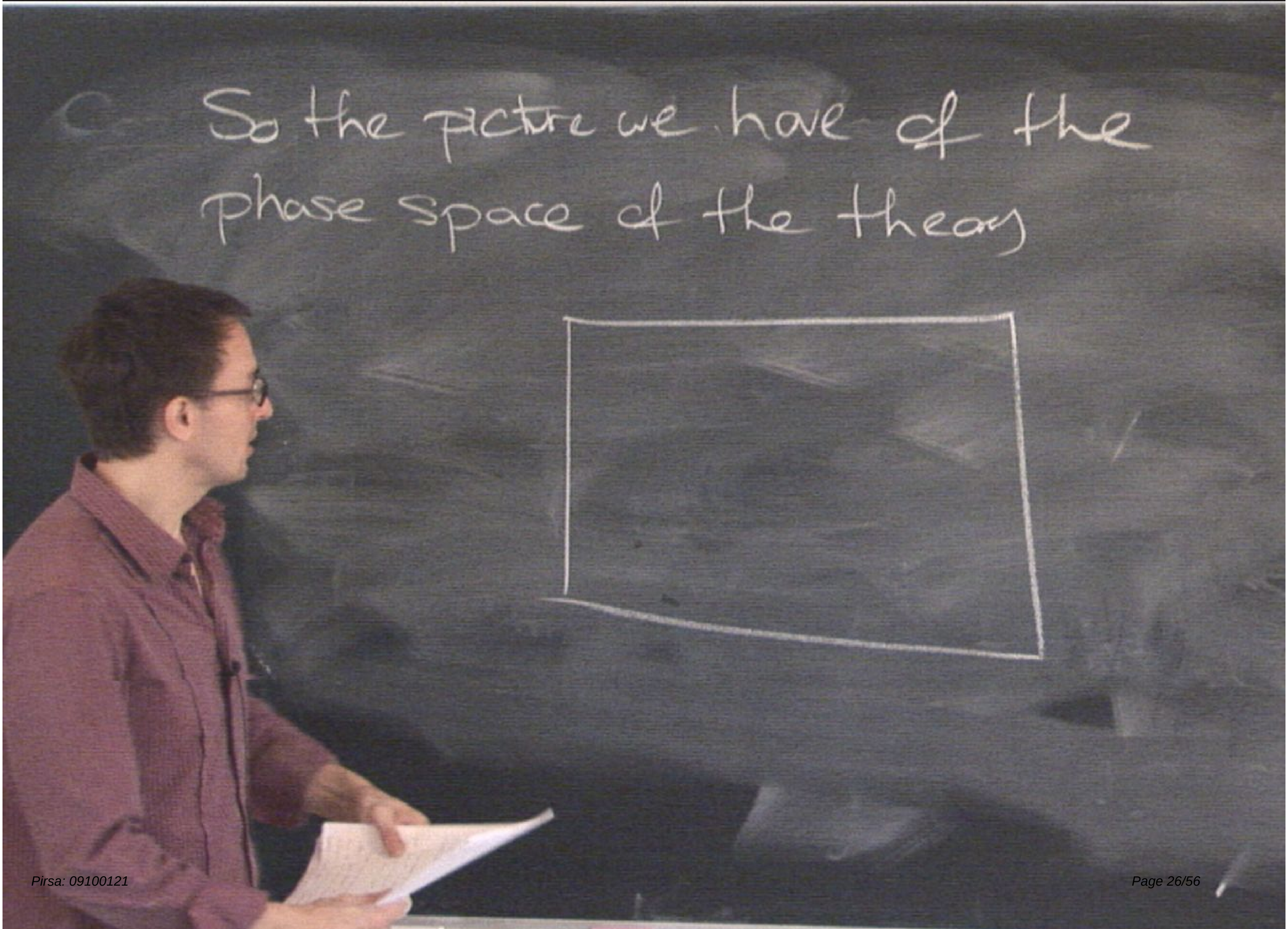
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