Title: Quantum Field Theory (PHYS 601) - Lecture 4

Date: Oct 01, 2009 09:00 AM

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Abstract:


Free Field Theory.

Free Field Theory

$$H = \frac{1}{2}\int d^3x \; \pi^2 + (\nabla\phi)^2 + m^2\phi^2$$

$$[\phi(\underset{\sim}{x}), \pi(\underset{\sim}{y})]$$

Free Field Theory

$$H = \frac{1}{2}\int d^3x \; \pi^2 + (\nabla\phi)^2 + m^2\phi^2$$

$$[\phi(\underset{\sim}{x}), \pi(\underset{\sim}{y})] = i\delta^{(3)}(\underset{\sim}{x} - \underset{\sim}{y})$$

Simple Harmonic Oscillater (Again!)

$$H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2 \quad \text{with} \quad [q,p] = i$$

write

$$\left.\begin{aligned} a &= \frac{i}{\sqrt{2\omega}}p + \sqrt{\frac{\omega}{2}}\,q \\ a^+ &= \frac{-i}{\sqrt{2\omega}}p + \sqrt{\frac{\omega}{2}}\,q \end{aligned}\right\} \Longrightarrow \begin{aligned} q &= \frac{1}{\sqrt{2\omega}}(a + a^+) \\ p &= -i\sqrt{\frac{\omega}{2}}\,(a - a^+) \end{aligned}$$

$$\Longrightarrow H = \omega\left(aa^+ + \frac{1}{2}\right)$$

We'll define $a_{\vec{k}}$ and $a^\dagger_{\vec{k}}$.

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$$\phi(\underset{\sim}{x}) = \int \frac{d^3 k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\underset{\sim}{k}}}} \left[a_{\underset{\sim}{k}} e^{i \underset{\sim}{k} \cdot \underset{\sim}{x}} + a^{\dagger}_{\underset{\sim}{k}} e^{-i \underset{\sim}{k} \cdot \underset{\sim}{x}} \right]$$

$$\pi(\underset{\sim}{x}) = \int \frac{d^3 k}{(2\pi)^3} (-i) \sqrt{\frac{\omega_k}{2}} \left[a_{\underset{\sim}{k}} e^{i \underset{\sim}{k} \cdot \underset{\sim}{x}} - a^{\dagger}_{\underset{\sim}{k}} e^{-i \underset{\sim}{k} \cdot \underset{\sim}{x}} \right]$$

We'll define $a_{\underset{\sim}{p}}$ and $a_{\underset{\sim}{p}}^{\dagger}$.

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operators

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$$\boxed{\omega_{\mathbf{k}}^2 = \mathbf{k}^2 + M^2}$$

We'll define $a_{\underline{k}}$ and $a^\dagger_{\underline{k}}$.

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$$\boxed{\omega_p = +\sqrt{\underline{k}^2 + M^2}}$$

We'll define a_k and $a^\dagger_k$.

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_k}} \left[a_k e^{ik\cdot x} + a^\dagger_k e^{-ik\cdot x} \right]$$

