Title: Quantum Field Theory (PHYS 601) - Lecture 5

Date: Oct 02, 2009 09:00 AM

URL: http://pirsa.org/09100108

Abstract:

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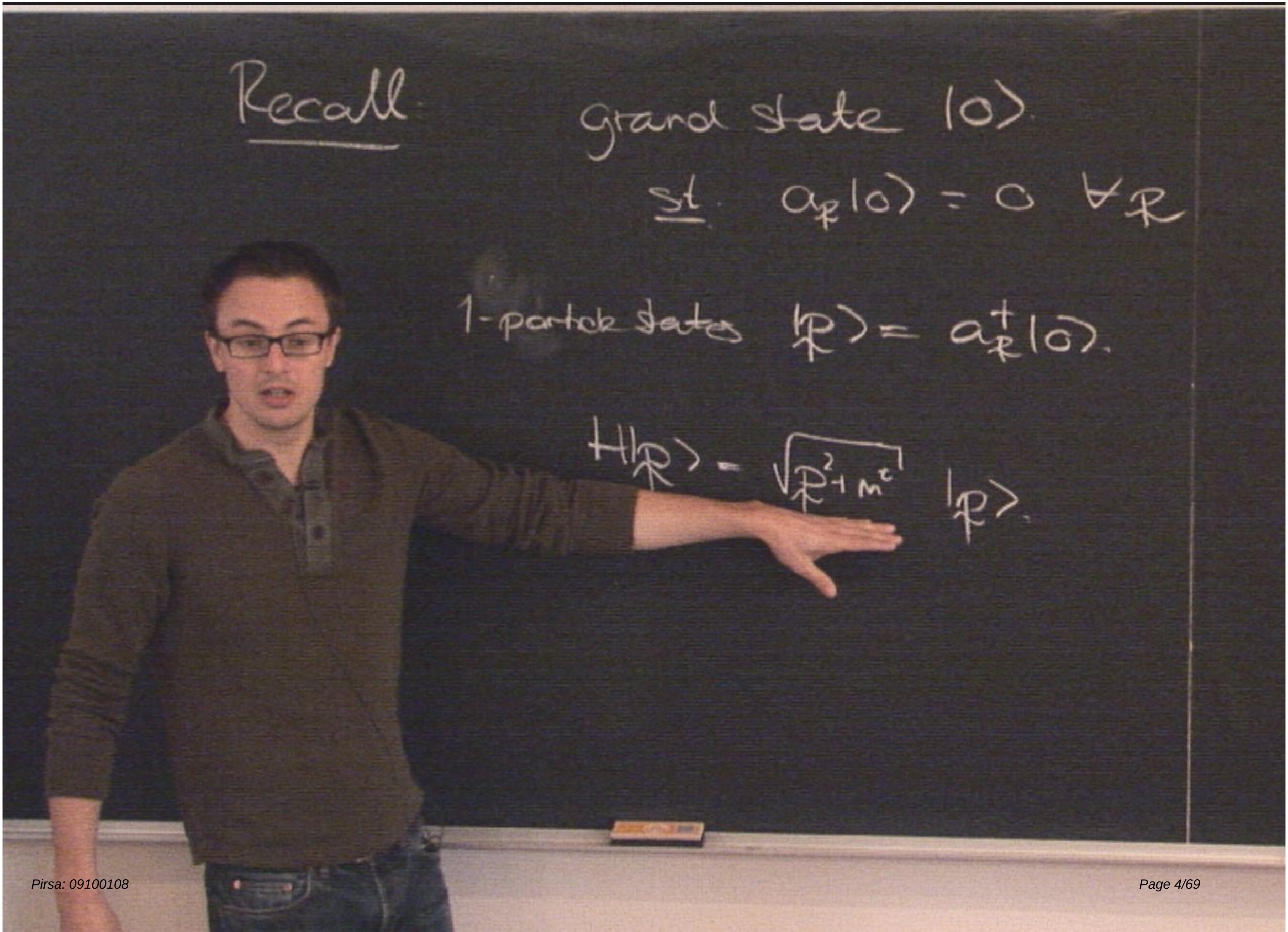
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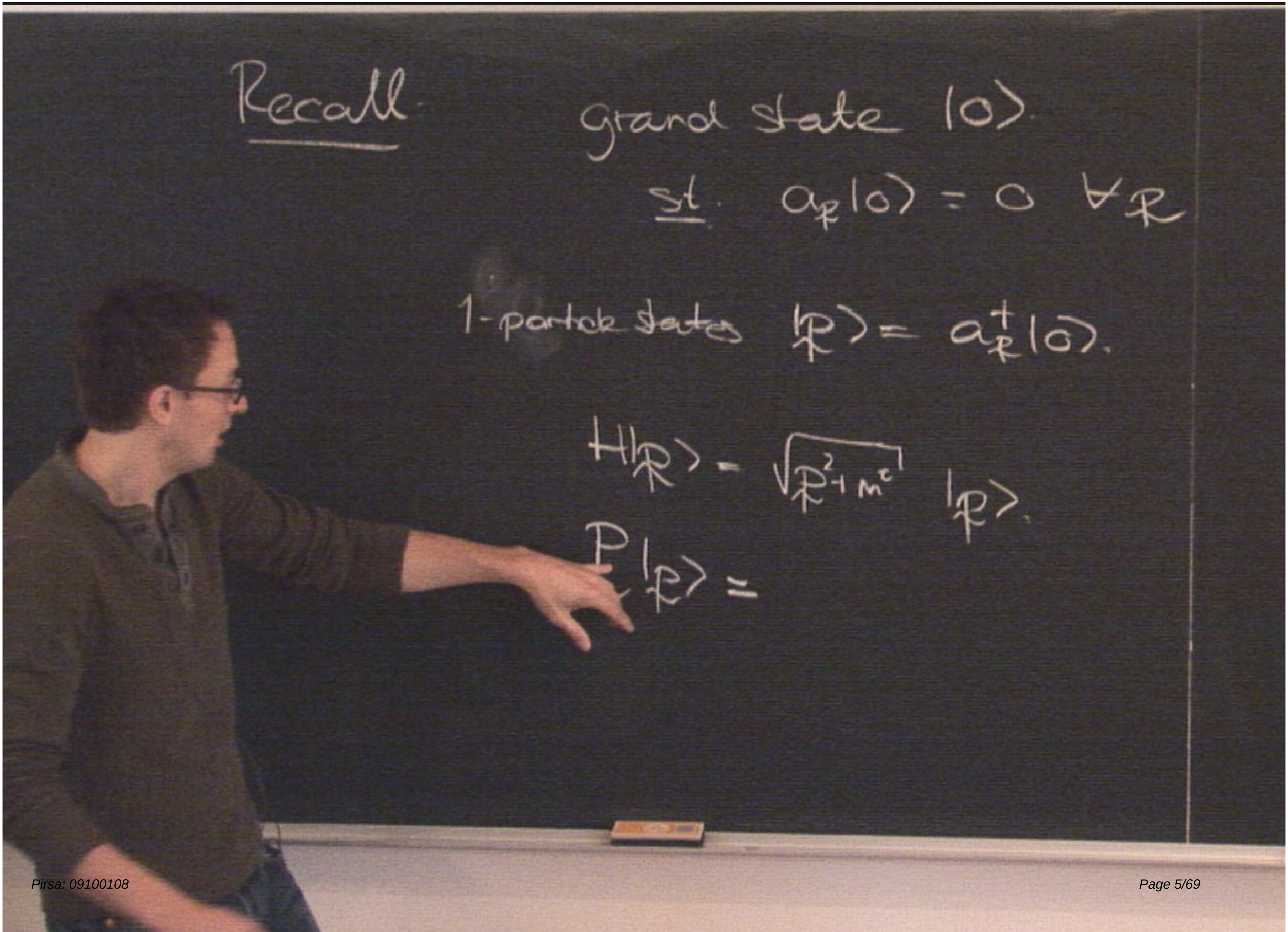
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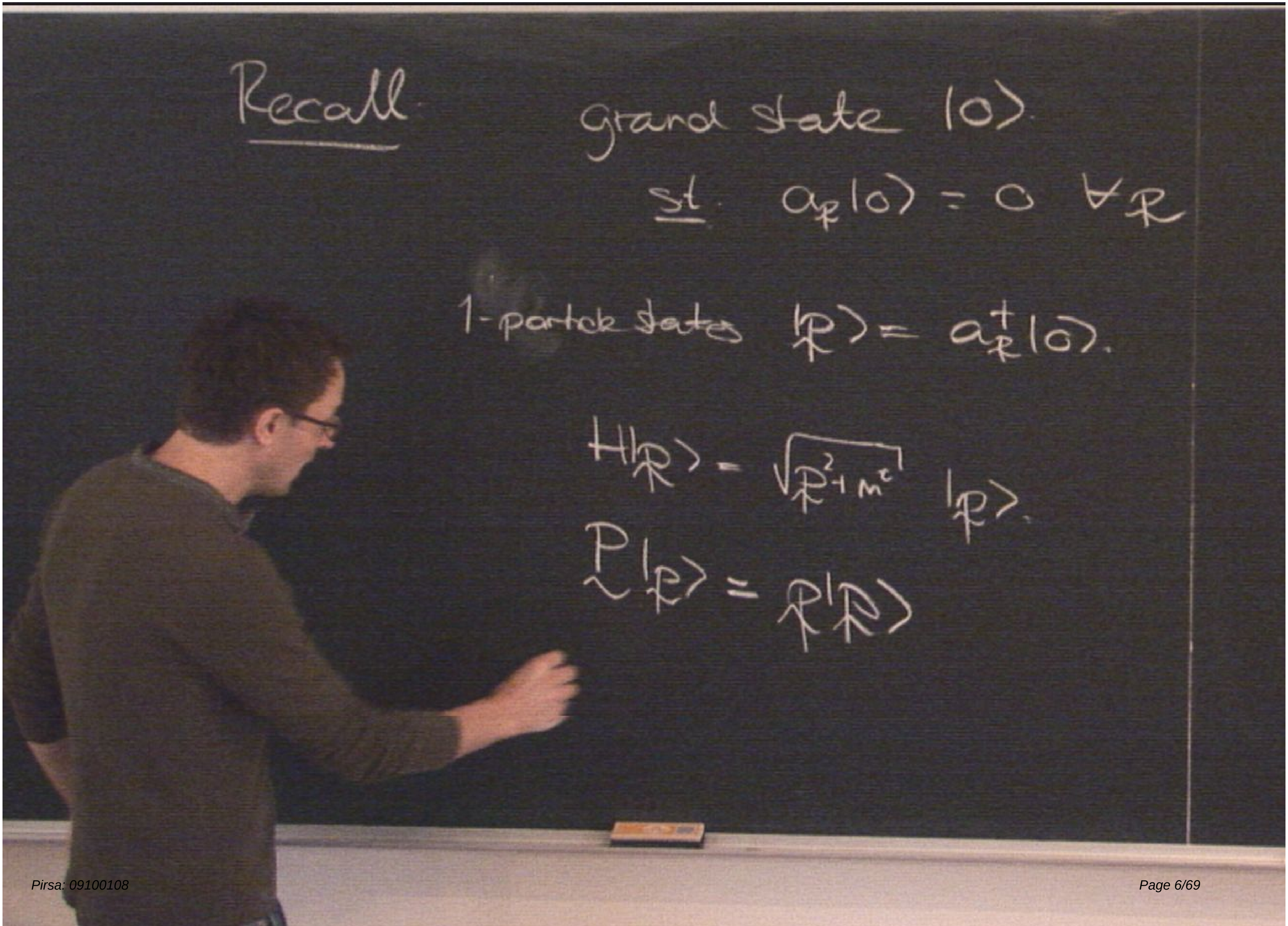
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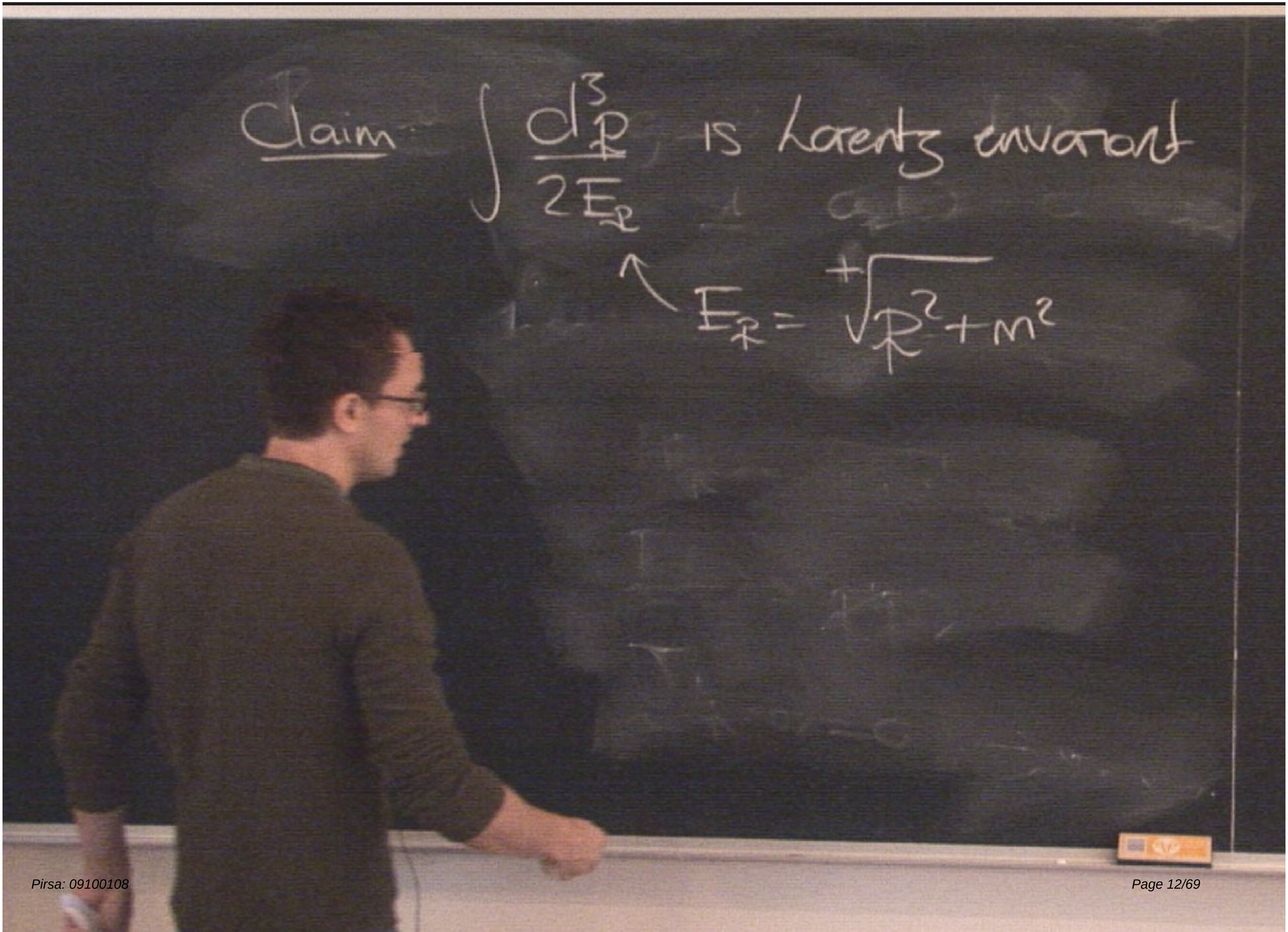
Problem: This $\delta$-fn is not Lorentz invariant.