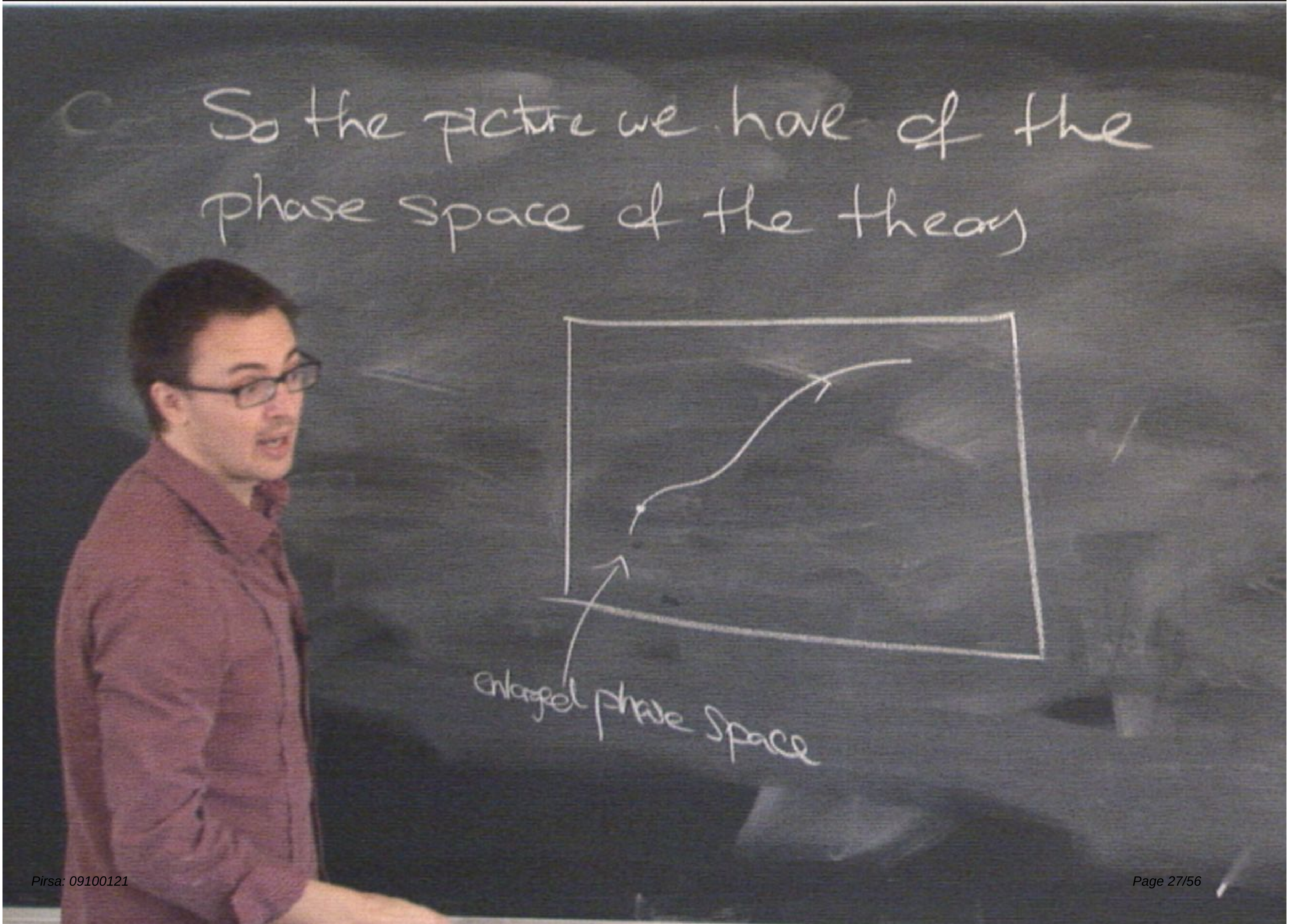
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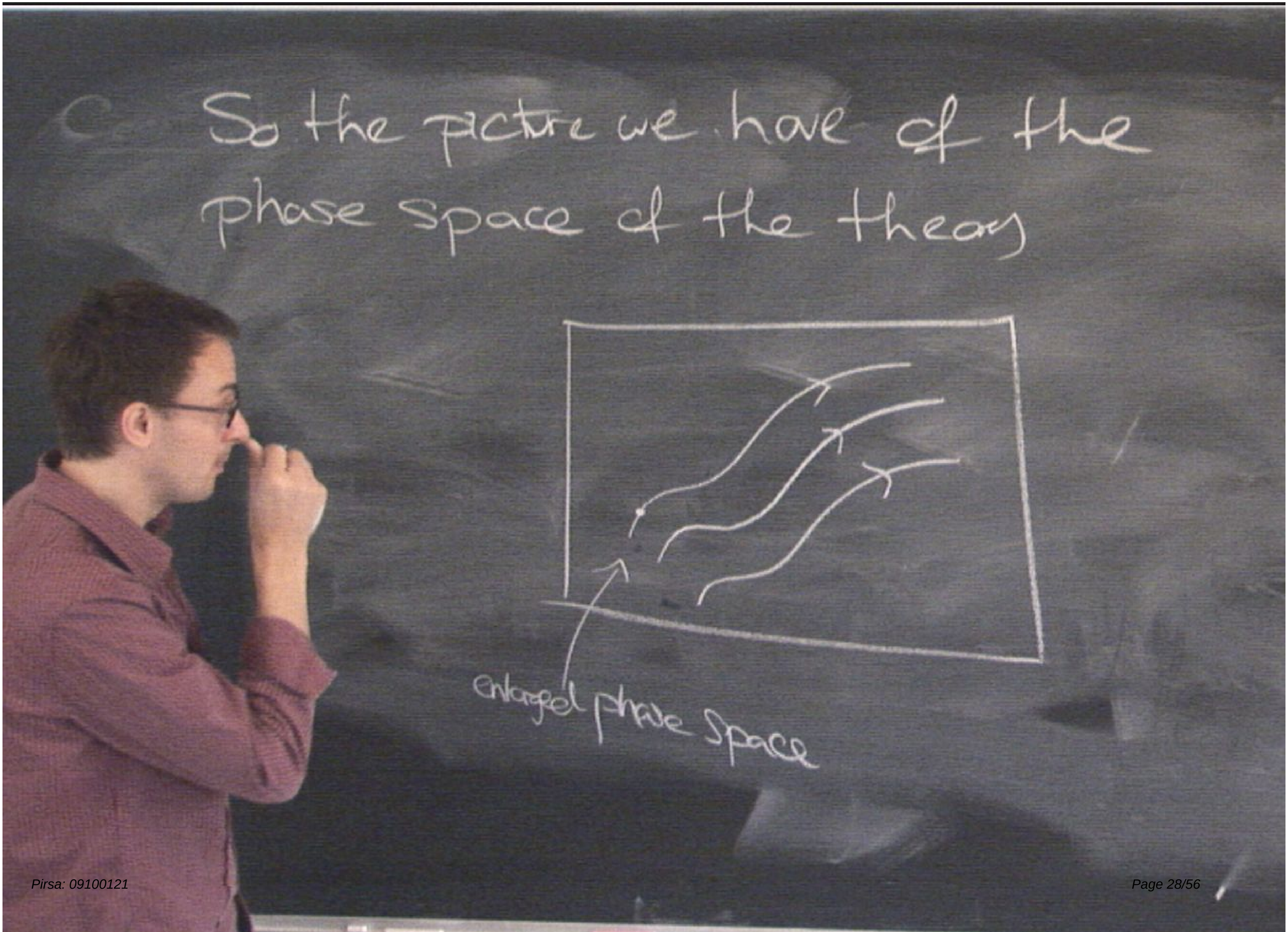
