Title: Quantum Field Theory (PHYS 601) - Lecture 10

Date: Oct 09, 2009 09:00 AM

URL: http://pirsa.org/09100115

Abstract:

Recall: $\Lambda = \exp\left(\frac{1}{2}\Omega_{\rho\sigma} M^{\rho\sigma}\right).$

Six 4×4 matrices

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six number. Six 4×4 matrices

$$[M^{\rho\sigma}, M^{\tau\nu}] = \eta^{\sigma\tau} M^{\rho\nu} - \eta^{\rho\tau} M^{\sigma\nu} + \eta^{\rho\nu} M^{\sigma\tau} - \eta^{\sigma\nu} M^{\rho\tau}$$

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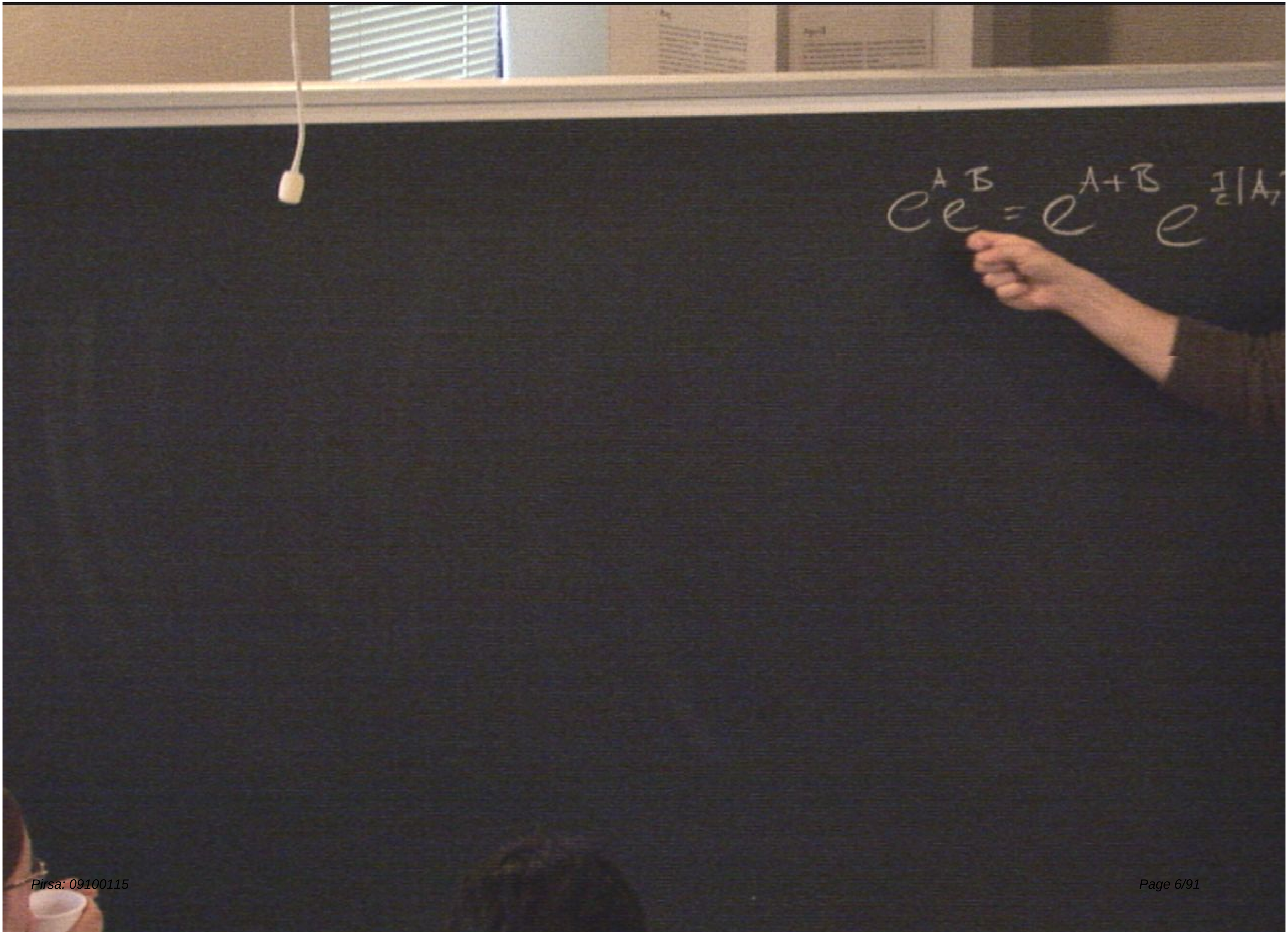
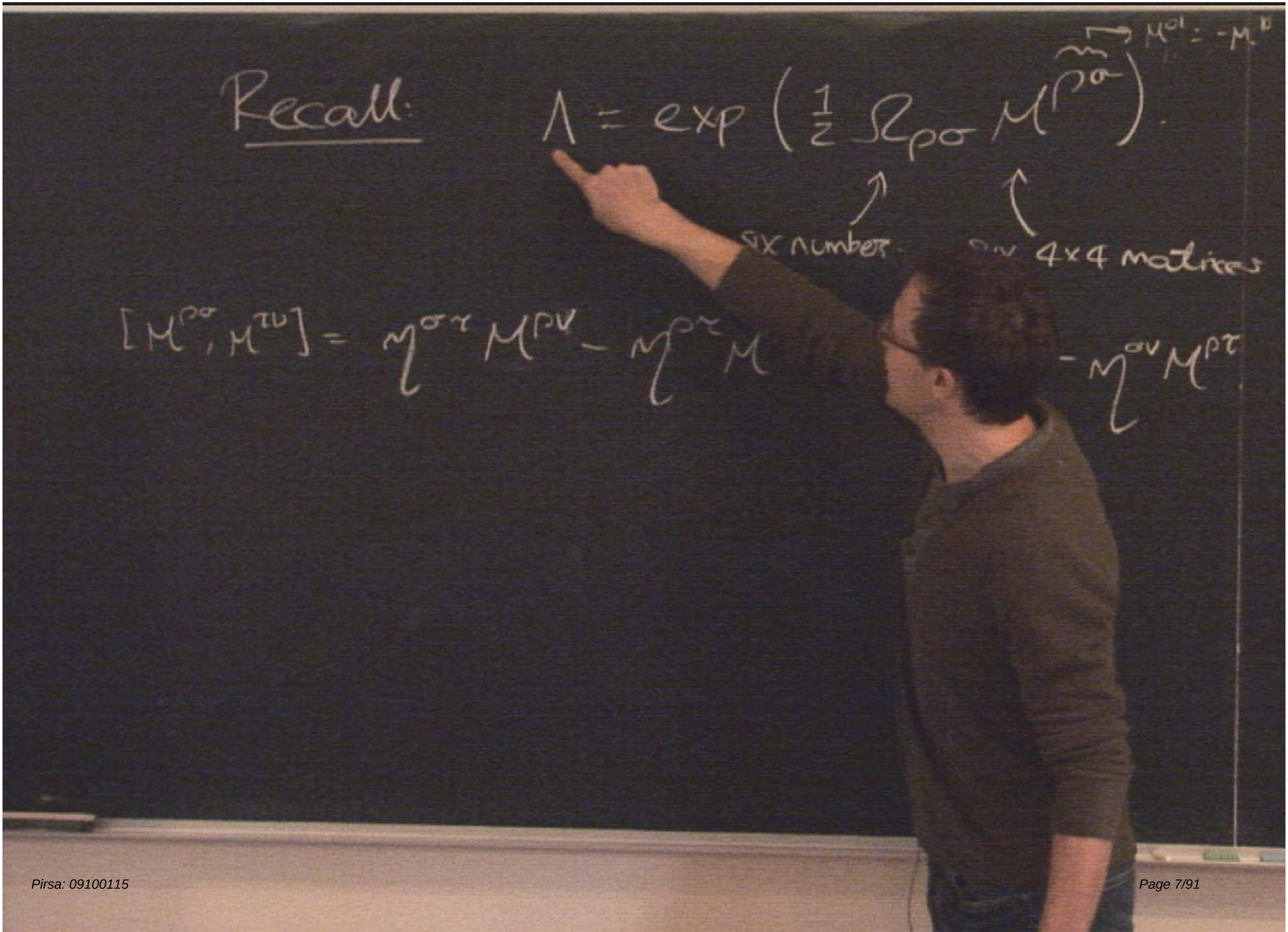

$e^{A}e^{B}=e^{A+B}\,e^{\frac{1}{2}[A,}$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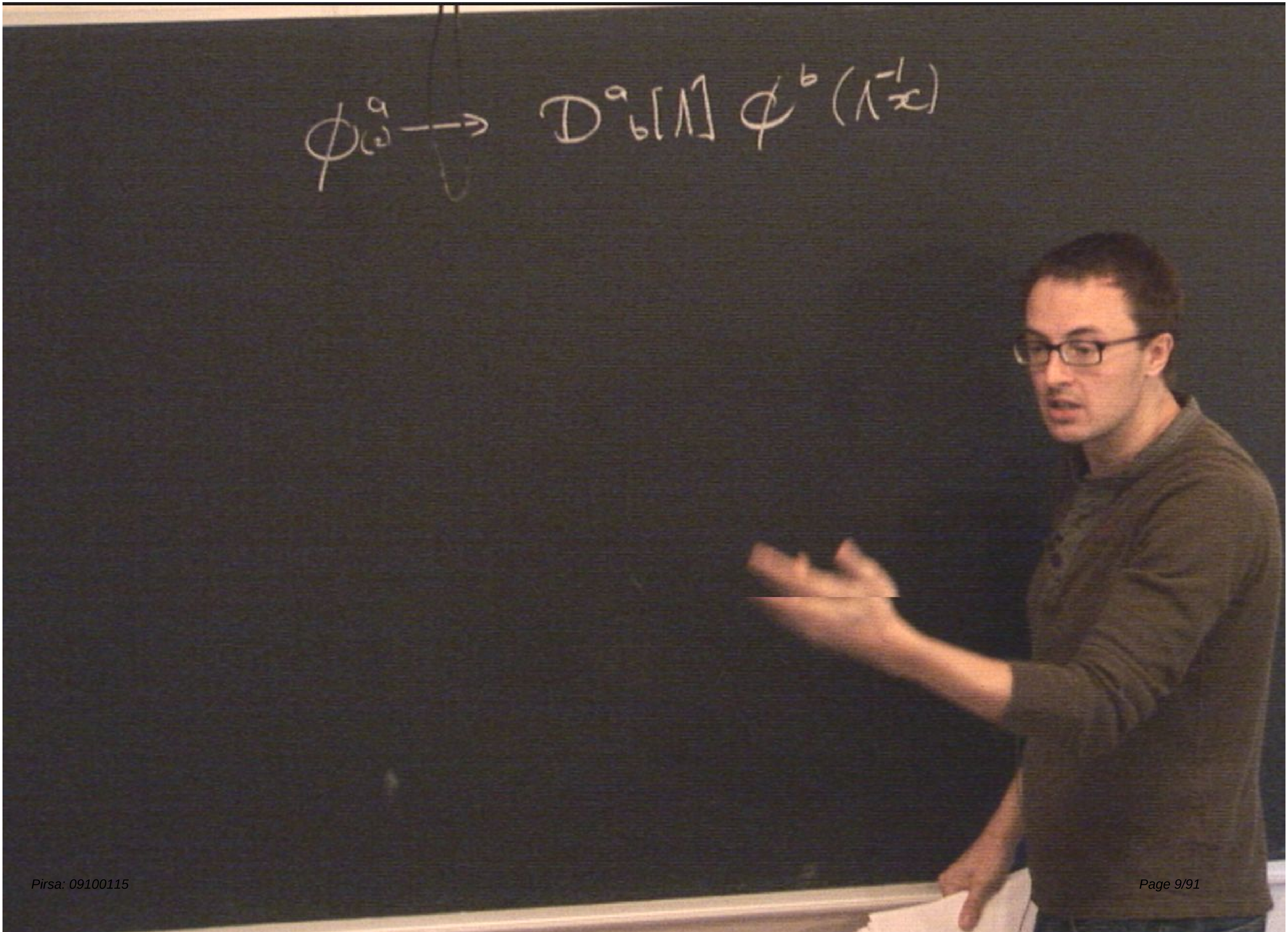
$\phi^a(x) \dashrightarrow D^a{}_b[\Lambda]\, \phi^b(\Lambda^{-1}x)$

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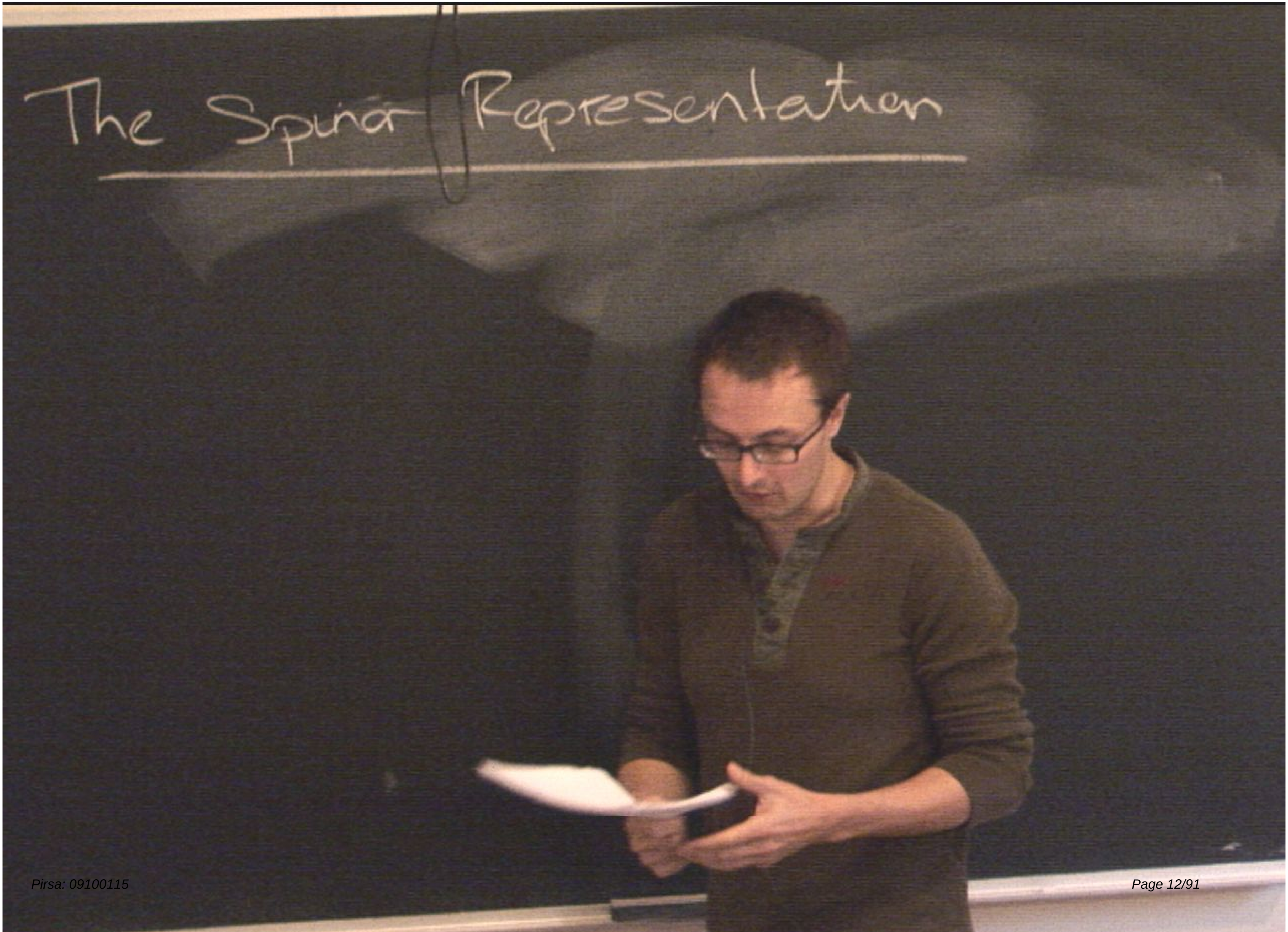
The Spinor Representation

