Title: Adiabatic quantum optimization fails for random instances of NP-complete problems

Date: Oct 07, 2009 04:00 PM

URL: http://pirsa.org/09100099

Abstract: Adiabatic quantum optimization has attracted a lot of attention because small scale simulations gave hope that it would allow to solve NP-complete problems efficiently. Later, negative results proved the existence of specifically designed hard instances where adiabatic optimization requires exponential time. In spite of this, there was still hope that this would not happen for random instances of NP-complete problems. This is an important issue since random instances are a good model for hard instances that can not be solved by current classical solvers, for which an efficient quantum algorithm would therefore be desirable. Here, we will show that because of a phenomenon similar to Anderson localization, an exponentially small eigenvalue gap appears in the spectrum of the adiabatic Hamiltonian for large random instances, very close to the end of the algorithm. This implies that unfortunately, adiabatic quantum optimization also fails for these instances by getting stuck in a local minimum, unless the computation is exponentially long.
Joint work with Boris Altshuler and Hari Krovi

Adiabatic quantum optimization and Anderson localization

Jérémie Roland

Joint work with:

Boris Altshuler
Hari Krovi

October 7, 2009

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Why quantum computing?

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Quantum computing provides speed-up for specific problems

- Factoring
- Discrete logarithms
- Simulation of quantum mechanics
- etc...

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What about NP-complete problems?

- 3-SAT:

$$(x_1 \vee \bar{x_2} \vee x_5) \wedge (\bar{x_1} \vee x_3 \vee \bar{x_5}) \wedge (\bar{x_2} \vee x_4 \vee x_5)$$

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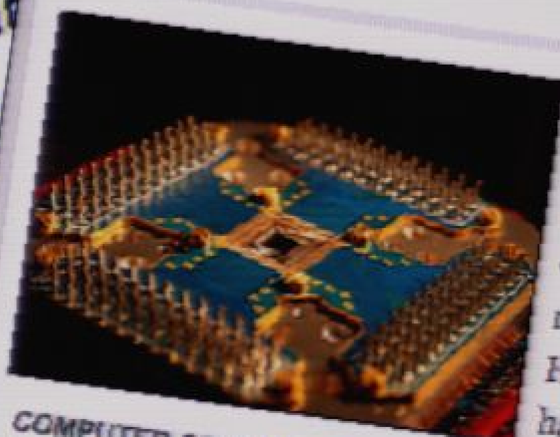
February 13, 2007 | 0 comments

First "Commercial" Quantum Computer Solves Sudoku Puzzles

Quantum computing company banks on a long-shot form of quantum computing

By JR Minkel



COMPUTER OF TOMORROW:
D-Wave Systems, a Ca
company, has anno

A Canadian firm today unveiled what it call
"the world's first commercially viable qua
computer." D-Wave Systems, Inc. "T
Quantum Computing Compa
much ballyhooed rollout
History Mus
hailed the
age of
the

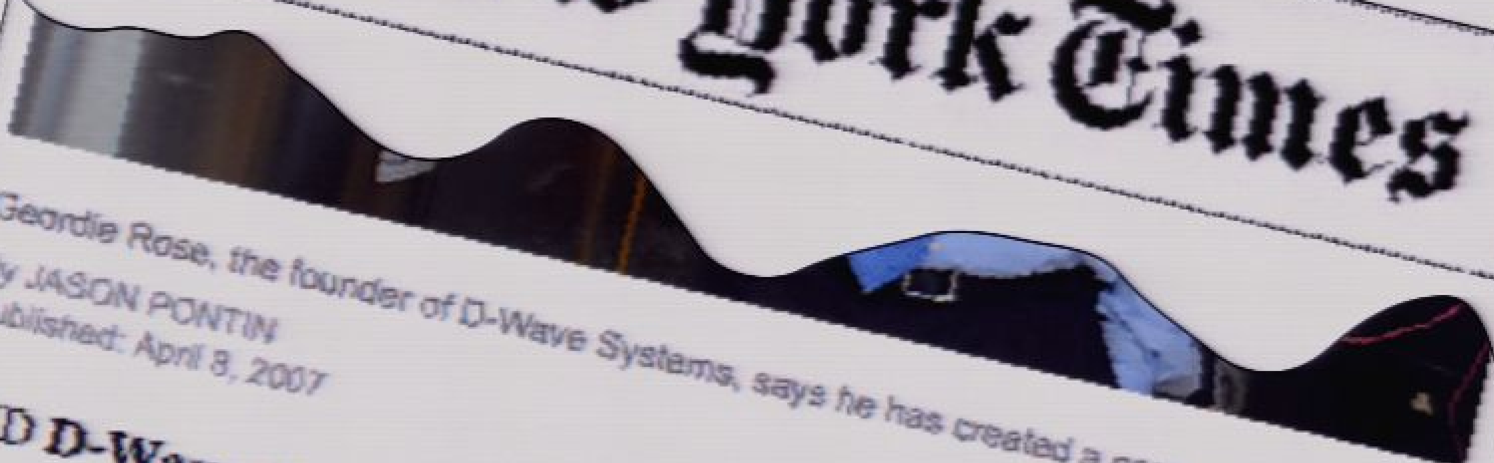
The Economist

On paper at least, quantum computers promise to reduce dramatically the time needed to solve a
range of mathematical tasks known as NP-complete problems. One famous example is the travelling
...man problem--finding the shortest route between several cities ... This is a puzzle that increases
...tially in com... with the num...

...008.

As it turns out quantum technology is particularly adept at tackling
what are known in mathematics as "NP-complete" problems. NP
standing for nondeterministic polynomial time, these are problems
where the massive volu... of comp...

The New York Times

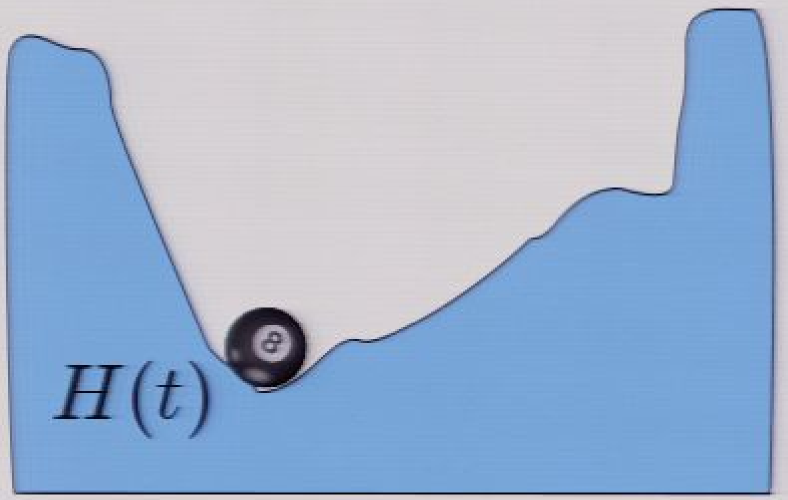


Geordie Rose, the founder of D-Wave Systems, says he has created a commercial quantum

By JASON PONTIN
Published: April 8, 2007

DID **D-Wave Systems** achieve the incredible -- a startling advance in computing that would radically expand human capacities for industrial activity and scientific discovery, long before experts believed it possible?

Adiabatic evolution



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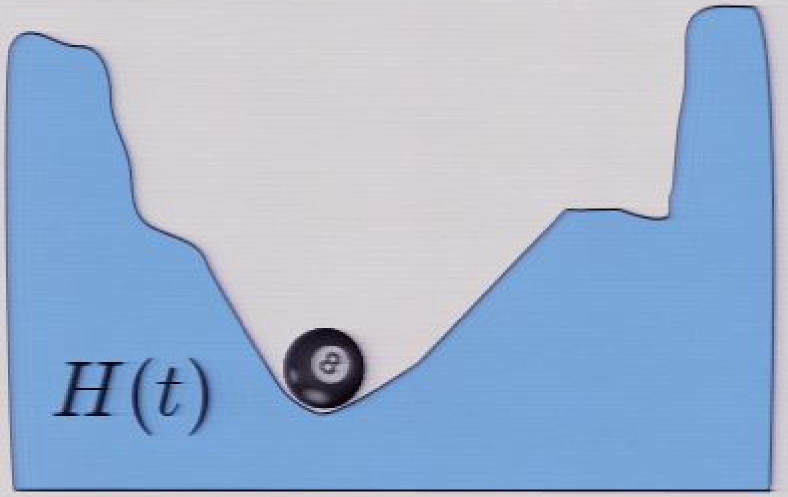

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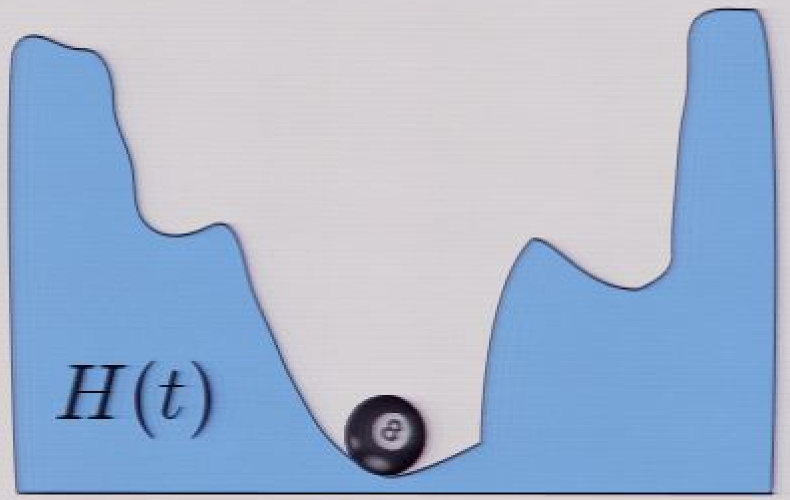

Slowly varying $H(t)$ → Stays close to ground state

Adiabatic evolution

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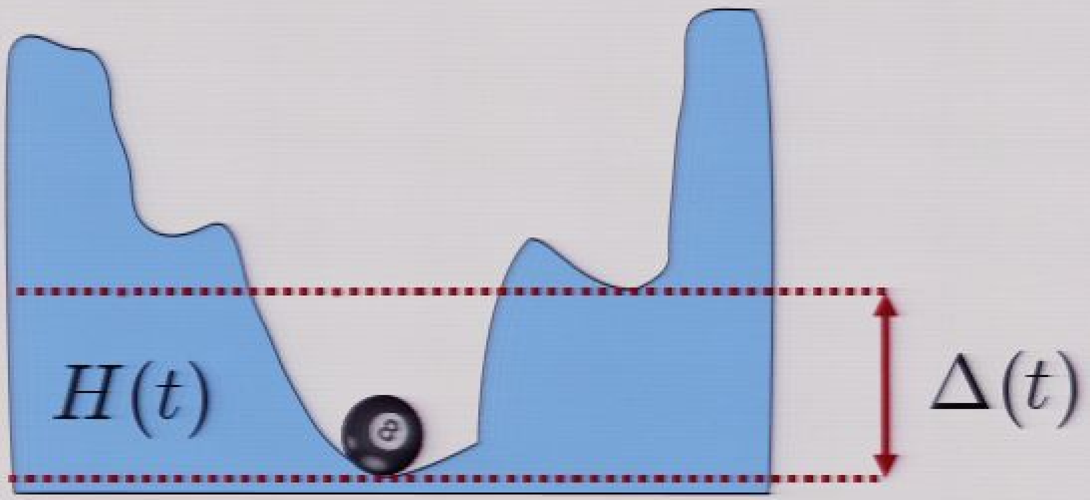

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Probability of excitation depends on

- Total time T (slower is better)
- Gap $\Delta(t)$ (larger gap is better)

Adiabatic evolution

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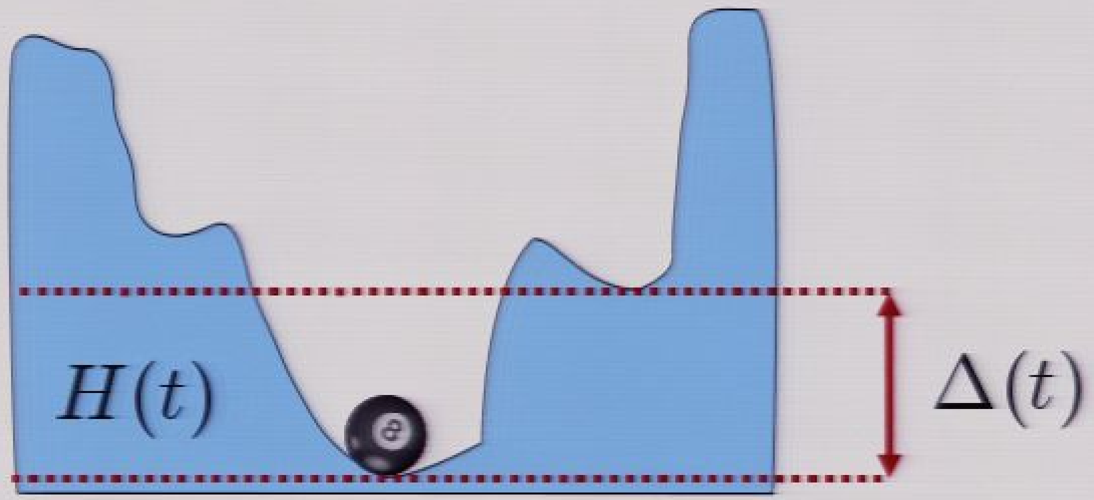

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$$T \gg \frac{1}{\Delta_{\min}^{k}}$$

Adiabatic quantum optimization

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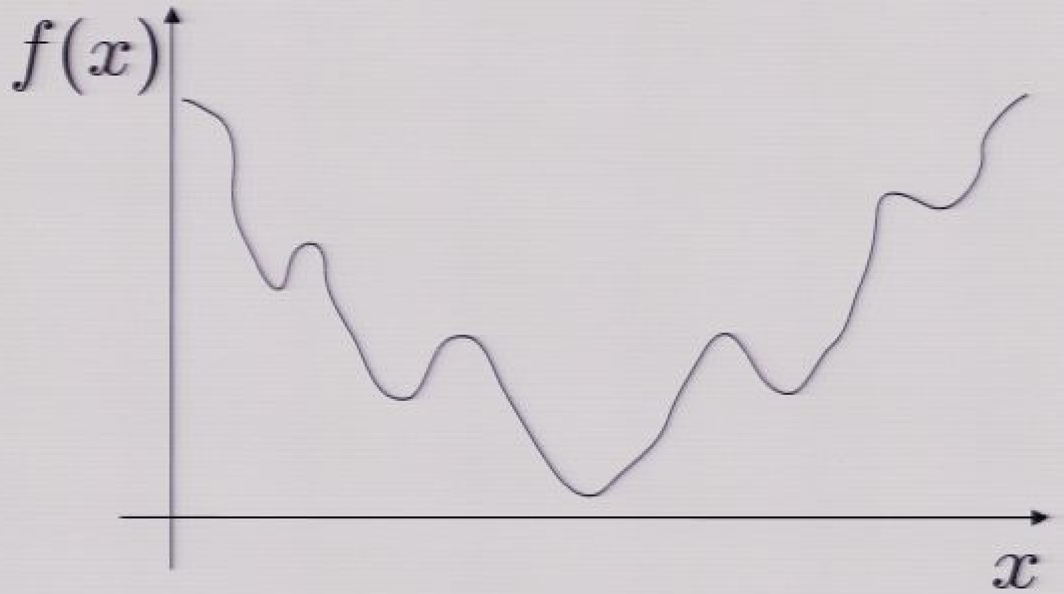
[Farhi *et al*. '00]

- Problem: Find minimum of a function *f(x)*



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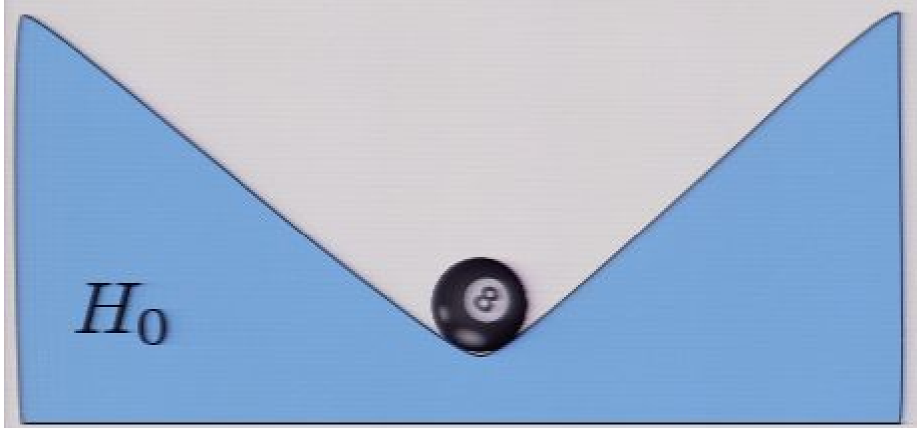

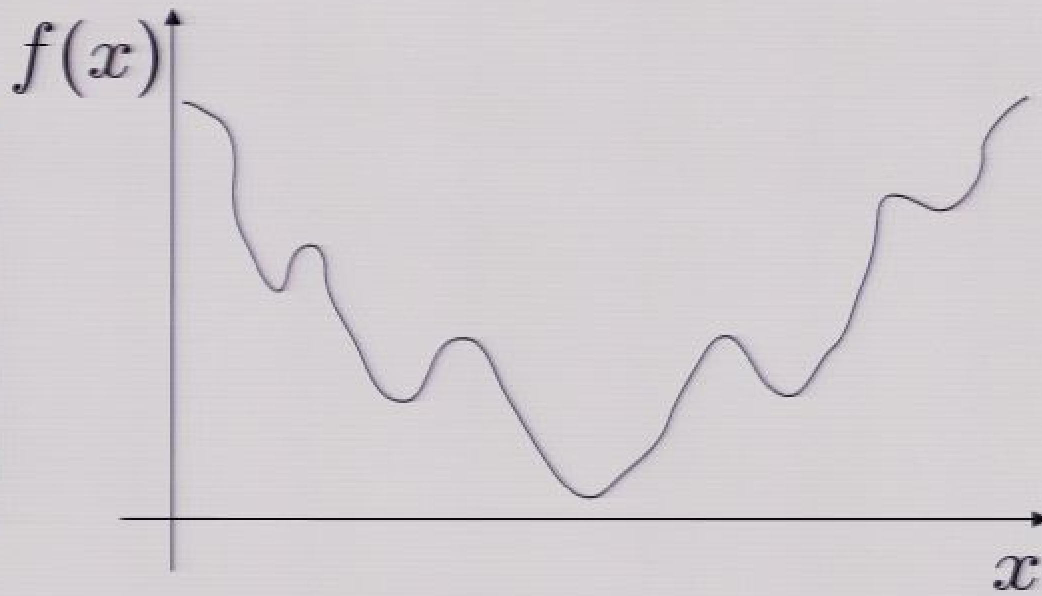

Adiabatic quantum optimization

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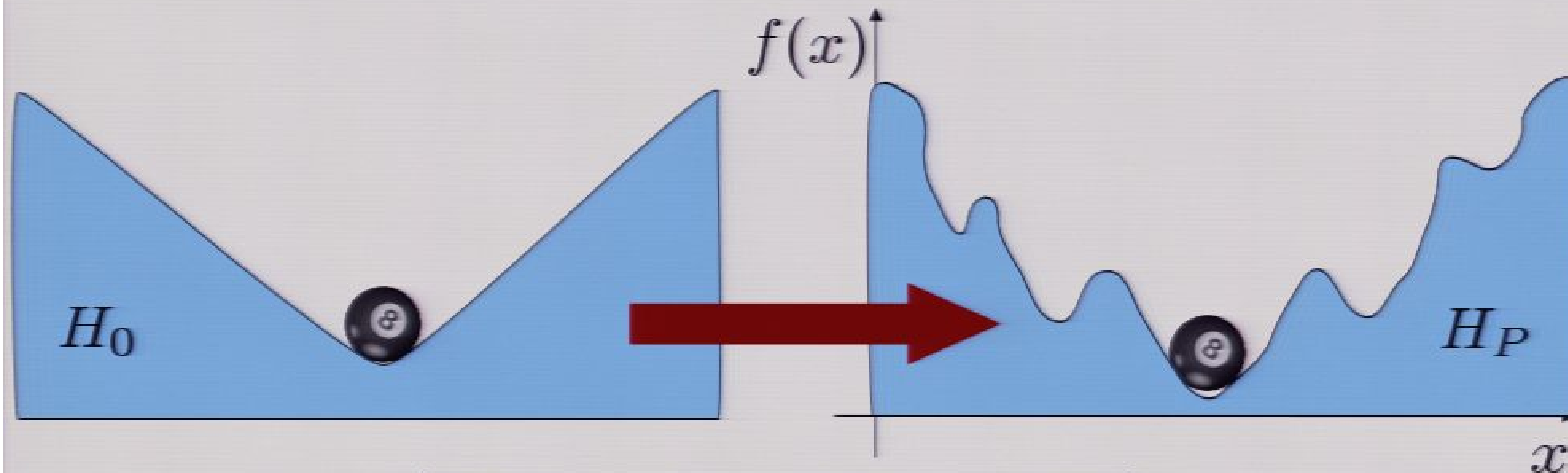

$$H(s) = (1-s)H_0 + sH_P$$

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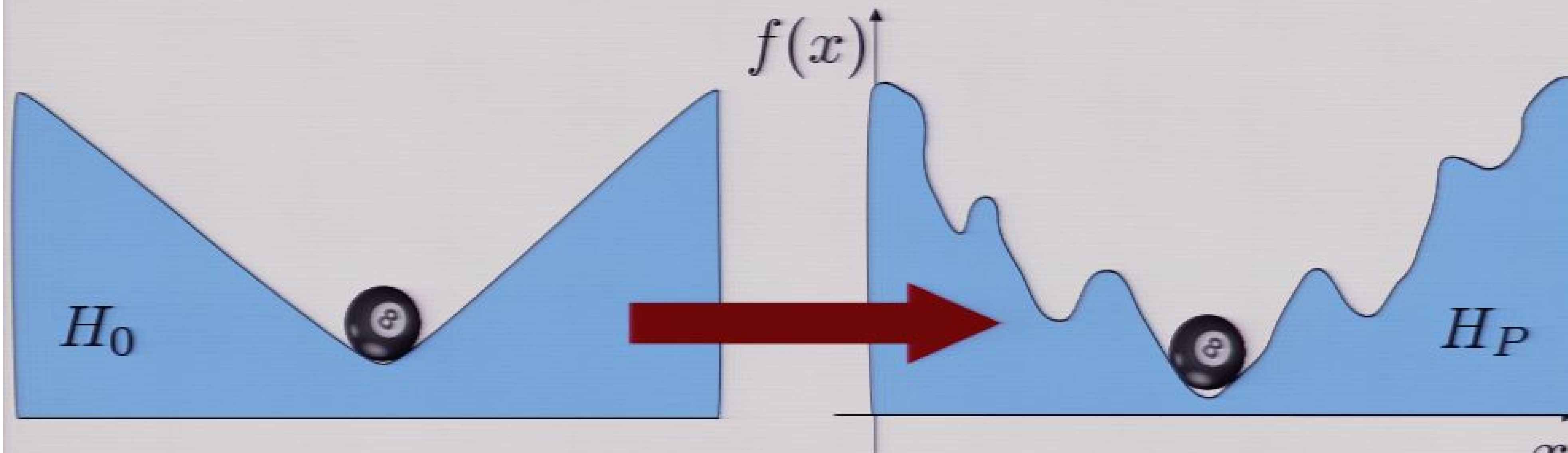

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  3) T large enough ⇒ measuring reveals the minimum

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How powerful is it?