Claim: $\int \frac{d^3 p}{2E_p}$ is Lorentz invariant
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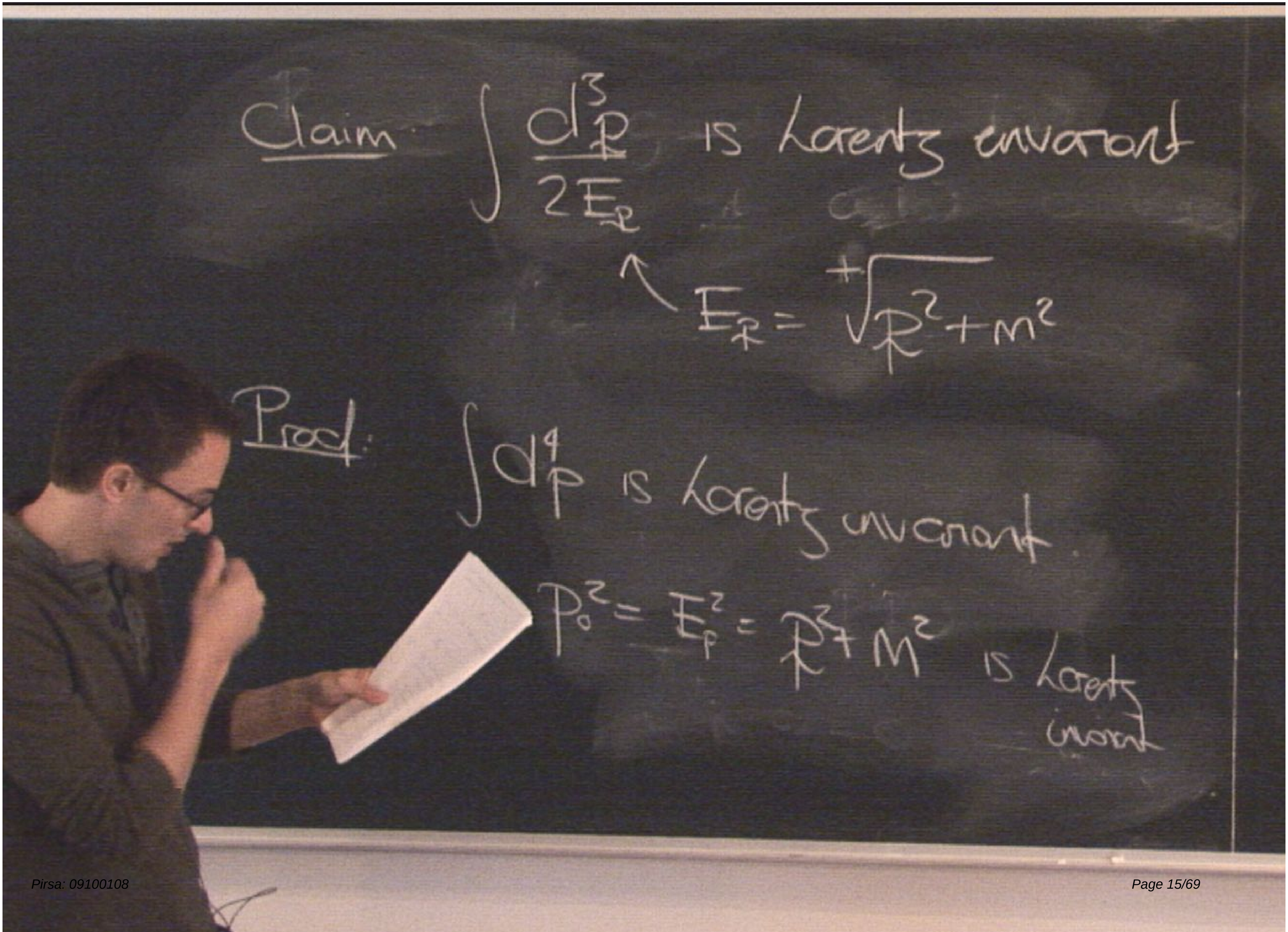
<u>Claim</u>: $\int \frac{d^3\vec{p}}{2E_{\vec{p}}}$ is Lorentz invariant

