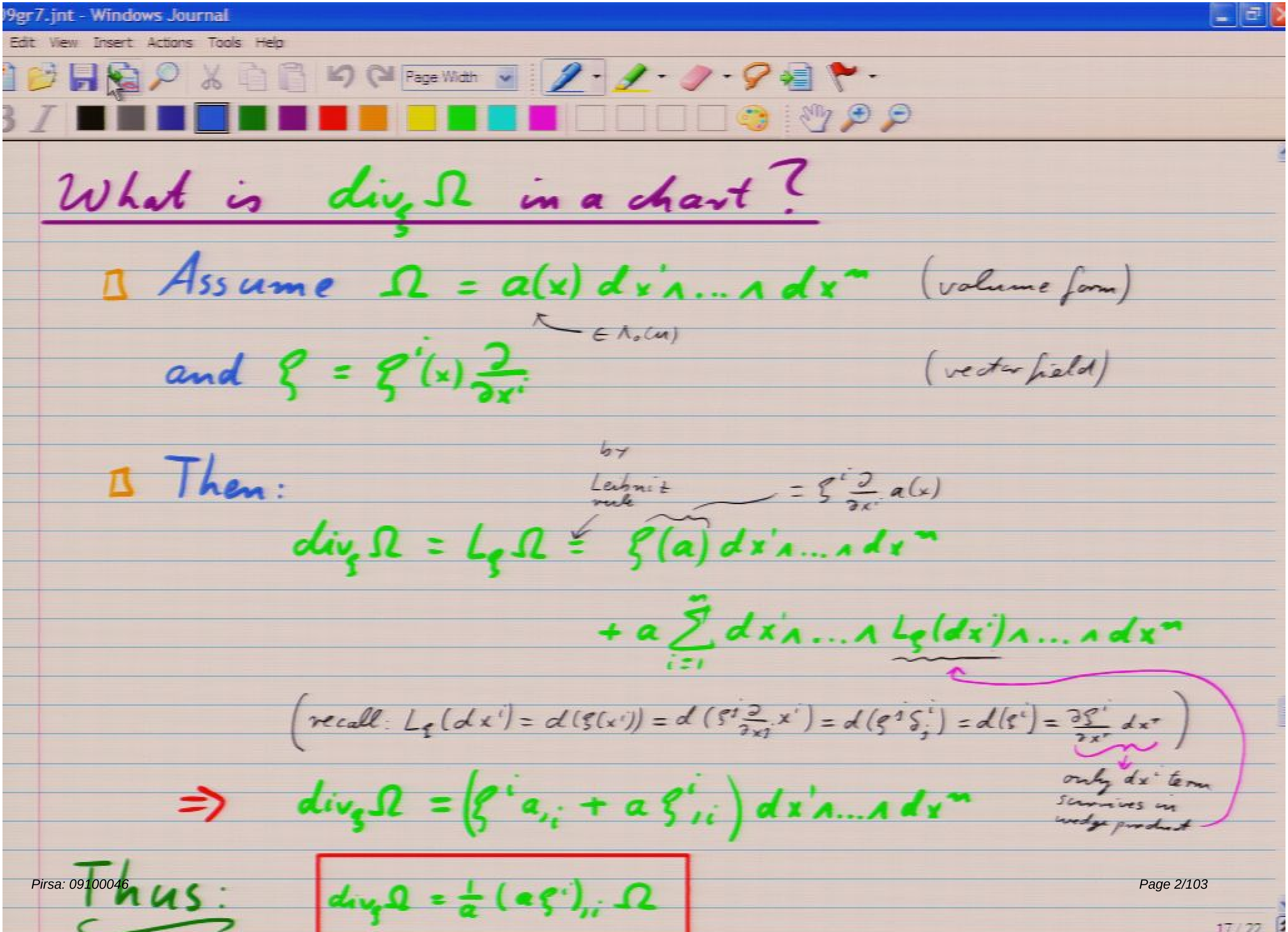
Title: General Relativity for Cosmology - Lecture 6

Date: Oct 08, 2009 04:00 PM

URL: http://pirsa.org/09100046

Abstract:



What is $\text{div}_\xi \Omega$ in a chart?

- Assume $\Omega = \underbrace{a(x)}_{\in \Lambda_0(M)} dx^1 \wedge \ldots \wedge dx^m$ (volume form)

  and $\xi = \xi^i(x) \frac{\partial}{\partial x^i}$ (vector field)

- Then:

$$\text{div}_\xi \Omega = L_\xi \Omega \overset{\text{by Leibniz rule}}{=} \overbrace{\xi(a)}^{= \xi^i \frac{\partial}{\partial x^i} a(x)} dx^1 \wedge \ldots \wedge dx^m$$

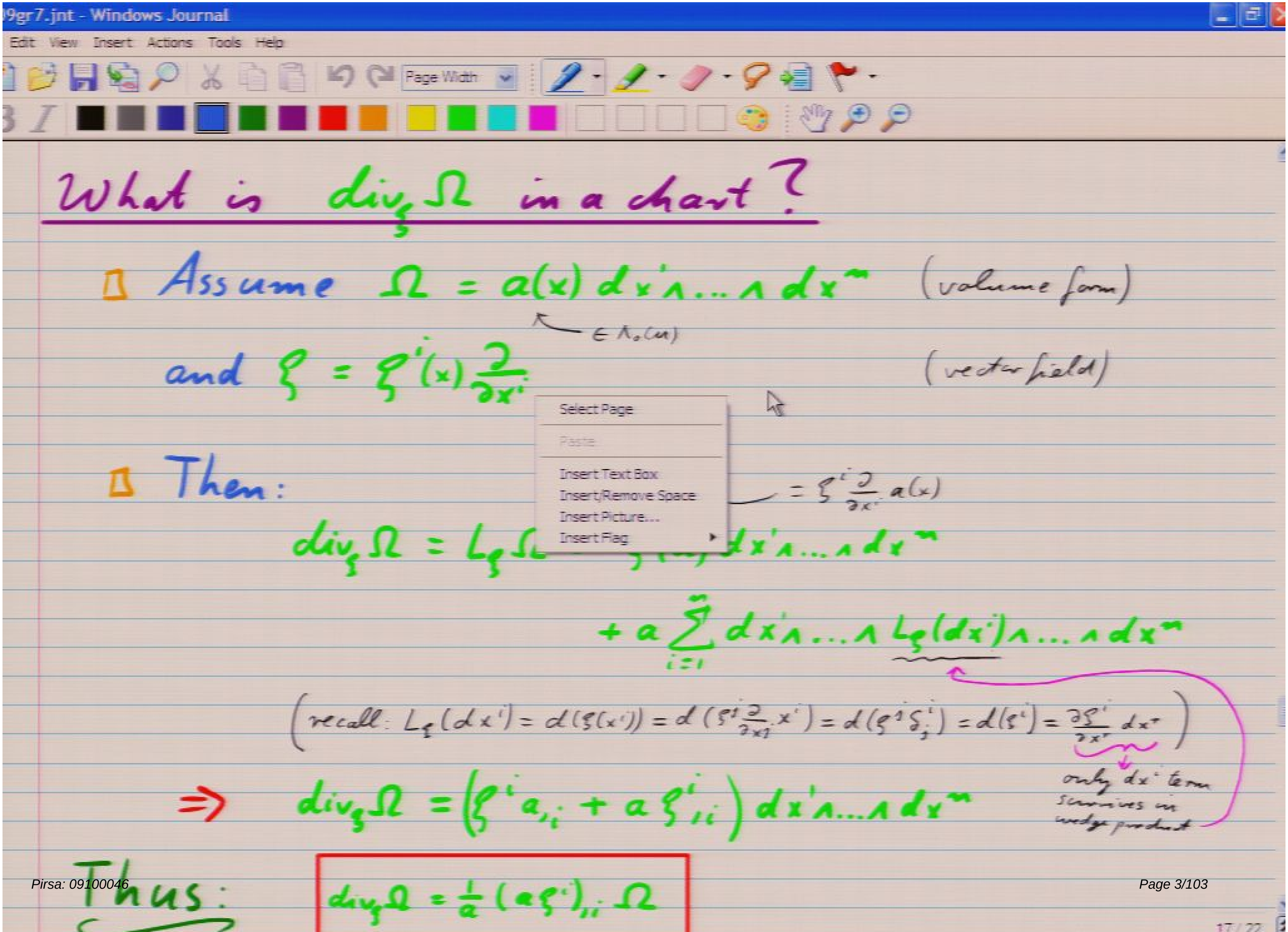
$$+ a \sum_{i=1}^{m} dx^1 \wedge \ldots \wedge \underbrace{L_\xi(dx^i)} \wedge \ldots \wedge dx^m$$

$$\left(\text{recall: } L_\xi(dx^i) = d(\xi(x^i)) = d\left(\xi^j \frac{\partial}{\partial x^j} x^i\right) = d(\xi^j \delta_j^i) = d(\xi^i) = \underbrace{\frac{\partial \xi^i}{\partial x^r} dx^r}_{\text{only } dx^i \text{ term survives in wedge product}} \right)$$

$$\Rightarrow \quad \text{div}_\xi \Omega = \left(\xi^i a_{,i} + a \xi^i_{,i} \right) dx^1 \wedge \ldots \wedge dx^m$$

Thus:

$$\boxed{\text{div}_\xi \Omega = \frac{1}{a} (a \xi^i)_{,i}\, \Omega}$$



What is $\mathrm{div}_\xi \Omega$ in a chart?

- Assume $\Omega = a(x)\, dx^1 \wedge \ldots \wedge dx^m$ (volume form)

  $a(x)$ ← $\in \Lambda_0(M)$

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- Then:

$$\mathrm{div}_\xi \Omega = L_\xi \Omega \quad \ldots \xi(a)\; dx^1 \wedge \ldots \wedge dx^m$$

$$= \xi^i \frac{\partial}{\partial x^i} a(x)$$

$$+ a \sum_{i=1}^{m} dx^1 \wedge \ldots \wedge \underbrace{L_\xi(dx^i)} \wedge \ldots \wedge dx^m$$

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only dx^i term survives in wedge product

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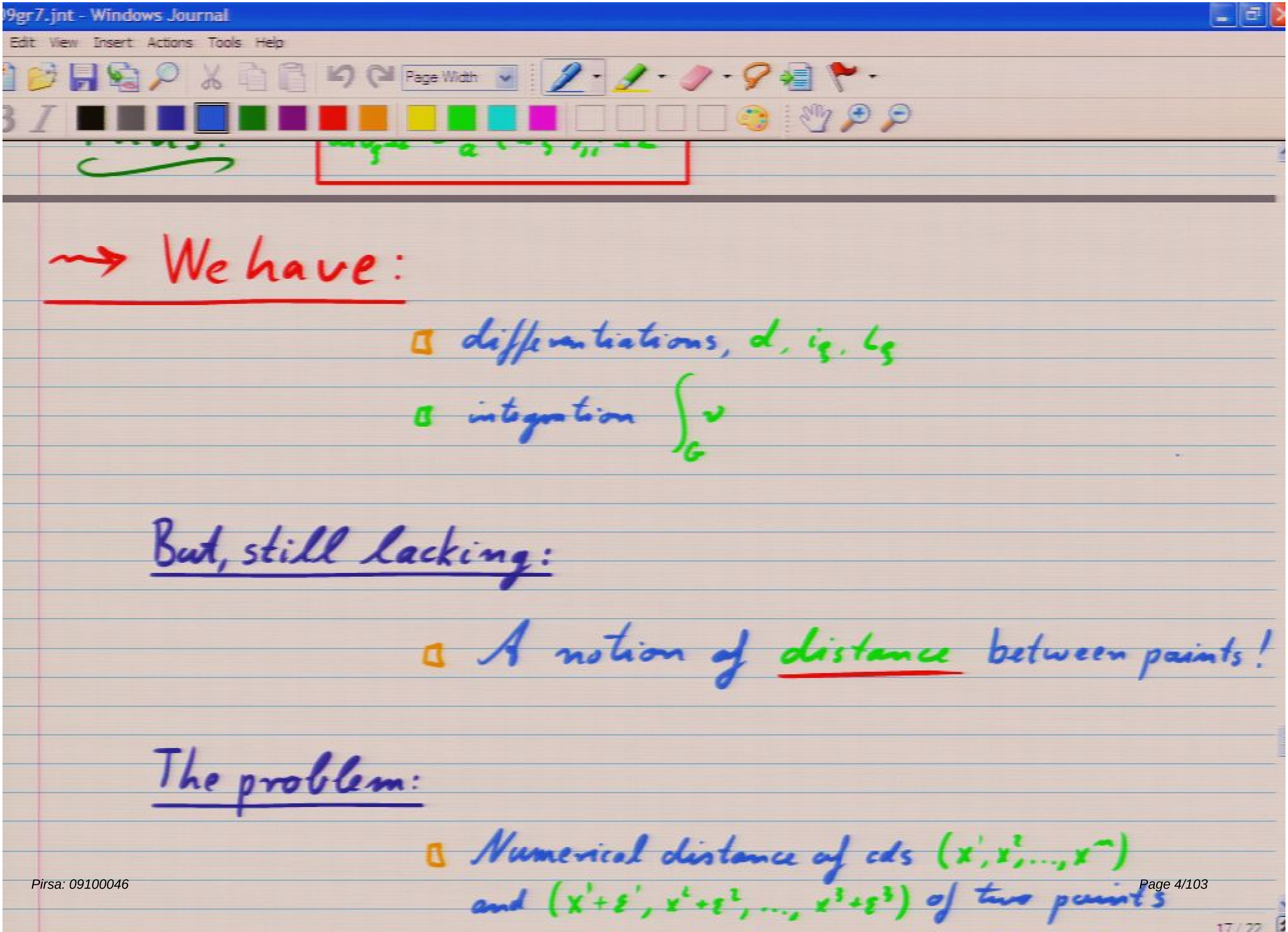


→ We have:

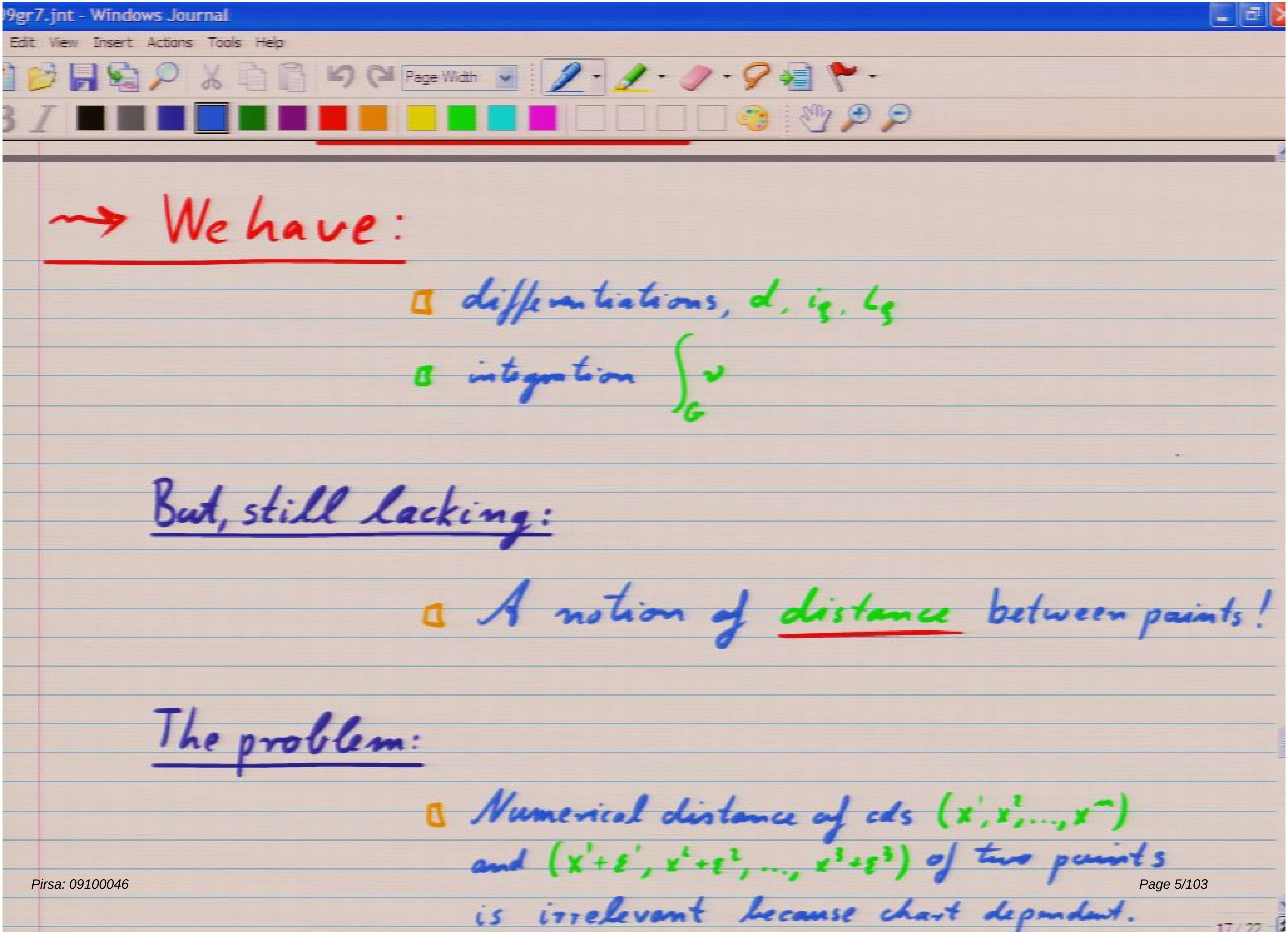
- differentiations, d, i_ξ, L_ξ
- integration $\int_G \nu$

But, still lacking:

- A notion of distance between points!

The problem:

- Numerical distance of cds $(x^1, x^2, \dots, x^n)$ and $(x^1+\varepsilon^1, x^2+\varepsilon^2, \dots, x^3+\varepsilon^3)$ of two points



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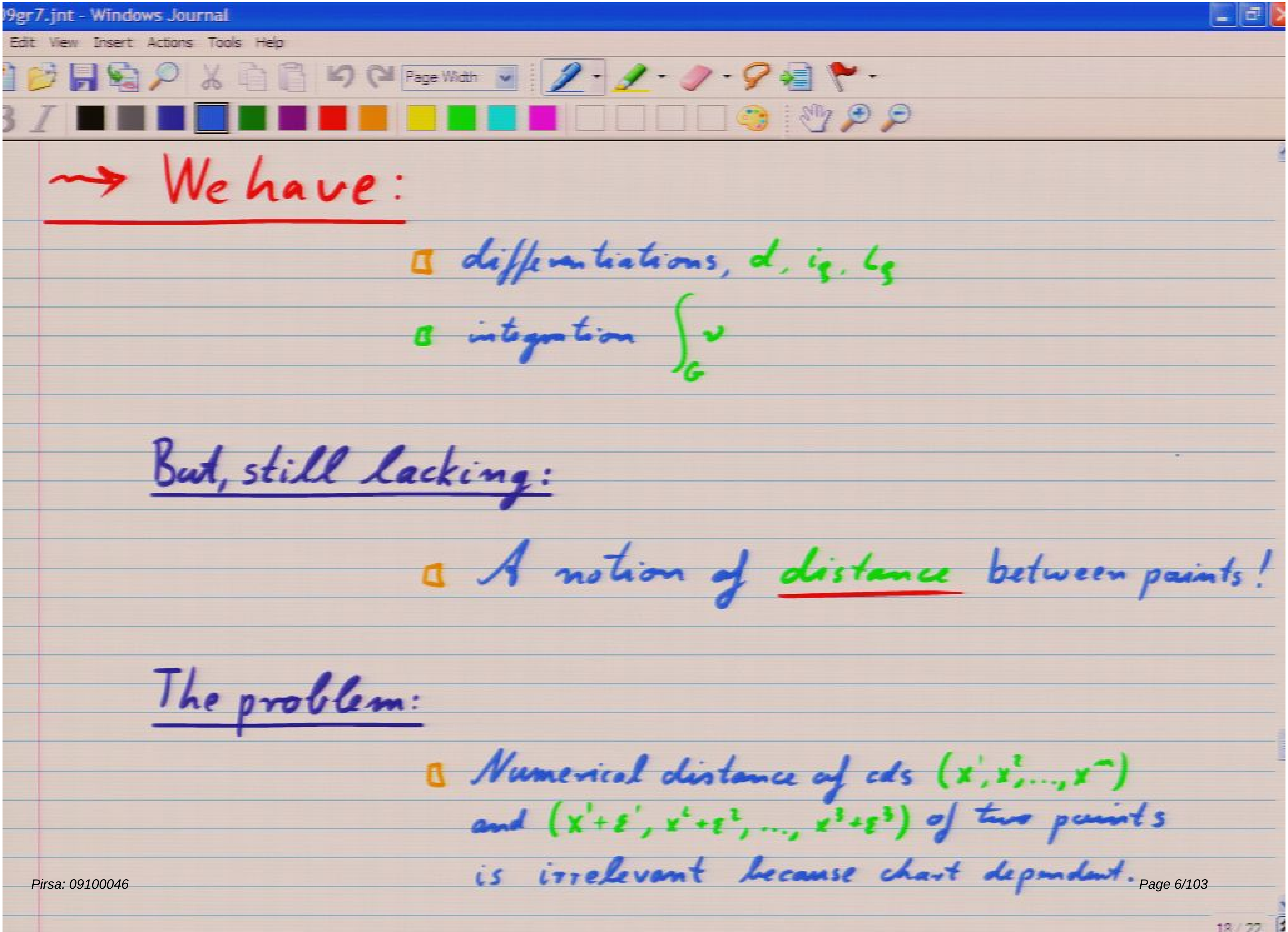

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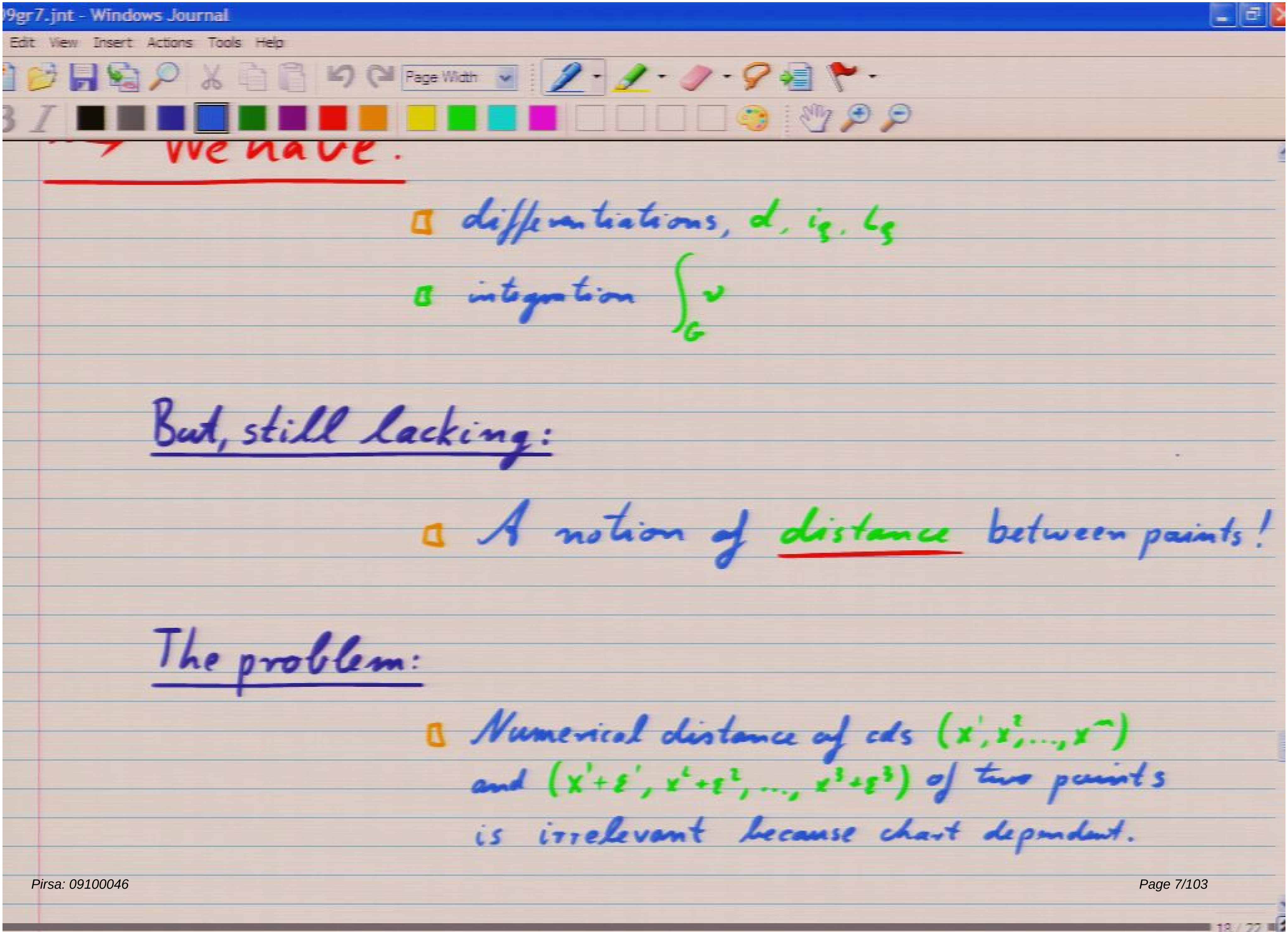