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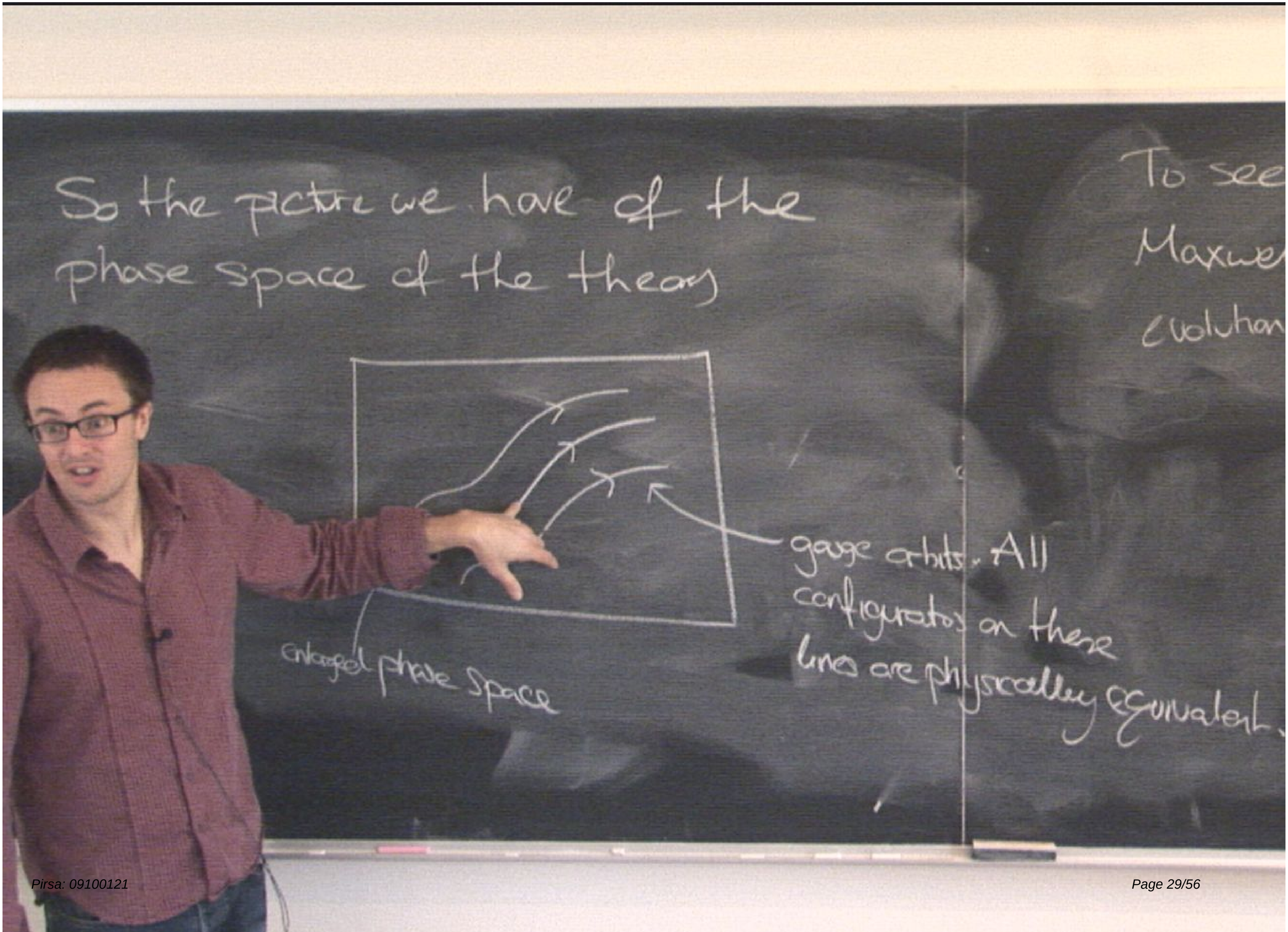