$$E_{\vec{p}} = +\sqrt{\vec{p}^{\,2}+m^2}$$

<u>Proof</u>: $\int d^4p$ is Lorentz invariant.

$$p_0^2 = E_p^2 = \vec{p}^{\,2} + m^2$$ is Lorentz invariant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$$\Rightarrow \int d^4p \; \delta\left(p_0^2 - \vec{p}^{\,2} - m^2\right)\Big|_{p_0 > 0.}$$

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$\square$.

Claim 2: $2E_p\,\delta^{(3)}(\underline{p}-\underline{q})$ is Lorentz invariant $\delta$-function

Proof. $\displaystyle\int \frac{d^3\underline{p}}{2E_p}\, 2E_p\,\delta^{(3)}(\underline{p}-\underline{q}) = 1$ $\square$

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Claim 2: $2E_p\, \delta^{(3)}(\vec{p} - \vec{q})$ is Lorentz invariant $\delta$-function

Proof. $\int \frac{d^3\vec{p}}{2E_p} \, 2E_p\, \delta^{(3)}(\vec{p} - \vec{q}) = 1$ $\square$

This means if we normalize our states as

$$|p\rangle = \sqrt{2E_{\vec{p}}}\,|\vec{p}\rangle$$

Then $$\langle p|q\rangle = (2\pi)^3\, 2E_p\, \delta^{(3)}(\vec{p}-\vec{q})$$

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We have the mode expansion

$$\psi(\underset{\sim}{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\underset{\sim}{p}}}} \left( b_{\underset{\sim}{p}} e^{+i\underset{\sim}{p}\cdot\underset{\sim}{x}} + c^{\dagger}_{\underset{\sim}{p}} e^{-i\underset{\sim}{p}\cdot\underset{\sim}{x}} \right)$$

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$$\pi = \partial \mathcal{L} / \partial \dot{\psi} = \dot{\psi}^{*}$$

$$\pi = \int \frac{d^3p}{(2\pi)^3}\, i \sqrt{\frac{E_p}{2}} \left( b_p^{+} e^{-i p \cdot x} - c_p e^{+i p \cdot x} \right)$$

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Commutation Relations

$$[\psi(x), \pi(x)] = i\delta^{(3)}(x-y)$$

all others zero.

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$$[c_p, c_q^+] = (2\pi)^3\delta^{(3)}(p-q)$$

$\Rightarrow$ we get two particles, each of mass $M$ and spin zero.

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The theory has a conserved charge

$$Q = i\int d^3x \; \dot{\psi}^* \psi - \psi^* \dot{\psi}$$

$$= i\int d^3x \; \pi \psi - \psi^* \pi^*$$

After normal ordering, we'll find

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= # c particles − # b particles

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$= \#$ c particles $-$ $\#$ b particles.

Note: $[Q, H] = 0$ and this will continue to hold in interacting theories. (with symmetry).

We interpret $b$ and $c$ as particles and anti-particles

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<u>Note</u>: For real scalar fields, the particle is its own anti-particle.

# The Heisenberg Picture

At the moment, our operators $\phi(\underline{x})$, depend on space, but not time. We can change this by going to the Heisenberg picture

$$O_H(t) = e^{iHt} O_S e^{-iHt}$$

Heisenberg

Schrödinger

$$\Rightarrow \frac{d O_H}{dt} = i[H, O]$$

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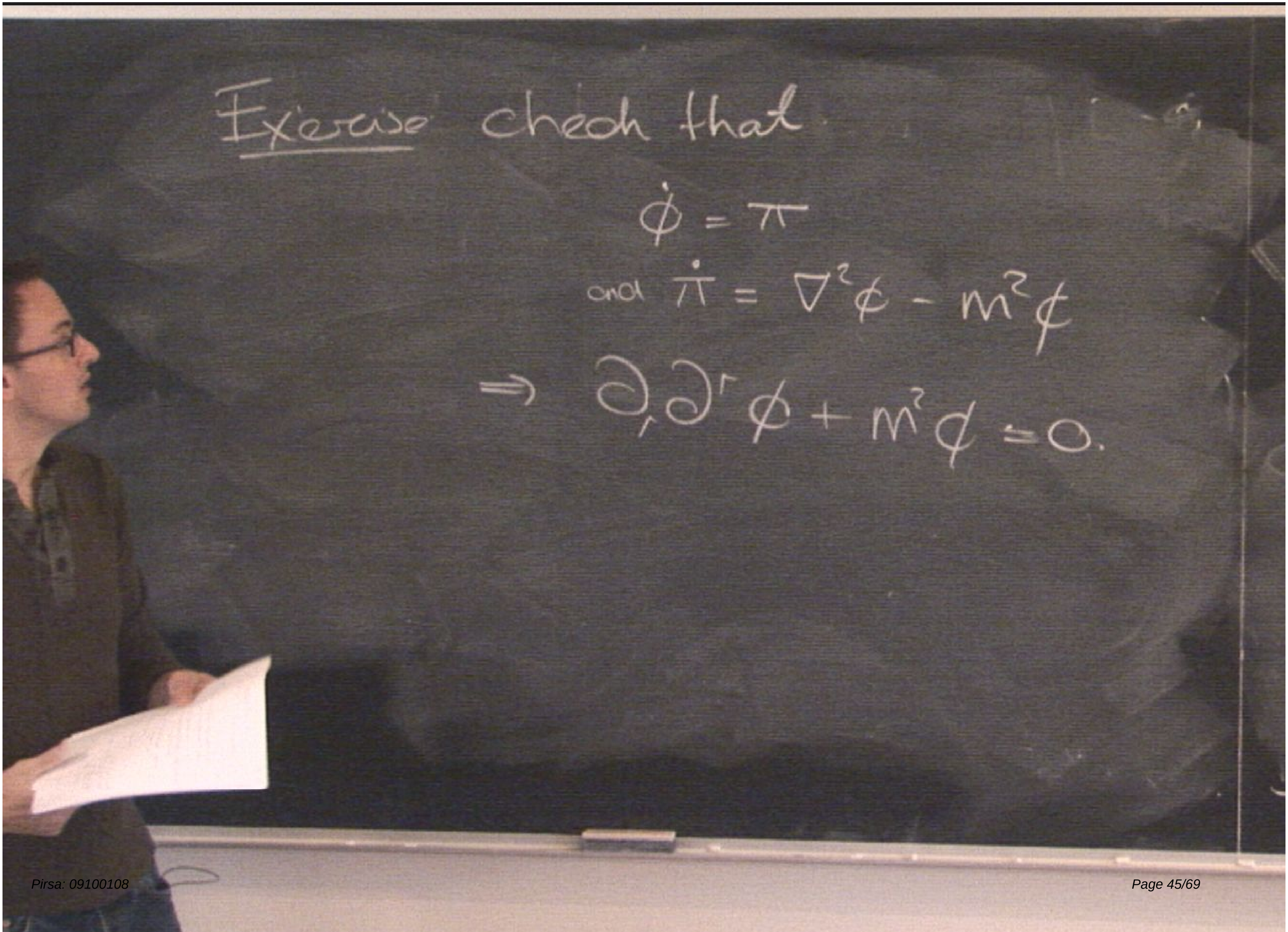
Exercise check that
$\dot{\phi} = \pi$
and $\dot{\pi} = \nabla^2\phi - m^2\phi$
$\Rightarrow \partial_\mu \partial^\mu \phi + m^2 \phi = 0.$

