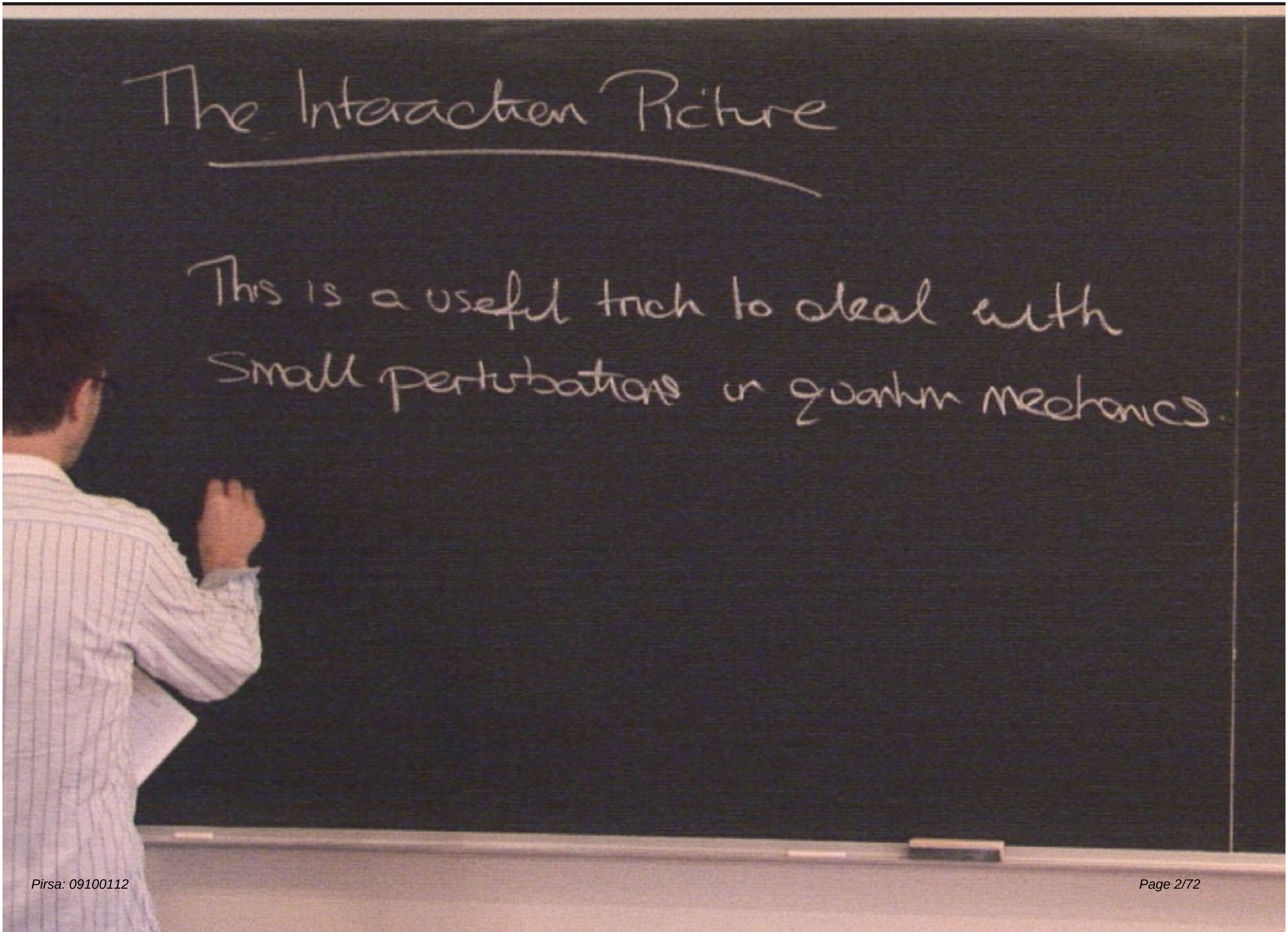
Title: Quantum Field Theory (PHYS 601) - Lecture 7

Date: Oct 06, 2009 09:00 AM

URL: http://pirsa.org/09100112

Abstract:


The Interaction Picture
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In Schrödinger picture $$i\frac{\partial}{\partial t}|\psi\rangle_S = H_S|\psi\rangle_S$$

while operators O_S are time independent.

In Heisenberg picture,

$$\mathcal{O}_H(t) = e^{iHt}\,\mathcal{O}_S\,e^{-iHt}$$

$$|\psi\rangle_H = e^{iHt}\,|\psi\rangle_S.$$

In Heisenberg picture,

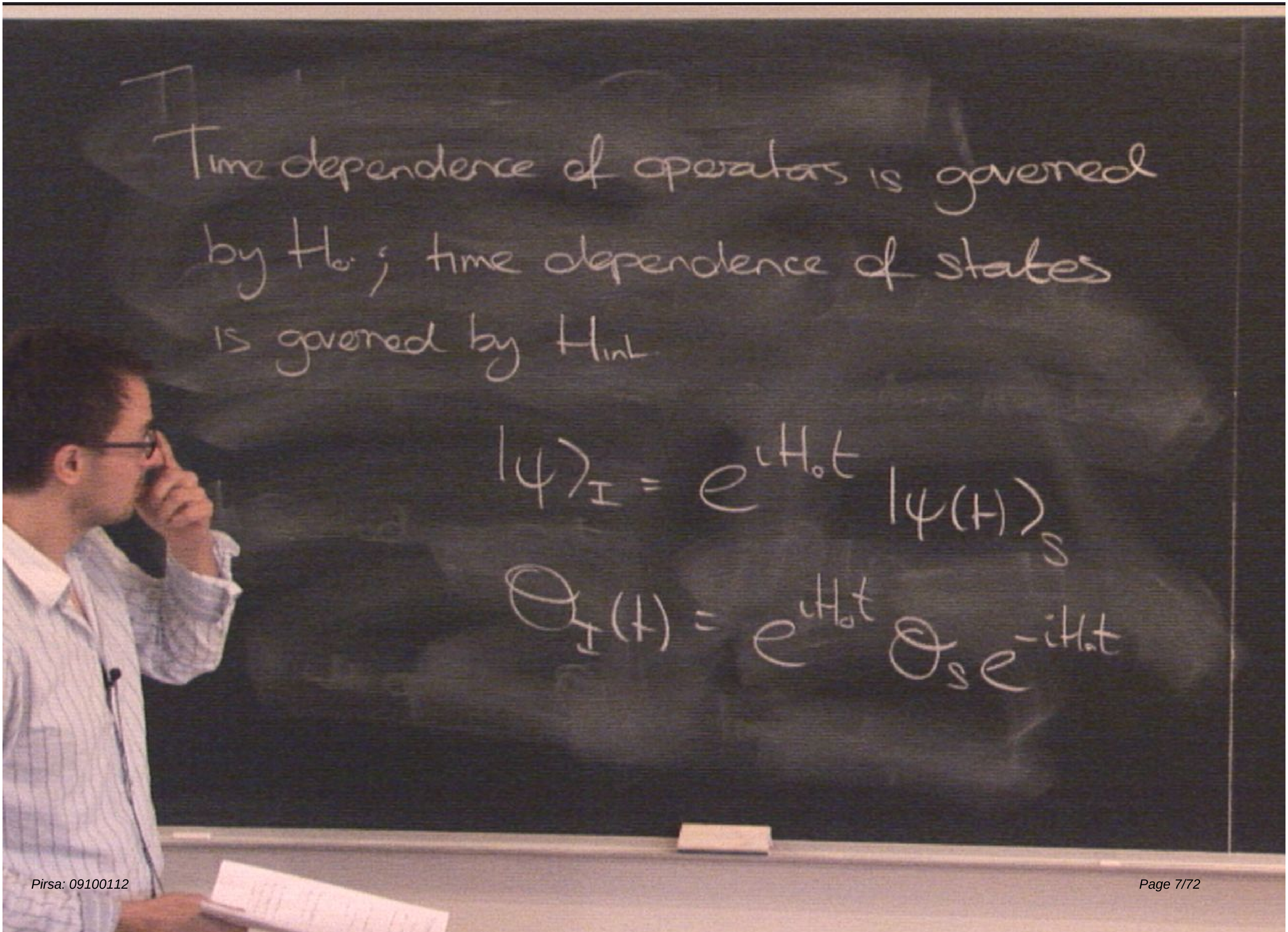
$$O_H(t) = e^{iHt} O_S e^{-iHt}$$

$$|\psi\rangle_H = e^{iHt} |\psi\rangle_S .$$

The interaction picture is a hybrid of the two;

We write the Hamiltonian as $H = H_0 + H_{int}$.

Time dependence of operators is governed by H_0.


