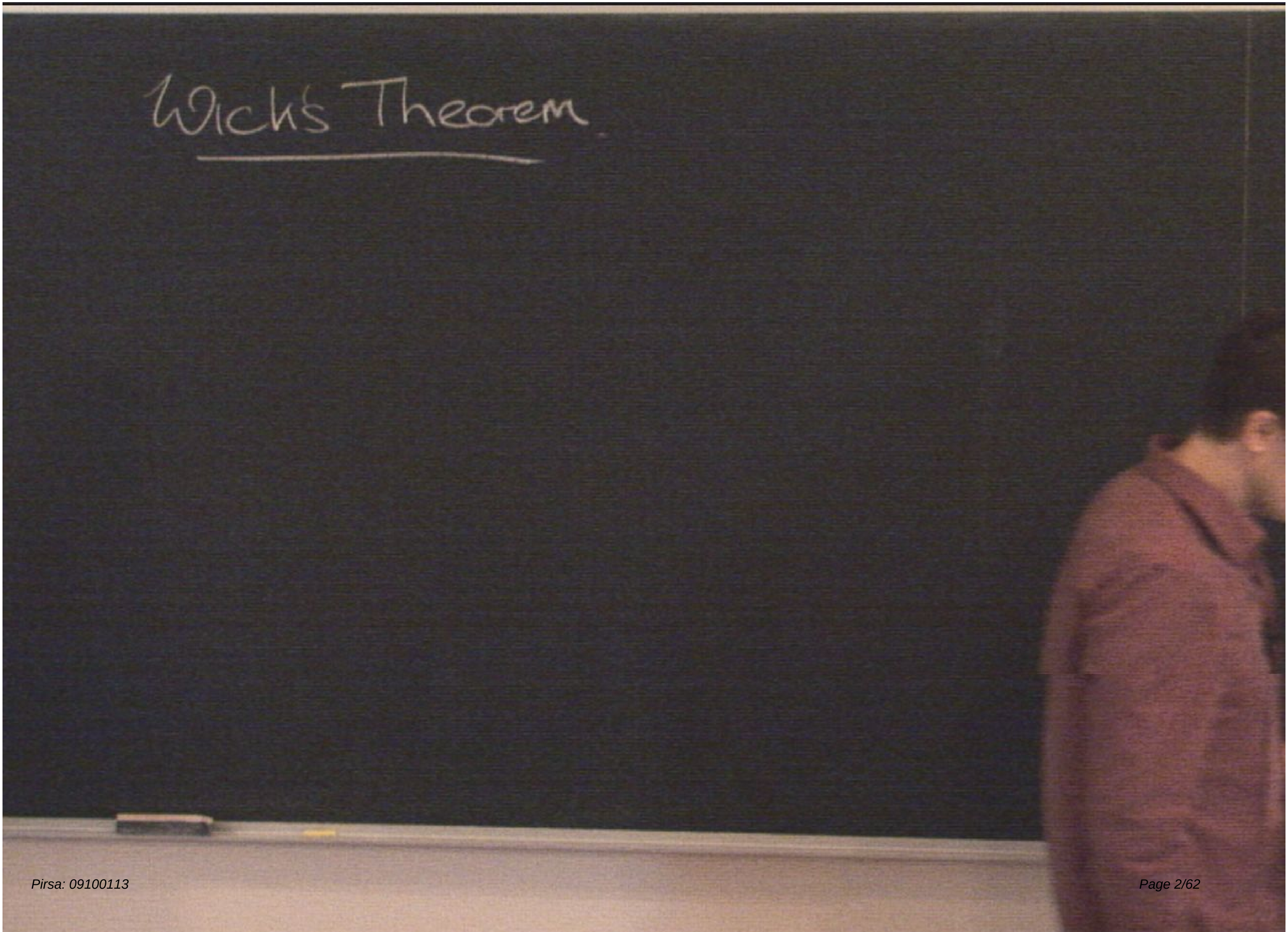
Title: Quantum Field Theory (PHYS 601) - Lecture 8

Date: Oct 07, 2009 09:00 AM

URL: http://pirsa.org/09100113

Abstract:


Wick's Theorem

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We'll need to compute things like

$$\langle f| T\left\{ H_I(x_1)\dots H_I(x_n)\right\} |i\rangle$$

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In a

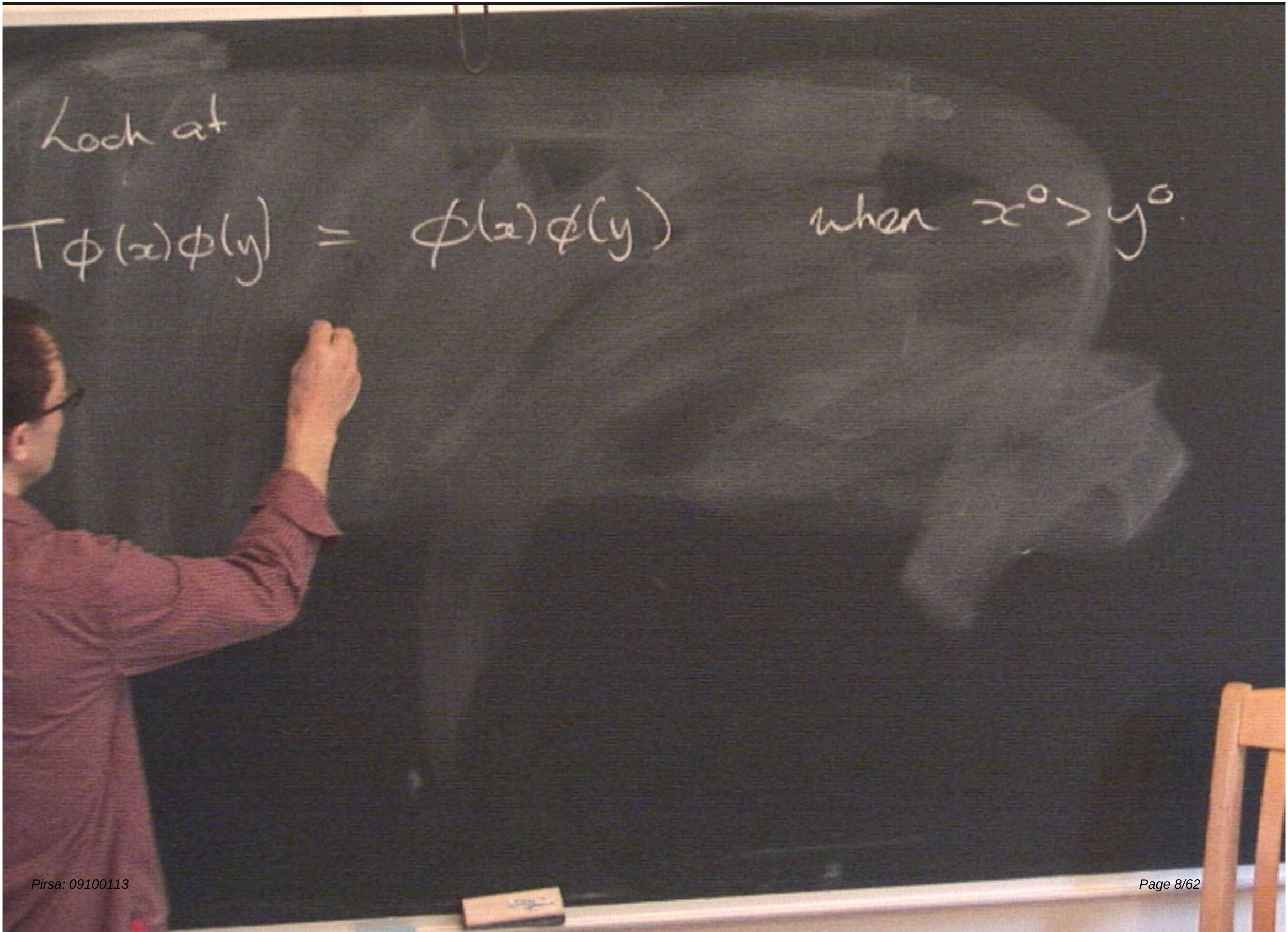
In any calculation, we'd like to rewrite the time ordered operators as normal ordered operators, where all annihilation operators are to the right.

Example. Real scalar field

$$\phi(x) = \phi^+(x) + \phi^-(x)$$

$$\phi^+(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} a_p e^{-ip\cdot x}$$

$$\phi^-(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} a_p^+ e^{+ip\cdot x}$$


Look at
$T\phi(x)\phi(y) = \phi(x)\phi(y)$ when $x^0 > y^0$.

Look at

$$T\phi(x)\phi(y) = \phi(x)\phi(y) \quad \text{when } x^0 > y^0$$

$$= \left(\phi^+(x) + \phi^-(x)\right)\left(\phi^+(y) + \phi^-(y)\right)$$

$$= \phi^+(x)\phi^+(y) + \phi^-(x)\phi^+(y)$$

$$+ \phi^-(y)\phi^+(x) + \left[\phi^+(x), \phi^-(y)\right]$$

Look at

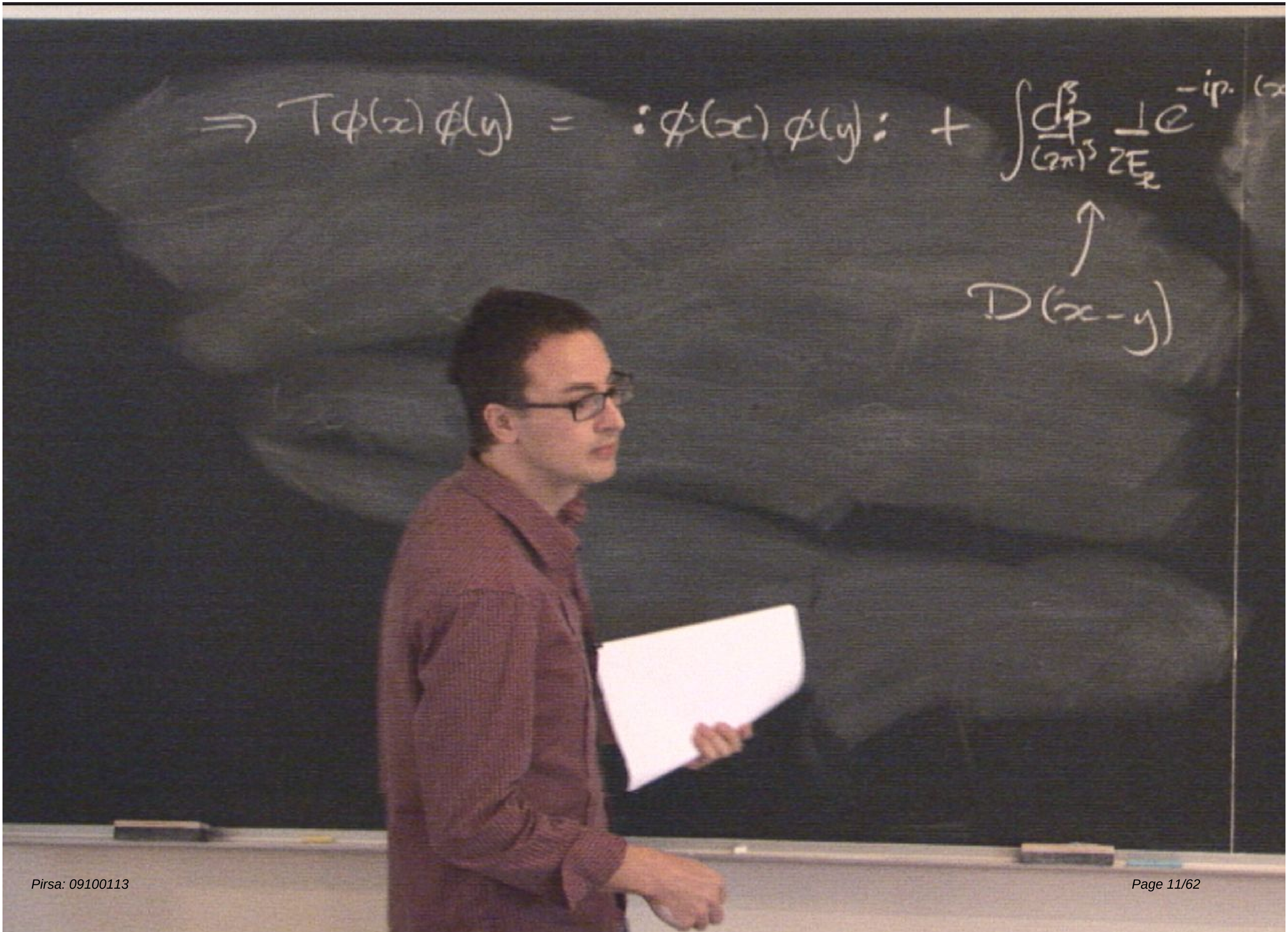
$$T\phi(x)\phi(y) = \phi(x)\phi(y) \quad \text{when } x^0 > y^0.$$