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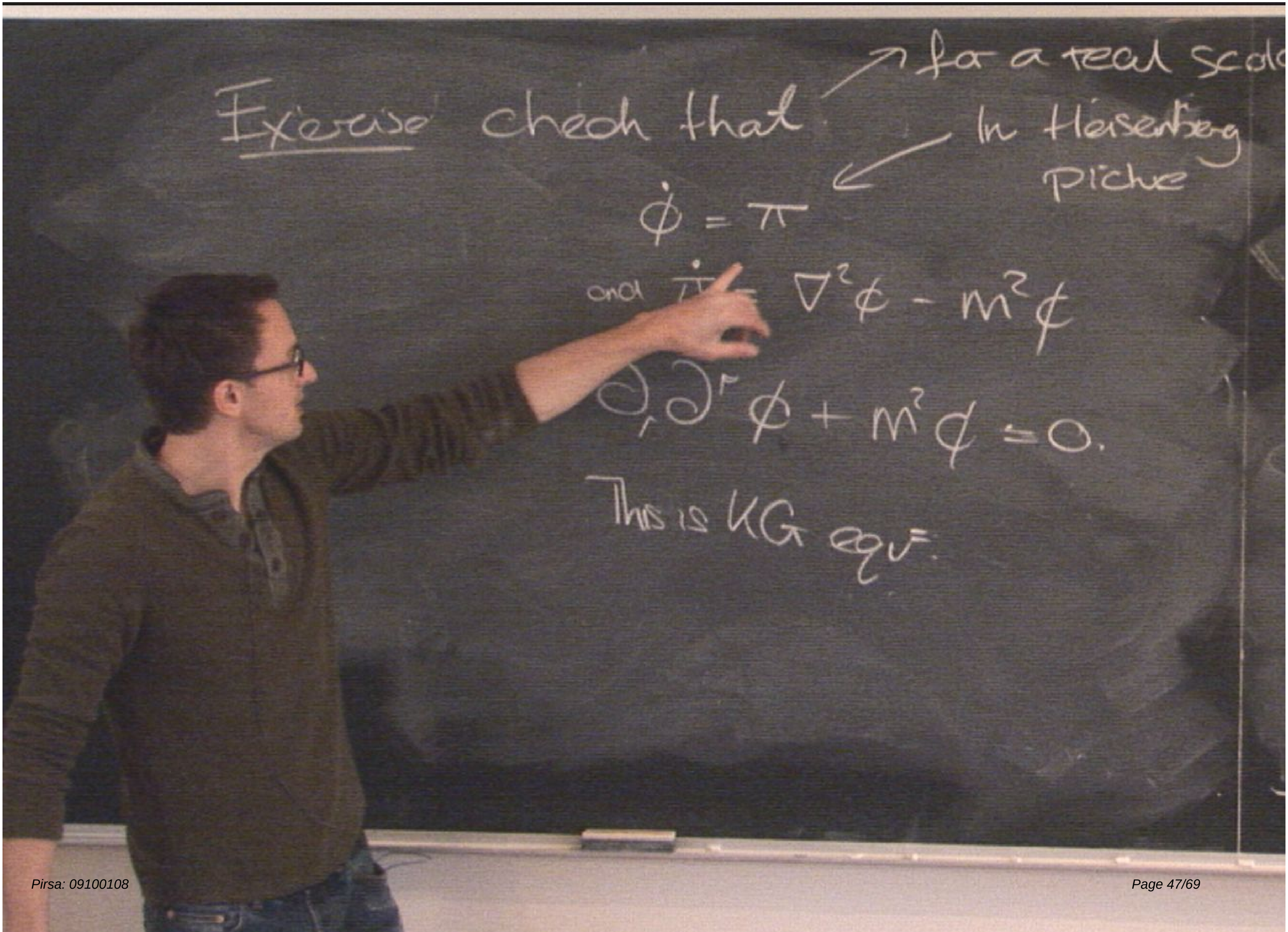
This is KG eqn.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for a real scalar

In Heisenberg picture

$O_H(t) = e^{iHt} O_S e$

Heisenberg

Sch

$\phi - m^2 \phi$

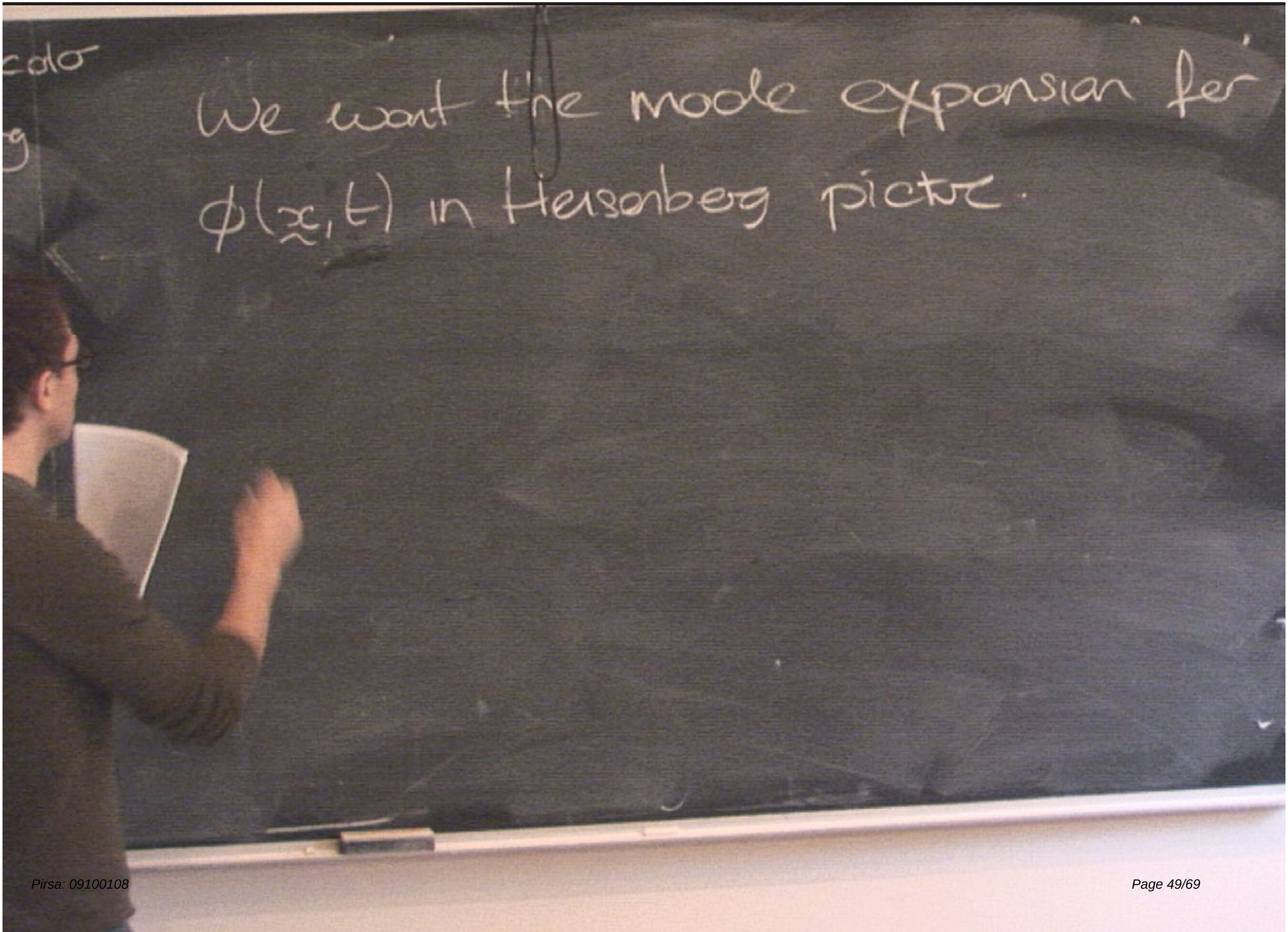
$- m^2 \phi = 0.$

$$\Rightarrow \frac{dO_H}{dt} = i[H, O_H]$$

$$\psi = \int b + c^\dagger$$

$$\boxed{\pi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = \dot{\psi}^*}$$