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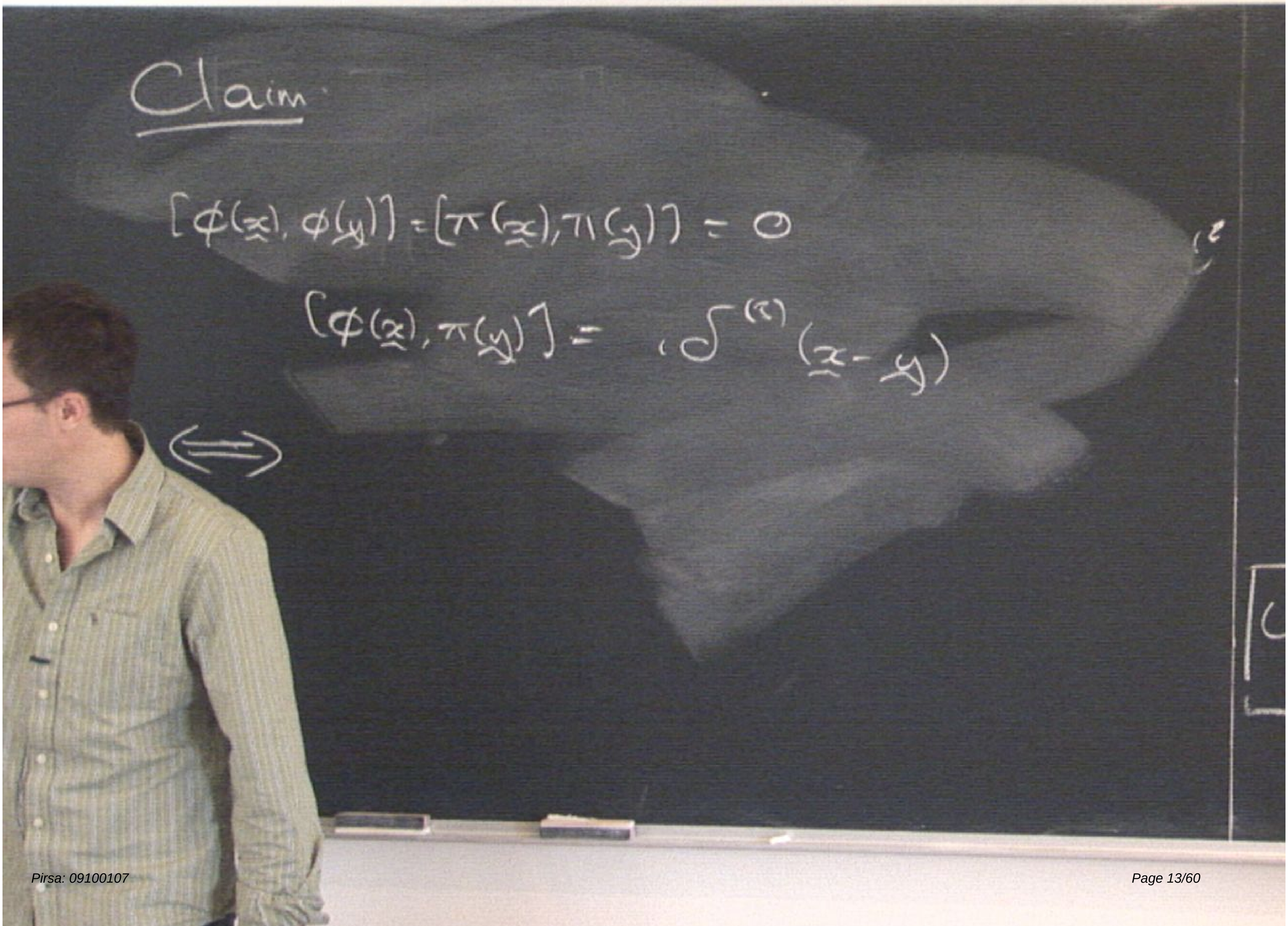
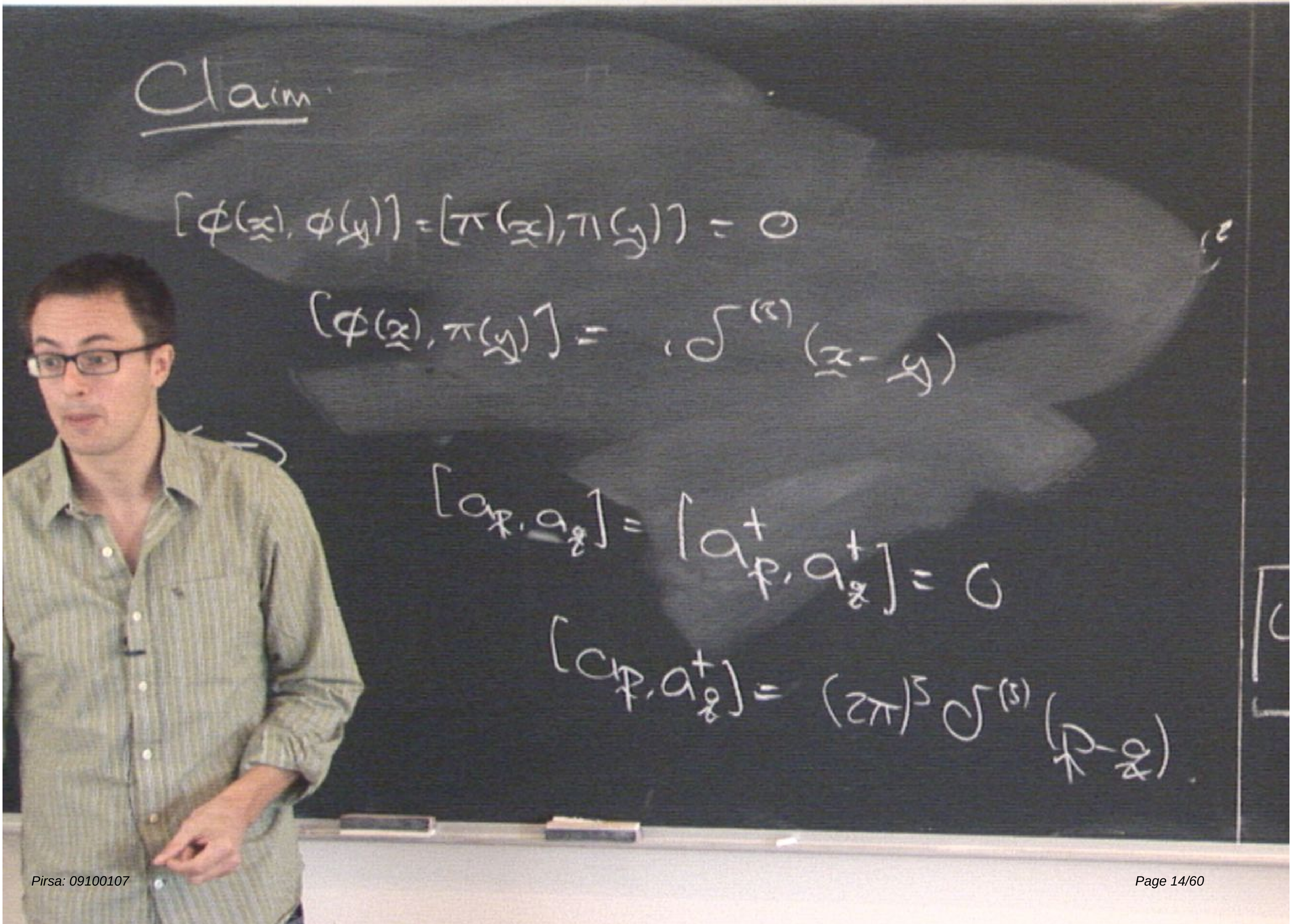
$$\boxed{\omega_p = +\sqrt{\underset{\sim}{p}^2 + M^2}}$$

This has $\phi^\dagger = \phi$ and $\pi^\dagger = \pi$.