So the picture we have of the phase space of the theory
gauge orbits. All configurations on these lines are physically equivalent.
enlarged phase space
To see

Examples

1) Lorentz Gauge $\qquad \partial_\mu A^\mu = 0$

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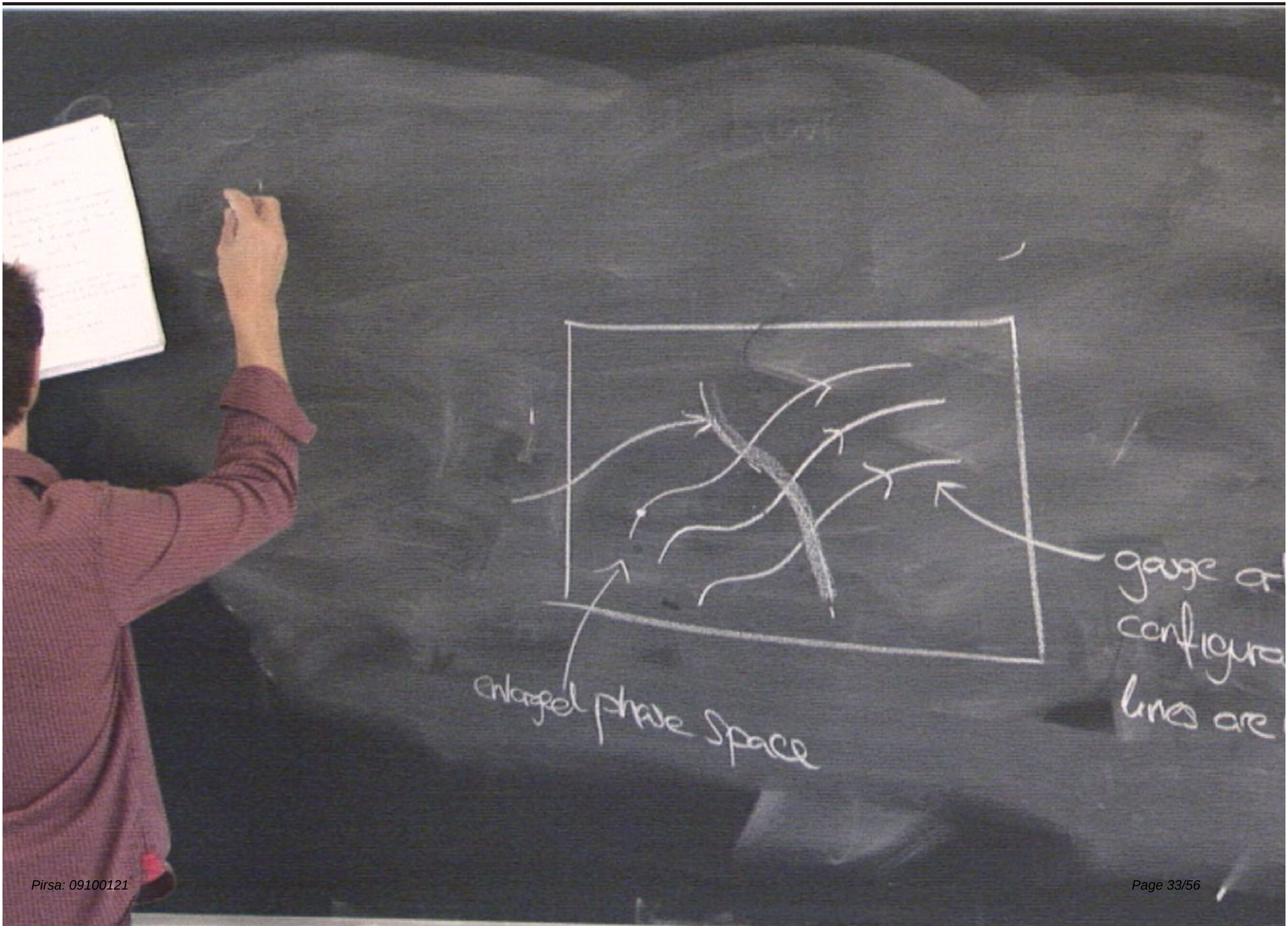
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s.t. $\partial_\mu \partial^\mu \lambda = 0$.