- It is quantum! Unstructured search in time *O(N)* (cf Grover)

[vanDam-Mosca-Vazirani'01,Roland-Cerf'02]

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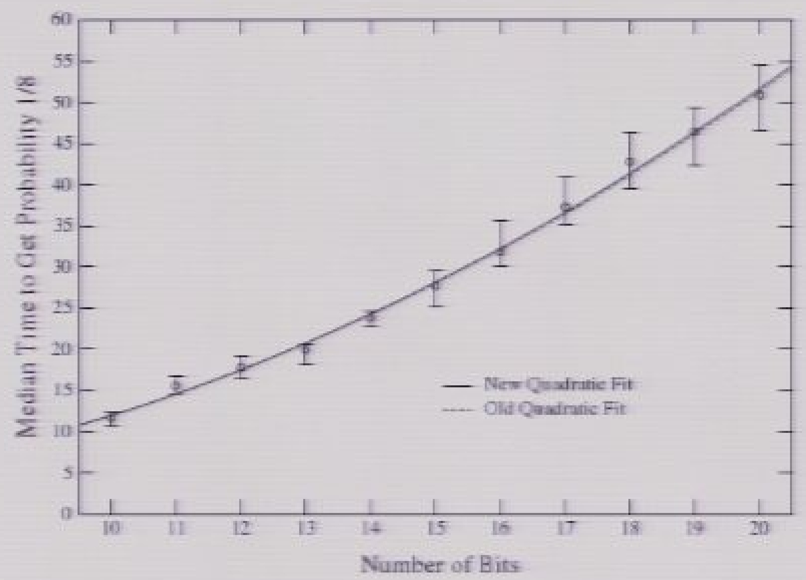

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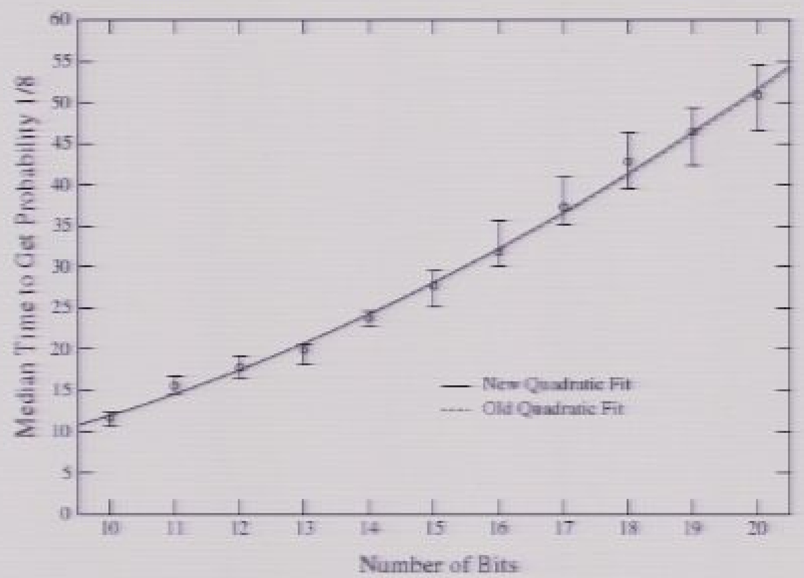

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- N bits $\vec{x} = (x_1, \ldots, x_N)$
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$$(x_{i_C}, x_{j_C}, x_{k_C}) \text{ satisfied } \Leftrightarrow x_{i_C} + x_{j_C} + x_{k_C} = 1$$
$$\Leftrightarrow 100, 010 \text{ or } 001$$

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➡ Minimize function
$$f(\vec{x}) = \sum_C (x_{i_C} + x_{j_C} + x_{k_C} - 1)^2$$
$$= M - \sum_i B_i x_i + 2 \sum_{i,j} J_{ij} x_i x_j$$

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#clauses

#clauses with bit i

#clauses with bits i,j

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Random instances

- Pick M clauses uniformly at random

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- Hardness depends on clauses-to-bit ratio: $\alpha = \dfrac{M}{N}$

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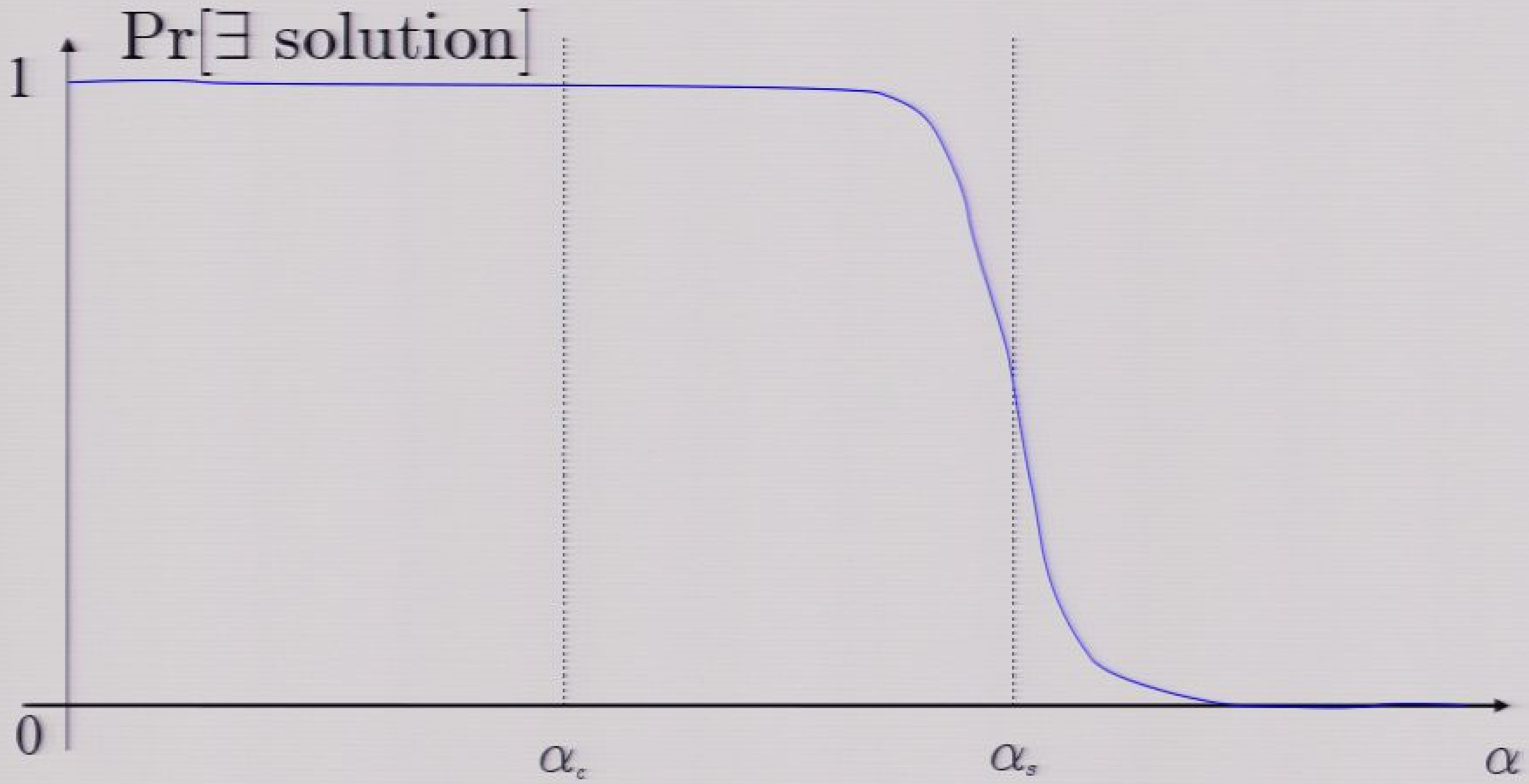

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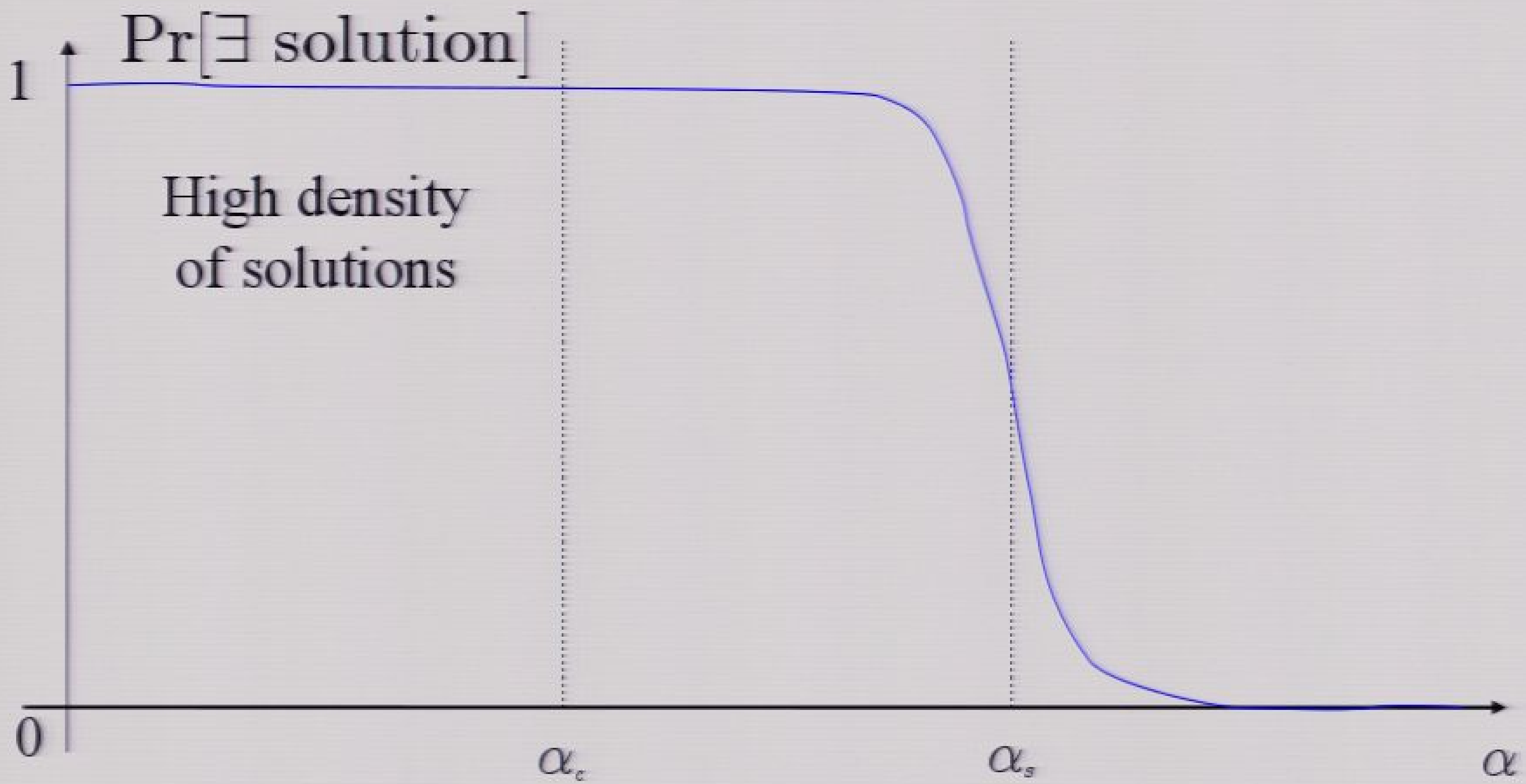

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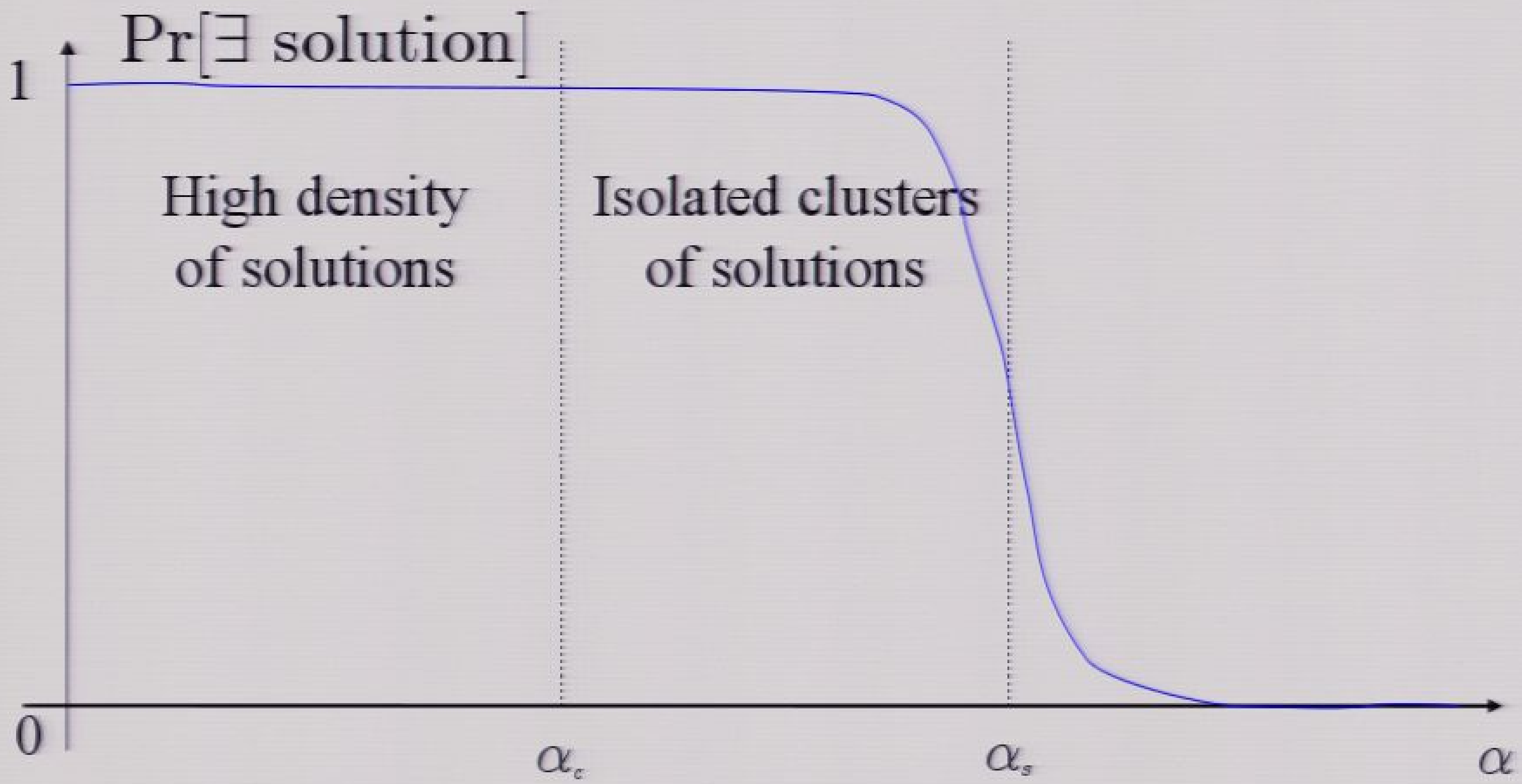

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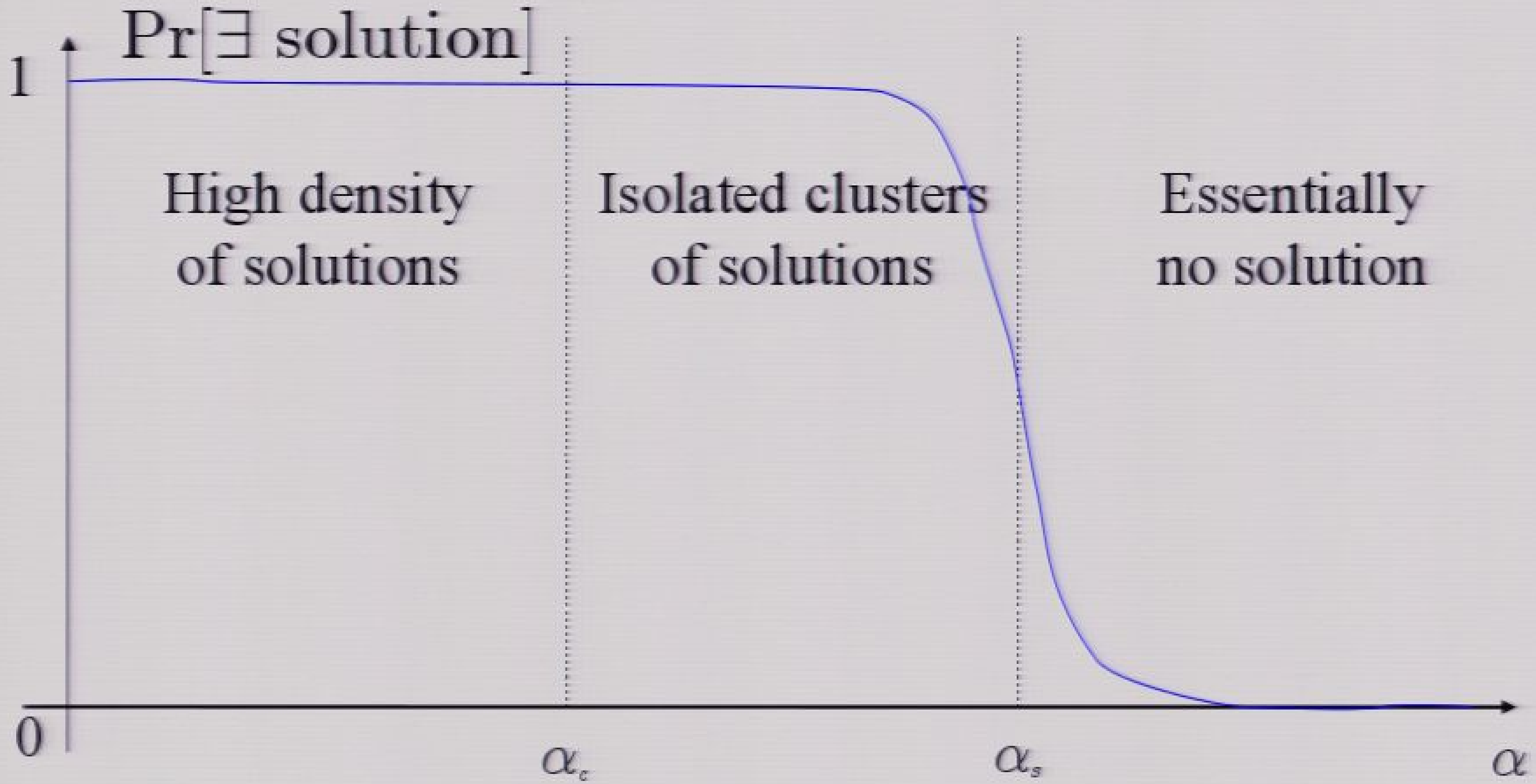

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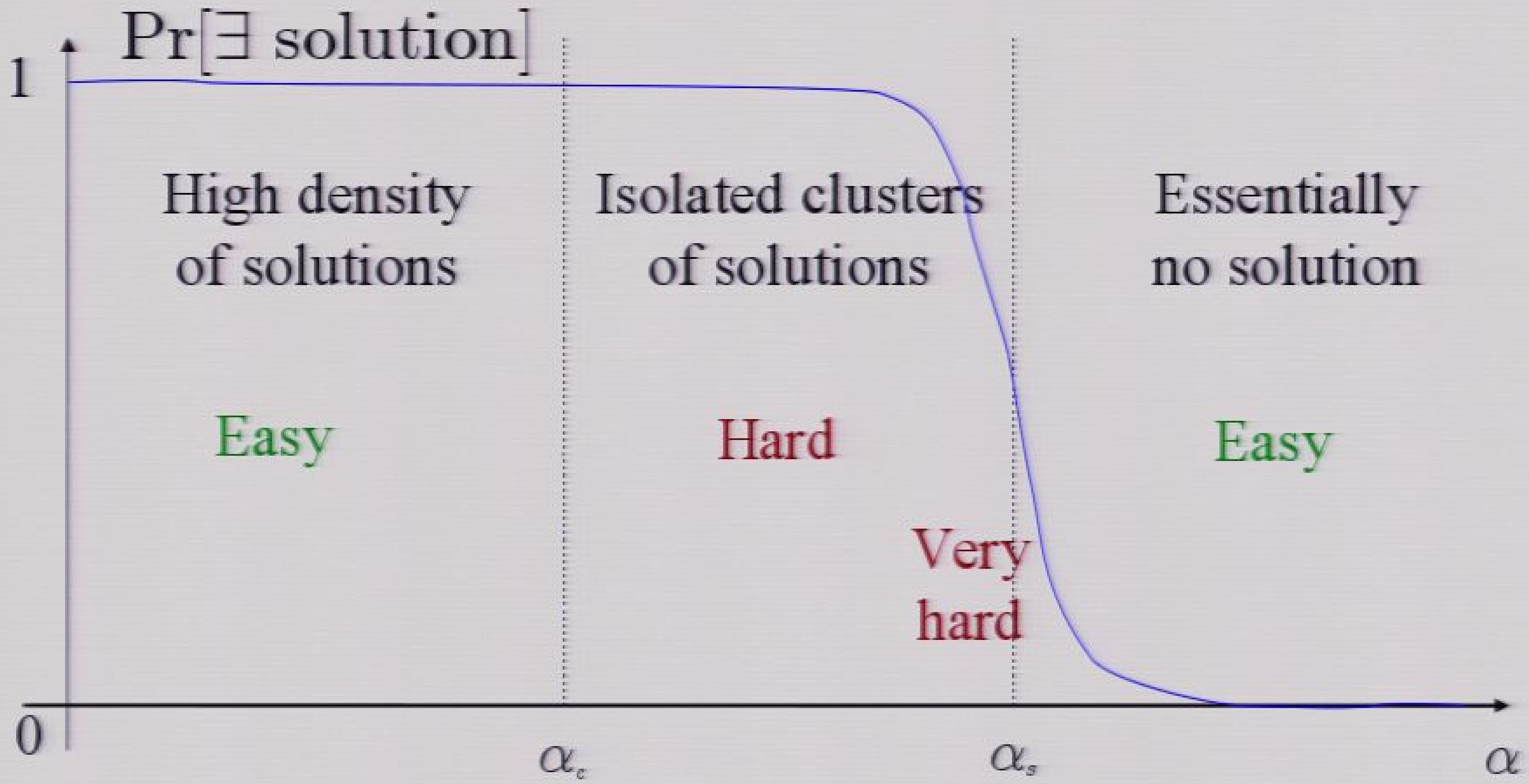

The adiabatic algorithm

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- Problem Hamiltonian $H_P = \sum_{\vec{x}} f(\vec{x}) |\vec{x}\rangle\langle\vec{x}|$

$$= M - \frac{1}{2}\sum_{i=1}^{N} B_i \sigma_z^{(i)} + \frac{1}{4}\sum_{i,j=1}^{N} J_{ij}\sigma_z^{(i)}\sigma_z^{(j)}$$

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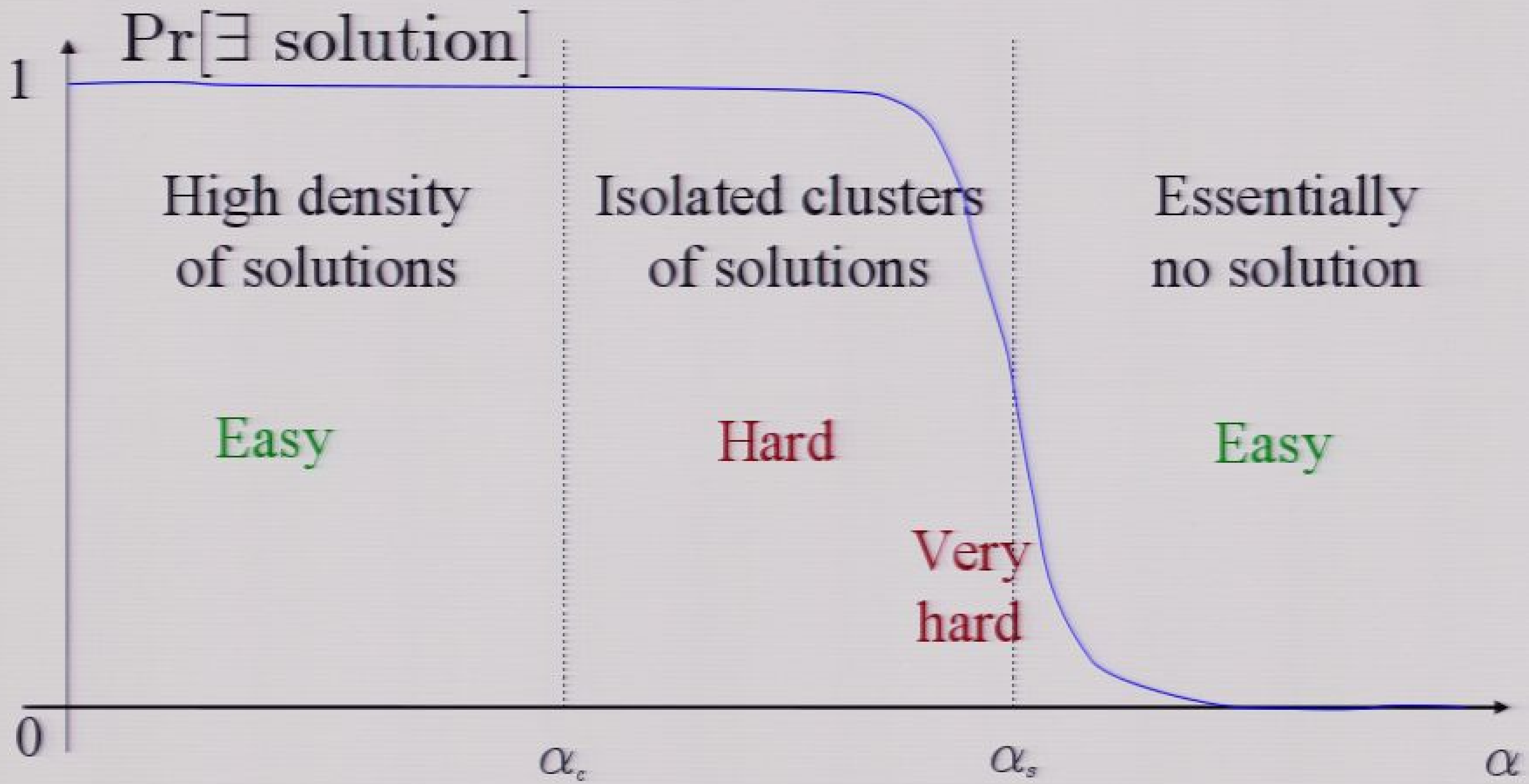

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Connection to Anderson localization

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To study the spectrum of H(s) close to $s = 1$, we consider

$$H(\lambda) = \frac{H(s)}{s} = H_P + \lambda H_0 \qquad \text{where} \qquad \lambda = \frac{1-s}{s}$$

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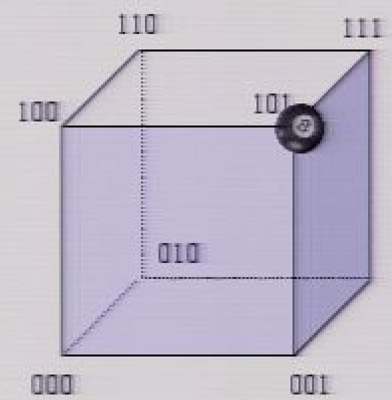

⇒ Particle hopping on a hypercube

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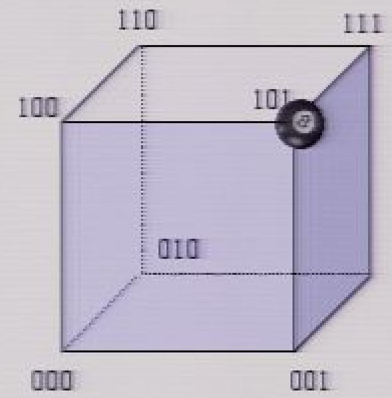
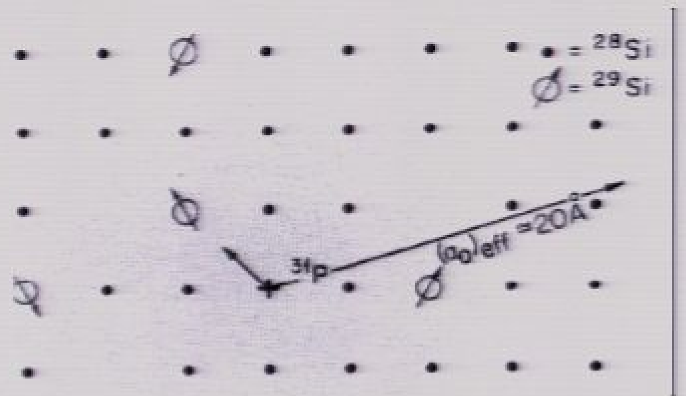