Claim:
$[\phi(\underline{x}), \phi(\underline{y})] = [\pi(\underline{x}), \pi(\underline{y})] = 0$
$[\phi(\underline{x}), \pi(\underline{y})] = i\delta^{(3)}(\underline{x} - \underline{y})$
$\Longleftrightarrow$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$[\phi(\underline{x}), \phi(\underline{y})] = [\pi(\underline{x}), \pi(\underline{y})] = 0$
$[\phi(\underline{x}), \pi(\underline{y})] = i\,\delta^{(3)}(\underline{x} - \underline{y})$
$[a_{\underline{p}}, a_{\underline{q}}] = [a^{\dagger}_{\underline{p}}, a^{\dagger}_{\underline{q}}] = 0$
$[a_{\underline{p}}, a^{\dagger}_{\underline{q}}] = (2\pi)^3\,\delta^{(3)}(\underline{p} - \underline{q})$

Claim:

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$$\Longleftrightarrow$$

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Proof: Exercise.

$$H = \frac{1}{2}\int d^3x \; \pi^2 + (\nabla\phi)^2 + m^2\phi^2$$

$$= \frac{1}{2}\int \frac{d^3x\, d^3p\, d^3q}{(2\pi)^6}$$

$$H = \frac{1}{2}\int d^3x \; \pi^2 + (\nabla\phi)^2 + m^2\phi^2$$

$$= \frac{1}{2}\int \frac{d^3x \, d^3p \, d^3q}{(2\pi)^6} \left[-\frac{\sqrt{\omega_p \omega_q}}{2} \left(a_{\underline{p}} e^{i\underline{p}\cdot\underline{x}} - a^\dagger_{\underline{p}} e^{-i\underline{p}\cdot\underline{x}}\right) \right.$$

$$\left(a_{\underline{q}} e^{i\underline{q}\cdot\underline{x}} - a^\dagger_{\underline{q}} e^{-i\underline{q}\cdot\underline{x}}\right)$$

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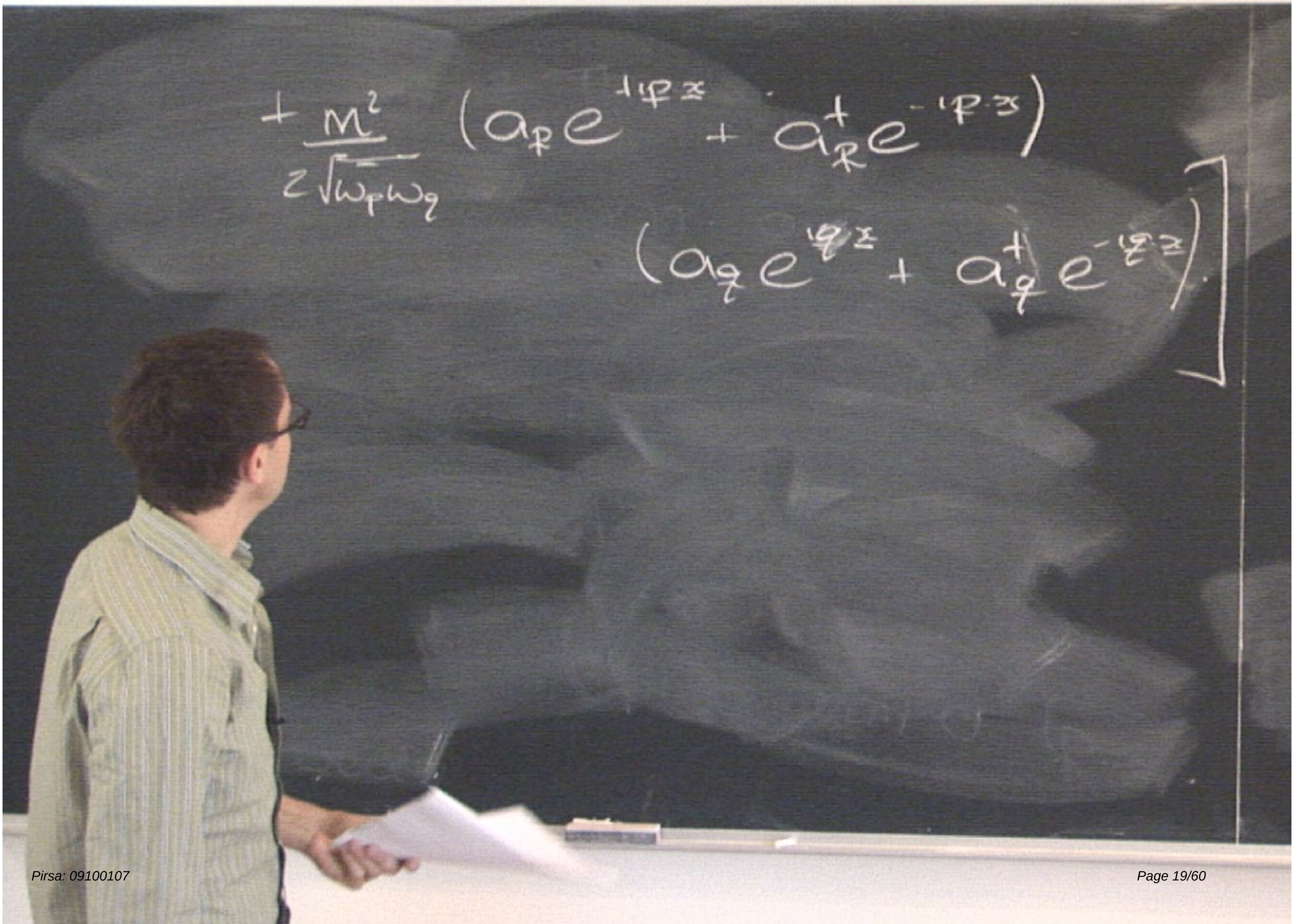
$$+ \frac{1}{2\sqrt{\omega_p\omega_q}}\left(i\mathbf{p}\, a_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}} - i\mathbf{p}\, a^{\dagger}_{\mathbf{p}} e^{-i\mathbf{p}\cdot\mathbf{x}}\right)\left(i\mathbf{q}\, a_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{x}} - i\mathbf{q}\, a^{\dagger}_{\mathbf{q}} e^{-i\mathbf{q}\cdot\mathbf{x}}\right)$$