Time dependence of operators is governed
by H_0; time dependence of states
is governed by H_{int}.
$|\psi\rangle_I = e^{iH_0t}|\psi(t)\rangle_S$
$\mathcal{O}_I(t) = e^{iH_0t}\,\mathcal{O}_S\,e^{-iH_0t}$

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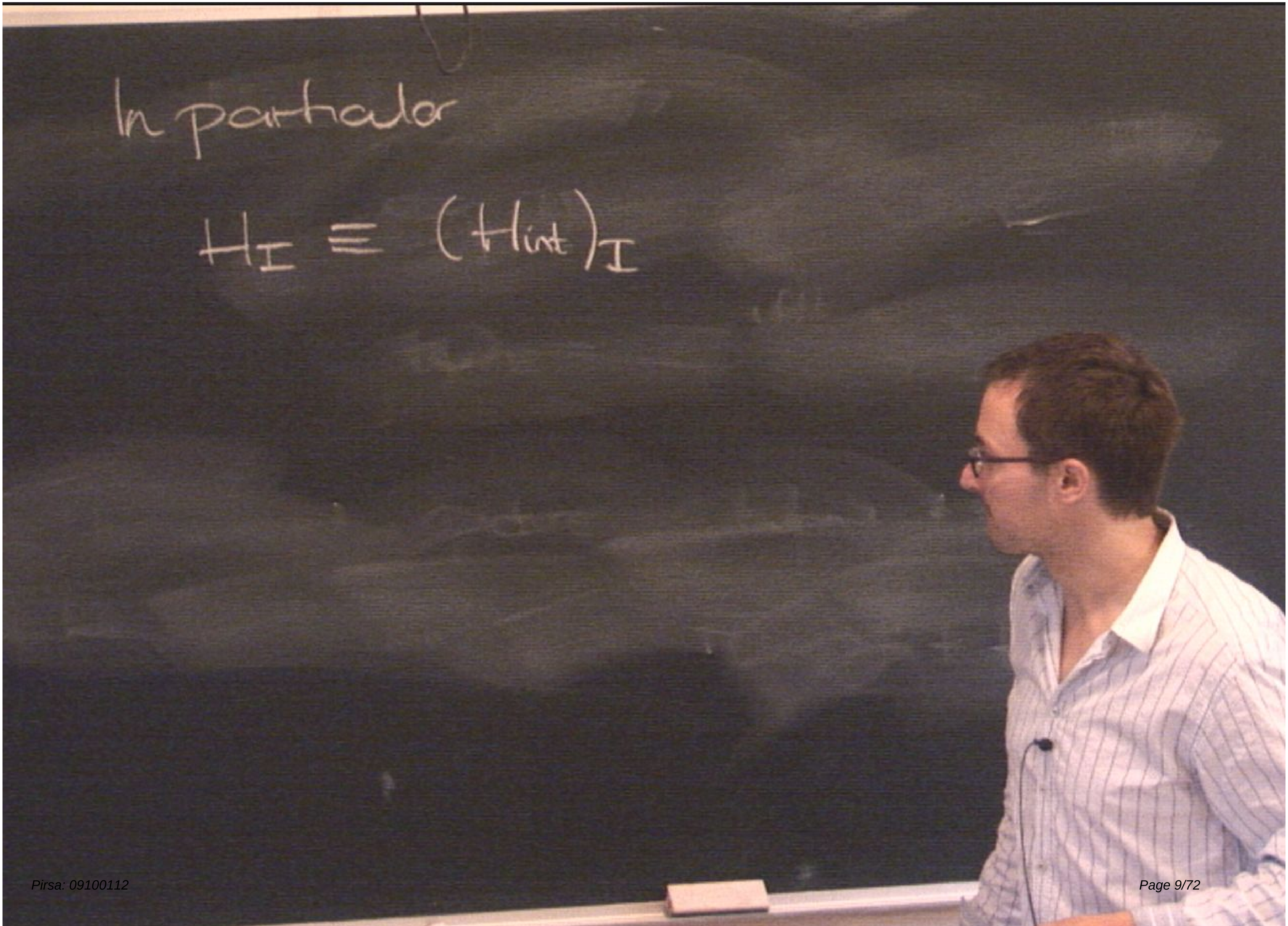
$$|\psi\rangle_I = e^{iH_0 t}\,|\psi(t)\rangle_S$$

$$O_I(t) = e^{iH_0 t}\, O_S\, e^{-iH_0 t}$$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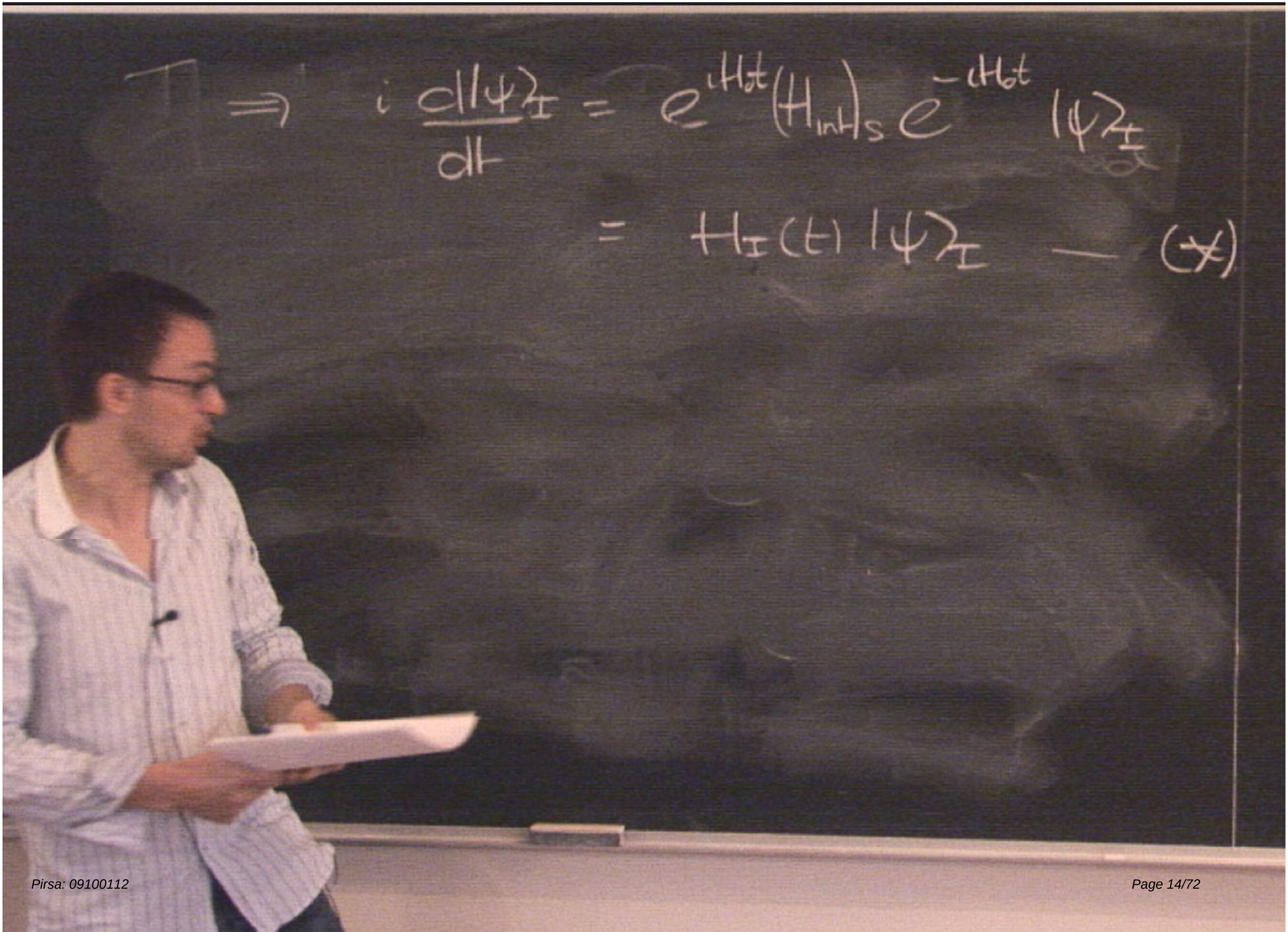
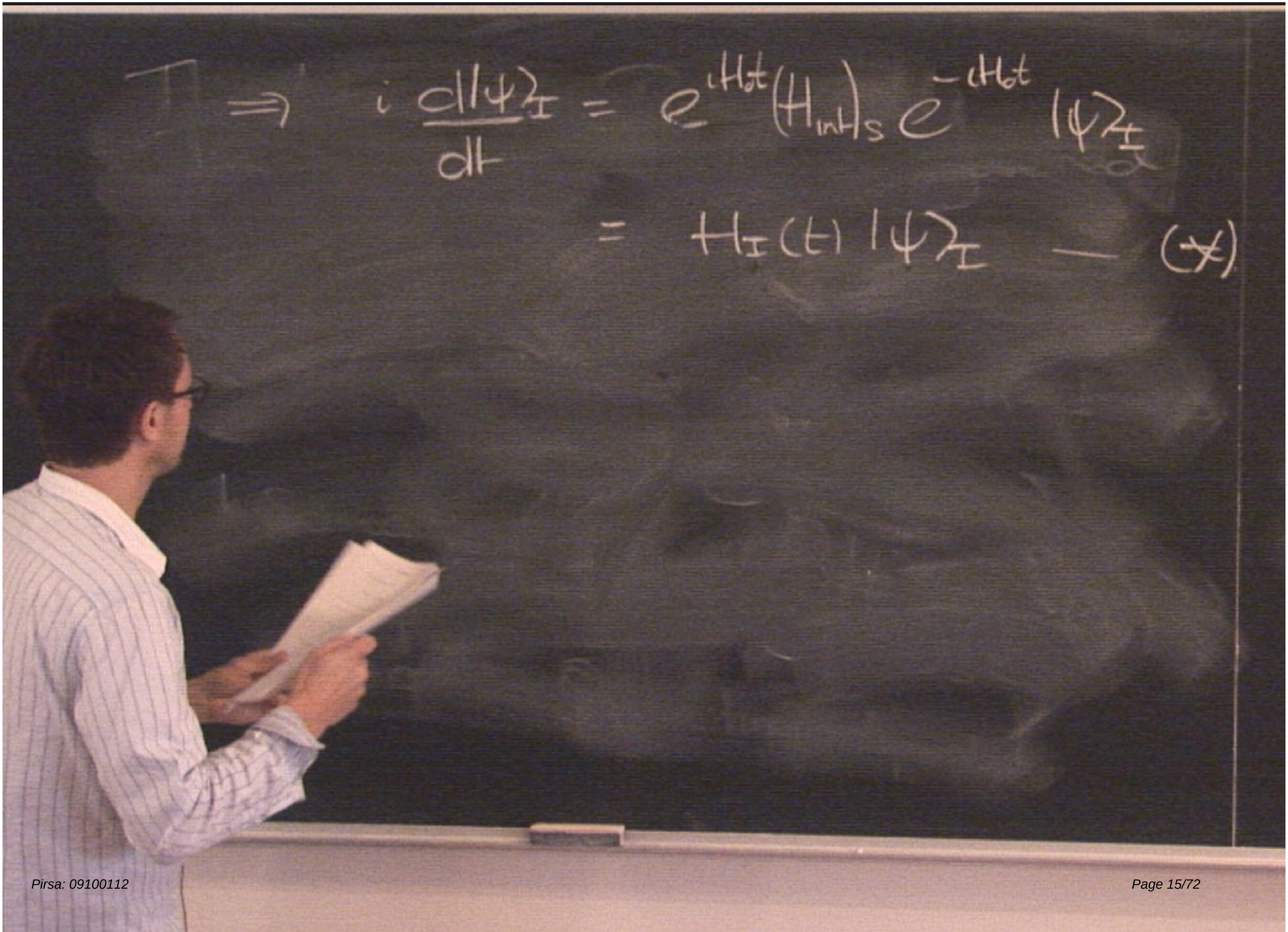
$$H_I \equiv (H_{int})_I = e^{iH_0 t} H_{int} e^{-iH_0 t}$$

