Title: Quantum Field Theory (PHYS 601) - Lecture 12

Date: Oct 14, 2009 09:00 AM

URL: http://pirsa.org/09100120

Abstract:

Recall:

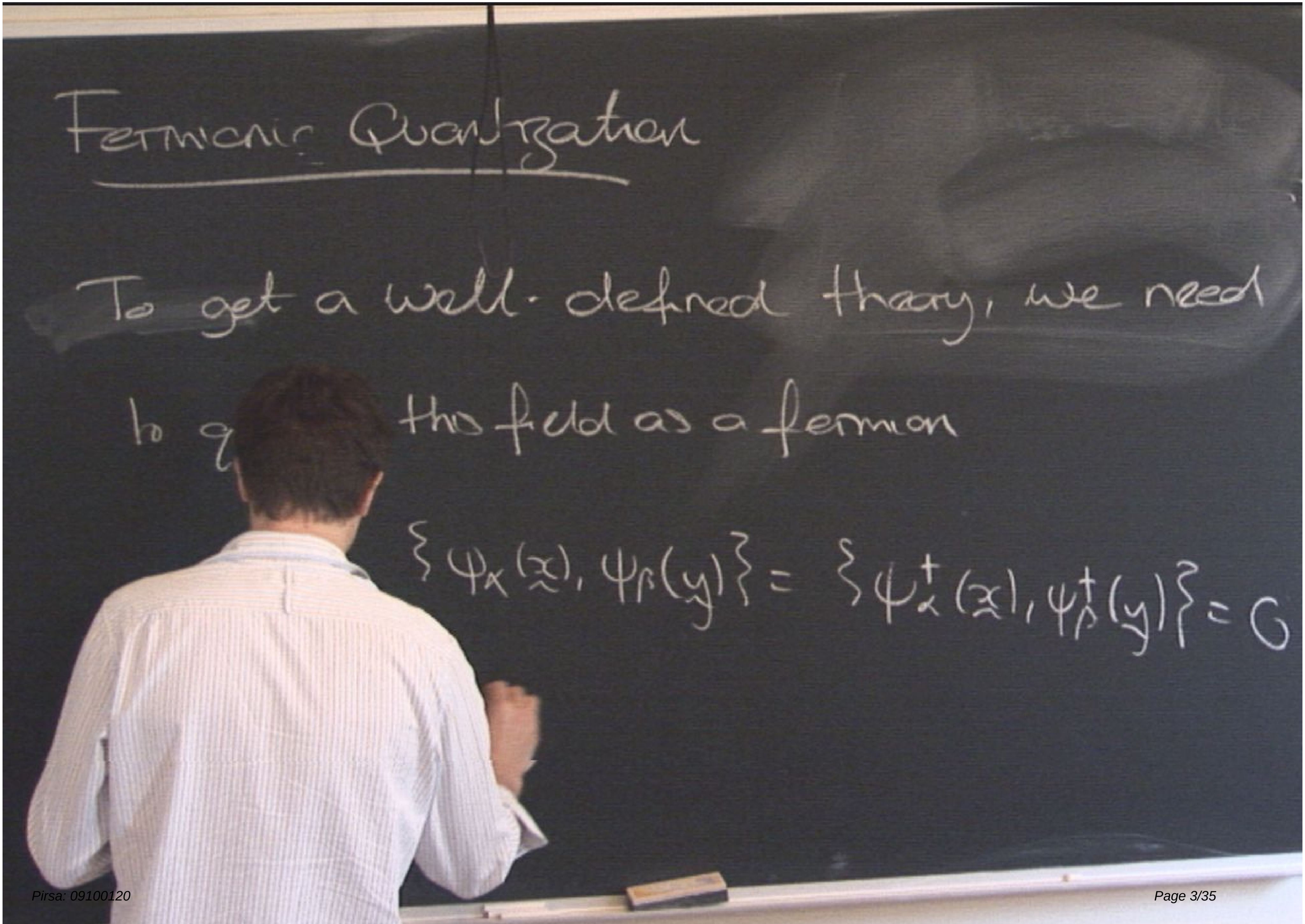
$$[\psi_\alpha(\underset{\sim}{x}), \psi_\beta^\dagger(\underset{\sim}{y})] = \delta^{(3)}(\underset{\sim}{x}-\underset{\sim}{y})\,\delta_{\alpha\beta}$$

$$\Longleftrightarrow \quad [b^r_{\underset{\sim}{p}}, b^{s\dagger}_{\underset{\sim}{q}}] = (2\pi)^3 \delta^{rs} \delta^{(3)}(\underset{\sim}{p}-\underset{\sim}{q})$$

$$[c^r_{\underset{\sim}{p}}, c^{s\dagger}_{\underset{\sim}{q}}] = -(2\pi)^3 \delta^{rs} \delta^{(3)}(\underset{\sim}{p}-\underset{\sim}{q})$$

and $$H = \int \frac{d^3 p}{(2\pi)^3} E_{\underset{\sim}{p}} \left(b^{s\dagger}_{\underset{\sim}{p}} b^s_{\underset{\sim}{p}} - c^{s\dagger}_{\underset{\sim}{p}} c_{\underset{\sim}{p}} \right)$$

BAD!