⇒ Particle hopping on a hypercube

⇒ Similar to Anderson's tight binding model

Anderson localization

“Extended states become localized due to disorder”

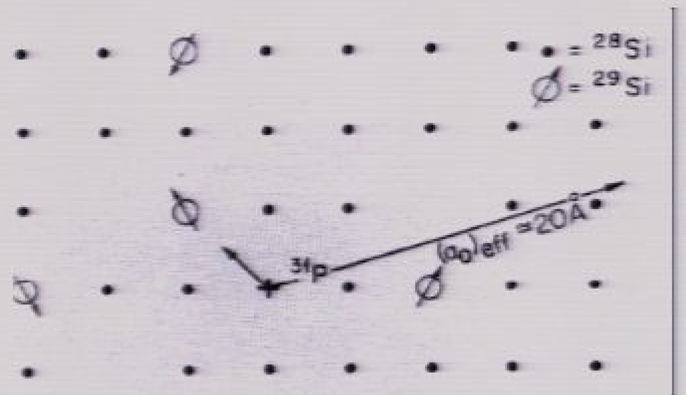


P. Anderson
Nobel Prize
Physics 1977

Anderson localization

"Extended states become localized due to disorder"

Model:
- Grid with coupling λ
- Random energies



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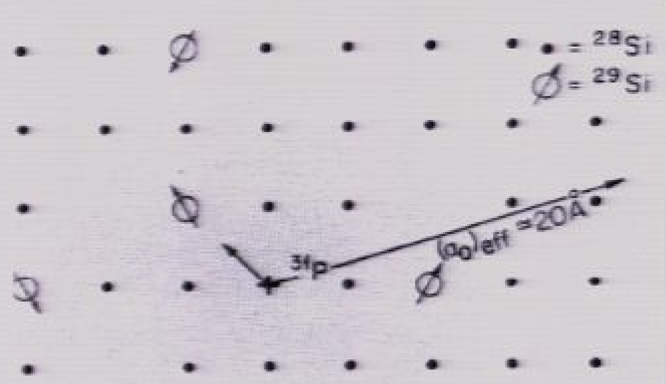

P. Anderson
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$\lambda > \lambda_c$ → Extended state → Metal

$\lambda < \lambda_c$ → Localized state → Insulator

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Anderson localization

"Extended states become localized due to disorder"

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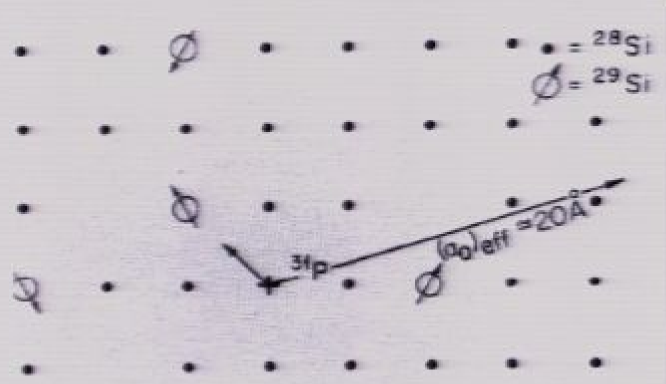


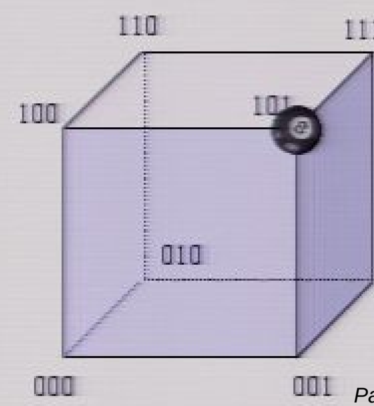
P. Anderson
Nobel Prize
Physics 1977

$\lambda > \lambda_c$ → Extended state → Metal

$\lambda < \lambda_c$ → Localized state → Insulator

In our case:

- Hypercube with coupling λ
- Energies from random Exact-Cover 3



Localized and extended states

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Localized and extended states

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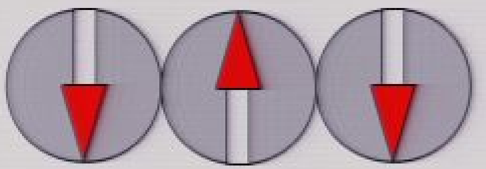
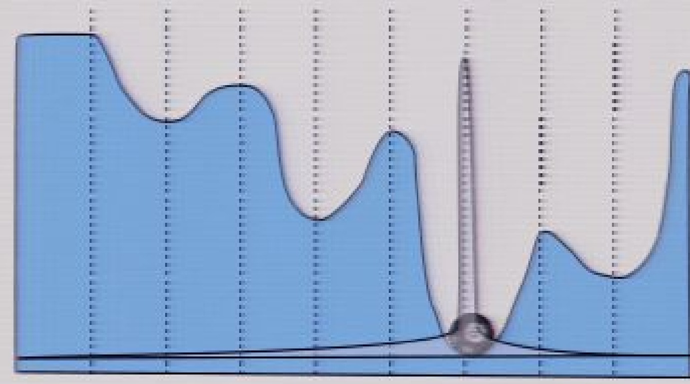
$\lambda = 0$

- State is localized





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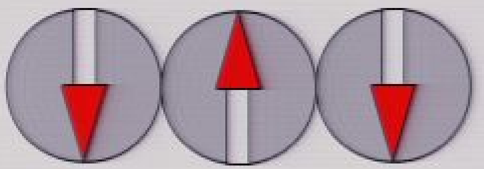
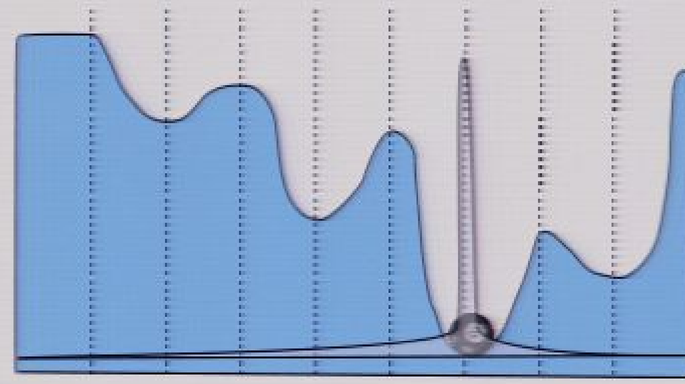
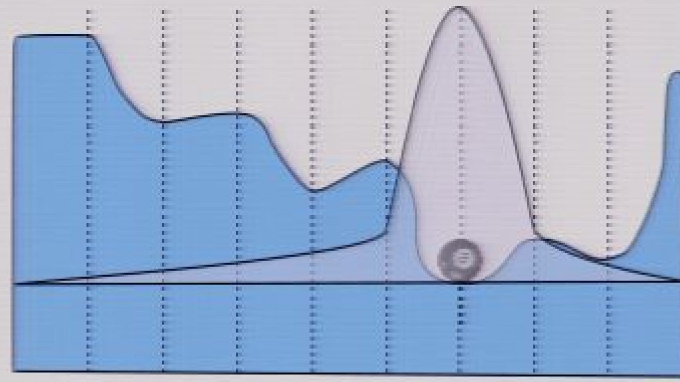




$\lambda > 0$

- Transverse field "spreads" the state



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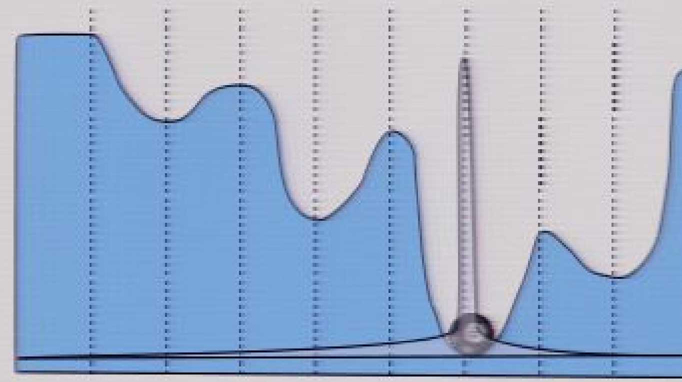

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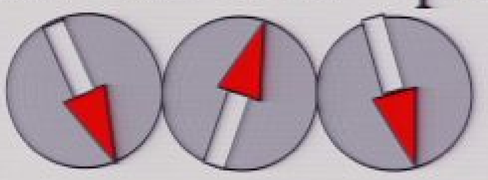
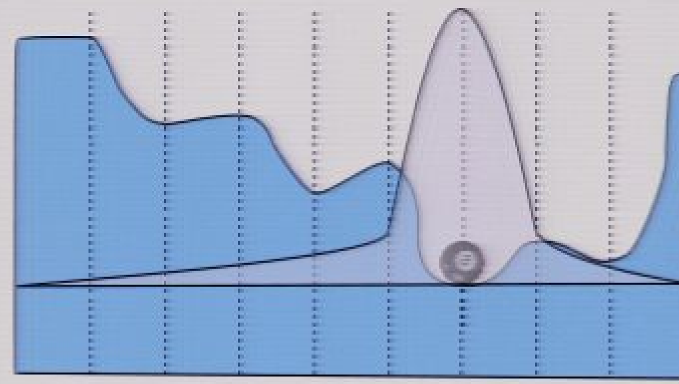
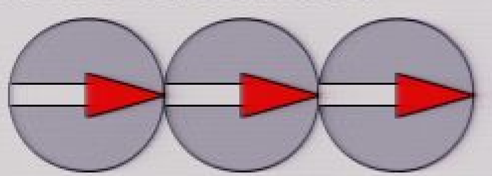
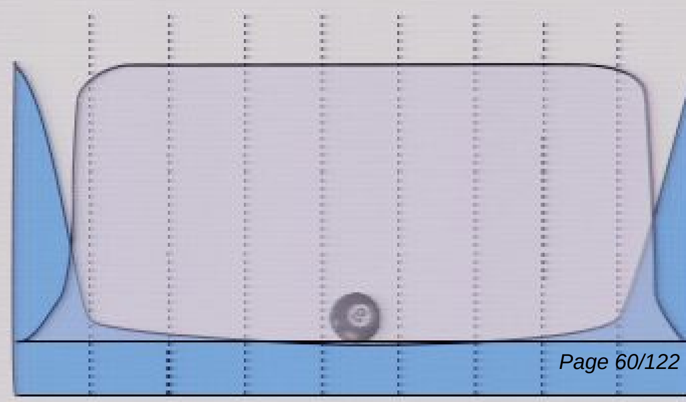




$\lambda \gg 1$

- State is extended

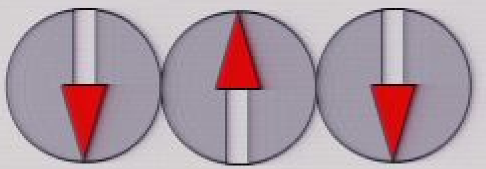




Localized and extended states

$\lambda = 0$

- State is localized



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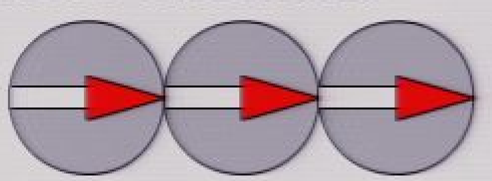

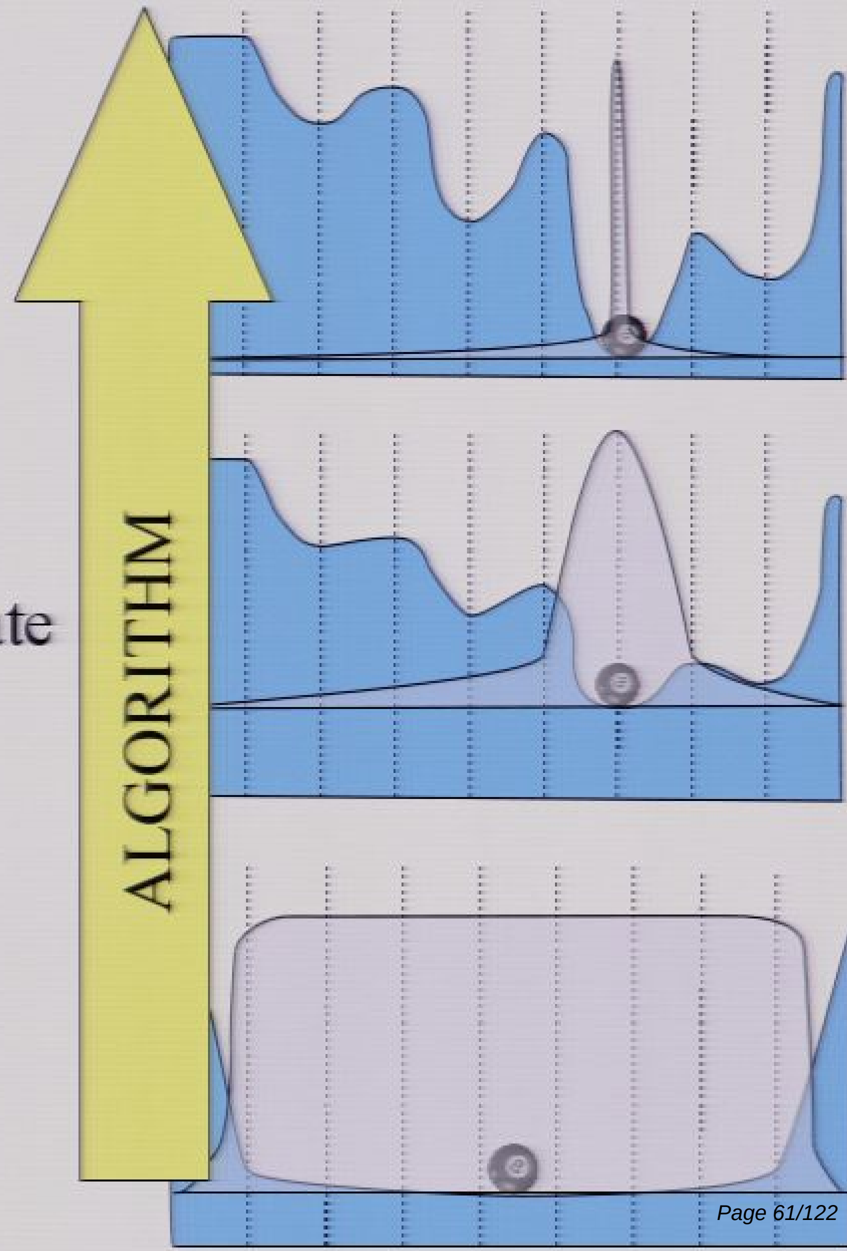

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Localized and extended states

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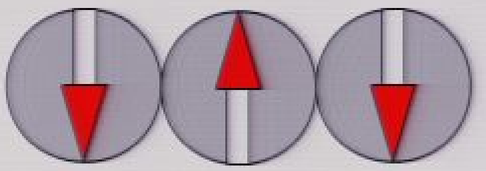
- State is localized



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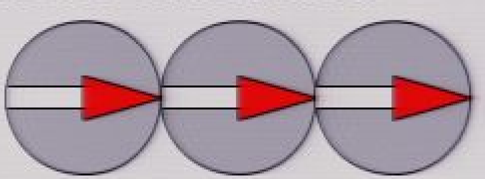

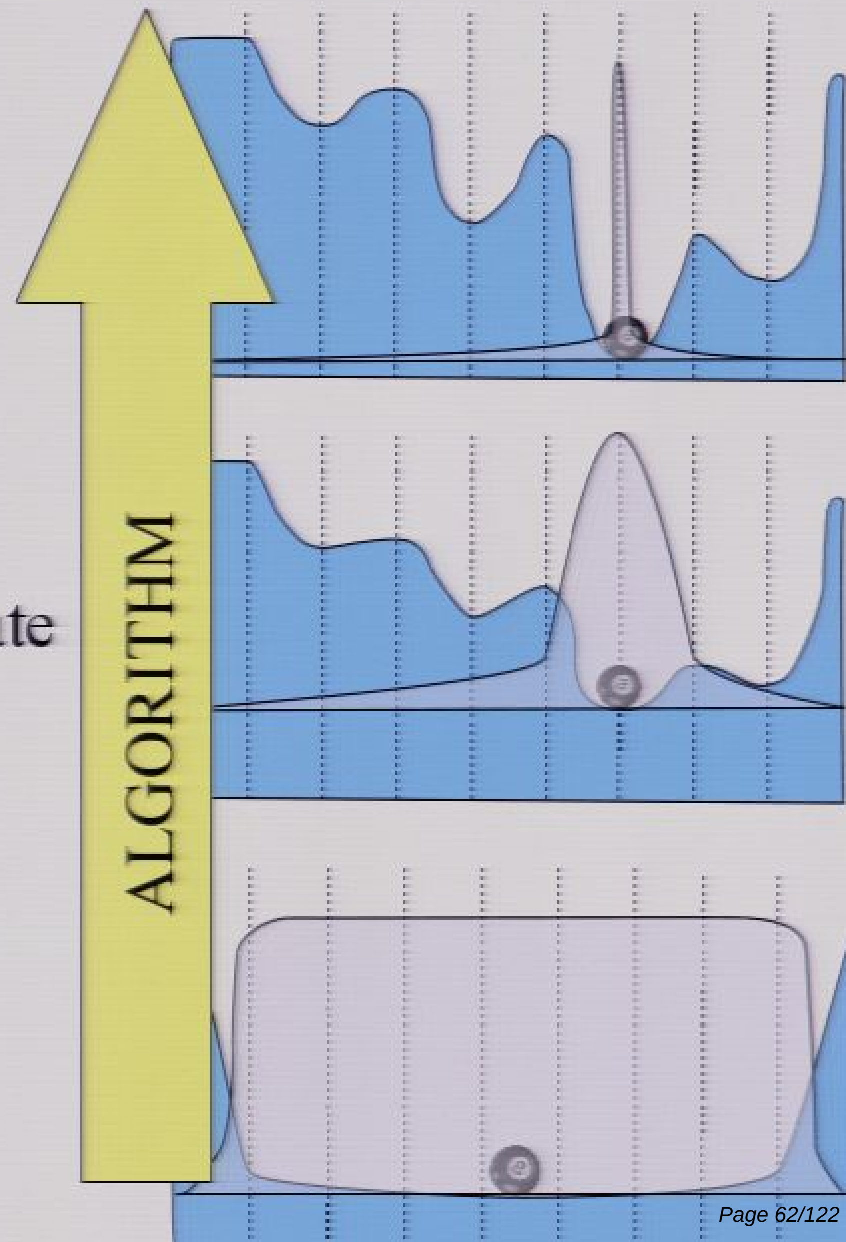

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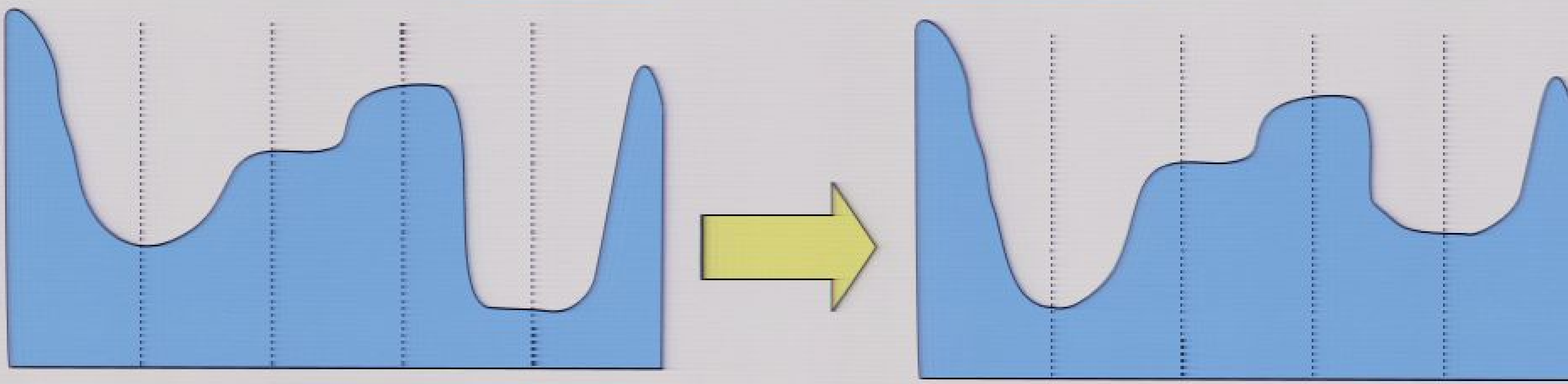
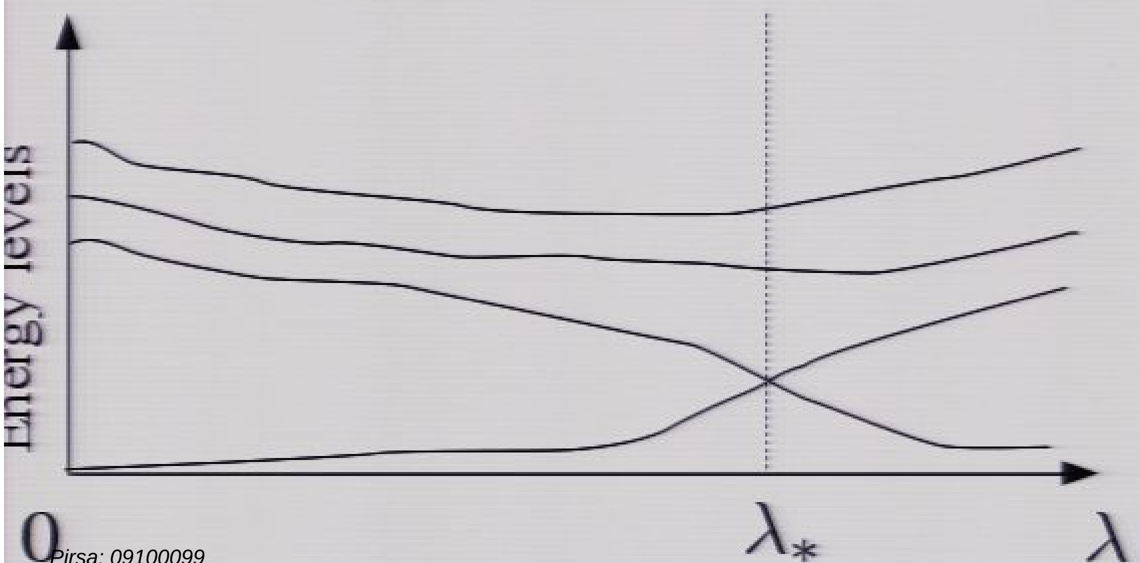
Tunneling: extended state

What if a local minimum later becomes the global minimum?

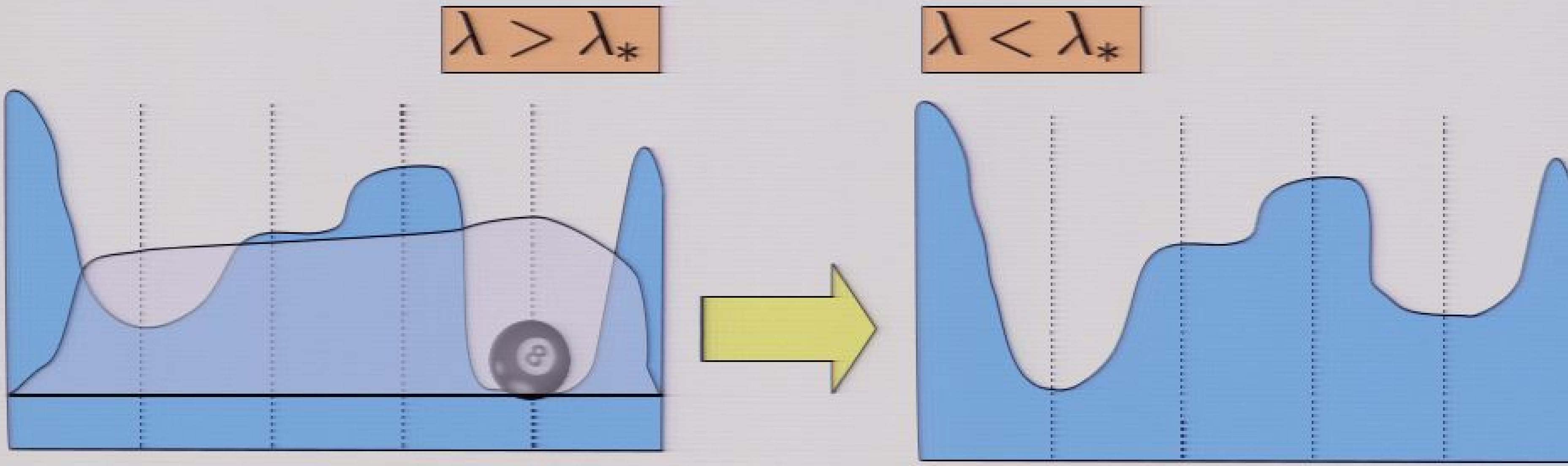
$\lambda > \lambda_*$ $\qquad$ $\lambda < \lambda_*$

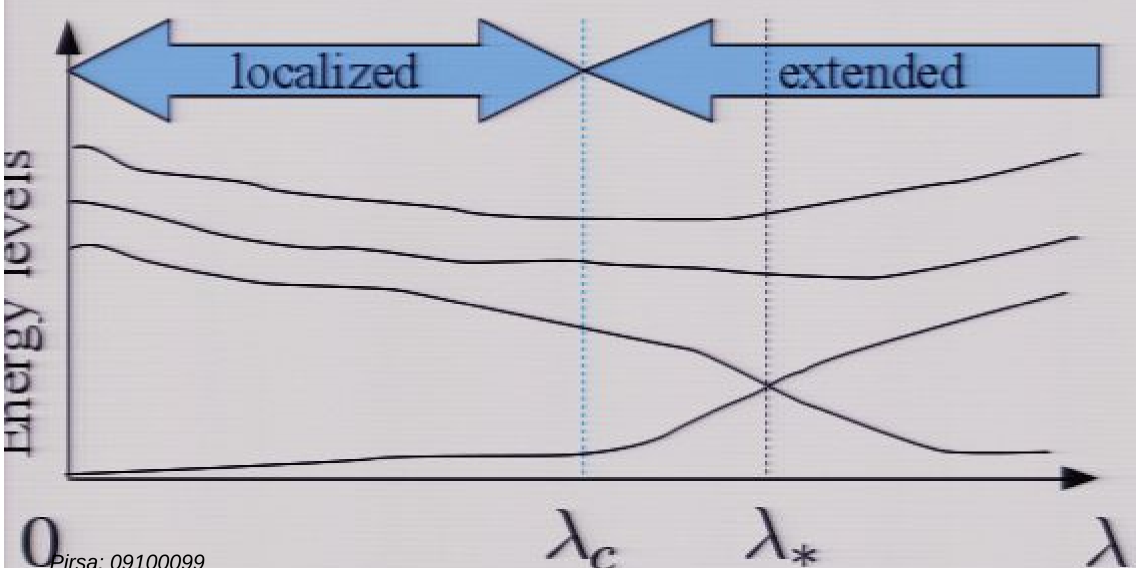




Tunneling: extended state

What if a local minimum later becomes the global minimum?