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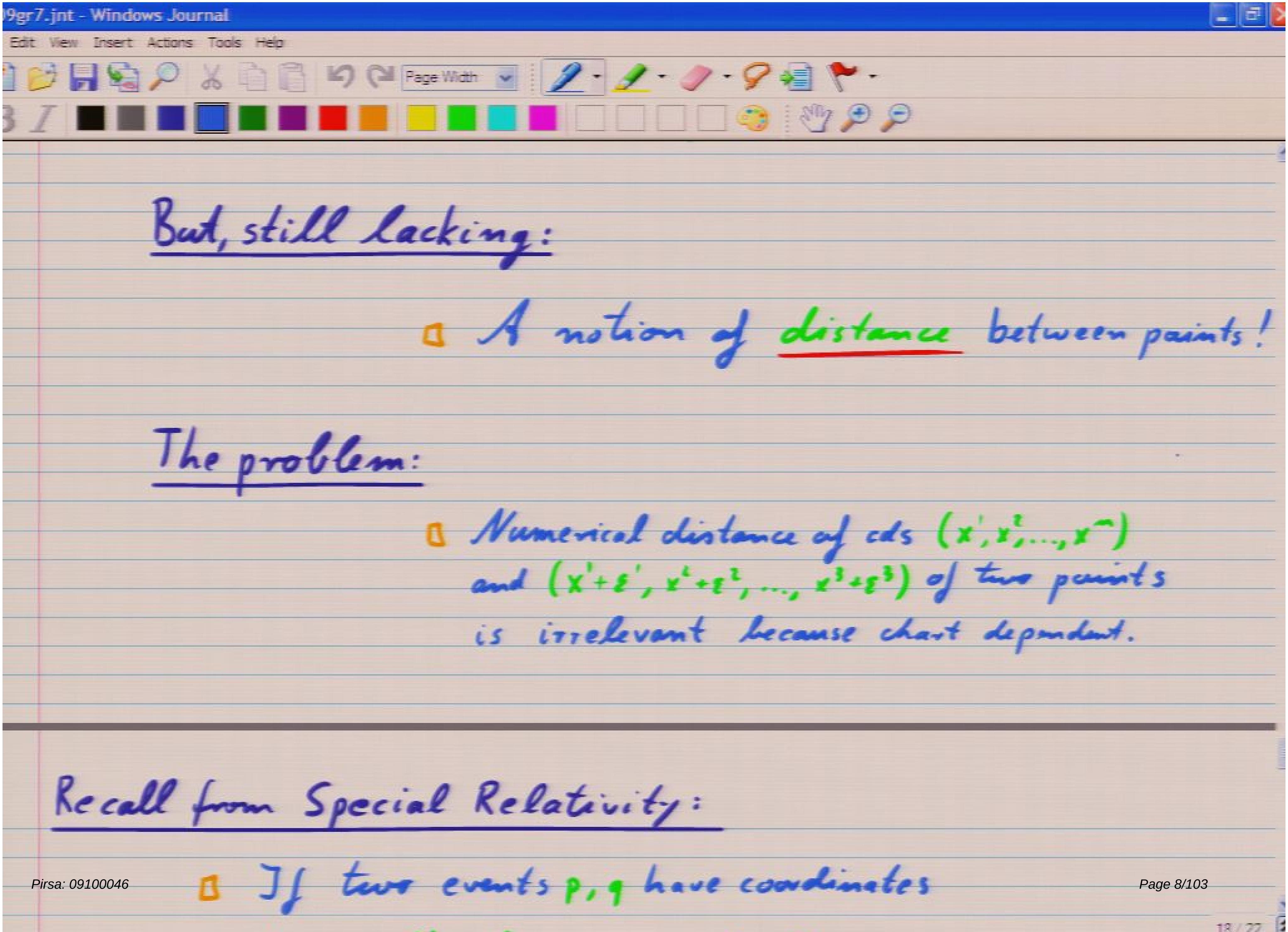

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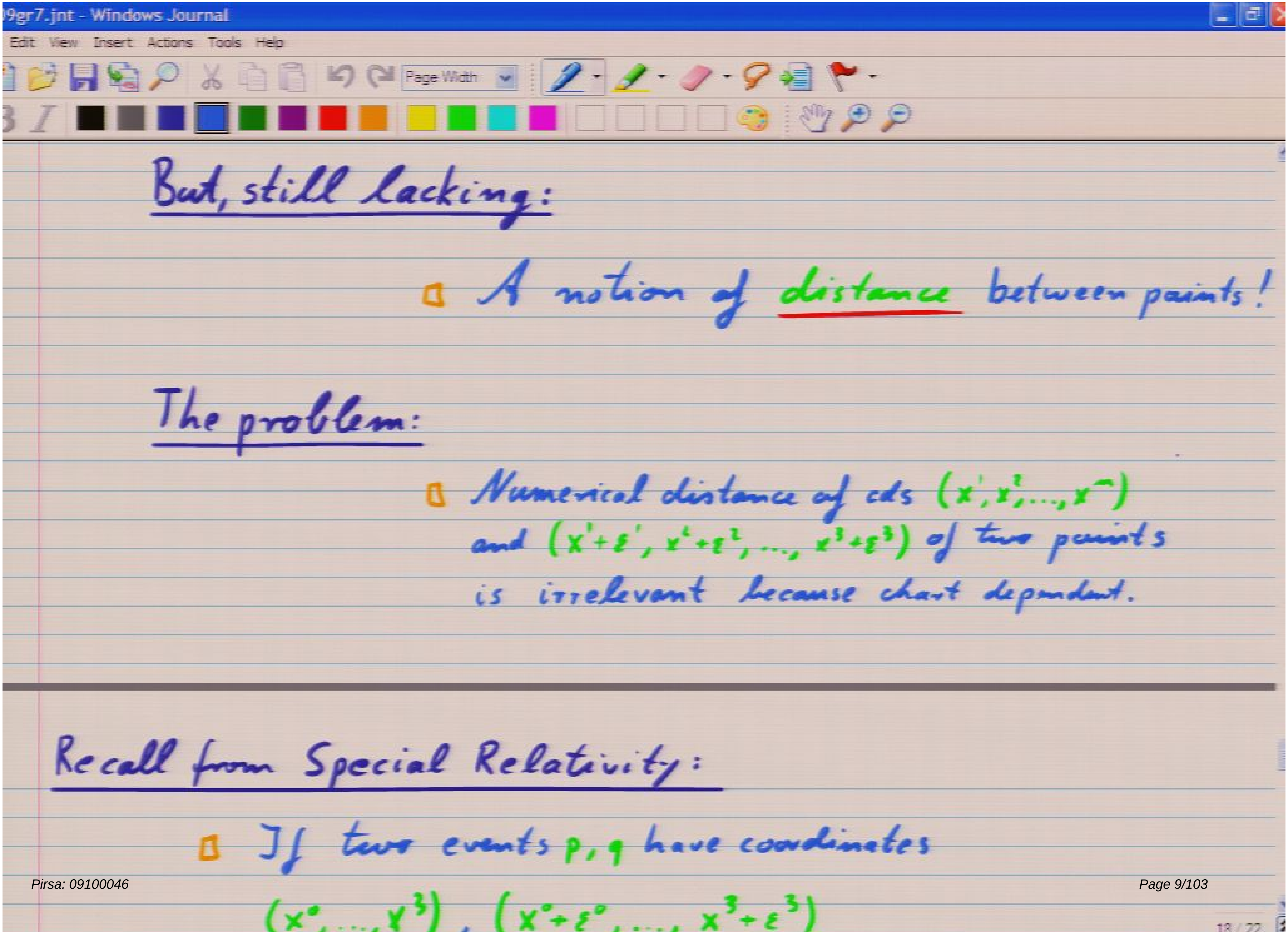
Recall from Special Relativity:

- If two events p, q have coordinates



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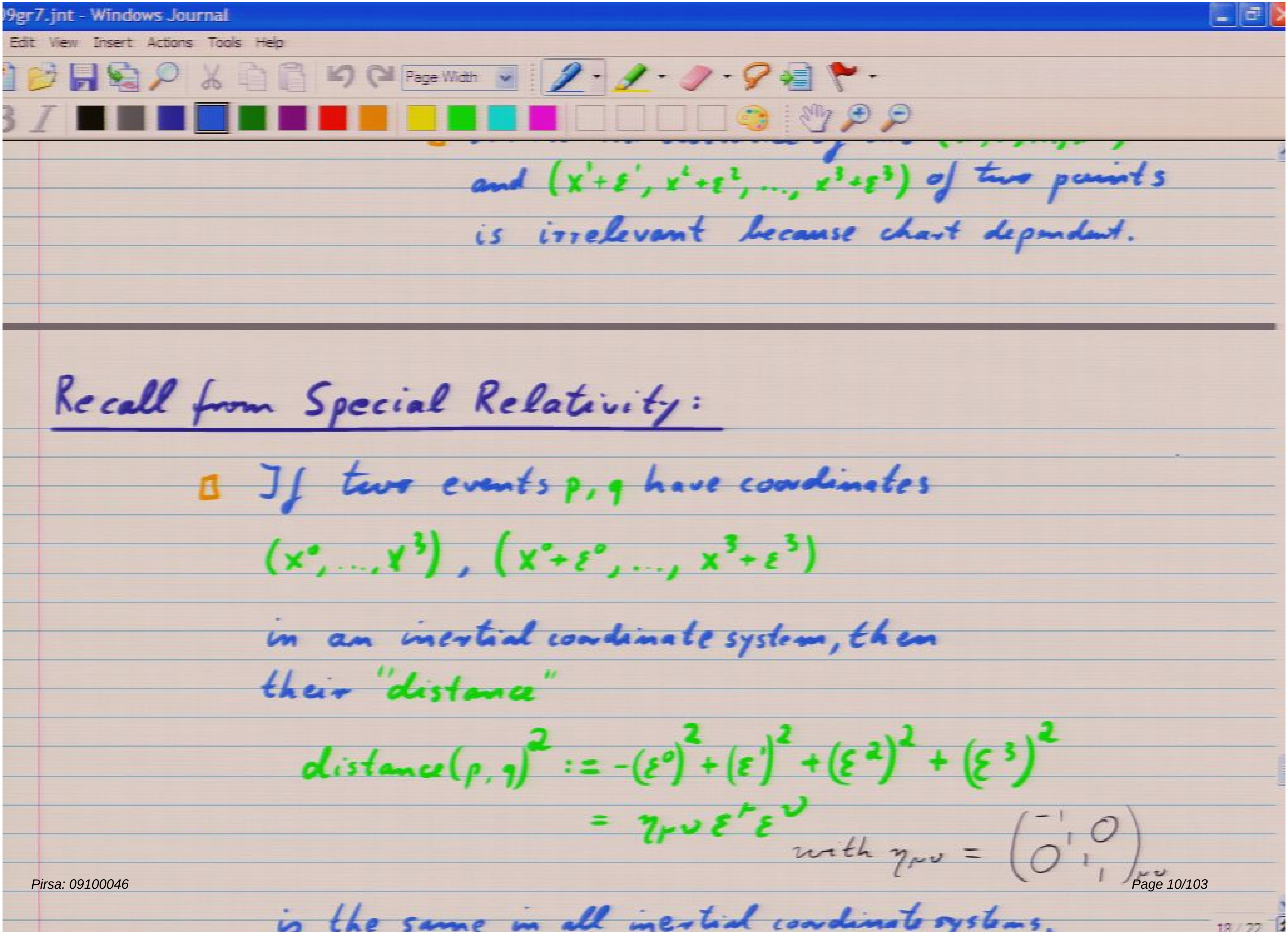


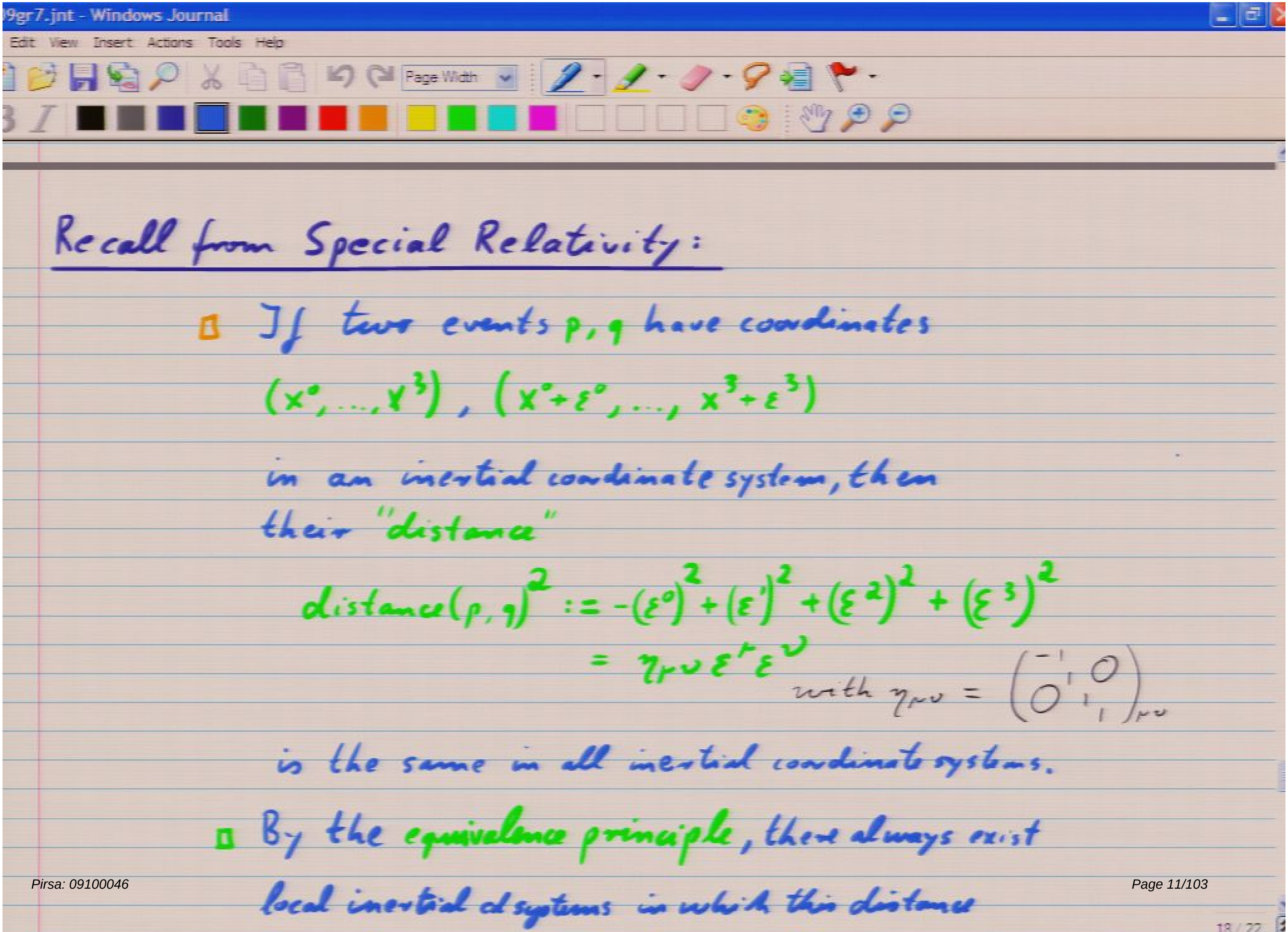
and $(x^1+\varepsilon^1, x^2+\varepsilon^2, \ldots, x^3+\varepsilon^3)$ of two points is irrelevant because chart dependent.

Recall from Special Relativity:

- If two events p, q have coordinates $(x^0, \ldots, x^3)$, $(x^0+\varepsilon^0, \ldots, x^3+\varepsilon^3)$ in an inertial coordinate system, then their "distance"

$$\text{distance}(p,q)^2 := -(\varepsilon^0)^2 + (\varepsilon^1)^2 + (\varepsilon^2)^2 + (\varepsilon^3)^2$$
$$= \eta_{\mu\nu}\,\varepsilon^\mu \varepsilon^\nu \quad \text{with } \eta_{\mu\nu} = \left(\begin{array}{c|c} -1 & 0 \\ \hline 0 & 1 \end{array}\right)_{\mu\nu}$$

is the same in all inertial coordinate systems.



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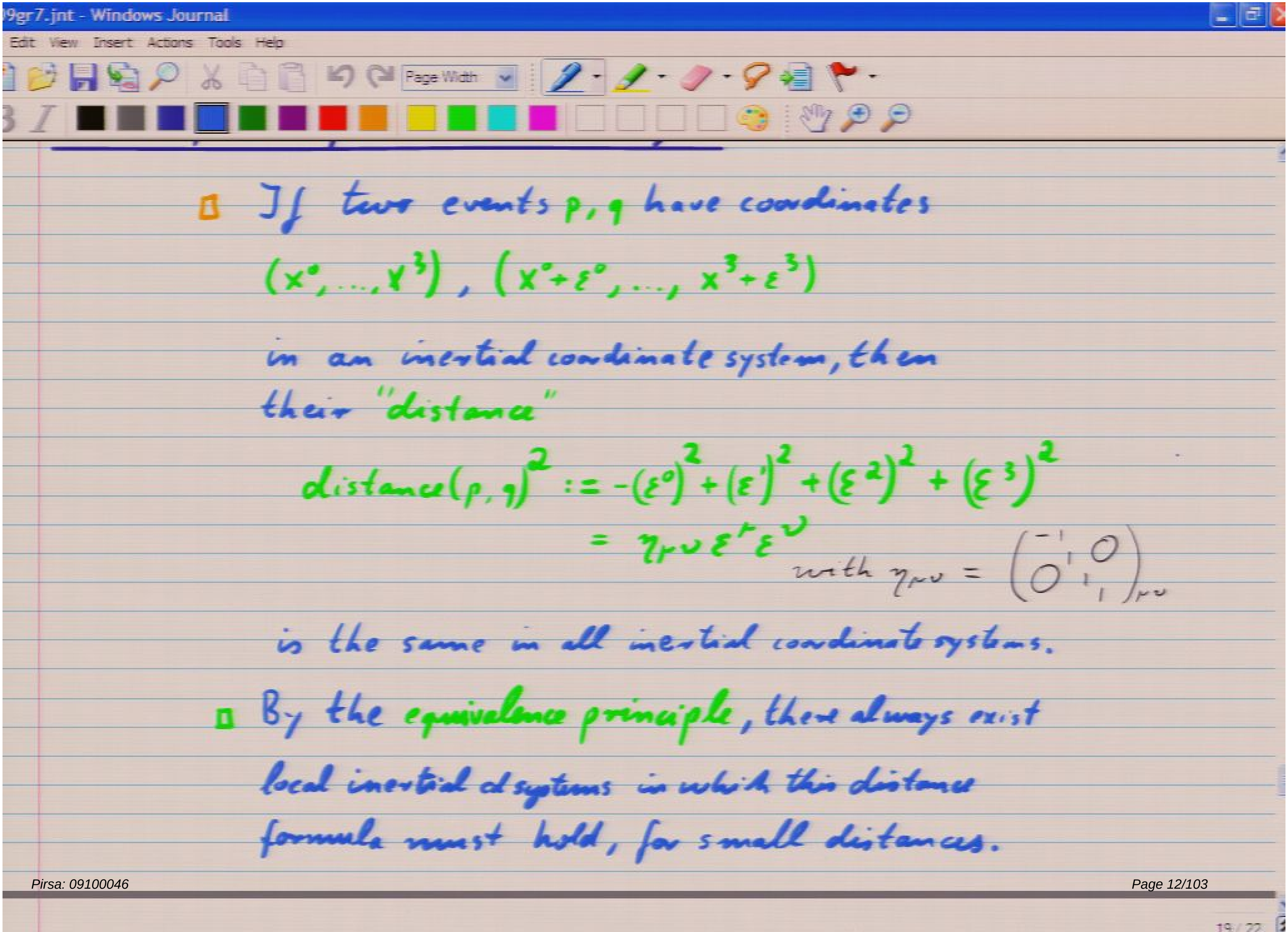
- ☐ By the equivalence principle, there always exist

  local inertial cd systems in which this distance



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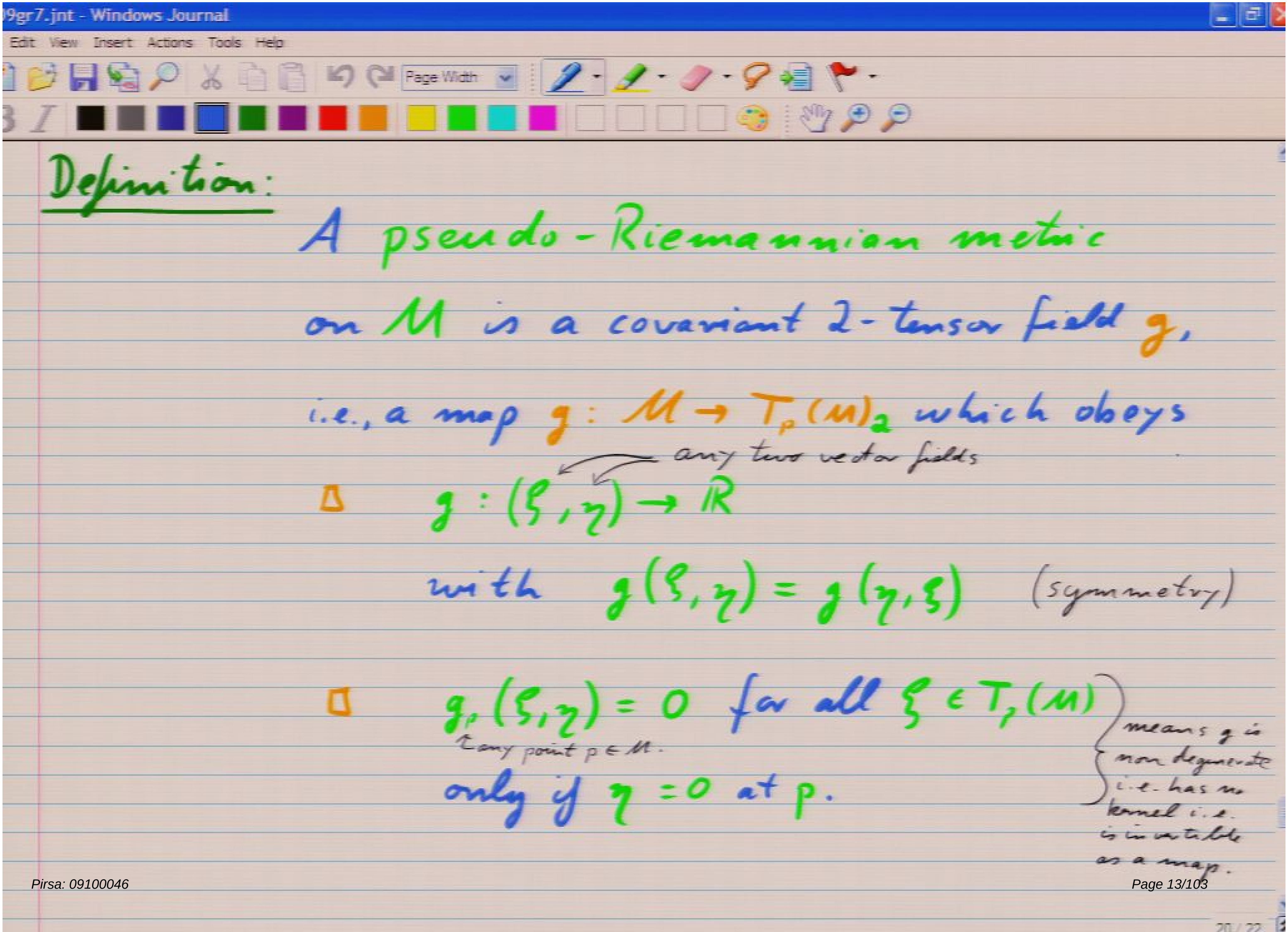
☐ By the equivalence principle, there always exist local inertial cd systems in which this distance formula must hold, for small distances.