The Schrödinger equ^{n} is $\quad i \dfrac{d|\psi\rangle_S}{dt} = H_S|\psi\rangle_S .$

$$\Rightarrow i\frac{d}{dt}\left(e^{-iH_0 t}|\psi\rangle_I\right) = (H_0 + H_{int})_S\, e^{-iH_0 t}|\psi\rangle_I$$



$\mathrm{I} \Rightarrow \quad i\frac{d|\psi\rangle_I}{dt} = e^{iH_0 t}(H_{int})_S\, e^{-iH_0 t}\,|\psi\rangle_I$
$= H_I(t)\,|\psi\rangle_I \quad — \quad (*)$


II $\Rightarrow$ $i \frac{d|\psi\rangle_I}{dt} = e^{iH_0 t}(H_{int})_S e^{-iH_0 t} |\psi\rangle_I$
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$\Rightarrow \quad i \frac{d|\psi\rangle_I}{dt} = e^{iH_0 t} (H_{int})_S \, e^{-iH_0 t} \; |\psi\rangle_I$

$= H_I(t) \, |\psi\rangle_I \quad — \quad (*)$

Dyson's Formula

We'll write $|\psi(t)\rangle_I = U(t, t_0) \, |\psi(t_0)\rangle_I$

$$\Rightarrow \quad i\frac{d|\psi\rangle_I}{dt} = e^{iH_0 t}(H_{int})_S\, e^{-iH_0 t}\, |\psi\rangle_I$$

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If U and H were functions, we could solve this by

$$U(t,t_0) = \exp\left(-i\int_{t_0}^{t} H_I(t')\,dt'\right).$$

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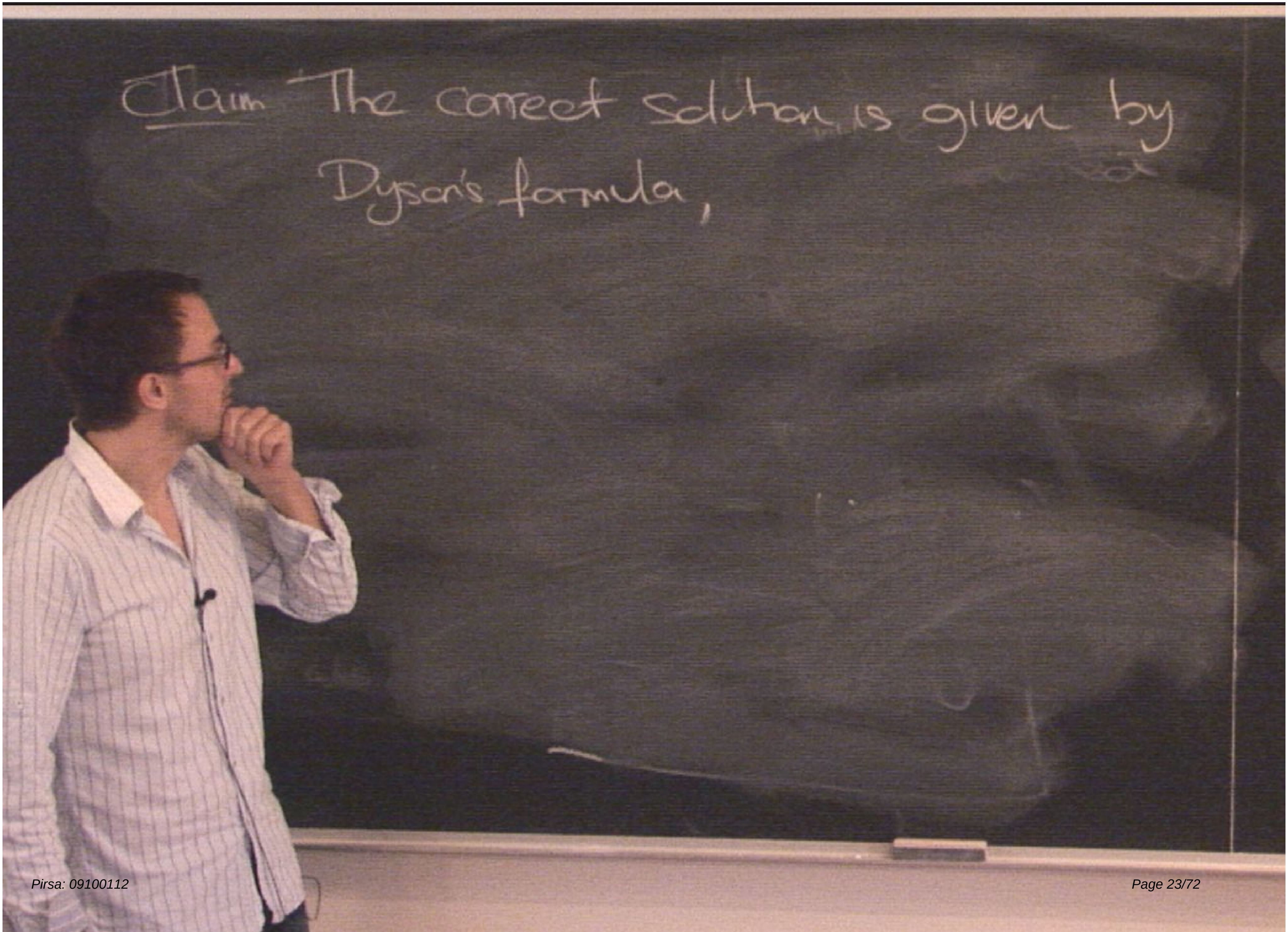
Claim The correct solution is given by
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$$U(t,t_0) = T\exp\left(-i\int_{t_0}^{t} H_I(t')\,dt'\right)$$

T is time ordering: $T[O(t_1)O(t_2)] = \begin{cases} O(t_1)O(t_2) & t_1>t_2 \\ O(t_2)O(t_1) & t_2<t_1 \end{cases}$

This means

$$(t,t_0) = 1 - i\int_{t_0}^{t} H_I(t')\,dt' + \frac{(-i)^2}{2}\left[\int_{t_0}^{t} dt' \int_{t'}^{t} dt'' \, H_I(t'')\,H_I(t') + \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' \, H_I(t')\,H_I(t'')\right]$$

$$+ \ldots$$

$t_1 > t_2$

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$$+ \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt''\, H_I(t') H_I(t'')\Big]$$

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$$\int_{t_0}^{t} dt'\, H_I(t') \int_{t_0}^{t} dt''\, H_I(t'')$$