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i.e. $\gamma^\mu \gamma^\nu = -\gamma^\nu \gamma^\mu \quad \mu \neq \nu$

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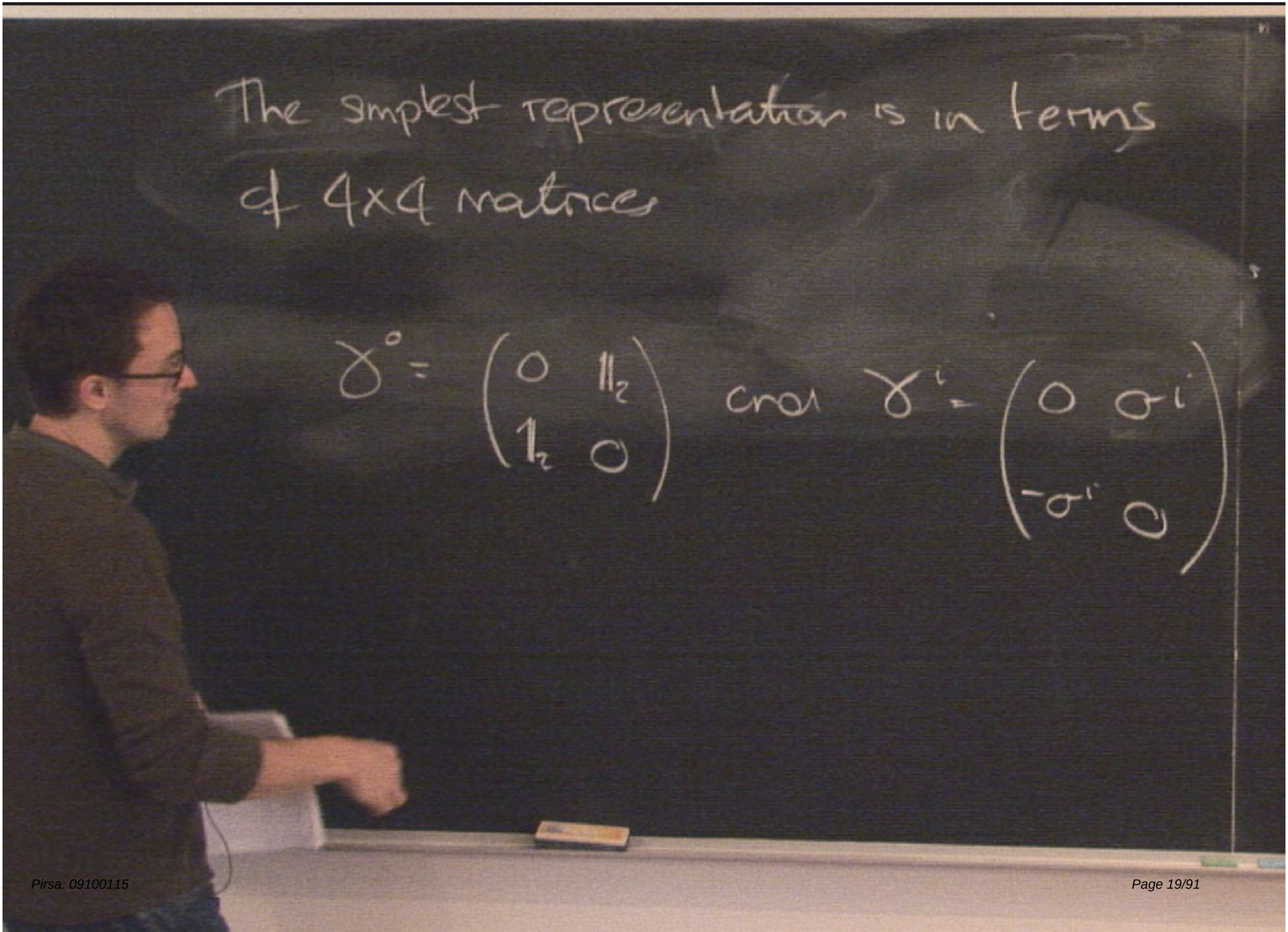
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$$\sigma^i = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

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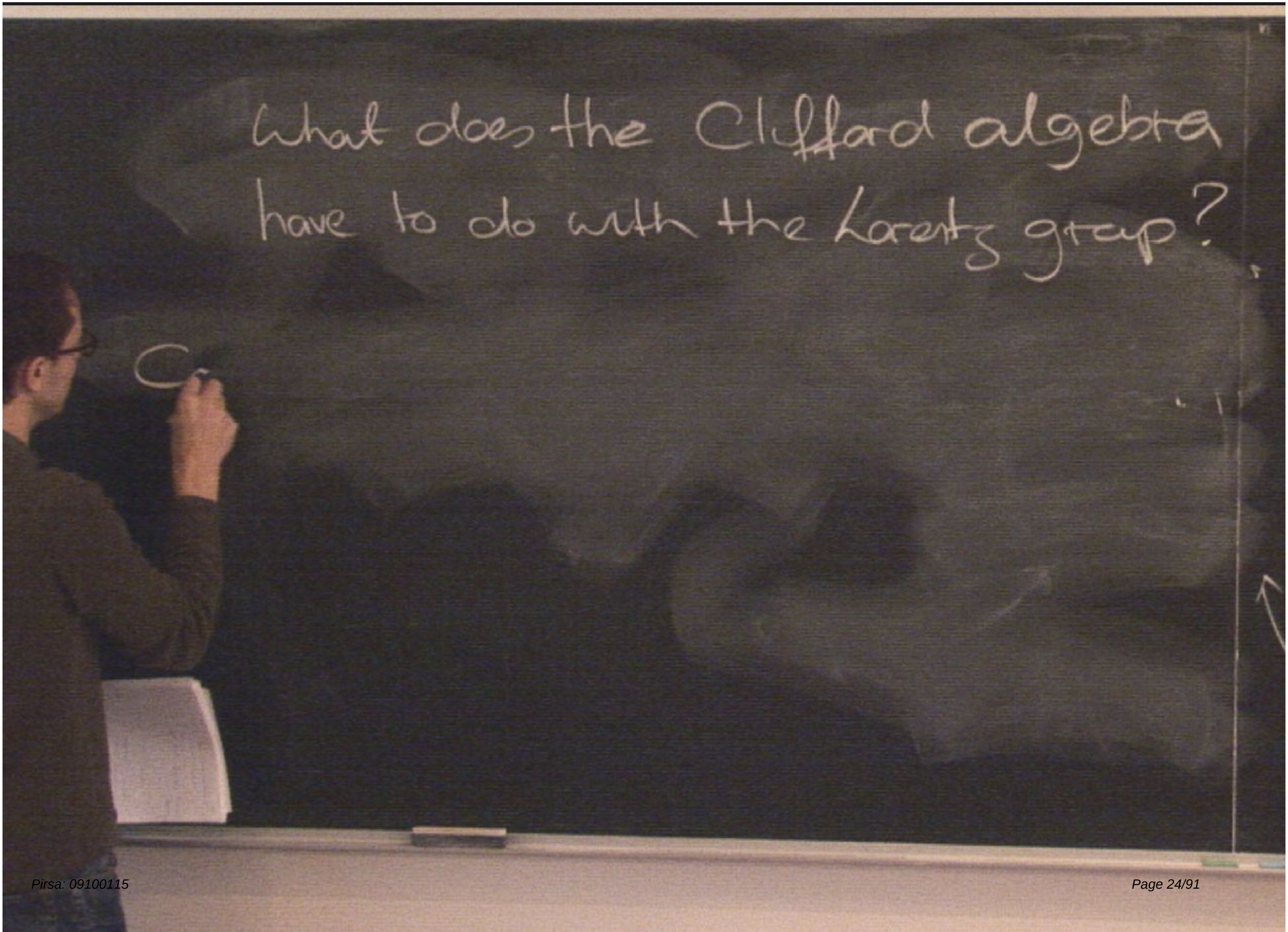
This choice of γ^μ is called chiral representation


What does the Clifford algebra
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What does the Clifford algebra have to do with the Lorentz group?

Consider the six matrices

$$S^{\rho\sigma} = \frac{1}{4}\left[\gamma^\rho, \gamma^\sigma\right]$$

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Proof: Exercise!

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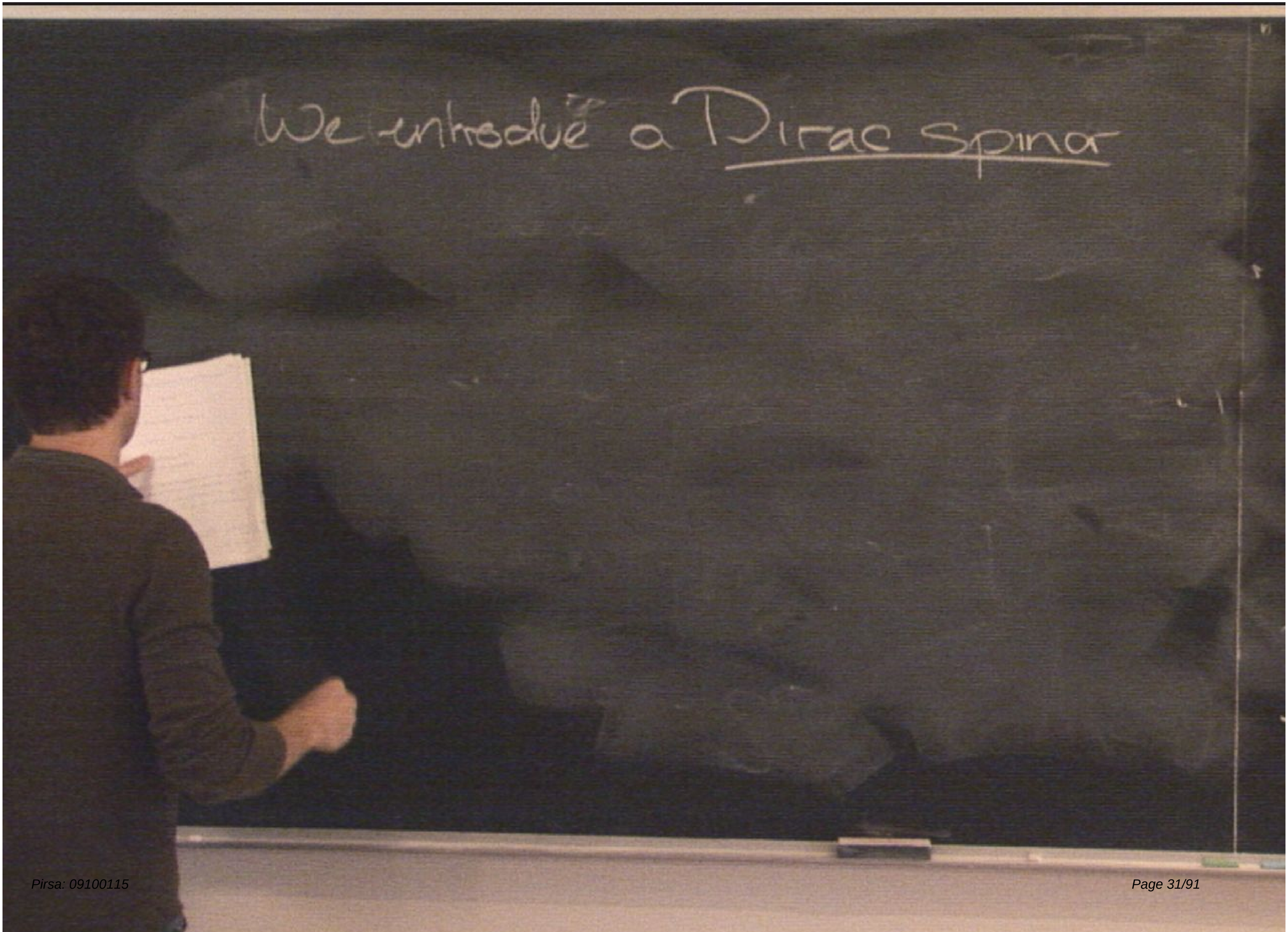
Proof: Exercise! First proof $[S^{\mu\nu}, \gamma^{\rho}] = \gamma^{\mu} \eta^{\nu\rho} - \gamma^{\nu} \eta^{\rho\mu}$