Definition:

A pseudo-Riemannian metric on $\mathcal{M}$ is a covariant 2-tensor field g, i.e., a map $g: \mathcal{M} \to T_p(\mathcal{M})_2$ which obeys

any two vector fields

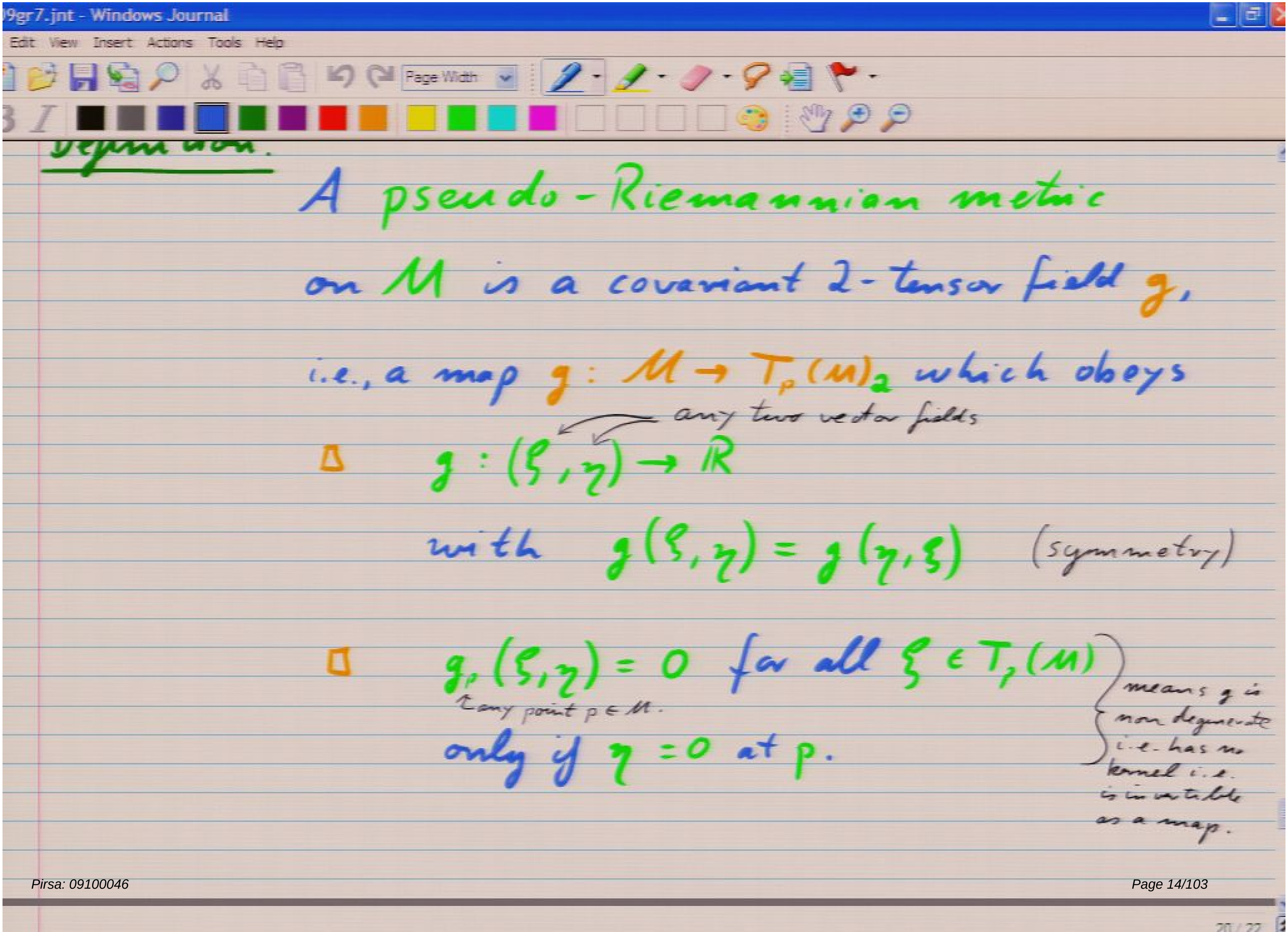
- $g: (\xi, \eta) \to \mathbb{R}$

  with $g(\xi,\eta) = g(\eta,\xi)$ (symmetry)

- $g_p(\xi,\eta) = 0$ for all $\xi \in T_p(\mathcal{M})$ only if $\eta = 0$ at p.

  any point $p \in \mathcal{M}$.

  means g is non degenerate i.e. has no kernel i.e. is invertible as a map.



Definition:

A pseudo-Riemannian metric on M is a covariant 2-tensor field g, i.e., a map $g: M \to T_p(M)_2$ which obeys

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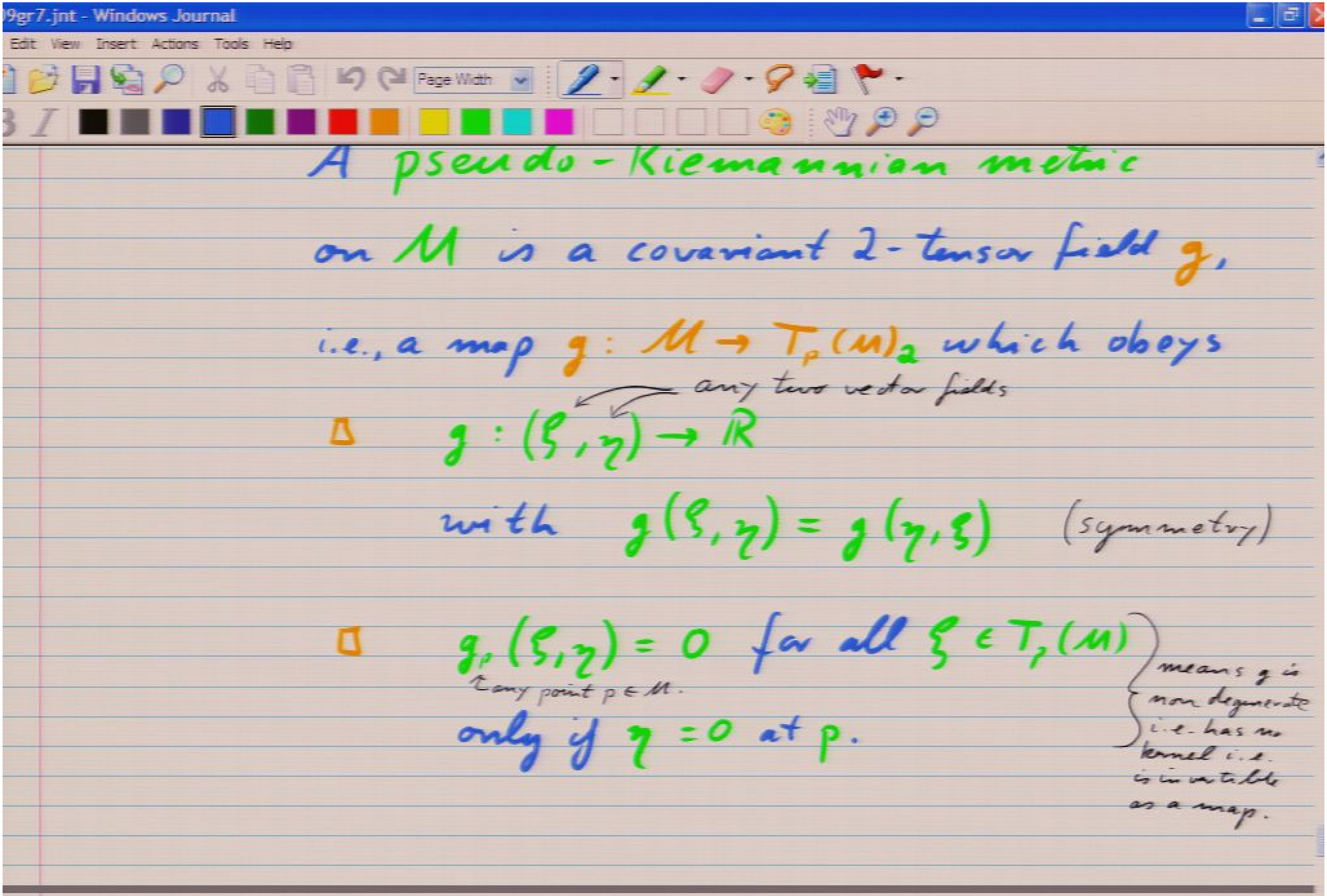

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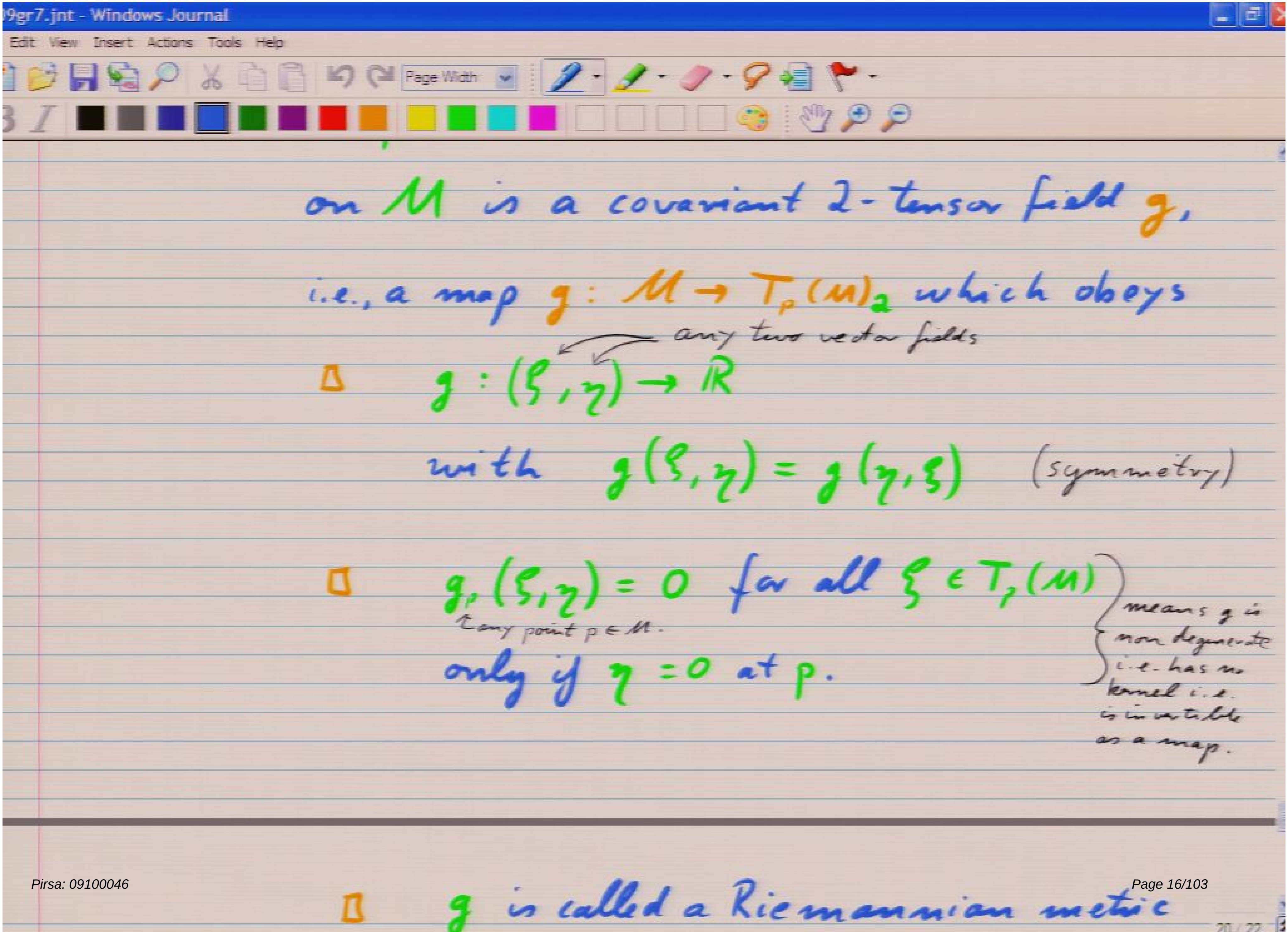

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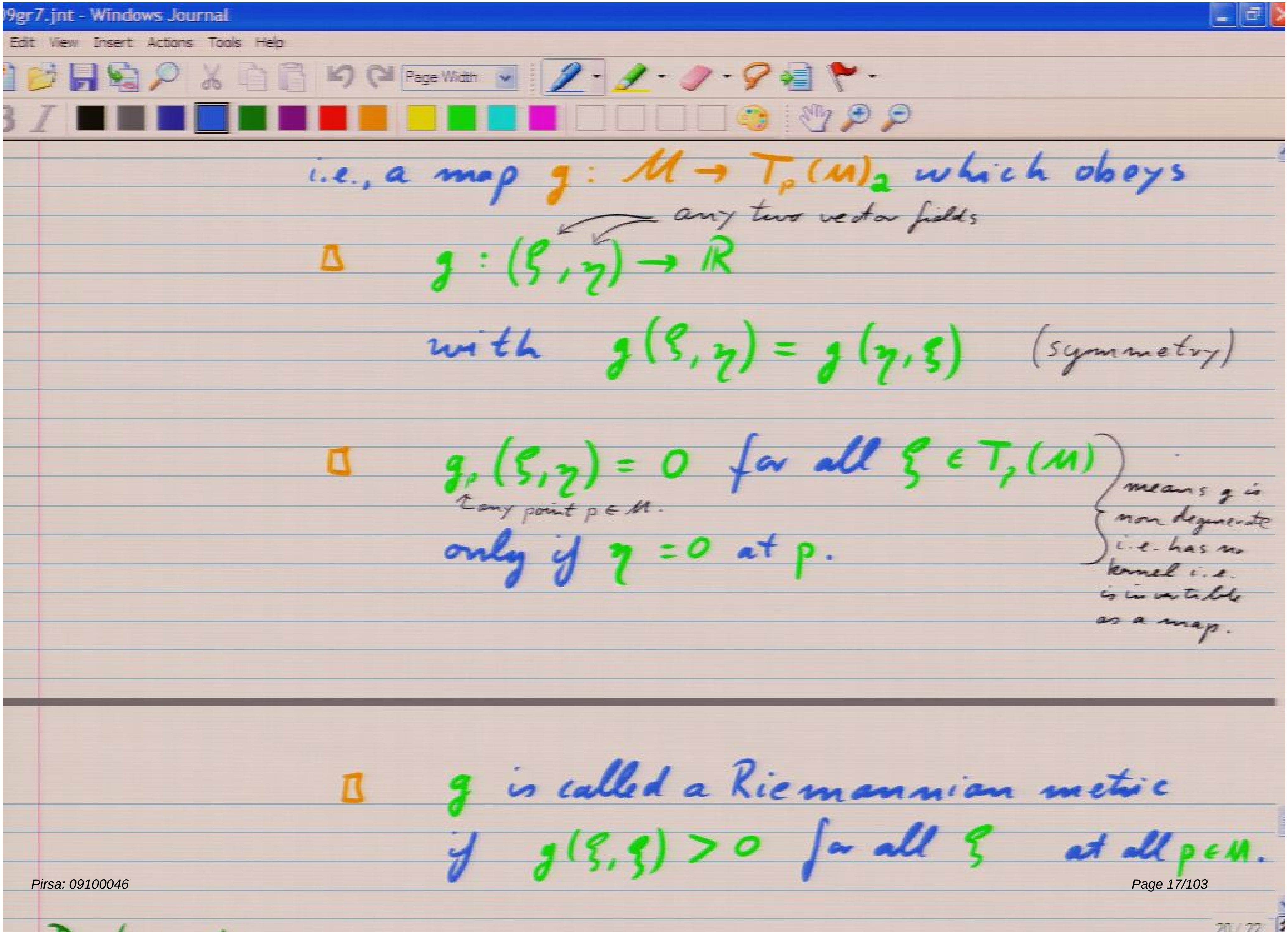
- g is called a Riemannian metric



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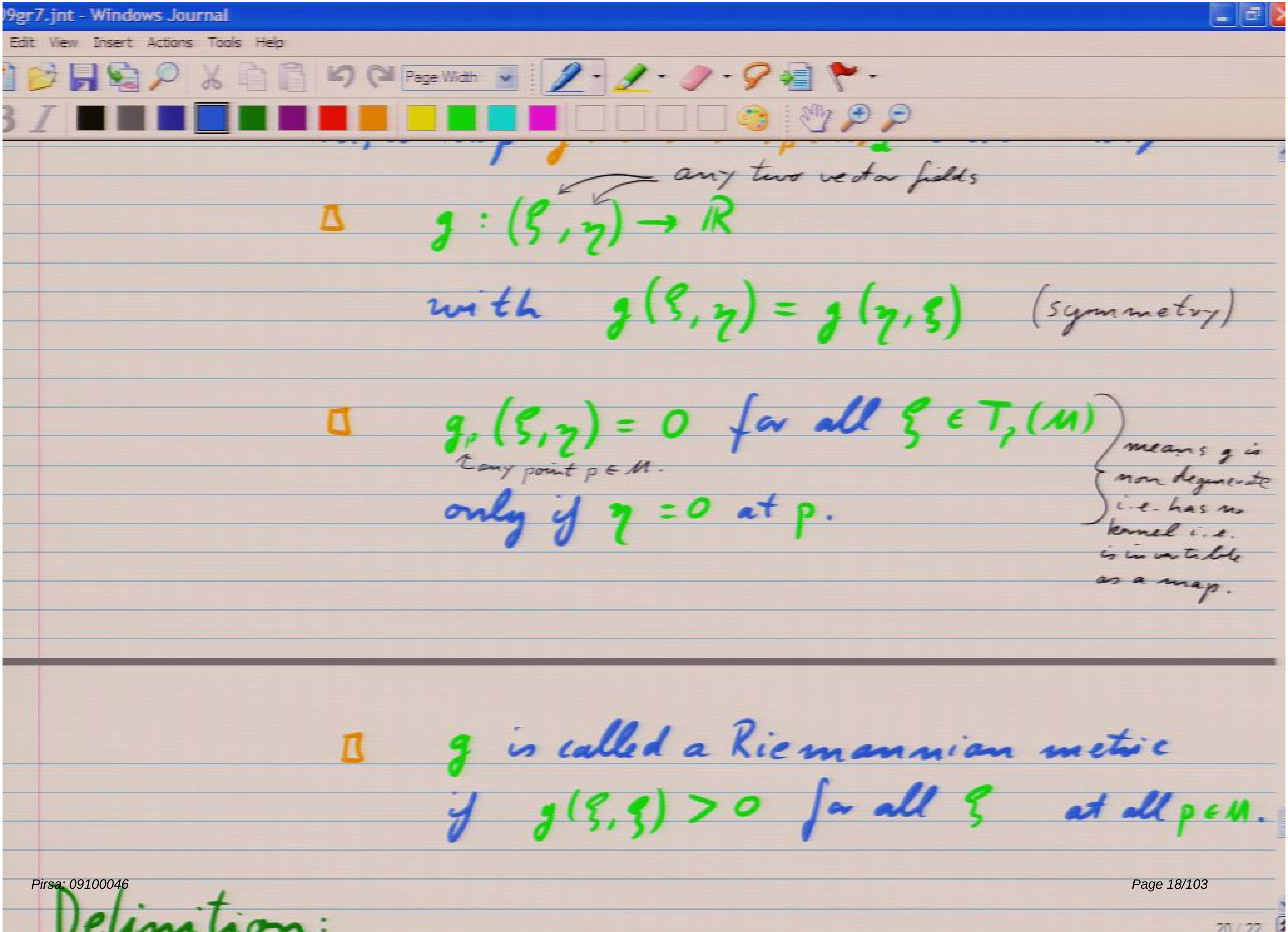
☐ g is called a Riemannian metric if $g(\xi, \xi) > 0$ for all ξ at all $p \in M$.



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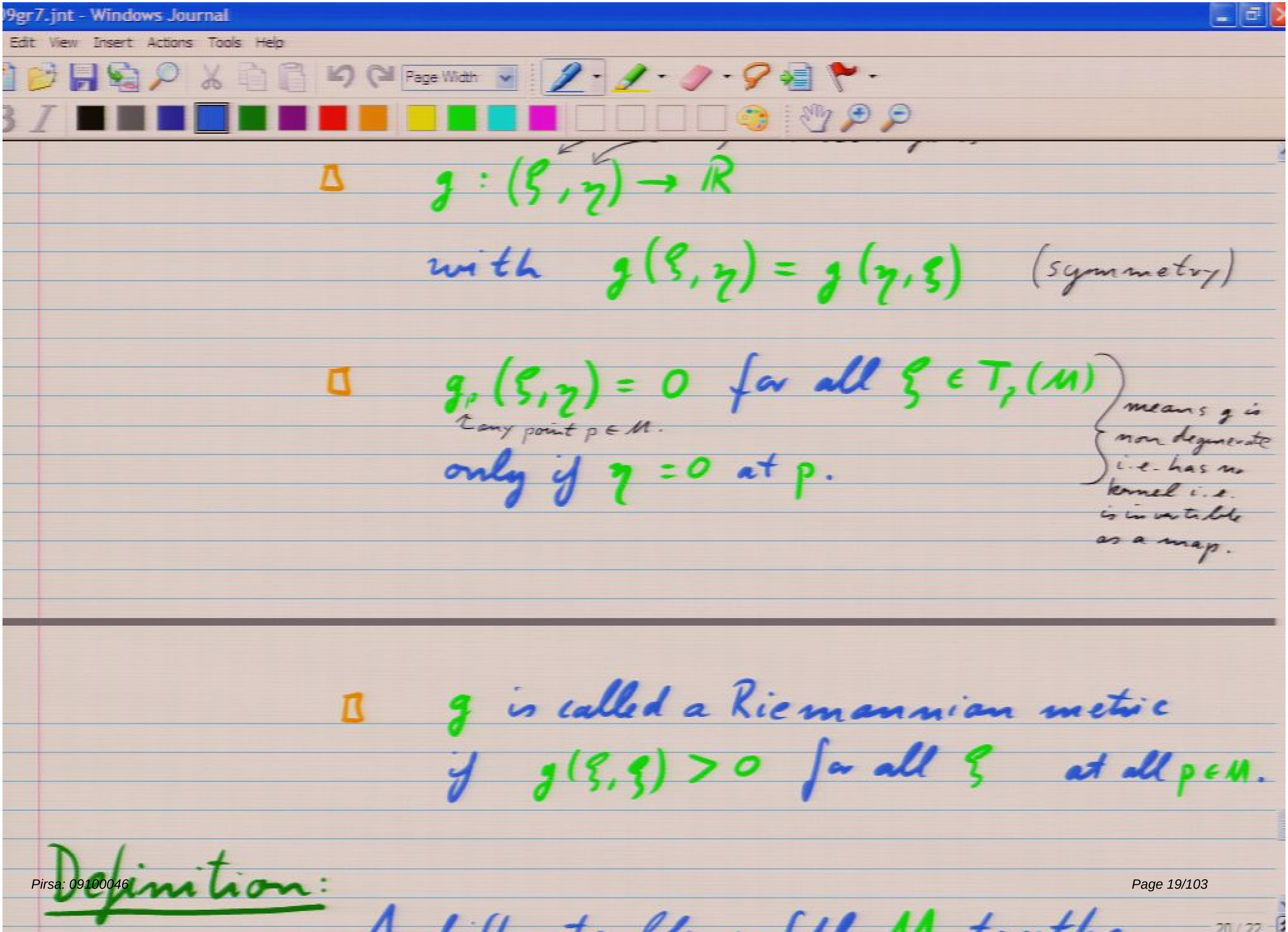

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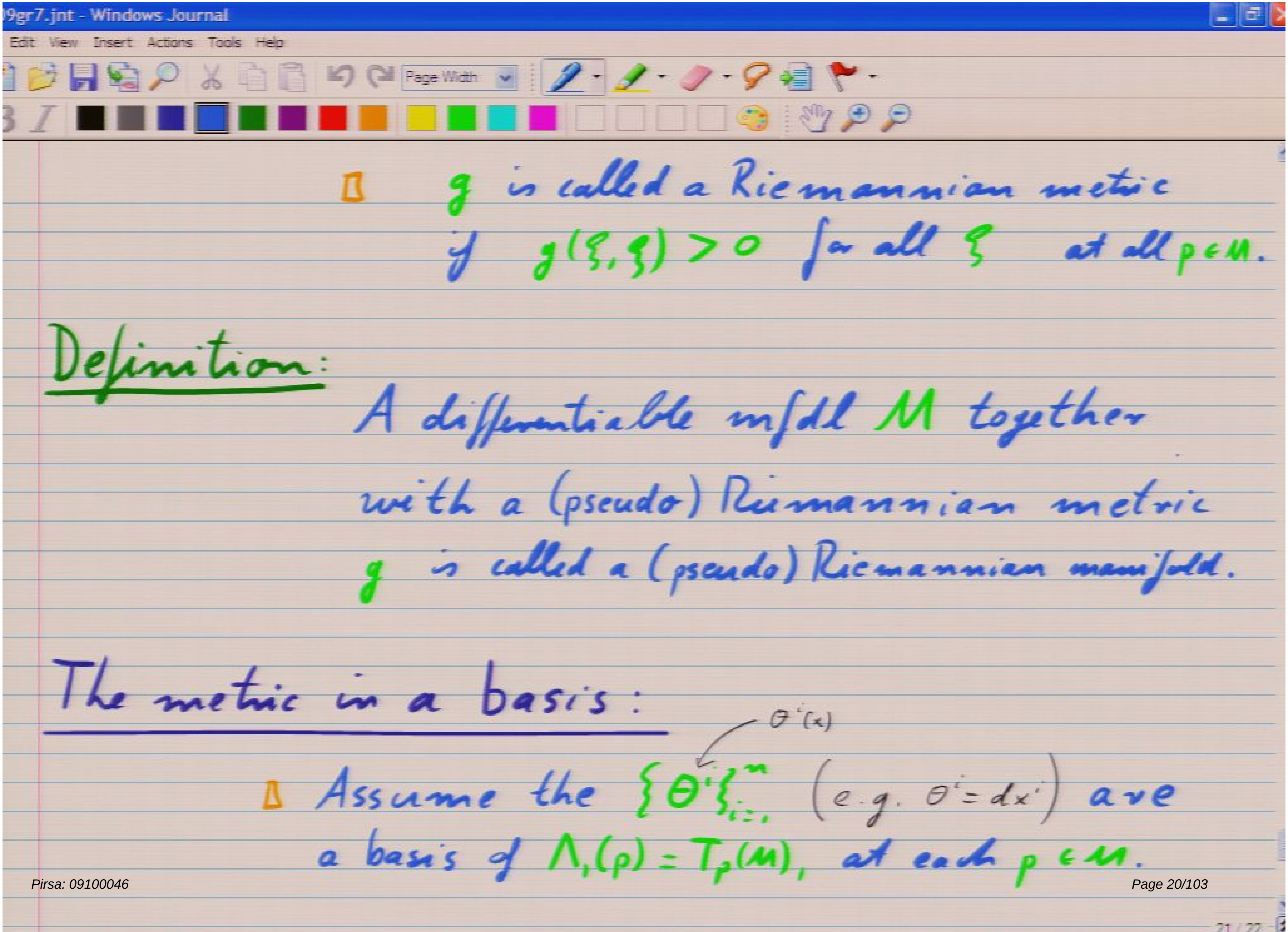