$+ \frac{M^2}{2\sqrt{\omega_p \omega_q}} \left(a_p e^{ip\cdot x} + a_p^\dagger e^{-ip\cdot x}\right)$
$\left(a_q e^{iq\cdot x} + a_q^\dagger e^{-iq\cdot x}\right)]$

$$+\frac{m^2}{2\sqrt{\omega_p\omega_q}}\left(a_p e^{+ip\cdot x} + a_p^\dagger e^{-ip\cdot x}\right)$$

$$\left(a_q e^{iq\cdot x} + a_q^\dagger e^{-iq\cdot x}\right)\Big]$$

$$= \frac{1}{4}\int \frac{d^3p}{(2\pi)^3}\frac{1}{\omega_p}\Big[\left(-\omega_p^2 + p^2 + m^2\right)\left(a_p a_{-p} + a_p^\dagger a\right.$$

$$+\left(\omega_p^2 + p^2 + m^2\right)\left(a_p a_p^\dagger + a_p^\dagger a\right.$$

$$+\frac{m^2}{2\sqrt{\omega_p\omega_q}}\left(a_p e^{ip\cdot x} + a_p^\dagger e^{-ip\cdot x}\right)$$

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$$+ \left(\omega_p^2 + p^2 + m^2\right)\left(a_p a_p^\dagger + a_p^\dagger a_p\right)\Big]$$

$$+\frac{m^2}{2\sqrt{\omega_p \omega_q}}\left(a_p e^{ip\cdot x} + a_p^\dagger e^{-ip\cdot x}\right)$$

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$$+\left(\omega_p^2 + p^2 + m^2\right)\left(a_p a_p^\dagger + a_p^\dagger a\ldots\right.$$

but $\boxed{\omega_p^2 = p^2 + m^2}$

$$\Rightarrow H = \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\, \omega_p \left(a_p a_p^\dagger + a_p^\dagger a_p \right).$$

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$$= \int \frac{d^3p}{(2\pi)^3}\,\omega_p\left(a_p^\dagger a_p + \frac{1}{2}(2\pi)^3 \delta^{(3)}(0)\right).$$