$$= (\phi^+(x) + \phi^-(x))(\phi^+(y) + \phi^-(y)).$$

$$= \phi^+(x)\phi^+(y) + \phi^-(x)\phi^+(y)$$

$$+ \phi^-(y)\phi^+(x) + [\phi^+(x), \phi^-(y)]$$

$$+ \phi^-(x)\phi^-(y)$$


$\Rightarrow T\phi(x)\phi(y) = :\phi(x)\phi(y): + \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip\cdot(x}$
$D(x-y)$

$$\Rightarrow T\phi(x)\phi(y) = :\phi(x)\phi(y): + \underbrace{\int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip\cdot(x-y)}}_{D(x-y)}$$

For $y^0 > x^0$, $T\phi(x)\phi(y) = :\phi(x)\phi(y): + D(y-x)$

$$\Rightarrow T\phi(x)\phi(y) = :\phi(x)\phi(y): + \Delta_F(x-y)$$

Feynman propagator

where $\Delta_F(x-y) = \int \frac{d^4k}{(2\pi)^4} \, \frac{i e^{ik\cdot(x-y)}}{k^2 - m^2 + i\epsilon}$

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Definition

where $$\Delta_F(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{i e^{ik\cdot(x-y)}}{k^2-m^2+i\epsilon}$$

<u>Definition</u> We define a <u>contraction</u> of a pair of fields in a string of operators

$$\cdots\cdots\ \phi(x_1) \cdots\cdots\ \phi(x_2) \cdots\cdots$$

to mean replacing the operator with $\Delta_F(x_1 - x_2)$

$$\phi(x_1)\,\phi(x_2) = \Delta_F(x_1 - x_2)$$

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$$\phi(x_1)\,\phi(x_2) = \Delta_F(x_1 - x_2)$$

For a complex field, $\psi(x)\,\psi^\dagger(y) = \Delta_F(x - y)$

$$\psi(x)\,\psi(y) = \psi^\dagger(x)\,\psi^\dagger(y) = 0$$

Wick's Thm

For any collection of fields, $\phi_1 \equiv \phi(x_1)$
$\phi_2 \equiv \phi(x_2)$ etc, we have

$$T(\phi_1 \cdots \phi_n) = \,:\phi_1 \cdots \phi_n: + \,:\text{all possible contractions}:$$

Example

$$T\phi_1\phi_2\phi_3\phi_4 = :\phi_1\phi_2\phi_3\phi_4: + \overbracket{\phi_1\phi_2}:\phi_3\phi_4:$$

$$+ \overbracket{\phi_1\phi_3}:\phi_2\phi_4: + 4 \text{ similar terms}$$

$$+ \overbracket{\phi_1\phi_2}\,\overbracket{\phi_3\phi_4} + \overbracket{\phi_1\phi_3}\,\overbracket{\phi_2\phi_4} + \overbracket{\phi_1\phi_4}\,\overbracket{\phi_2\phi_3}$$

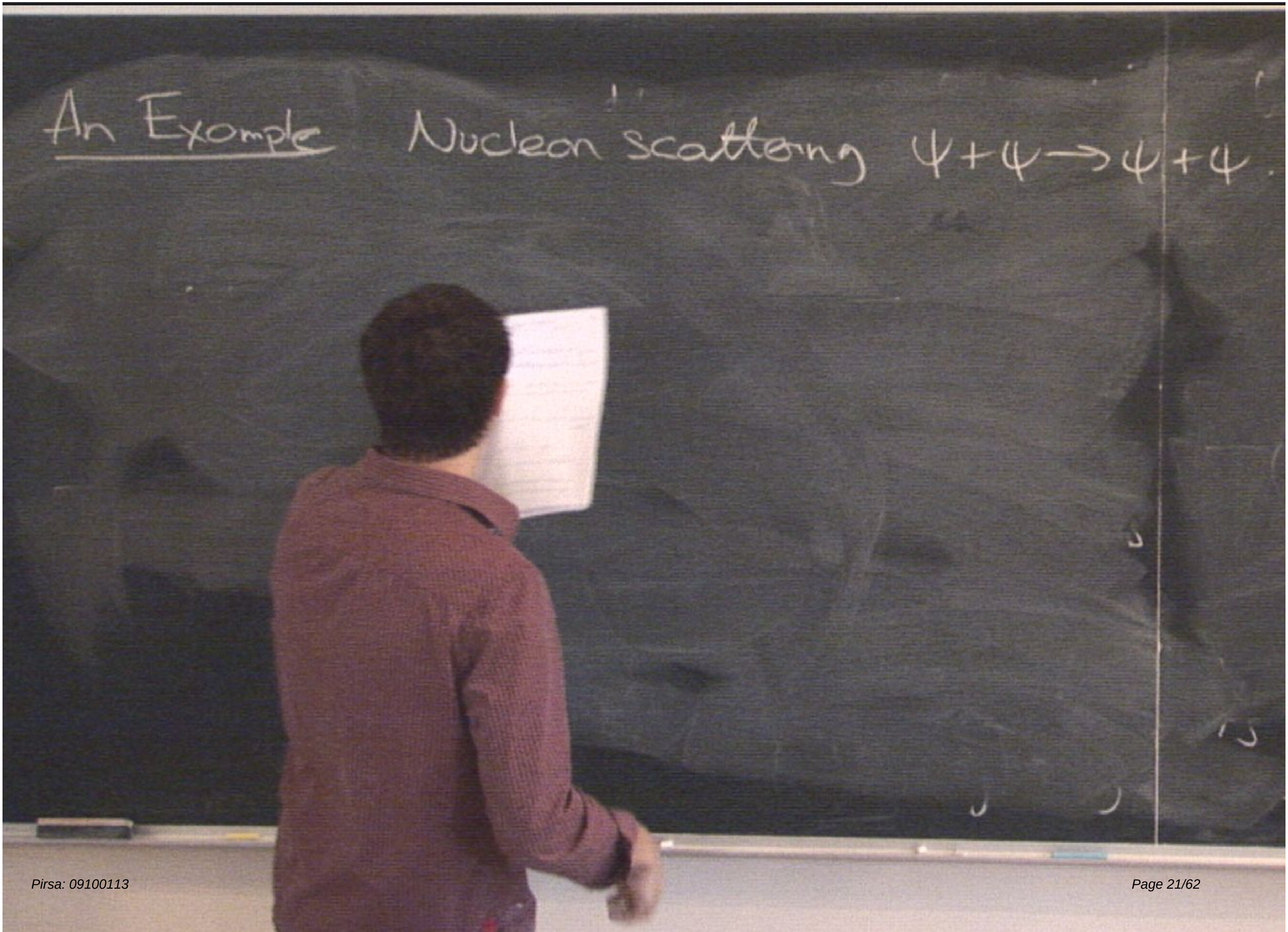
Wick's Thm

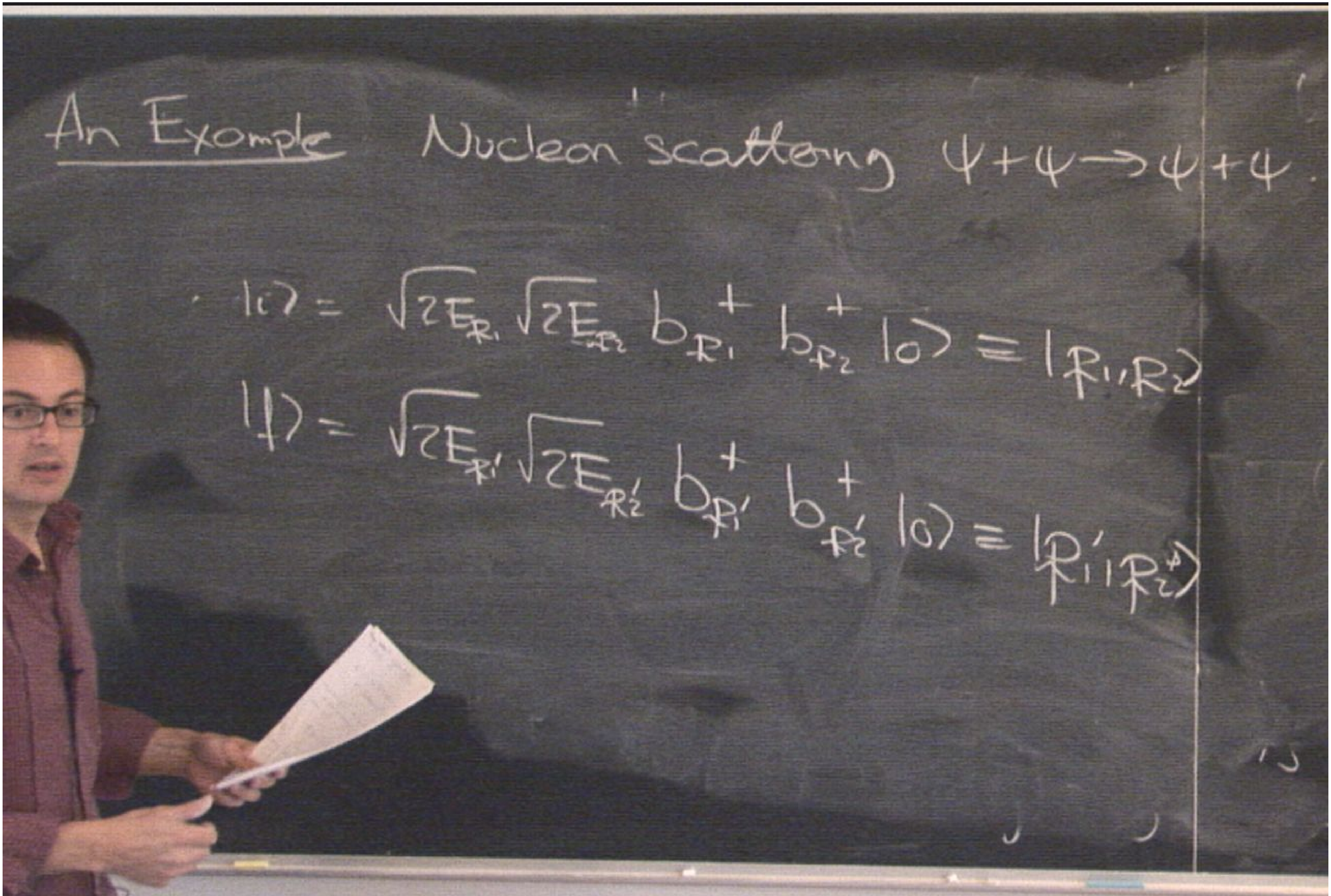
For any collection of fields, $\phi_1 \equiv \phi(x_1)$, $\phi_2 \equiv \phi(x_2)$ etc, we have

$$T(\phi_1 \cdots \phi_n) = \,:\phi_1 \cdots \phi_n: + \,:\text{all possible contractions}:$$