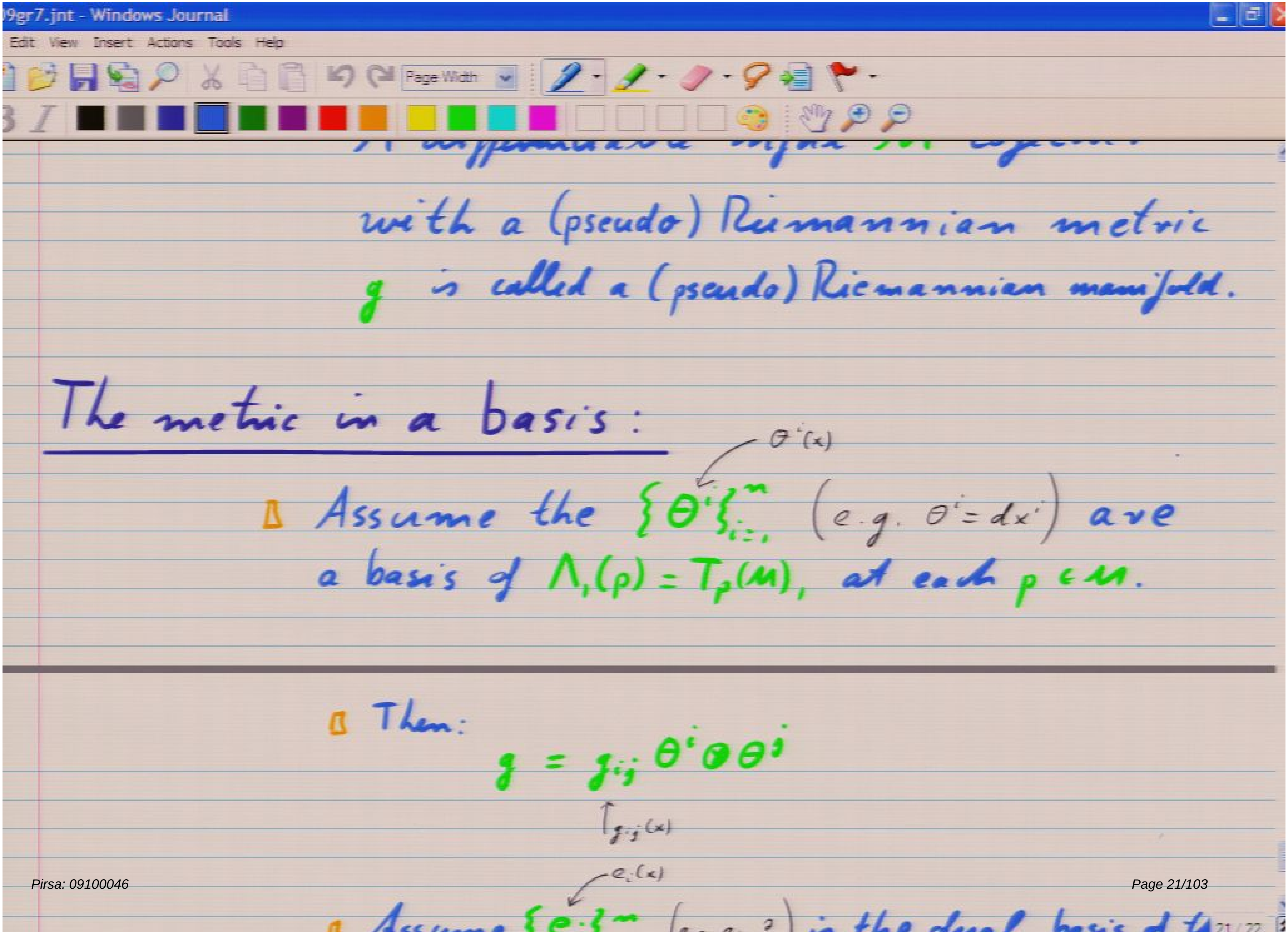
- g is called a Riemannian metric if $g(\xi,\xi) > 0$ for all ξ at all $p \in M$.

Definition:

A differentiable mfdl M together with a (pseudo) Riemannian metric g is called a (pseudo) Riemannian manifold.

The metric in a basis:

- Assume the $\{\overset{\theta^i(x)}{\theta^i}\}_{i=1}^{n}$ (e.g. $\theta^i = dx^i$) are a basis of $\Lambda_1(p) = T_p(M)$, at each $p \in M$.



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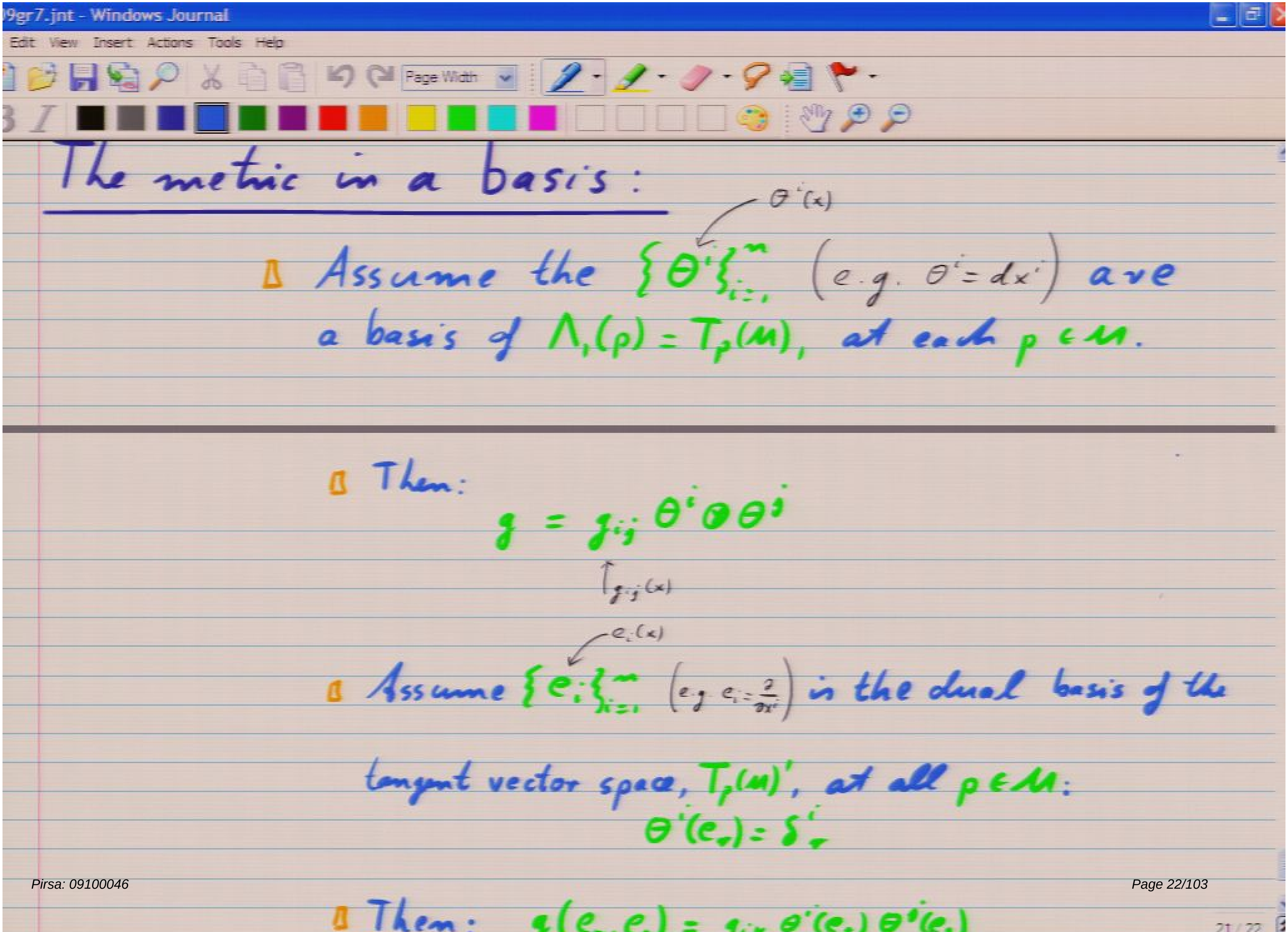
- Then:

$$g = \underset{\uparrow g_{ij}(x)}{g_{ij}}\, \theta^i \otimes \theta^j$$

- Assume $\{\overset{e_i(x)}{e_i}\}^n$ ( ) is the dual basis of



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- Then:

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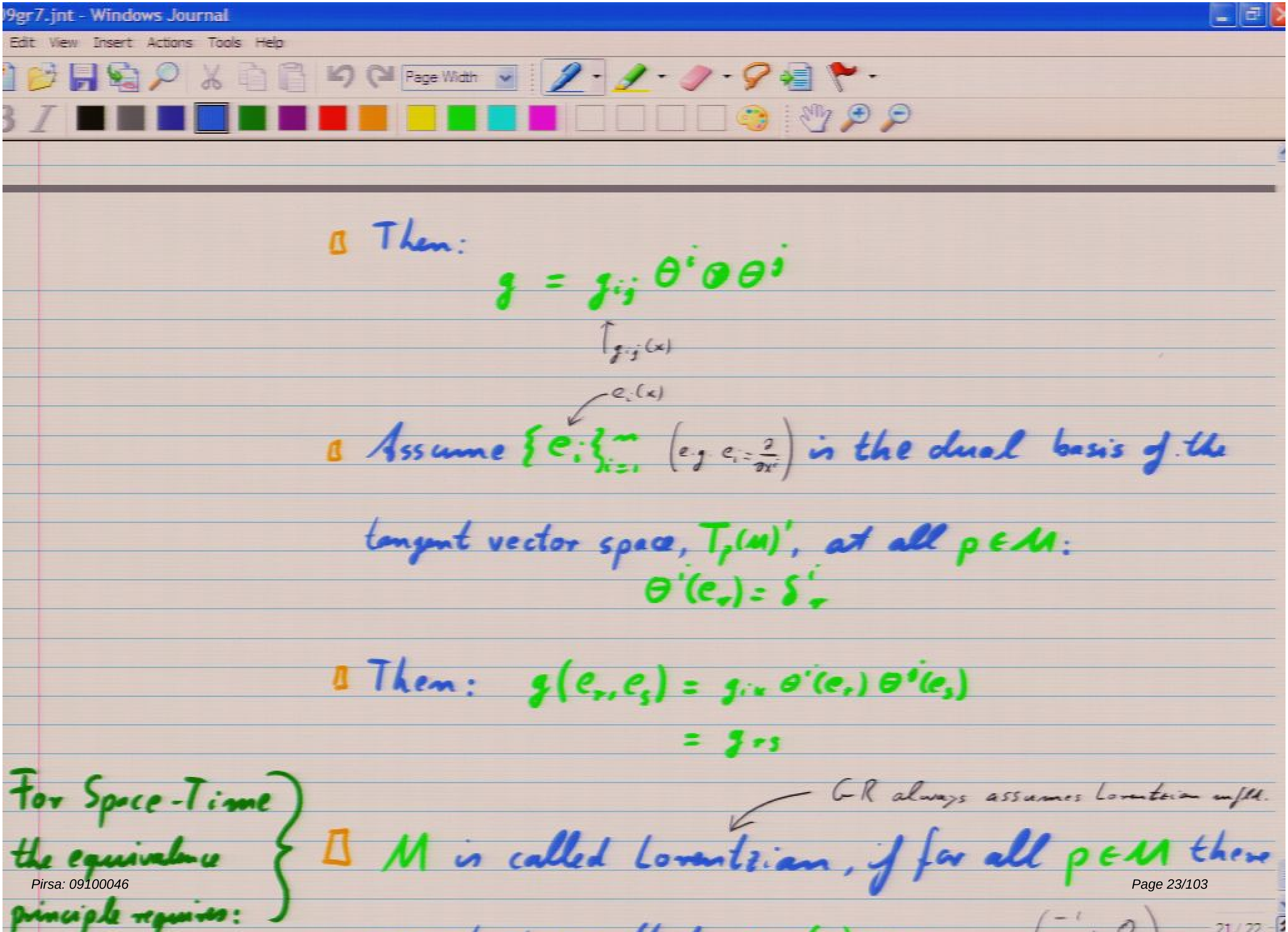
- Assume $\{e_i\}_{i=1}^{n}$ $\overset{e_i(x)}{}$ (e.g. $e_i = \frac{\partial}{\partial x^i}$) is the dual basis of the tangent vector space, $T_p(M)'$, at all $p \in M$:

$$\theta^i(e_r) = \delta^i_r$$

- Then: $g(e_r, e_s) = g_{ij}\, \theta^i(e_r)\, \theta^j(e_s)$



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$\uparrow g_{ij}(x)$

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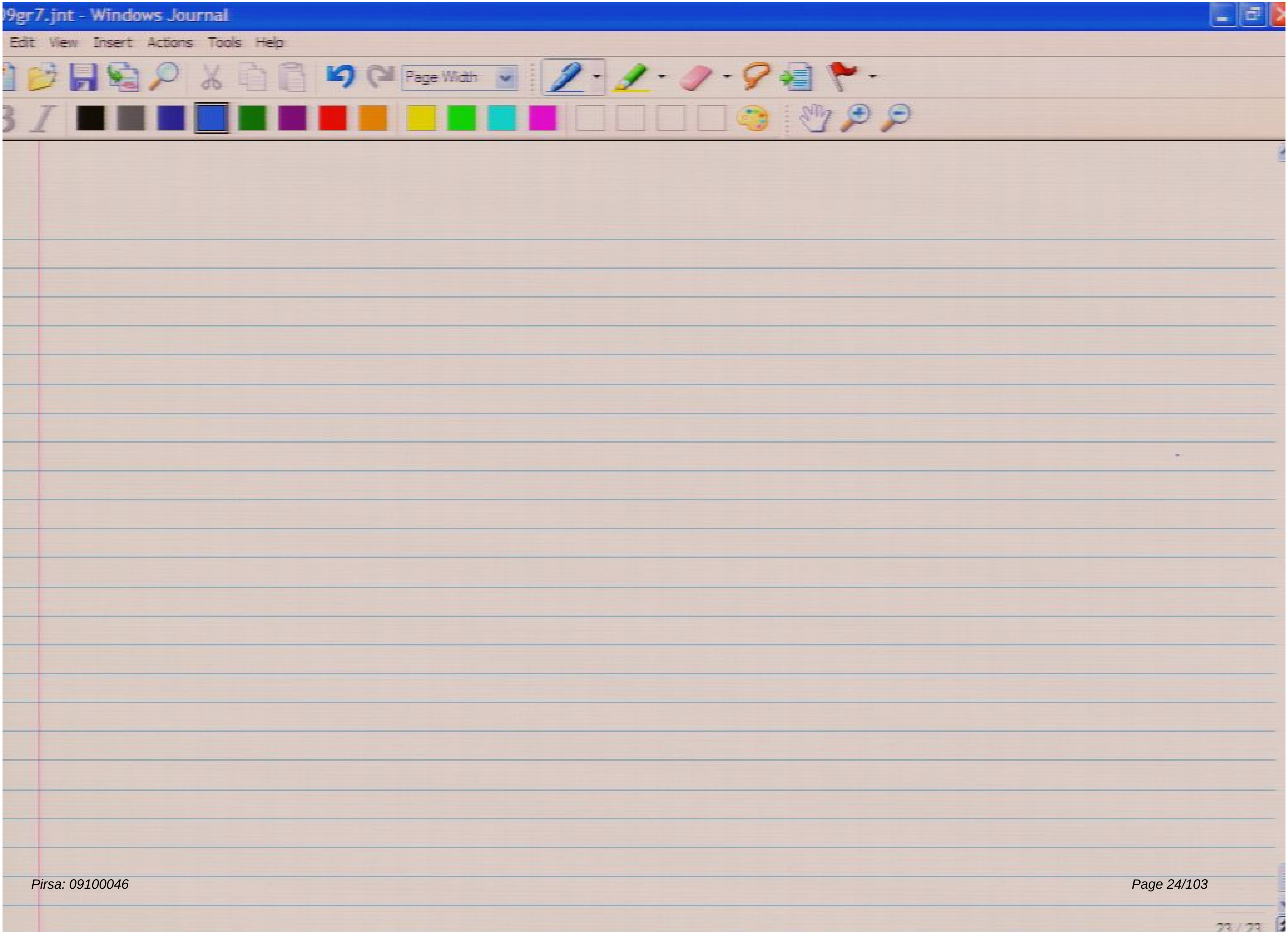
- Then: $g(e_r, e_s) = g_{ij}\,\theta^i(e_r)\,\theta^j(e_s)$

$$= g_{rs}$$

GR always assumes Lorentzian mfld.

For Space-Time the equivalence principle requires: $\}$ - M is called Lorentzian, if for all $p \in M$ there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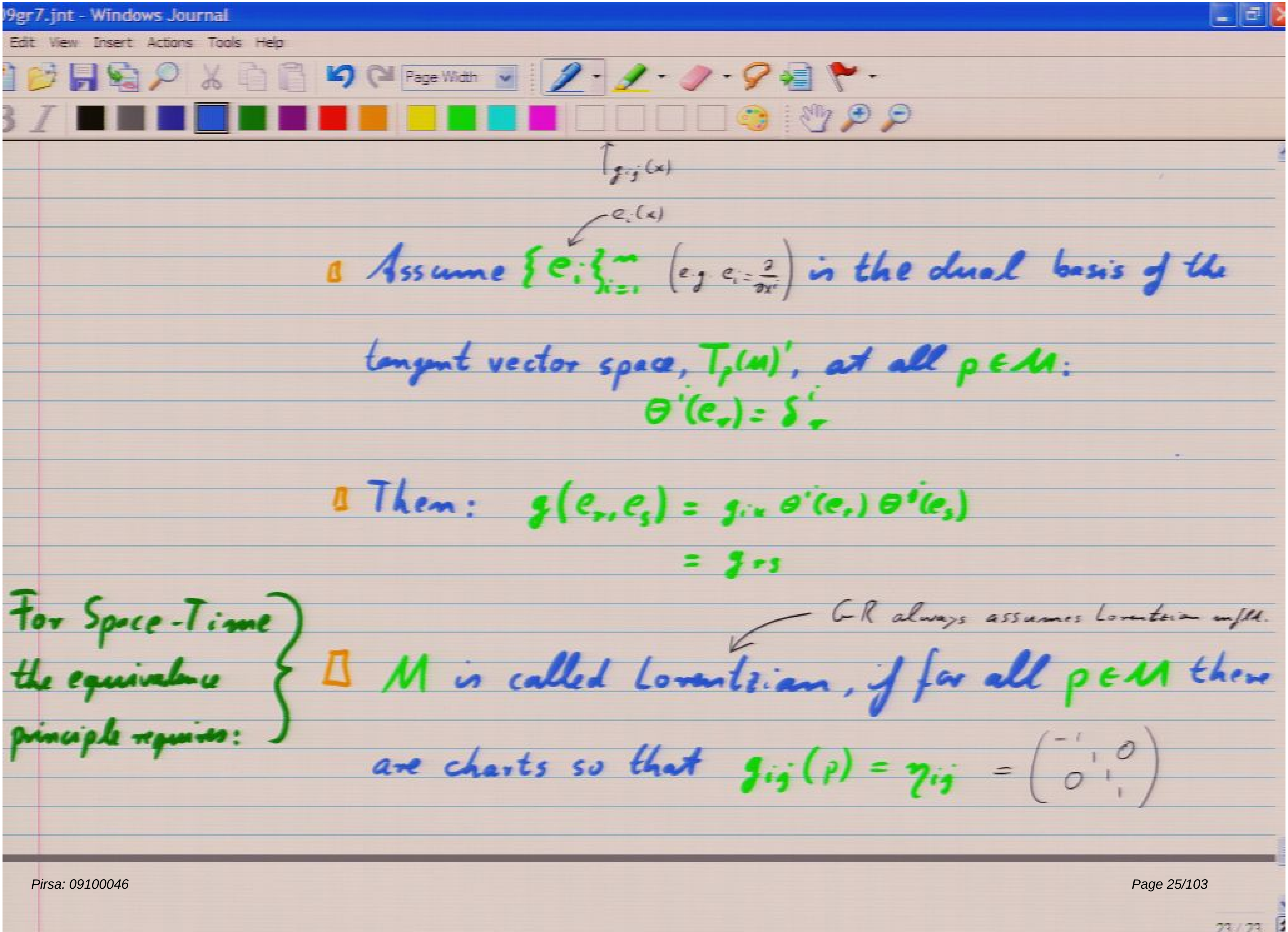

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For Space-Time the equivalence principle requires:

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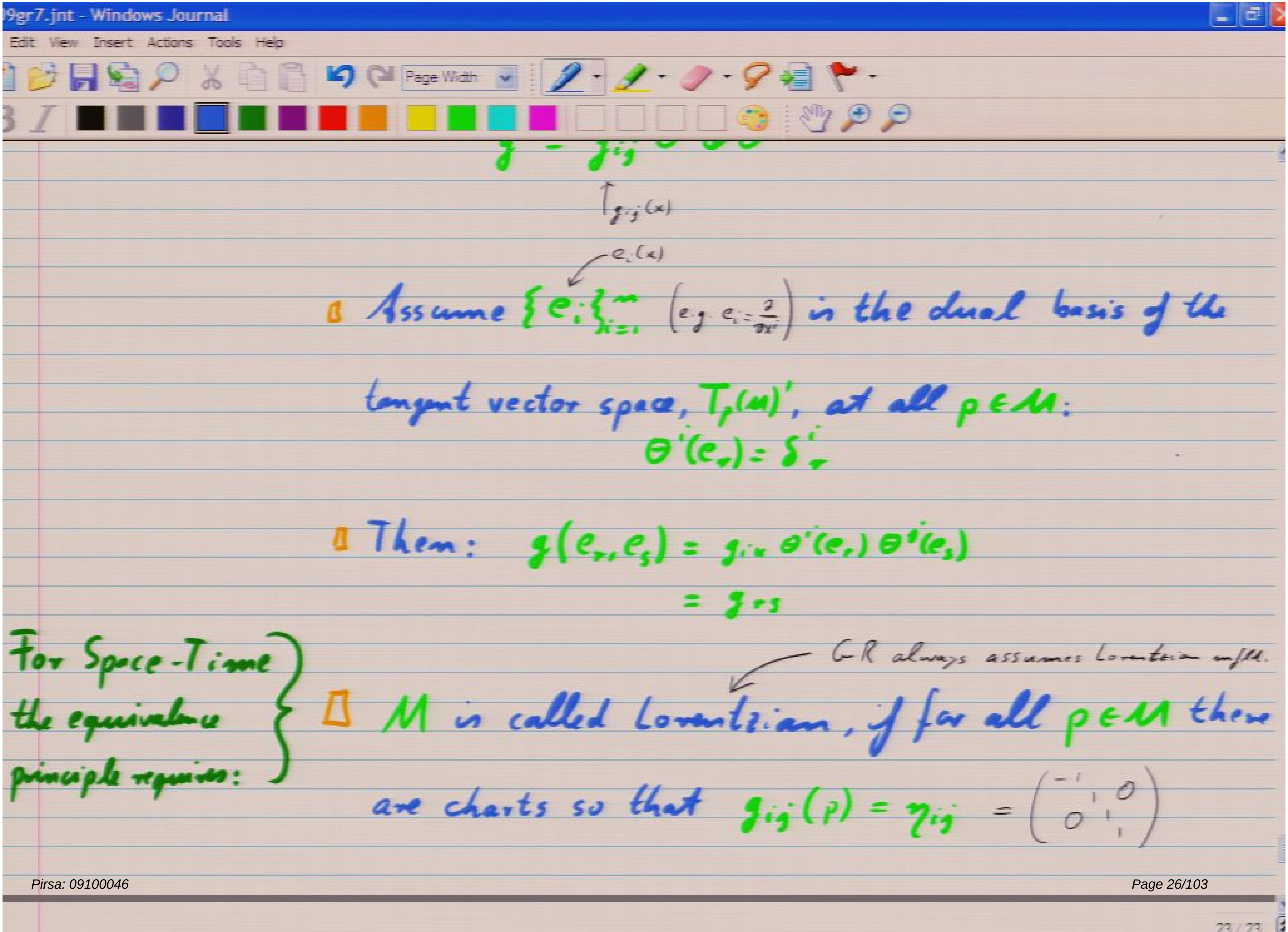
- M is called Lorentzian, if for all $p \in M$ there are charts so that $g_{ij}(p) = \eta_{ij} = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$