Fermionic Quantization
To get a well-defined theory, we need
the field as a fermion
$\{\psi_\alpha(\underset{\sim}{x}), \psi_\beta(\underset{\sim}{y})\} = \{\psi^\dagger_\alpha(\underset{\sim}{x}), \psi^\dagger_\beta(\underset{\sim}{y})\} = 0$

Where $\{A, B\} = AB + BA$.

$$\Rightarrow \quad \{b_k^r, b_q^{s+}\} = (2\pi)^3 \delta^{(3)}(k - q)$$

$$\{c_k^r, c_k^{s+}\} = (2\pi)^3 \delta^{(3)}(k - q)$$

and $\{b_k, b_q\} = \{c_k, c_q\} = \{b_k^+, b_q^+\} = \{c_k^+, c_q^+\}$

We can now compute

$$H = \sum_{s=1}^{2} \int \frac{d^3p}{(2\pi)^3} \, E_p \left(b_p^{s\dagger} b_p^{s} + c_p^{s\dagger} c_p^{s} - (2\pi)^3 \delta^{(3)}(0) \right].$$

throw away by normal ordering

$= 0$

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The Foch Space

The vacuum is defined as

$$b_k^s|0\rangle = c_k^s|0\rangle = 0 \qquad \forall\, k, \; s = 1,2$$

$= 1 \ldots$

The Fock Space

The vacuum is defined as

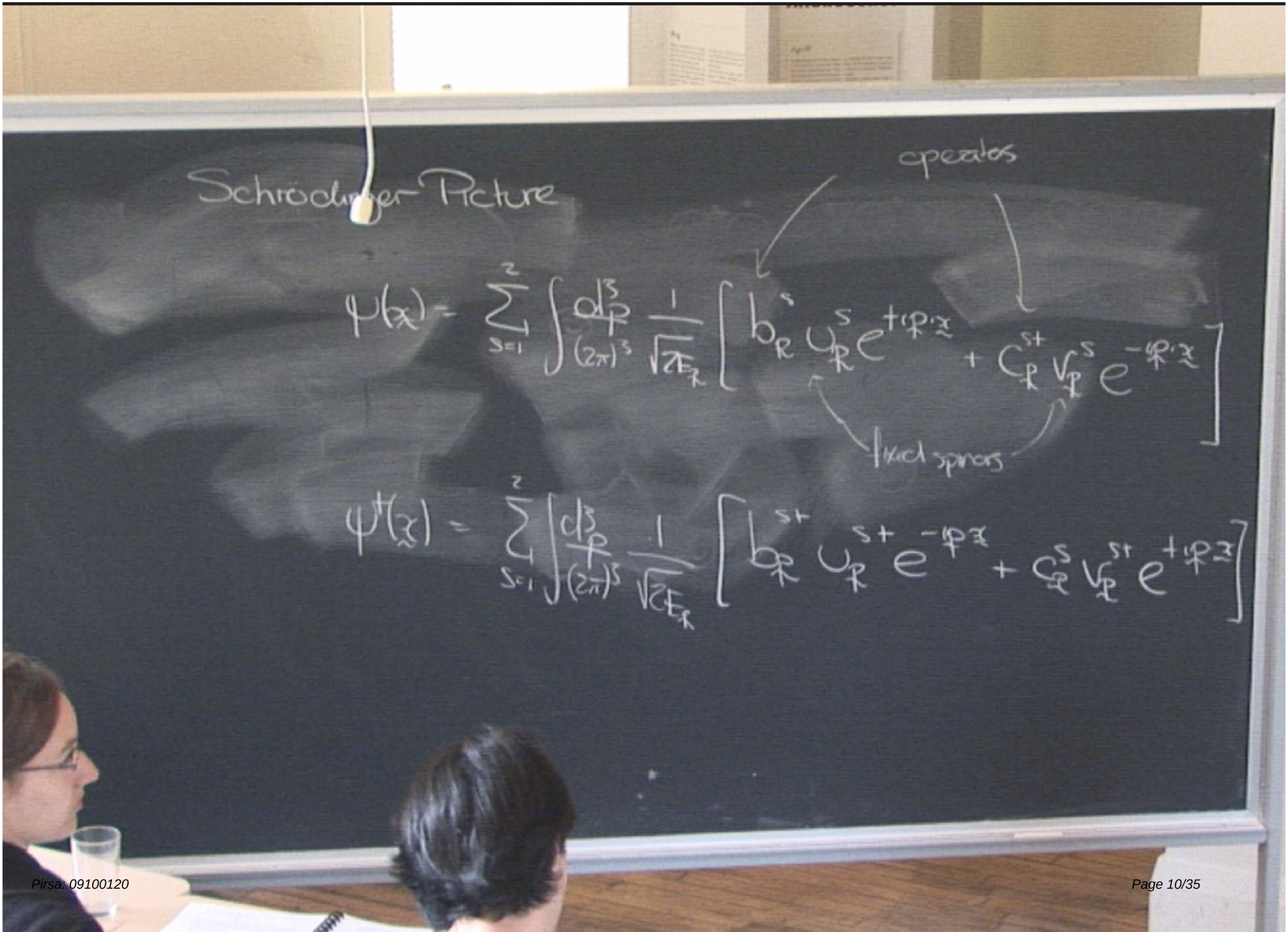
$$b_k^s|0\rangle = c_k^s|0\rangle = 0 \qquad \forall\, k,\ s=1,2$$