We introduce a Dirac spinor

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This is a complex-valued object

$$\psi_\alpha(x)$$

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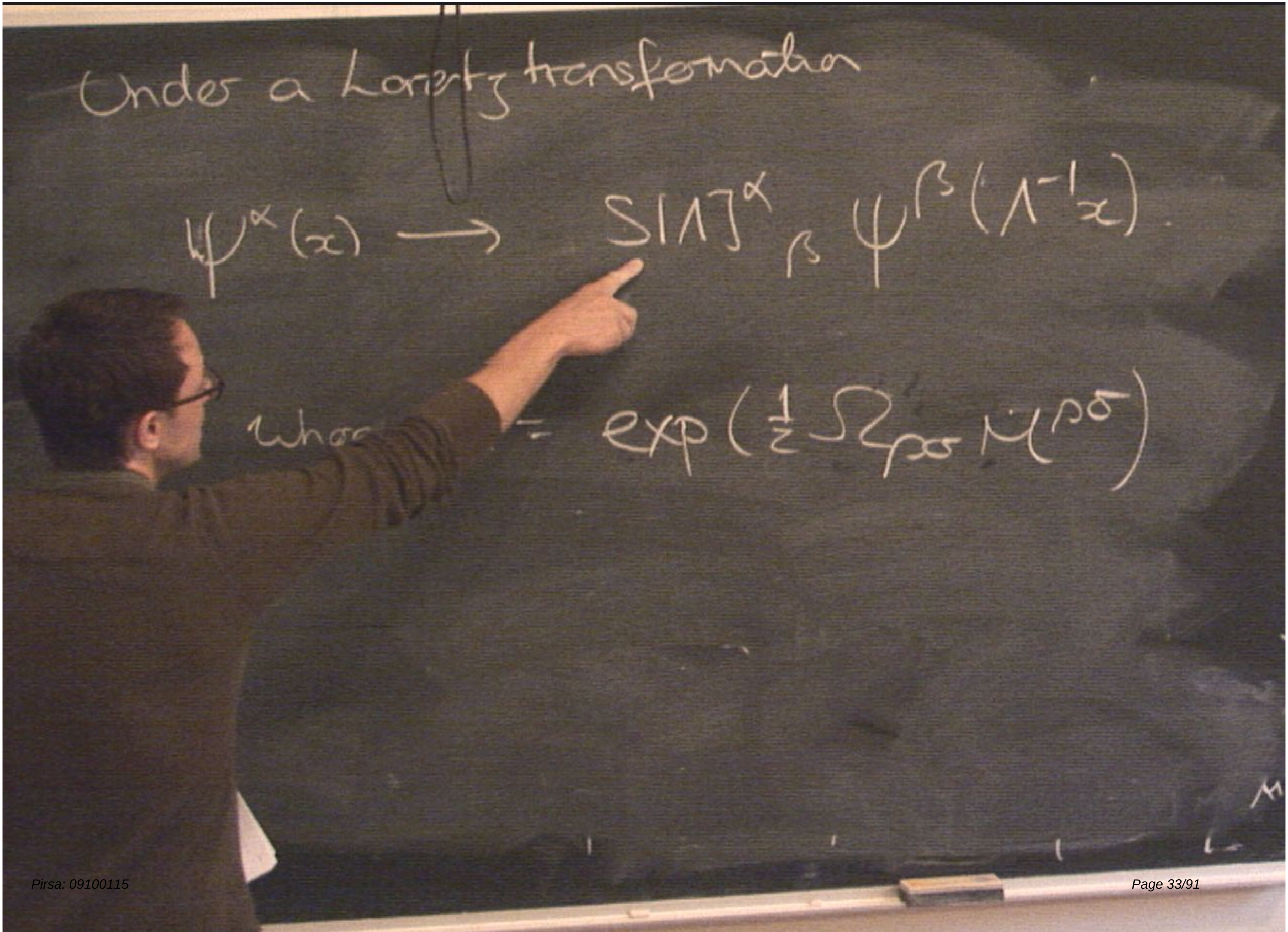
Under a Lorentz transformation
$\psi^\alpha(x) \longrightarrow S[\Lambda]^\alpha{}_\beta\, \psi^\beta(\Lambda^{-1}x).$
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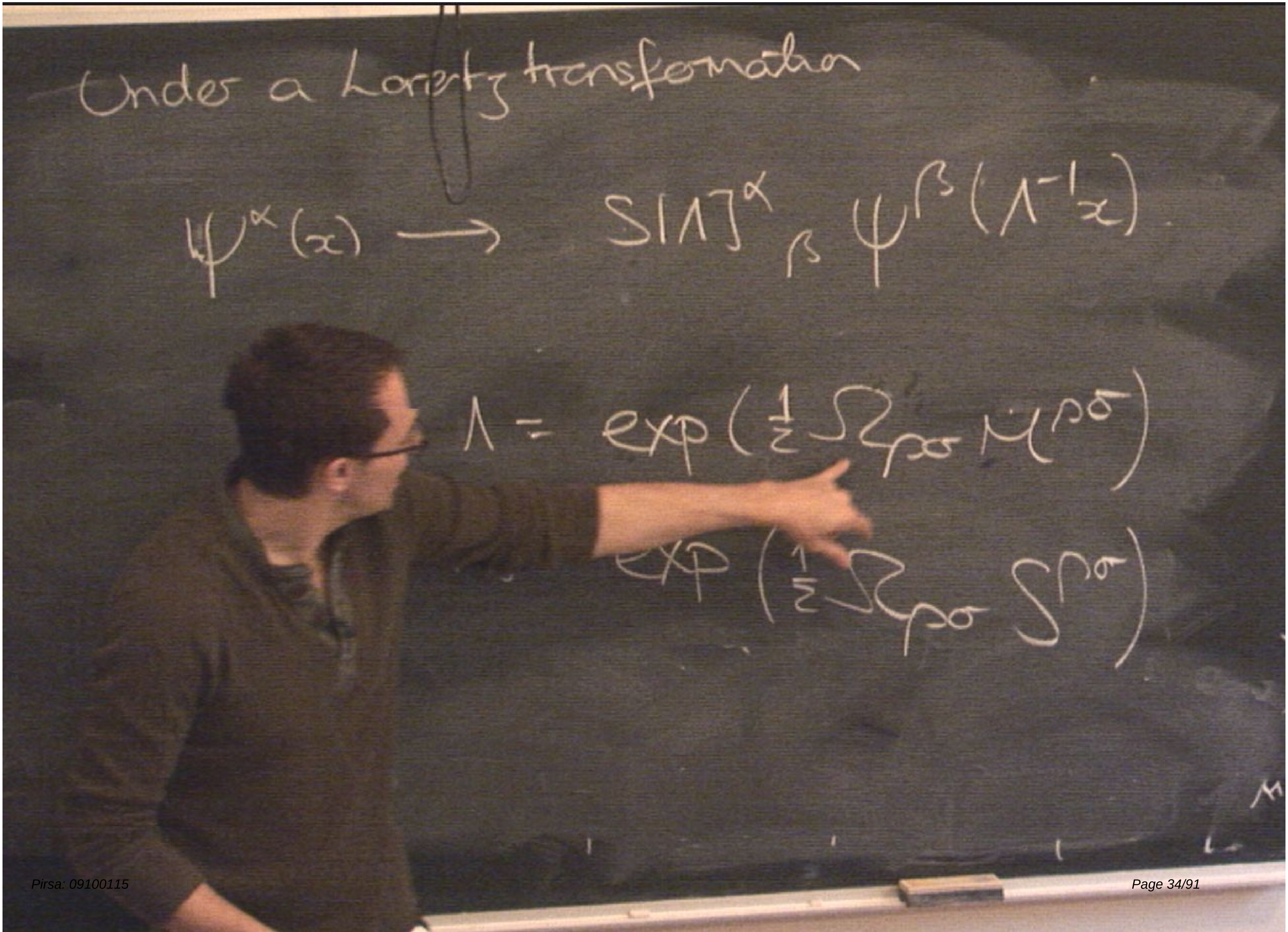
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$$S[\Lambda_1]S[\Lambda_2] = S[\Lambda_1\Lambda_2]$$

We introduce a <u>Dirac spinor</u>.

This is a complex-valued object

$$\psi^{\alpha}(x)$$

$\alpha = 1,2,3,4.$

$$\Gamma^{\mu} = \begin{pmatrix} \gamma^{\mu} & 0 \\ 0 & \gamma^{\mu} \end{pmatrix}$$

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Rotations.

$$S^{ij} = \frac{1}{4}[\gamma^i, \gamma^j] \qquad i,j = 1,2,3$$

$$= -\frac{i}{2}\epsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}$$

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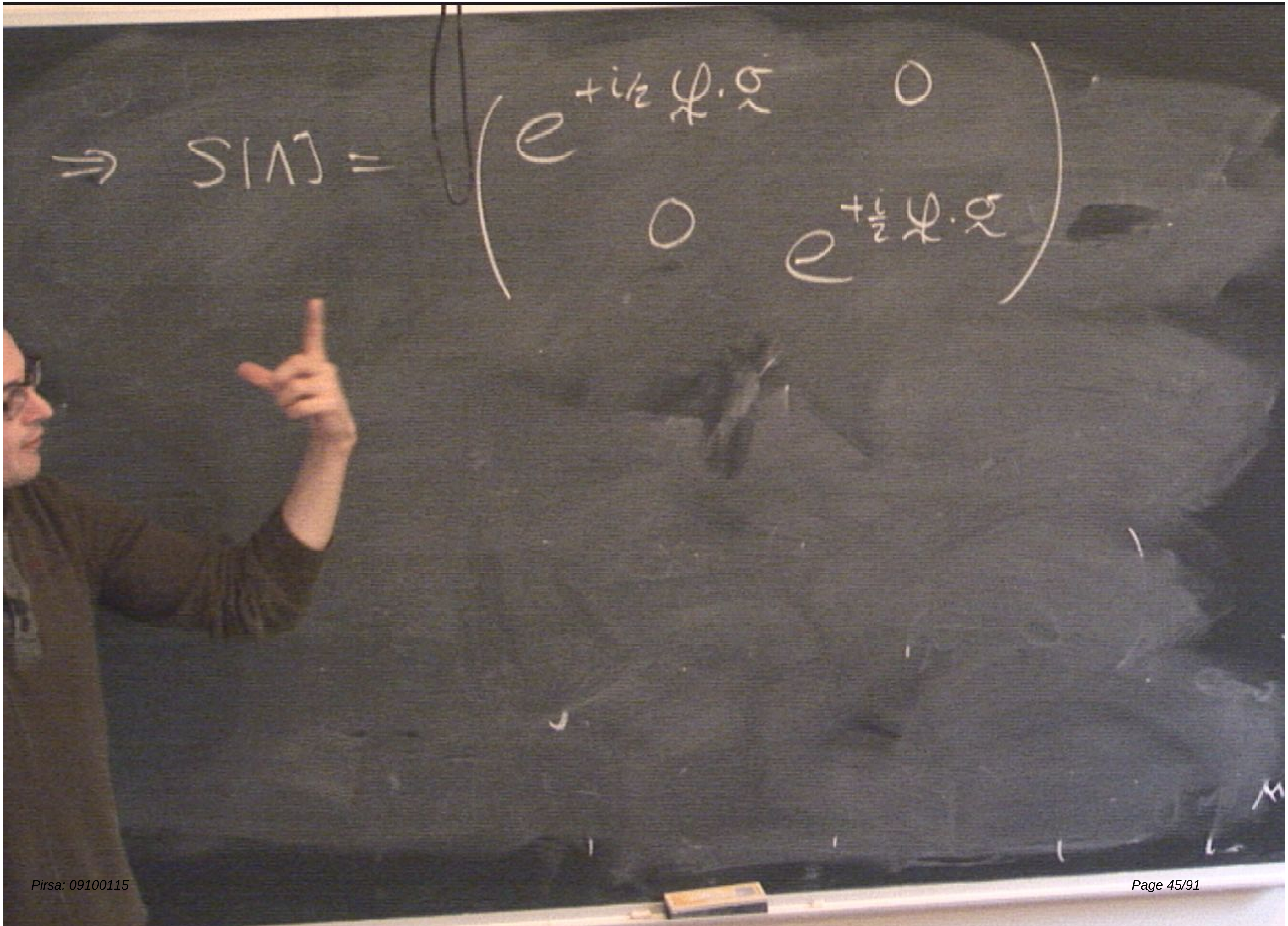
Define $\Omega_{ij} = -\epsilon_{ijk}\varphi^k$


$\Rightarrow \quad S[\Lambda] = \begin{pmatrix} e^{+\frac{i}{2}\underline{\varphi}\cdot\underline{\sigma}} & 0 \\ 0 & e^{+\frac{1}{2}\underline{\varphi}\cdot\underline{\sigma}} \end{pmatrix}$

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Consider a rotation by 2π around z-axis.