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We can get to this by making use of the residual gauge symmetry in Lorentz gauge.

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2) Coulomb Gauge $\quad \underline{\nabla} \cdot \underline{A} = 0.$

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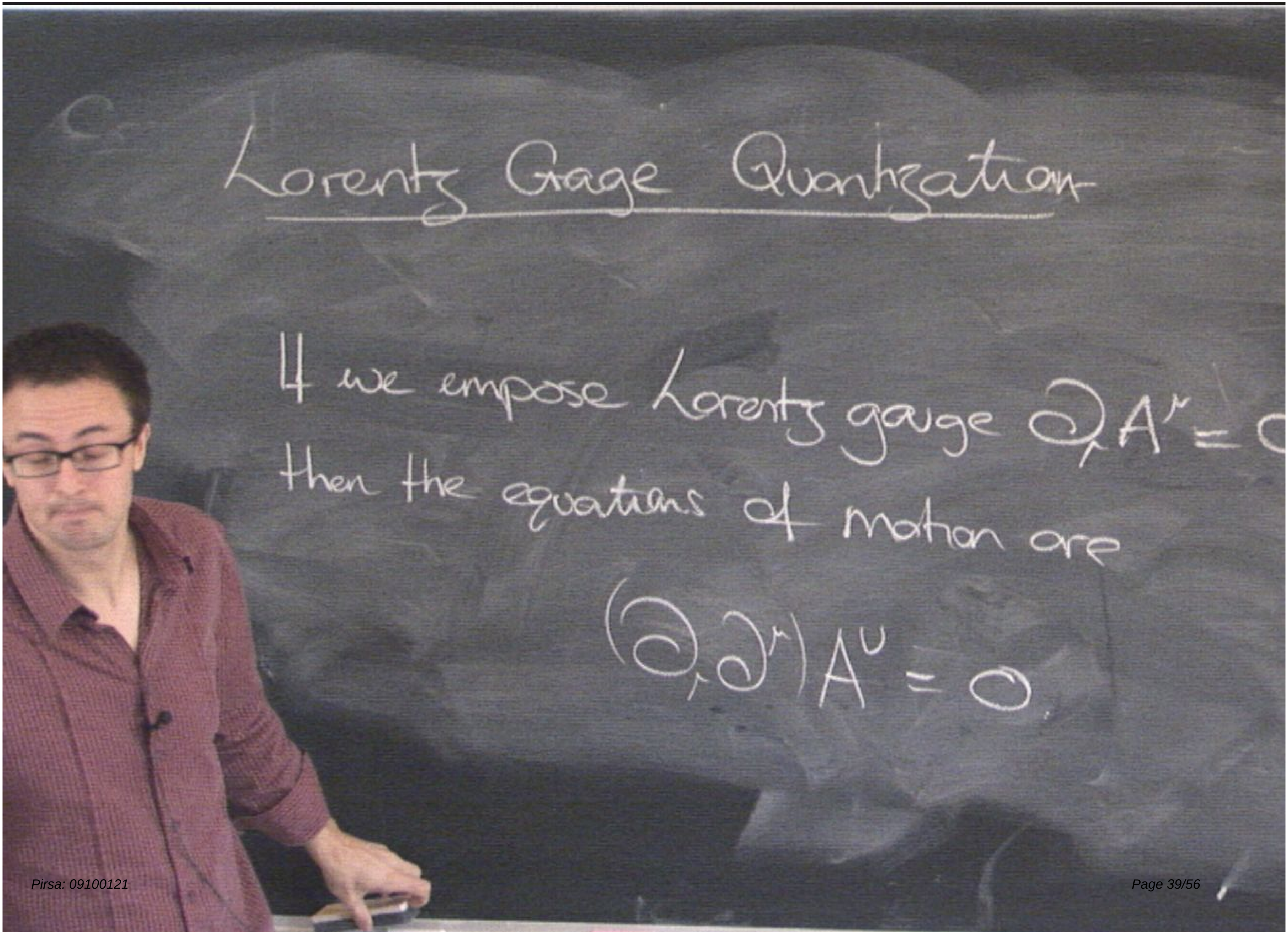
$$A_\mu \to A_\mu + \partial_\mu \lambda$$