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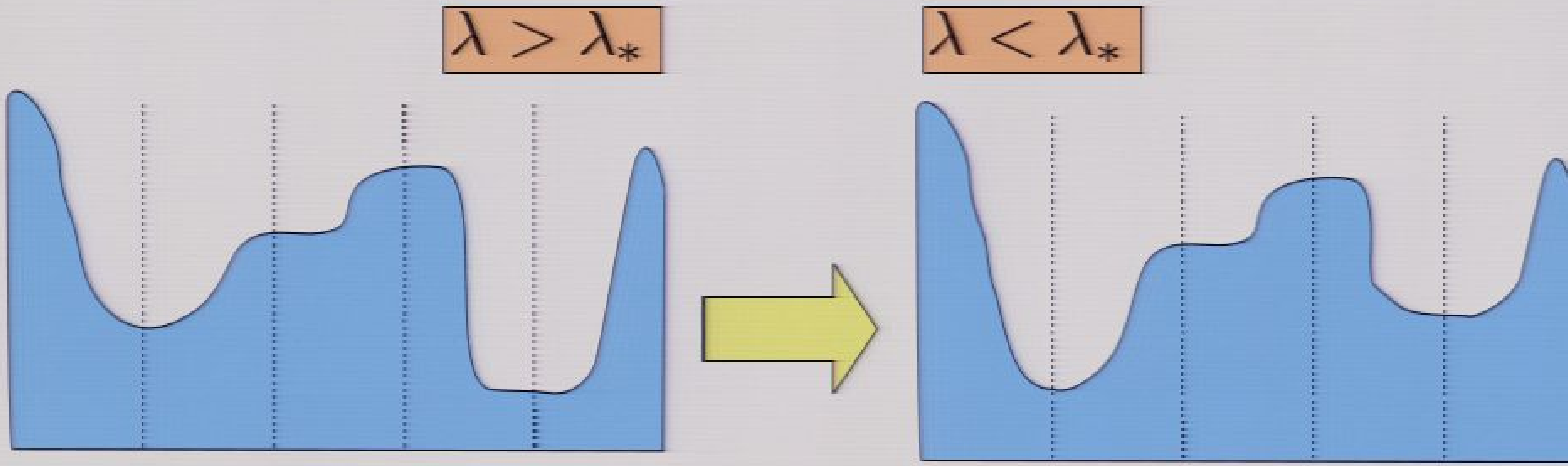

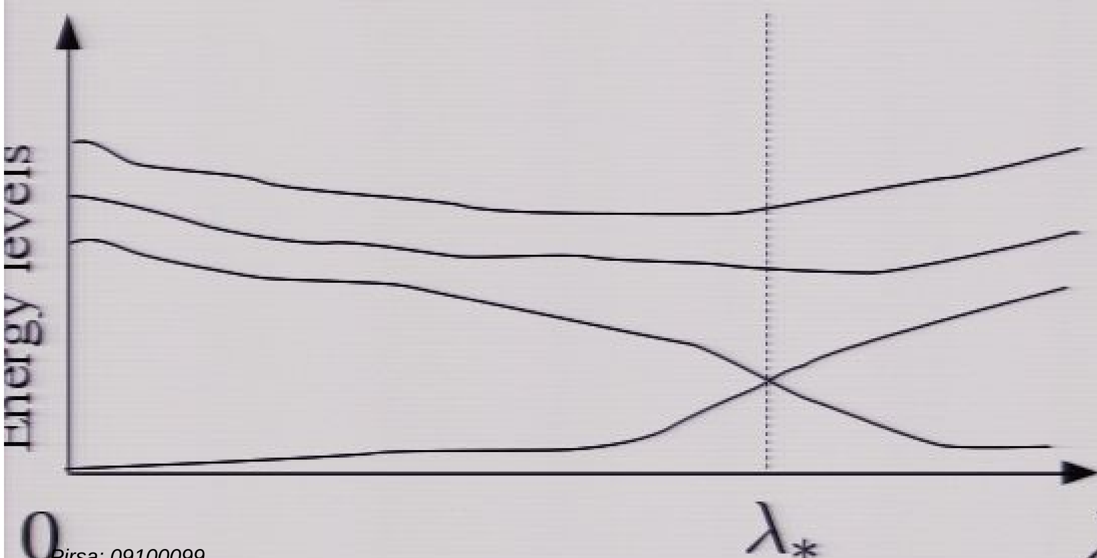

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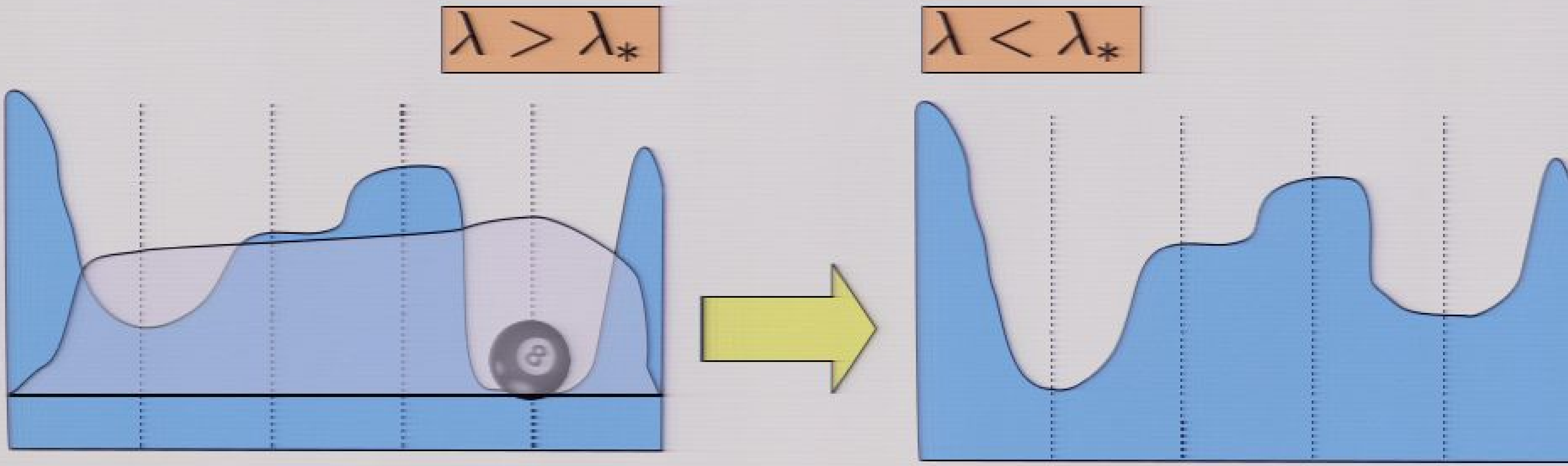

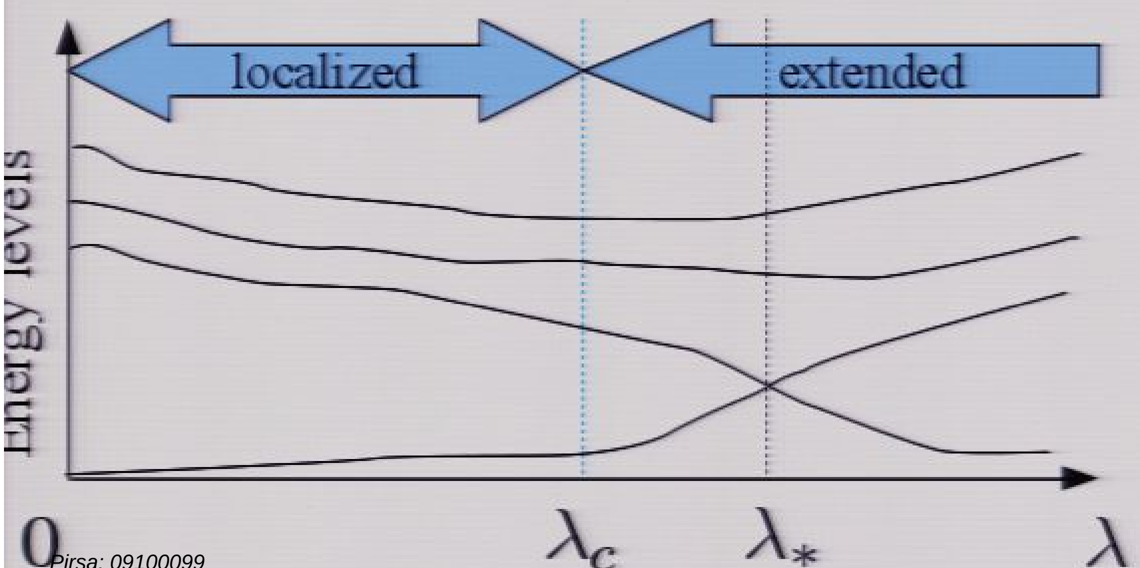

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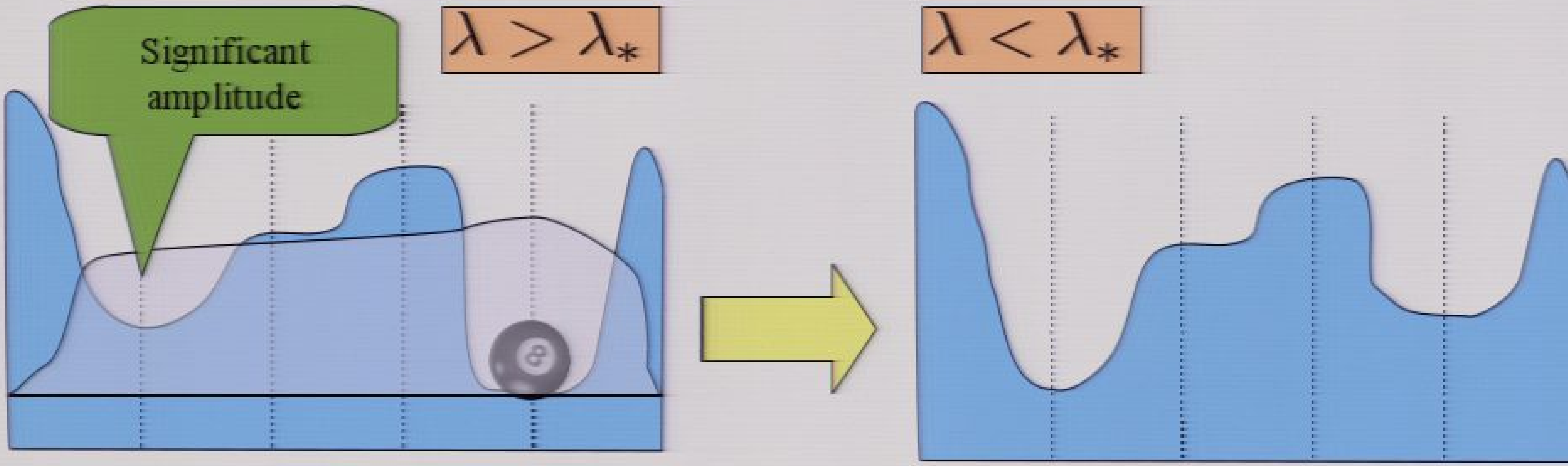

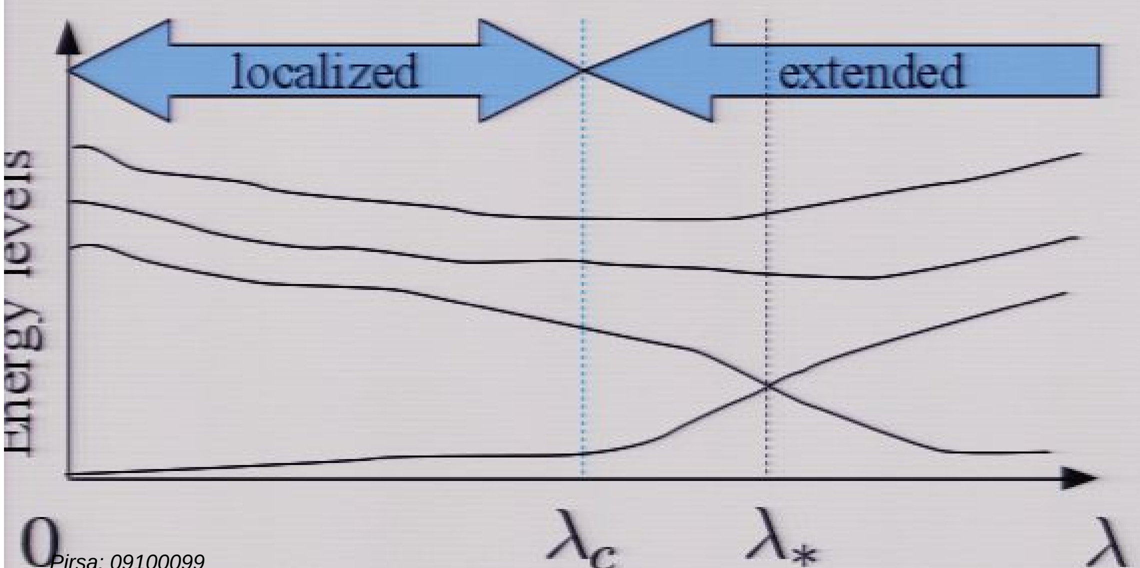

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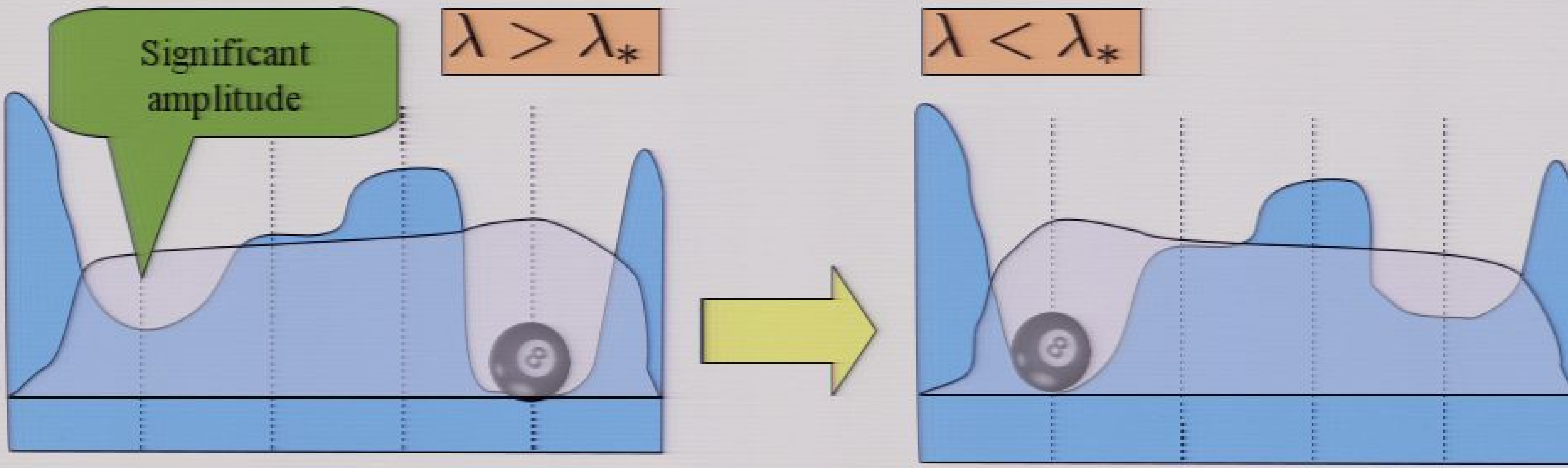

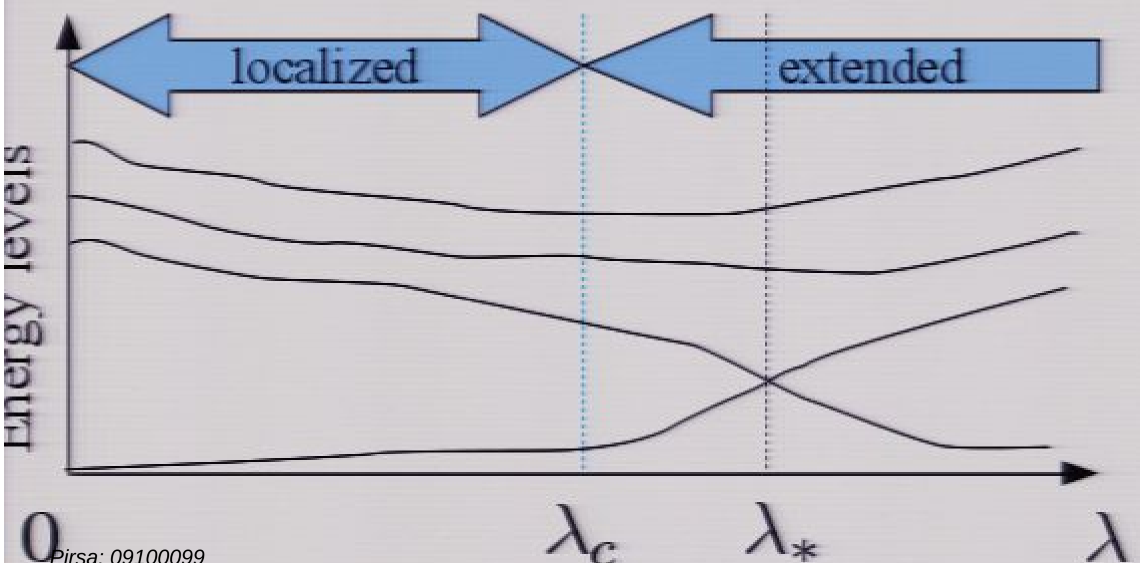

Tunneling: extended state

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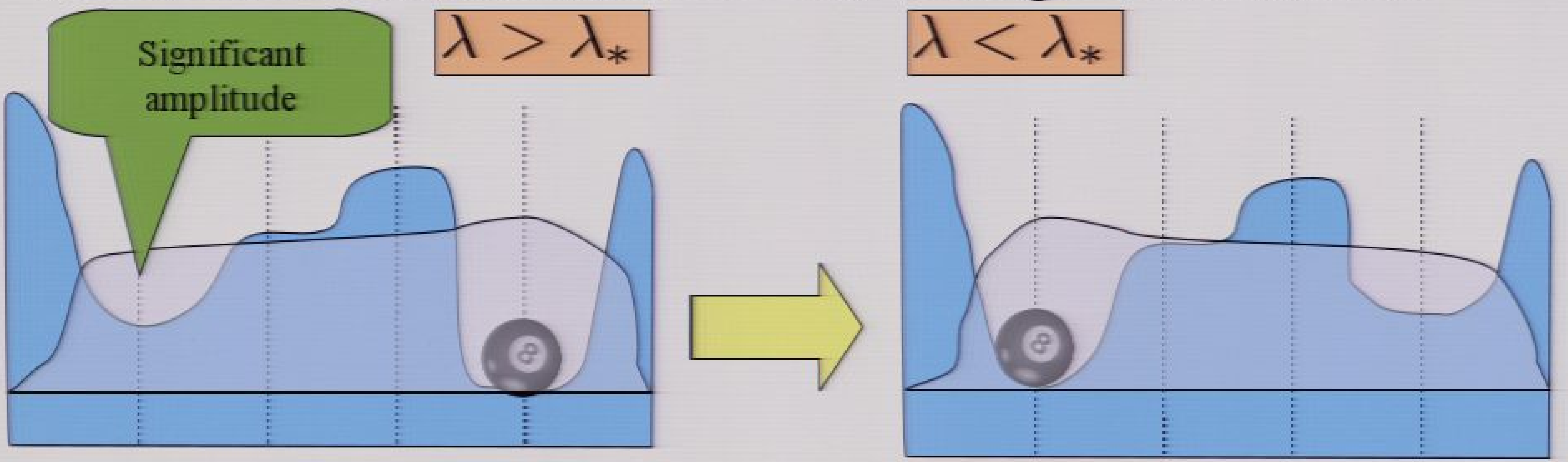
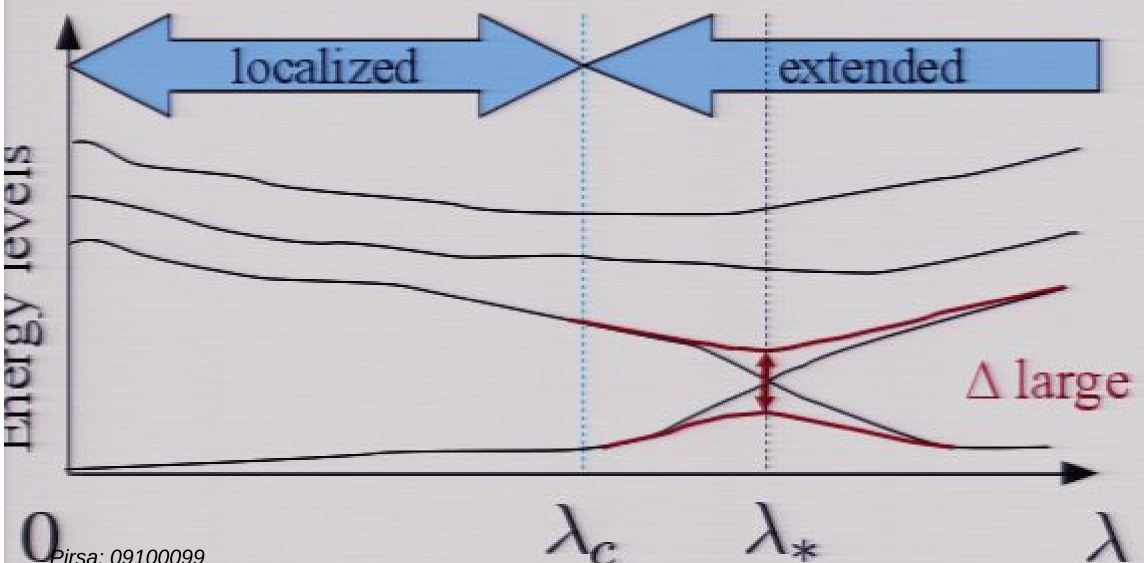
What if a local minimum later becomes the global minimum?

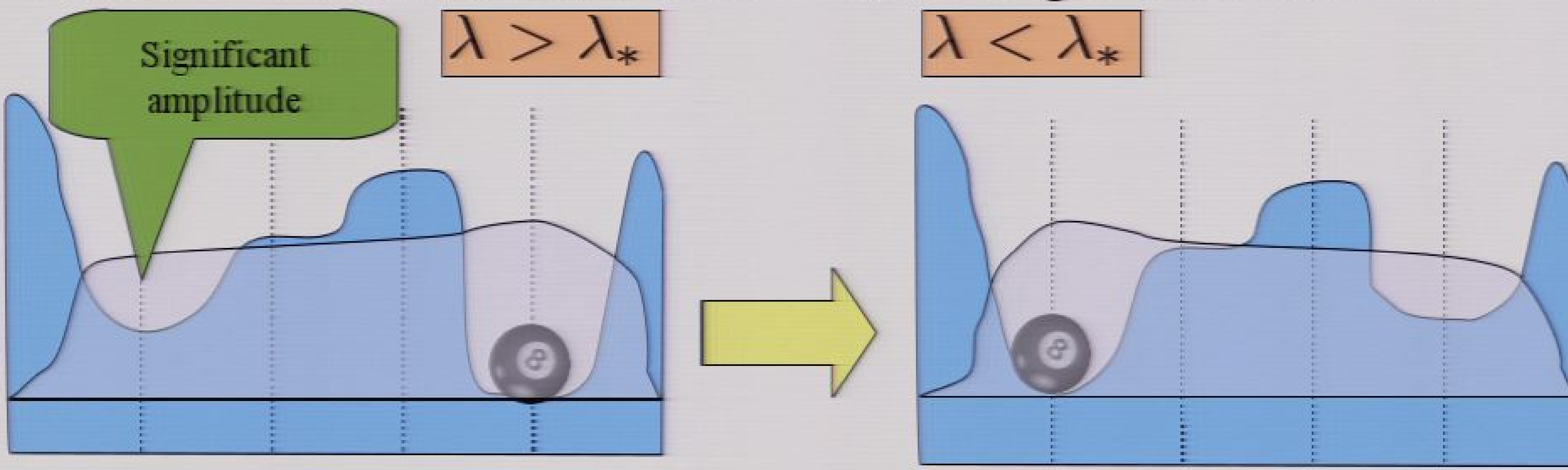


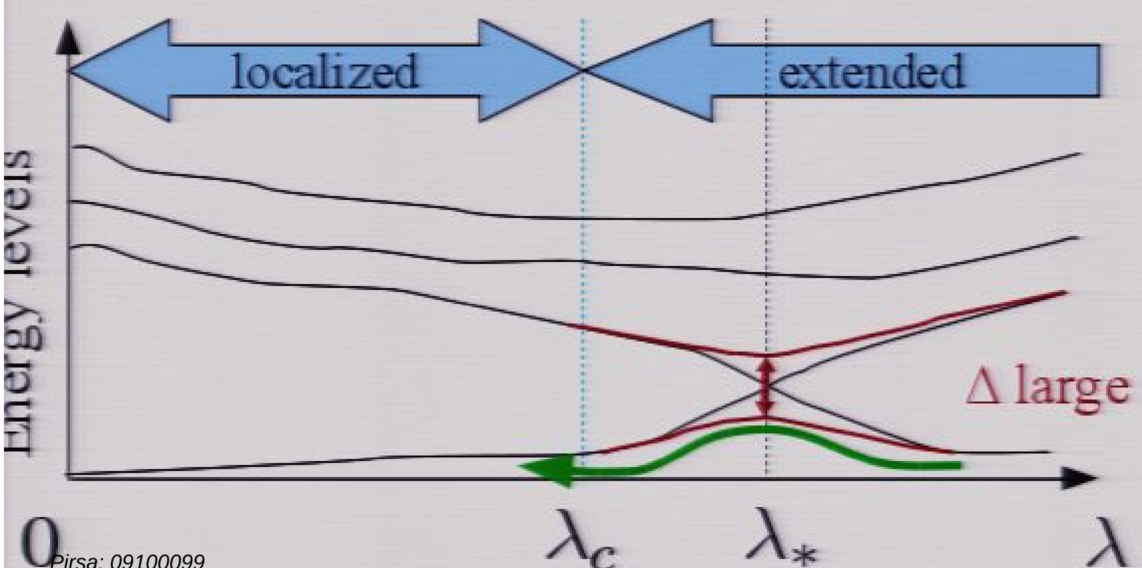


Large anti-crossing gap

Tunneling: extended state

What if a local minimum later becomes the global minimum?