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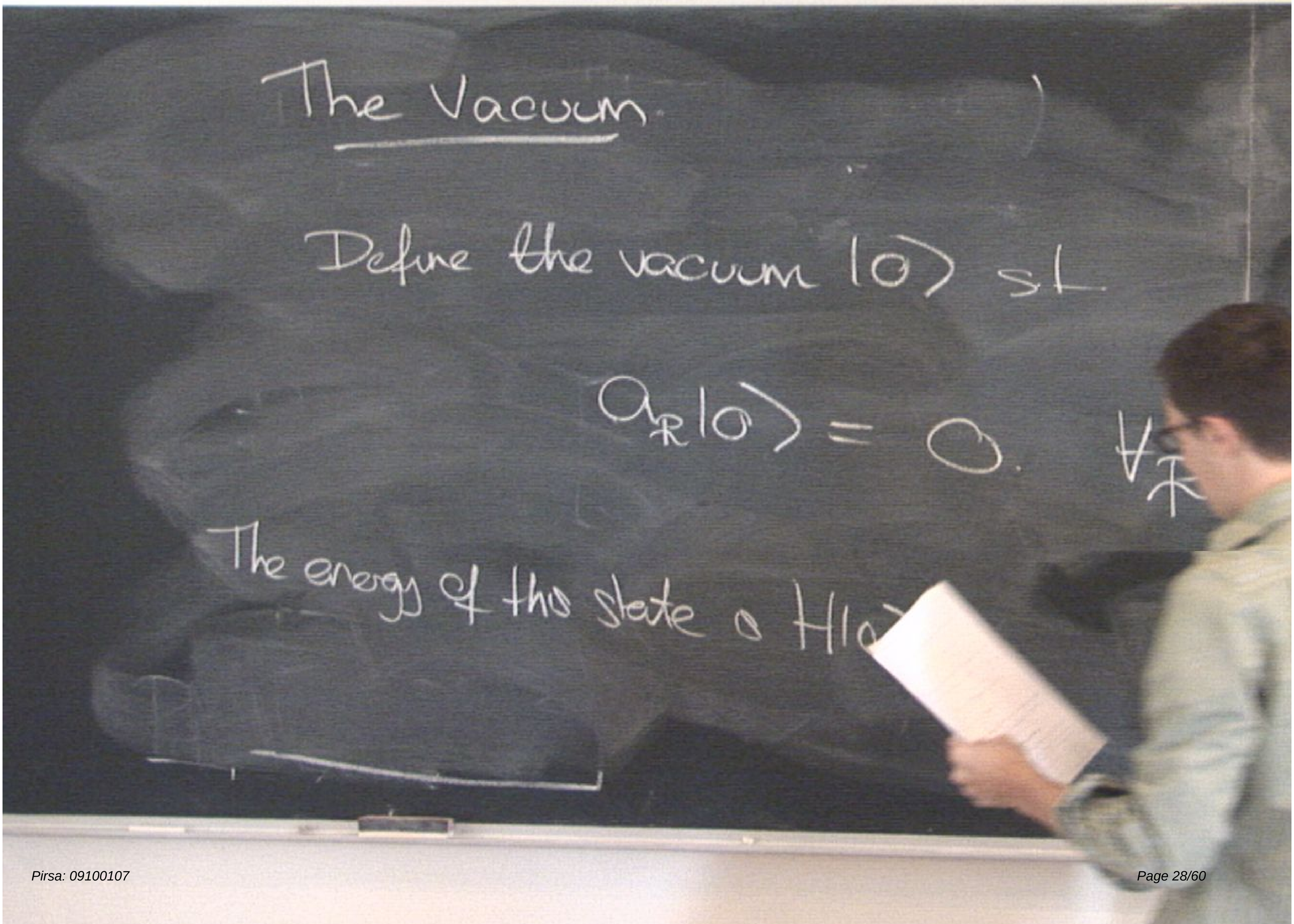
$$= \int \frac{d^3p}{(2\pi)^3}\, \omega_p \left(a_p^\dagger a_p + \frac{1}{2} (2\pi)^3 \times \delta^{(3)}(0) \right).$$

$$\boxed{[a_p, a_q^\dagger] = (2\pi)^3 \delta^{(3)}(p-q)}$$

The Vacuum

Define the vacuum $|0\rangle$ s.t.

$$a_k|0\rangle = 0 \quad \forall k.$$


The Vacuum
Define the vacuum |0⟩ s.t.
a_k|0⟩ = 0 ∀k
The energy of this state is H|0

Vacuum.

fine the vacuum $|0\rangle$ s.t.

$$a_{\vec{k}}|0\rangle = 0 \quad \forall \vec{k}$$

gy of the state: $H|0\rangle = \left[\int d^3p \, \omega_p \, \delta^{(3)}(0)\right]|0\rangle$

There are two infinities in this expression. The first is because space is big.

$(k_3)(0)\,]\,|0\rangle$

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Consider a box of size L.

$$(2\pi)^3 \delta^{(3)}(0) = \lim_{L\to\infty} \int_{-L/2}^{L/2} d^3x \, e^{i p\cdot x}\Big|_{p=0}$$

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$$(2\pi)^3 \delta^{(3)}(0) = \lim_{L\to\infty} \int_{-L/2}^{L/2} d^3x \, e^{i\mathbf{p}\cdot\mathbf{x}} \Big|_{\mathbf{p}=0} = \text{Volume of box.}$$