Note. When $\underline{\nabla} \cdot \underline{A} = 0 \Longrightarrow A_0 = \text{const}$

$$\lambda = (\text{const})\, t.$$


Lorentz Gauge Quantization
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Our strategy is to quantize a theory with eq^{n}'s of motion (‡) and <u>then</u> impose $\partial_\mu A^\mu = 0$ together with residual gauge symmetry. We work with the Lagrangian

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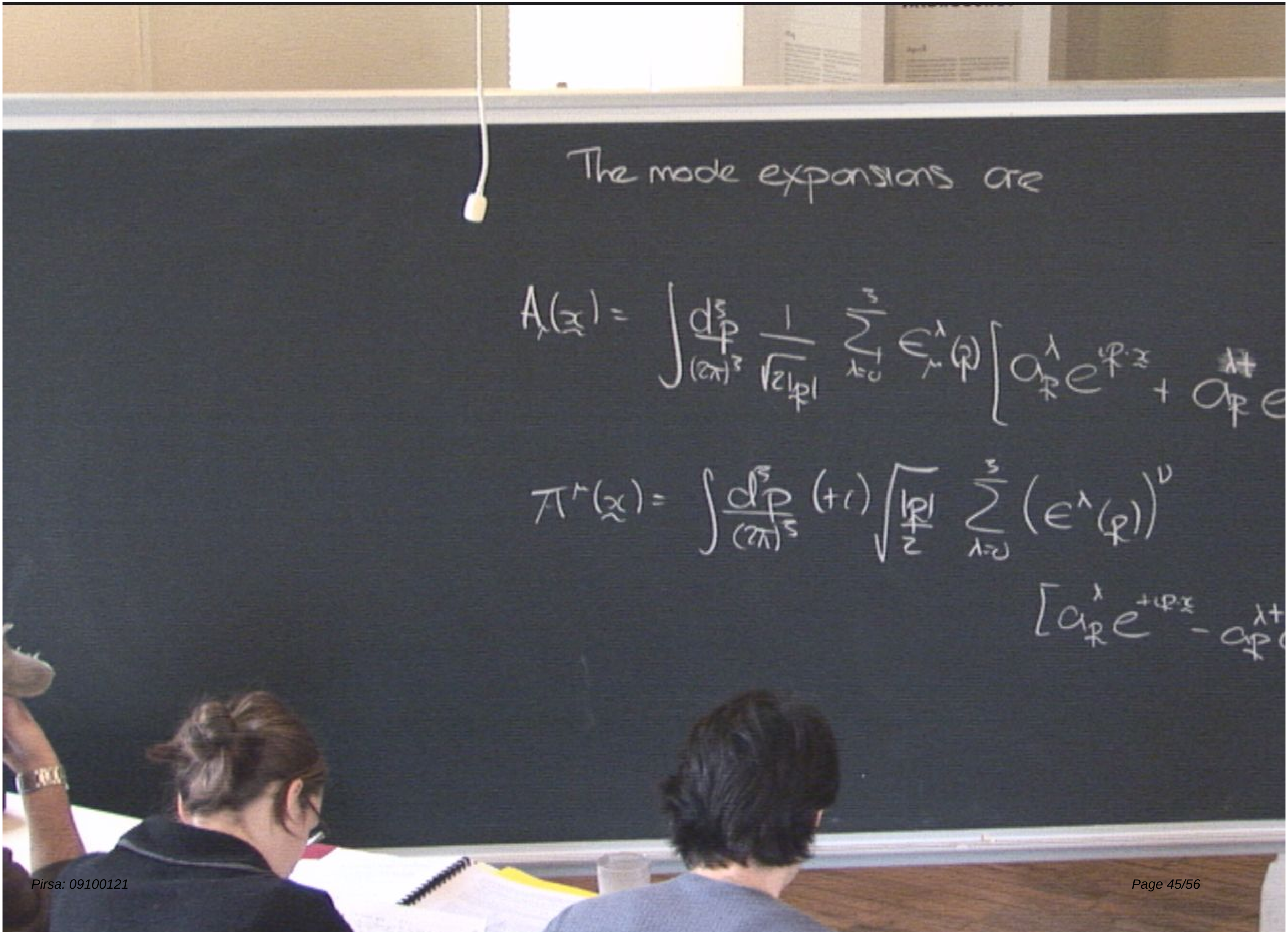
We have canonical momenta