Can check: $[H, b_k^{s\dagger}] = E_k b_k^{s\dagger}$

$[H, b_k] = -E_k b_k$ $= \ldots$

We have one-particle states

$$b_{k}^{s\dagger}|0\rangle \quad \text{and} \quad c_{k}^{s\dagger}|0\rangle. \quad s=1,2$$


Schrödinger Picture
operators
$\psi(\underset{\sim}{x}) = \sum_{s=1}^{2} \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left[b_p^s u_p^s e^{+ip\cdot x} + c_p^{s+} v_p^s e^{-ip\cdot x} \right]$
fixed spinors
$\psi^+(\underset{\sim}{x}) = \sum_{s=1}^{2} \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left[b_p^{s+} u_p^{s+} e^{-ip\cdot x} + c_p^s v_p^{s+} e^{+ip\cdot x} \right]$

We have one-particle states

$$b_{\mathbf{p}}^{s\dagger}|0\rangle \quad \text{and} \quad c_{\mathbf{p}}^{s\dagger}|0\rangle \qquad s=1,2$$

The $s=1,2$ index tells us the spin:

$$u_{\mathbf{p}}^{s} = \begin{pmatrix} \sqrt{p\cdot\sigma}\ \xi^{s} \\ \sqrt{p\cdot\bar{\sigma}}\ \xi^{s} \end{pmatrix}$$

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If we pick $\xi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\xi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

the $s=1$ $(s=2)$ particles have up (dow

in z-direction

Note that

$$b_{R_1}^{s_1 +} \, b_{R_2}^{s_2 +} |0\rangle \equiv |R_1, s_1 ; R_2, s_2\rangle$$

$$= - |R_2, s_2 ; R_1, s_1\rangle .$$

i.e. Fermi-Dirac statistics.

Dirac's Hole Interpretation

Dirac originally wrote his equation as

$$(i\not\partial - m)\psi = 0 \implies i\frac{\partial\psi}{\partial t} = -i\underset{\sim}{\alpha}\cdot\underset{\sim}{\nabla}\psi + m\beta\psi$$

$$\underset{\sim}{\alpha} = -\gamma^0\underset{\sim}{\gamma} \qquad \beta = \gamma^0$$