$\overline{\phi_2 \phi_3}$

Proof. Works for $n=2$


An Example
Nucleon scattering ψ + ψ → ψ + ψ


An Example
Nucleon scattering $\psi + \psi \to \psi + \psi$.
$|i\rangle = \sqrt{2E_{\vec{p}_1}}\sqrt{2E_{\vec{p}_2}}\, b^{\dagger}_{\vec{p}_1} b^{\dagger}_{\vec{p}_2} |0\rangle \equiv |\vec{p}_1, \vec{p}_2\rangle$
$|f\rangle = \sqrt{2E_{\vec{p}_1'}}\sqrt{2E_{\vec{p}_2'}}\, b^{\dagger}_{\vec{p}_1'} b^{\dagger}_{\vec{p}_2'} |0\rangle \equiv |\vec{p}_1', \vec{p}_2'\rangle$

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$$|i\rangle = \sqrt{2E_{\vec{p}_1}}\sqrt{2E_{\vec{p}_2}}\, b^{+}_{\vec{p}_1} b^{+}_{\vec{p}_2} |0\rangle \equiv |p_1, p_2\rangle$$

$$|f\rangle = \sqrt{2E_{\vec{p}_1'}}\sqrt{2E_{\vec{p}_2'}}\, b^{+}_{\vec{p}_1'} b^{+}_{\vec{p}_2'} |0\rangle \equiv |p_1', p_2'\rangle$$

We want to compute $\langle f | S - 1 | i \rangle$

at order g^2

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Since we're not interested in nothing happening.

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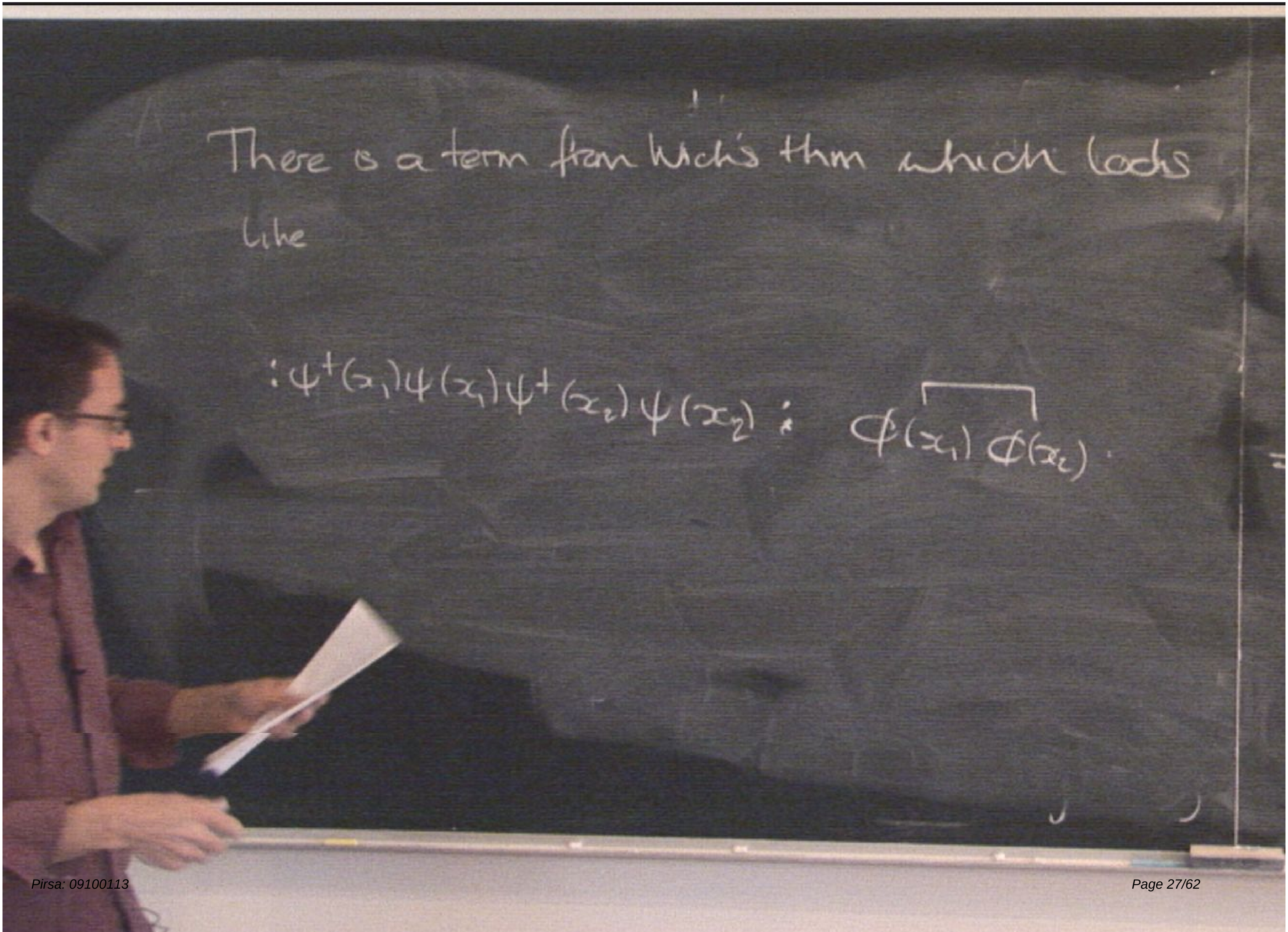
$$\frac{(-ig)^2}{2} \int d^4x_1 \, d^4x_2 \, T\left(\psi^\dagger(x_1)\psi(x_1)\phi(x_1)\,\psi^\dagger(x_2)\psi(x_2)\phi(x_2) \right)$$


There is a term from Wick's thm which looks like
$:\psi^\dagger(x_1)\psi(x_1)\psi^\dagger(x_2)\psi(x_2):\ \phi(x_1)\phi(x_2).$

Look at

$$\langle p_1', p_2'| :\psi^\dagger(x_1)\,\psi(x_1)\,\psi^\dagger(x_2)\,\psi(x_2): |p_1, p_2\rangle$$

Look at

$$\langle p_1', p_2' | :\psi^\dagger(x_1)\,\psi(x_1)\,\psi^\dagger(x_2)\,\psi(x_2): | p_1, p_2 \rangle$$

$$= e^{ix_1(p_1'-p_1)+ix_2\cdot(p_2'-p_1)} + e^{ix_1\cdot(p_2'-p_1)+ix_2(p_1'-p_2)} + (x_1 \rightarrow x_2)$$

Look at

$$\langle p_1', p_2' | : \psi^\dagger(x_1)\psi(x_1)\psi^\dagger(x_2)\psi(x_2) : | p_1, p_2 \rangle$$

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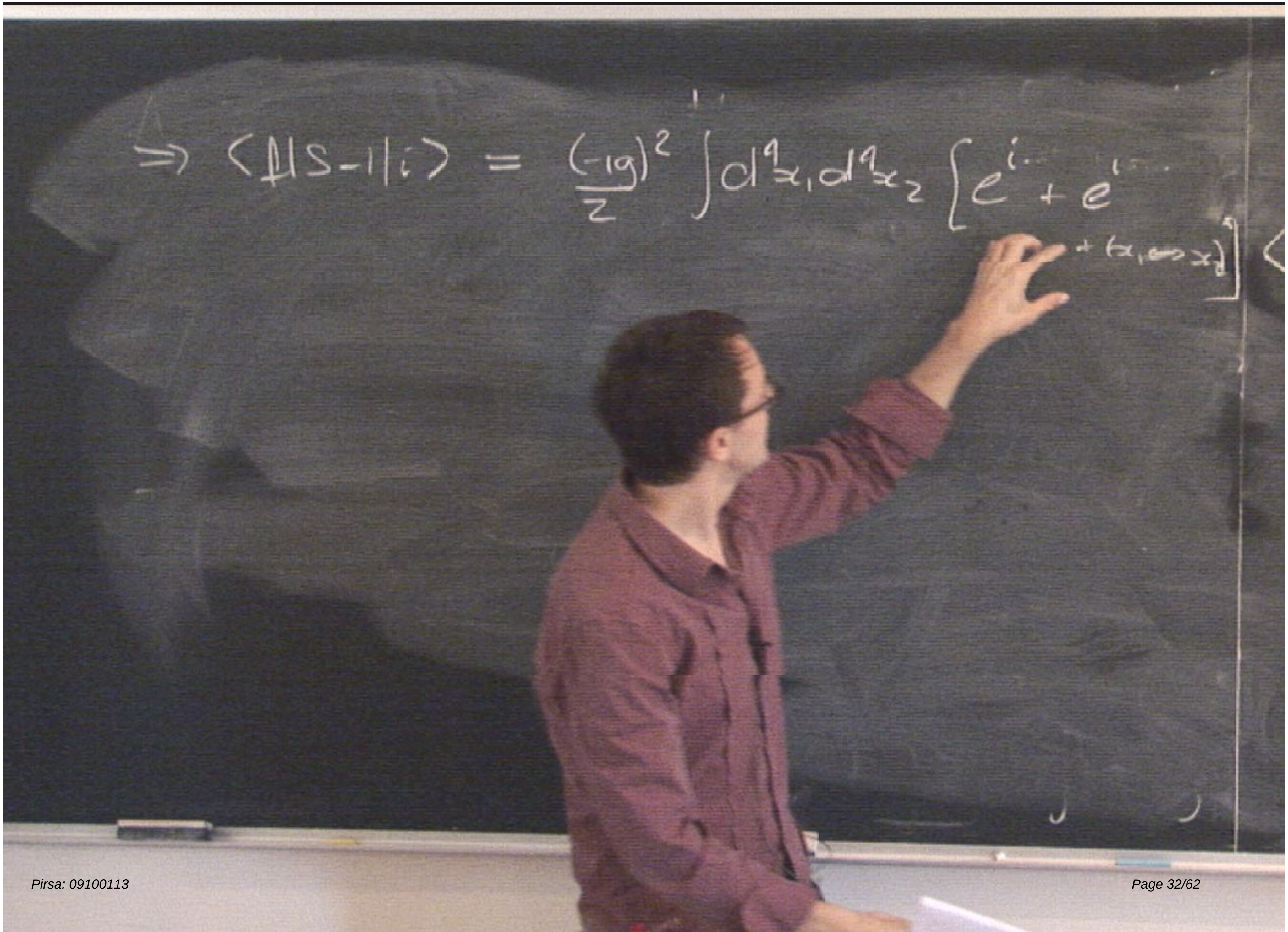
Look at