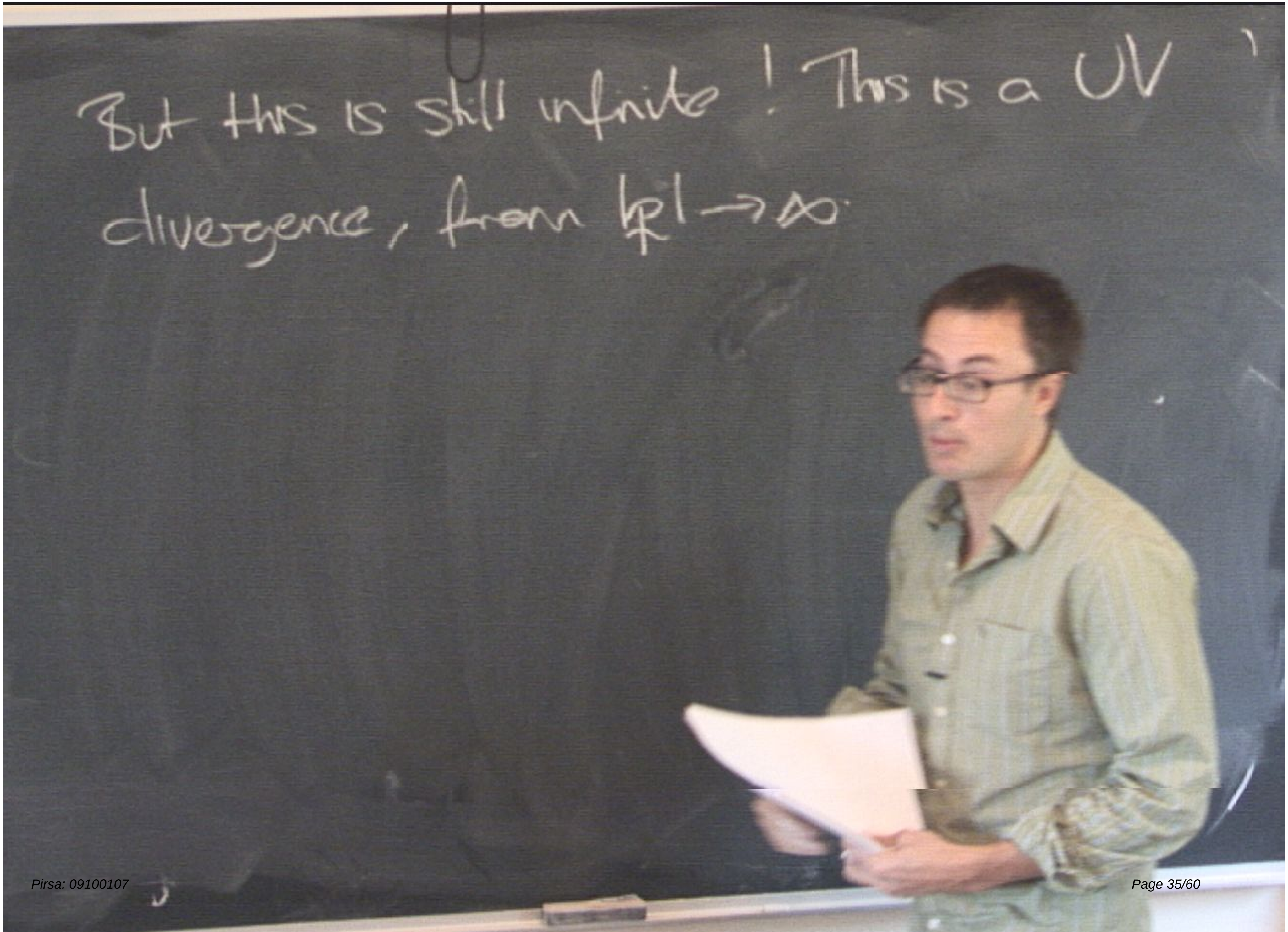
$$^{(3)}(0) \Big] |0\rangle$$

So grand state energy is

$$E_0 = V \int \frac{d^3p}{(2\pi)^3} \frac{\omega_p}{2}$$

we should work with energy densities

$$\mathcal{E}_0 = E_0/V = \int \frac{d^3p}{(2\pi)^3} \frac{\omega_p}{2}$$


But this is still infinite! This is a UV
divergence, from |k| → ∞.

energy densities

$$\int \frac{d^3p}{(2\pi)^3} \frac{\omega_p}{2} \qquad \omega_p = +\sqrt{p^2+M^2}.$$

But this is still infinite! This is a <u>UV divergence</u>, from $|\underline{k}| \to \infty$. This arises because we've assumed our theory is valid on arbitrarily short distance scales.

$+M^2$,

For this particular infinity, we are only interested in energy differences, so lets just remove this infinity and redefine