With the interpretation of ψ as a one-part

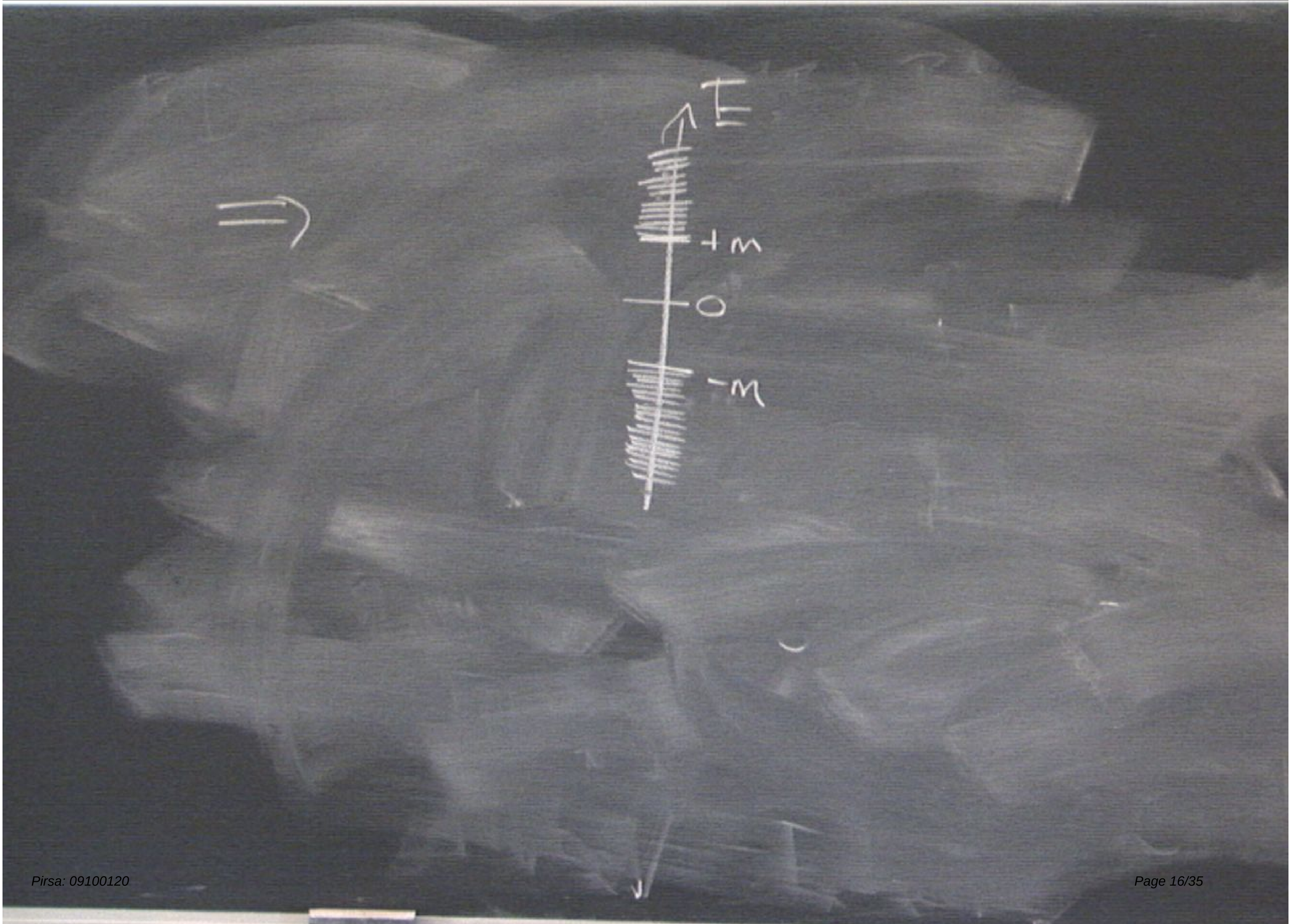
wavefunction,