Large anti-crossing gap

→ Tunneling

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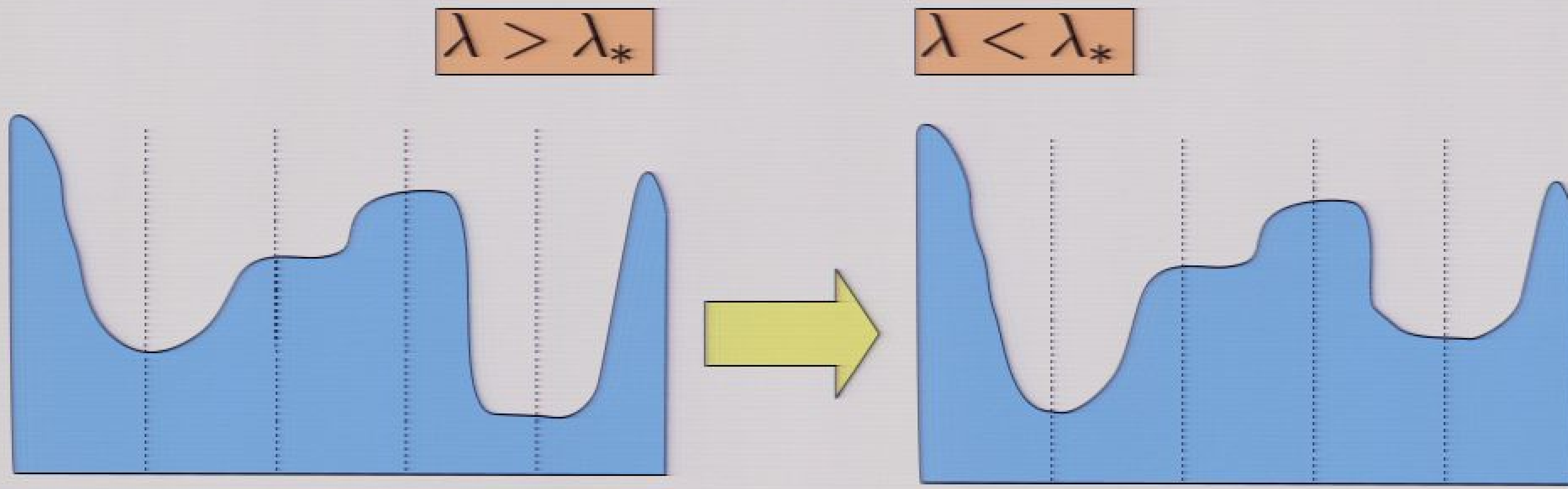
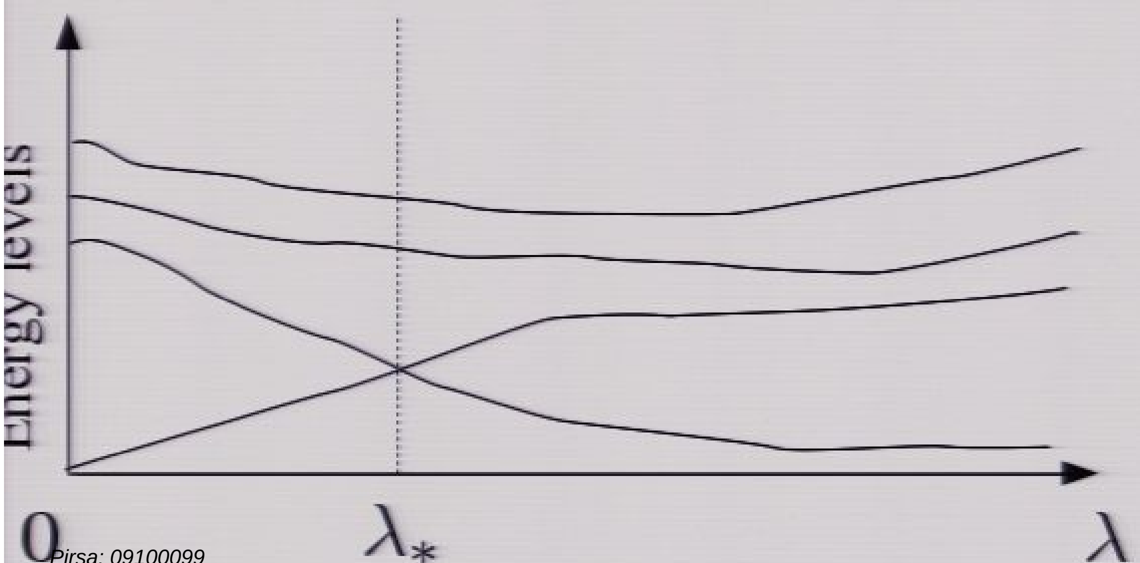
Tunneling: localized state

What if a local minimum later becomes the global minimum?





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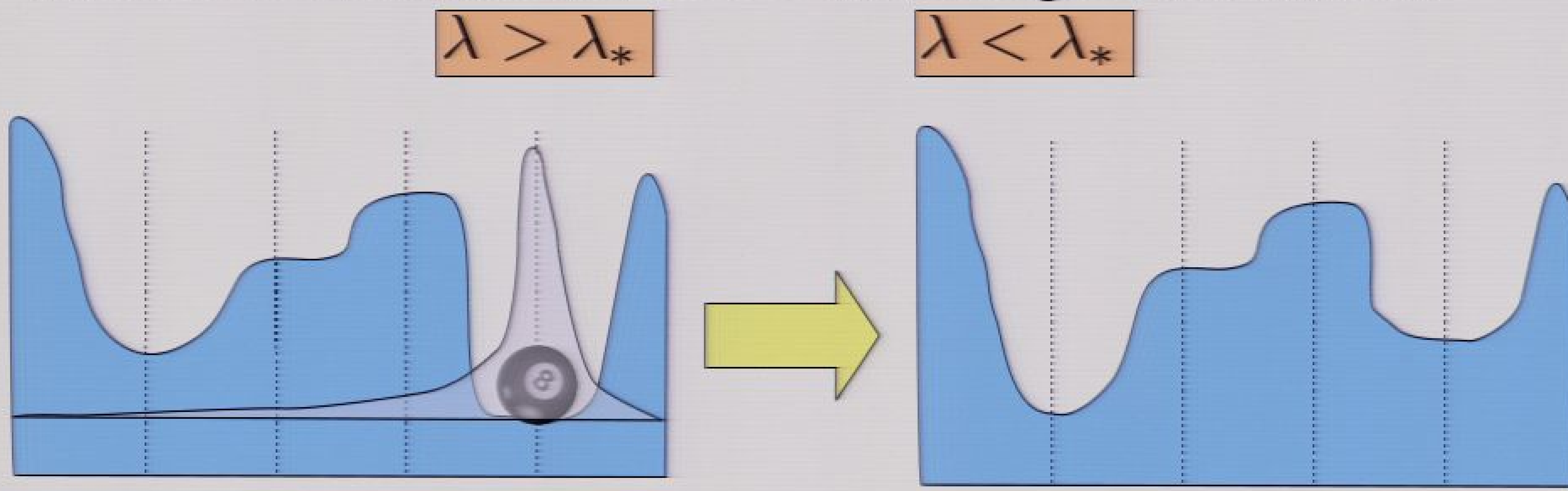

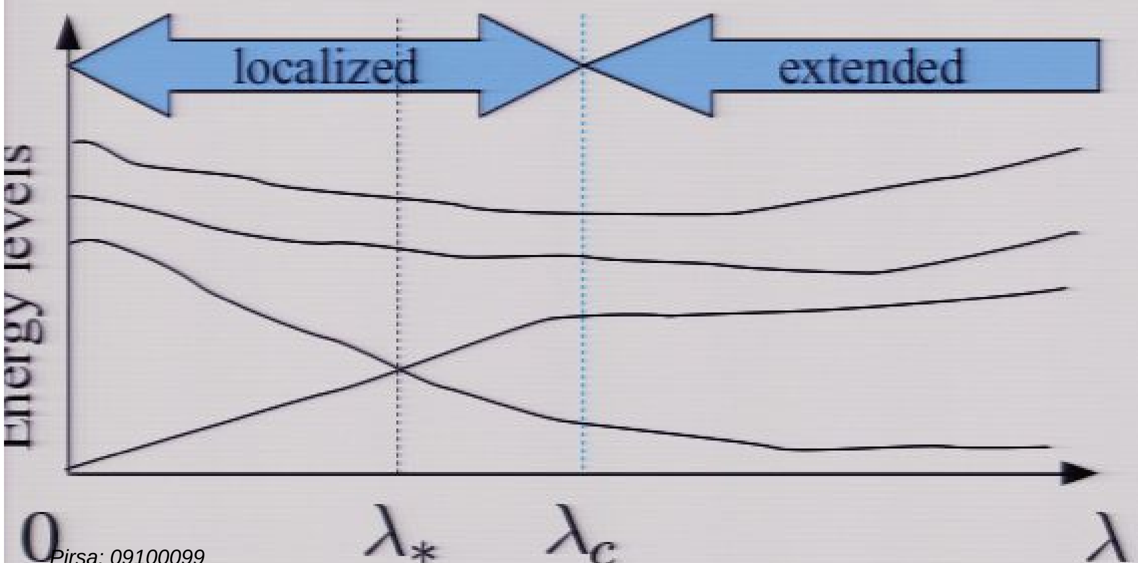

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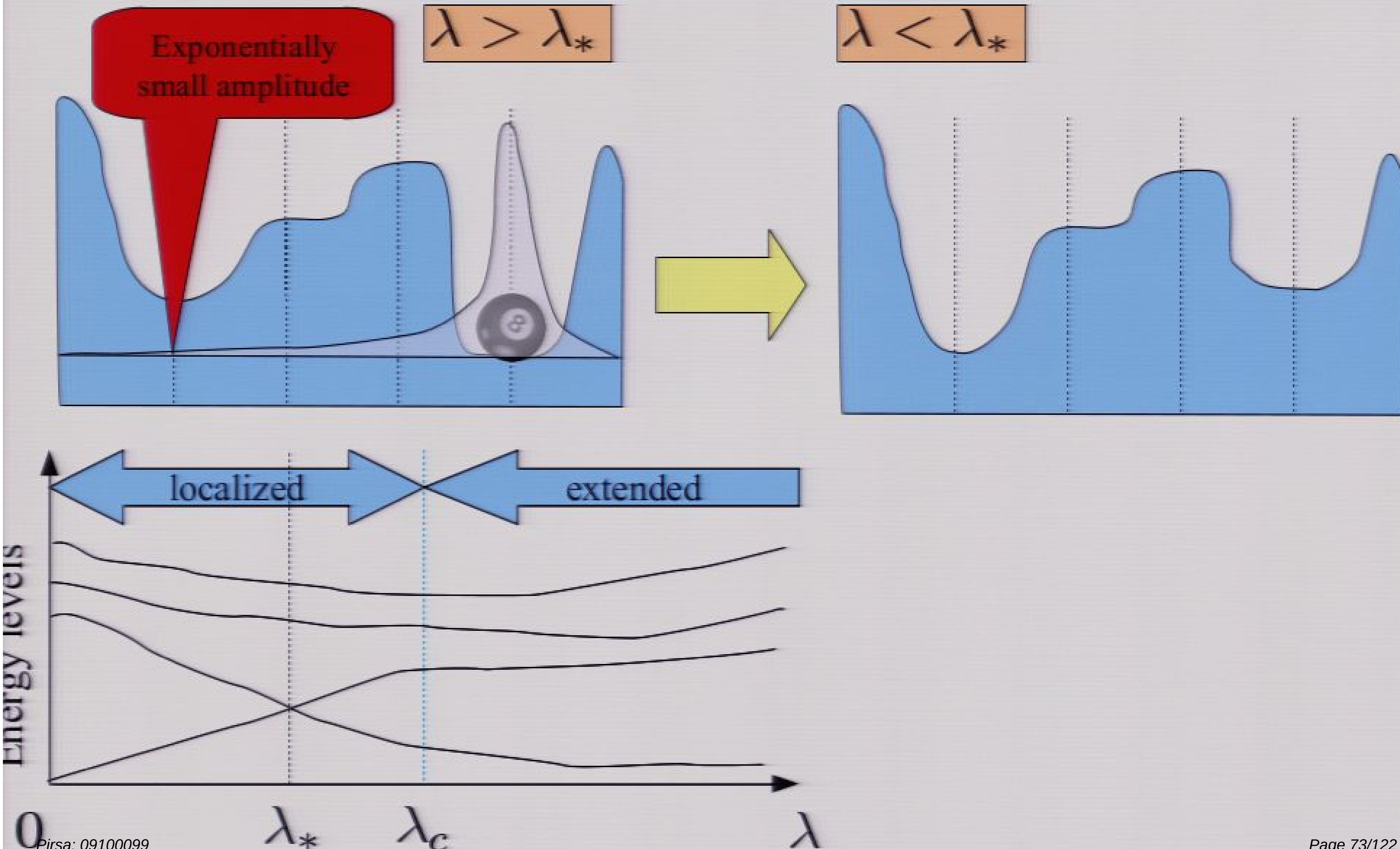

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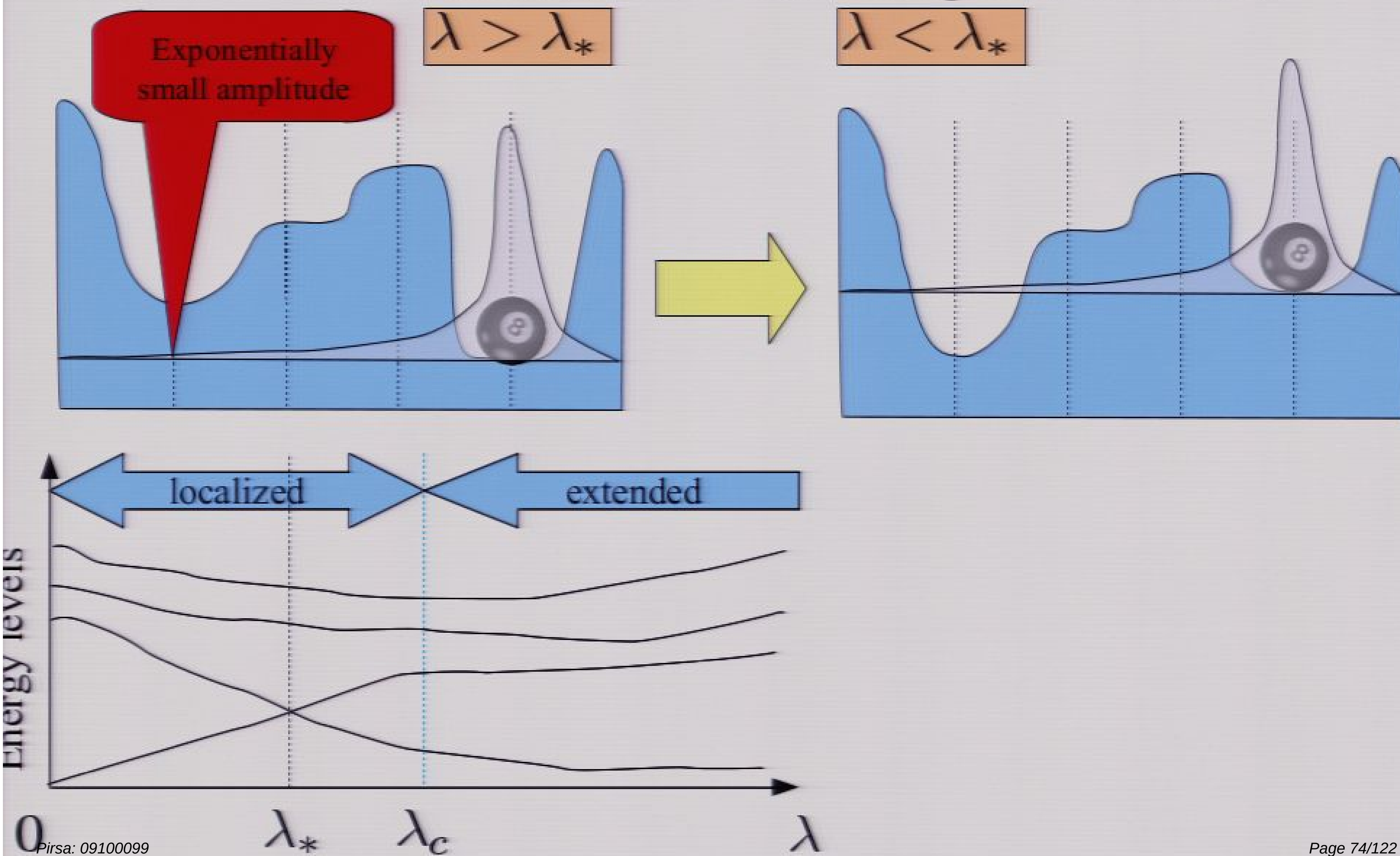

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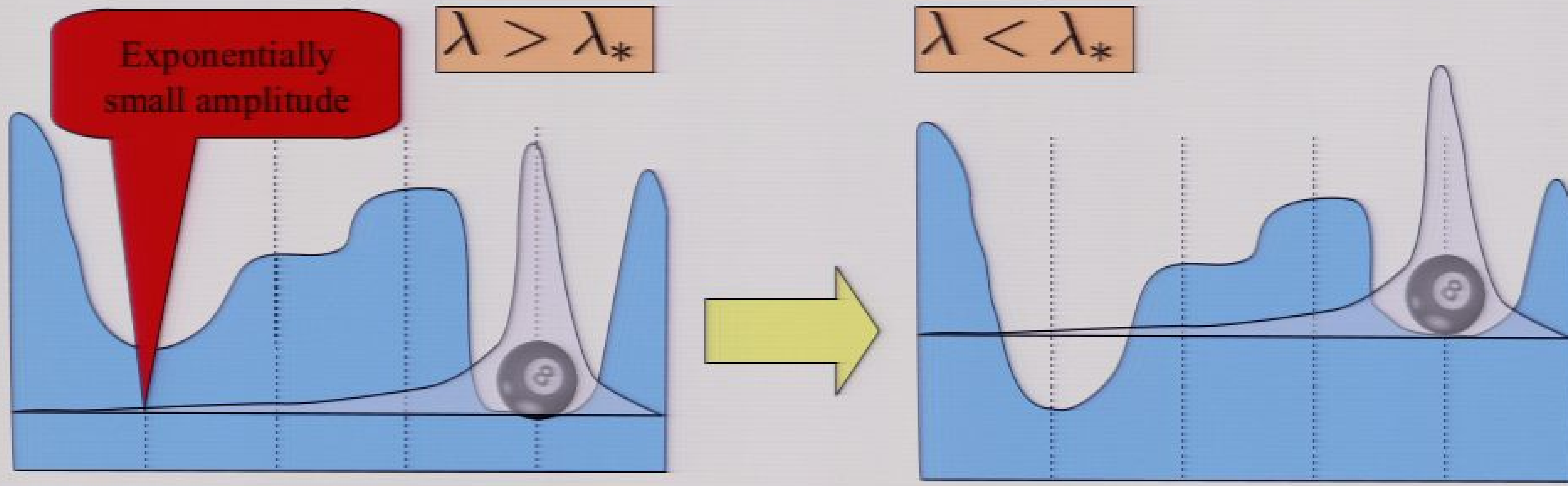

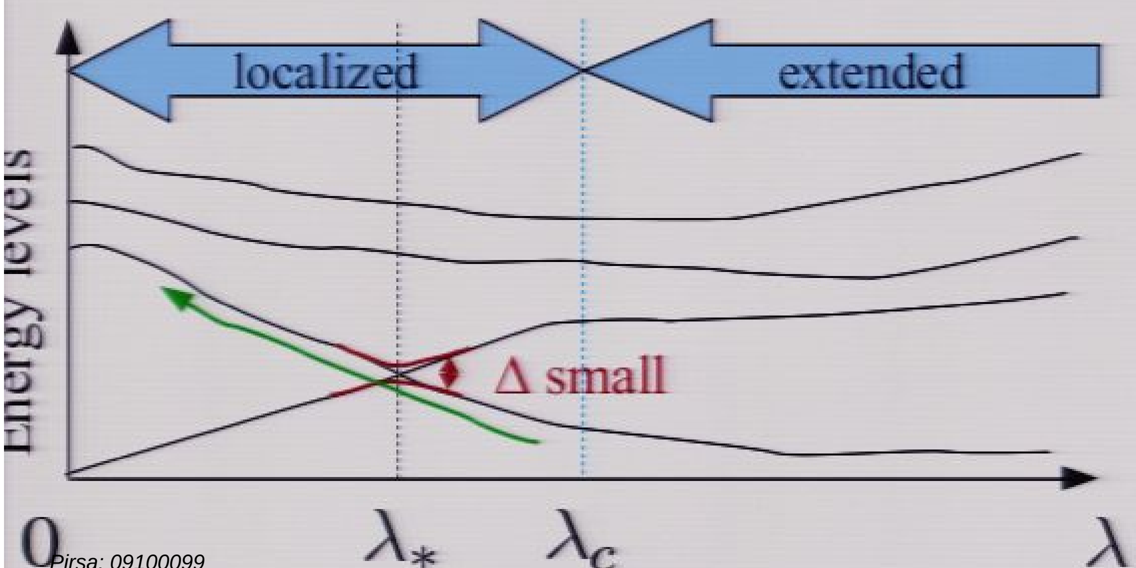

Small anti-crossing gap

→ Landau-Zener transition

Our result

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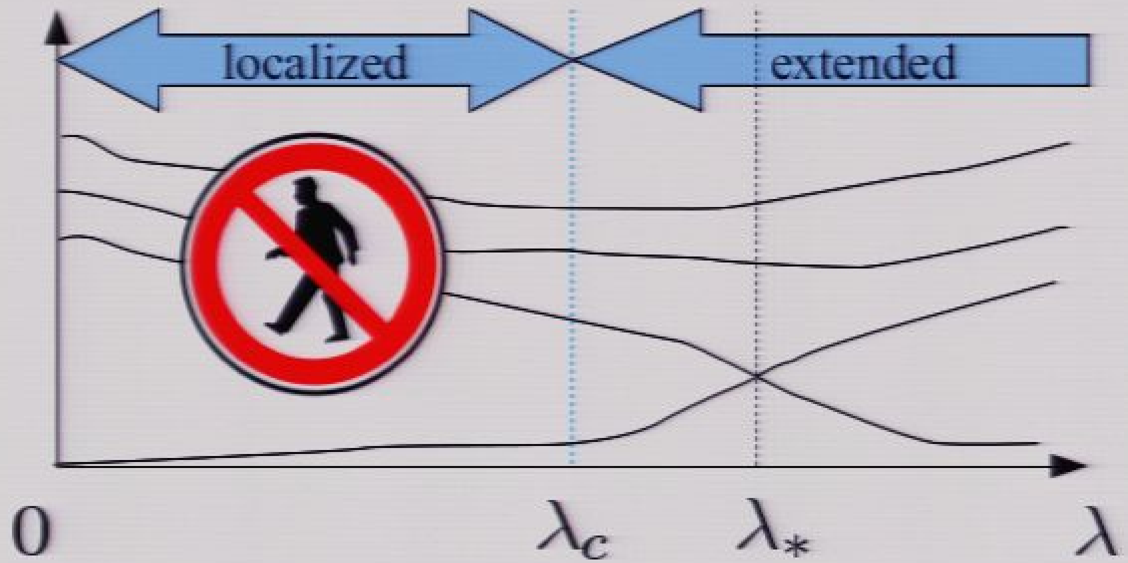
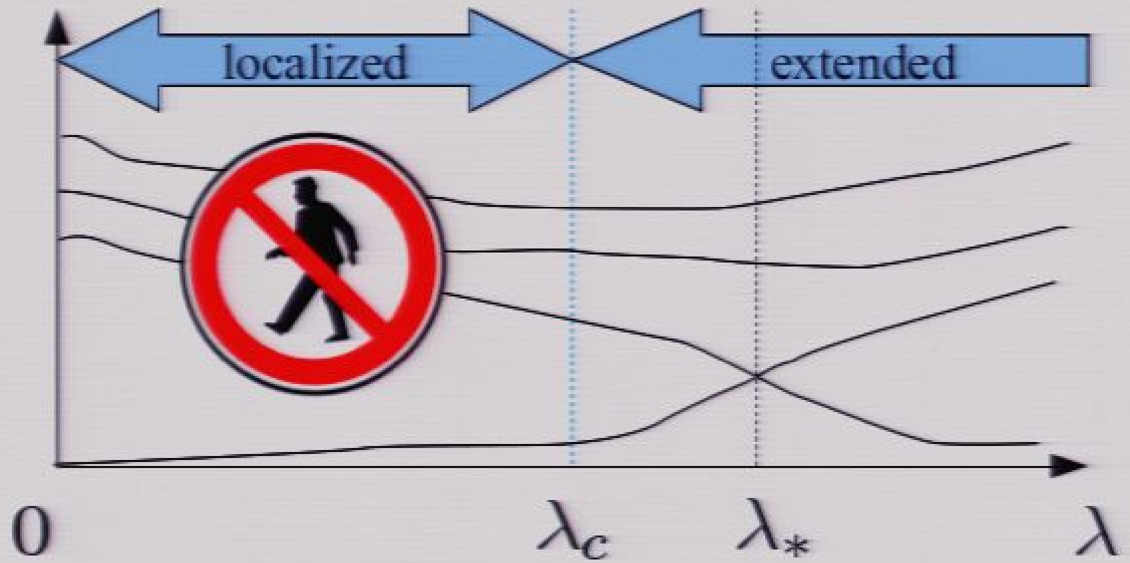