$$H = \int \frac{d^3p}{(2\pi)^3} \, \omega_p \, a_p^\dagger a_p$$

Simple Harmonic Oscillater

$$H = \tfrac{1}{2}(\omega q - ip)(\omega q + ip).$$

$$H = \tfrac{1}{2}p^2 + \tfrac{1}{2}\omega^2 q^2 \quad \text{with } [q,$$

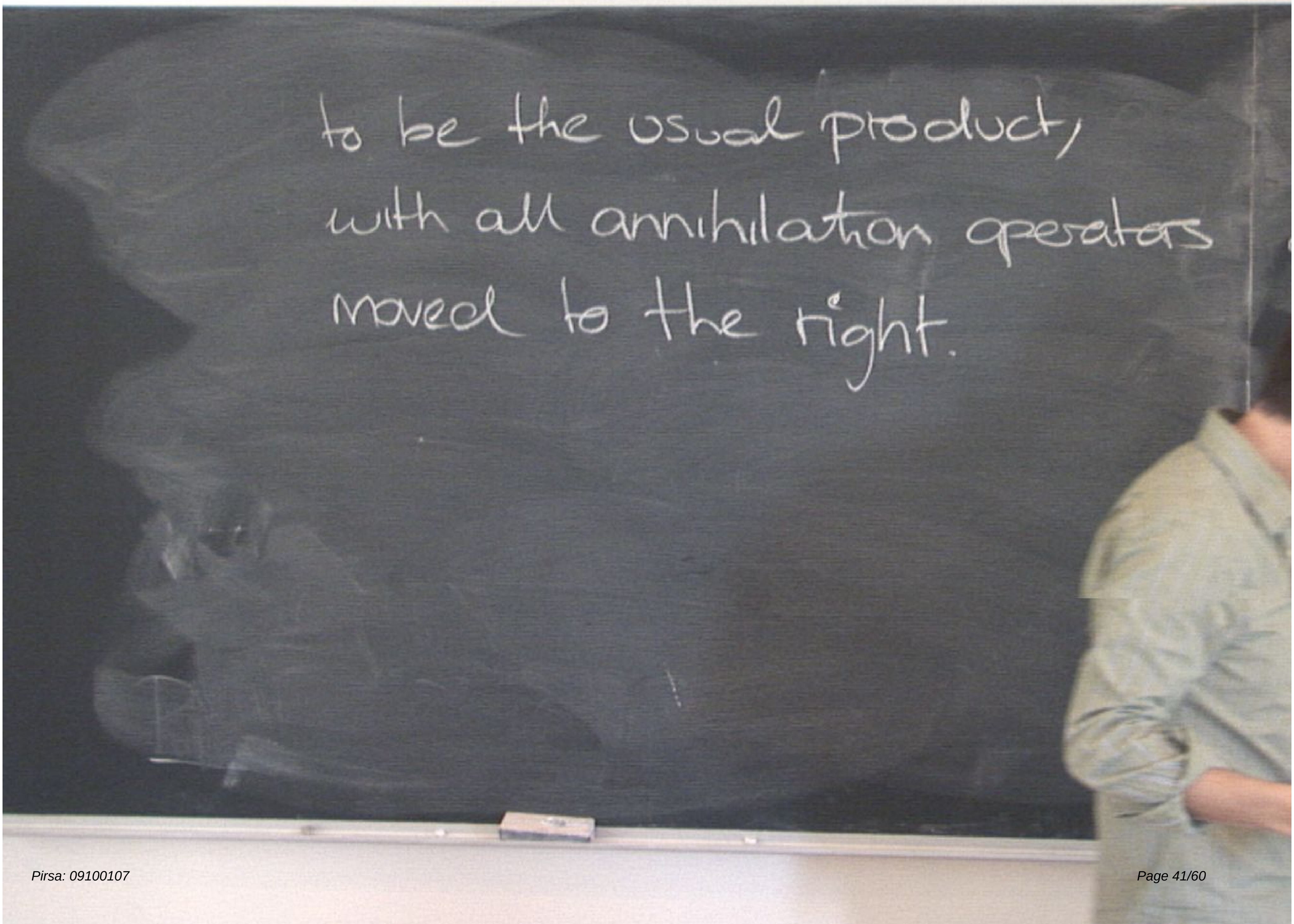
write
$$a = \frac{i}{\sqrt{2\omega}}p + \sqrt{\frac{\omega}{2}}\,q$$
$$a^+ = \frac{-i}{\sqrt{2\omega}}p + \sqrt{\frac{\omega}{2}}\,q$$
$$\Rightarrow \quad q = \frac{1}{\sqrt{2\omega}} \qquad p = -i$$

$$\Rightarrow H = \omega\left(aa^+ + \tfrac{1}{2}\right)$$

$\Rightarrow H|0\rangle = 0.$

Defⁿ In free field theory, we define a normal ordered string of operators $\phi_1(x_1) \ldots \phi_n(x_n)$

$$:\phi_1(x_1) \ldots \phi_n(x_n):$$


to be the usual product,
with all annihilation operators
moved to the right.

to be the usual product, with all annihilation operators moved to the right.

In particular, we're using the Hamiltonian

$$:H:$$

Recovering Particles

Recovering Particles

It's simple

Recovering Particles

It's simple to check that

$$[H, a^\dagger_{\underline{k}}] = \omega_{\underline{k}} a^\dagger_{\underline{k}}$$

$$[H, a_{\underline{k}}] = -\omega_{\underline{k}} a_{\underline{k}}$$

Define $|k\rangle = a_k^\dagger |0\rangle$

which has energy $H|k\rangle = \omega_p |k\rangle$.

with $\omega_p^2 = k^2 + m^2$

Define $|\vec{k}\rangle = a^\dagger_{\vec{k}}|0\rangle$

which has energy $H|\vec{k}\rangle = \omega_p|\vec{k}\rangle$.

with $\omega_p^2 = \vec{k}^2 + m^2$.

This is the relativistic dispersion relation for a particle of mass m and 3-momentum $\vec{k}$

$$E_{\vec{p}}^{2} = \vec{p}^{\,2} + m^2 .$$

This motivates us to interpret $|\vec{p}\rangle$ as the state of a single particle with momentum $\vec{p}$ and mass m.

we have a momentum operator

$$P^i = \int d^3x \, T^{0i}$$

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or $$\underset{\sim}{P} = -\int d^3x \; \pi(x) \, \underset{\sim}{\nabla} \phi(x)$$

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$$\underset{\sim}{P} = -\int d^3x \, \pi(x) \, \underset{\sim}{\nabla}\phi(x)$$