$$\pi = \int b^\dagger - c$$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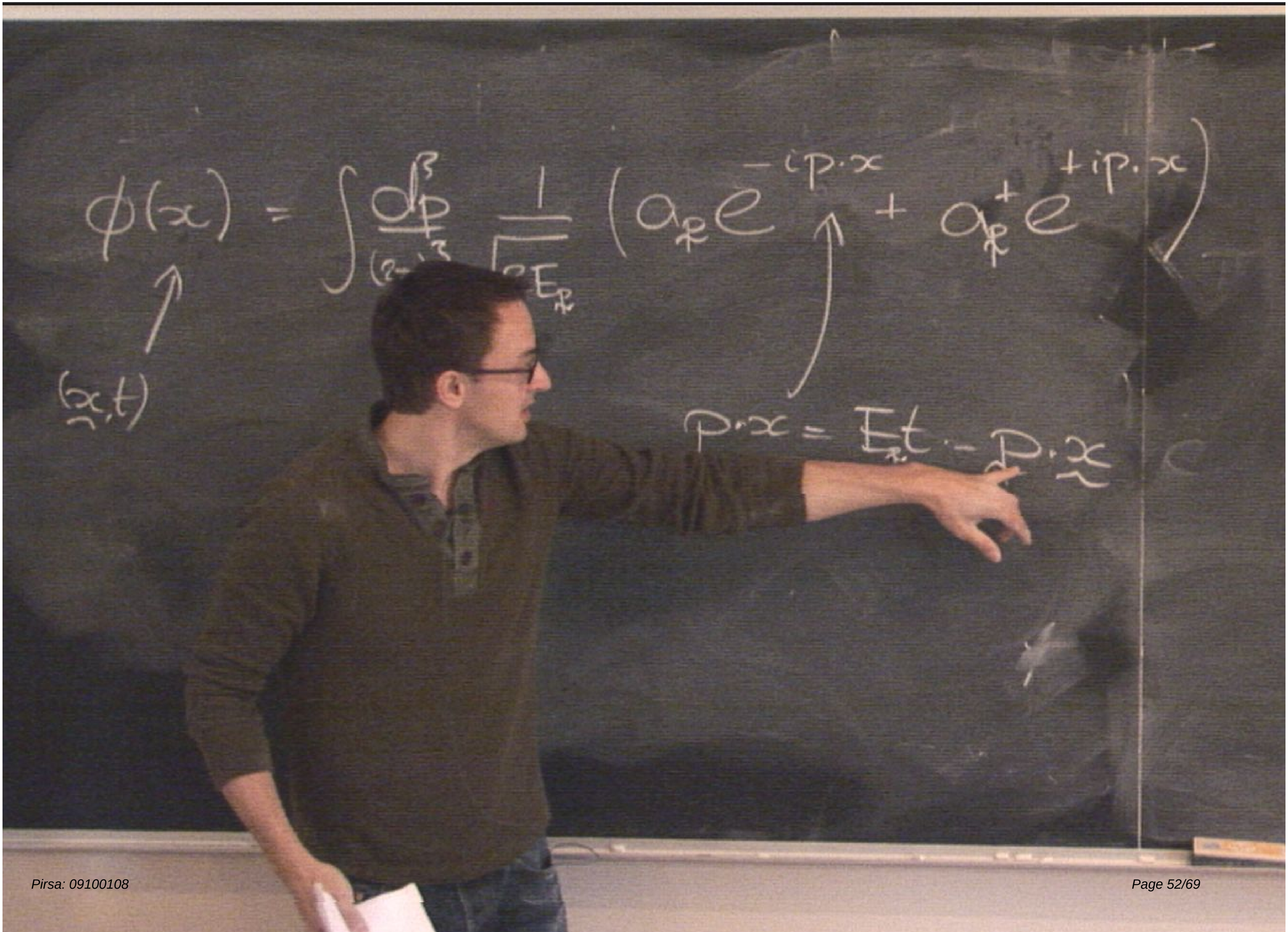
We need $$e^{iHt} a_{\underset{\sim}{k}} e^{-iHt} = a_{\underset{\sim}{k}} e^{-iE_p t}$$

and $$e^{iHt} a^{+}_{\underset{\sim}{k}} e^{-iHt} = a^{+}_{\underset{\sim}{k}} e^{-iE_p t}$$

$$\phi(x) = \int \frac{d^3 \underset{\sim}{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\underset{\sim}{p}}}} \left( a_{\underset{\sim}{p}} e^{-ip\cdot x} + a^{+}_{\underset{\sim}{p}} e^{+ip\cdot x} \right)$$