As the size of the problem N increases

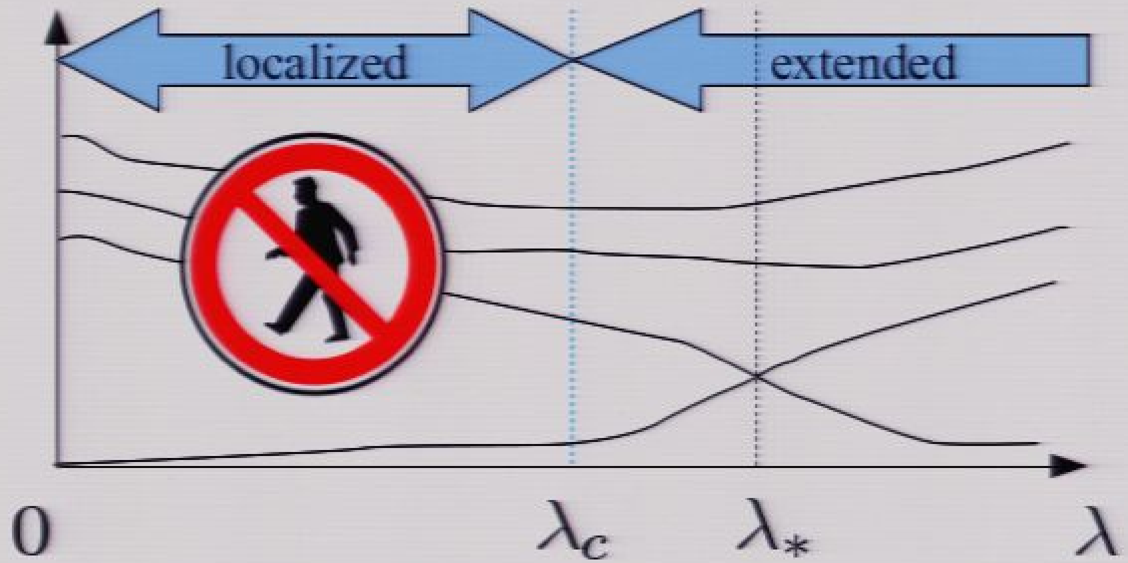
Our result



As the size of the problem N increases

1) Anderson localization would imply $\lambda_c = \Omega(1/\log N)$

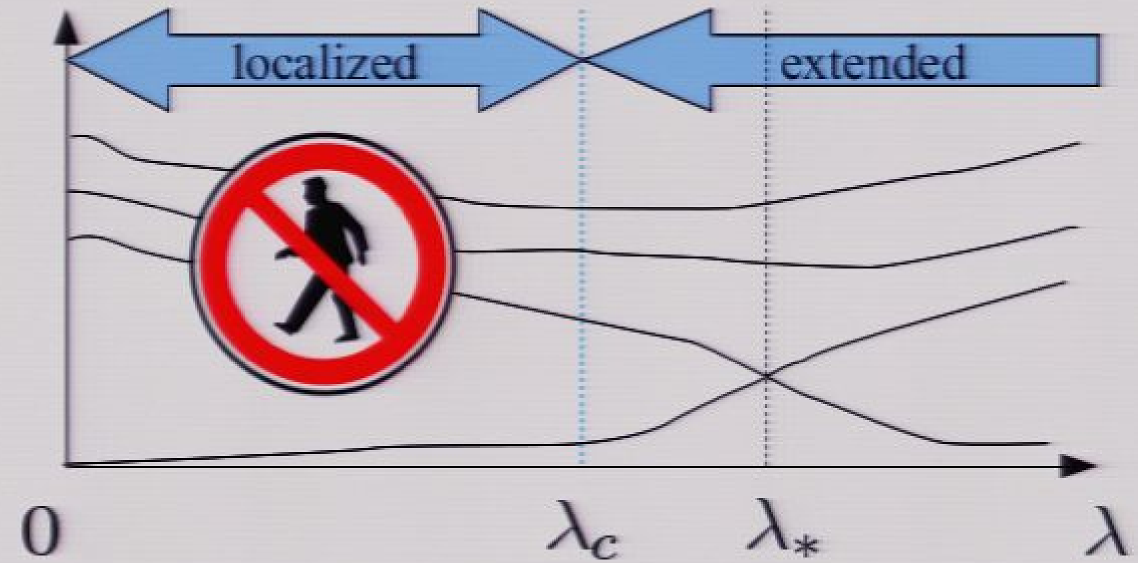
Our result



As the size of the problem N increases

1) Anderson localization would imply $\lambda_c = \Omega(1/\log N)$
2) Level crossings for smaller and smaller $\lambda_* = (CN)^{-1/8}$

Our result



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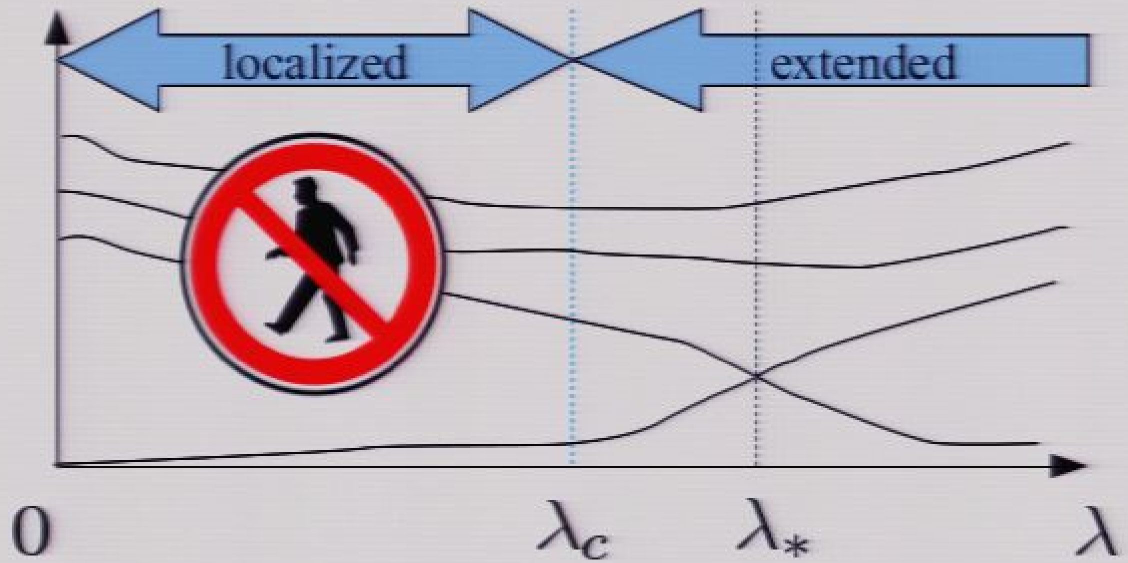
Our result



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The algorithm fails (stuck in a local minimum)

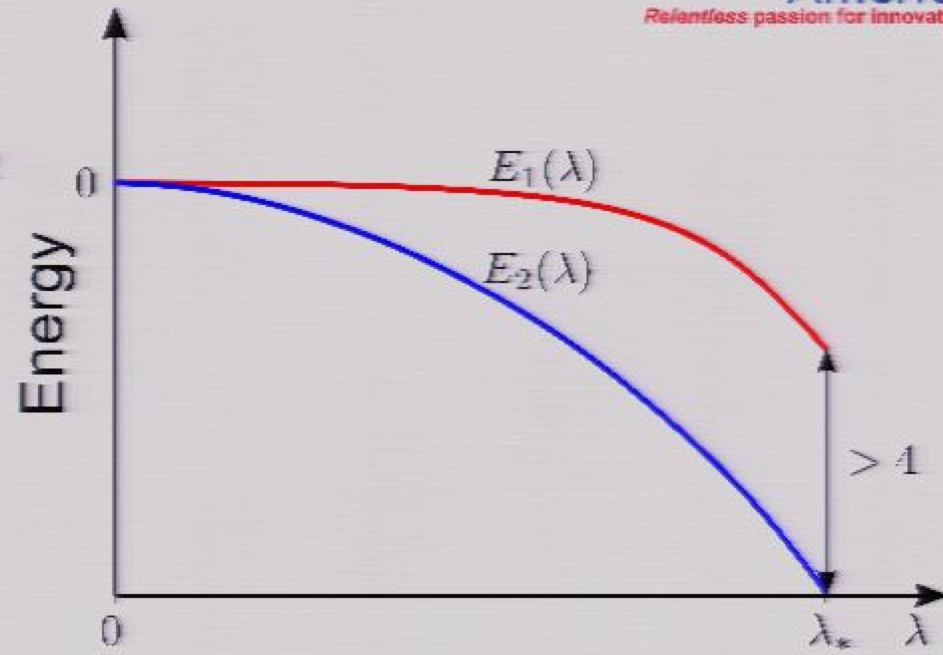
Level crossings

Consider EC3 instance with 2 solutions $\vec{x}_1, \vec{x}_2$

$$E_1(0) = E_2(0) = 0$$

Suppose

$$E_1(\lambda_*) - E_2(\lambda_*) > 4$$



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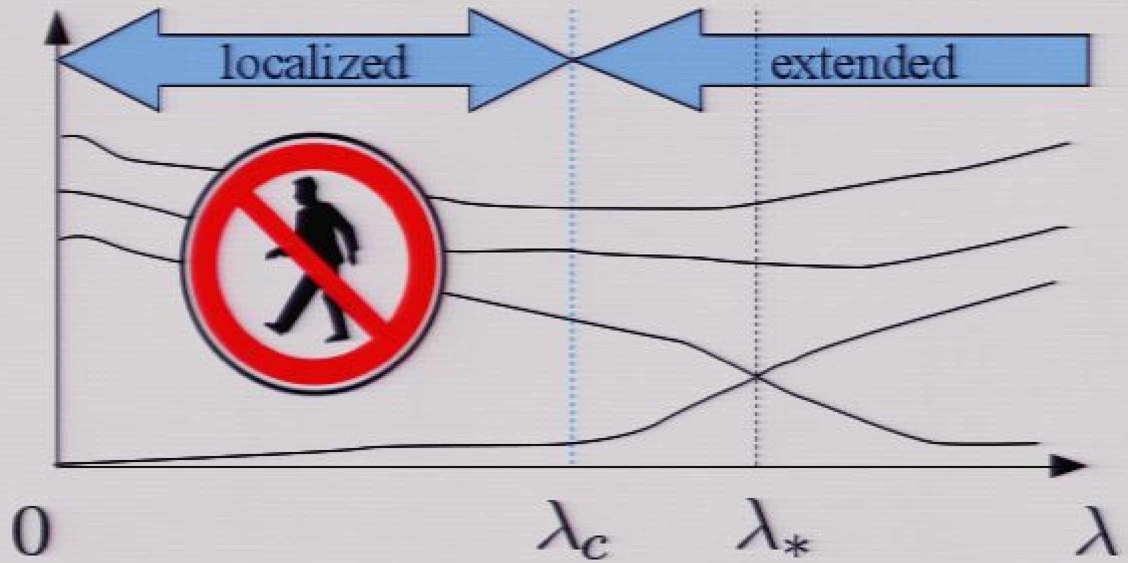
Our result



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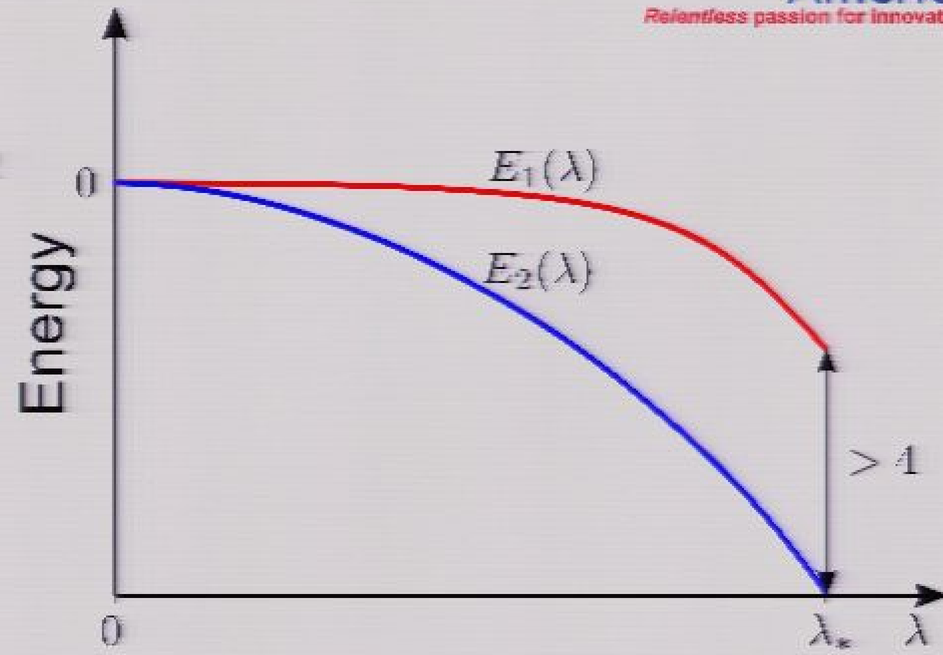

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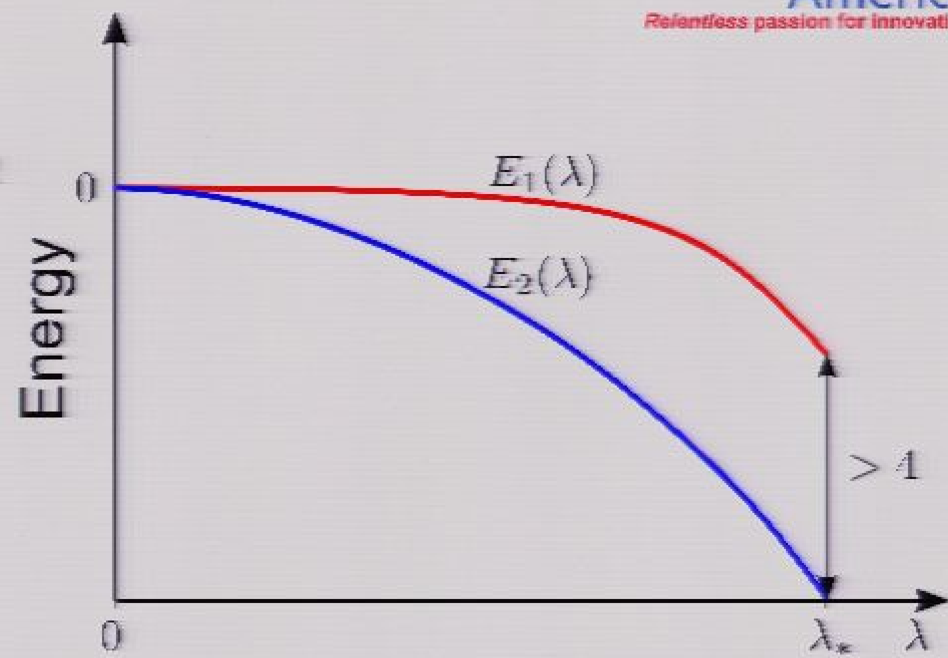
Add a clause

- satisfied by $\vec{x}_1$ $\tilde{E}_1(0) = 0$
- violated by $\vec{x}_2$ $\tilde{E}_2(0) = 4$



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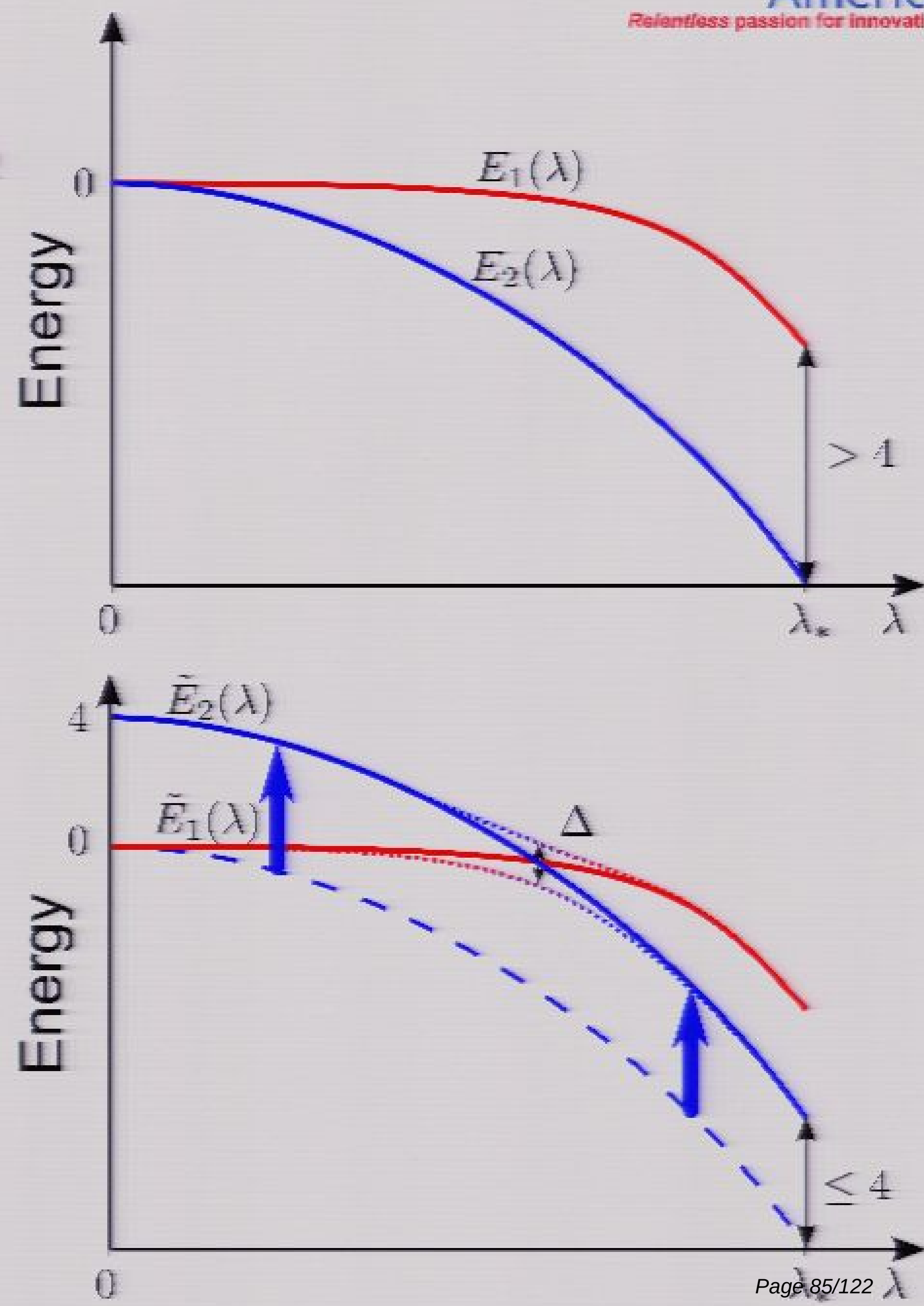
Level crossing



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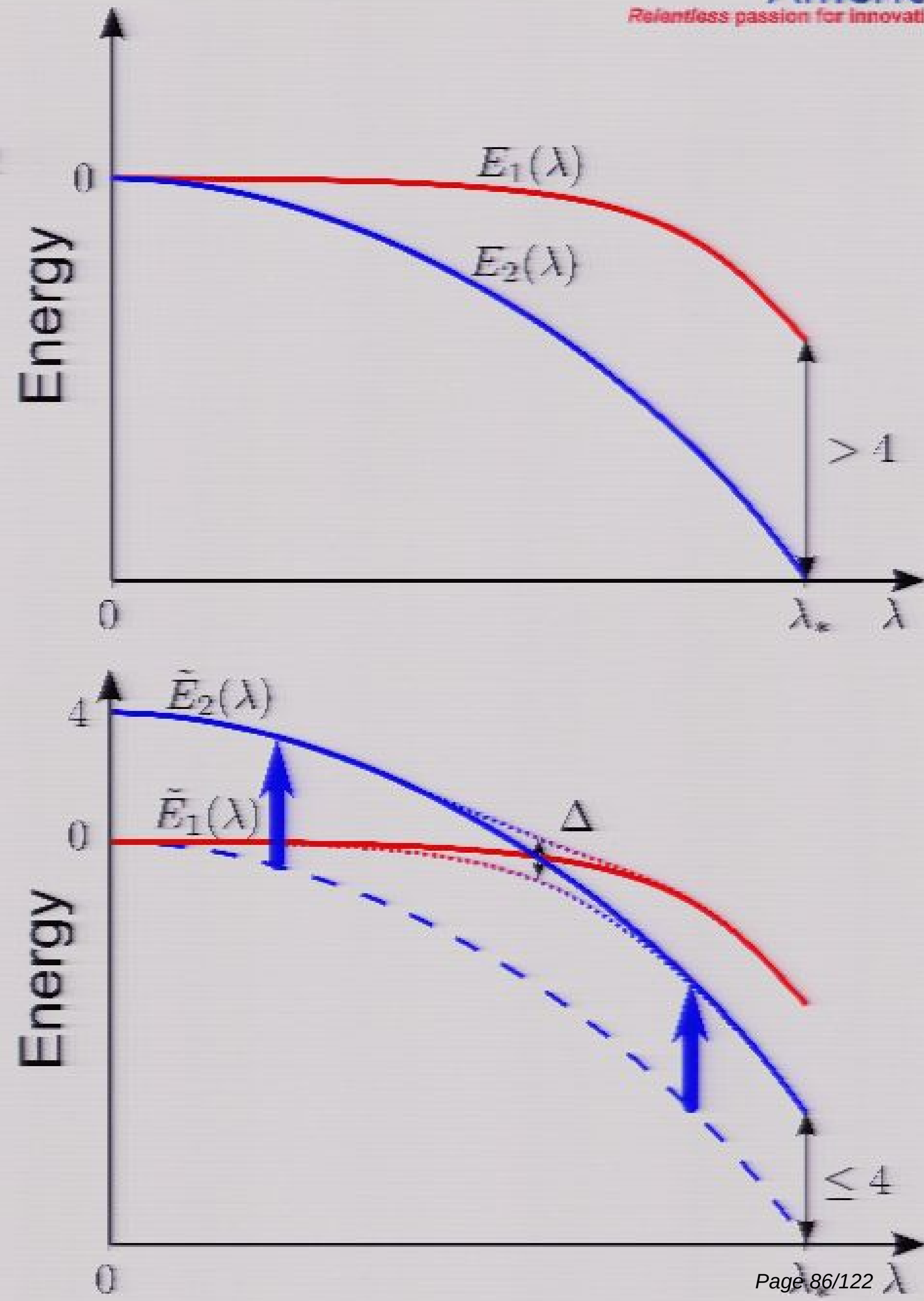
Level crossing

$$d(\vec{x}_1, \vec{x}_2) = n$$

$\rightarrow$ Gap $\Delta \sim \lambda_*^n$



What if the wrong solution is killed?