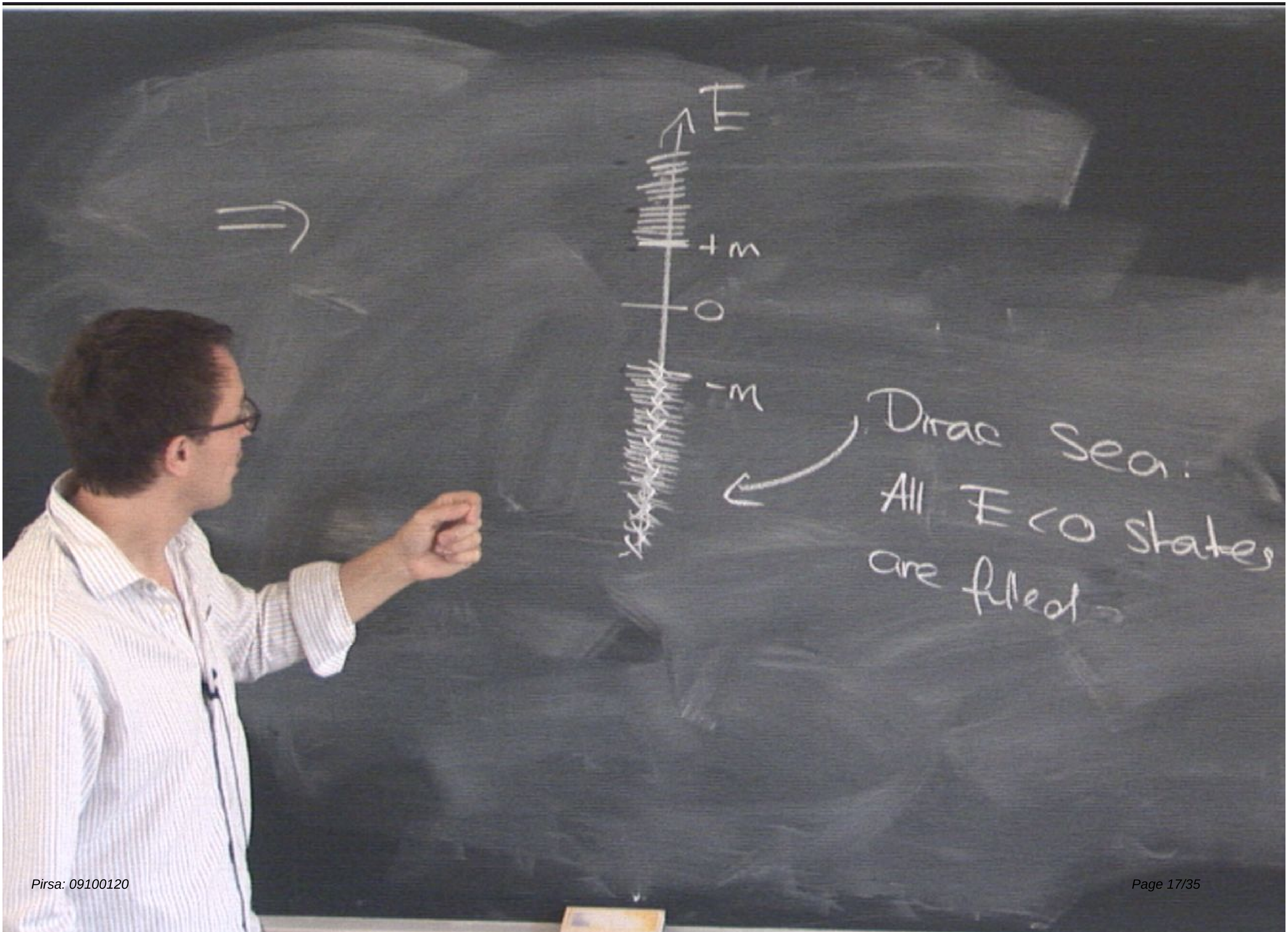
$$\psi = u_p e^{-ip\cdot x} \implies i\frac{\partial\psi}{\partial t} = E_p\psi$$