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$$i\frac{\partial}{\partial t} T\exp\left(-i\int_{t_0}^{t} H_I(t')\,dt'\right)$$

$$= T\exp\left(H_I(t)\exp\left(-i\int_{t_0}^{t} H_I(t')\,dt'\right)\right)$$

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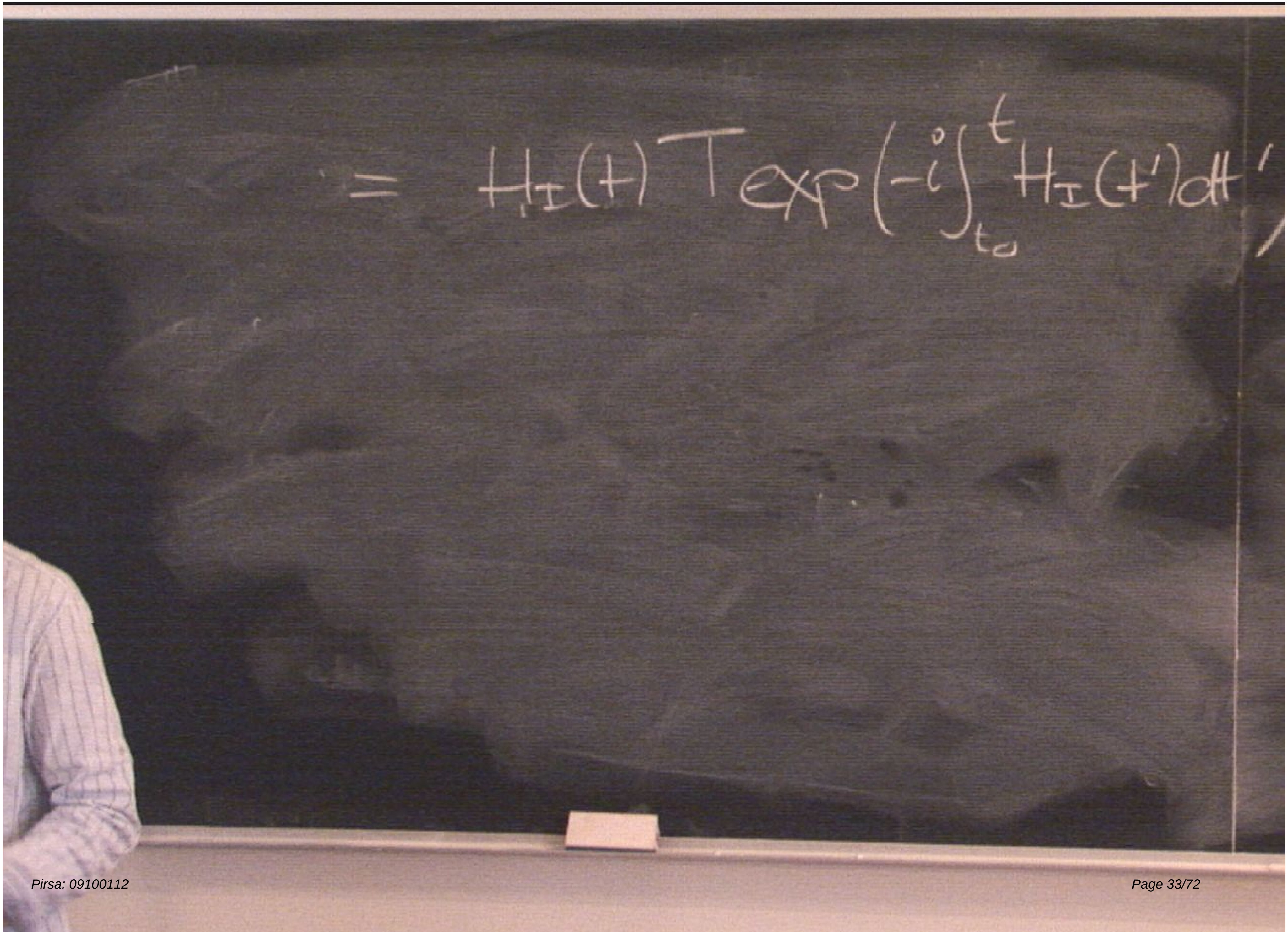
$$= T\left(H_I(t) \exp\left(-i \int_{t_0}^{t} H_I(t')\,dt'\right)\right)$$

$$= H_I(t)\, T\exp\left(-\int_{t_0}^{t} H_I(t')\,dt'\right)$$



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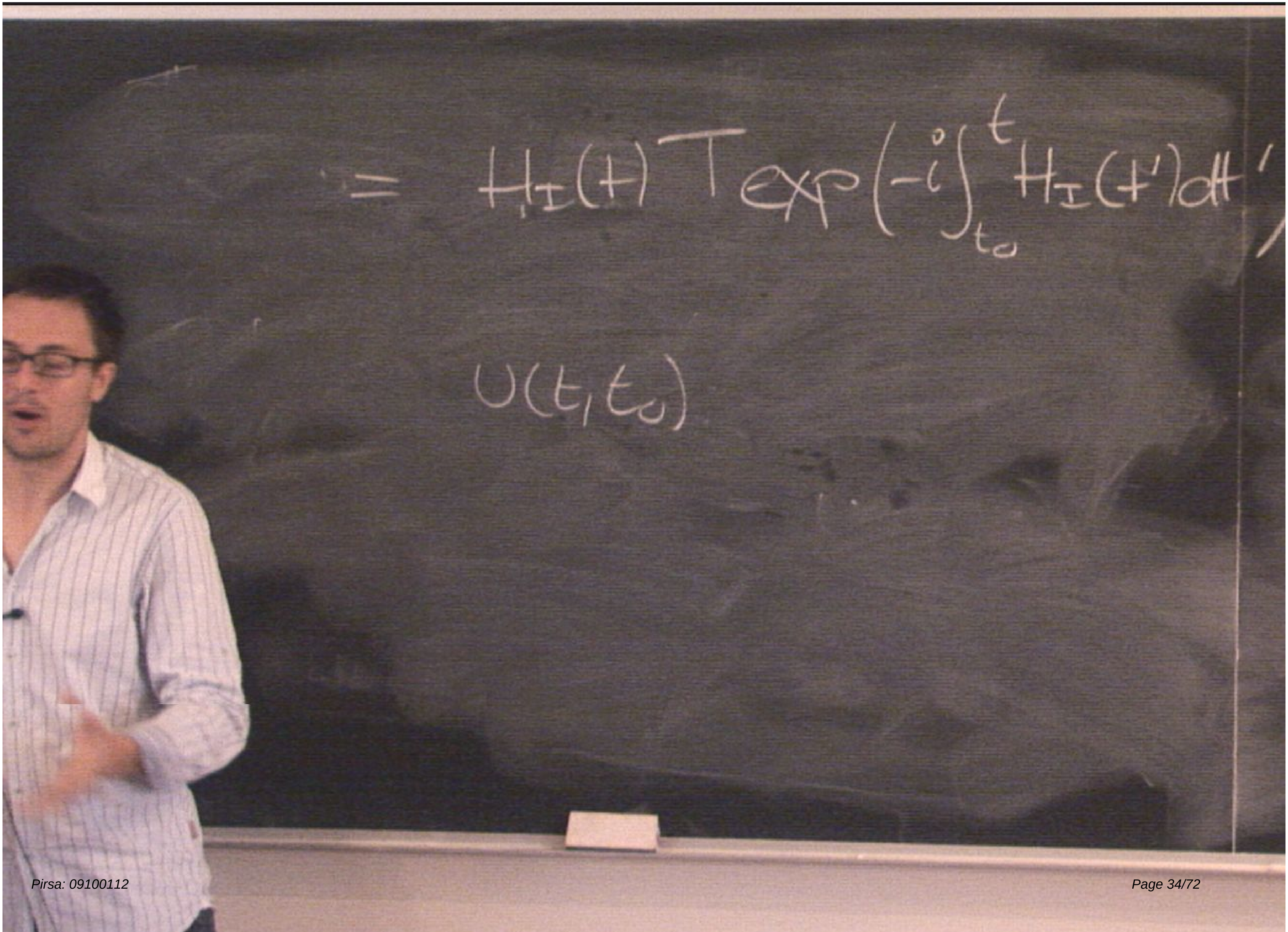
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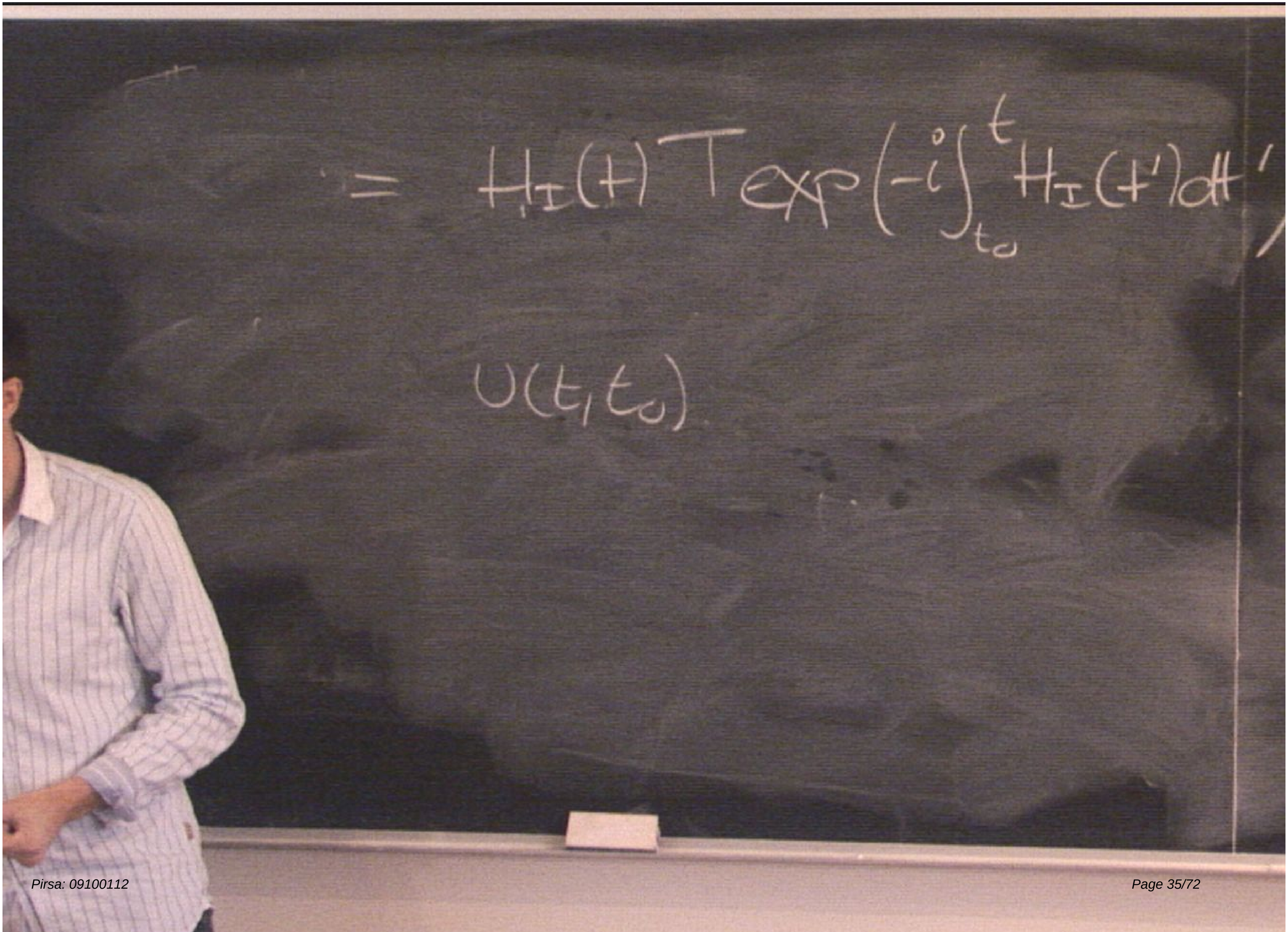
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A First look at Scatterings