$g = g_{ij}\, \theta \otimes \theta$

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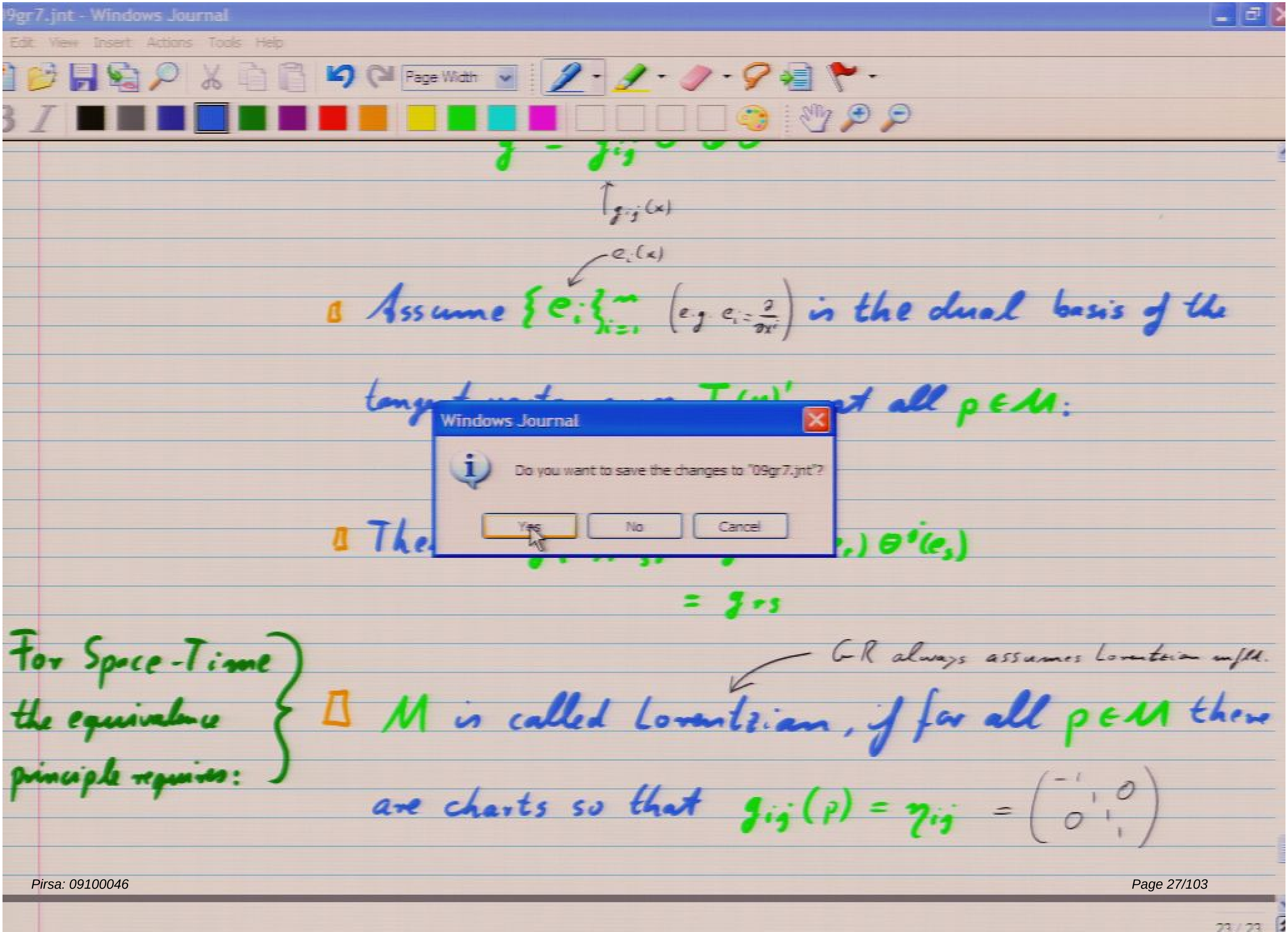

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$g = g_{ij} \theta^i \otimes \theta^j$

$\uparrow g_{ij}(x)$

$e_i(x)$

Assume $\{e_i\}_{i=1}^{m}$ $\left(\text{e.g. } e_i = \frac{\partial}{\partial x^i}\right)$ is the dual basis of the tangent vectors [illegible] $T(M)'$ at all $p \in M$:

The [illegible] $e_r) \theta^j(e_s)$

$= g_{rs}$

For Space-Time the equivalence principle requires:

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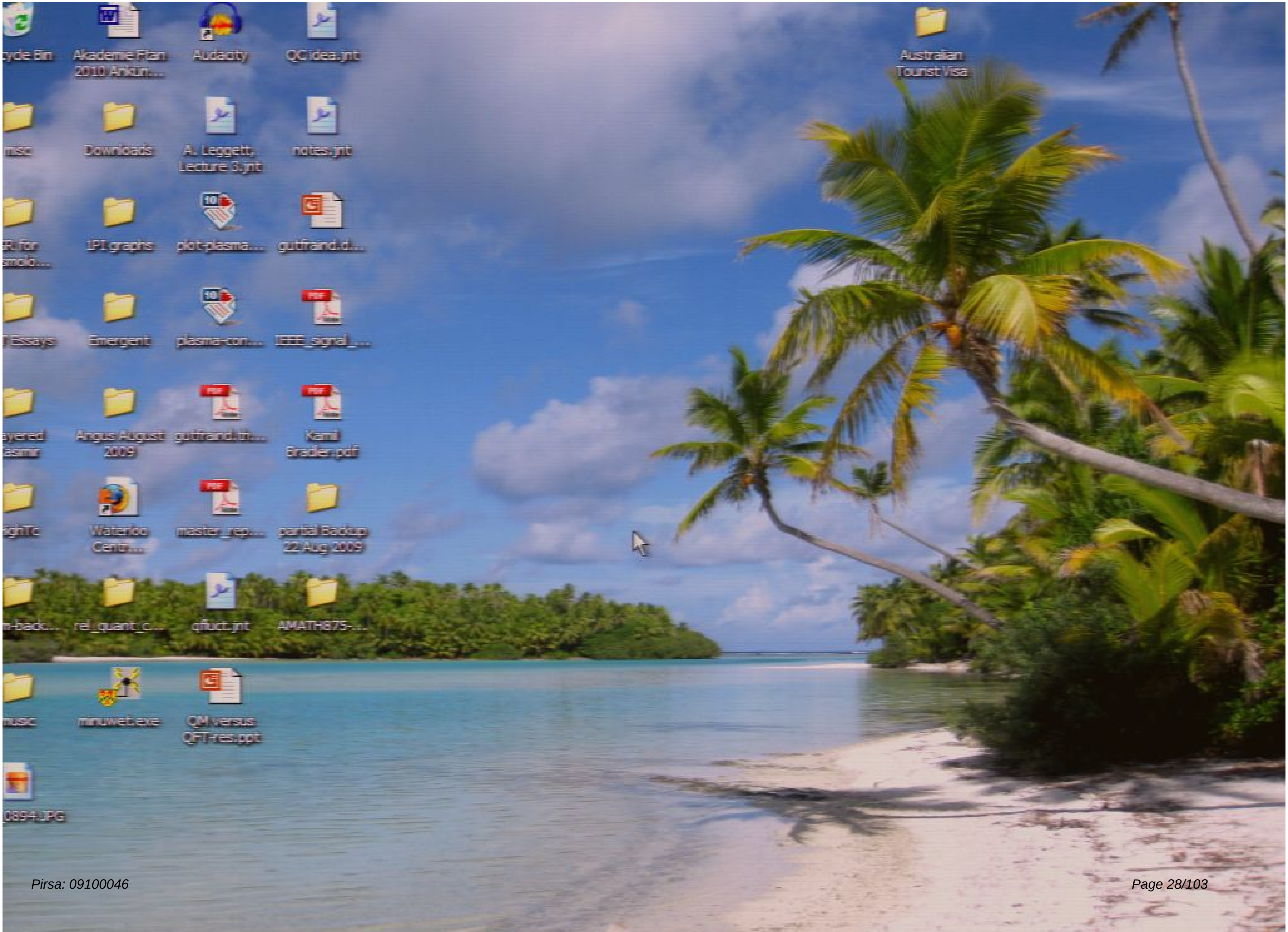
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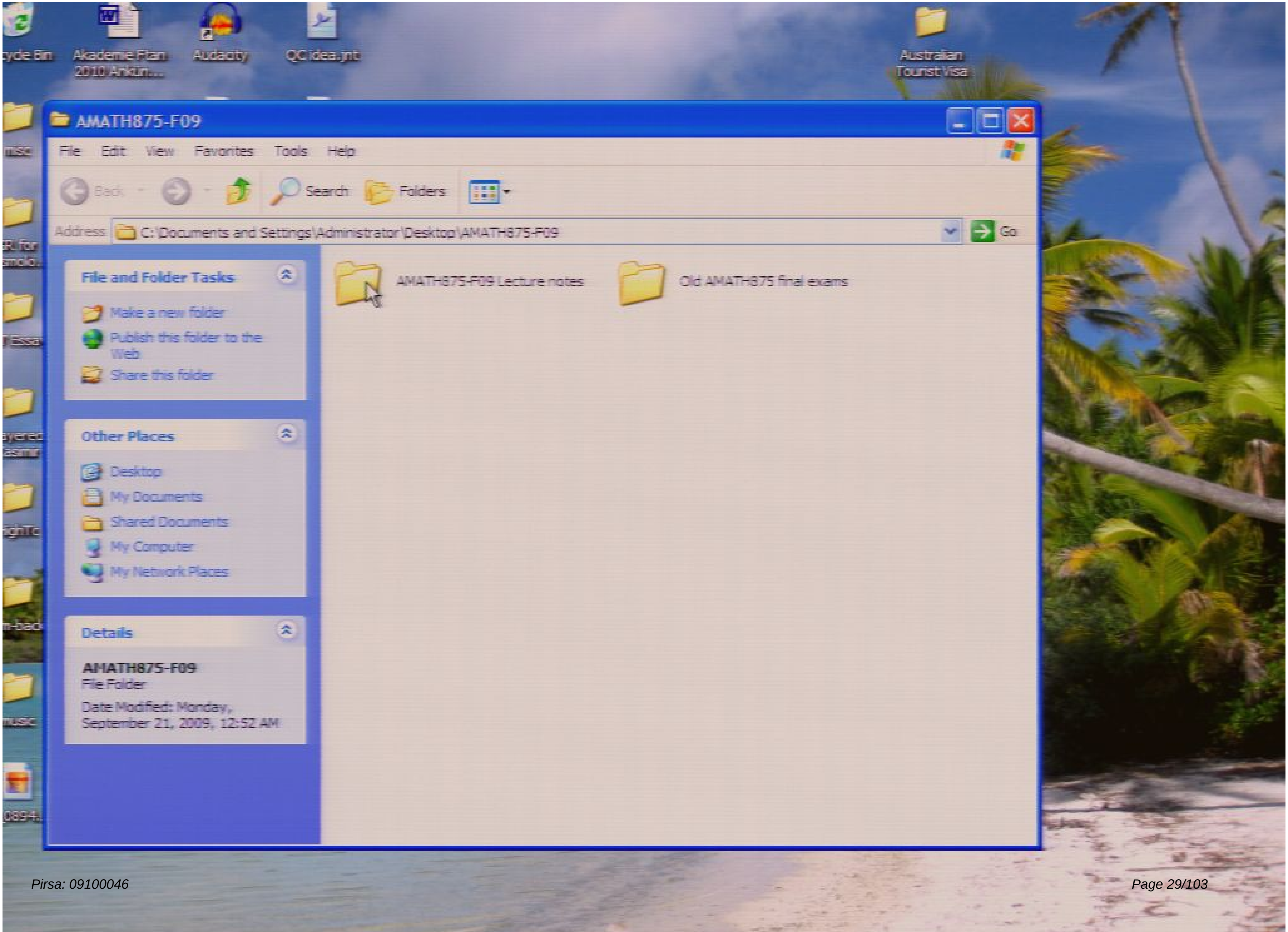

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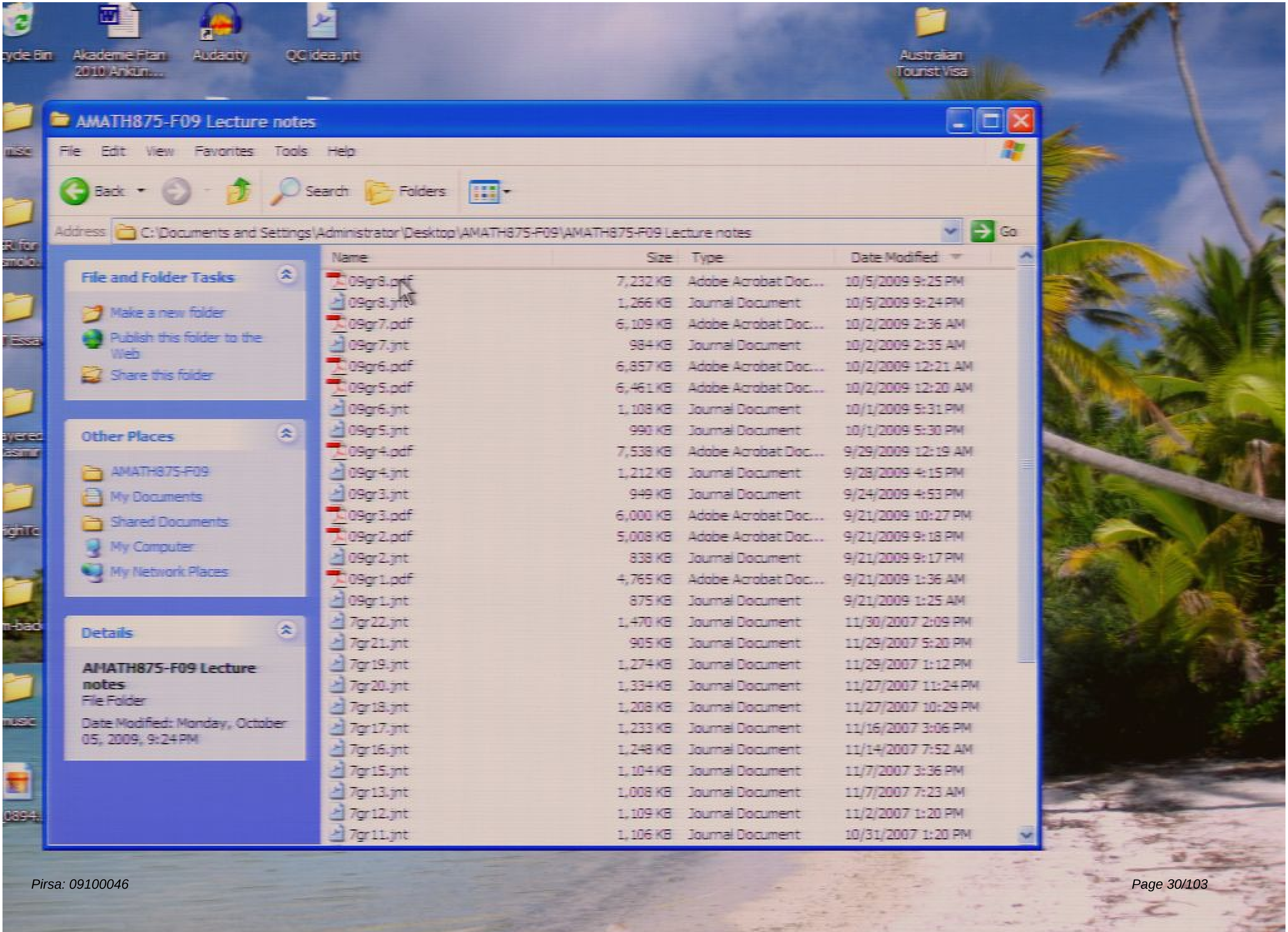

AMATH875-F09 Lecture notes

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AMATH875-F09 Lecture notes
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09gr5.pdf	6,461 KB	Adobe Acrobat Doc...	10/2/2009 12:20 AM
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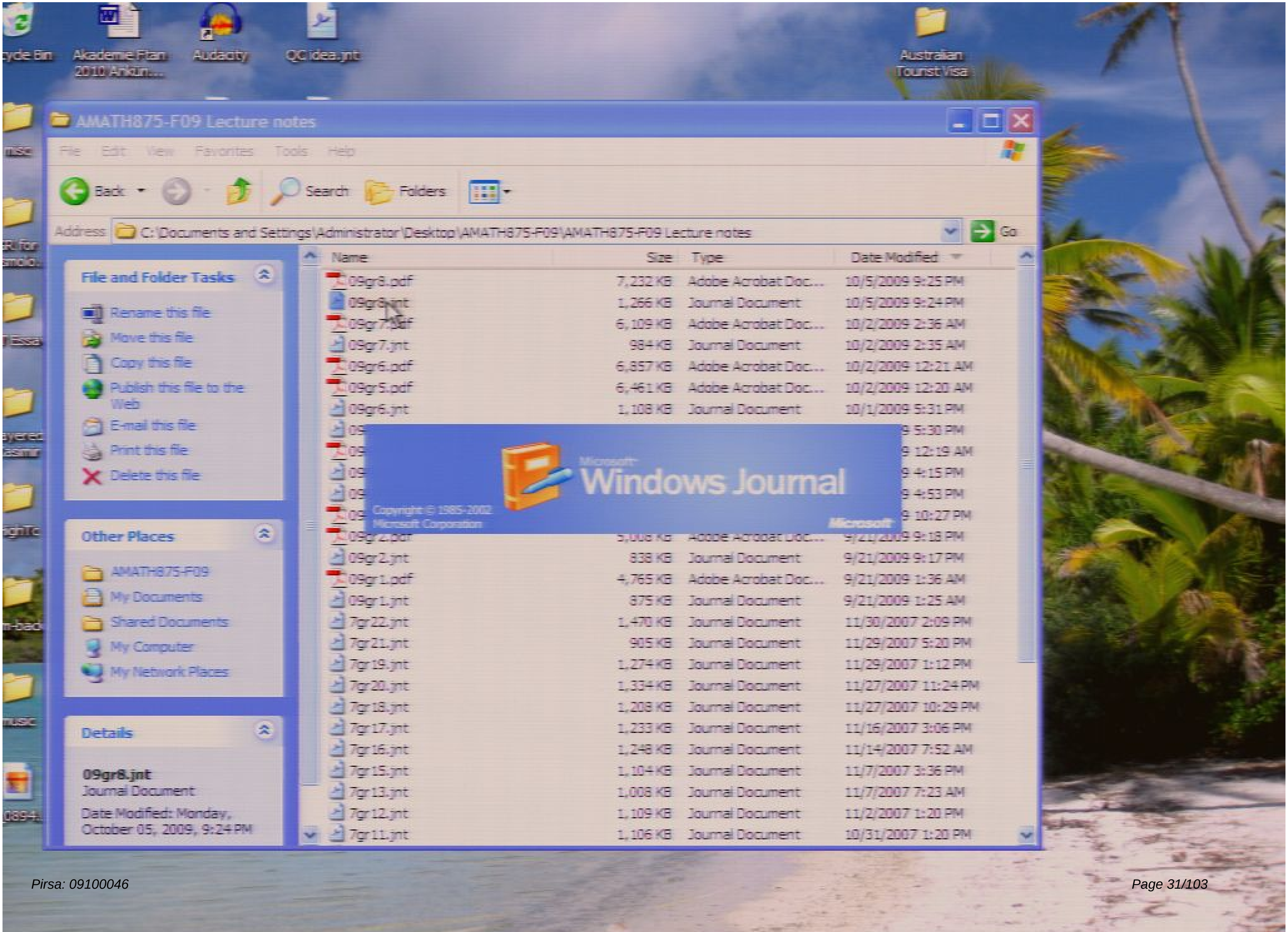

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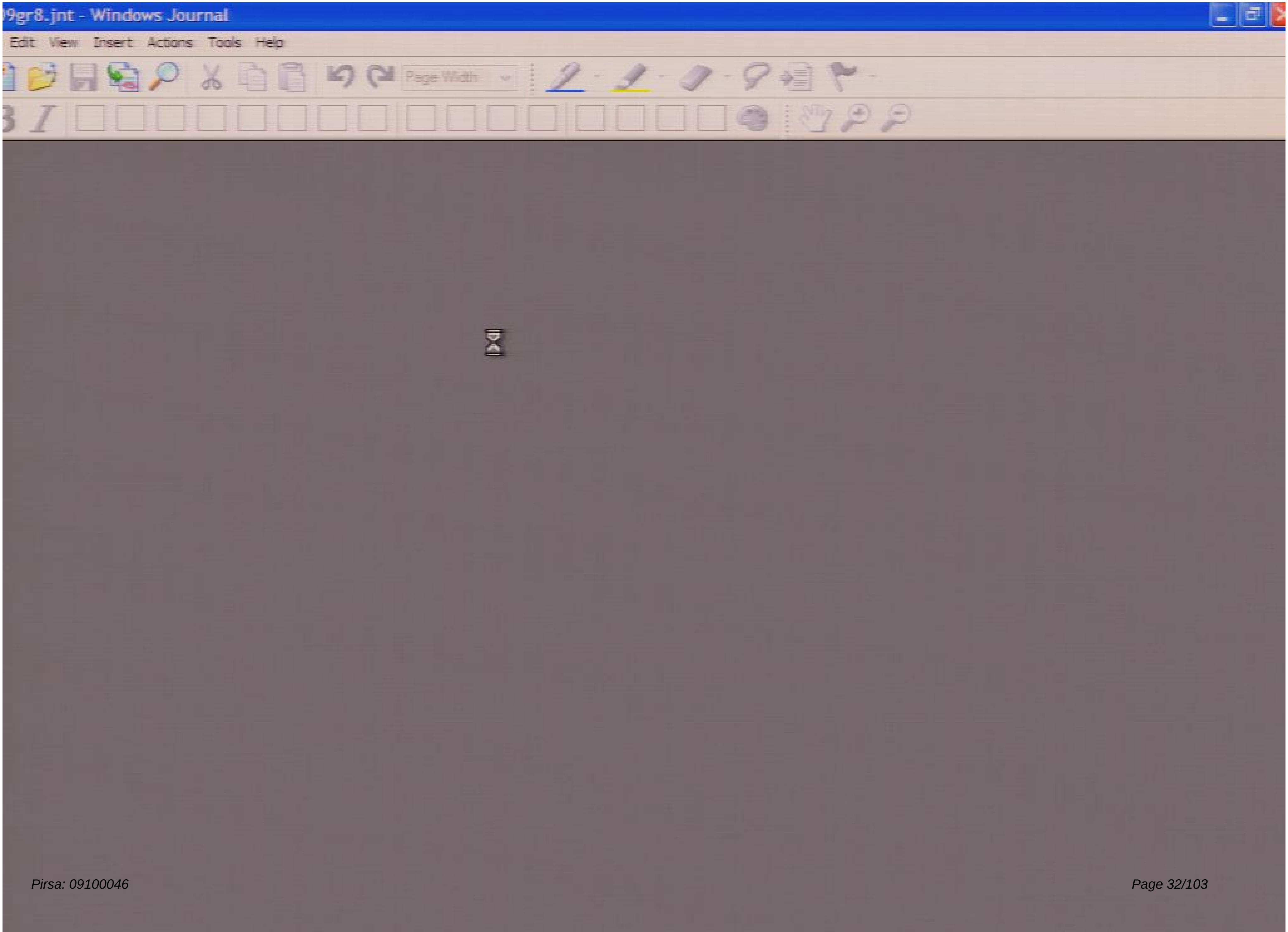
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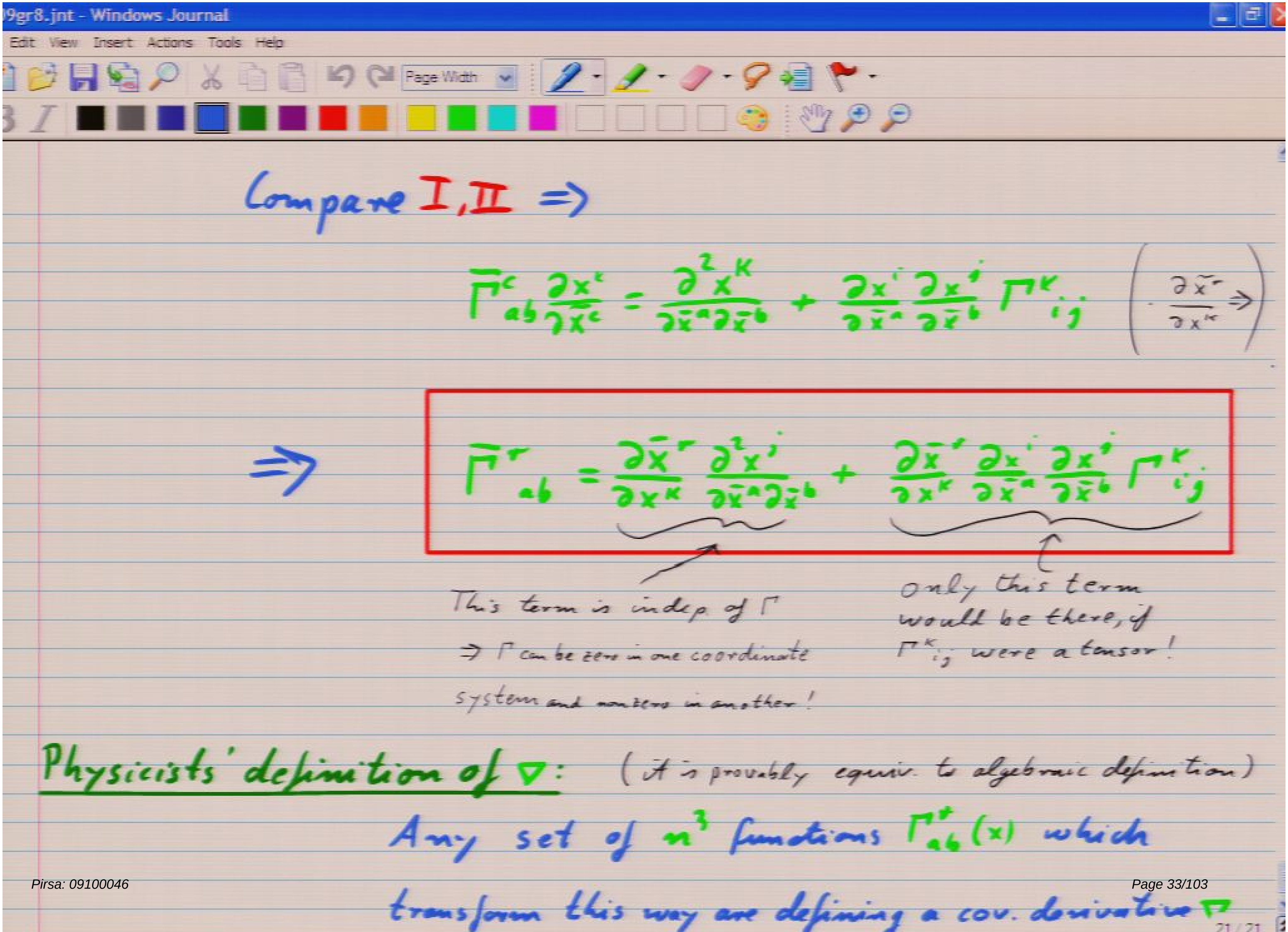