$$\psi = v_p e^{+ip\cdot x} \implies i\frac{\partial\psi}{\partial t} = -E_p\psi$$

$$\psi \equiv \hat{H}\psi$$

$$\beta = \gamma^0$$


E
+m
0
-m


E
+m
0
-m
Dirac Sea:
All E<0 states
are filled.

~~Schrödinger Picture~~

Heisenberg

operators

$$\psi(x) = \sum_{s=1}^{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left[b_p^s u_p^s e^{-ip\cdot x} + c_p^{s\dagger} v_p^s e^{+ip\cdot x} \right]$$

fixed spinors

$$\psi^\dagger(x) = \sum_{s=1}^{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left[b_p^{s\dagger} u_p^{s\dagger} e^{+ip\cdot x} + c_p^s v_p^{s\dagger} e^{-ip\cdot x} \right]$$

~~Schrödinger Picture~~

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Can show that

$$\{\psi(x), \psi(y)\} = 0$$

when $(x-y)^2 < 0.$

Feynman Propagator

Define $(S_F)_{\alpha\beta}(x-y) =$

Feynman Propagator

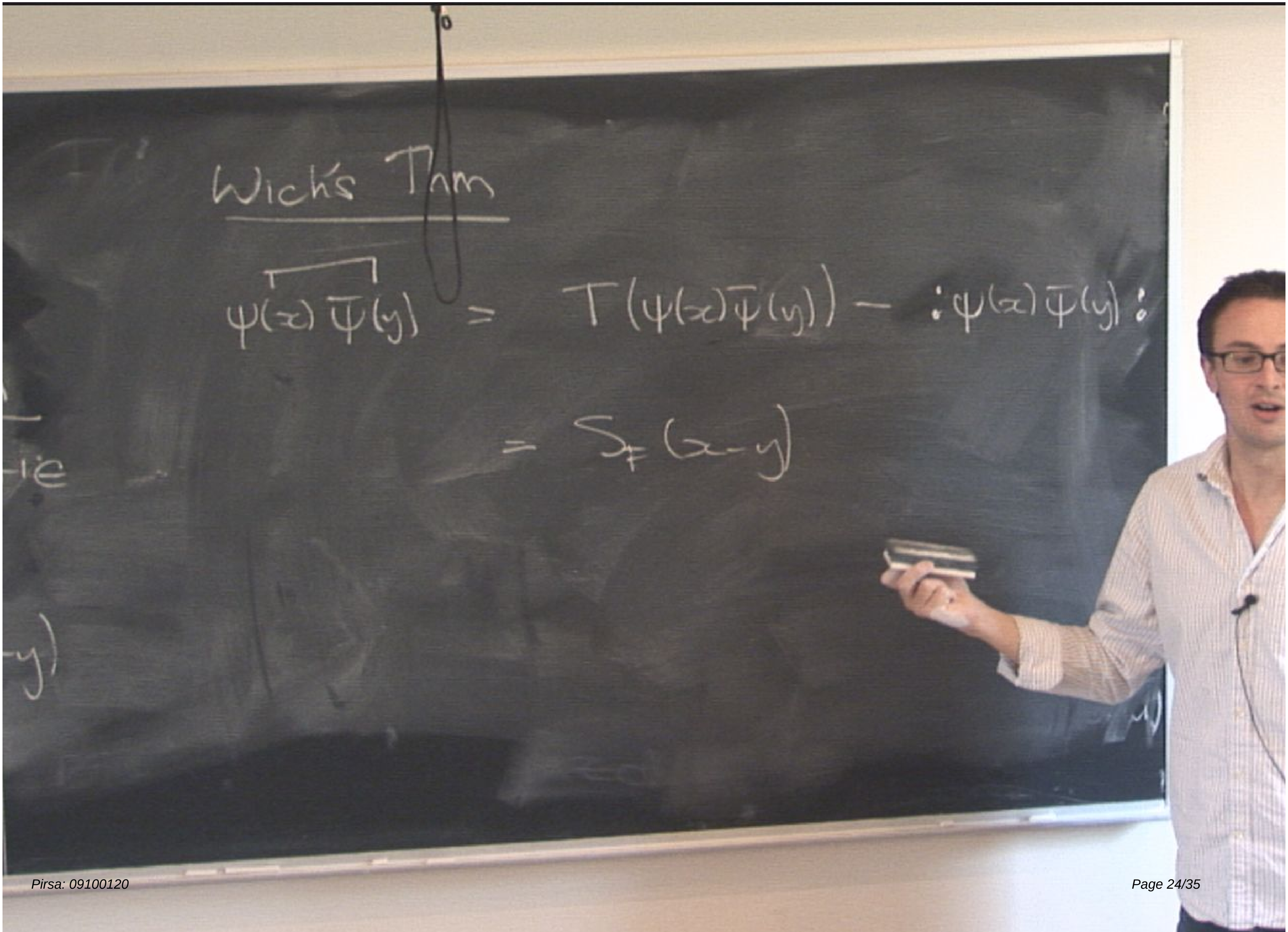
Define $(S_F)_{\alpha\beta}(x-y) = \langle 0|T\,\psi_\alpha(x)\,\bar{\psi}_\beta(y)|0\rangle$

$$\equiv \begin{cases} \langle 0|\psi_\alpha(x)\,\bar{\psi}_\beta(y)|0\rangle & x^0 > y^0 \\ \langle 0|-\bar{\psi}_\beta(y)\,\psi_\alpha(x)|0\rangle & x^0 < y^0 \end{cases}$$

"We" can show that

$$S_F(x-y) = i\int \frac{d^4p}{(2\pi)^4}\, e^{-ip\cdot(x-y)}\, \frac{\not{p}+m}{p^2-m^2+i\epsilon}$$