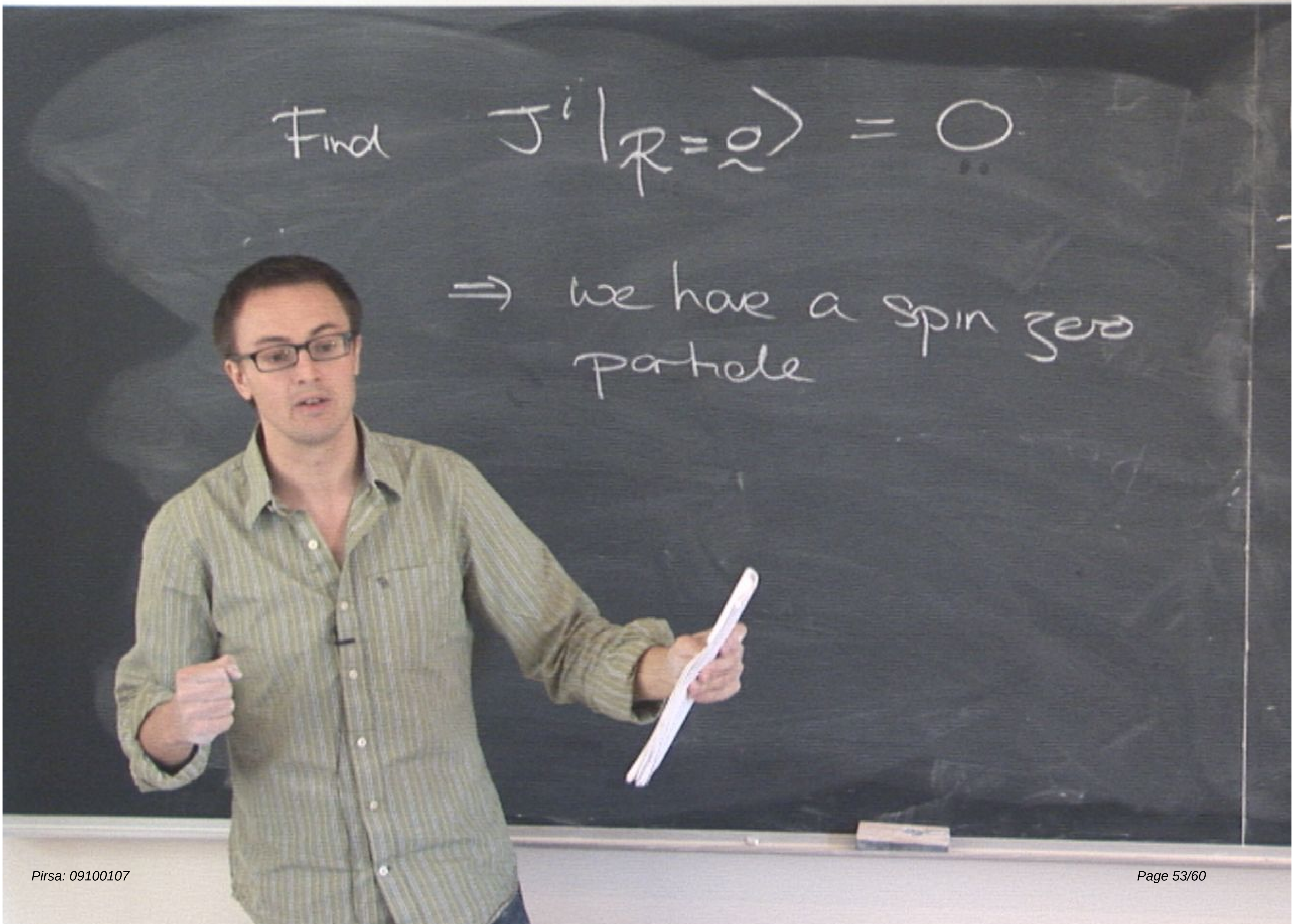
$$= \int \frac{d^3p}{(2\pi)^3} \, \underset{\sim}{p} \, a^\dagger_{\underset{\sim}{p}} a_{\underset{\sim}{p}}$$

e $\underset{\sim}{P}|\underset{\sim}{p}\rangle = \underset{\sim}{p}|\underset{\sim}{p}\rangle$ (Exercise)

ed.

We can also compute the spin of the particle using the angular momentum operator

$$J^i = \epsilon^{ijk} \int d^3x \, (M^0)^{jk}$$


Find $J^i|\underset{\sim}{R}=\underset{\sim}{0}\rangle = 0$
⇒ we have a spin zero particle

We can create multi-particle states

by:

$$|\vec{p}_1, \cdots, \vec{p}_n\rangle = a^\dagger_{\vec{p}_1} \cdots a^\dagger_{\vec{p}_n} |0\rangle.$$

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$$|\underline{p}_1, \dots, \underline{p}_n\rangle = a^\dagger_{\underline{p}_1} \cdots a^\dagger_{\underline{p}_n} |0\rangle.$$

Note that $|\underline{p}, \underline{q}\rangle = |\underline{q}, \underline{p}\rangle$

$\Longrightarrow$ these particles are <u>bosons</u>

The full Hilbert Space of the theory is spanned by

$$|0\rangle, \quad a^\dagger_k|0\rangle, \quad a^\dagger_k a^\dagger_q |0\rangle, \ldots$$

This is a <u>Fock space</u>.

There is an important operator which counts the number of particles.

$$N = \int \frac{d^3k}{(2\pi)^3}\, a_k^\dagger a_k$$

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$$N = \int \frac{d^3k}{(2\pi)^3} a^\dagger_k a_k$$

So $N|k_1, \ldots, k_n\rangle = n|k_1, \ldots, k_n\rangle$.

Notice that $[N, H] = 0$.

$\Rightarrow$ particle # is conserved.

This will no longer be true in interacting theories.