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We'll work with scalar Yukawa theory.

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi + \partial_\mu\psi^*\partial^\mu\psi - \frac{1}{2}m^2\phi^2$$

real complex

$$- M^2\psi^*\psi$$

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real

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$$- M^2\psi^*\psi - g\psi^*\psi\phi$$

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$$H_{int} = g\psi^{*}\psi\phi$$

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$\phi \sim a + a^{\dagger}$ can create/destroy ϕ-particles (mesons)

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$\psi \sim b + c^\dagger$ can destroy ψ-particles and create $\psi^\dagger$-particles (<u>nucleons</u>)

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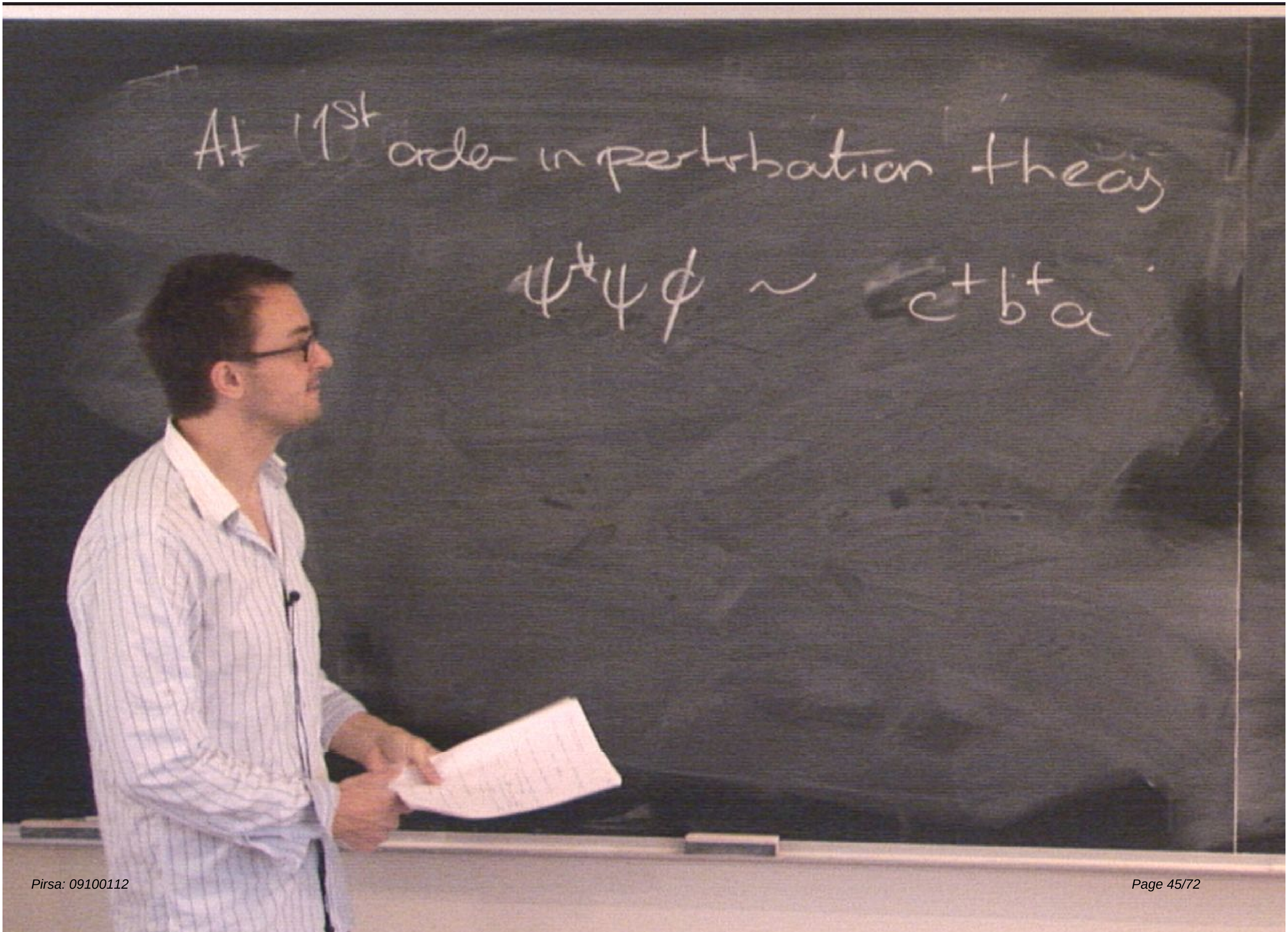
$\psi^\dagger \sim b^\dagger + c$ similarly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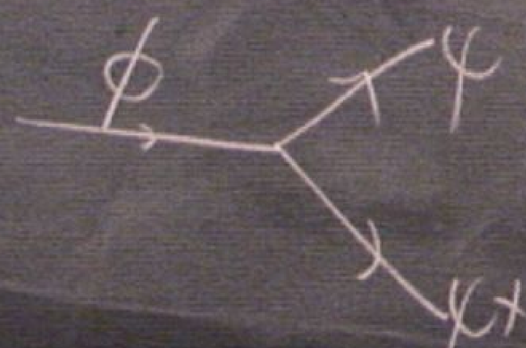

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includes terms like

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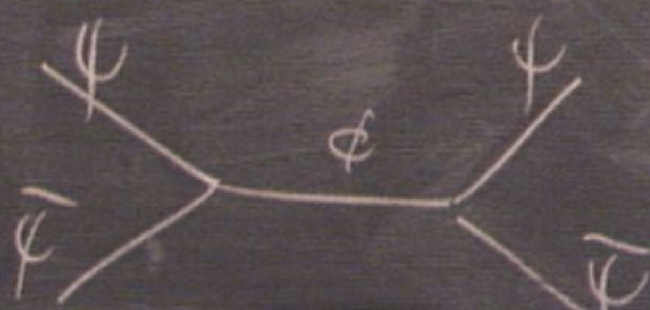

To calculate amplitudes for these processes, we first need an important (and slightly dodgy) assumption.