Compare I, II $\Rightarrow$

$$\bar{\Gamma}^c{}_{ab}\frac{\partial x^k}{\partial \bar{x}^c} = \frac{\partial^2 x^k}{\partial \bar{x}^a \partial \bar{x}^b} + \frac{\partial x^i}{\partial \bar{x}^a}\frac{\partial x^j}{\partial \bar{x}^b}\Gamma^k{}_{ij} \qquad \left(\cdot \frac{\partial \bar{x}^r}{\partial x^k} \Rightarrow\right)$$

$$\Rightarrow \quad \boxed{\bar{\Gamma}^r{}_{ab} = \underbrace{\frac{\partial \bar{x}^r}{\partial x^k}\frac{\partial^2 x^j}{\partial \bar{x}^a \partial \bar{x}^b}} + \underbrace{\frac{\partial \bar{x}^r}{\partial x^k}\frac{\partial x^i}{\partial \bar{x}^a}\frac{\partial x^j}{\partial \bar{x}^b}\Gamma^k{}_{ij}}}$$

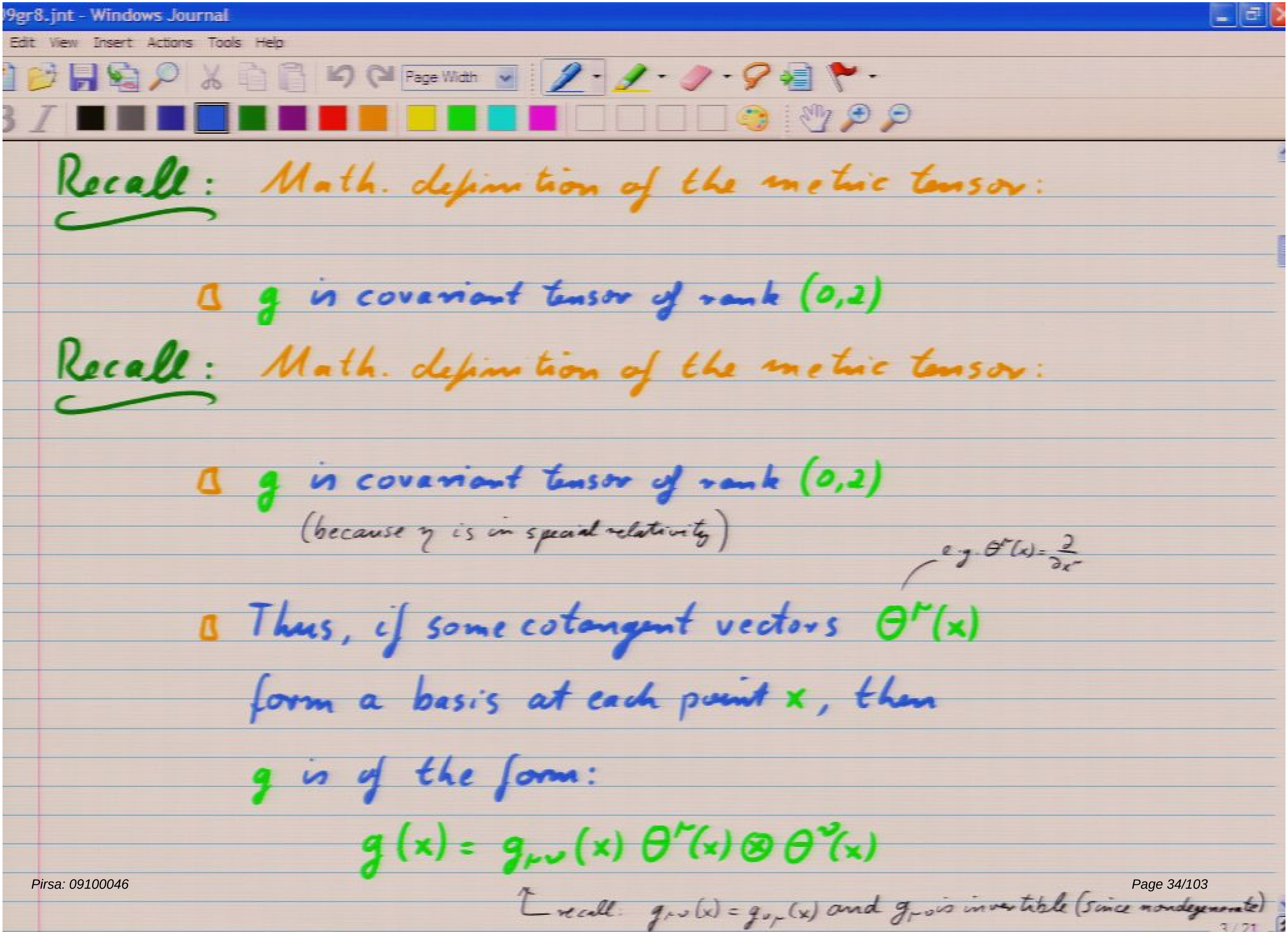
This term is indep. of Γ

$\Rightarrow$ Γ can be zero in one coordinate system and nonzero in another!

only this term would be there, if $\Gamma^k{}_{ij}$ were a tensor!

Physicists' definition of ∇: (it is provably equiv. to algebraic definition)

Any set of n^3 functions $\Gamma^r{}_{ab}(x)$ which transform this way are defining a cov. derivative ∇



Recall: Math. definition of the metric tensor:

- g is covariant tensor of rank $(0,2)$

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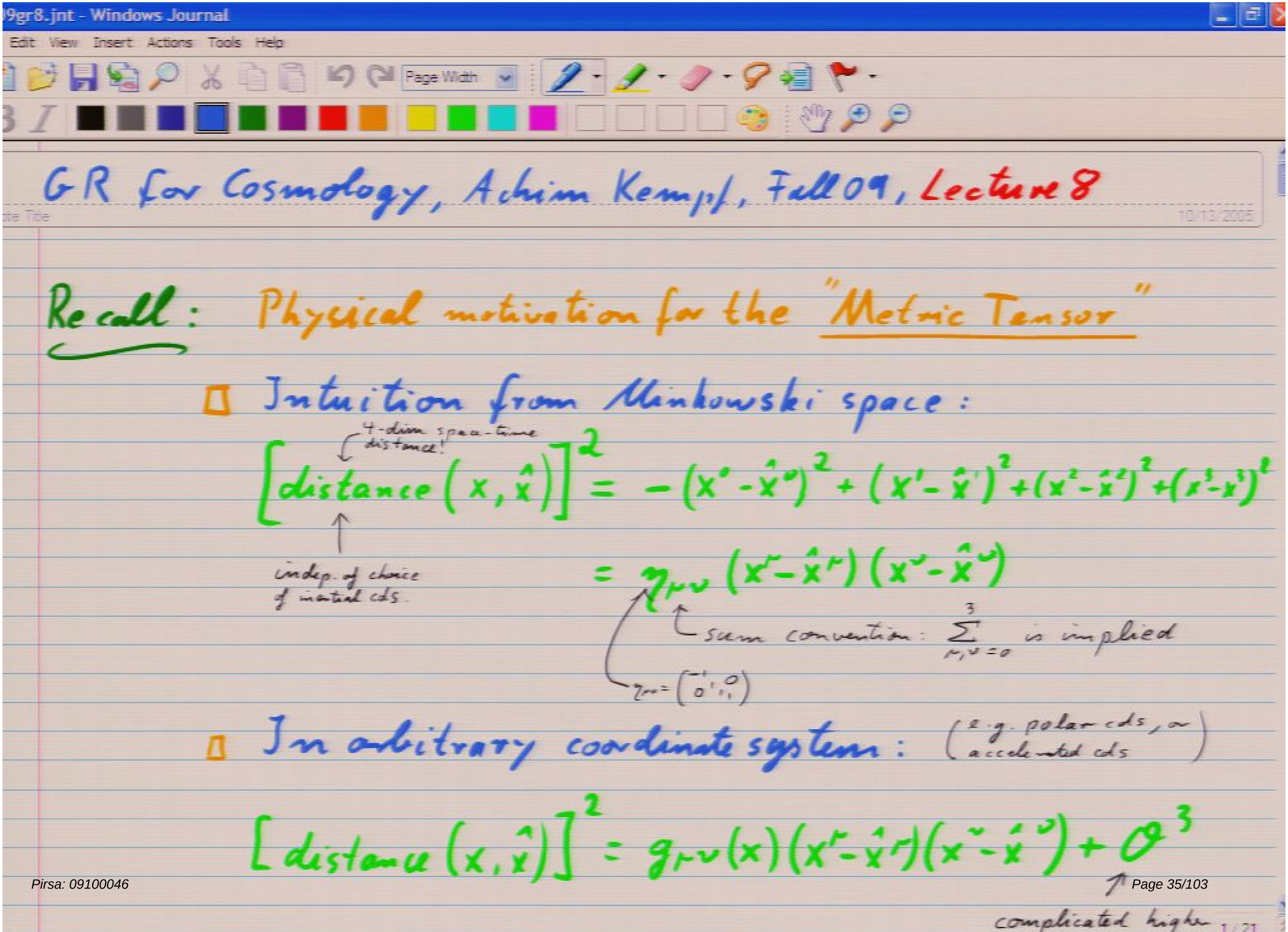
(because η is in special relativity)

e.g. $\theta^\mu(x) = \frac{\partial}{\partial x^\mu}$

- Thus, if some cotangent vectors $\theta^\mu(x)$ form a basis at each point x, then g is of the form:

$$g(x) = g_{\mu\nu}(x)\, \theta^\mu(x) \otimes \theta^\nu(x)$$

recall: $g_{\mu\nu}(x) = g_{\nu\mu}(x)$ and $g_{\mu\nu}$ is invertible (since nondegenerate)



GR for Cosmology, Achim Kempf, Fall 09, Lecture 8

Recall: Physical motivation for the "Metric Tensor"

☐ Intuition from Minkowski space:

4-dim space-time distance!

$$\left[\text{distance}(x,\hat{x})\right]^2 = -(x^0-\hat{x}^0)^2+(x^1-\hat{x}^1)^2+(x^2-\hat{x}^2)^2+(x^3-\hat{x}^3)^2$$

indep. of choice of inertial cds.

$$= \eta_{\mu\nu}(x^\mu-\hat{x}^\mu)(x^\nu-\hat{x}^\nu)$$

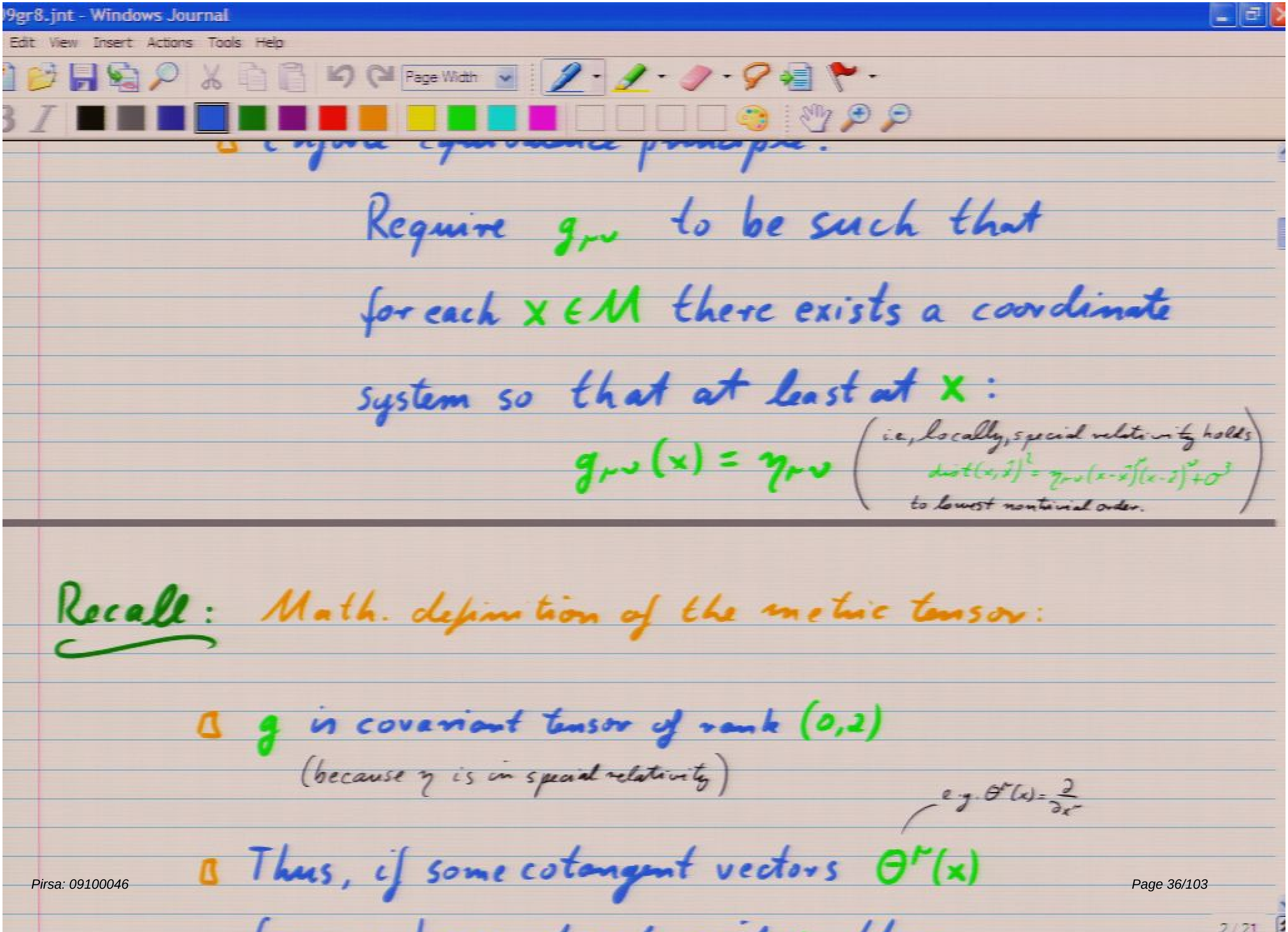
sum convention: $\sum_{\mu,\nu=0}^{3}$ is implied

$\eta_{\mu\nu} = \begin{pmatrix} -1 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix}$

☐ In arbitrary coordinate system: (e.g. polar cds, or accelerated cds)

$$\left[\text{distance}(x,\hat{x})\right]^2 = g_{\mu\nu}(x)(x^\mu-\hat{x}^\mu)(x^\nu-\hat{x}^\nu)+\mathcal{O}^3$$

complicated highe



Enforce equivalence principle:

Require $g_{\mu\nu}$ to be such that for each $x \in \mathcal{M}$ there exists a coordinate system so that at least at x:

$$g_{\mu\nu}(x) = \eta_{\mu\nu} \quad \left(\begin{array}{c} \text{i.e., locally, special relativity holds} \\ dist(x,\tilde{x})^2 = \eta_{\mu\nu}(x-\tilde{x})^\mu(x-\tilde{x})^\nu + O^3 \\ \text{to lowest nontrivial order.} \end{array} \right)$$

Recall: Math. definition of the metric tensor:

- g is covariant tensor of rank $(0,2)$
  (because η is in special relativity)

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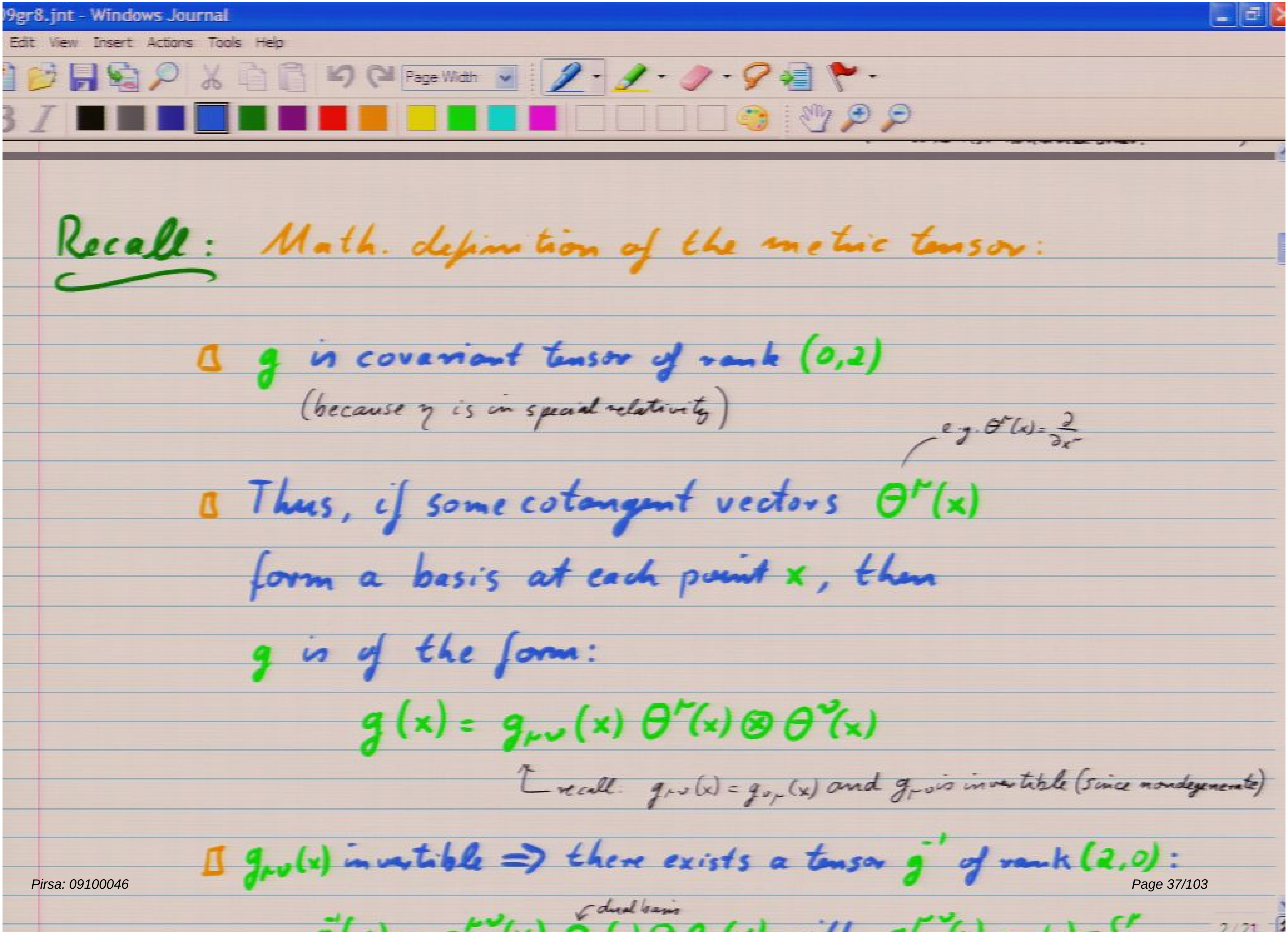

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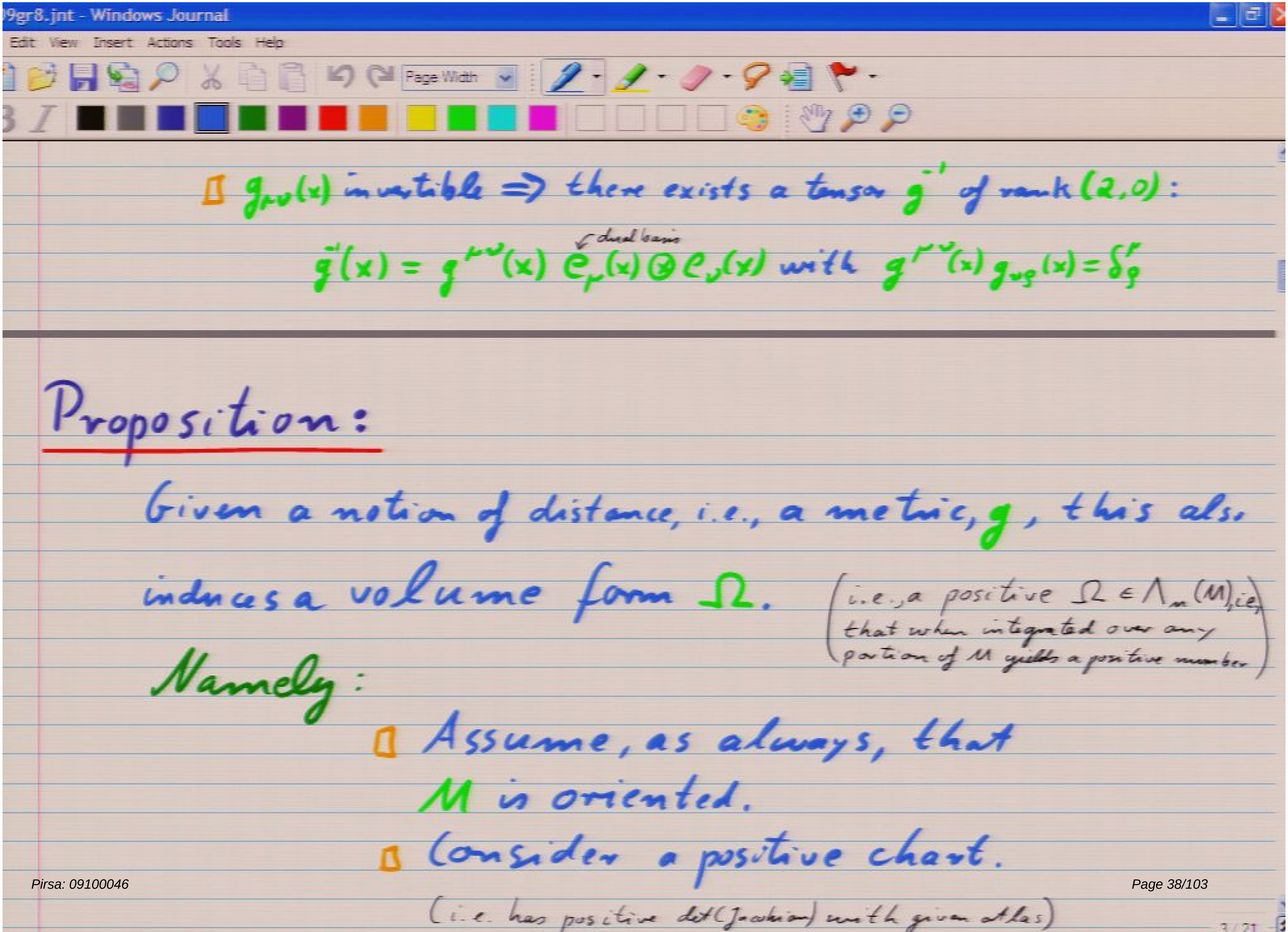
recall: $g_{\mu\nu}(x) = g_{\nu\mu}(x)$ and $g_{\mu\nu}$ is invertible (since nondegenerate)

- $g_{\mu\nu}(x)$ invertible $\Rightarrow$ there exists a tensor g^{-1} of rank $(2,0)$:

dual basis



- $g_{\mu\nu}(x)$ invertible $\Rightarrow$ there exists a tensor g^{-1} of rank $(2,0)$:

$$g^{-1}(x) = g^{\mu\nu}(x)\, \overset{\swarrow\text{dual basis}}{e_\mu(x)} \otimes e_\nu(x) \text{ with } g^{\mu\nu}(x)\, g_{\nu\rho}(x) = \delta^\mu_\rho$$

Proposition:

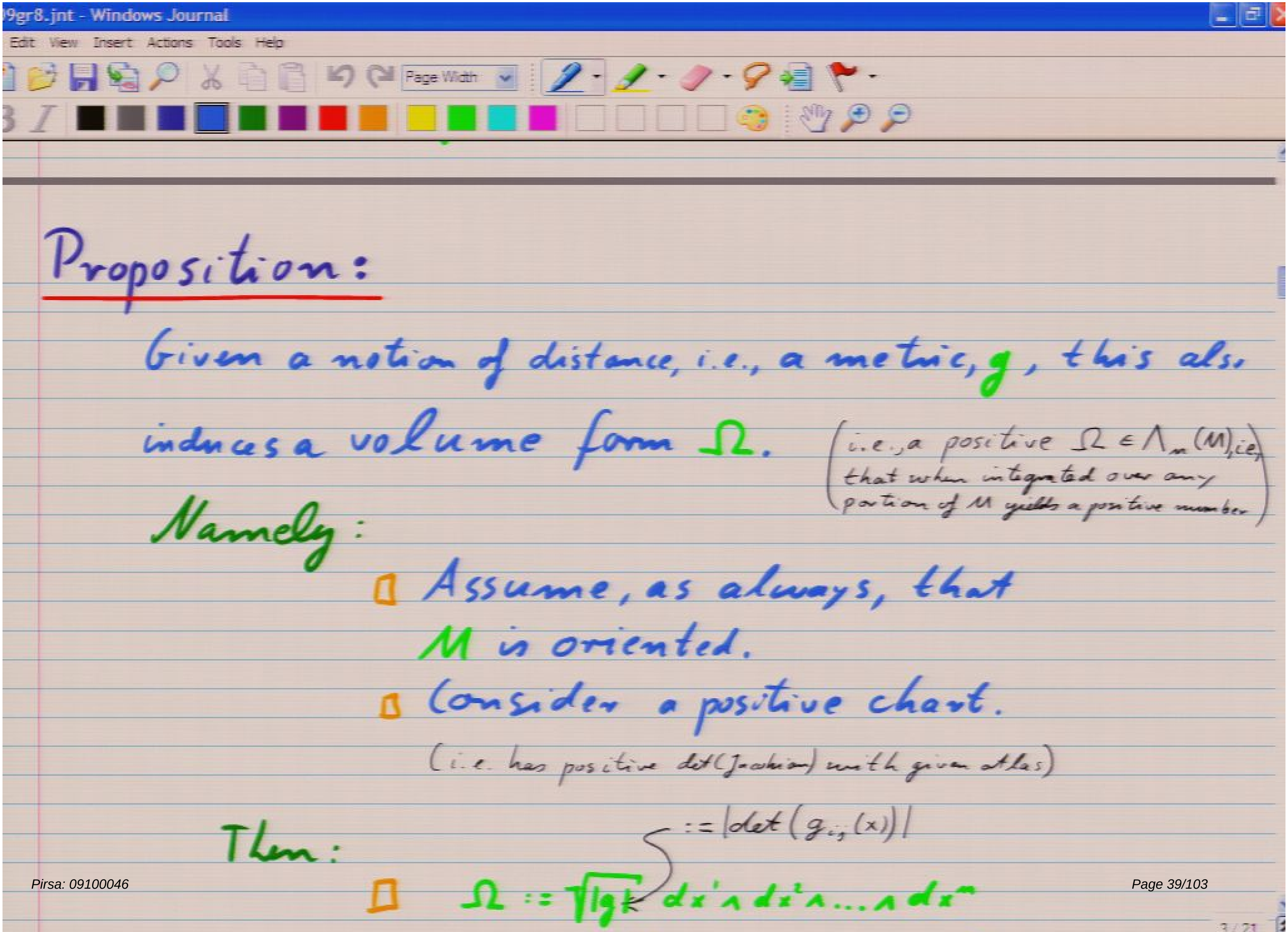
Given a notion of distance, i.e., a metric, g, this also induces a volume form Ω.