$$\pi^0 = \frac{\partial \mathcal{L}}{\partial \dot{A}_0} = -\partial_\mu A^\mu$$

$$\pi^i = \frac{\partial \mathcal{L}}{\partial \dot{A}_i} = \partial^i A^0 - \dot{A}^i$$

Quantize $\Rightarrow$ $[A_\mu(\underset{\sim}{x}), A_\nu(\underset{\sim}{y})] = [\pi^\mu(\underset{\sim}{x}), \pi^\nu(\underset{\sim}{y})] = 0$

and $[A_\mu(\underset{\sim}{x}), \pi^\nu(\underset{\sim}{y})] = i\delta^{(3)}(\underset{\sim}{x}-\underset{\sim}{y})\,\delta_\mu^{\ \nu}$


The mode expansions are
$A_\mu(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2|p|}} \sum_{\lambda=0}^{3} \epsilon^\lambda_\mu(p) \left[a^\lambda_p e^{ip\cdot x} + a^{\lambda+}_p e \right.$
$\pi^\mu(x) = \int \frac{d^3p}{(2\pi)^3} (+i) \sqrt{\frac{|p|}{2}} \sum_{\lambda=0}^{3} \left(\epsilon^\lambda(p) \right)^\nu$
$\left[a^\lambda_p e^{+ip\cdot x} - a^{\lambda+}_p \right.$

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polarization vectors

$$A_\mu(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2|p|}} \sum_{\lambda=0}^{3} \epsilon_\mu^\lambda(p) \left[a_p^\lambda e^{ip\cdot x} + a_p^{\lambda\dagger} e \right.$$