$\phi(x)$: $(\underset{\sim}{x}, t)$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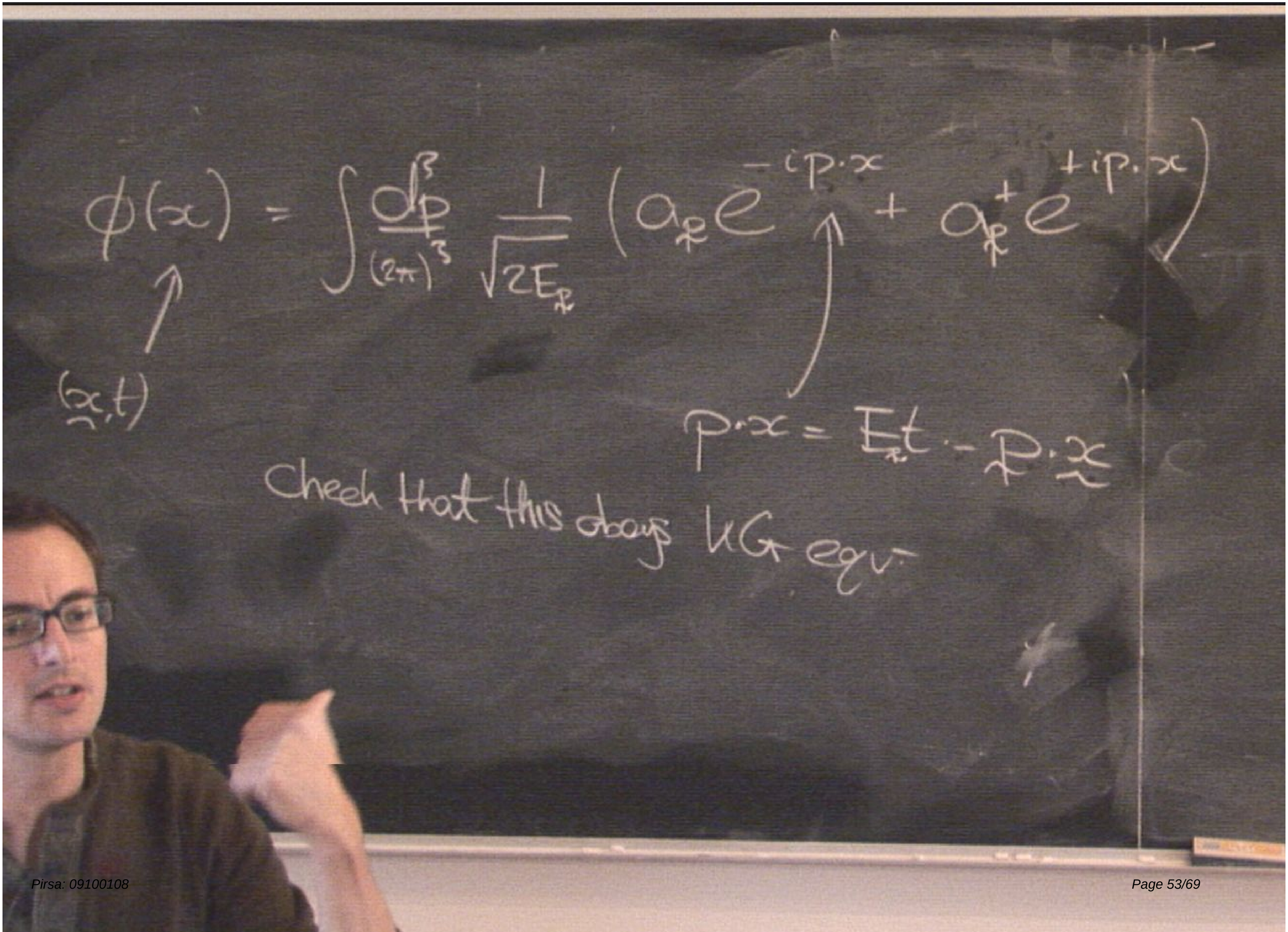
$(\underline{x}, t)$
$p\cdot x = E_{\underline{p}} t - \underline{p}\cdot\underline{x}$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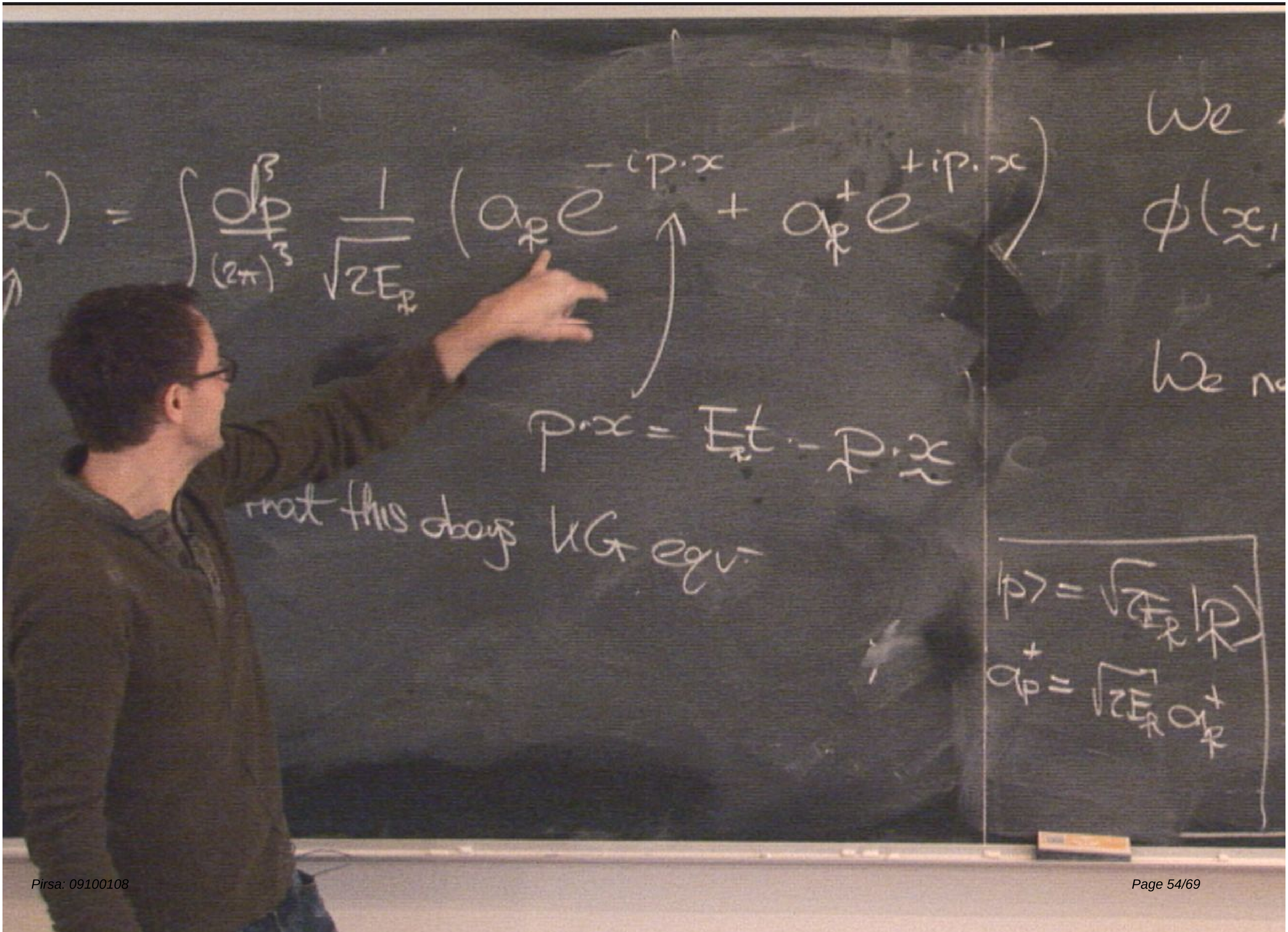
$(\underset{\sim}{x}, t)$