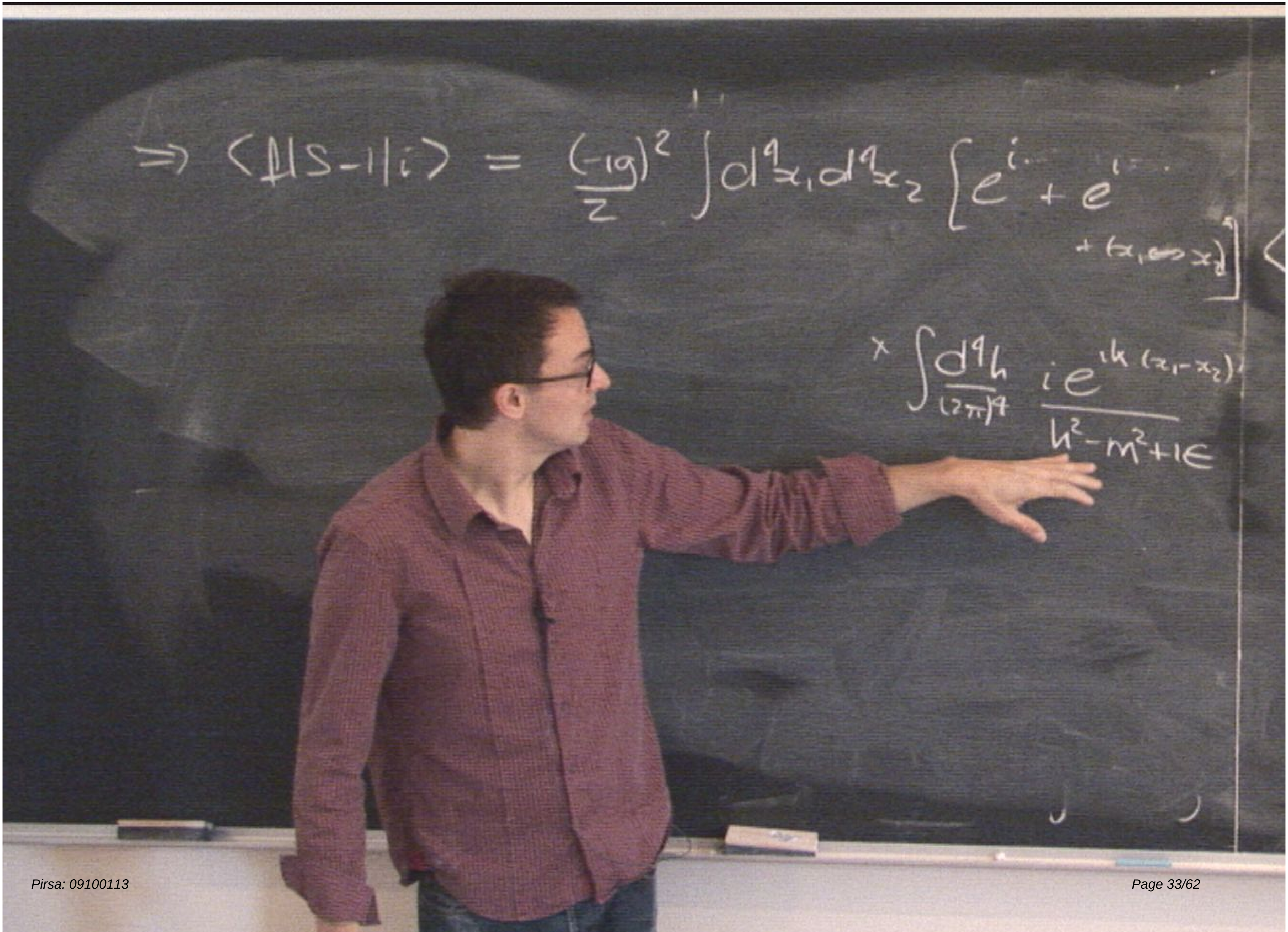
$$\langle p_1', p_2' | :\psi^\dagger(x_1)\psi(x_1)\psi^\dagger(x_2)\psi(x_2): | p_1, p_2 \rangle$$

$$= e^{i x_1 (p_1' - p_1) + i x_2 \cdot (p_2' - p_2)}$$

$$+ e^{i x_1 \cdot (p_2' - p_1) + i x_2 (p_1' - p_2)}$$

$$+ (x_1 \leftrightarrow x_2)$$


$\Rightarrow \langle f|S-1|i\rangle = \left(\frac{-ig}{2}\right)^2 \int d^4x_1 d^4x_2 \left\{ e^{i\cdots} + e^{i\cdots} + (x_1 \leftrightarrow x_2) \right\}$



$\Rightarrow \langle f|S-1|i\rangle = \frac{(-ig)^2}{2}\int d^4x_1 d^4x_2 \{e^{i\ldots} + e^{i\ldots}$
$+ (x_1 \leftrightarrow x_2)\}$
$\times \int \frac{d^4k}{(2\pi)^4} \frac{i e^{ik(x_1-x_2)}}{k^2-m^2+i\epsilon}$

$$\Rightarrow \langle f|S-1|i\rangle = \frac{(-ig)^2}{2}\int d^4x_1\, d^4x_2 \left[e^{i\cdots} + e^{i\cdots} + (x_1 \leftrightarrow x_2) \right]$$

$$\times \int \frac{d^4k}{(2\pi)^4} \frac{i e^{ik(x_1-x_2)}}{k^2-m^2+i\epsilon}$$

$$= i(-ig)^2 \left[\frac{1}{(p_1-p_2)^2-m^2} + \frac{1}{(p_1-p_2')^2-m^2} \right] (2\pi)^4 \delta^{(4)}(p_1+p_2$$

$$\Rightarrow \langle f|S-1|i\rangle = \frac{(-ig)^2}{2}\int d^4x_1\, d^4x_2 \left[e^{i\cdots} + e^{i\cdots} + (x_1 \leftrightarrow x_2) \right] \langle p_1', p_2'$$

$$\times \int \frac{d^4k}{(2\pi)^4} \frac{i e^{ik(x_1-x_2)}}{k^2-m^2+i\epsilon}$$

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Loch

$$\langle f|S-1|i\rangle = \frac{(-ig)^2}{2}\int d^4x_1 d^4x_2 \Big[e^{i\cdots} + e^{i\cdots} + (x_1 \leftrightarrow x_2)\Big] \langle p_1', p_2'| :\psi$$

$$\times \int \frac{d^4k}{(2\pi)^4} \frac{i e^{ik(x_1-x_2)}}{k^2-m^2+i\epsilon}$$

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Look at

$$= e$$

Feynman Diagrams

Instead of working with Wick's theorem, we'll instead draw pretty pictures.

Feynman Diagrams

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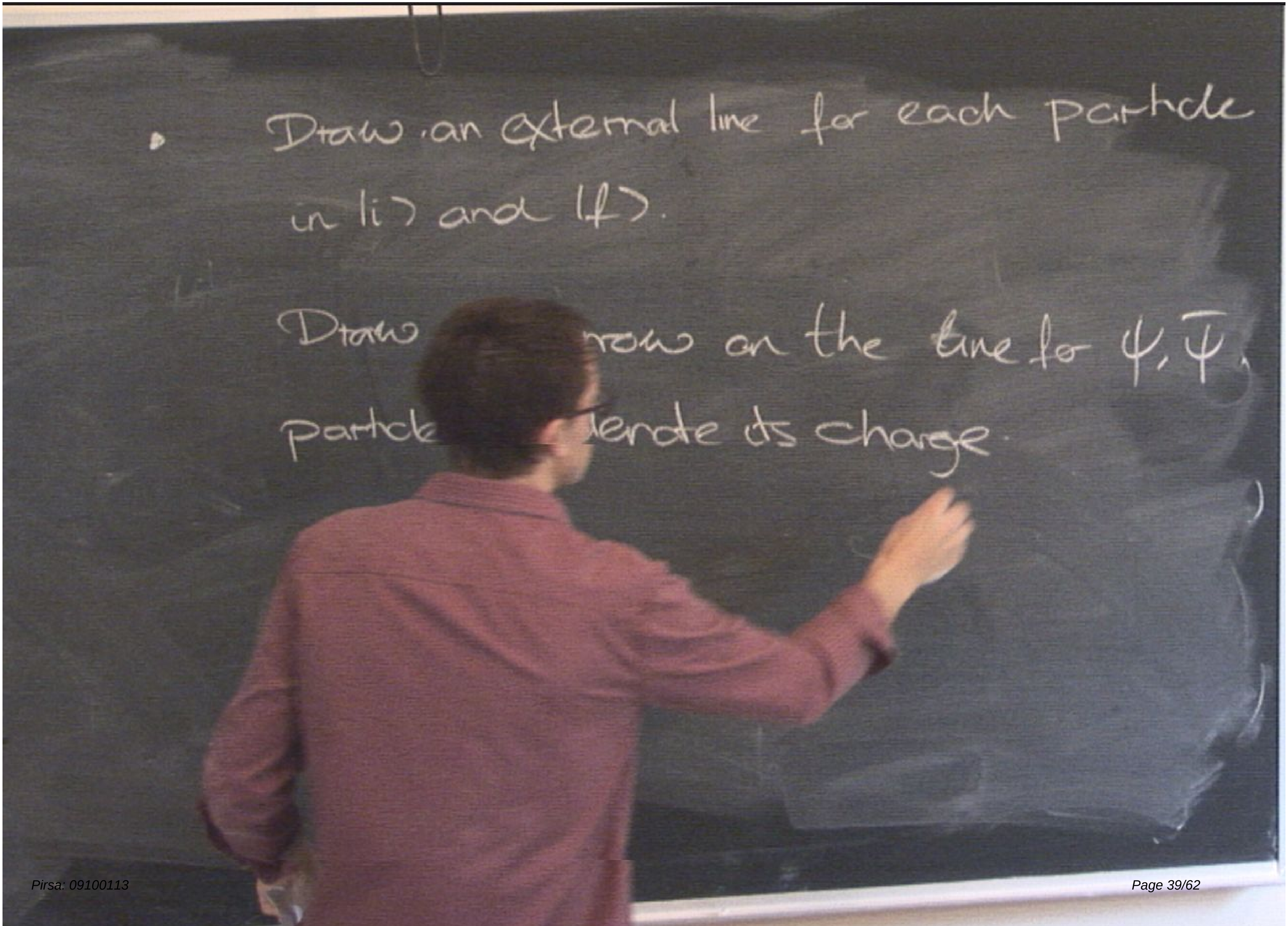
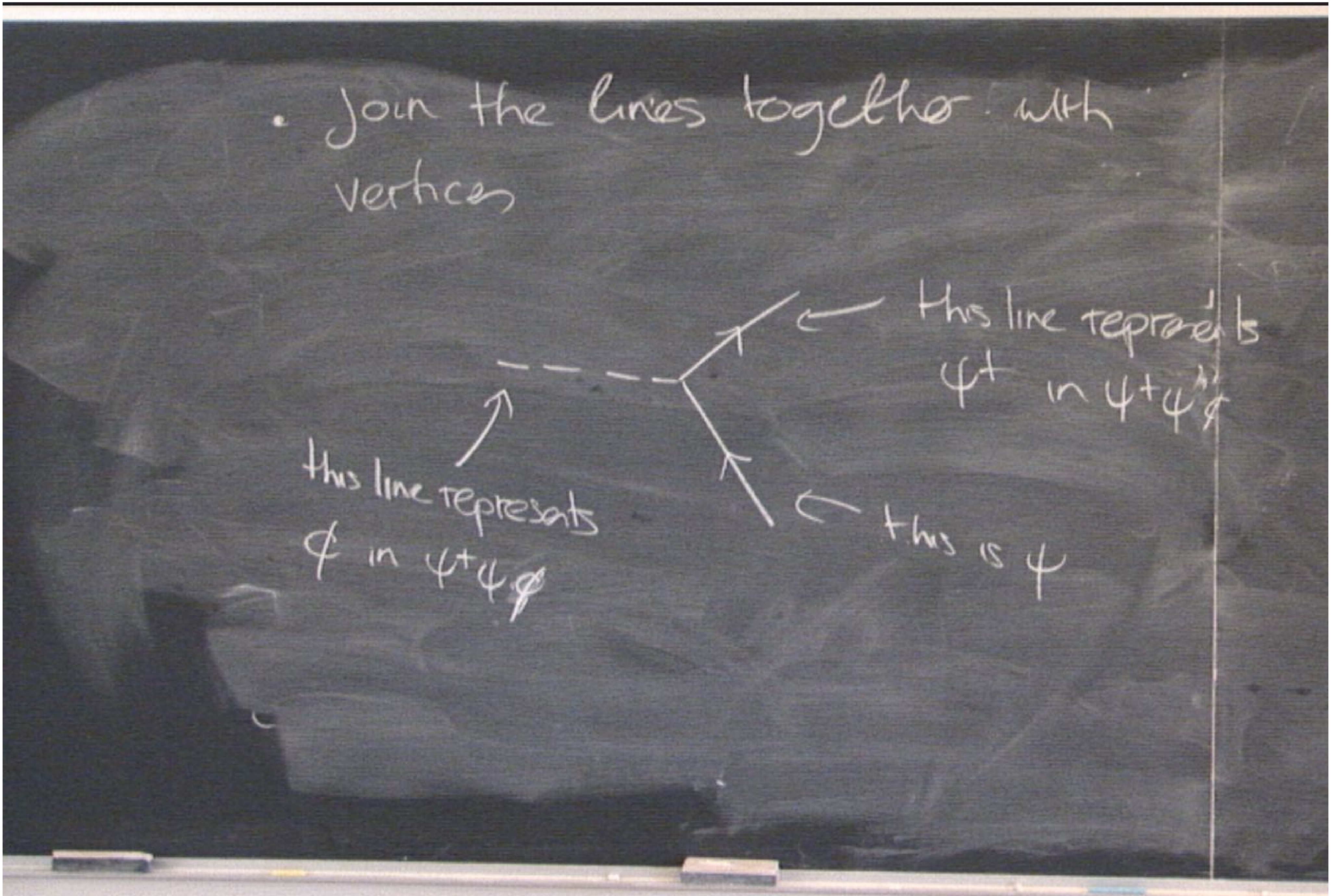
The terms that arise in $\langle f|S-1|i\rangle$ are in 1-1 correspondance with the following diagrams.