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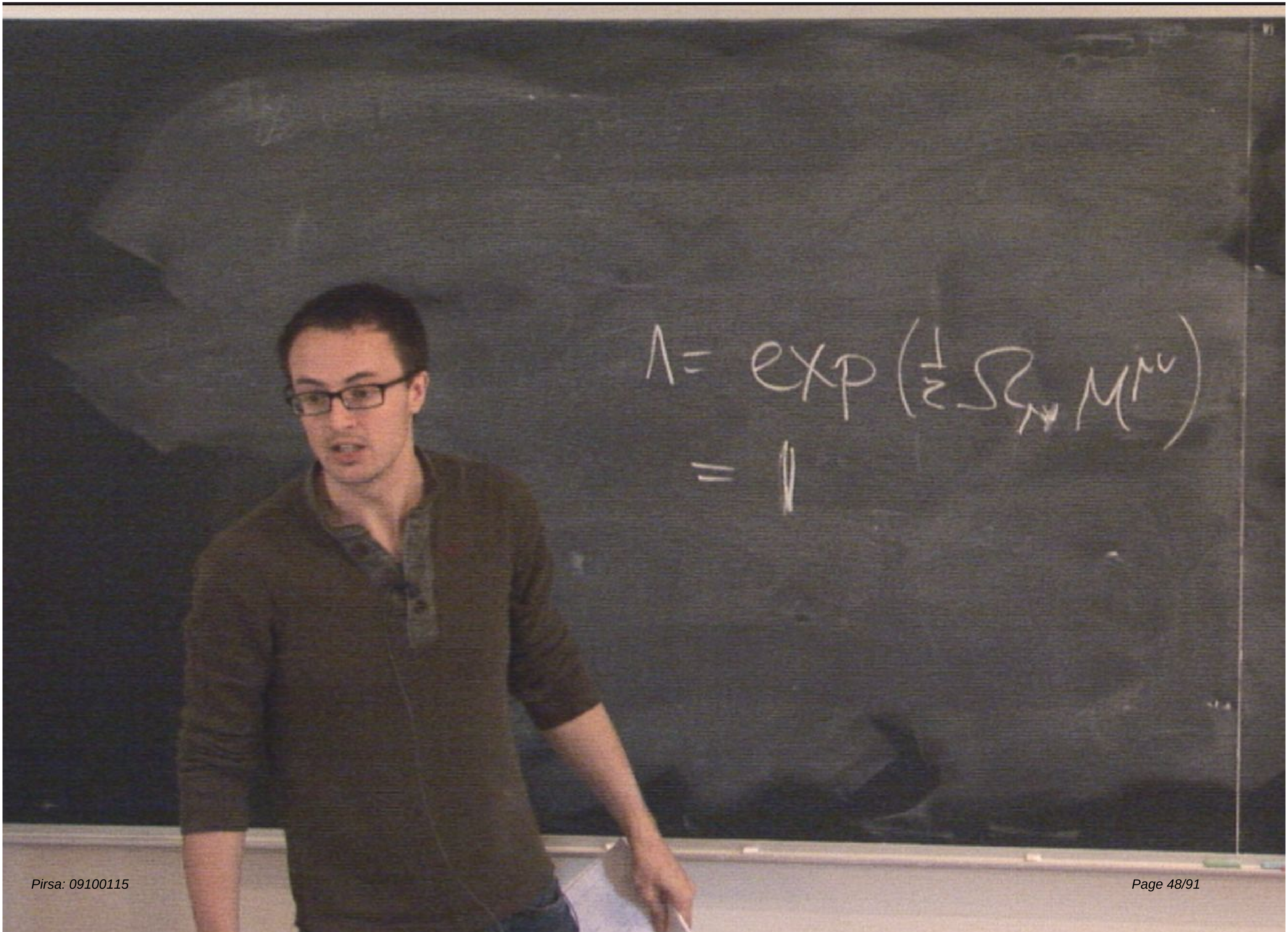
$$\Rightarrow \quad S[\Lambda] = -\mathbb{1} \quad \Rightarrow \quad \psi^{\alpha}(x) \longrightarrow -\psi^{\alpha}(x)$$


$\Lambda = \exp\left(\frac{1}{2}\Omega_{\mu\nu} M^{\mu\nu}\right)$
$= 1$

Boosts. $S^{0i} = \frac{1}{2}[\gamma^0, \gamma^i]$

$= \frac{1}{2}\begin{pmatrix} -\sigma^i & \\ & \sigma^i \end{pmatrix}$

Write $\Omega_{i0} = -\Omega_{0i} = \chi_i$

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Note: For rotations

$$S[\Lambda]^\dagger S[\Lambda] = \mathbb{1}$$

But for boosts

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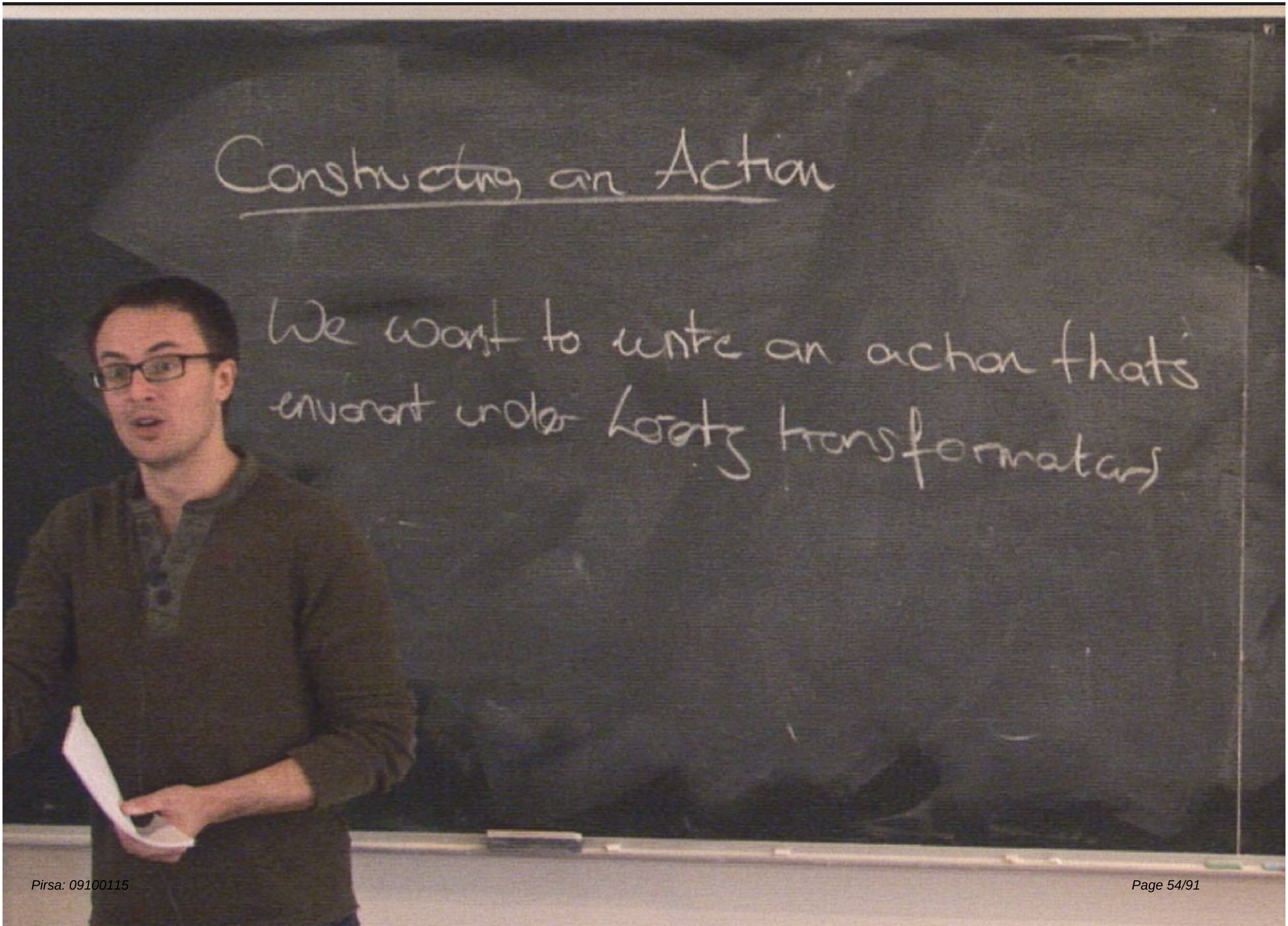
In fact, there are no finite dimensional unitary representations of Lorentz group.

Constructing an Action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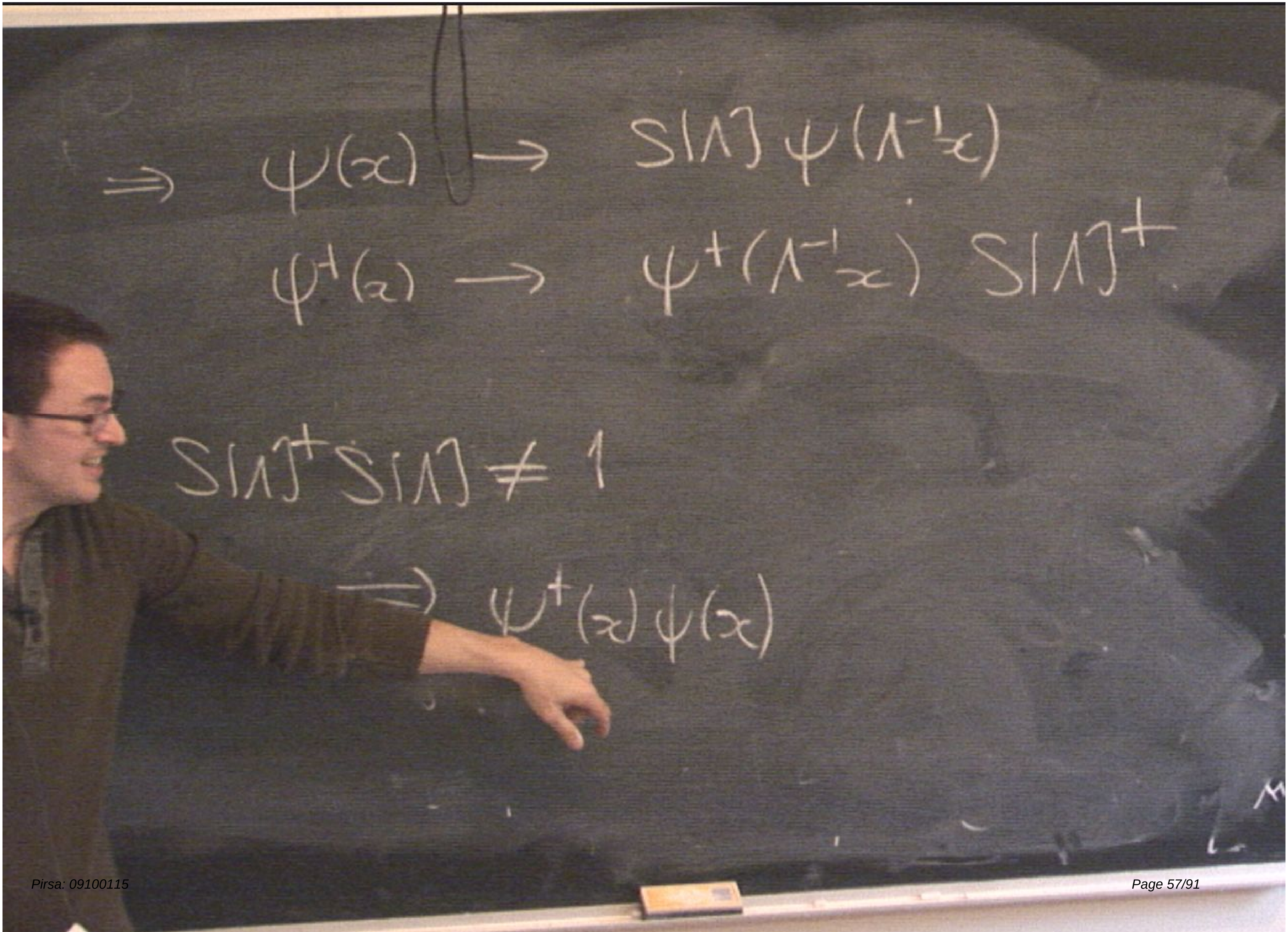
Define $\psi^{\dagger}(x) = (\psi^{*})^{T}(x)$

$$\Rightarrow \quad \psi(x) \longrightarrow S[\Lambda]\,\psi(\Lambda^{-1}x)$$

$$\psi^{+}(x) \longrightarrow \psi^{+}(\Lambda^{-1}x)\, S[\Lambda]^{+}$$


$\Rightarrow \quad \psi(x) \longrightarrow S[\Lambda]\,\psi(\Lambda^{-1}x)$
$\psi^{+}(x) \longrightarrow \psi^{+}(\Lambda^{-1}x)\, S[\Lambda]^{+}$
$S[\Lambda]^{+} S[\Lambda] \neq 1$
$\Rightarrow \quad \psi^{+}(x)\,\psi(x)$

$$\Rightarrow \quad \psi(x) \longrightarrow S[\Lambda]\,\psi(\Lambda^{-1}x)$$

$$\psi^\dagger(x) \longrightarrow \psi^\dagger(\Lambda^{-1}x)\,S[\Lambda]^\dagger$$

But $S[\Lambda]^\dagger S[\Lambda] \neq 1$

$\Rightarrow$ $\psi^\dagger(x)\,\psi(x)$ is not a Lorentz scalar

Recall that $S[\Lambda] = \exp\left(\frac{1}{2}\Omega_{\rho\sigma}S^{\rho\sigma}\right)$

So $S[\Lambda]$ unitary, we would need

$$(S^{\mu\nu})^{\dagger} = -S^{\mu\nu}$$

But $(S^{\mu\nu})^{\dagger} = -\frac{1}{4}\left[(\gamma^{\mu})^{\dagger}, (\gamma^{\nu})^{\dagger}\right]$

so $(S^{\mu\nu})^\dagger = -S^{\mu\nu}$ $\underline{\underline{\text{iff}}}$

$$(\gamma^\mu)^\dagger = \gamma^\mu \quad \forall \mu$$

or $(\gamma^\mu)^\dagger = -\gamma^\mu \quad \forall \mu$

so $(S^{\mu\nu})^\dagger = -S^{\mu\nu}$!!