$p\cdot x = E_{\underset{\sim}{p}} t - \underset{\sim}{p}\cdot\underset{\sim}{x}$
Check that this obeys KG eqn.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that this obeys KG eqn.
We
$\phi(x,$
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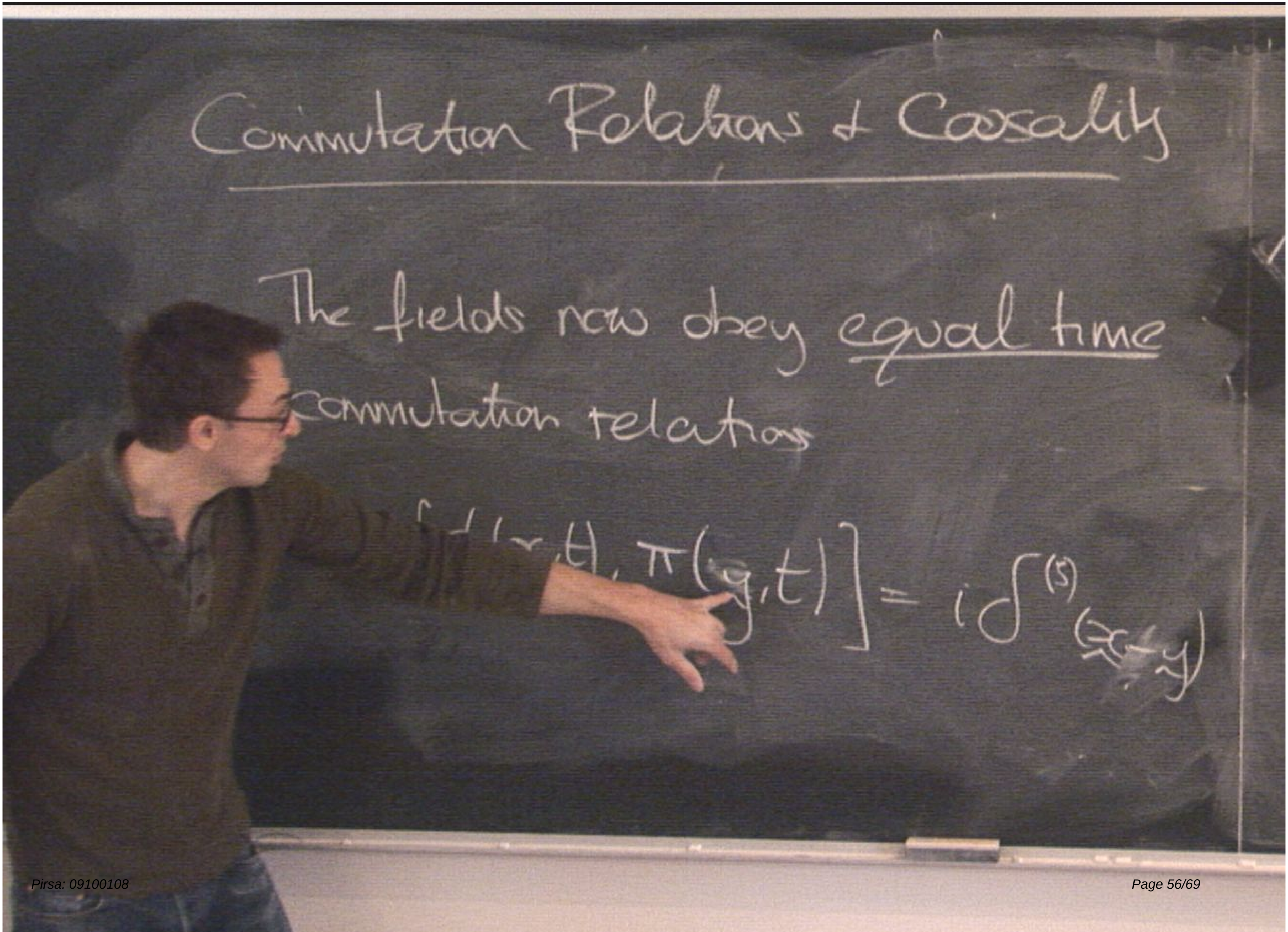
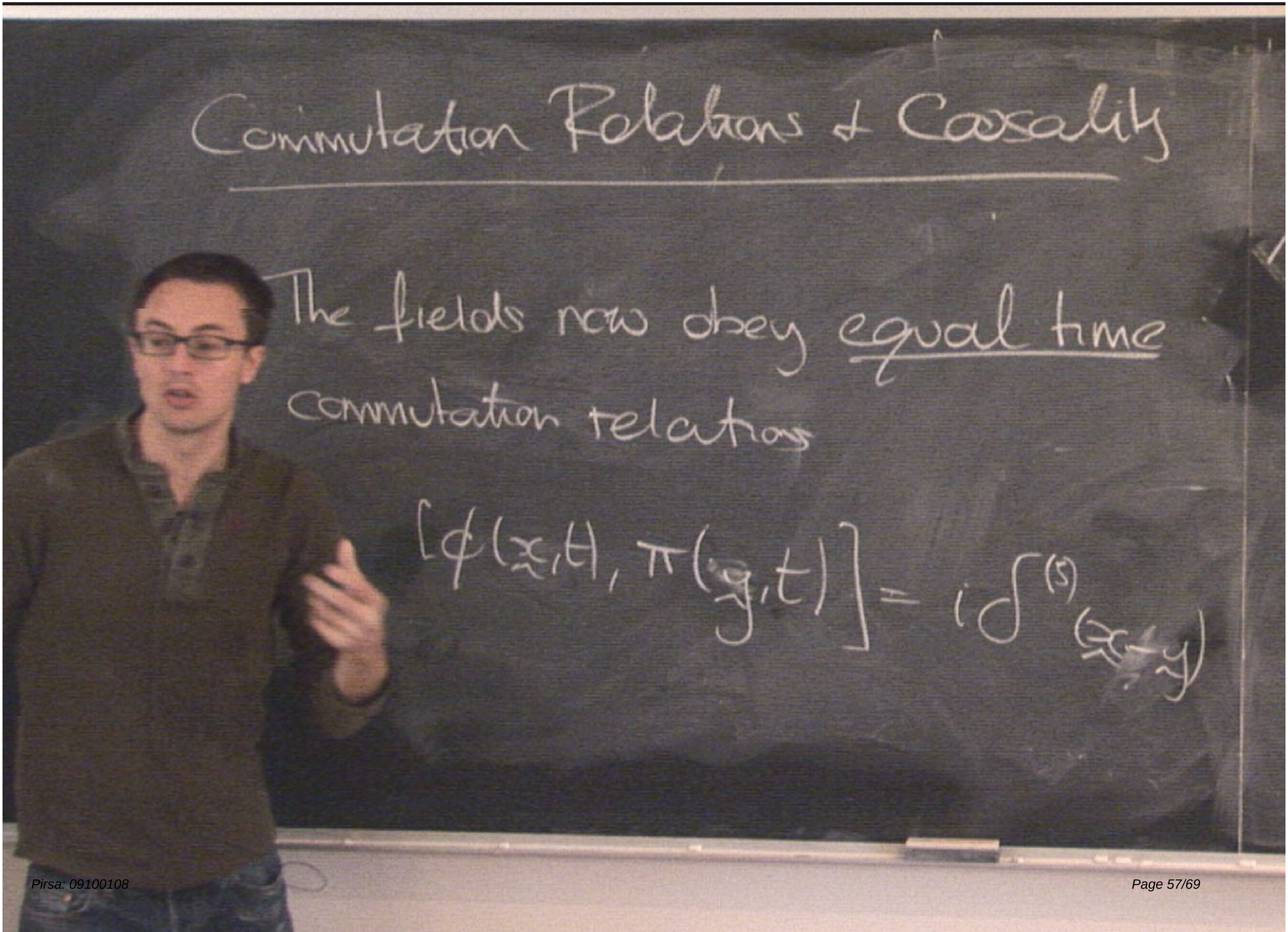
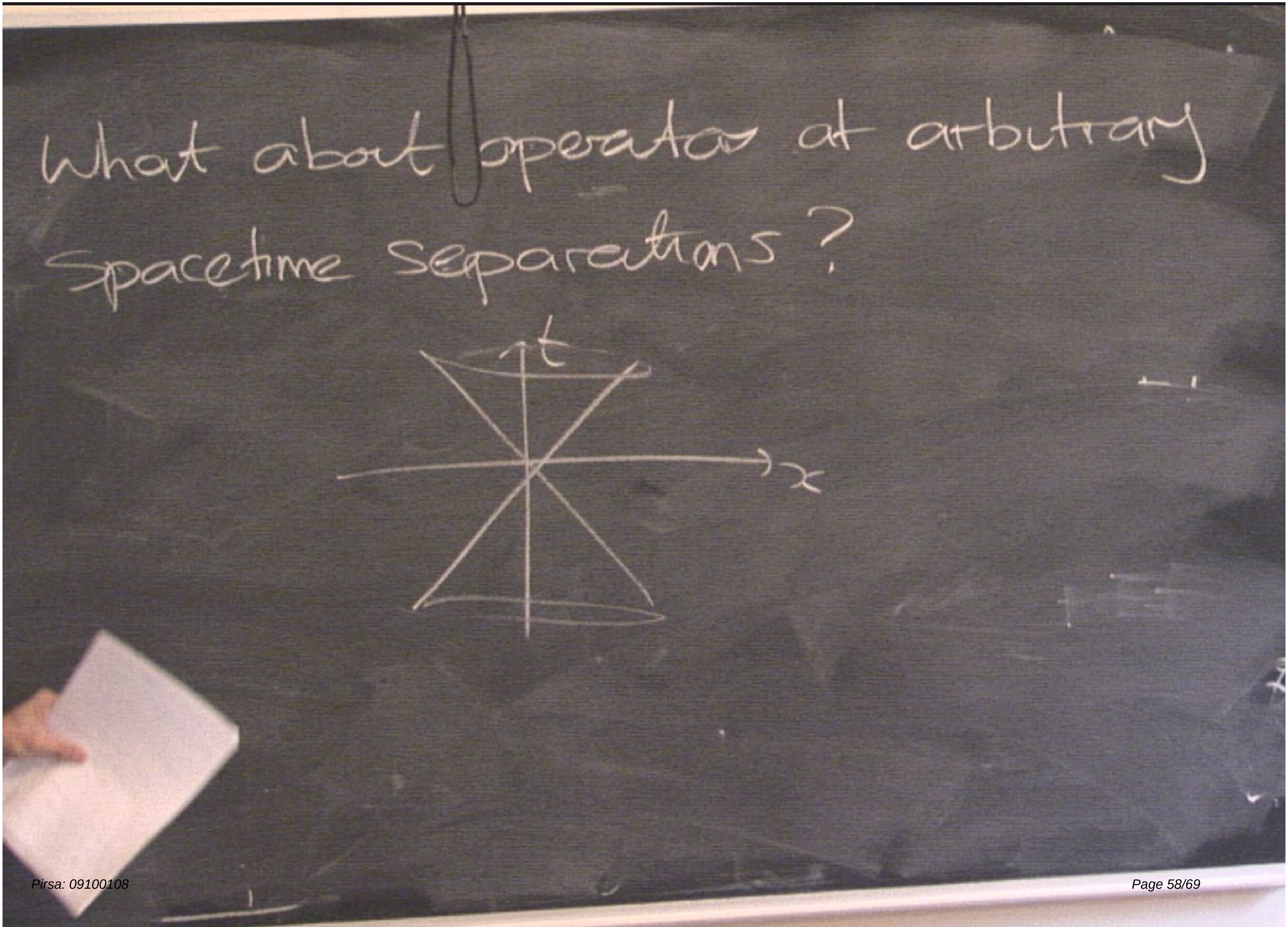
$a_p^+ = \sqrt{2E_p}\,a_p^+$