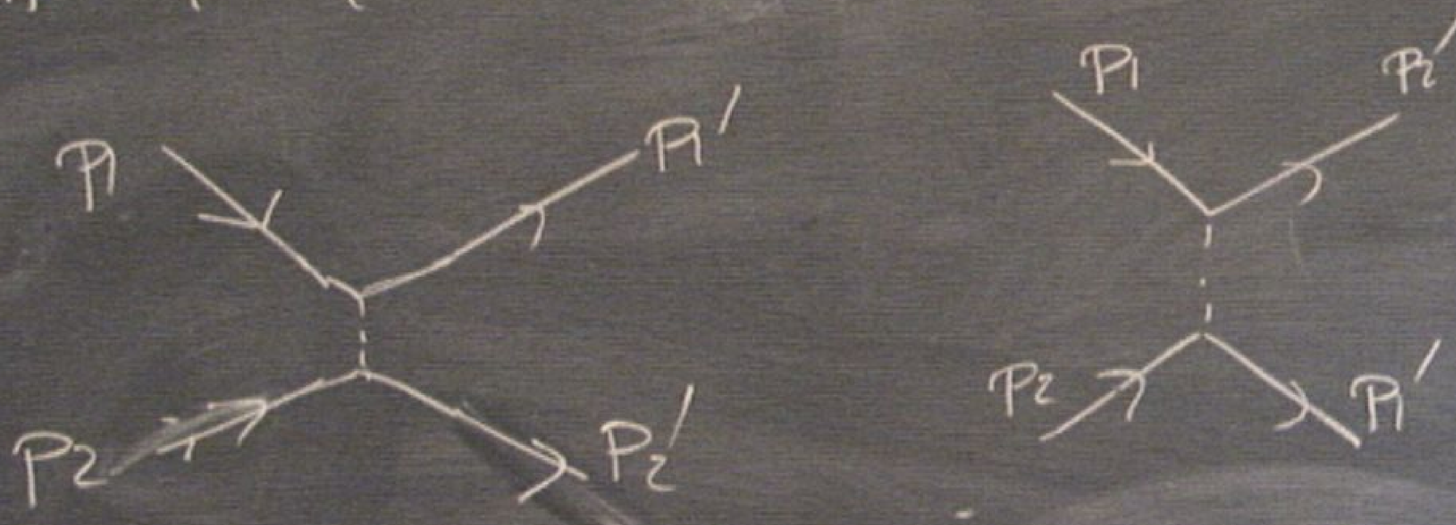

Draw an external line for each particle
in |i> and |f>.
Draw rrow on the line for ψ, ψ̄,
partic denote its charge.

- Draw an external line for each particle in $|i\rangle$ and $|f\rangle$.

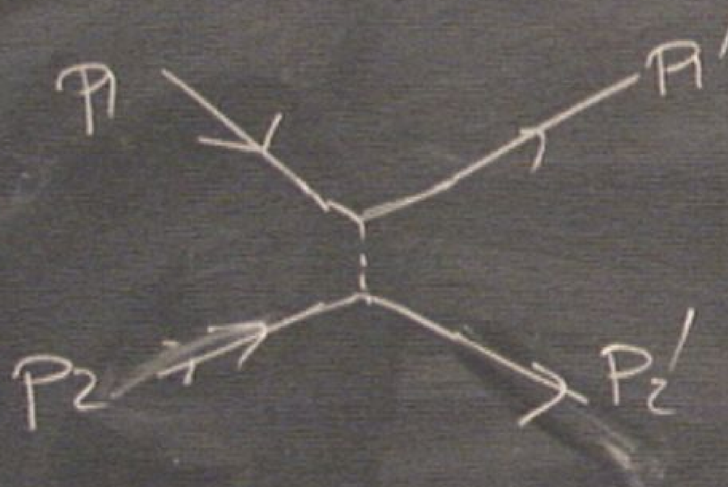
Draw an arrow on the line for $\psi, \bar{\psi}$ particles to denote its charge. Choose incoming (outgoing) arrows for ψ $(\bar{\psi})$ particles in $|i\rangle$, and opposite for $|f\rangle$.

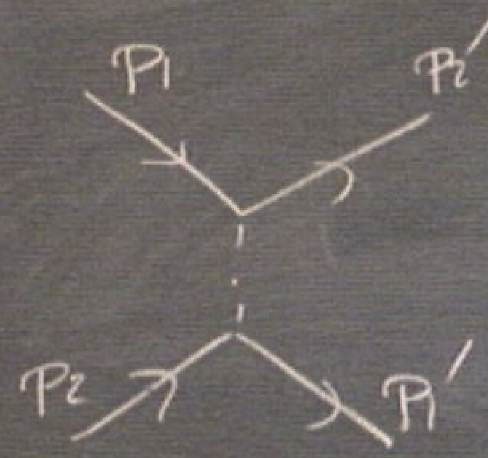

• Join the lines together with vertices
this line represents $\psi^\dagger$ in $\psi^\dagger\psi\phi$
this line represents ϕ in $\psi^\dagger\psi\phi$
this is ψ

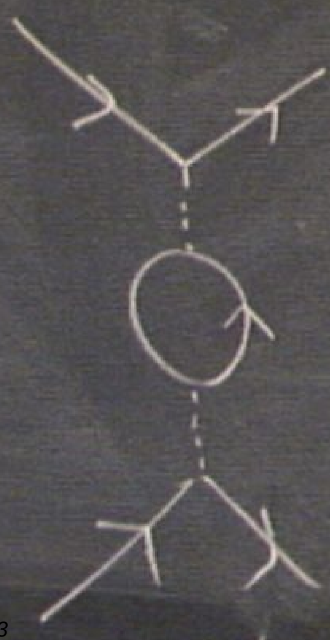
e.g. for $\psi + \psi \to \psi + \psi$, we have



e.g. for $\psi + \psi \rightarrow \psi + \psi$, we have







or

or . . .