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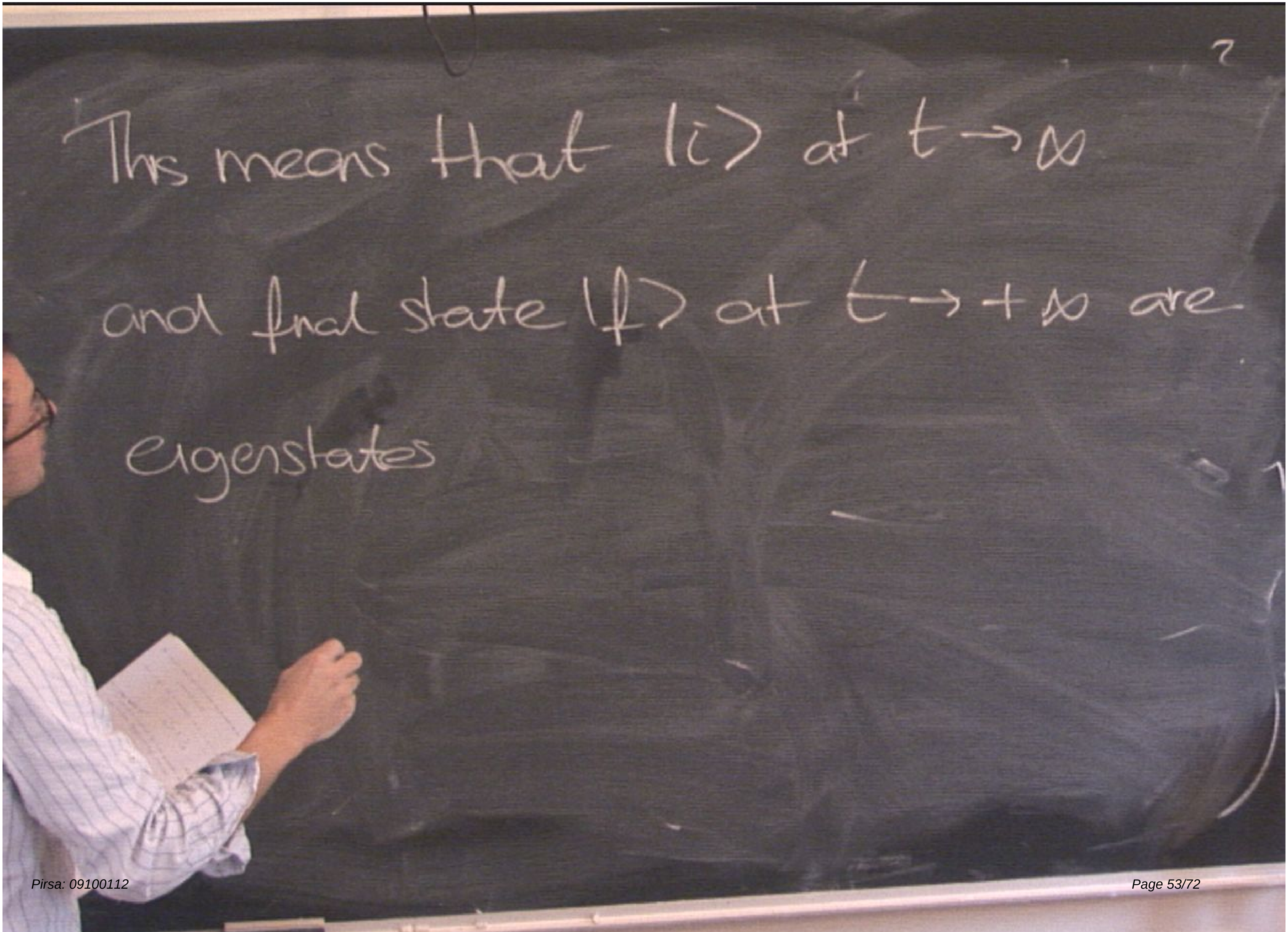
Initial and Final States look like non-interacting Particles


This means that $|i\rangle$ at $t \to \infty$
and final state $|f\rangle$ at $t \to +\infty$ are
eigenstates

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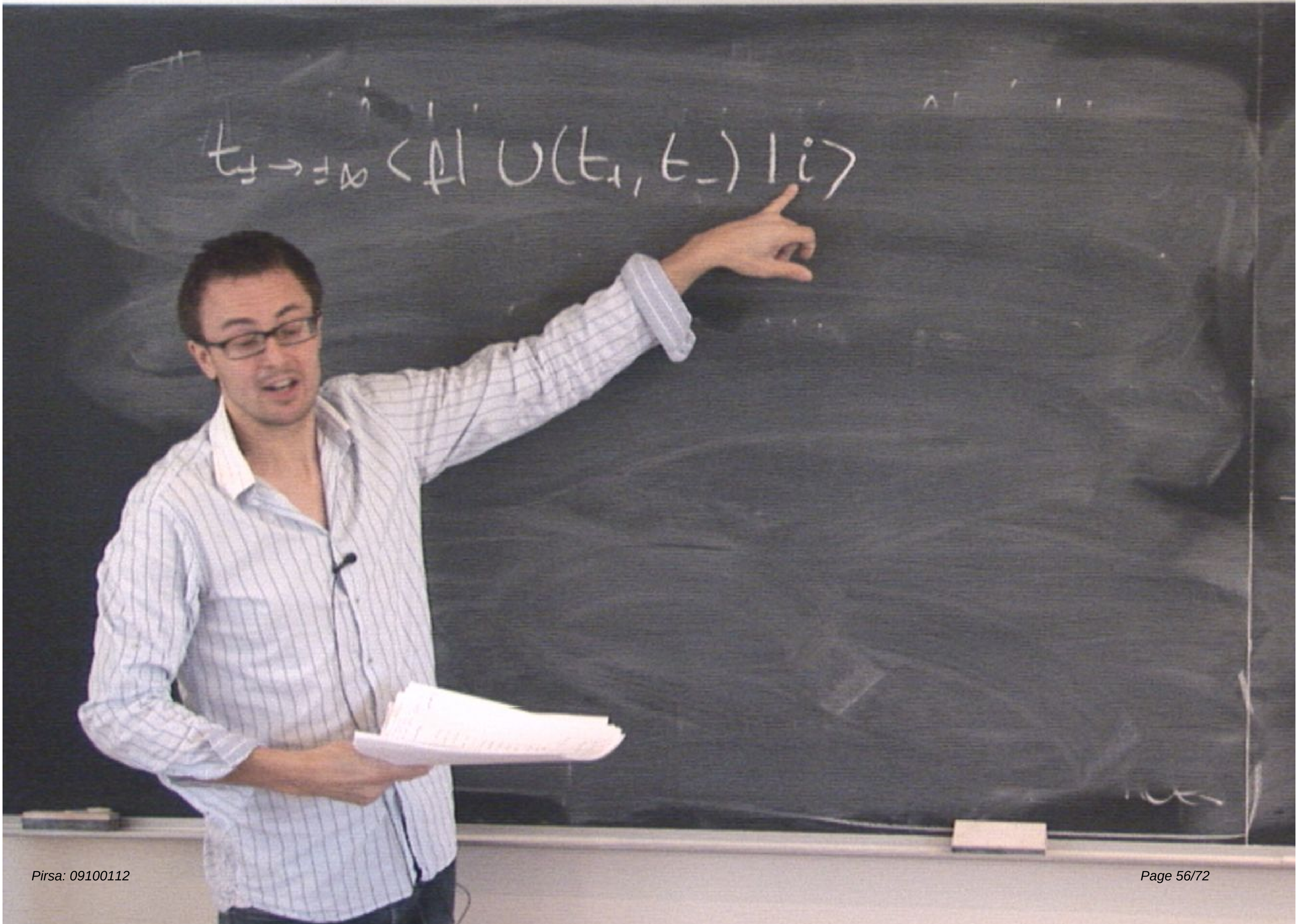
The amplitude to go from $|i\rangle$ to $|f\rangle$