Commutation Relations & Causality


Commutation Relations & Causality
The fields now obey equal time commutation relations


Commutation Relations & Causality
The fields now obey equal time commutation relations
$[\phi(\underset{\sim}{x},t), \pi(\underset{\sim}{y},t)] = i\delta^{(3)}(\underset{\sim}{x}-\underset{\sim}{y})$


What about operators at arbitrary
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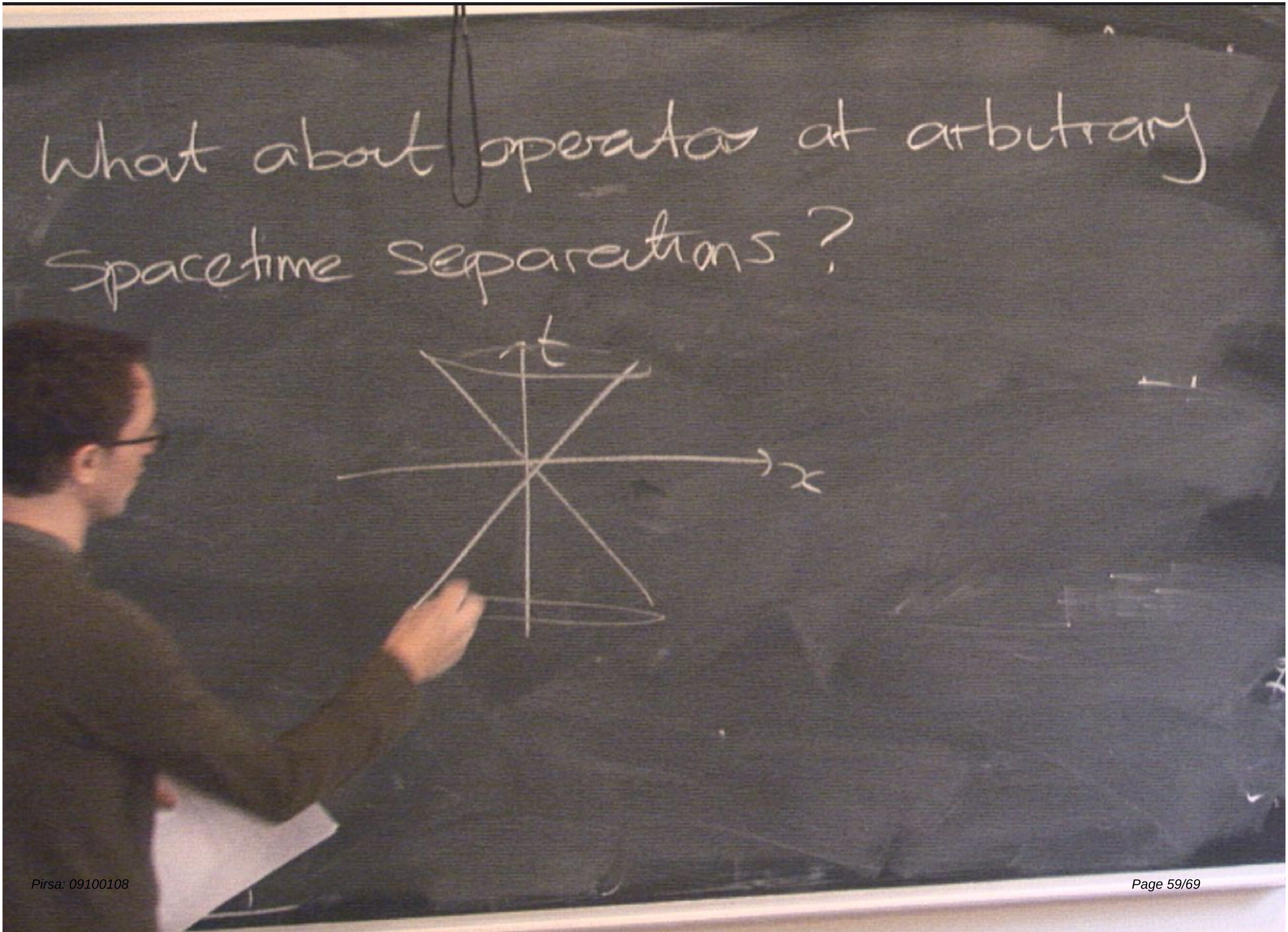
t
x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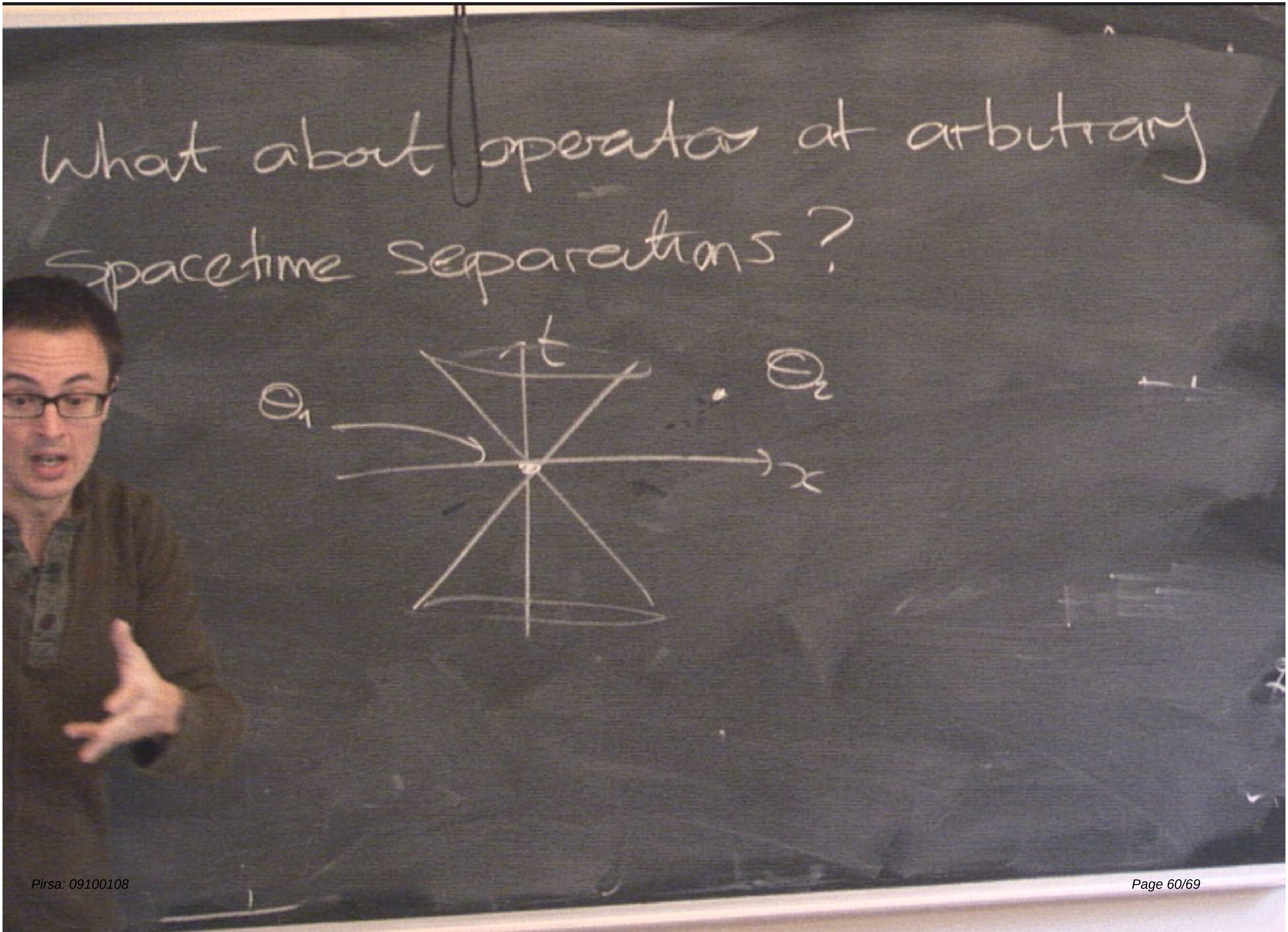
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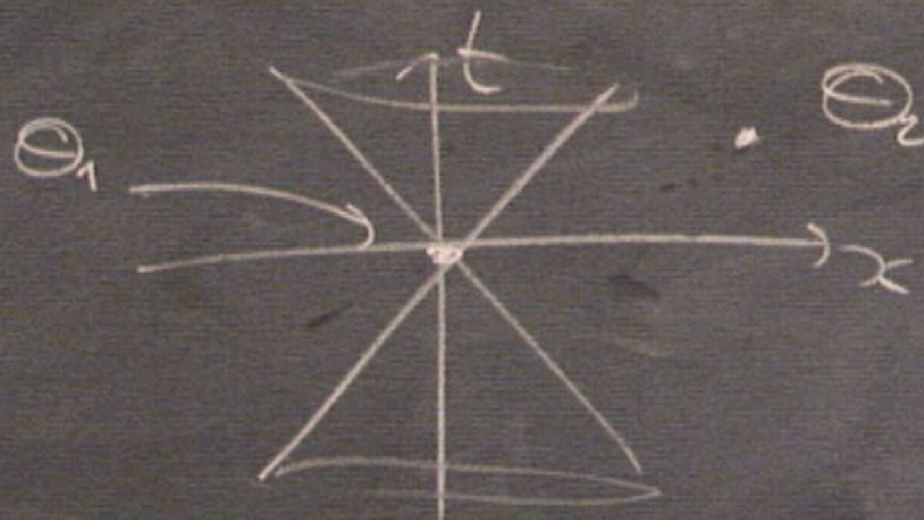


We would really like $[\mathcal{O}_1(x), \mathcal{O}_2(y)] = 0 \quad \forall \ (x^2-y^2) < 0$

What about operators at arbitrary spacetime separations?


We would really like $[\mathcal{O}_1(x), \mathcal{O}_2(y)] = 0 \quad \forall \ (x^2-y^2) < 0$

This ensures that a measurement at $y$ cannot signal to a measurement at $x$

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(we know $[\phi(\underline{x},t), \phi(\underline{y},t)] = 0$.)

Define $\Delta(x-y) = [\phi(x), \phi(y)]$

One can check that

$$\Delta(x-y) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \left( e^{-ip\cdot(x-y)} - e^{+ip\cdot(x-y)} \right).$$

One can check that.

$$\Delta(x-y) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \left( e^{-ip\cdot(x-y)} - e^{+ip\cdot(x-y)} \right)$$

Note... The RHS is a c-number function.

Lorentz invariant.

It doesn't vanish

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Note: The RHS is a c-number function.

- It's Lorentz invariant.
- It doesn't vanish for time-like separation

e.g. $(x-y) \sim (t, 0, 0, d)$

$$[\phi(\underset{\sim}{x}, d), \phi(\underset{\sim}{x}, t)] \sim (e^{-imt} - e^{+imt})$$

$$\neq 0$$

- It does vanish at spacelike separation.

e.g. $(x-y) \sim (t,0,0,d)$

$$[\phi(\underset{\sim}{x},d),\ \phi(\underset{\sim}{x},t)] \sim (e^{-imt} - e^{+imt})$$

$$\neq 0$$

- It does vanish at spacelike separation.

It vanishes at equal times, and is Lorentz invariant

$\Rightarrow$ theory is causal!