(i.e., a positive $\Omega \in \Lambda_n(M)$, i.e., that when integrated over any portion of M yields a positive number)

Namely:

- Assume, as always, that M is oriented.
- Consider a positive chart.

(i.e. has positive det(Jacobian) with given atlas)



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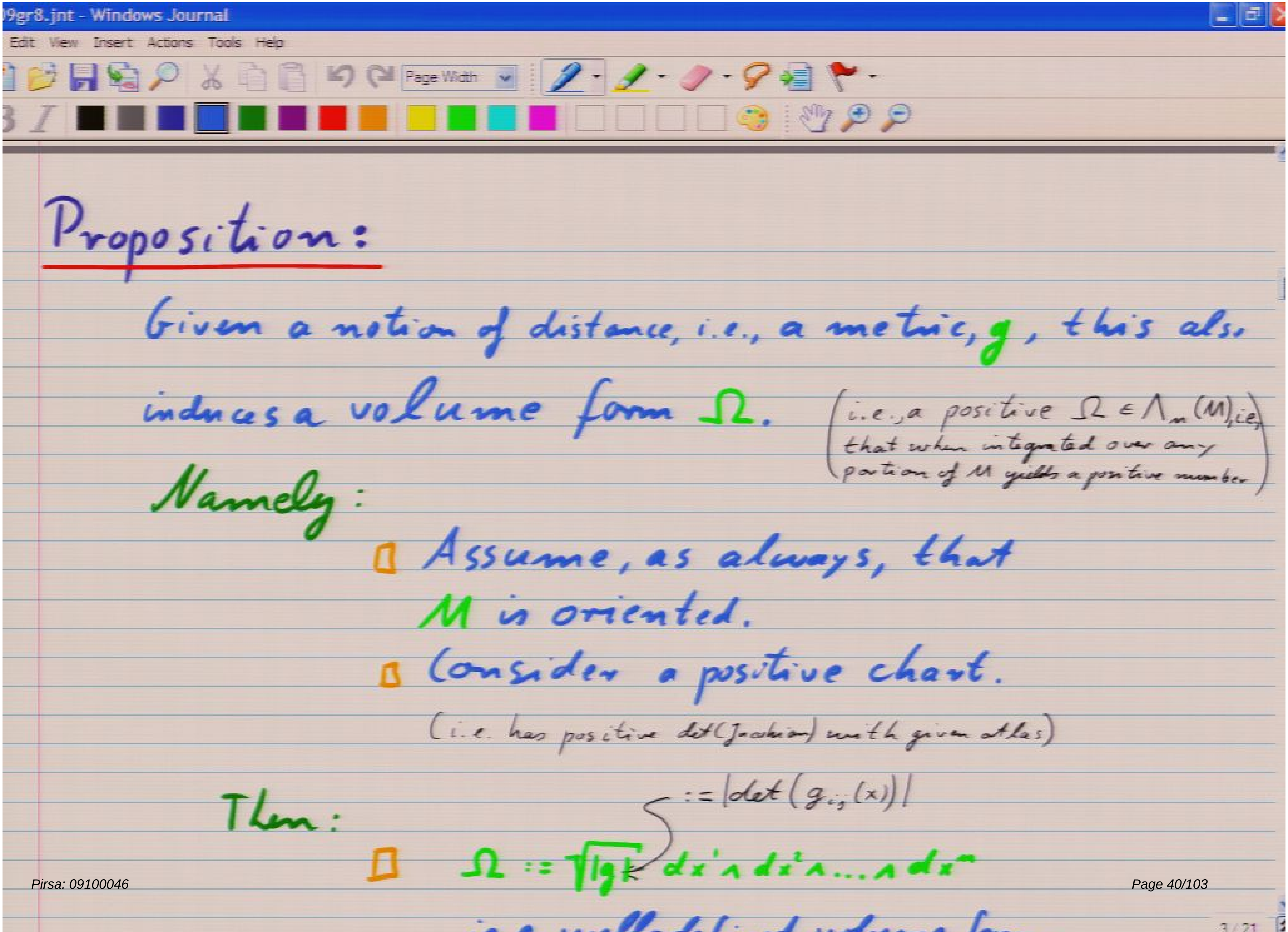
Then:

$:= |\det(g_{ij}(x))|$

- $\Omega := \sqrt{|g|}\, dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n$



Proposition:

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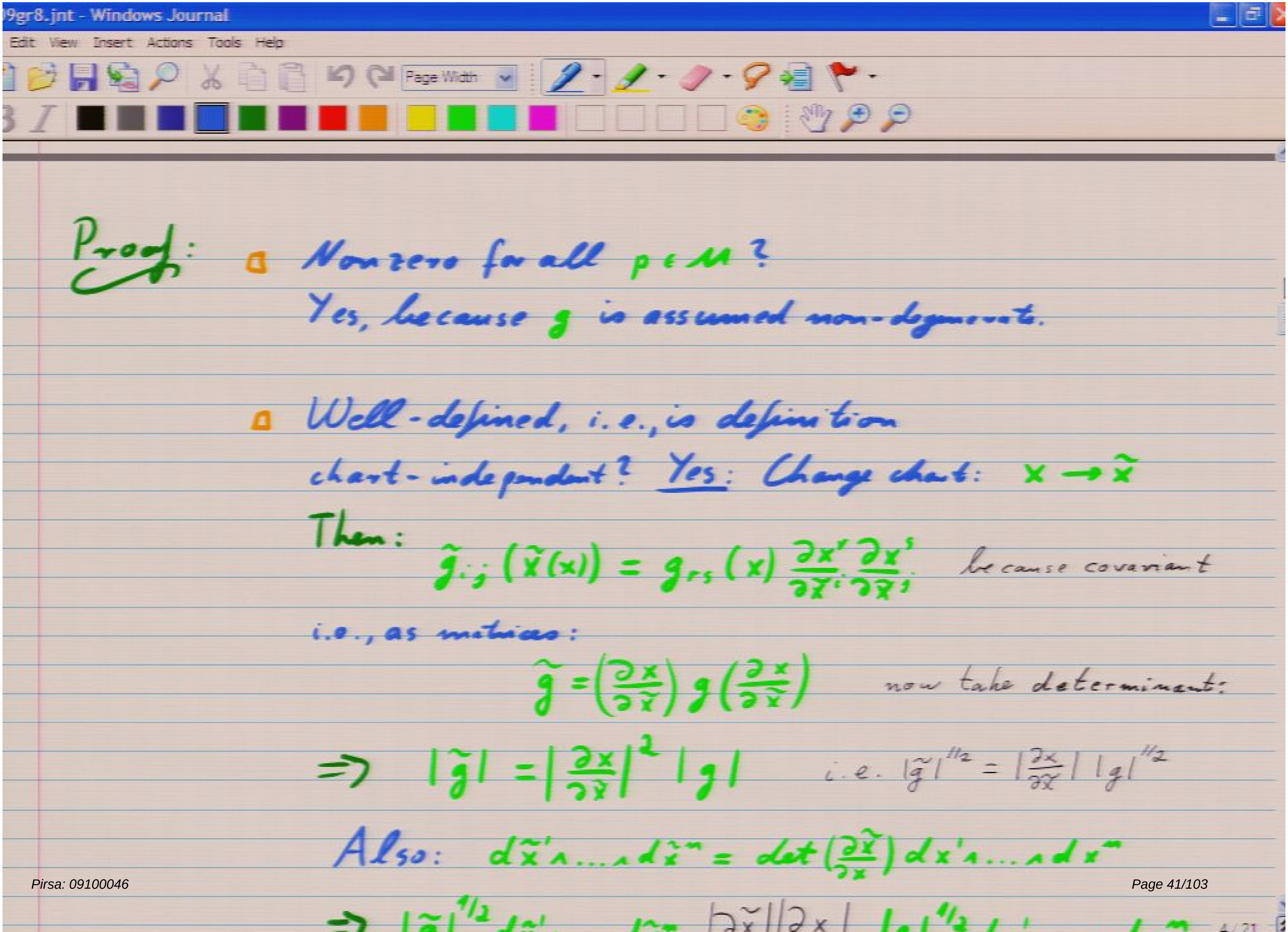


Proof:

☐ Nonzero for all $p \in M$?

Yes, because g is assumed non-degenerate.

☐ Well-defined, i.e., is definition chart-independent? Yes: Change chart: $x \to \tilde{x}$

Then:

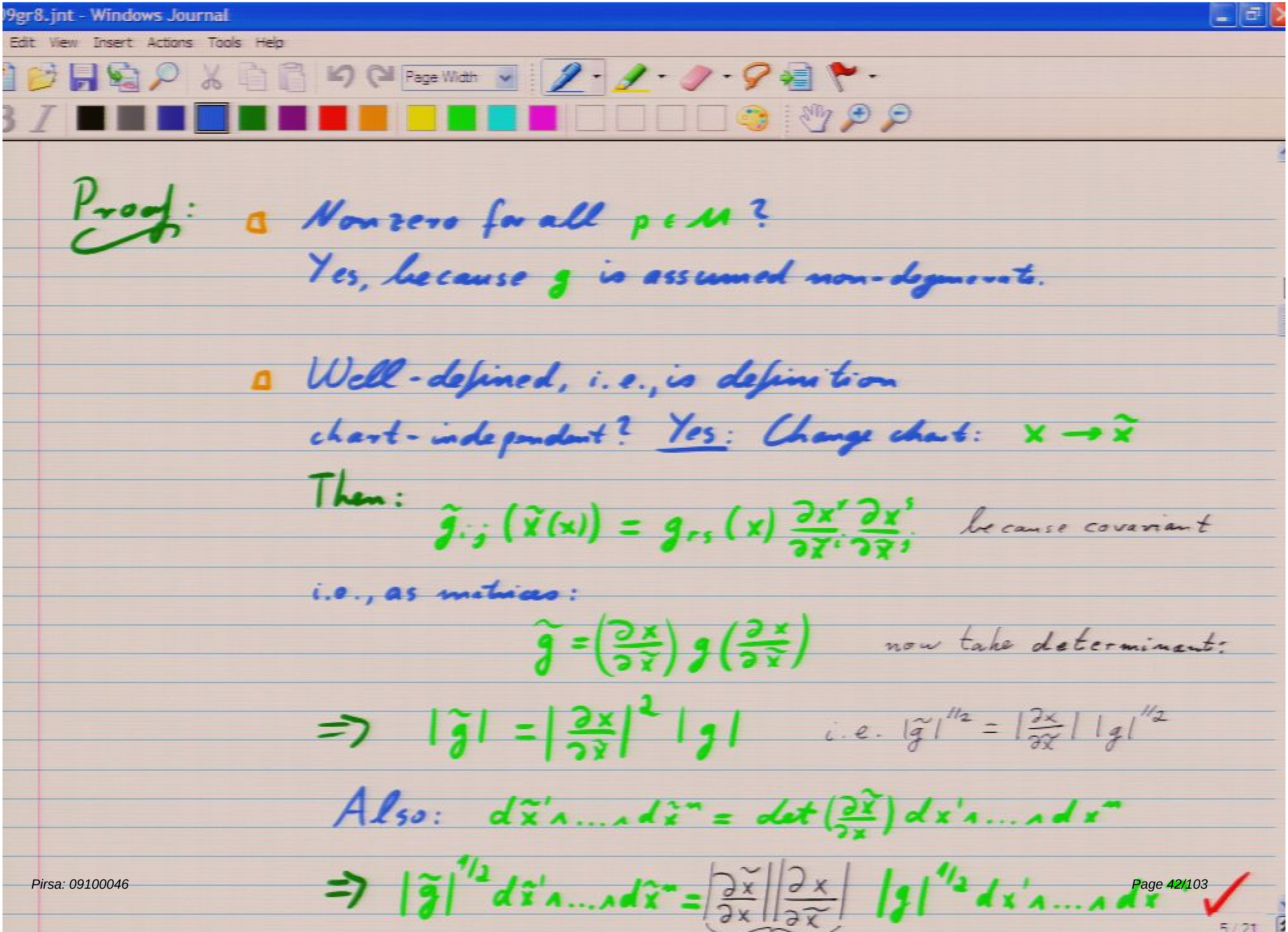
$$\tilde{g}_{ij}(\tilde{x}(x)) = g_{rs}(x) \frac{\partial x^r}{\partial \tilde{x}^i} \frac{\partial x^s}{\partial \tilde{x}^j}$$ because covariant

i.e., as matrices:

$$\tilde{g} = \left(\frac{\partial x}{\partial \tilde{x}}\right) g \left(\frac{\partial x}{\partial \tilde{x}}\right)$$ now take determinant:

$$\Rightarrow \quad |\tilde{g}| = \left|\frac{\partial x}{\partial \tilde{x}}\right|^2 |g|$$ i.e. $|\tilde{g}|^{1/2} = \left|\frac{\partial x}{\partial \tilde{x}}\right| |g|^{1/2}$

Also: $d\tilde{x}^1 \wedge \ldots \wedge d\tilde{x}^n = \det\left(\frac{\partial \tilde{x}}{\partial x}\right) dx^1 \wedge \ldots \wedge dx^n$



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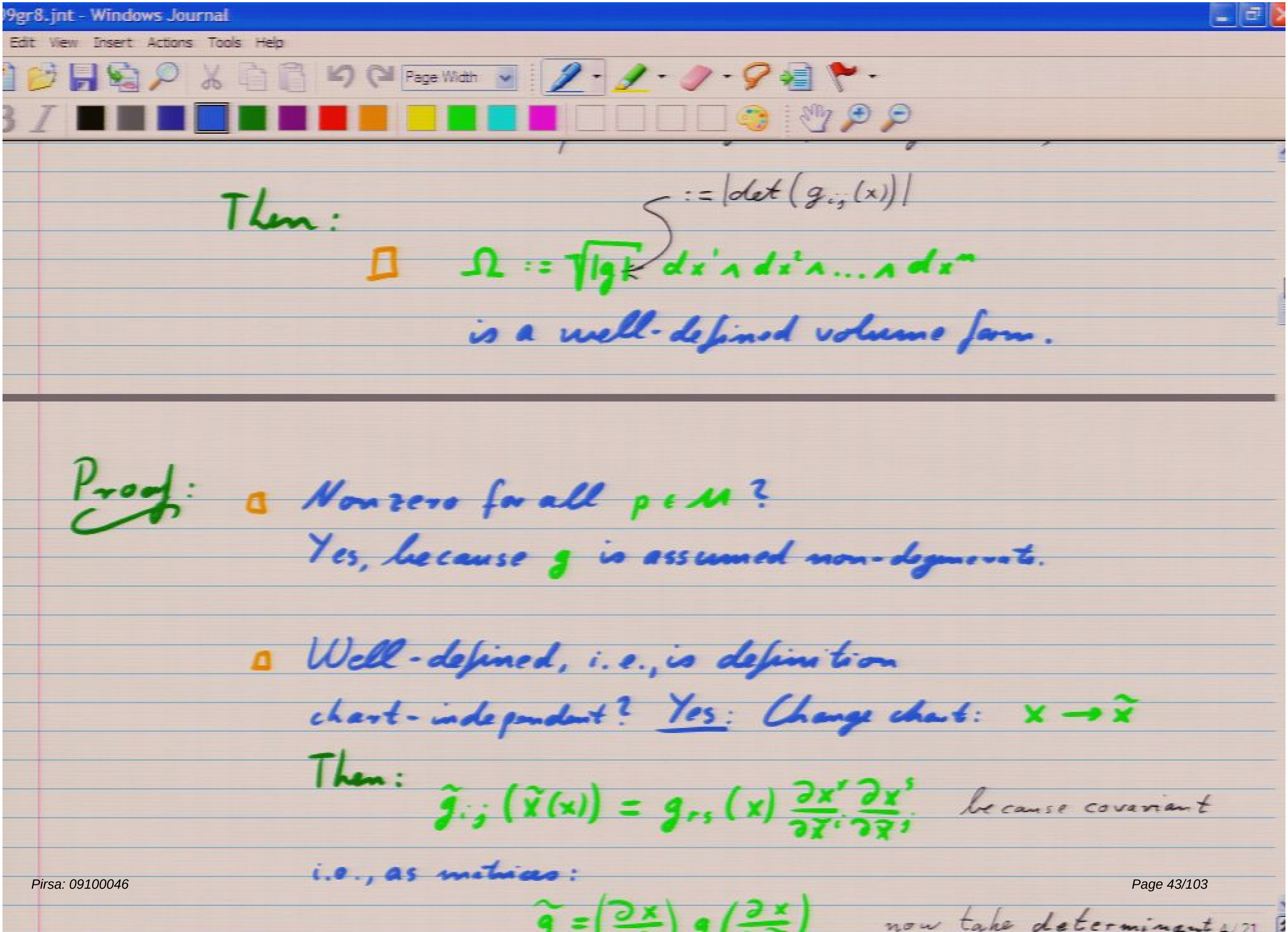

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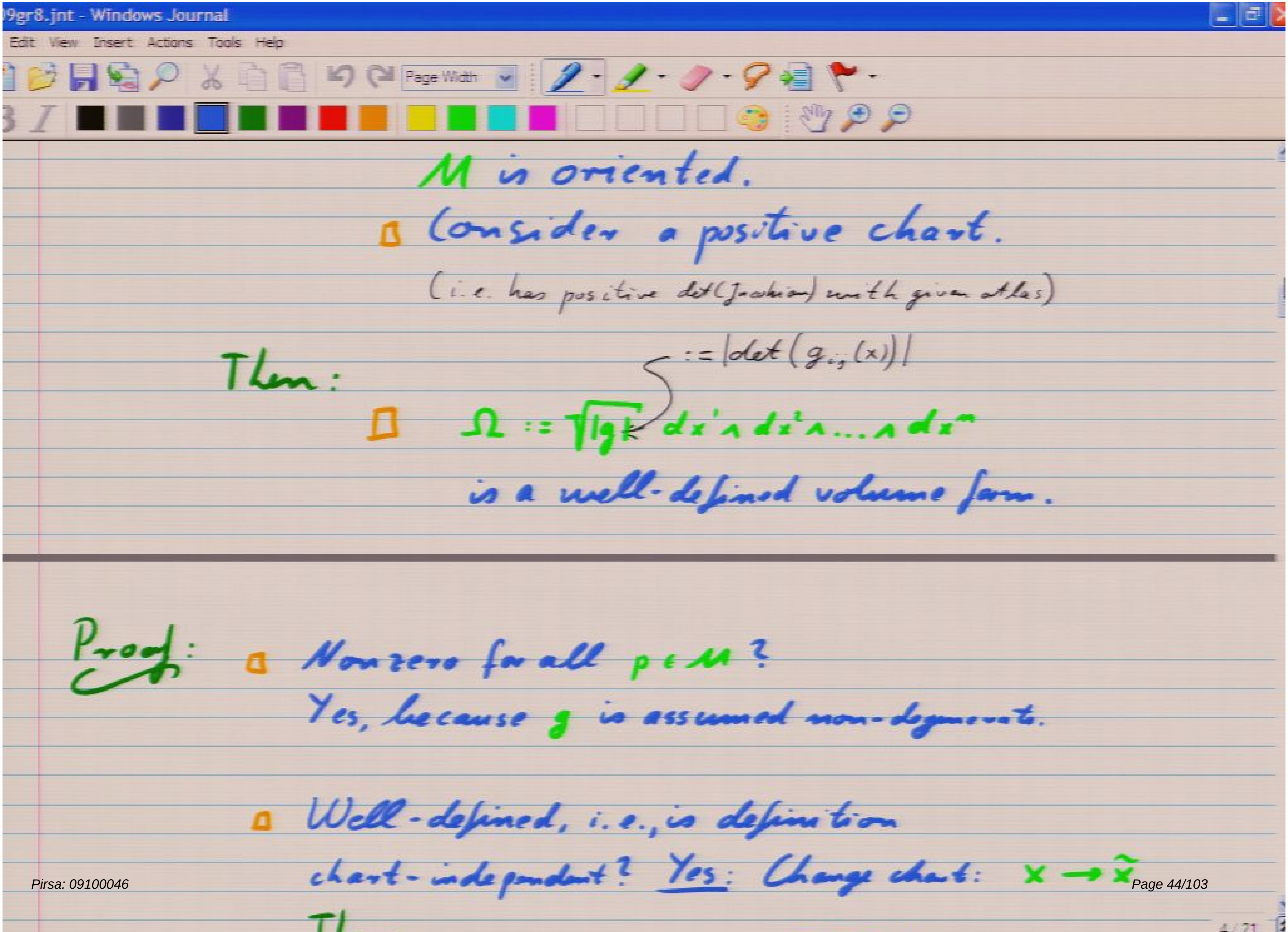

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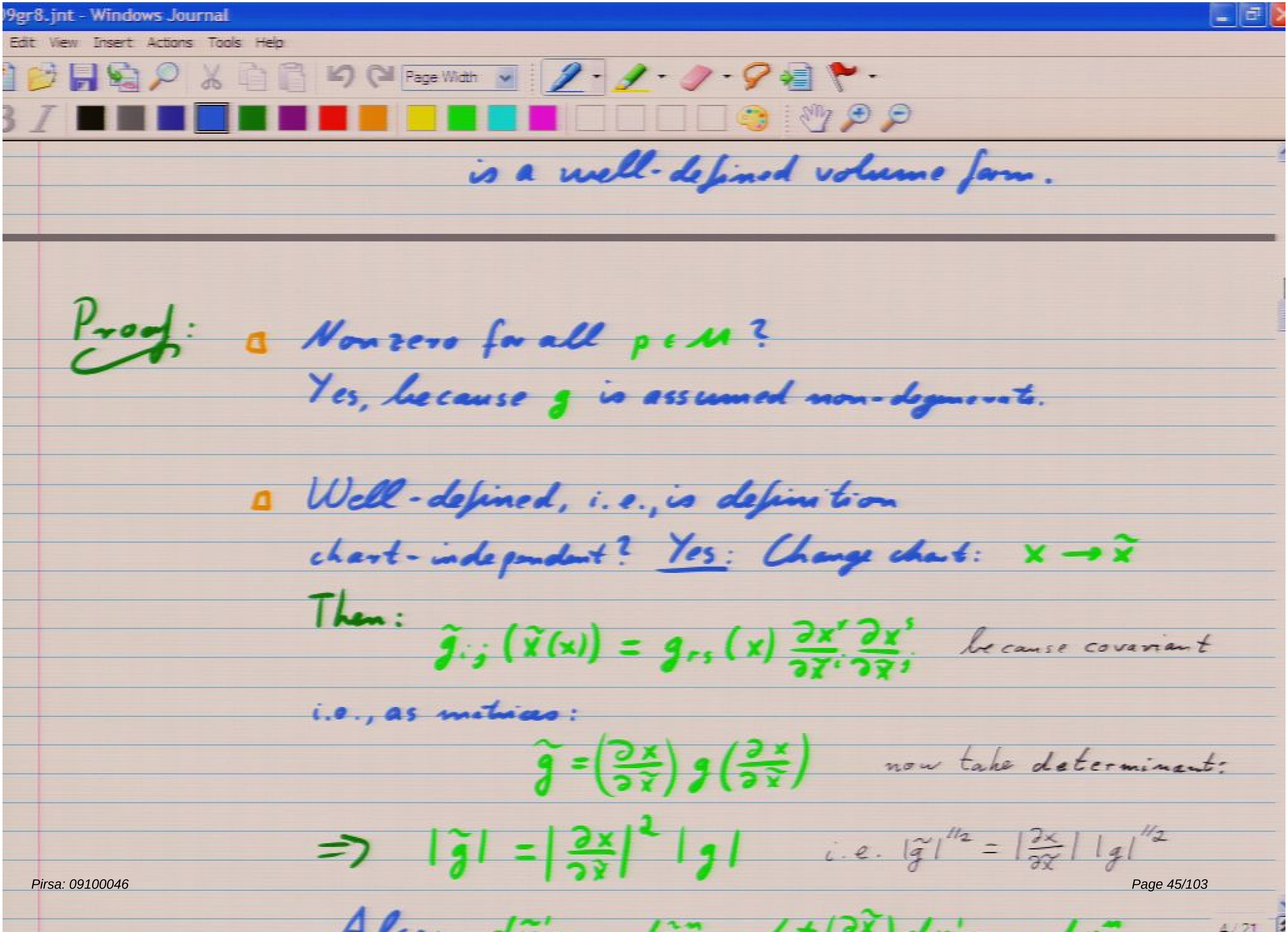