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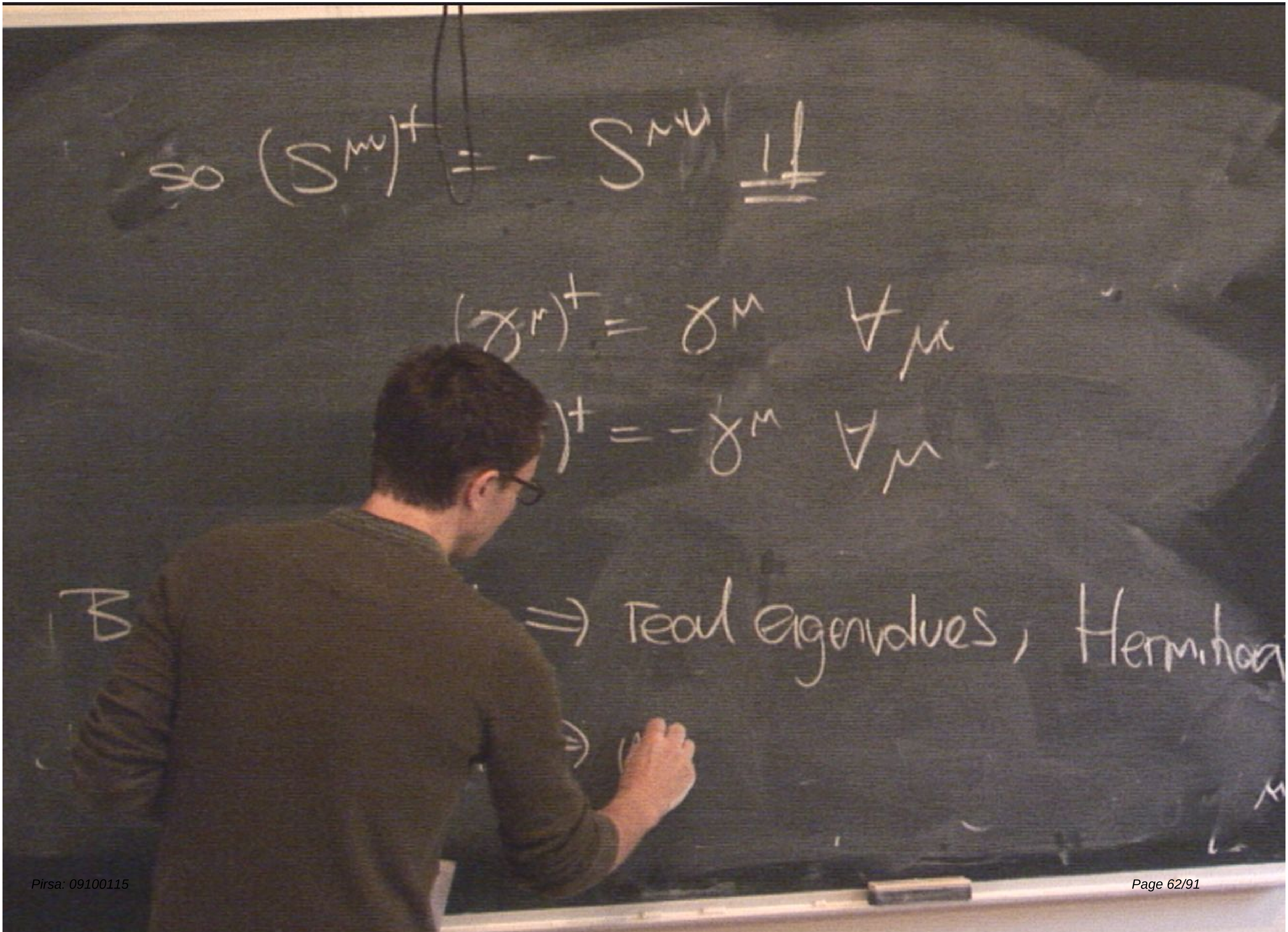
...t $(\gamma^a)^2 = 1 \Rightarrow$ real eigenvalues, Hermitian

$(\gamma$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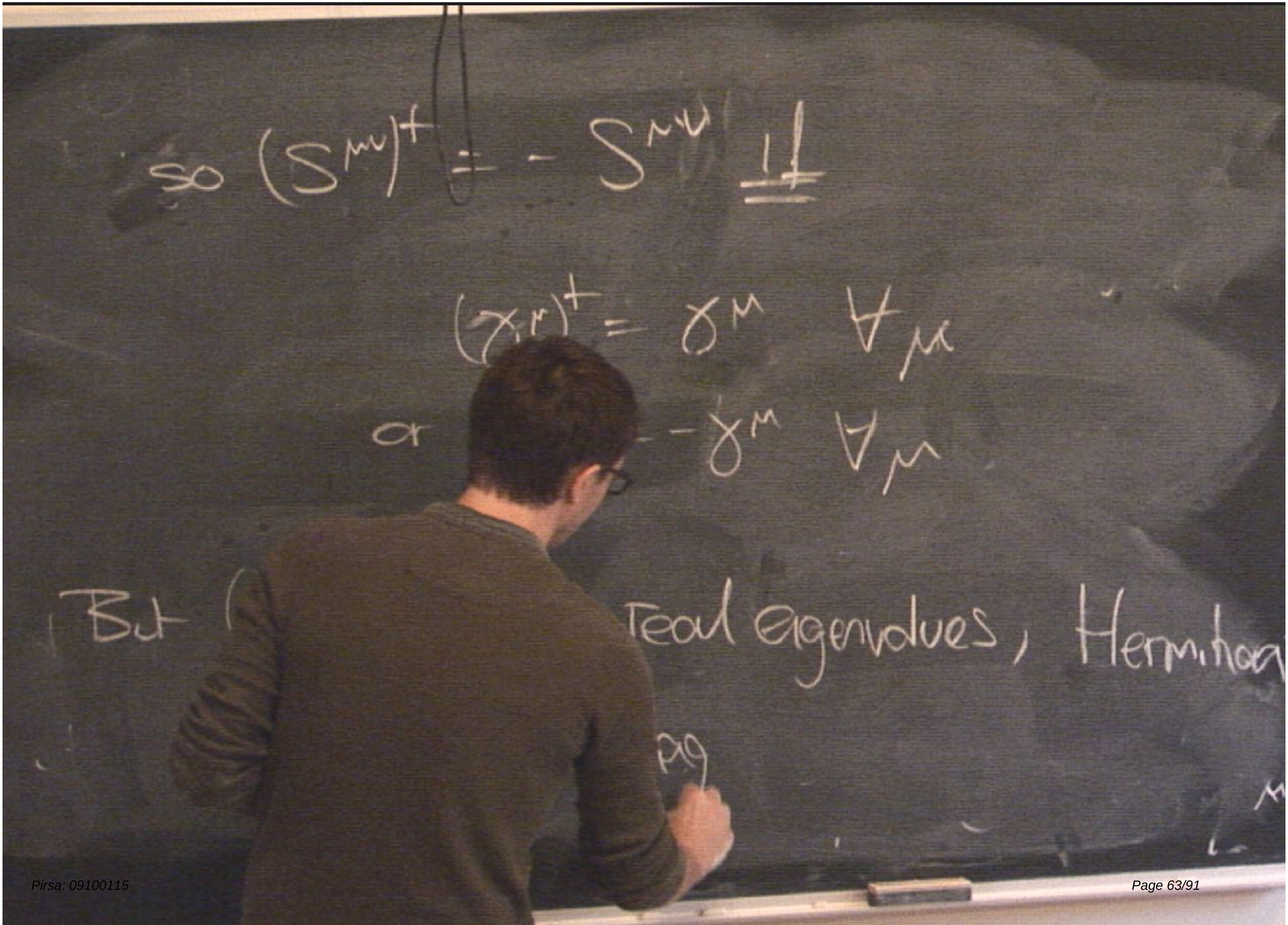