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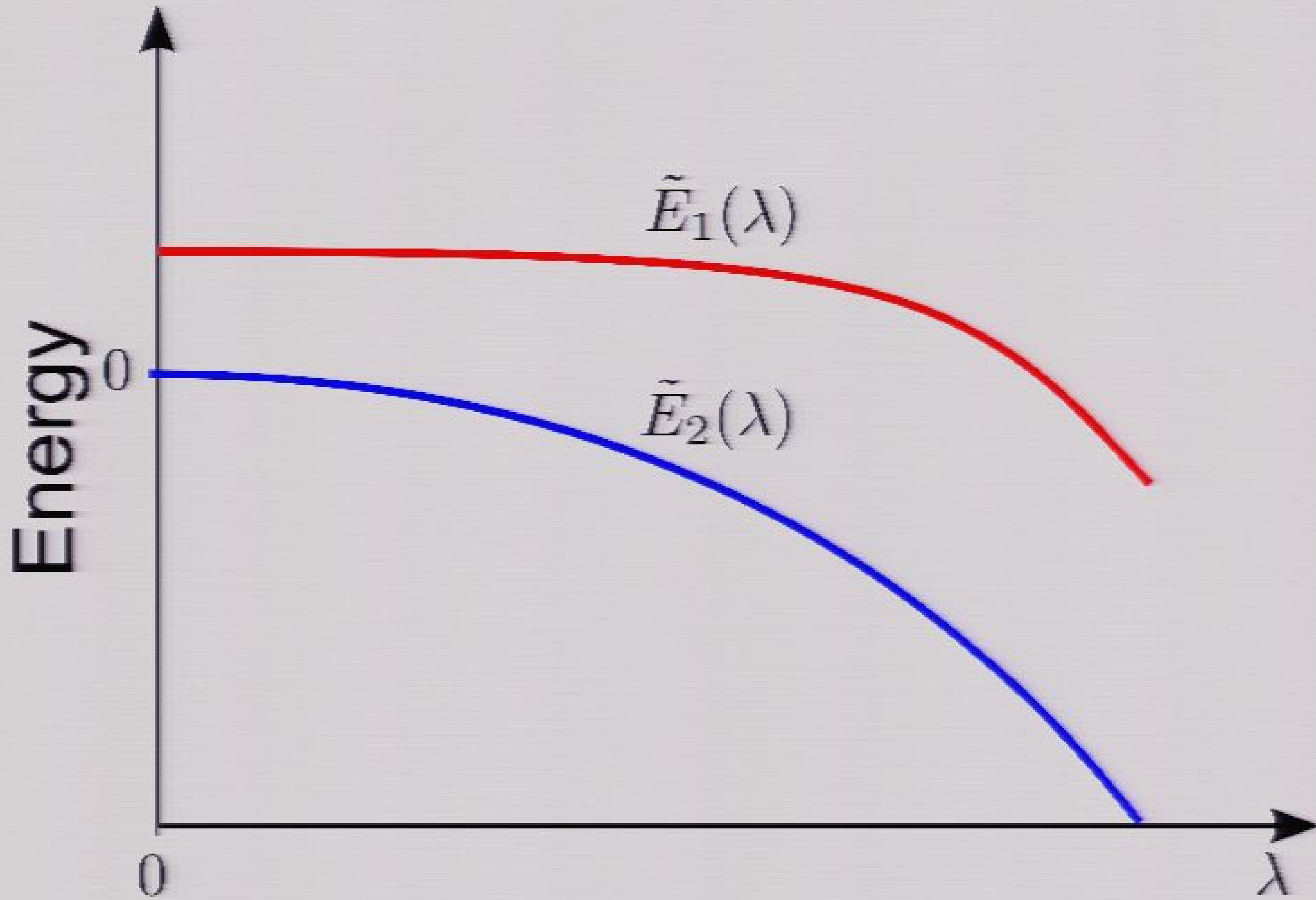

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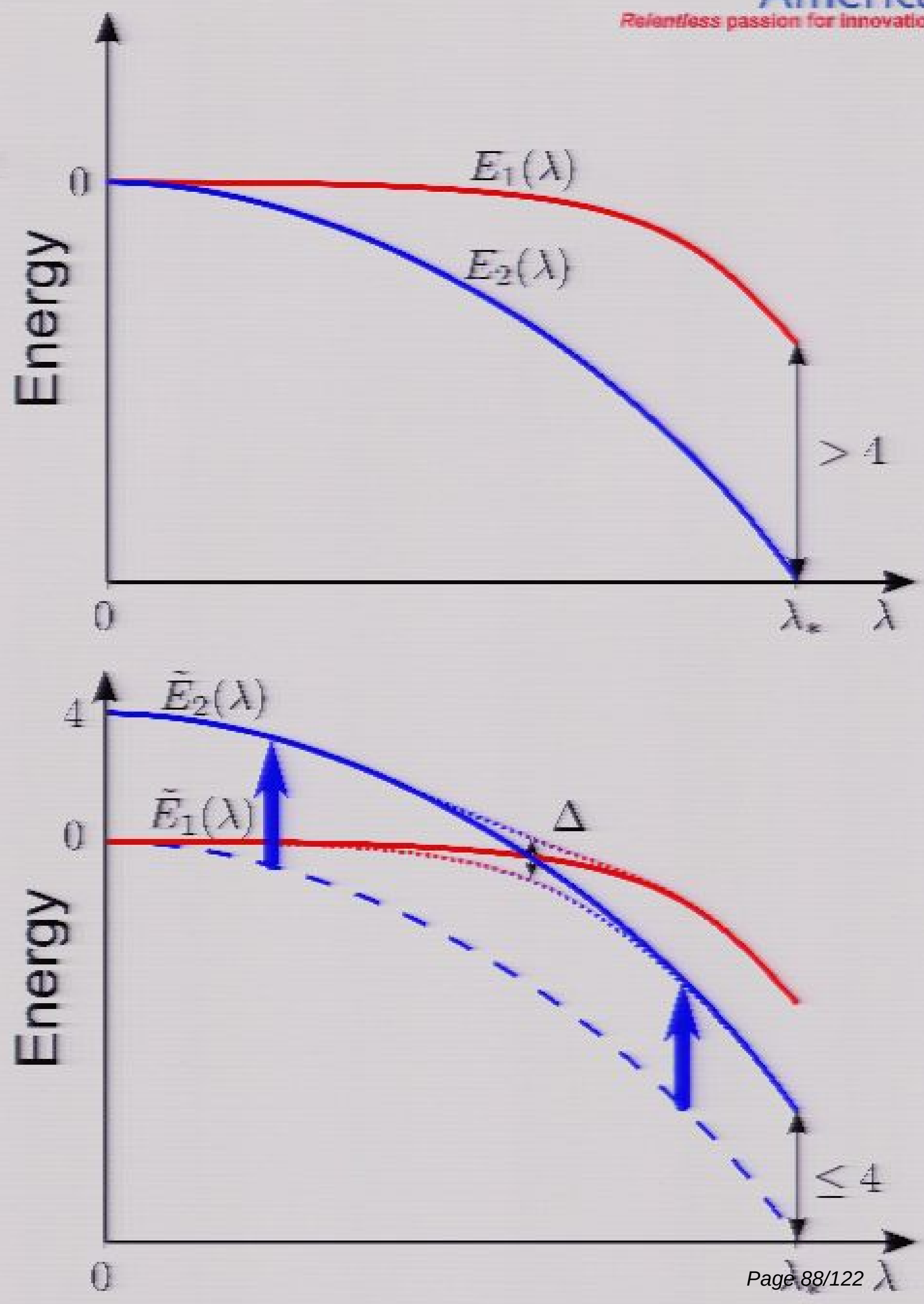
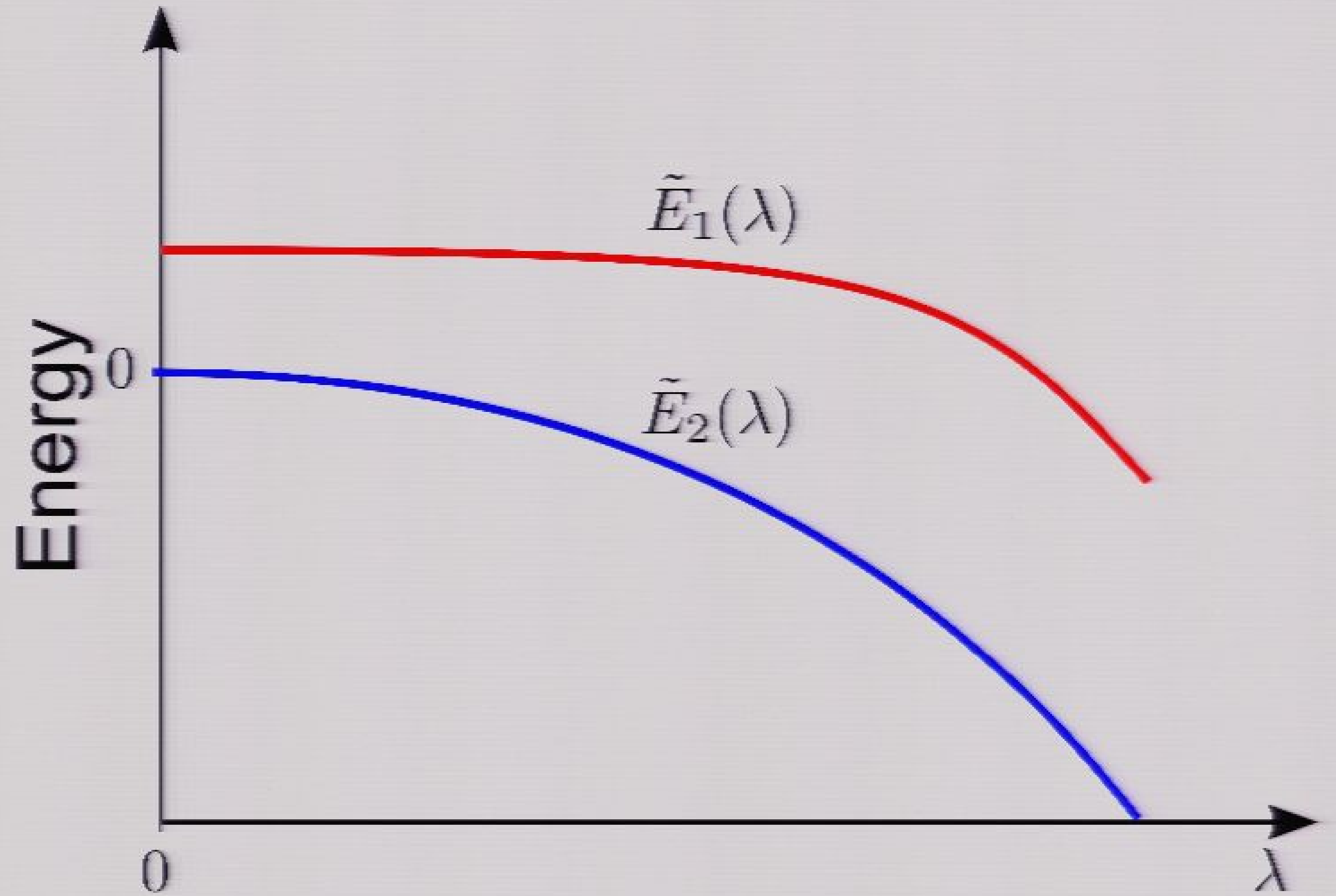
Level crossing

$$d(\vec{x}_1, \vec{x}_2) = n$$

$\rightarrow$ Gap $\quad \Delta \sim \lambda_*^n$



What if the wrong solution is killed?



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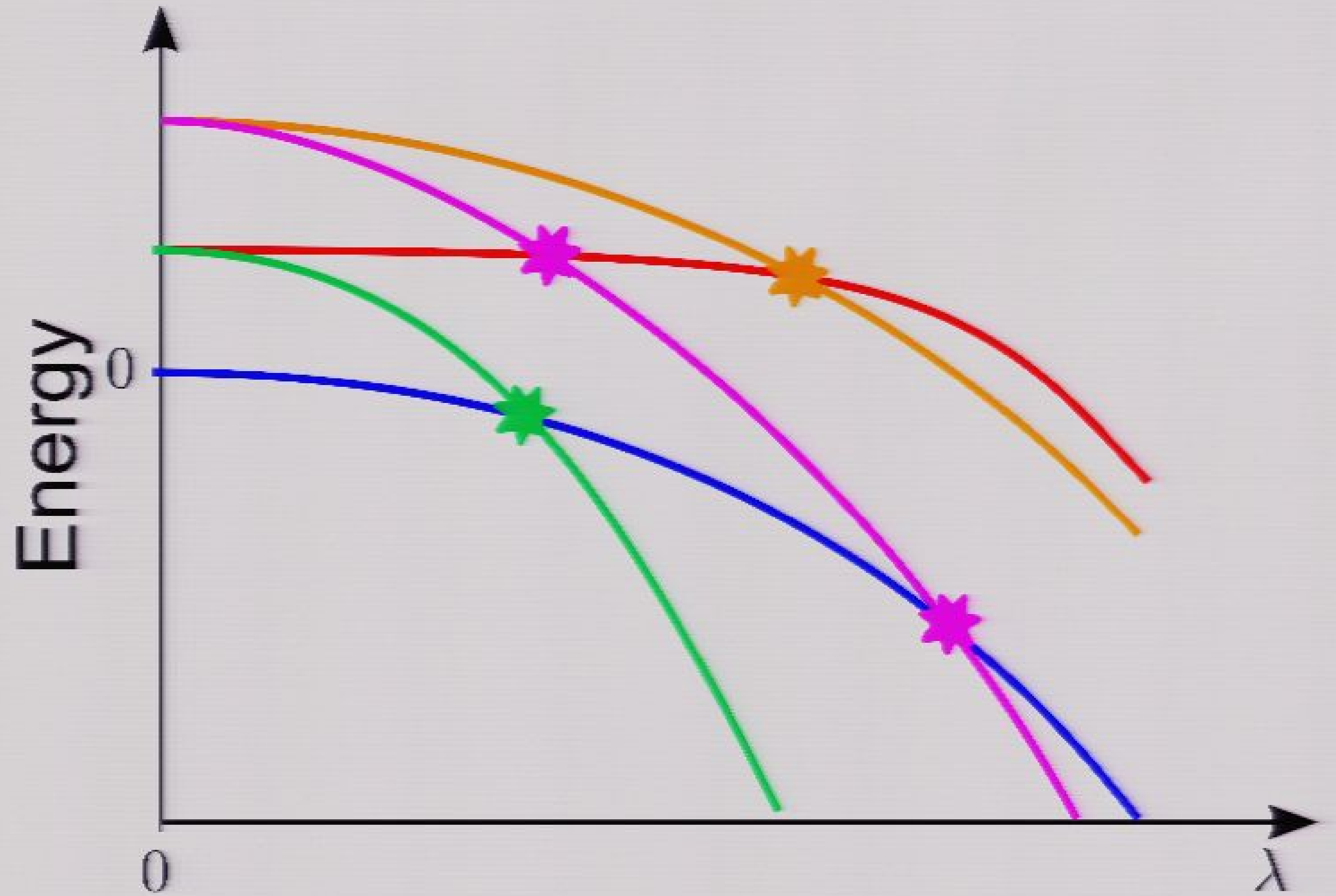

There are crossings with other levels!

Perturbation theory

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We compute $E_{1,2}(\lambda)$ by perturbation theory

$$E_{\vec{x}}(\lambda) = E_{\vec{x}}(0) + \sum_{m=1}^{\infty} \lambda^{2m} F_{\vec{x}}^{(m)}$$

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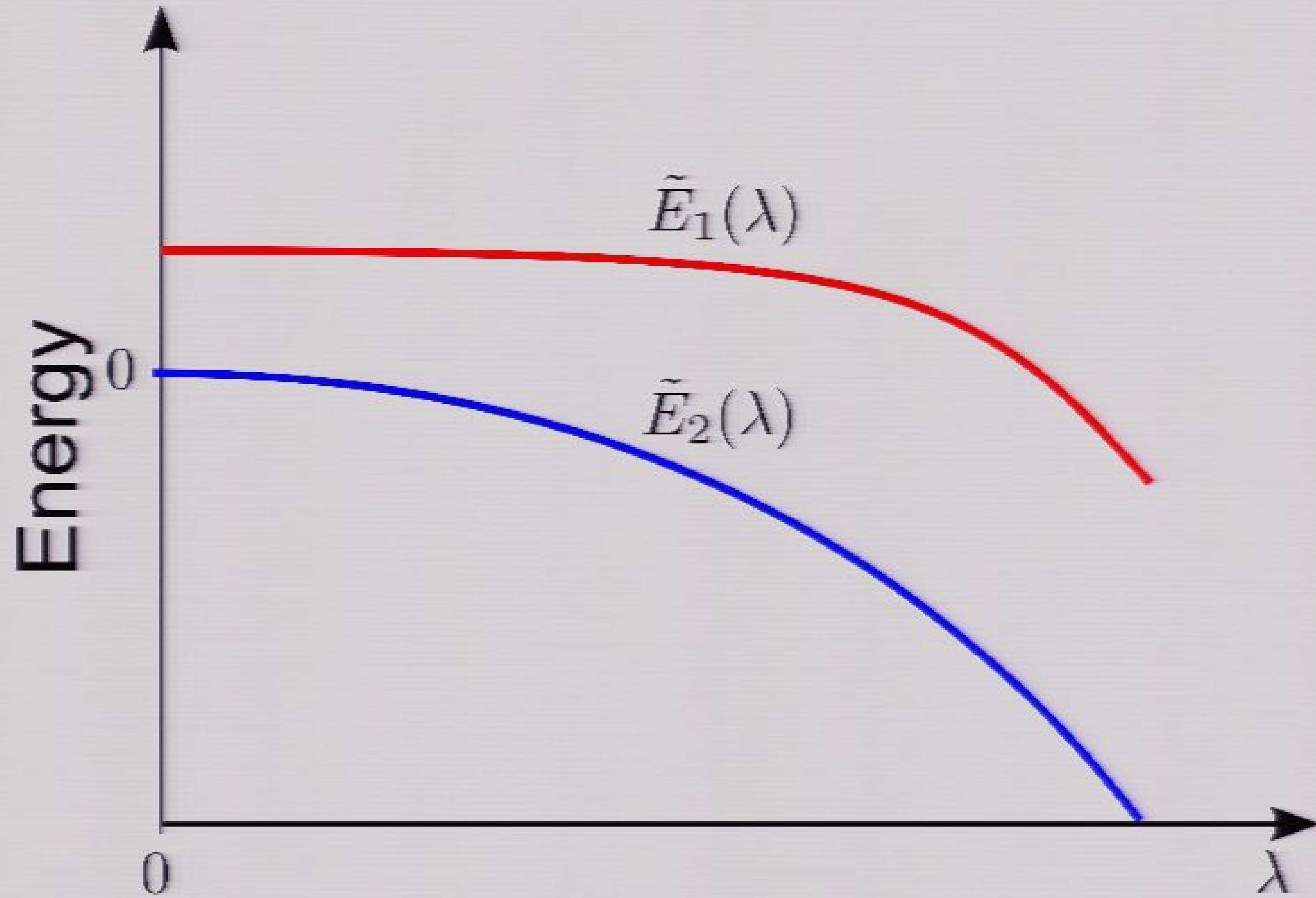

Level crossings

Consider EC3 instance with 2 solutions $\vec{x}_1, \vec{x}_2$

$$E_1(0) = E_2(0) = 0$$

Suppose

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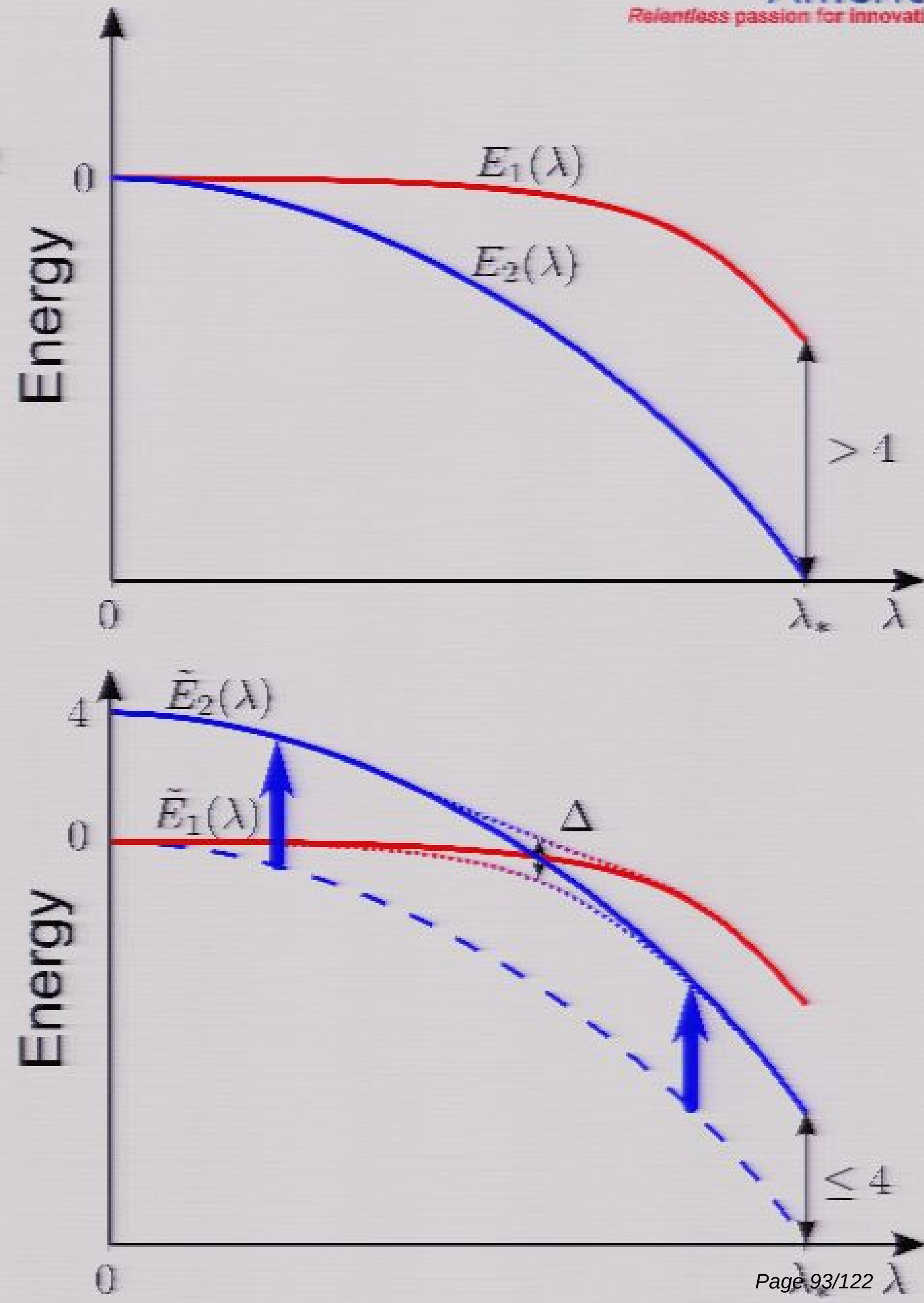

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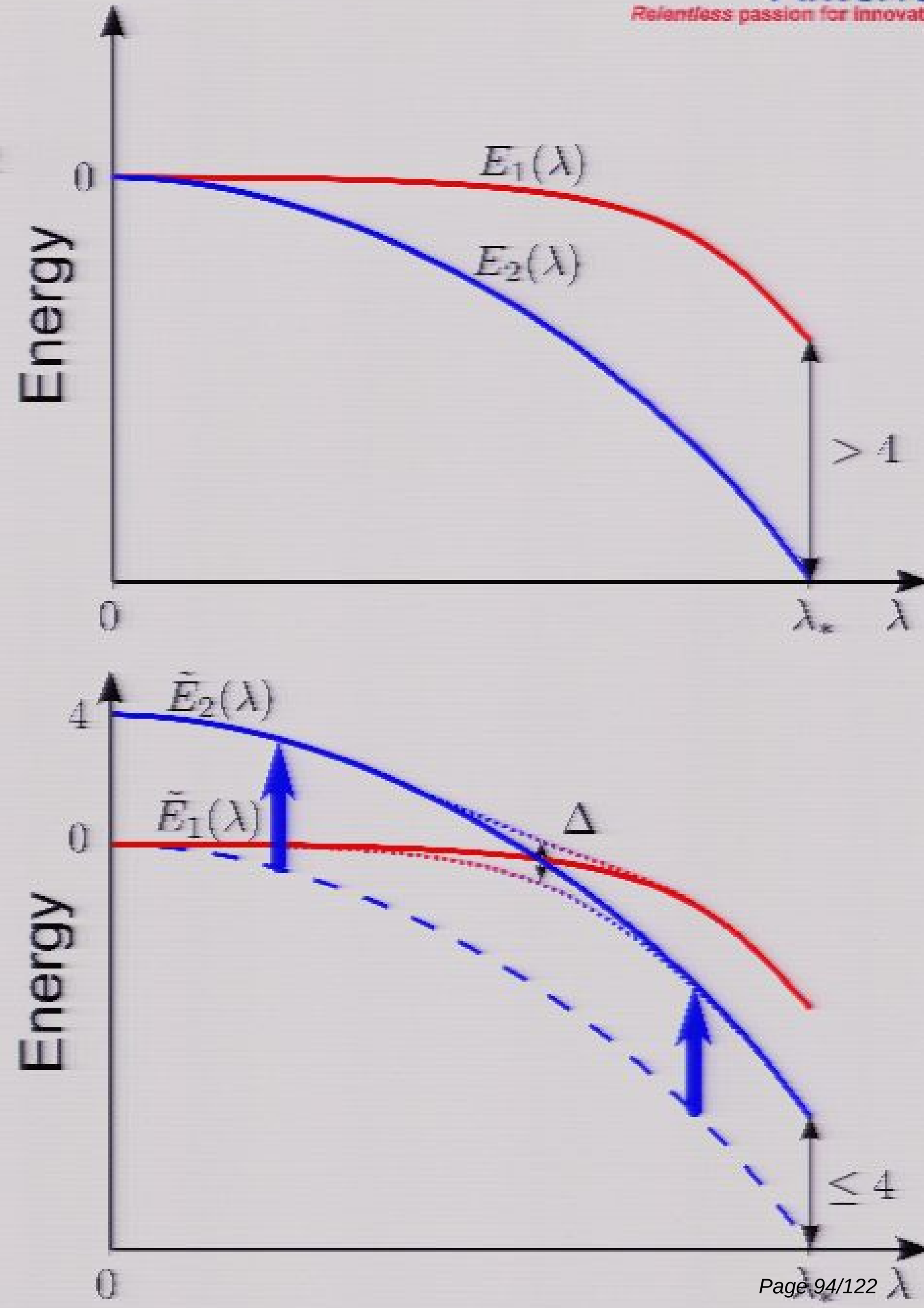