$t_\pm \to \pm\infty \langle f | U(t_+, t_-) | i \rangle$

$$\lim_{t_\pm \to \pm\infty} \langle f | U(t_+, t_-) | i \rangle$$

$$\equiv \langle f | S | i \rangle$$

This is a unitary operator, called the S(cattering)-Matrix

Meson Decay

$$|i\rangle = \sqrt{2E_{k}}\, a_{k}^{\dagger}|0\rangle$$

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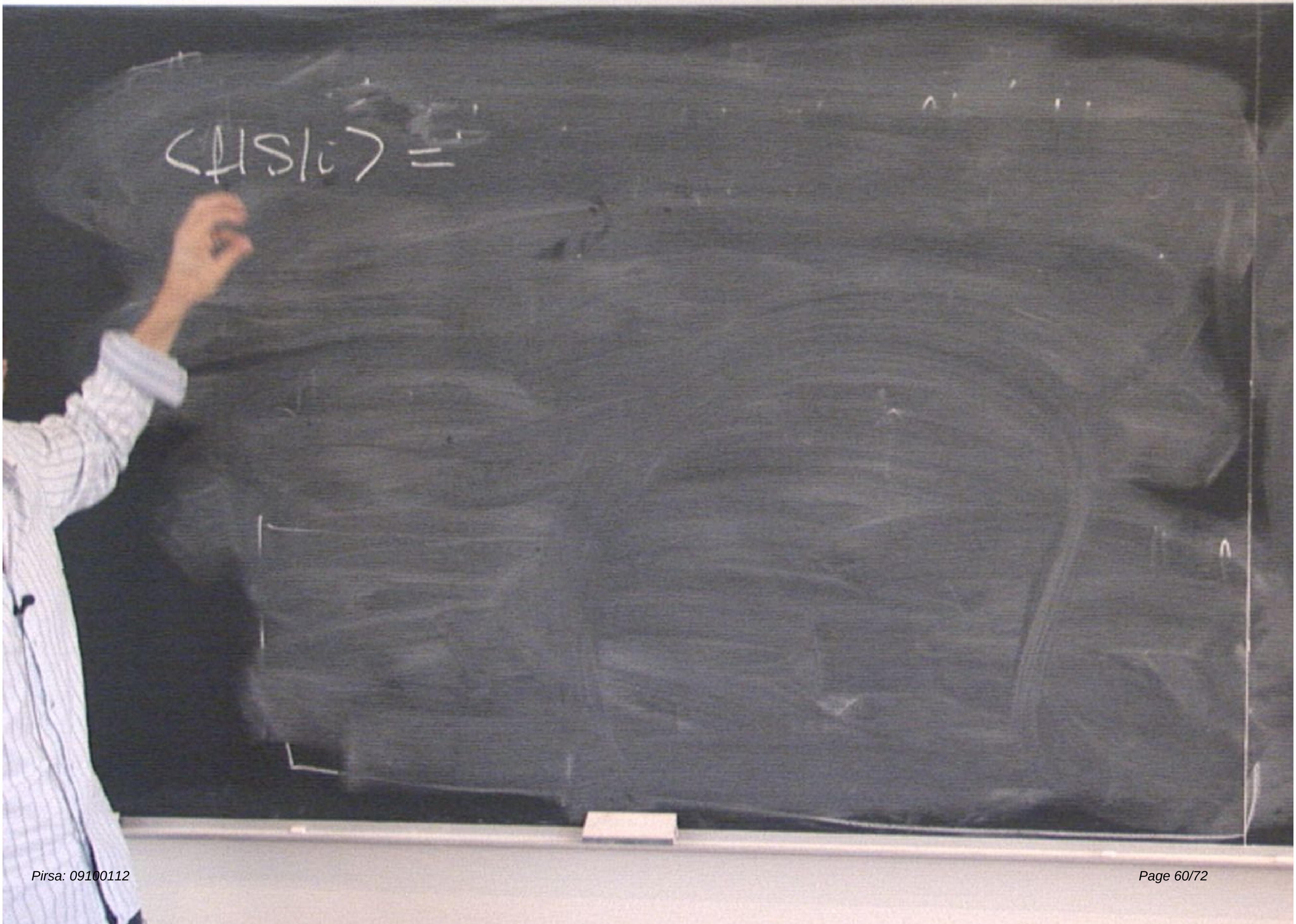
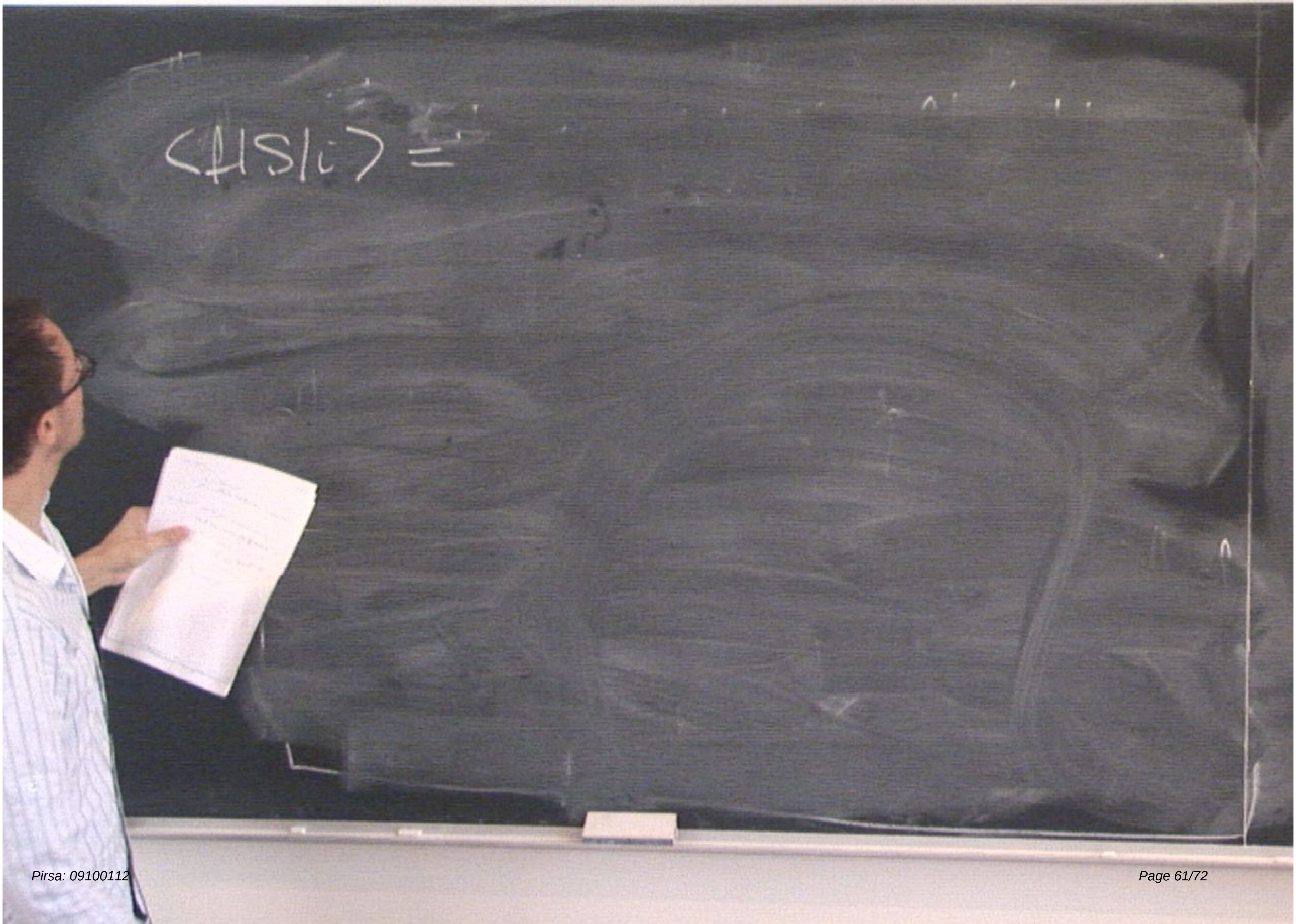

$\langle f|S|i\rangle =$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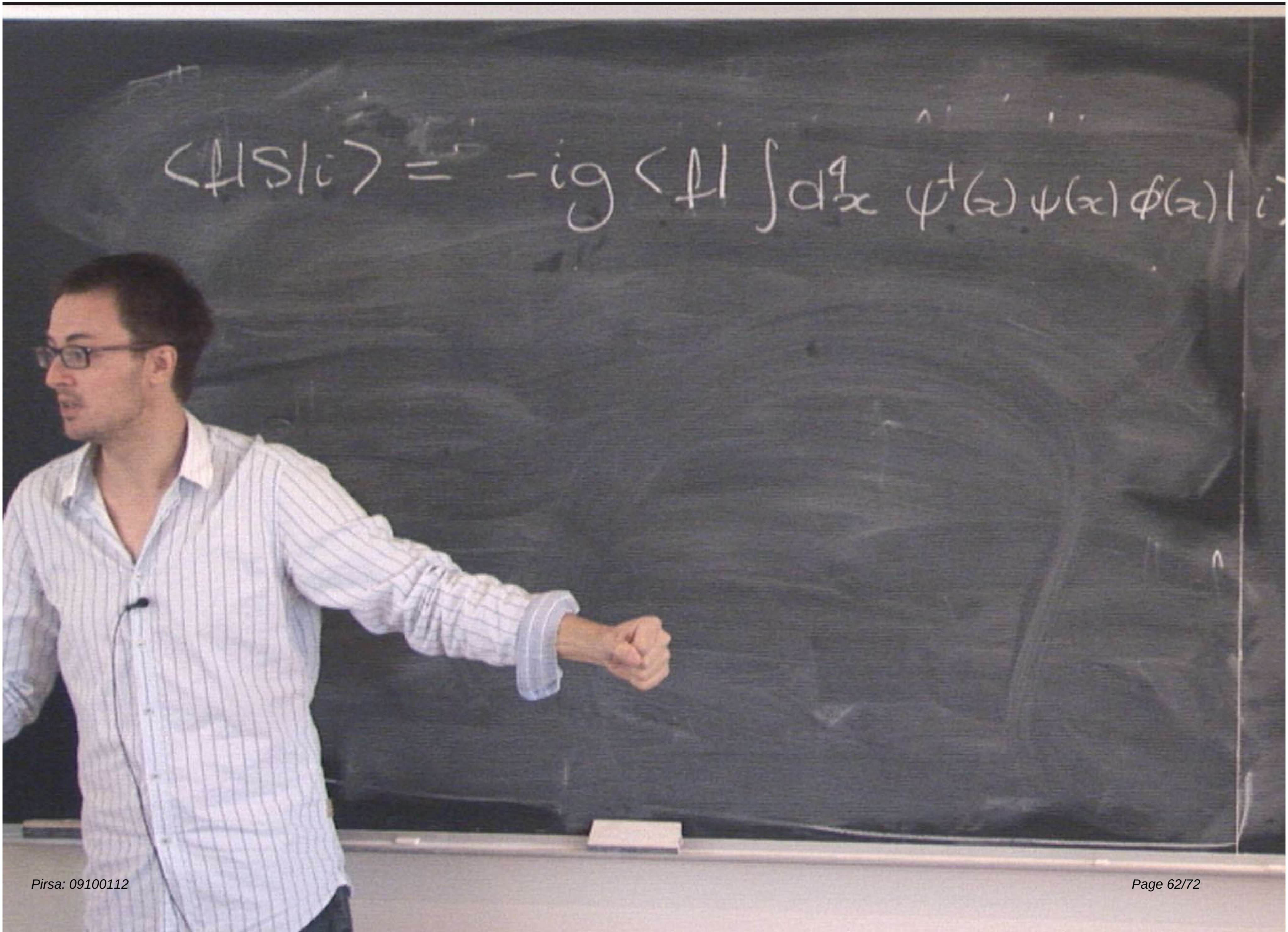
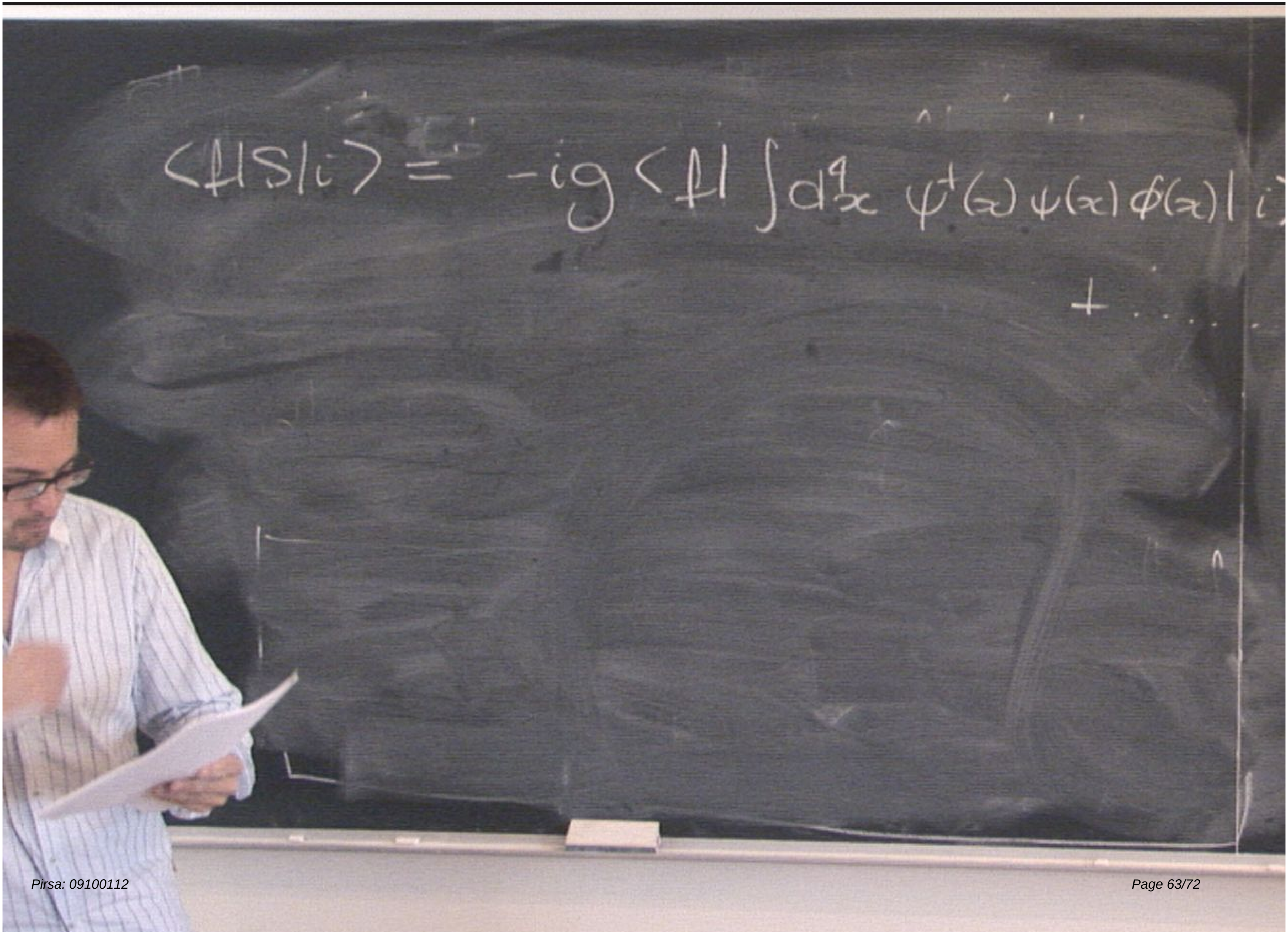