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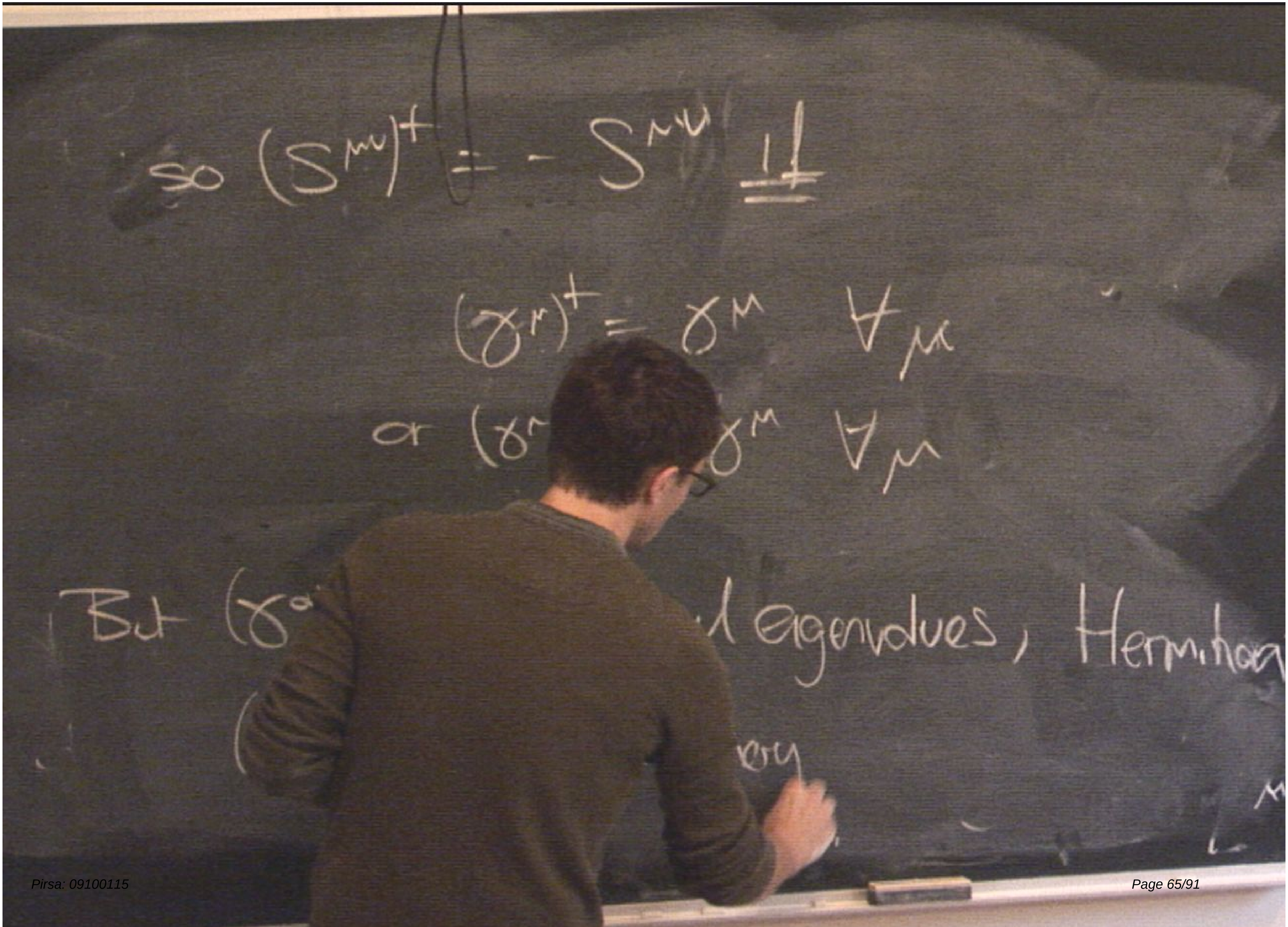
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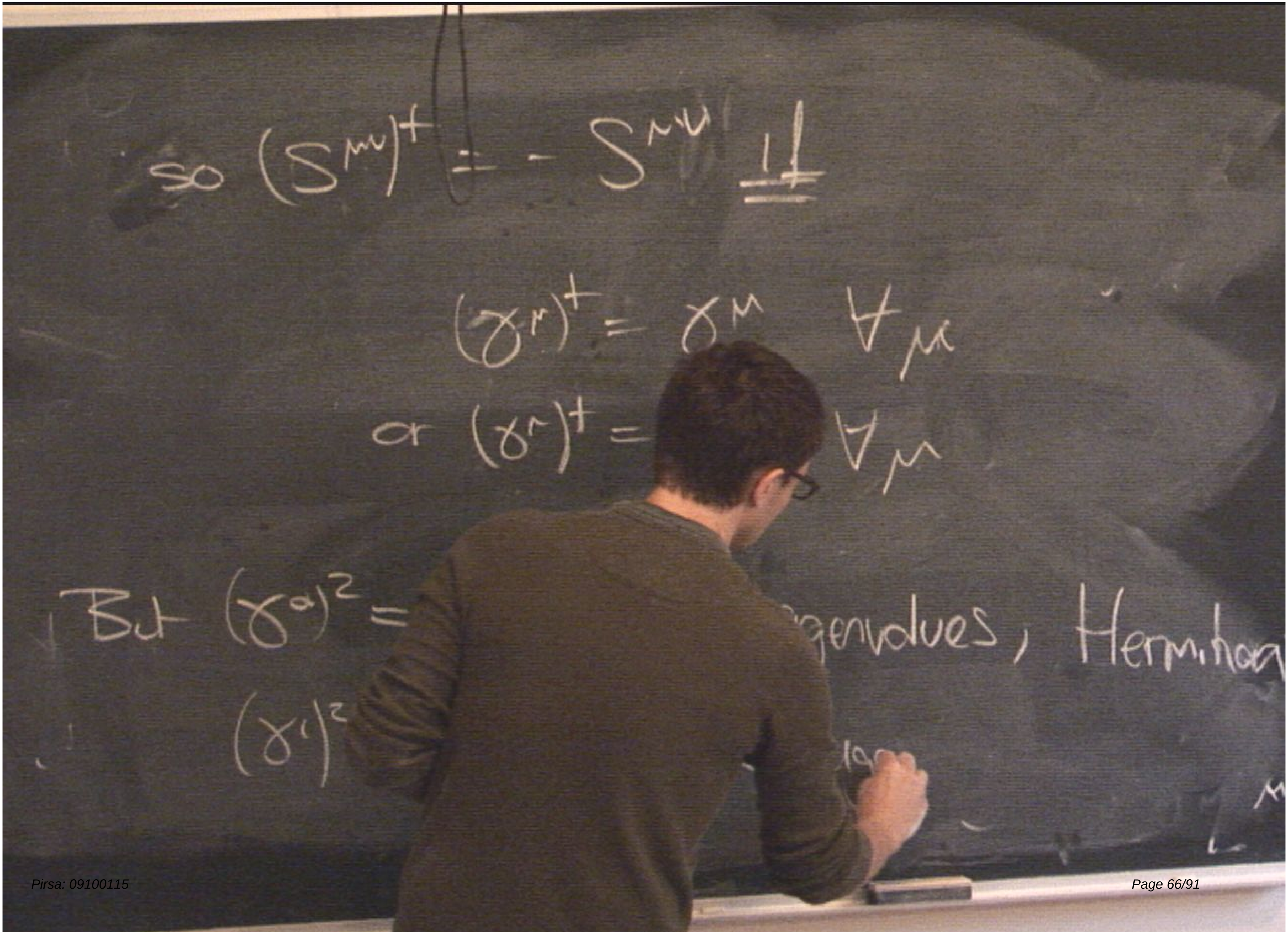
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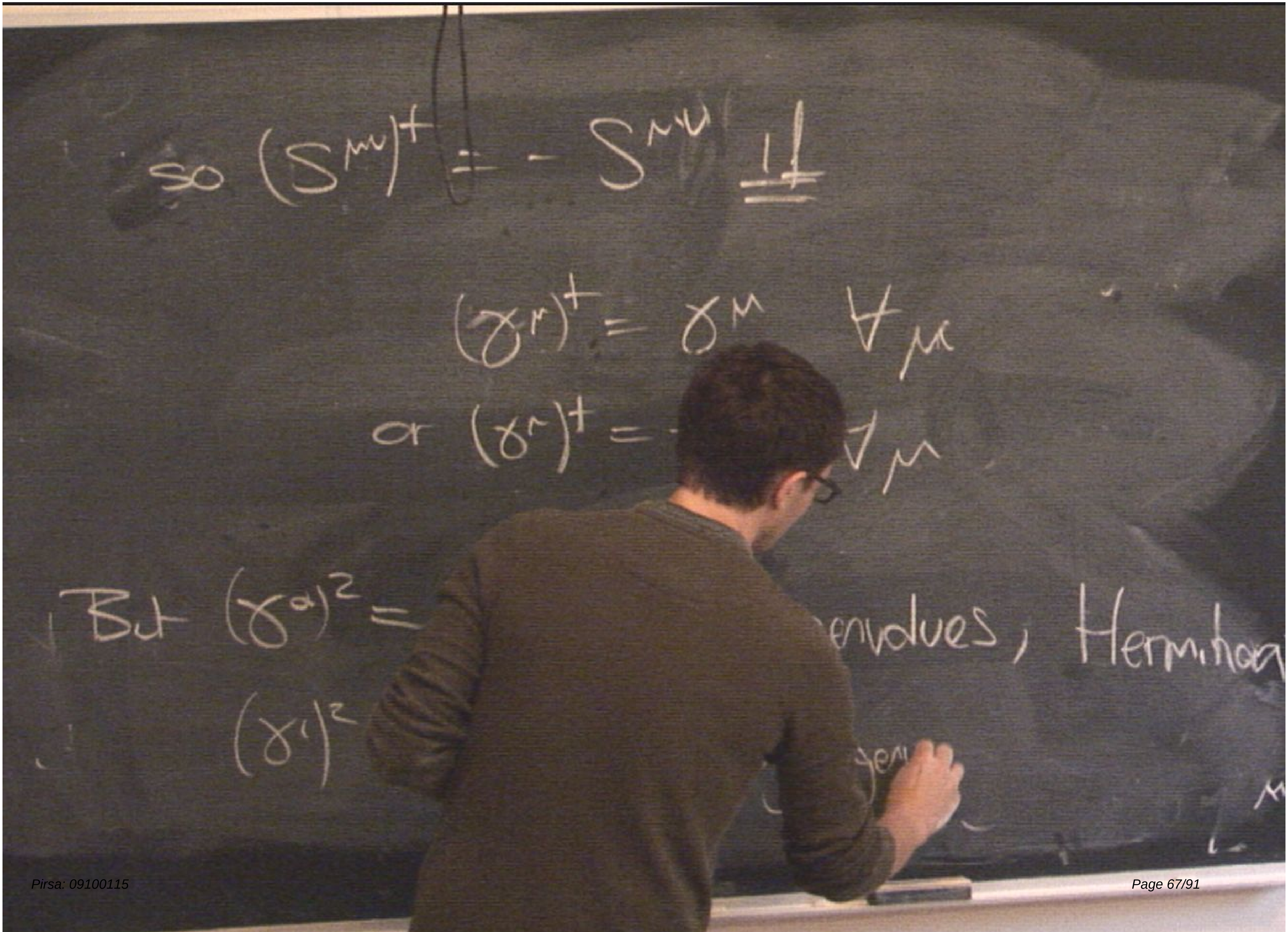
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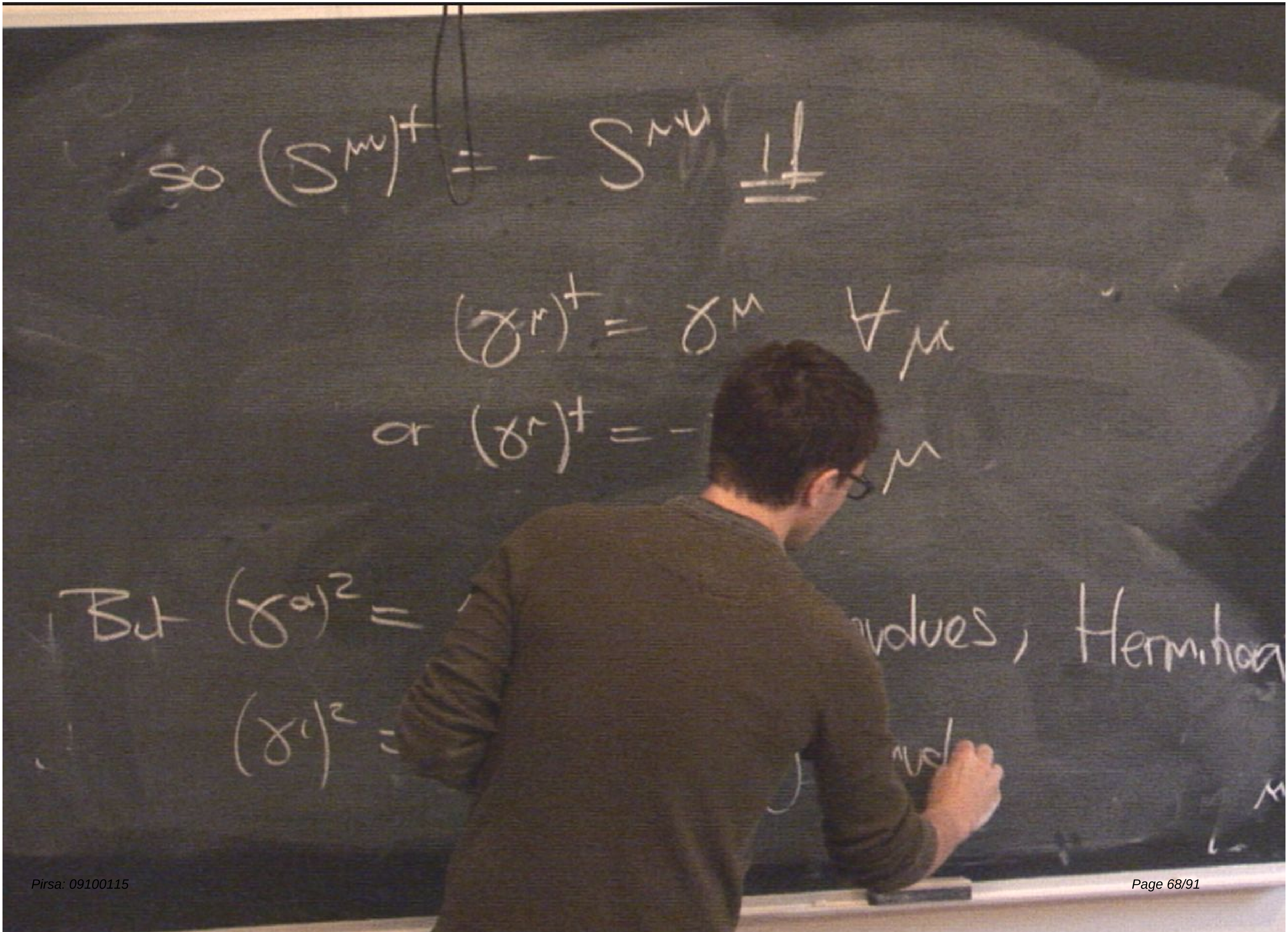
so (S^μν)† = − S^μν !!
(γ^μ)† = γ^μ ∀ μ
or (γ^μ)† = −
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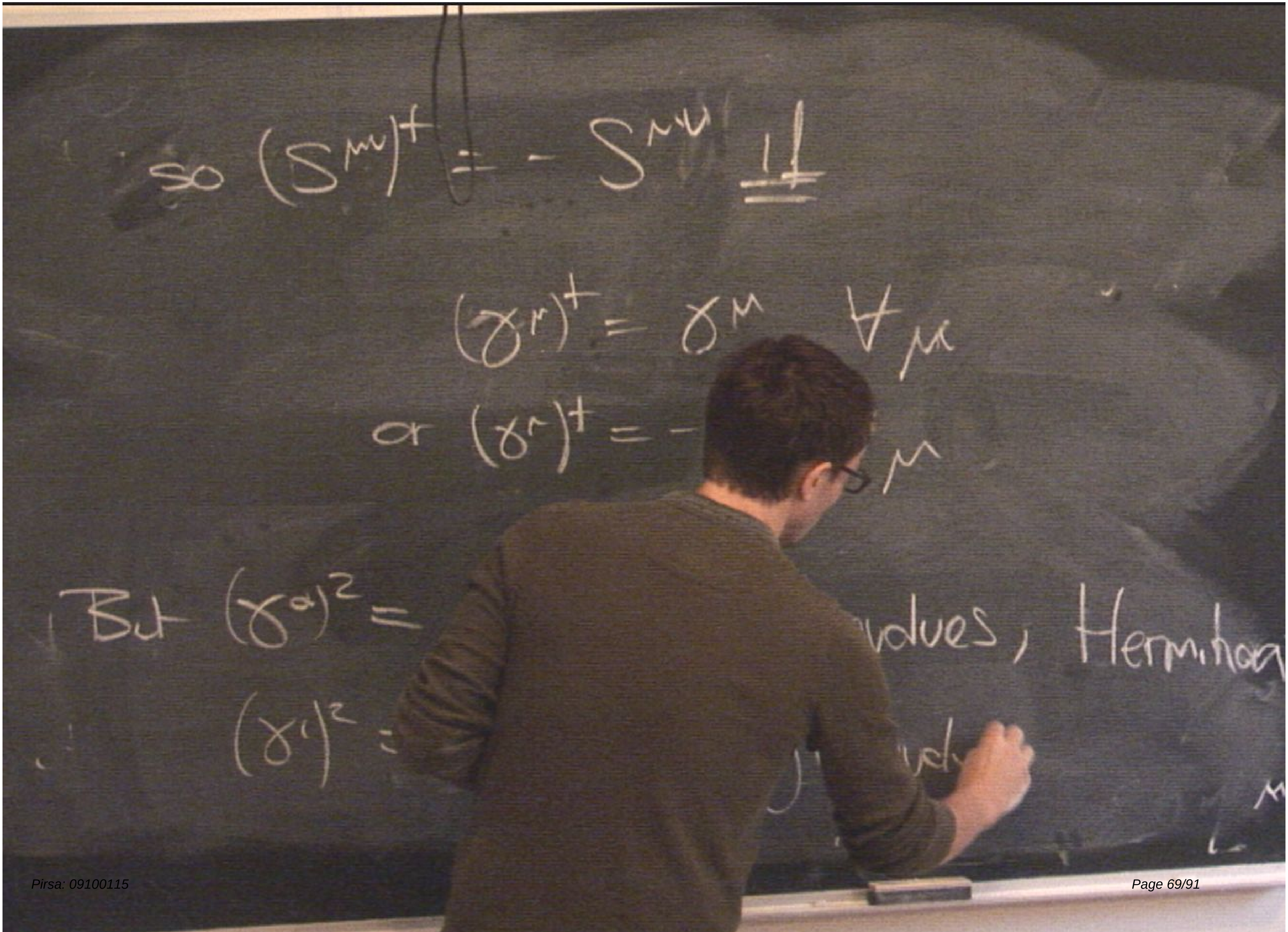
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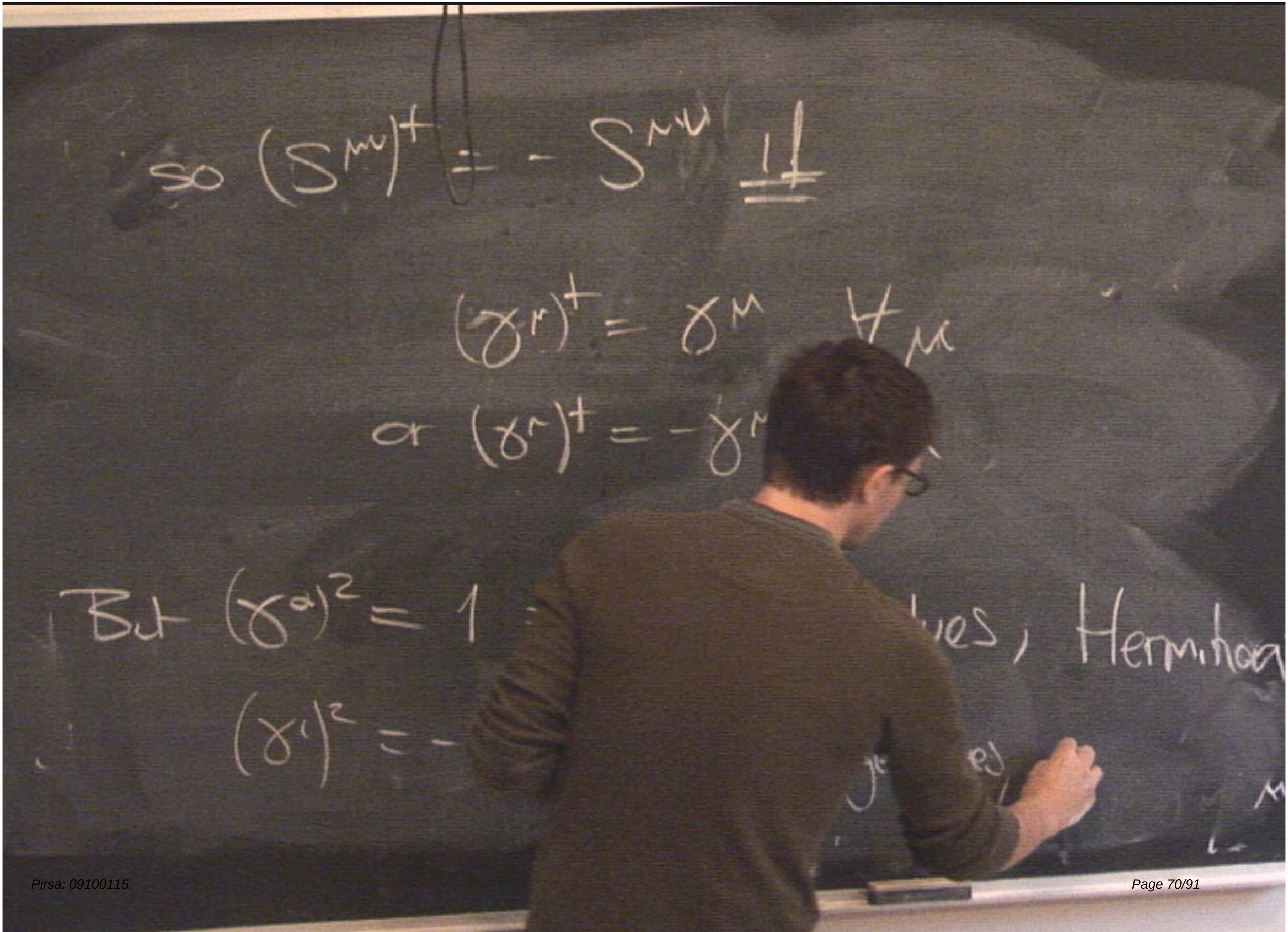