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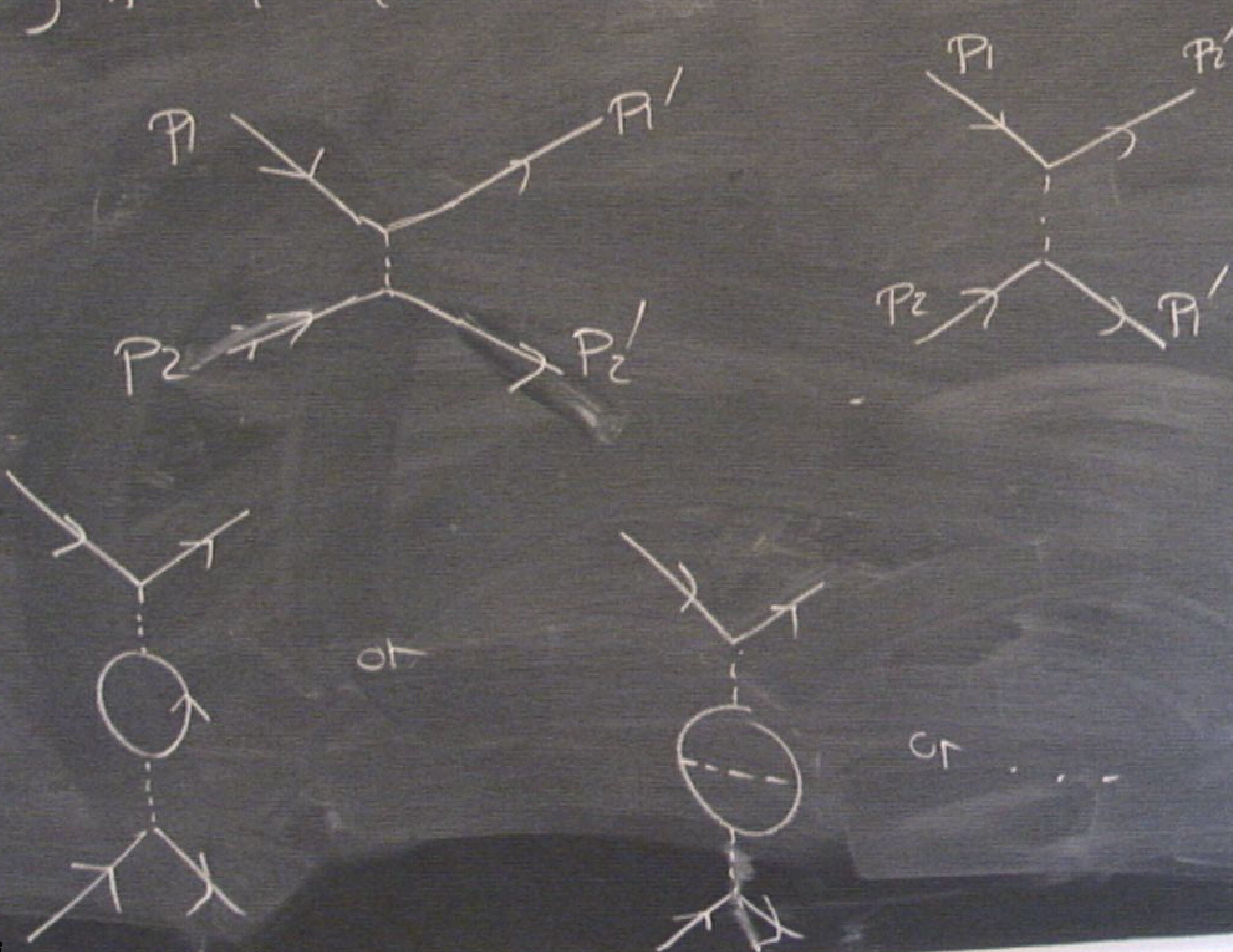
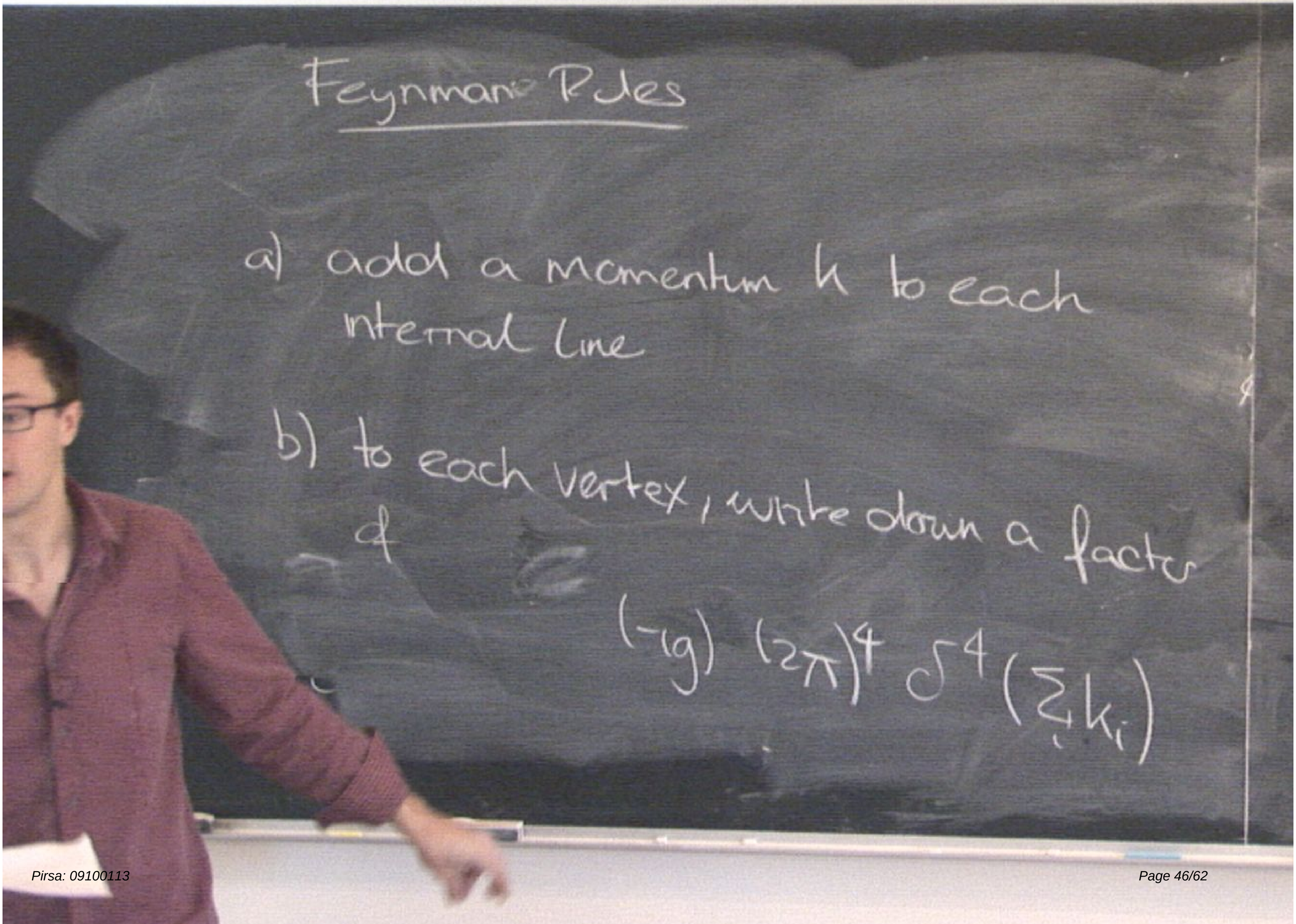
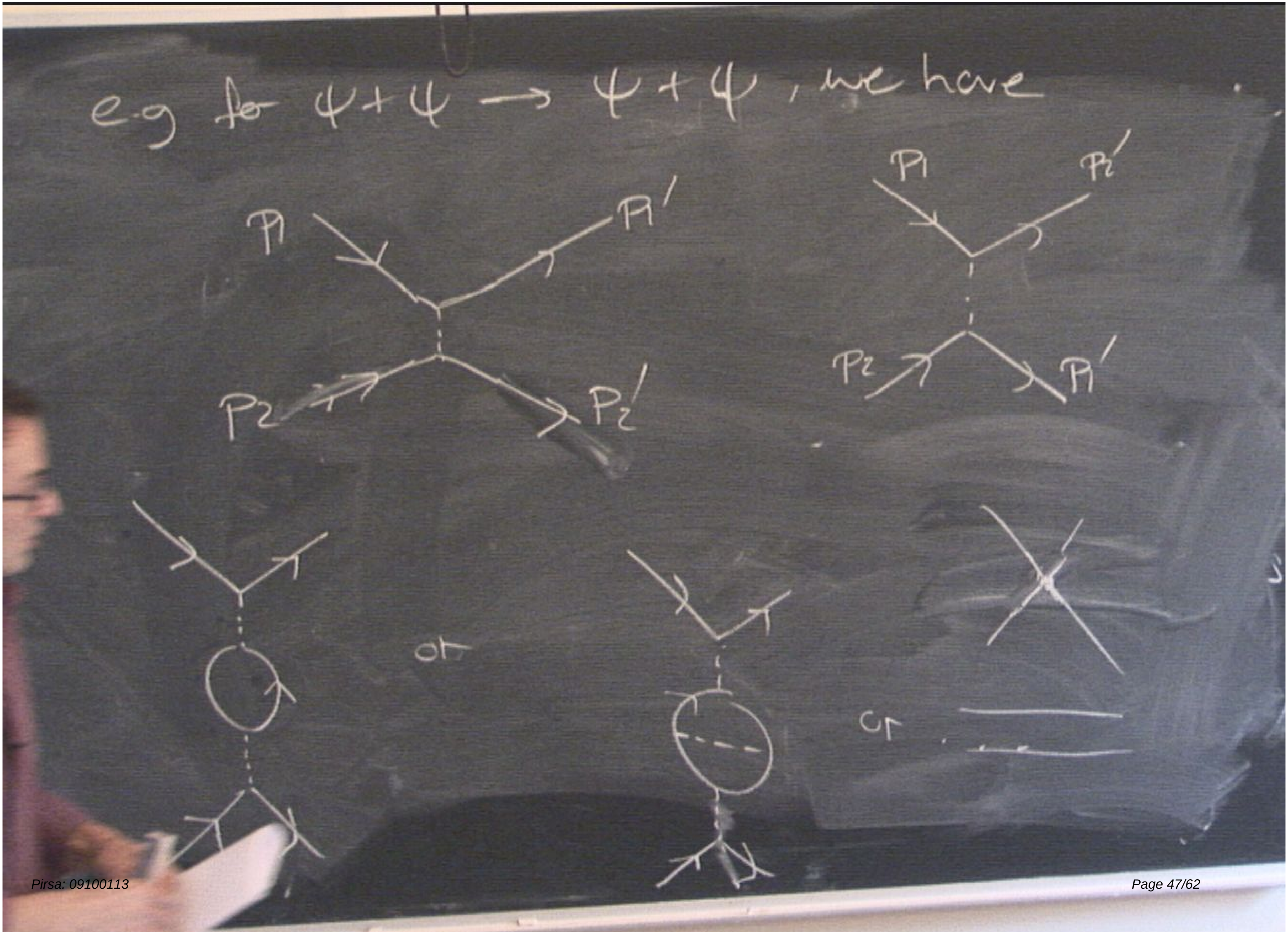


Feynman Rules

a) add a momentum k to each internal line


Feynman Rules
a) add a momentum k to each internal line
b) to each vertex, write down a factor
$(-ig)(2\pi)^4\delta^4(\sum_i k_i)$



e.g for ψ + ψ → ψ + ψ , we have
P1
P1'
P2
P2'
P1
P2'
P2
P1'
or
or

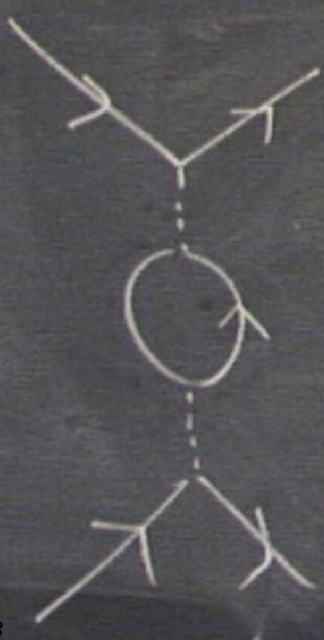
c) For each internal line, with momentum k;

write down a factor of

$$\int \frac{d^4k}{(2\pi)^4}\, D(k^2)$$

$$D = \frac{i}{k^2 - m^2 + i\epsilon} \quad \text{for } \varphi$$

$$D = \frac{i}{k^2 - M^2 + i\epsilon} \quad \text{for } \psi$$



or

or ...

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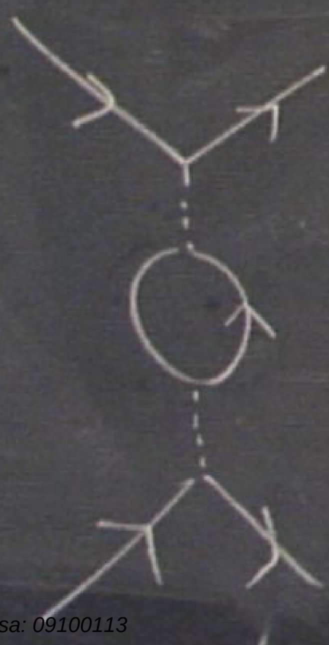

or

or .