and this obeys $(i\not{\partial}-m)\,S_F = i\,\delta^{(4)}(x-y)$

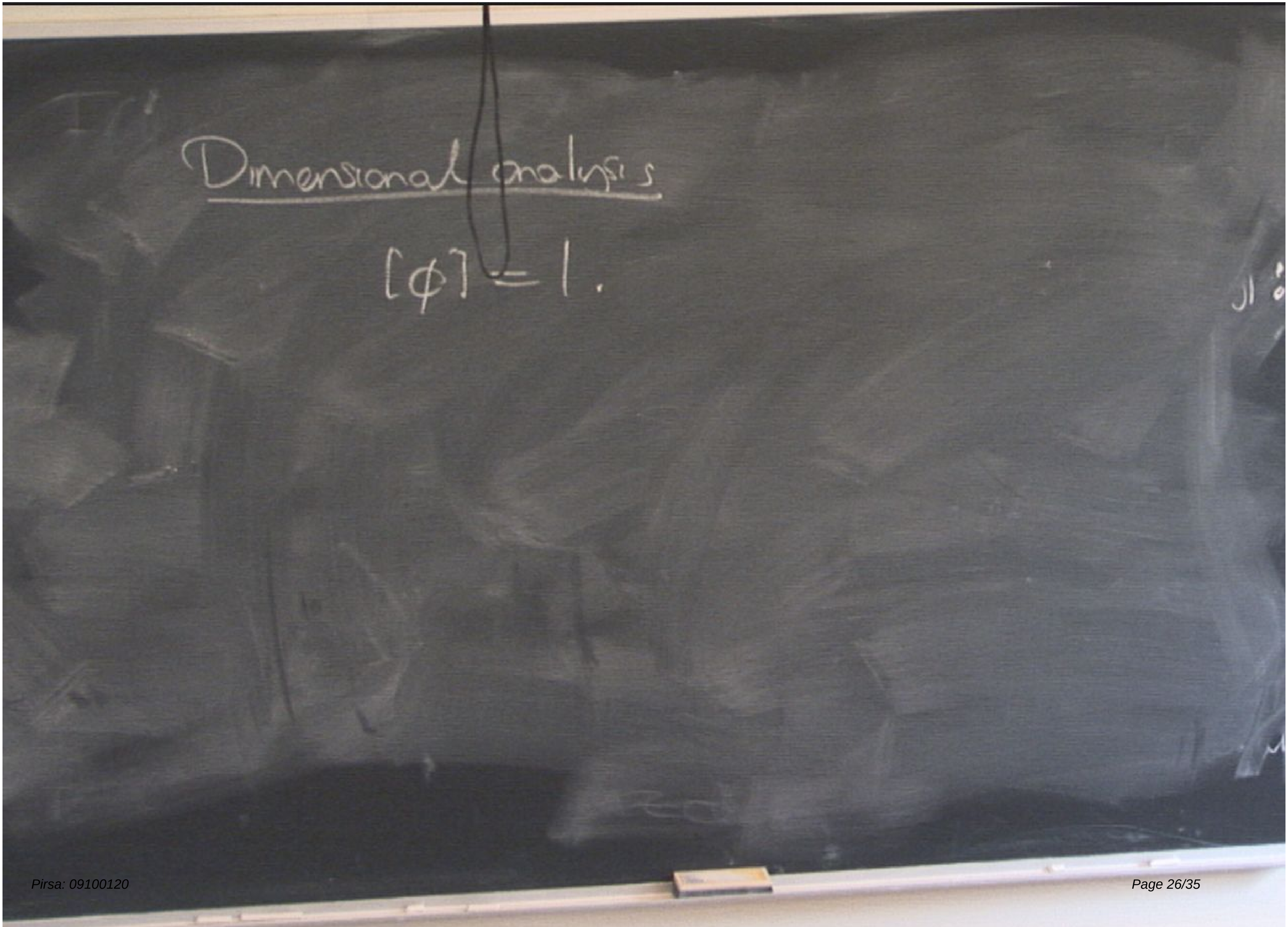

Wick's Thm
$\psi(x)\bar{\psi}(y) = T(\psi(x)\bar{\psi}(y)) - :\psi(x)\bar{\psi}(y):$
$= S_F(x-y)$

Yukawa Theory

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}\mu^2\phi^2$$

mass μ

$$+\bar{\psi}(i\not{\partial} - m)\psi - \lambda\bar{\psi}\psi\phi$$


Dimensional analysis
$[\phi] = 1.$

Dimensional analysis

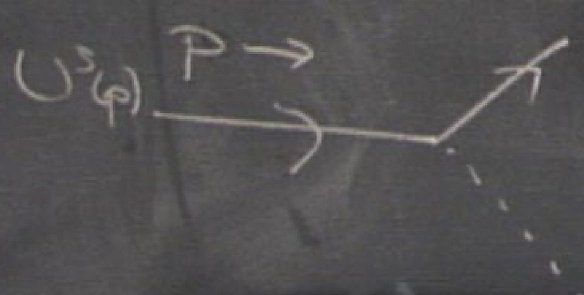
$$[\phi] = 1.$$

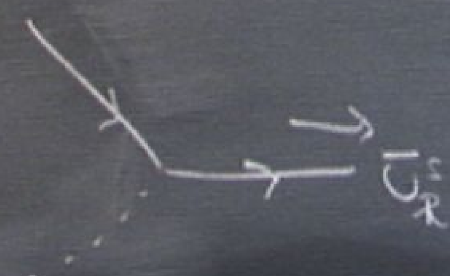
$$[\psi] = 3/2$$

Feynman Rules.

- To each incoming fermion, with momentum p and spin s, we associate u_p^s

(cf outgoing $\bar{u}_p^s$)





- Incoming (outgoing) anti-fermions,
  add $\overline{v}_R^{s}$ (v_R^{s}).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p
$\bar{v}_r^s$
p
v_r^s