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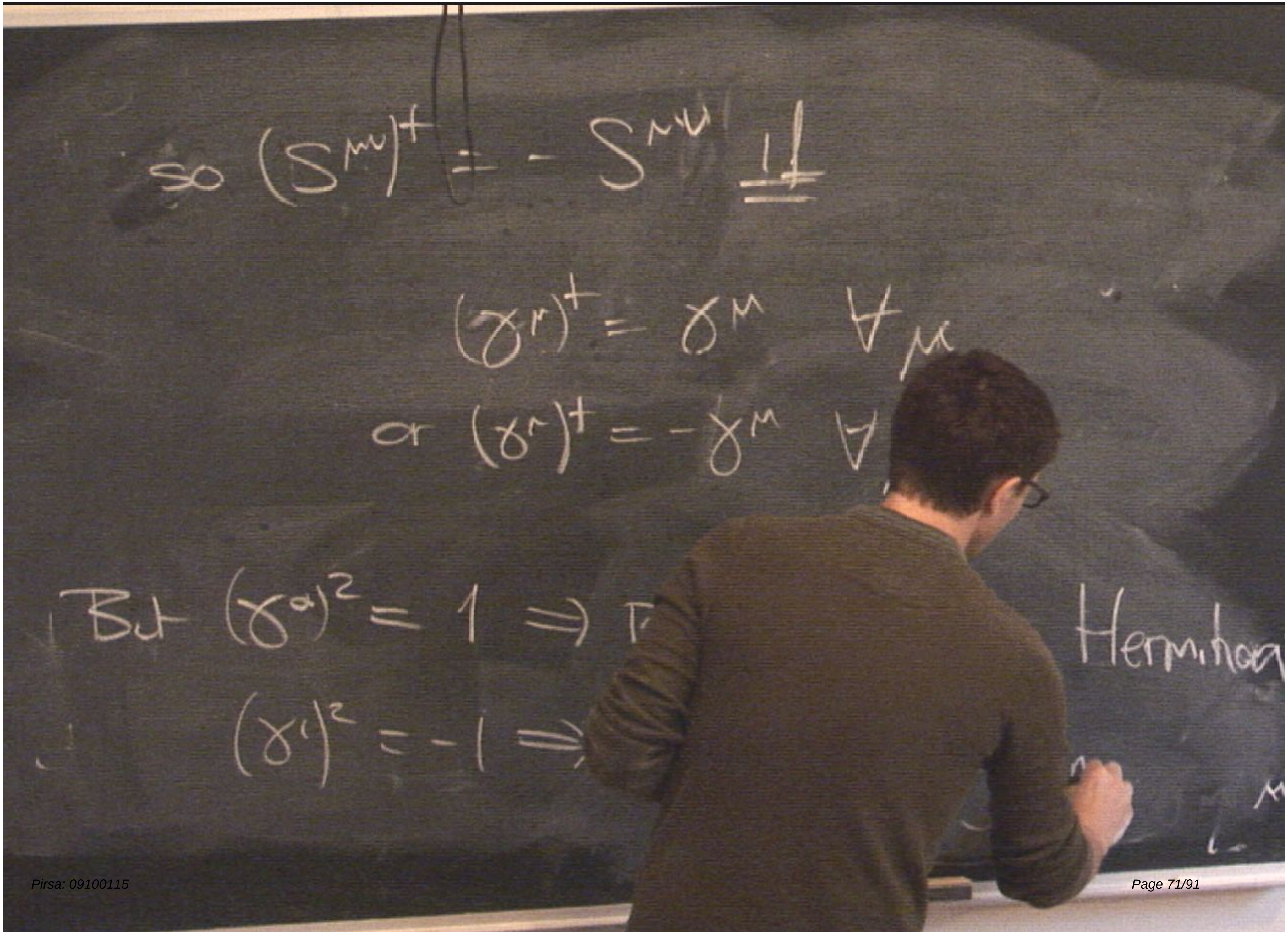
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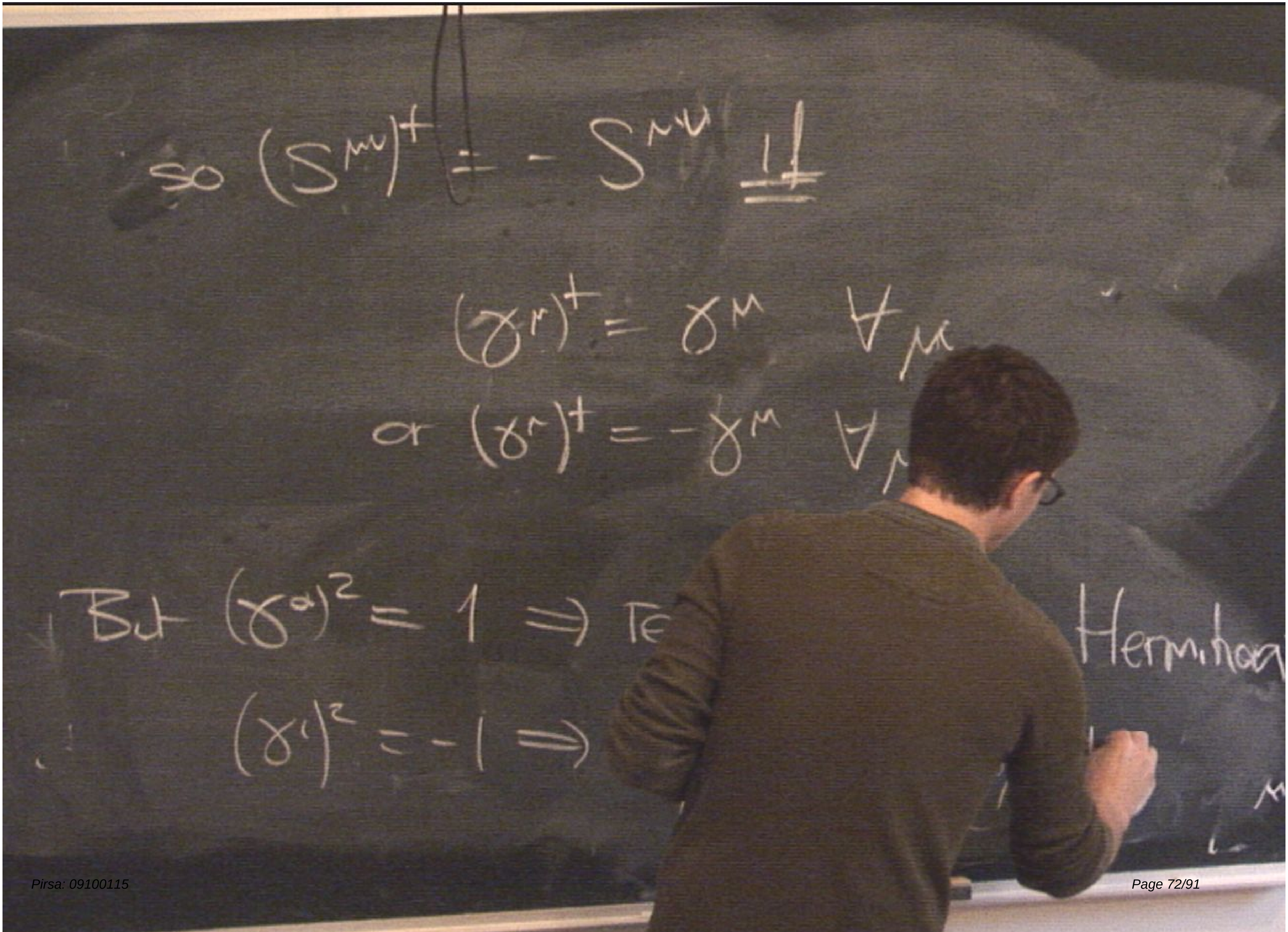
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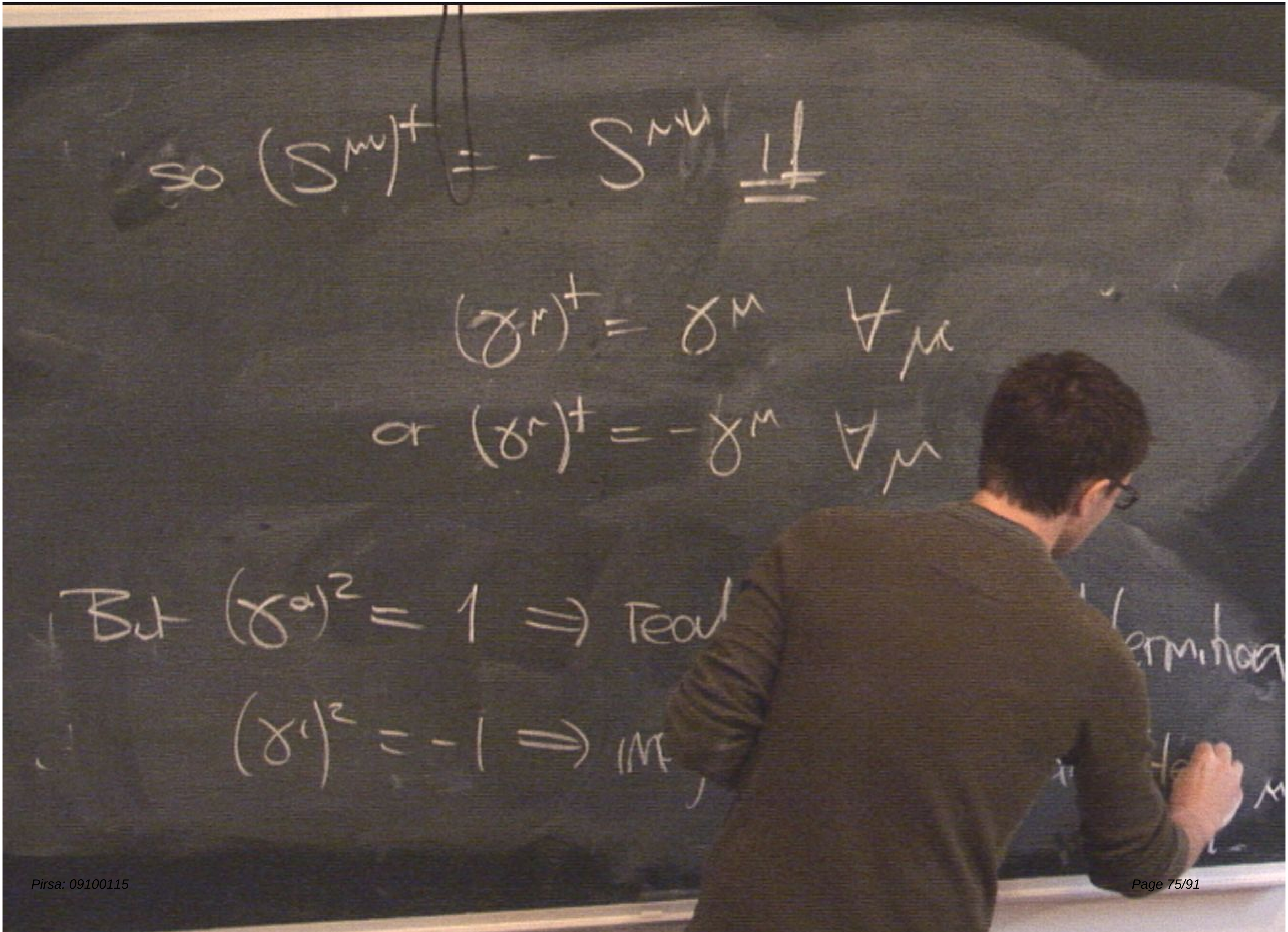
so $(S^{\mu\nu})^\dagger = -S^{\mu\nu}$!!
$(\gamma^\mu)^\dagger = \gamma^\mu \quad \forall \mu$
or $(\gamma^\mu)^\dagger = -\gamma^\mu \quad \forall \mu$
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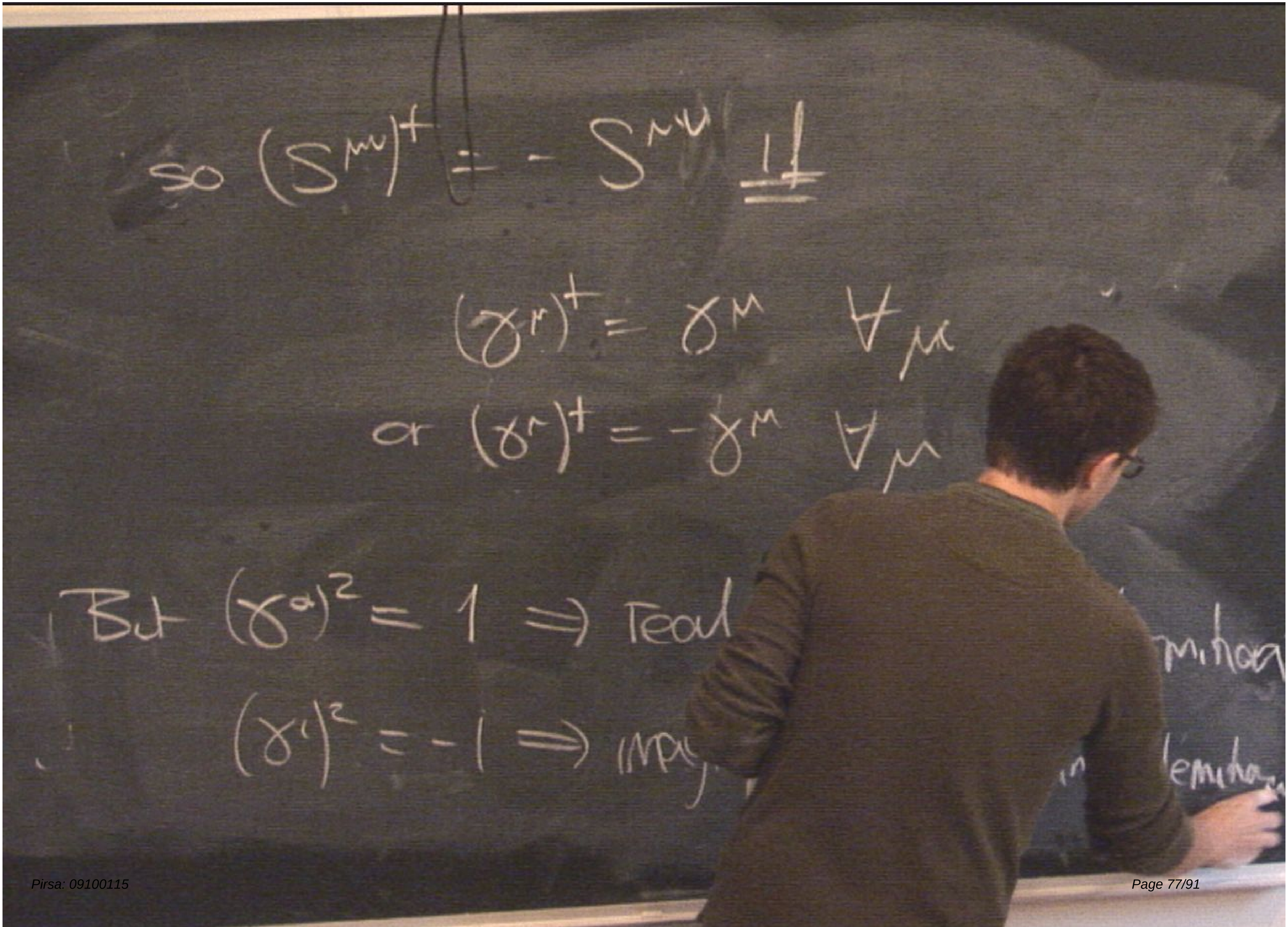
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But, if $(\gamma^0)^\dagger = \gamma^0$