$\langle f|S|i\rangle = -ig\langle f|\int d^4x\, \psi^\dagger(x)\psi(x)\phi(x)|i\rangle$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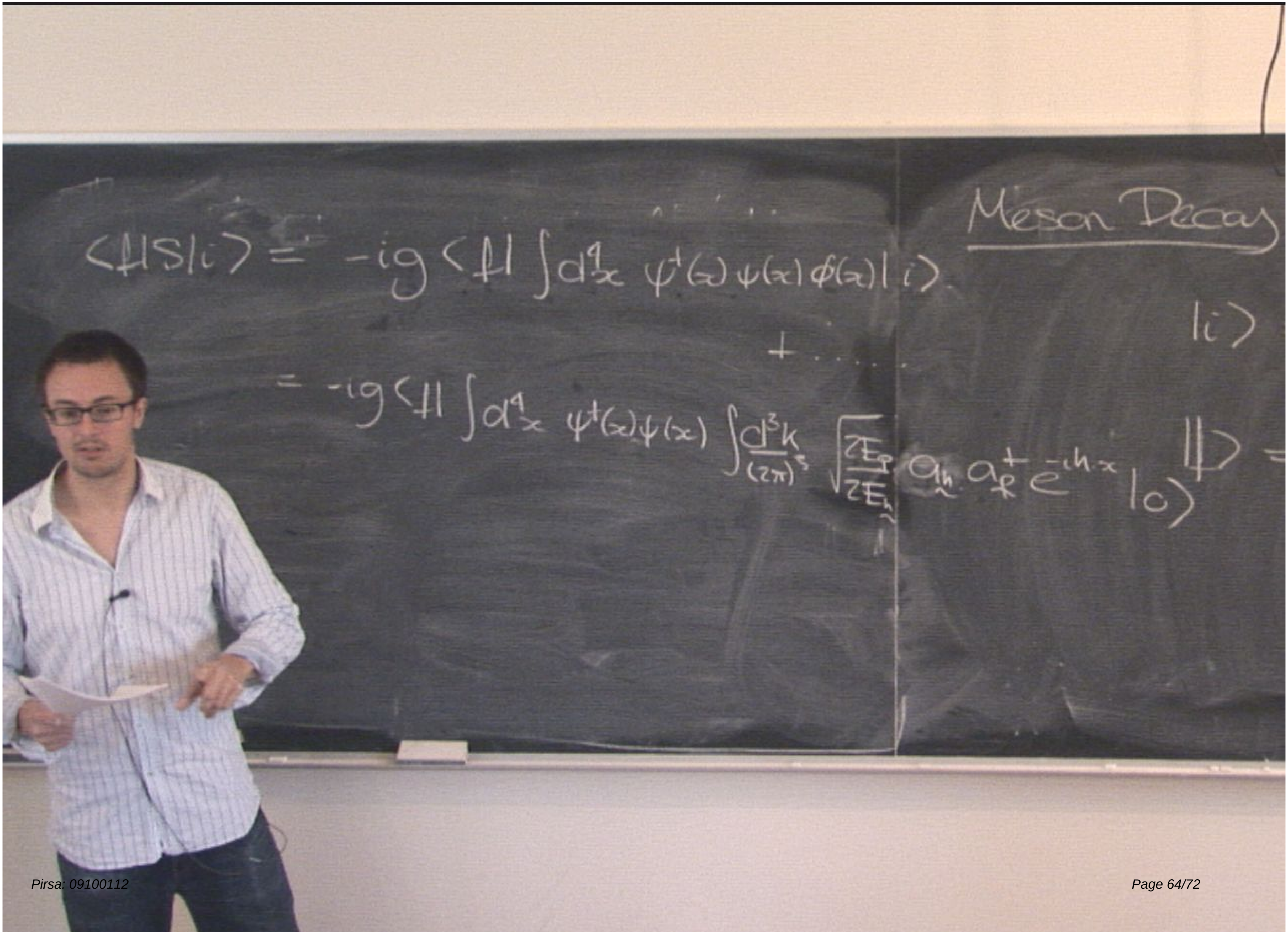
$+ \ldots$


Meson Decay
$\langle f|S|i\rangle = -ig\langle f|\int d^4x\, \psi^\dagger(x)\psi(x)\phi(x)|i\rangle$
$+ \ldots\ldots$
$= -ig\langle f|\int d^4x\, \psi^\dagger(x)\psi(x)\int \frac{d^3k}{(2\pi)^3}\sqrt{\frac{2E_p}{2E_k}}\, a_k a_p^\dagger e^{-ik\cdot x}|0\rangle$
$|i\rangle$
$|p\rangle =$

$$\langle f|S|i\rangle = -ig\langle f|\int d^4x\, \psi^\dagger(x)\psi(x)\phi(x)|i\rangle$$

$$+ \dots$$

$$= -ig\langle f|\int d^4x\, \psi^\dagger(x)\psi(x)\int\frac{d^3k}{(2\pi)^3}\sqrt{\frac{2E_p}{2E_k}}\, a_k\, a_p^\dagger\, e$$

$$= -ig\langle f|\int d^4x\, \psi^\dagger(x)\psi(x)\, e^{-ip\cdot x}|0\rangle$$

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$$= -ig\langle f|\int d^4x\, \psi^\dagger(x)\psi(x)\, e^{-ip\cdot x}|0\rangle$$

$$= -ig \langle f \int d^4x \int \frac{d^3k_1 \, d^3k_2'}{(2\pi)^6} \frac{1}{\sqrt{4E_{k_2}E_{k_1}}} c^\dagger_{k_1} b^\dagger_{k_2}$$

$$e^{+i(k_1+k_2-p)x} \, |0\rangle$$

$$a^\dagger_k e^{-ik\cdot x} |0\rangle$$

$$|0\rangle$$

$$= -ig \langle f| \int d^4x \int \frac{d^3k_1 \, d^3k_2}{(2\pi)^6} \frac{1}{\sqrt{4E_{k_2}E_{k_1}}} c^{\dagger}_{k_1} b^{\dagger}_{k_2} e^{+i(k_1+k_2-p)x} |0\rangle$$

$$= -ig \langle d| \int d^4x \, e^{+i(q_1+q_2-p)\cdot x} |0\rangle$$

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