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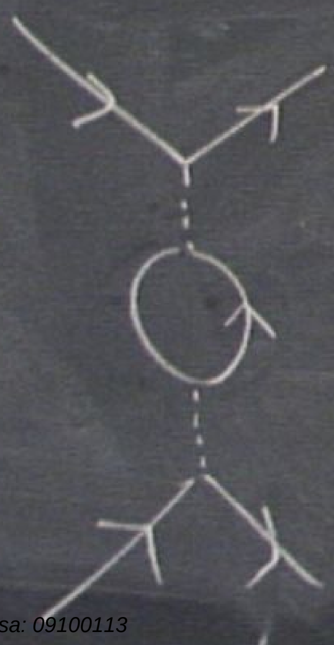
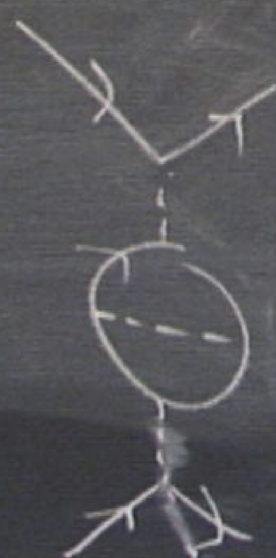
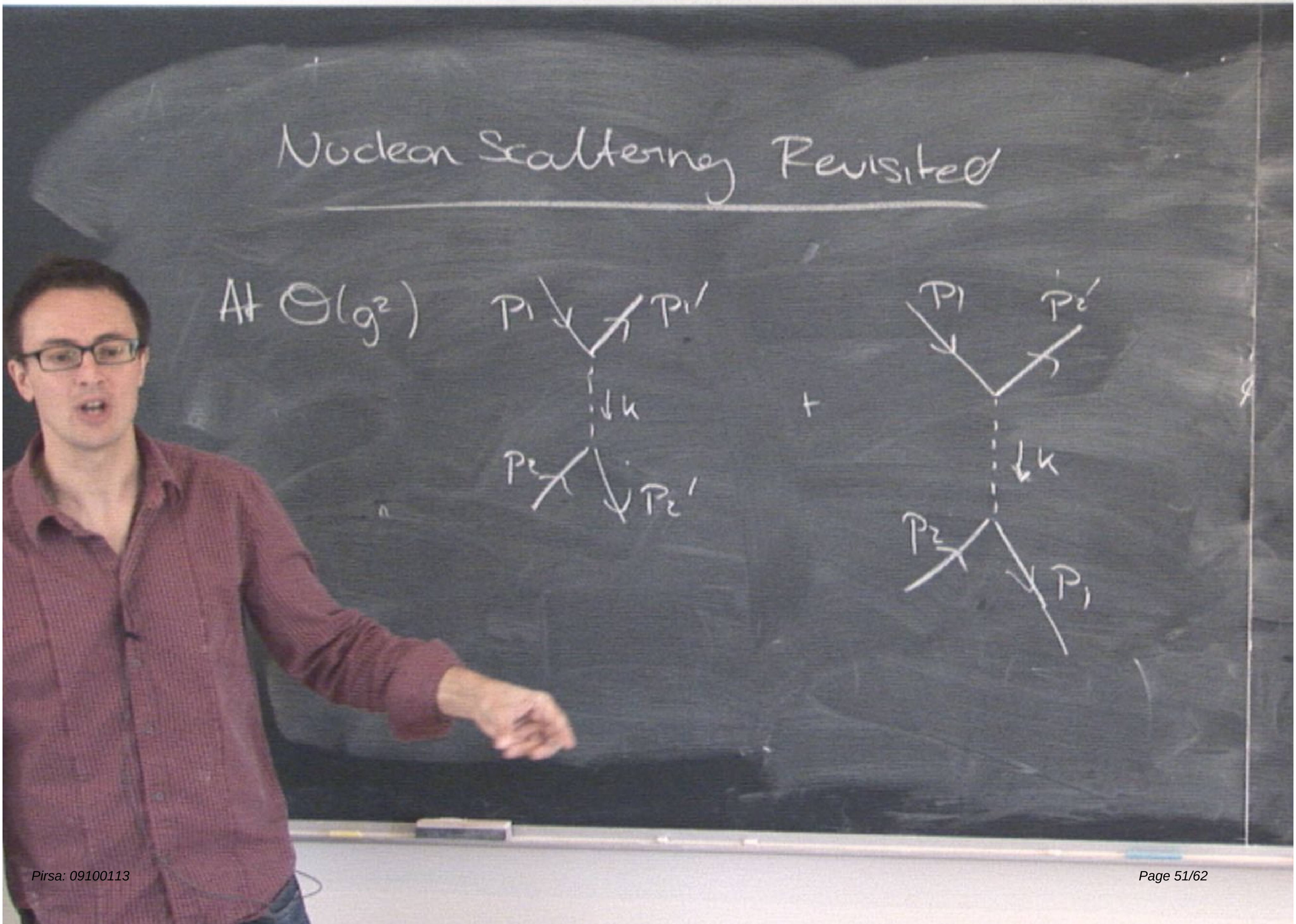
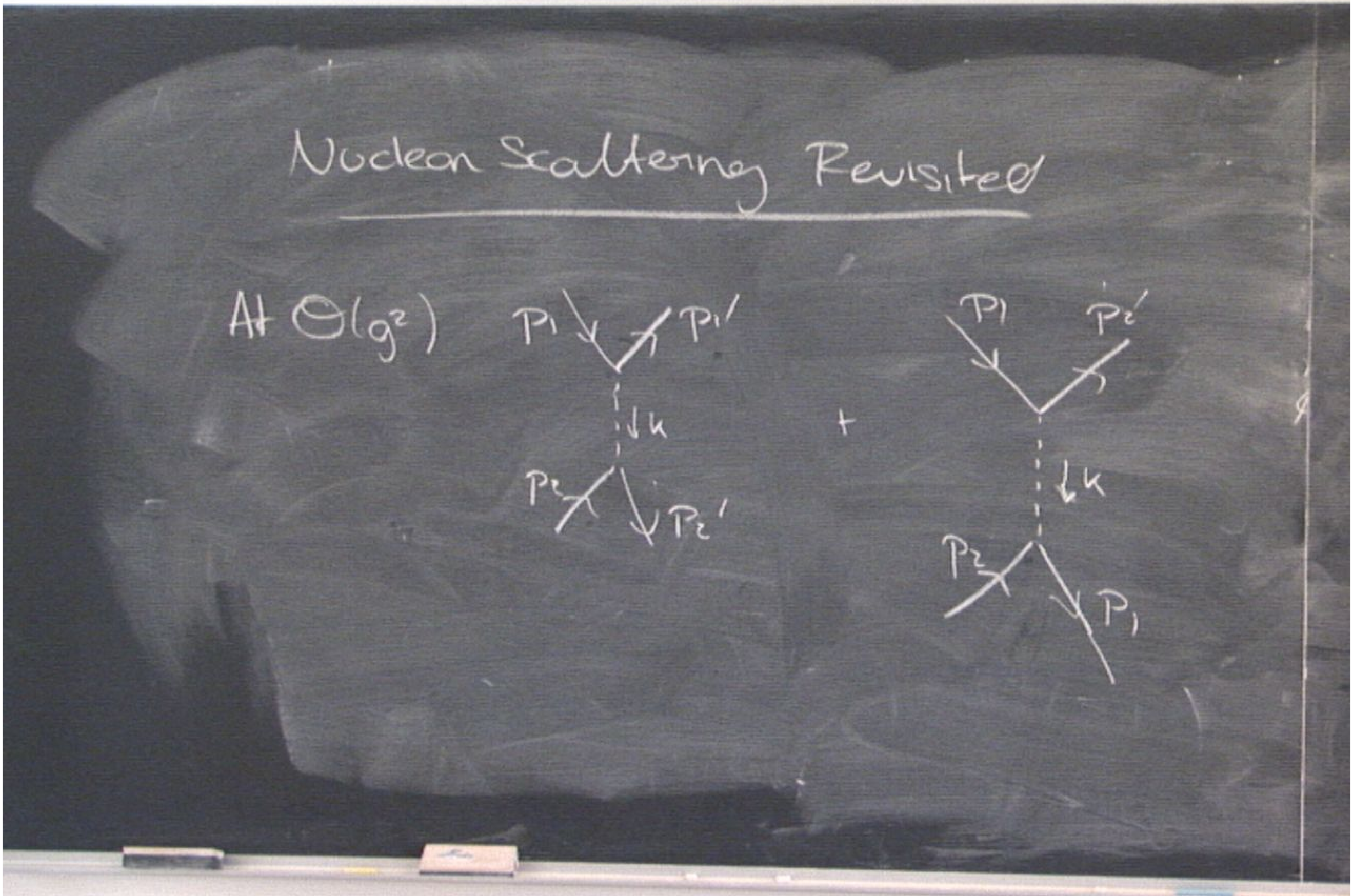


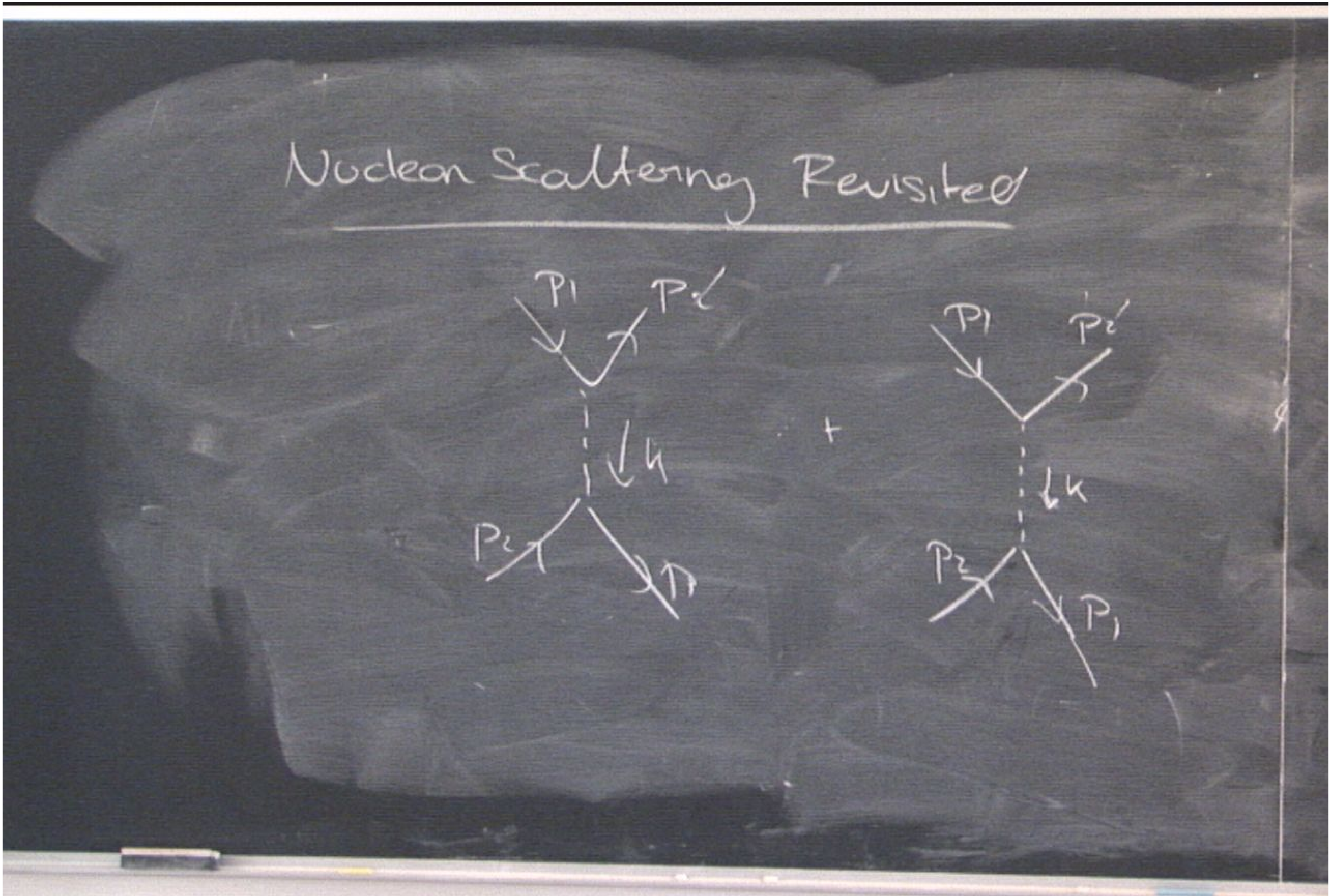
or



or .