- Each vertex get $(-i\lambda)$

- Internal lines $\xrightarrow{p}$ $\quad \dfrac{i}{p^2-\mu^2+i\epsilon}$

  $\xrightarrow{p}$ $\quad \dfrac{i(\not p+m)}{p^2-m^2+i\epsilon}$

- Momentum conservation and $\displaystyle\int \frac{d^4k}{(2\pi)^4}$

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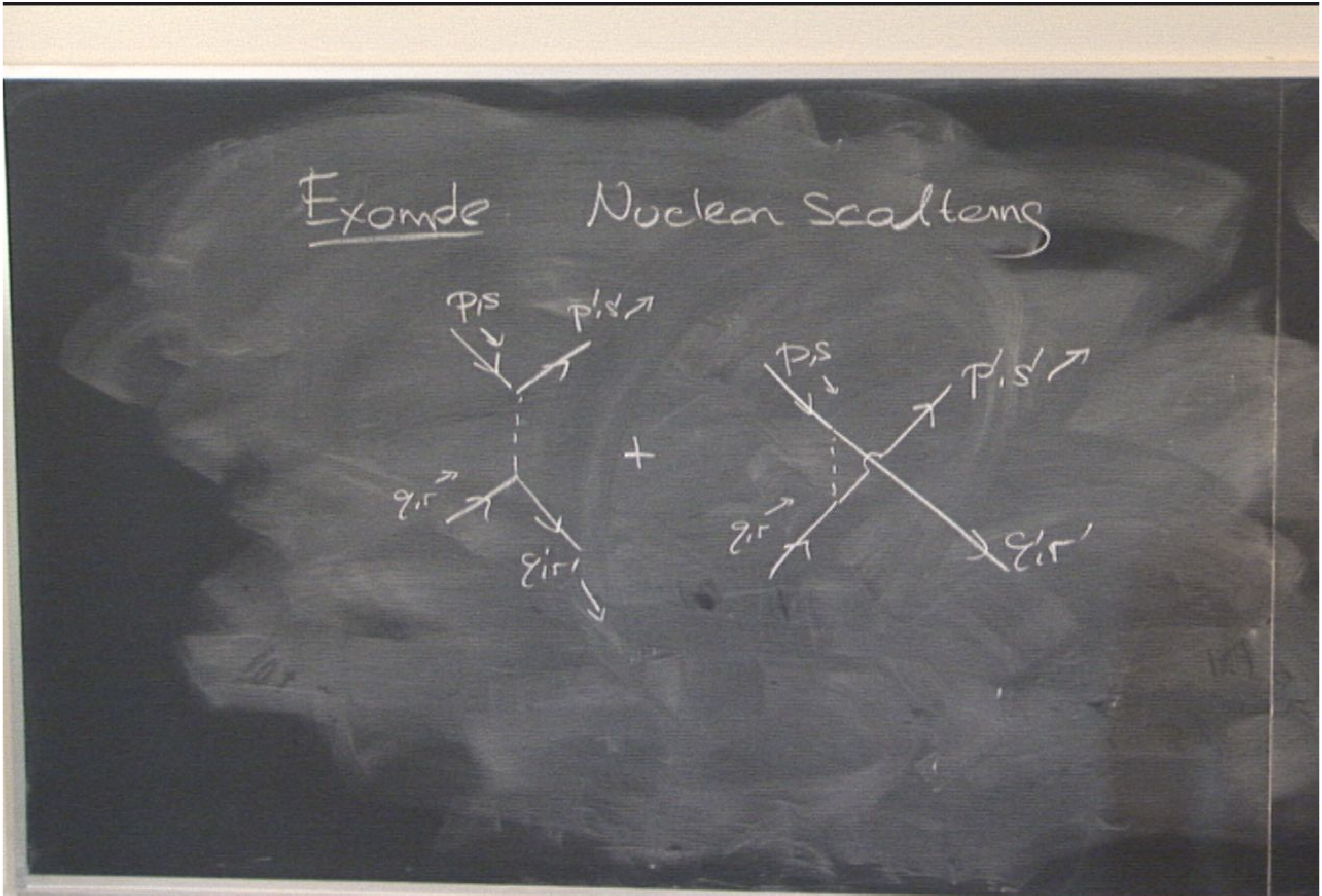
- Momentum conservation and $\displaystyle\int \frac{d^4k}{(2\pi)^4}$
- Extra minus signs !


Example Nucleon scattering
p,s
p',s'
q,r
q',r'
+
p,s
p',s'
q,r
q',r'

$$\mathcal{A} = (-i\lambda)^2 \left(\frac{[\bar{u}^{s'}_{p'} \cdot u^{s}_{p}]\,[\bar{u}^{r'}_{q'} \cdot u^{r}_{q}]}{(p-p')^2-\mu^2} - \frac{[\bar{u}^{s'}_{p'} \cdot u^{r}_{q}]\,[\bar{u}^{r'}_{q'} \cdot u^{s}_{p}]}{(p-q')^2-\mu^2} \right)$$

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$$u^{s}_{p} = \begin{pmatrix} \sqrt{p\cdot\sigma}\;\xi^{s} \\ \sqrt{p\cdot\bar{\sigma}}\;\xi^{s} \end{pmatrix}$$