$$(\gamma^i)^\dagger = -\gamma^i$$

$$\Longrightarrow \quad \gamma^0 \gamma^\mu \gamma^0 = (\gamma^\mu)^\dagger$$

$$\Longrightarrow \quad S[\Lambda]^\dagger = \gamma^0 S[\Lambda]^{-1} \gamma^0$$

Introduce the Dirac conjugate

$$\overline{\psi}(x) = \psi^{\dagger}(x)\,\gamma^{0}$$

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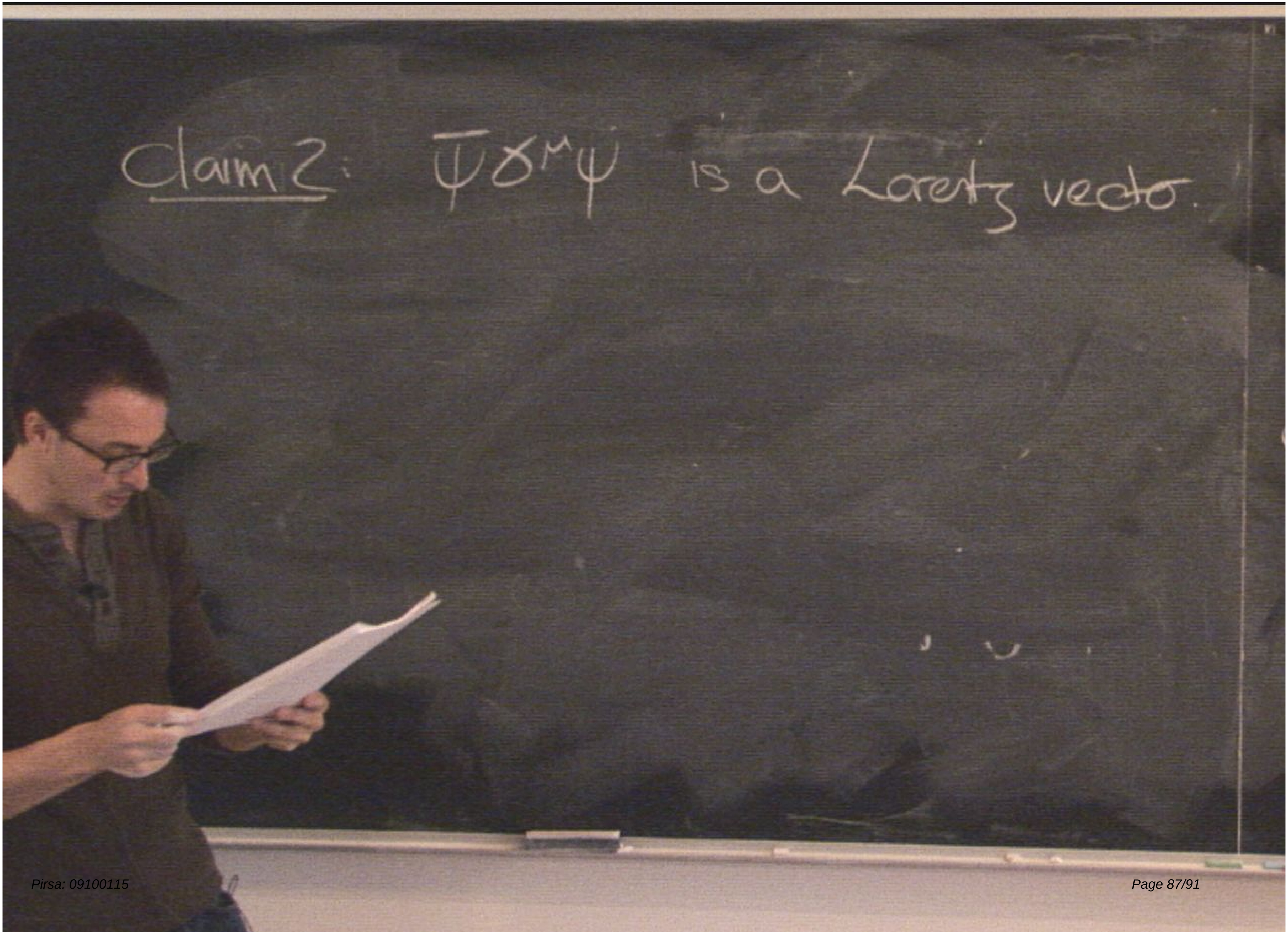
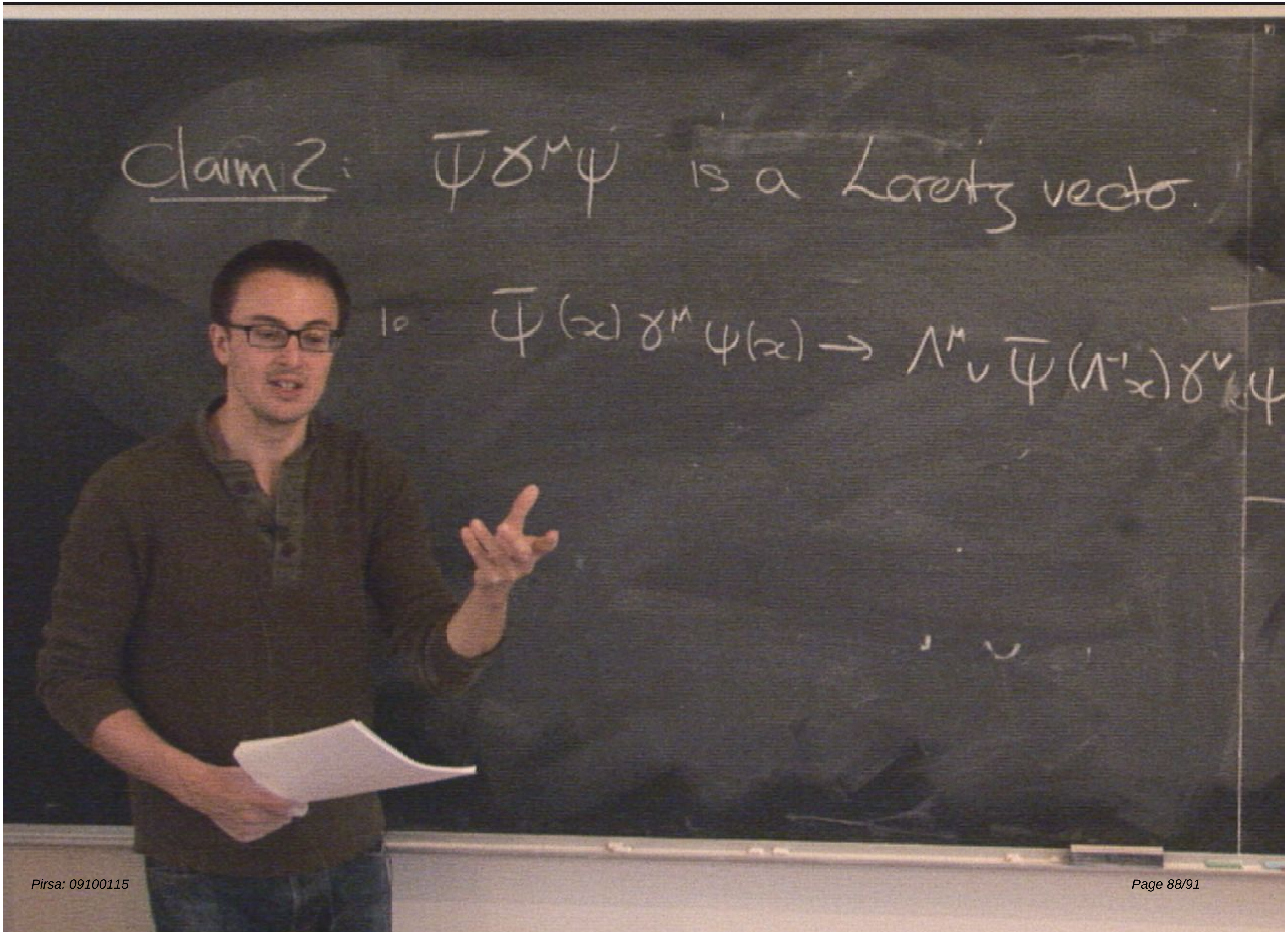
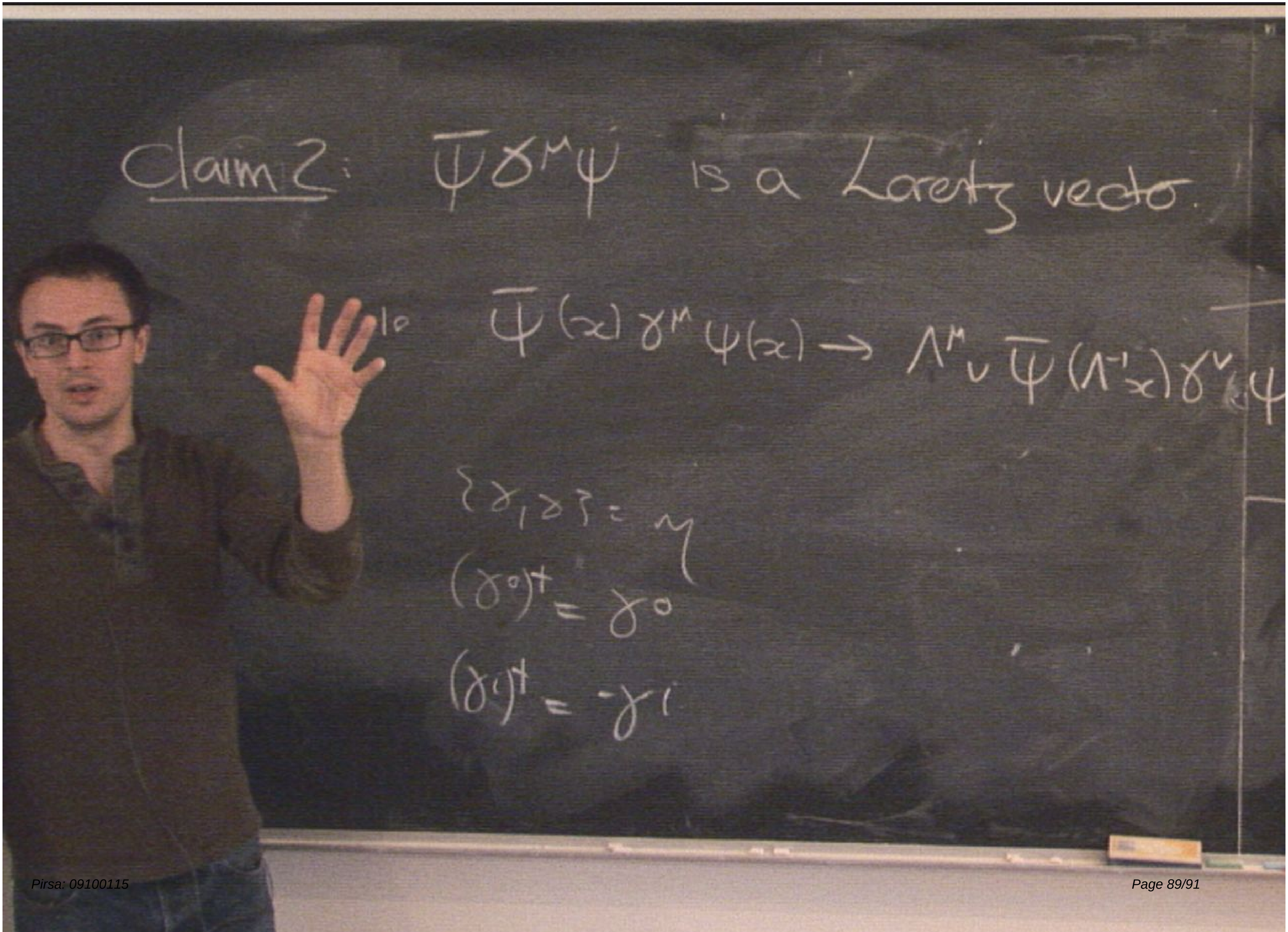
$$\bar{\psi}(x) = \psi^{\dagger}(x)\,\gamma^{0}$$

n 1: $\bar{\psi}(x)\,\psi(x)$ is a Lorentz scalar

i.e. $\bar{\psi}(x)\psi(x) \longrightarrow \bar{\psi}(\Lambda^{-1}x)\,\psi(\Lambda^{-1}x)$


claim 2: $\bar{\psi}\gamma^\mu\psi$ is a Lorentz vecto.


Claim 2: $\bar{\psi}\gamma^\mu\psi$ is a Lorentz vecto.
1. $\bar{\psi}(x)\gamma^\mu\psi(x) \to \Lambda^\mu{}_\nu \bar{\psi}(\Lambda^{-1}x)\gamma^\nu \psi$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The Dirac Equation

The following action is invariant under Lorentz transformations

$$S = \int d^4x \, \bar{\psi}(x) \left(i\gamma^\mu \partial_\mu - m \right) \psi(x).$$

The equation of motion:

$$\boxed{(i\gamma^\mu \partial_\mu - m)\psi = 0}$$

This is the Dirac equation