Nucleon Scattering Revisited
At O(g²)
p1
p1'
k
p2
p2'
+
p1
p2'
k
p2
p1


Nucleon Scattering Revisited
At $\mathcal{O}(g^2)$
p_1 p_1'
k
p_2 p_2'
+
p_1 p_2'
k
p_2 p_1


Nucleon Scattering Revisited
p1
p2'
k
p2
p1'
+
p1
p2'
k
p2
p1

$$= (-ig)^2 \int \frac{d^4k}{(2\pi)^4} \, \frac{i\,(2\pi)^8}{k^2 - m^2 + i\epsilon} \left[\delta^{(4)}(p_1 - p_1' - k) \; \delta^{(4)}(p_2 + k - p_2') \right.$$

$$
= (-ig)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i\,(2\pi)^8}{k^2 - m^2 + i\epsilon} \Big[\delta^{(4)}(p_1 - p_1' - k)\, \delta^{(4)}(p_2 + k - p_2') + \delta^{(4)}(p_1 - p_2' - k)\, \delta^{(4)}(p_2 - p_1' + k) \Big]
$$

$$= (-ig)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i\,(2\pi)^8}{k^2 - m^2 + i\epsilon} \Big[\delta^{(4)}(p_1 - p_1' - k)\, \delta^{(4)}(p_2 + k - p_2') + \delta^{(4)}(p_1 - p_2' - k)\, \delta^{(4)}(p_2 - p_1' + k) \Big]$$

$$= i(-ig)^2 \left[\frac{1}{(p_1 - p_1')^2 - m^2} + \frac{1}{(p_1 - p_2')^2 - m^2} \right] (2\pi)^4 \delta^{(4)}(p_1 + p_2 - p_1' - p_2')$$

$$= (-ig)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i\,(2\pi)^8}{k^2 - m^2 + i\epsilon} \Big[\delta^{(4)}(p_1 - p_1' - h)\, \delta^{(4)}(p_2 + h - p_2')$$

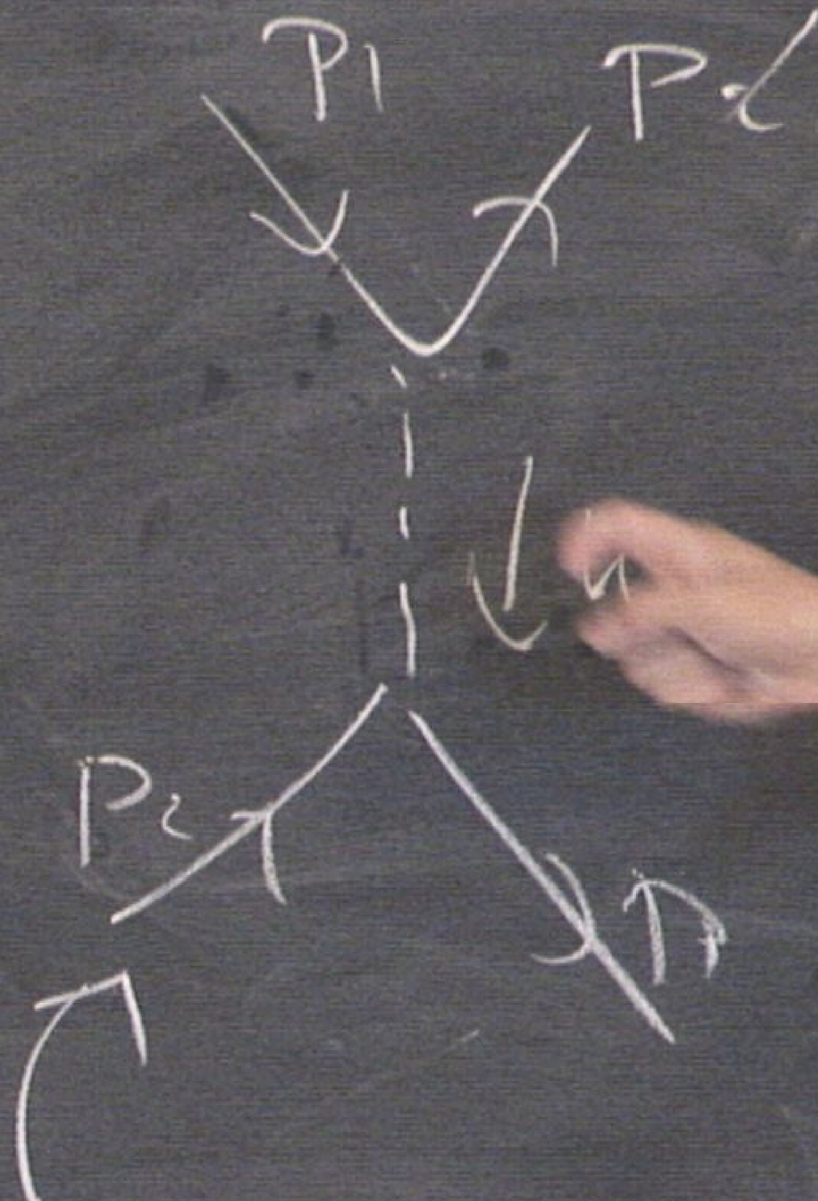
$$+ \delta^{(4)}(p_1 - p_2' - h)\, \delta^{(4)}(p_2 - p_1' + h) \Big]$$

$$= i(-ig)^2 \left[\frac{1}{(p_1 - p_1')^2 - m^2} + \frac{1}{(p_1 - p_2') - m^2} \right] (2\pi)^4 \delta^{(4)}(p_1 + p_2 - p_1' - p_2')$$

$$= (-ig)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i\,(2\pi)^8}{k^2 - m^2 + i\epsilon} \Big[\delta^{(4)}(p_1 - p_1' - k)\, \delta^{(4)}(p_2 + k - p_2') + \delta^{(4)}(p_1 - p_2' - k)\, \delta^{(4)}(p_2 - p_1' + k) \Big]$$

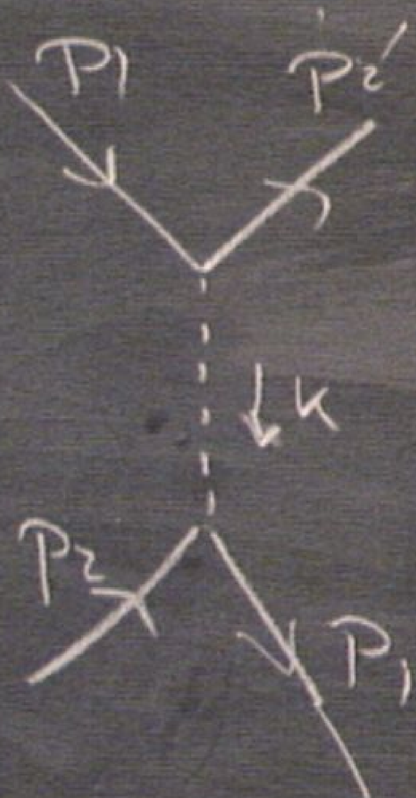
$$= i(-ig)^2 \left[\frac{1}{(p_1 - p_1')^2 - m^2} + \frac{1}{(p_1 - p_2')^2 - m^2} \right] (2\pi)^4 \delta^{(4)}(p_1 + p_2 - p_1' - p_2')$$

$$k^2 \neq m^2$$

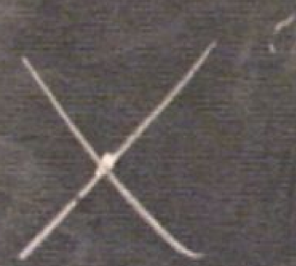


$$p^2 = M^2$$

visited



$$= (-ig) \int \frac{d^4k}{(2\pi)^4}$$

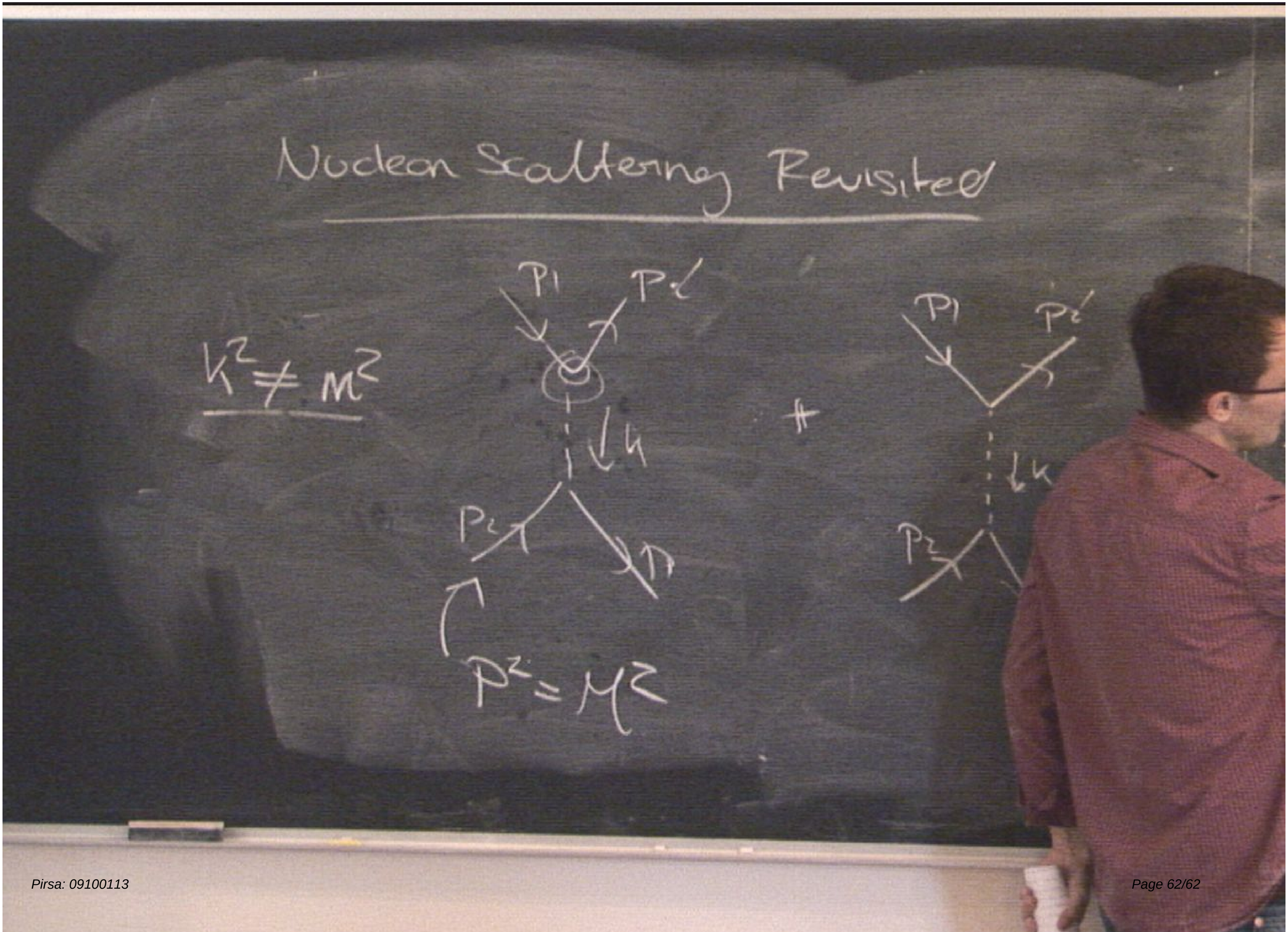


$$= i(-ig)^2 \left[\frac{1}{(p_1-p_1')^2} \right.$$

$$= (-ig)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i\,(2\pi)^8}{k^2-m^2+i\epsilon} \Big[\delta^{(4)}(p_1-p_1'-k)\ \delta^{(4)}(p_2+k \ldots$$

$$+\ \delta^{(4)}(p_1-p_2'-k \ldots \ \delta^{(4)}(p_2- \ldots$$

$$= i(-ig)^2 \left[\frac{1}{(p_1-p_1')^2-m^2} + \frac{1}{(p_1-p_2')^2-m^2} \right] (2\pi)^4 \delta^{(4} \ldots$$


Nucleon Scattering Revisited
k² ≠ M²
p1
p2'
k
p2
p1'
p² = M²