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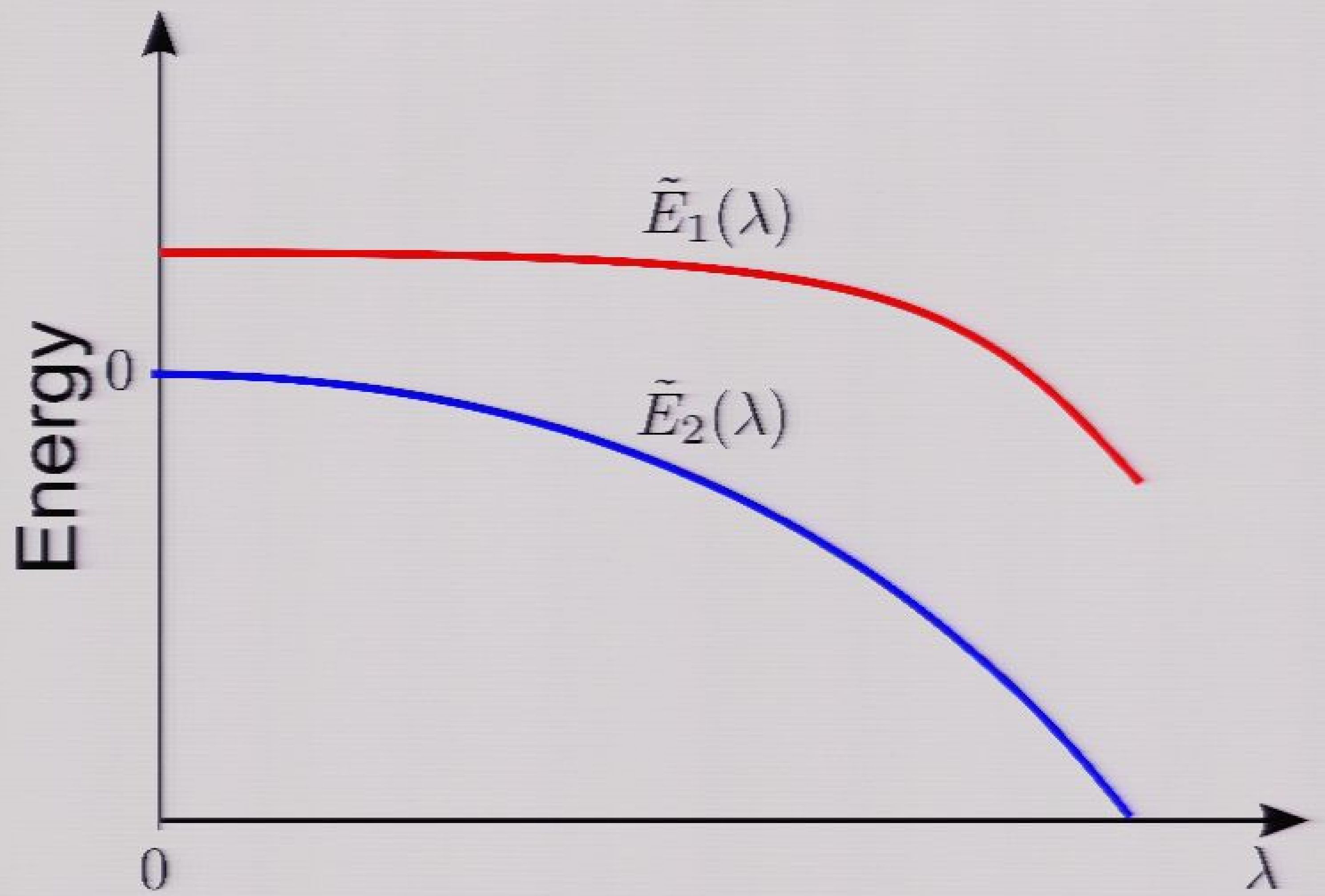

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We prove: $\boxed{F_{\vec{x}}^{(m)} = O(N)\ \forall m}$

Proof based on statistical properties of random instances

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For 2 solutions, the difference has zero mean, so

$$\boxed{(F_1^{(m)} - F_2^{(m)})^2 = O(N) \ \forall m}$$

Numerical simulations

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- We generated EC3 random instances with >2 solutions
- then computed $E_1(\lambda) - E_2(\lambda)$ by order 4 perturbation theory

Leading order because:
- Odd orders are zero
- Order 2 is solution-independent for EC3

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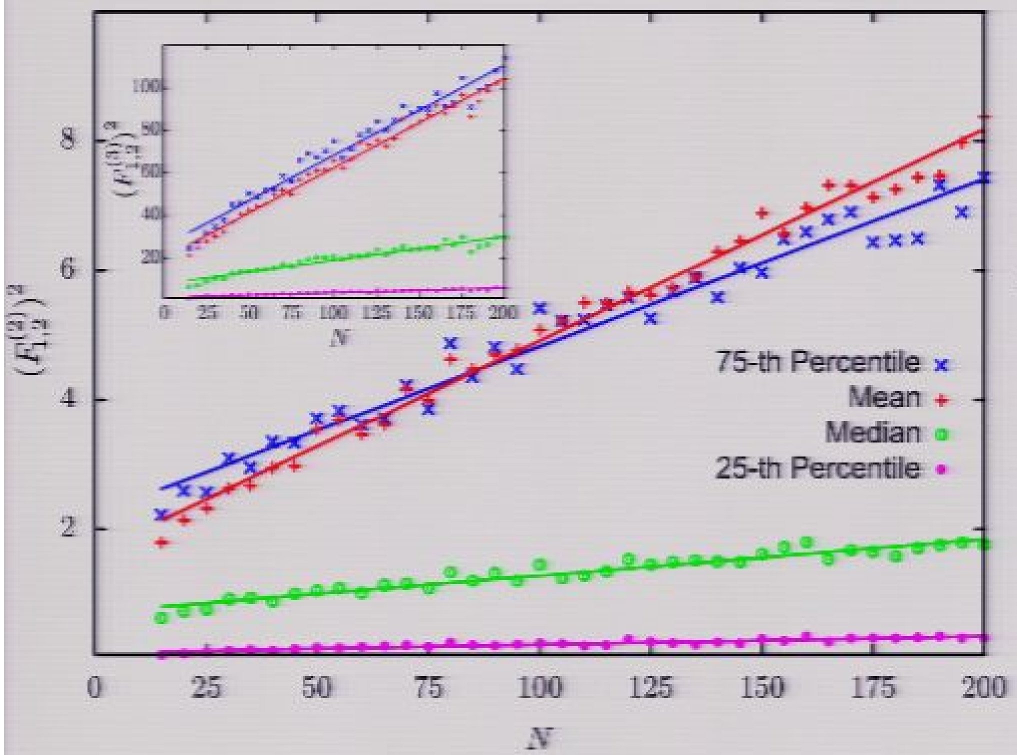

Each data point computed from 2500 instances

$$(E_1(\lambda) - E_2(\lambda))^2 \approx C N \lambda^8$$

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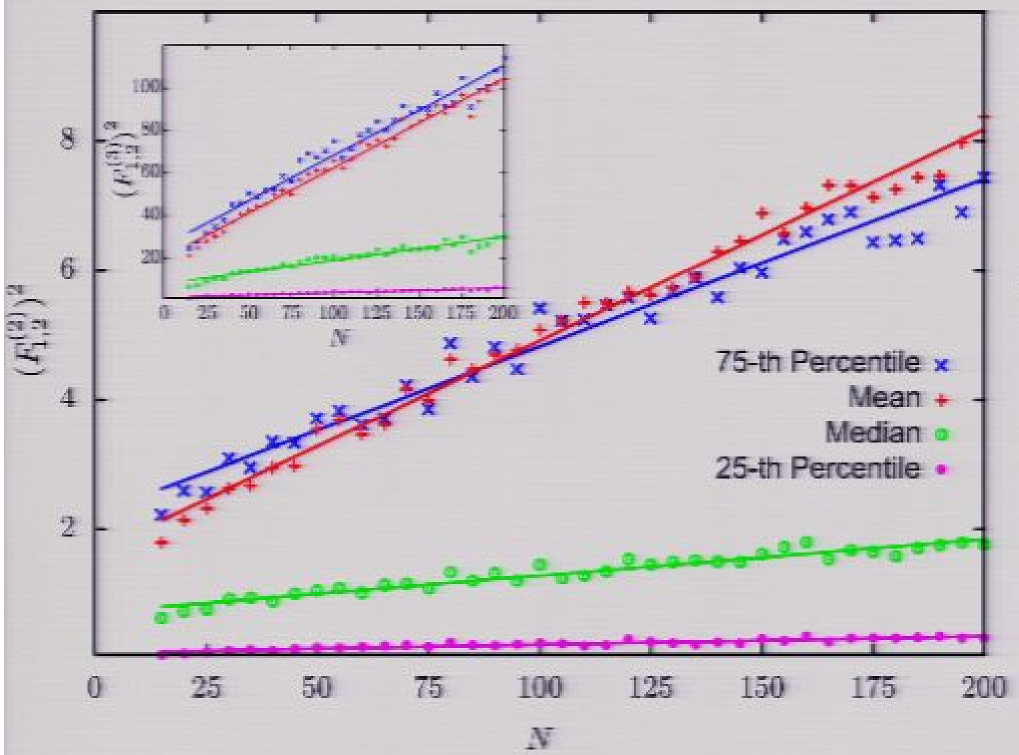

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We have $E_1(\lambda) - E_2(\lambda) > 4$ for $\lambda > \sqrt{2}(CN)^{-1/8}$

How small is the gap?

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We show that up to leading order in perturbation theory:

$$\boxed{\Delta < (2\lambda_*)^n}$$

Proof by reduction to the "Agree" problem:
2-bit clauses $(x_{i_C} = x_{j_C})$

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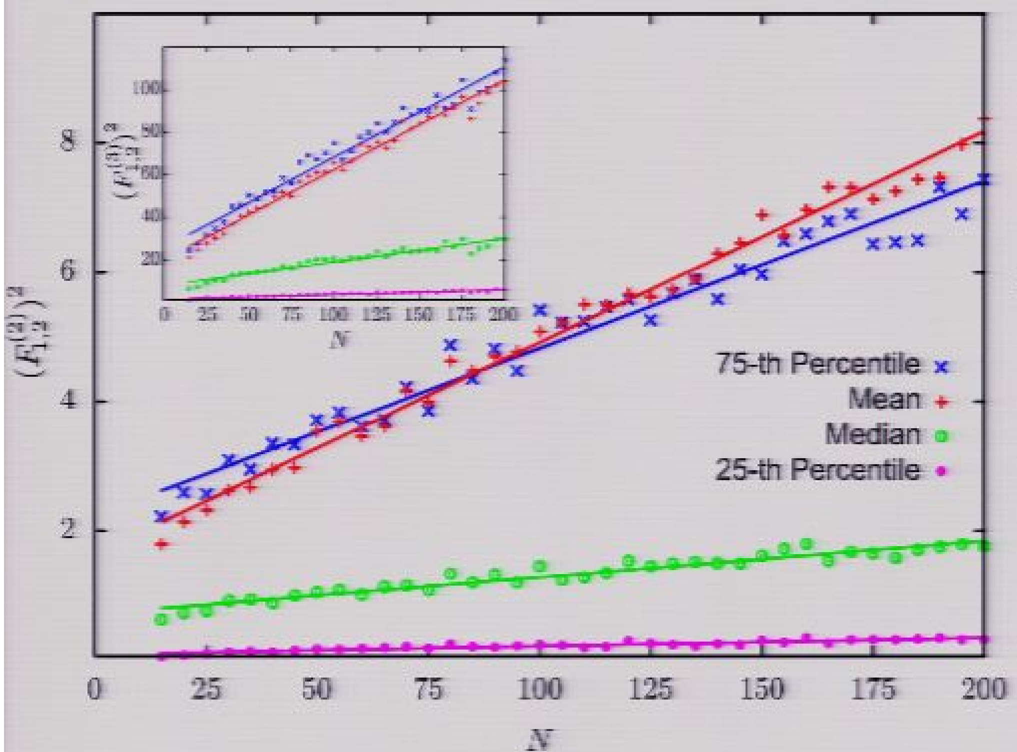

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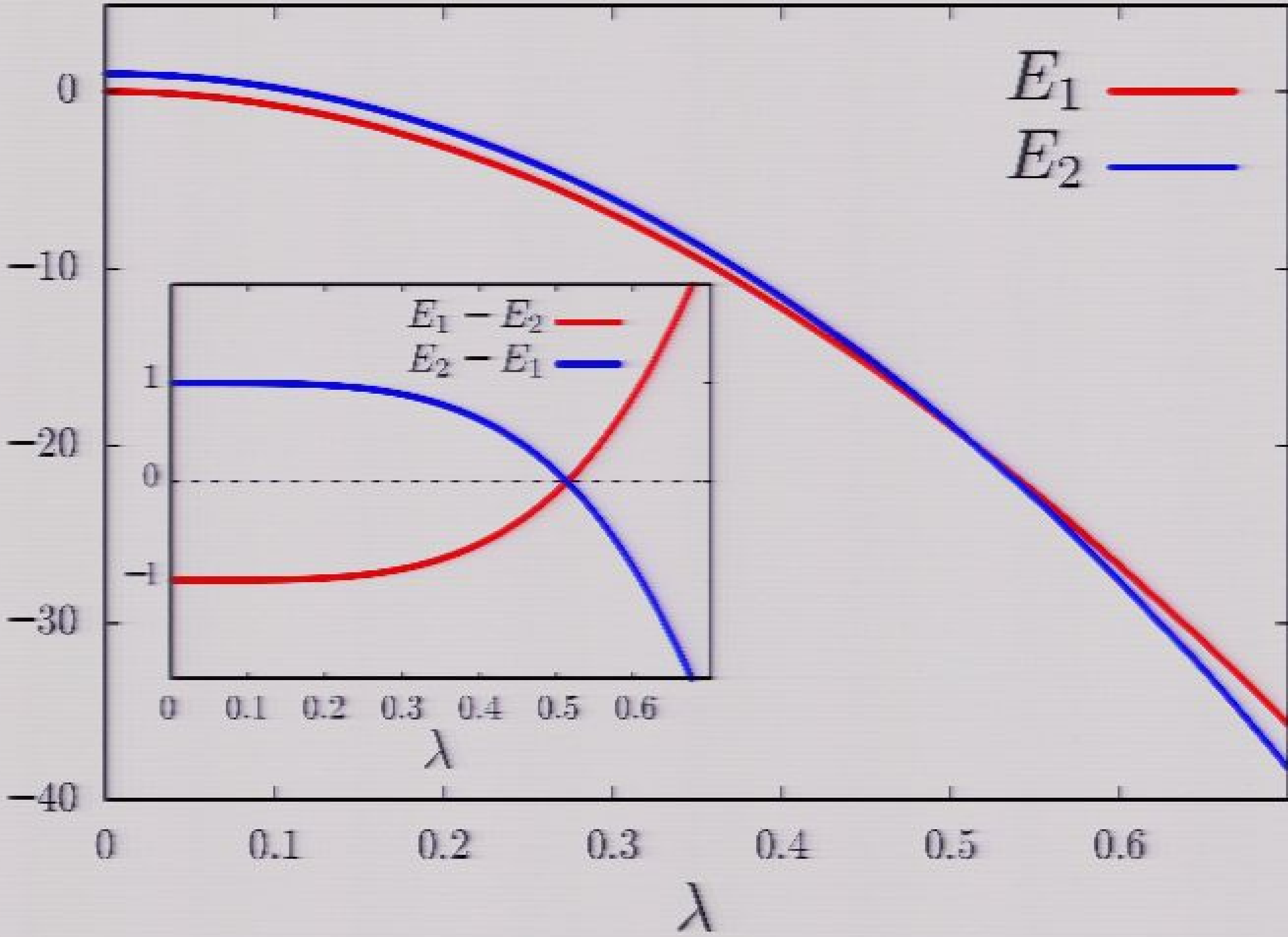
We have: $$\boxed{\Delta = O(\exp(-N \log N))}$$

Simulated anti-crossing



Simulation of an anti-crossing by 4th order perturbation theory:

- Number of bits: $N = 200$
- Distance between the assignments: $n = 60$

Simulated anti-crossing

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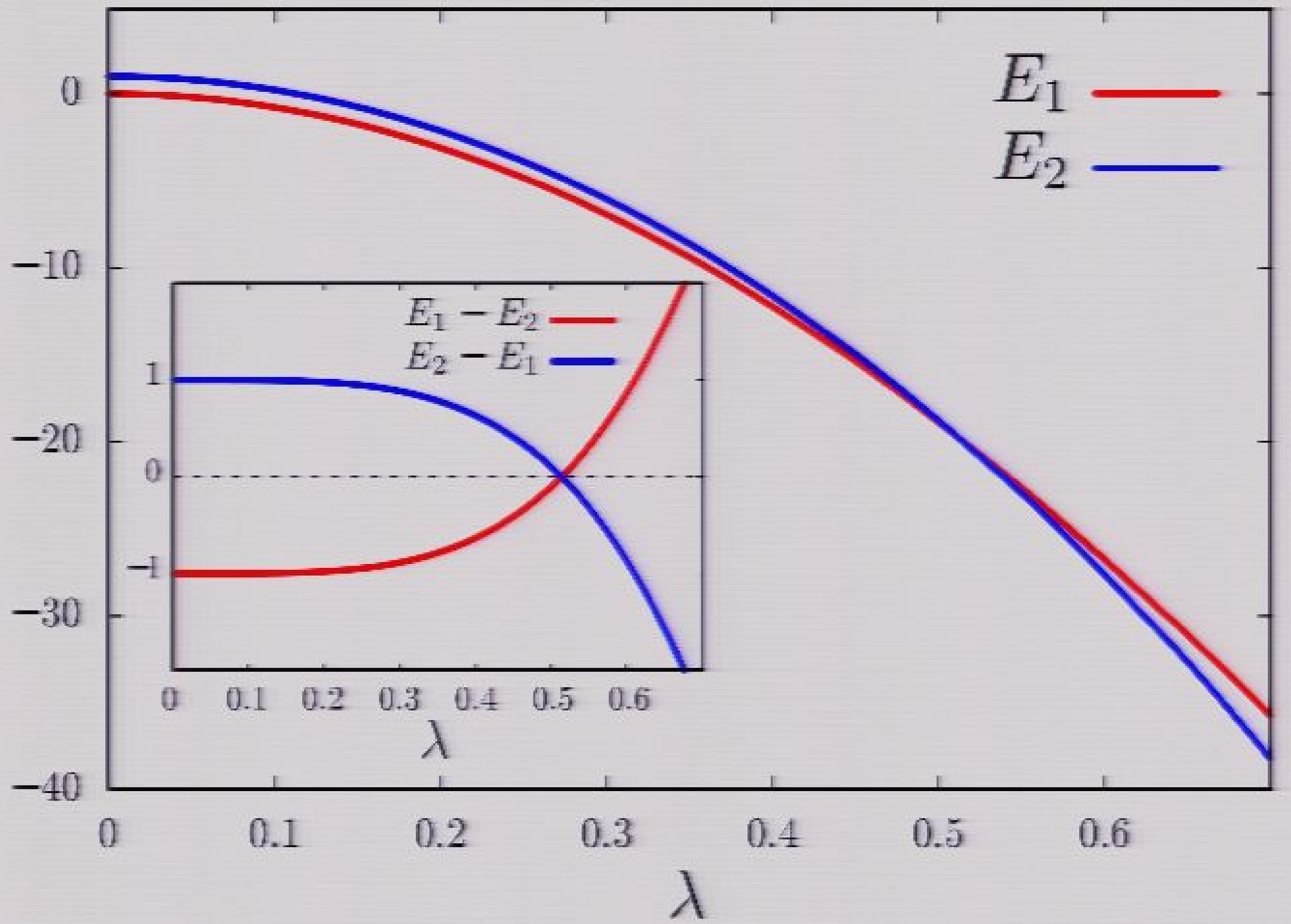

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Can we trust perturbation theory?

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Anderson localization theory

$\Rightarrow$ Perturbation theory valid as long as states are localized

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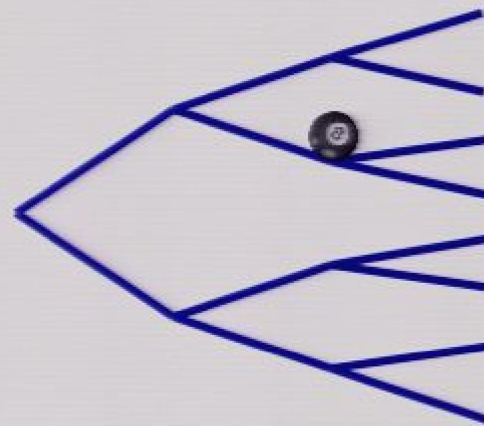
Cayley tree with branching number K :

$$\lambda_c = \Theta\left(\frac{E}{K \log K}\right)$$



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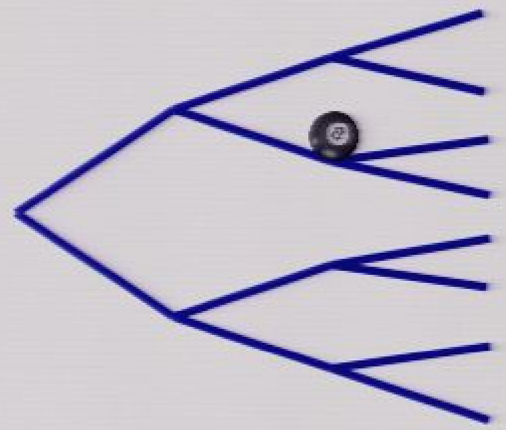

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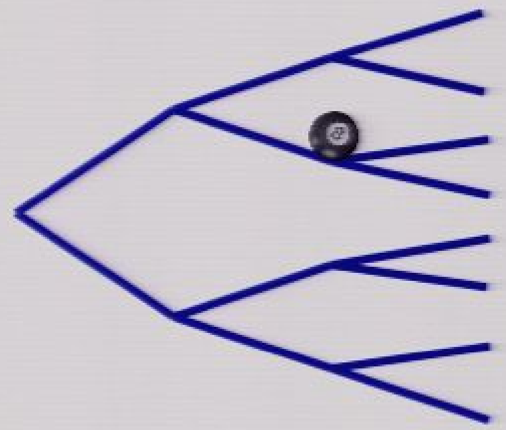

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Conclusion

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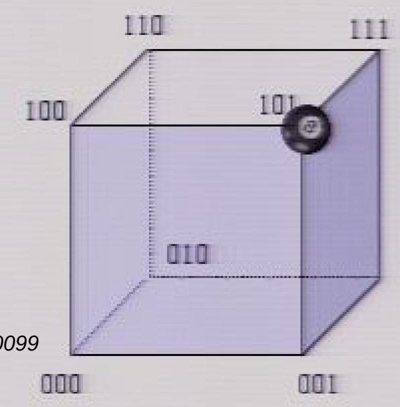
- Anderson localization causes exponentially small gaps in adiabatic quantum optimization



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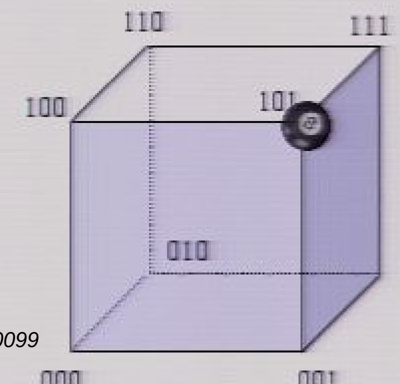

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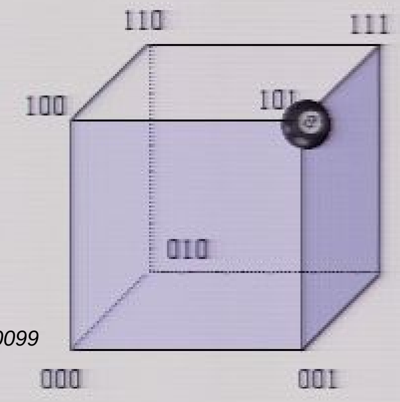

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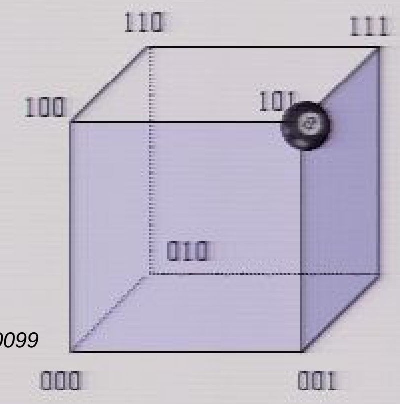

⇒ should be studied more closely


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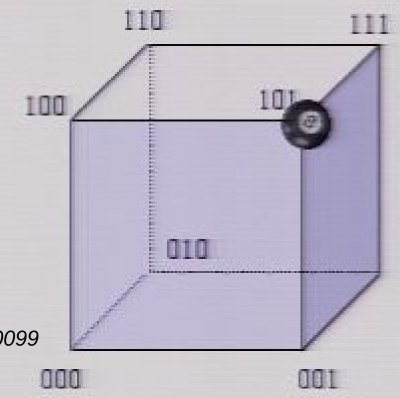

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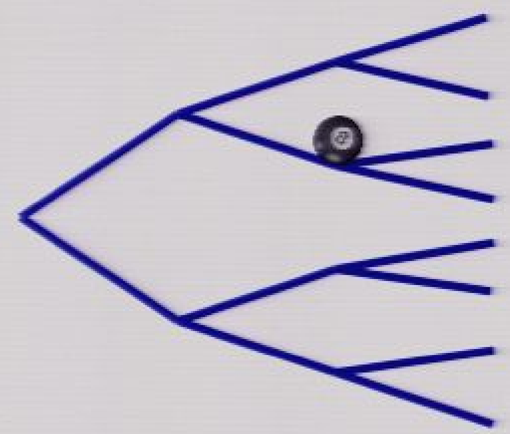

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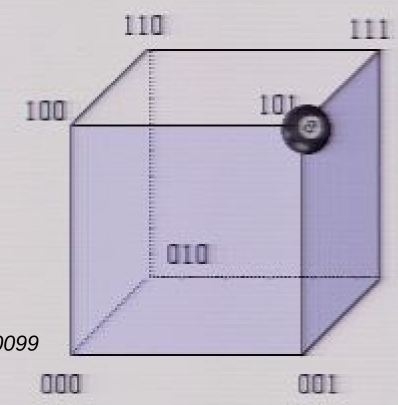

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