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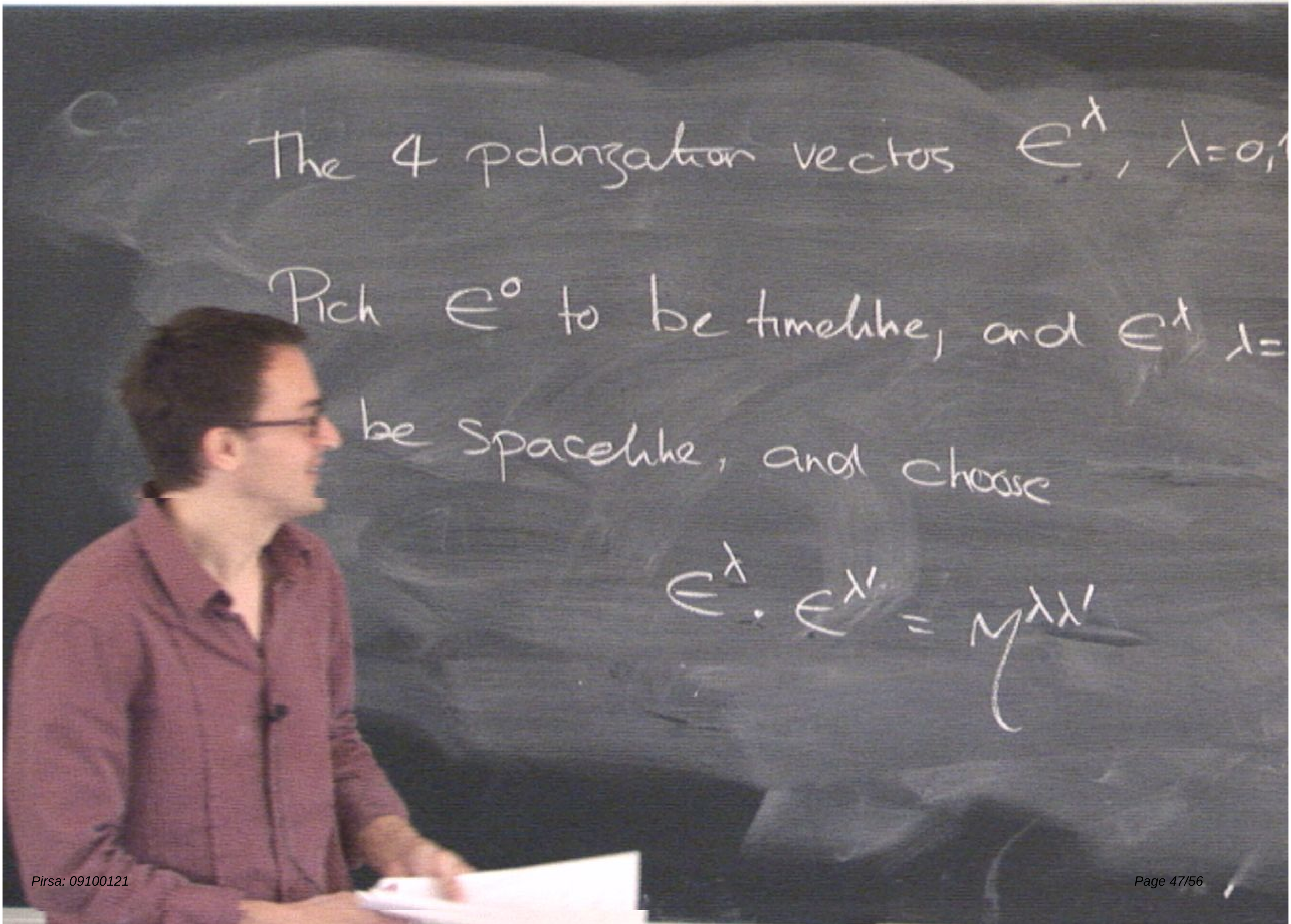
The 4 polarization vectors ϵ^{λ}, $\lambda=0,1$
Pick ϵ^{0} to be timelike, and ϵ^{λ} $\lambda=$
be spacelike, and choose
$\epsilon^{\lambda}\cdot\epsilon^{\lambda'}=\eta^{\lambda\lambda'}$

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Example: When $p \sim (1,0,0,1)$, we have

$$\epsilon^0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \epsilon^1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\epsilon^2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad \epsilon^3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$[A_\mu(\underset{\sim}{x}), \pi_\nu(\underset{\sim}{y})] = i\eta_{\mu\nu}\,\delta^{(3)}(\underset{\sim}{x}-\underset{\sim}{y})$$

$$\Longleftrightarrow \quad [a_{\underset{\sim}{p}}^{\lambda}, a_{\underset{\sim}{q}}^{\lambda' +}] = -\eta^{\lambda\lambda'}(2\pi)^3\,\delta^{(3)}(\underset{\sim}{p}-\underset{\sim}{q})$$

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all others zero

For $\lambda = 1,2,3$

$$[a_{\vec{k}}^{\lambda}, a_{\vec{q}}^{\lambda' +}] = +(2\pi)^3 \delta^{\lambda\lambda'} \delta^{(3)}(\vec{k}-\vec{q}$$

is the usual relation

But $[a_{\vec{k}}^{0}, a_{\vec{q}}^{0+}] = -(2\pi)^3 \delta^{(3)}(\vec{k}-\vec{q})$

So, if we define the vacuum

$$a_{\vec{k}}^{\lambda}|0\rangle = 0$$

Then we can create 1-particle states

$$a_{\vec{k}}^{\lambda\dagger}|0\rangle \equiv |\vec{k},\lambda\rangle$$

But timelike polarization states have <u>negative norm</u>

$$\langle p, \lambda=0 | q, \lambda=0 \rangle = \langle 0 | a^0_p \, a^{0\dagger}_q | 0 \rangle$$

$$= -(2\pi)^3 \delta^{(3)}(p - \ldots$$