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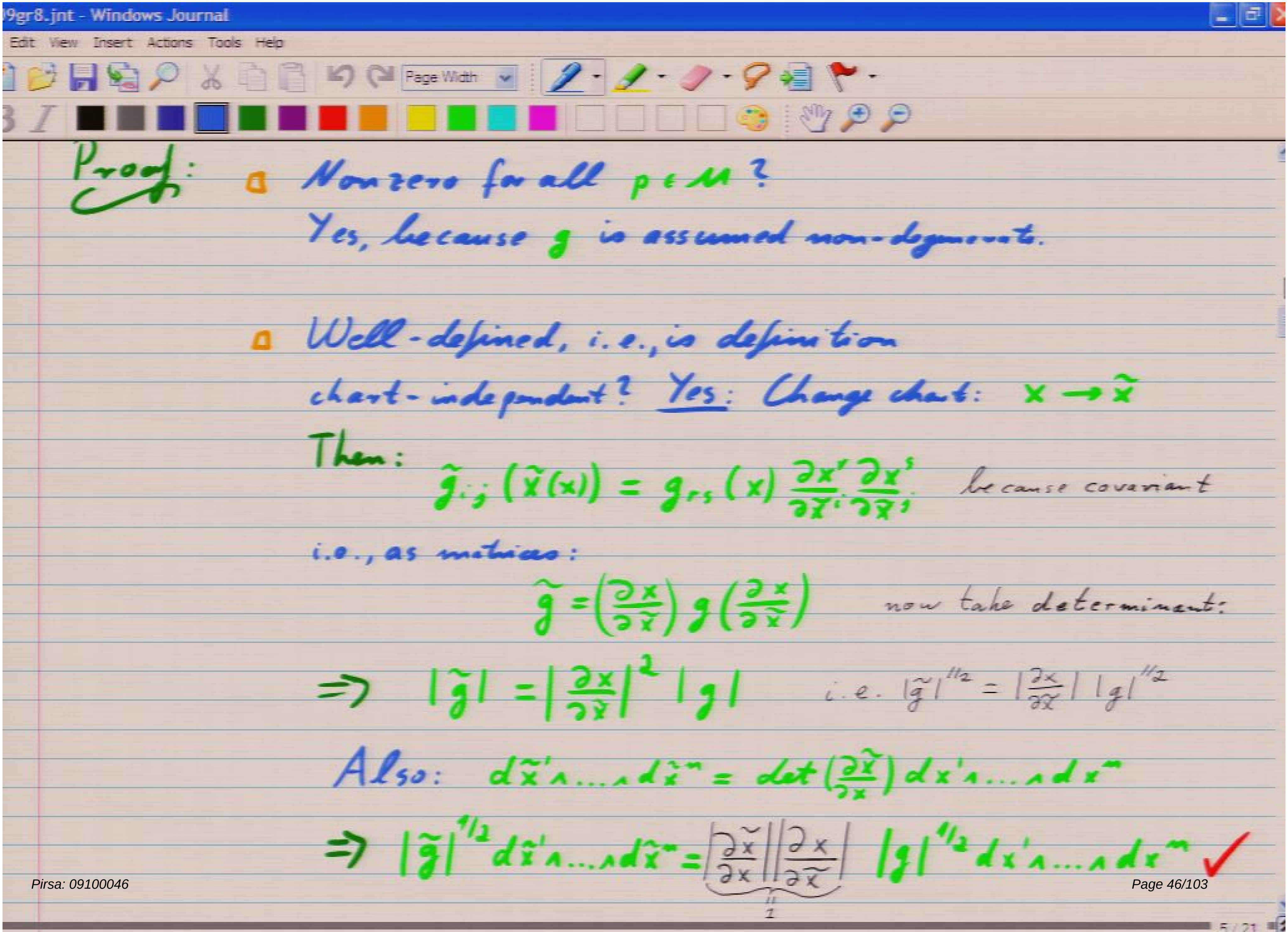

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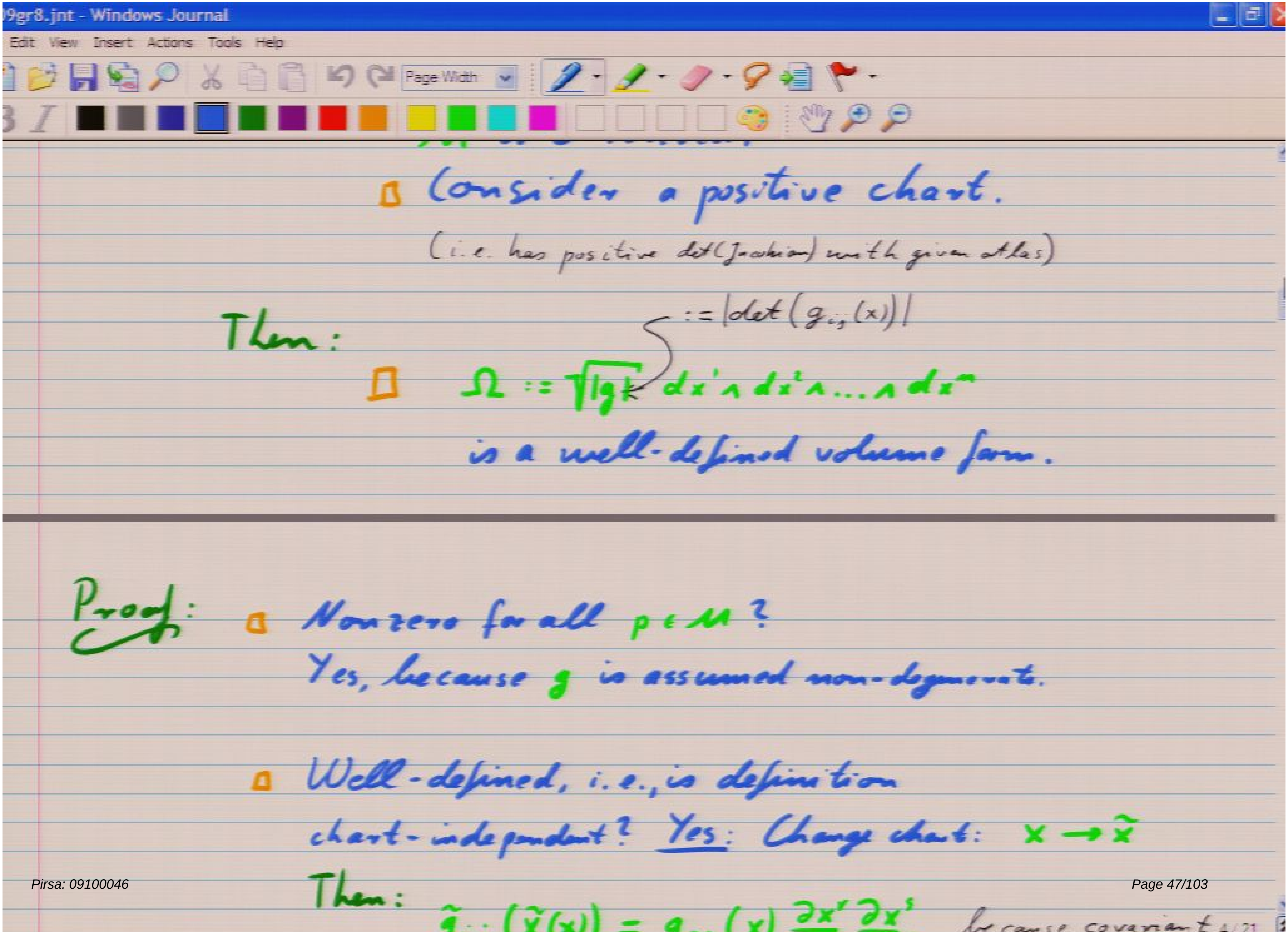

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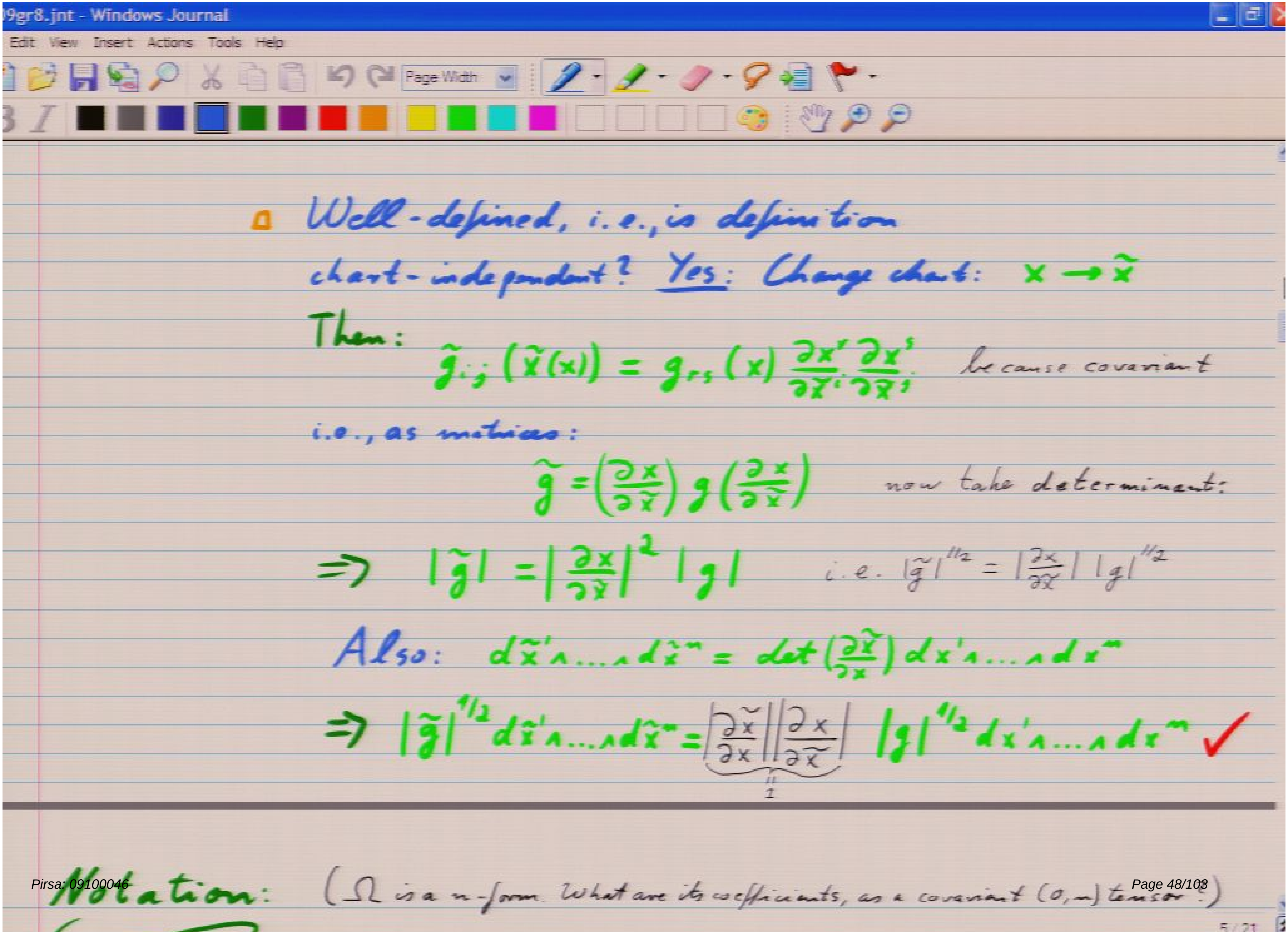

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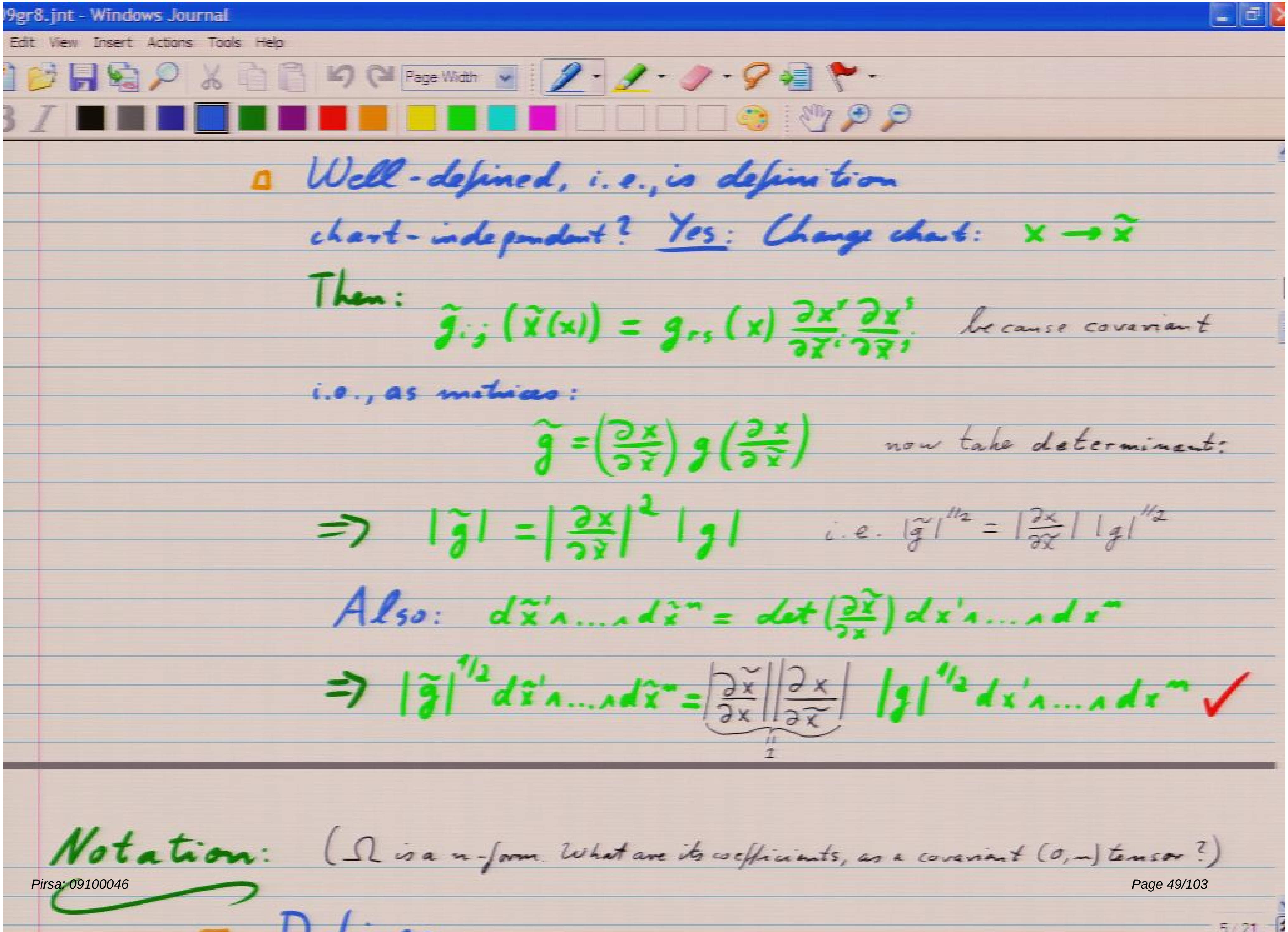
Notation: (Ω is a n-form. What are its coefficients, as a covariant $(0,n)$ tensor?)



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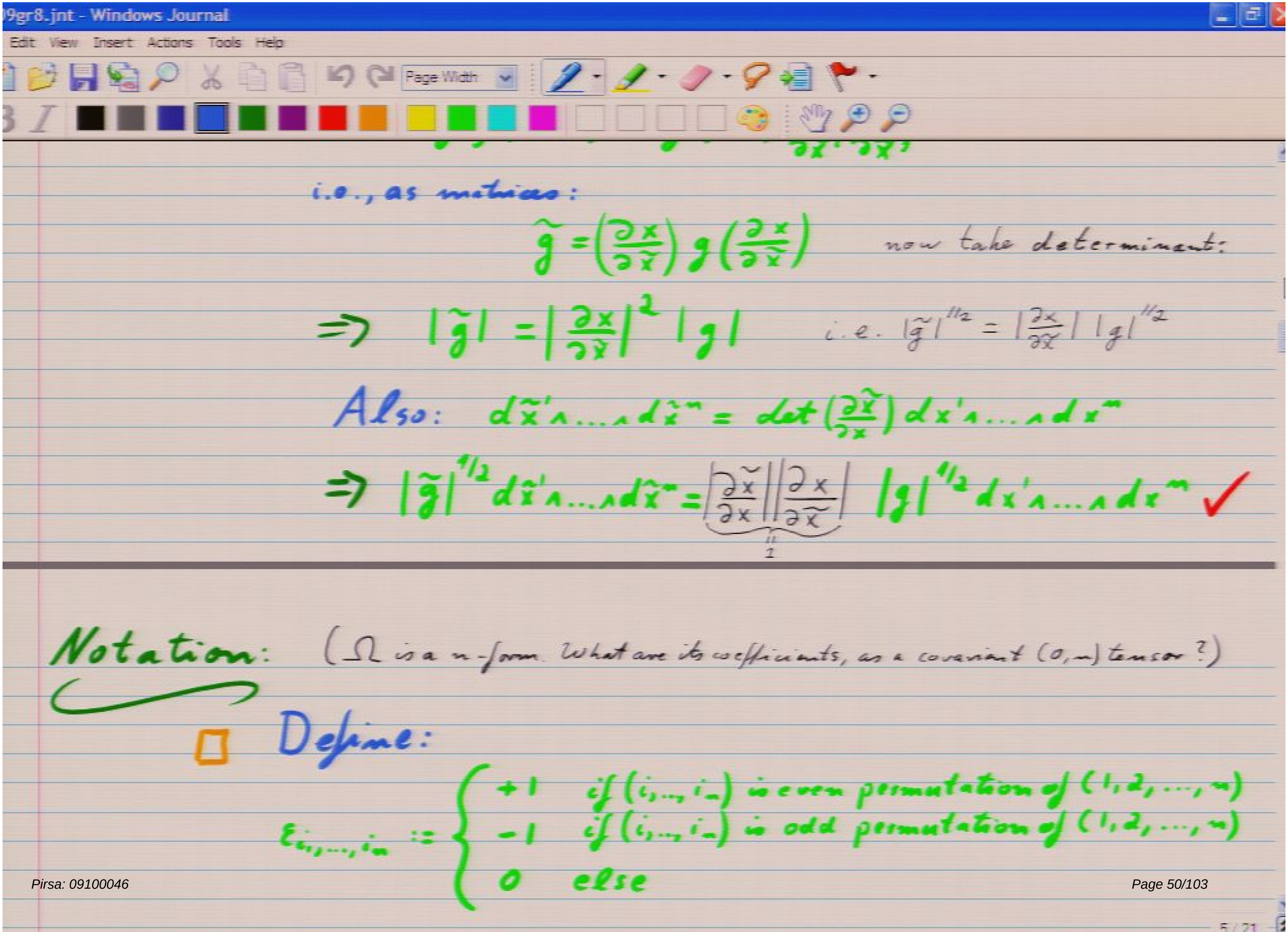
$$\Rightarrow |\tilde{g}|^{1/2} d\tilde{x}^1\wedge\ldots\wedge d\tilde{x}^n = \underbrace{\left|\frac{\partial \tilde{x}}{\partial x}\right|\left|\frac{\partial x}{\partial \tilde{x}}\right|}_{\overset{\shortparallel}{1}} |g|^{1/2} dx^1\wedge\ldots\wedge dx^n \ \checkmark$$

Notation: (Ω is a n-form. What are its coefficients, as a covariant $(0,n)$ tensor?)



i.e., as matrices:

$$\tilde g = \left(\frac{\partial x}{\partial \tilde x}\right) g \left(\frac{\partial x}{\partial \tilde x}\right)$$

now take determinant:

$$\Rightarrow \quad |\tilde g| = \left|\frac{\partial x}{\partial \tilde x}\right|^2 |g|$$

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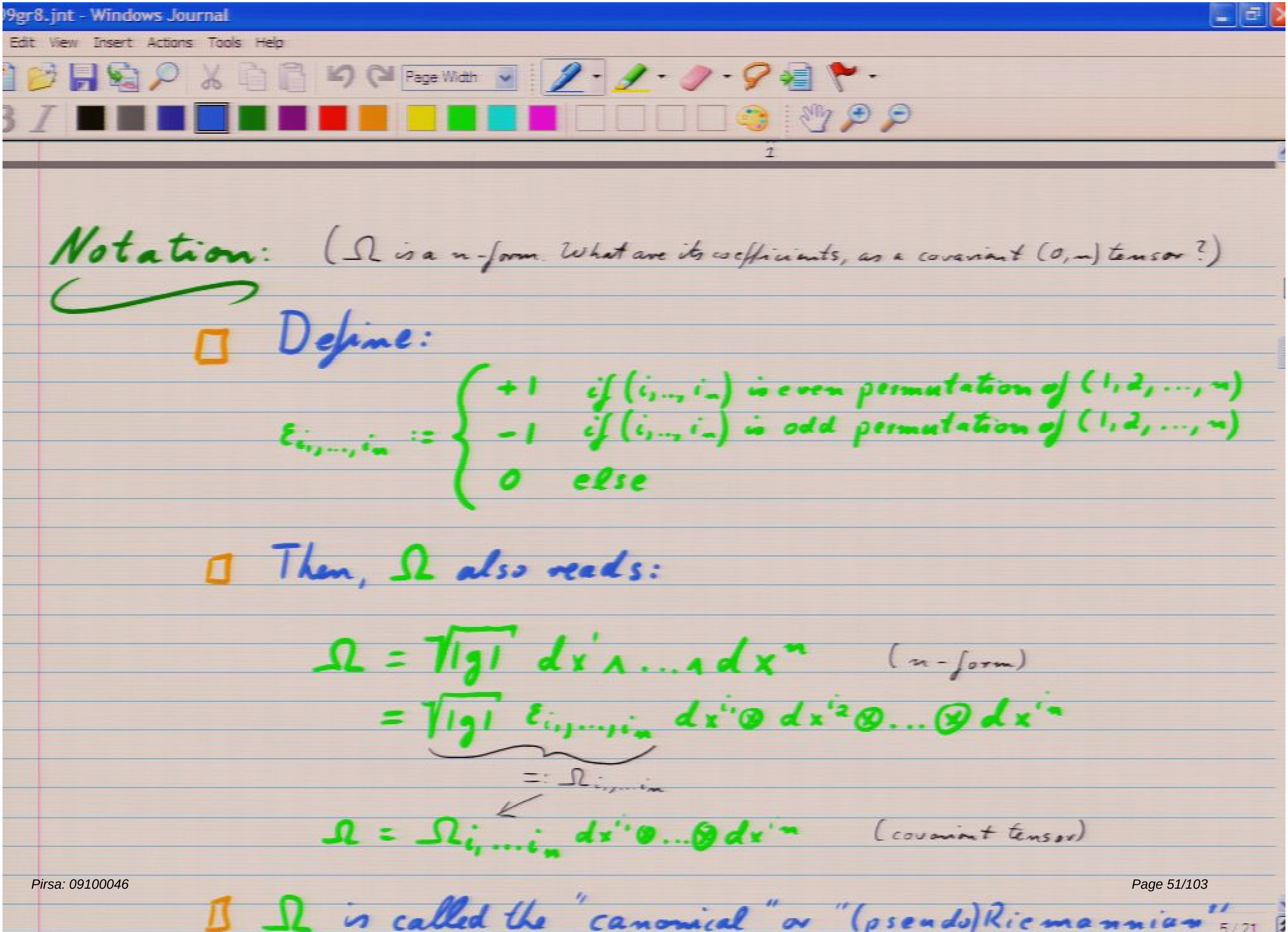

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$$\Omega = \sqrt{|g|}\, dx^1 \wedge \dots \wedge dx^n \qquad (n\text{-form})$$

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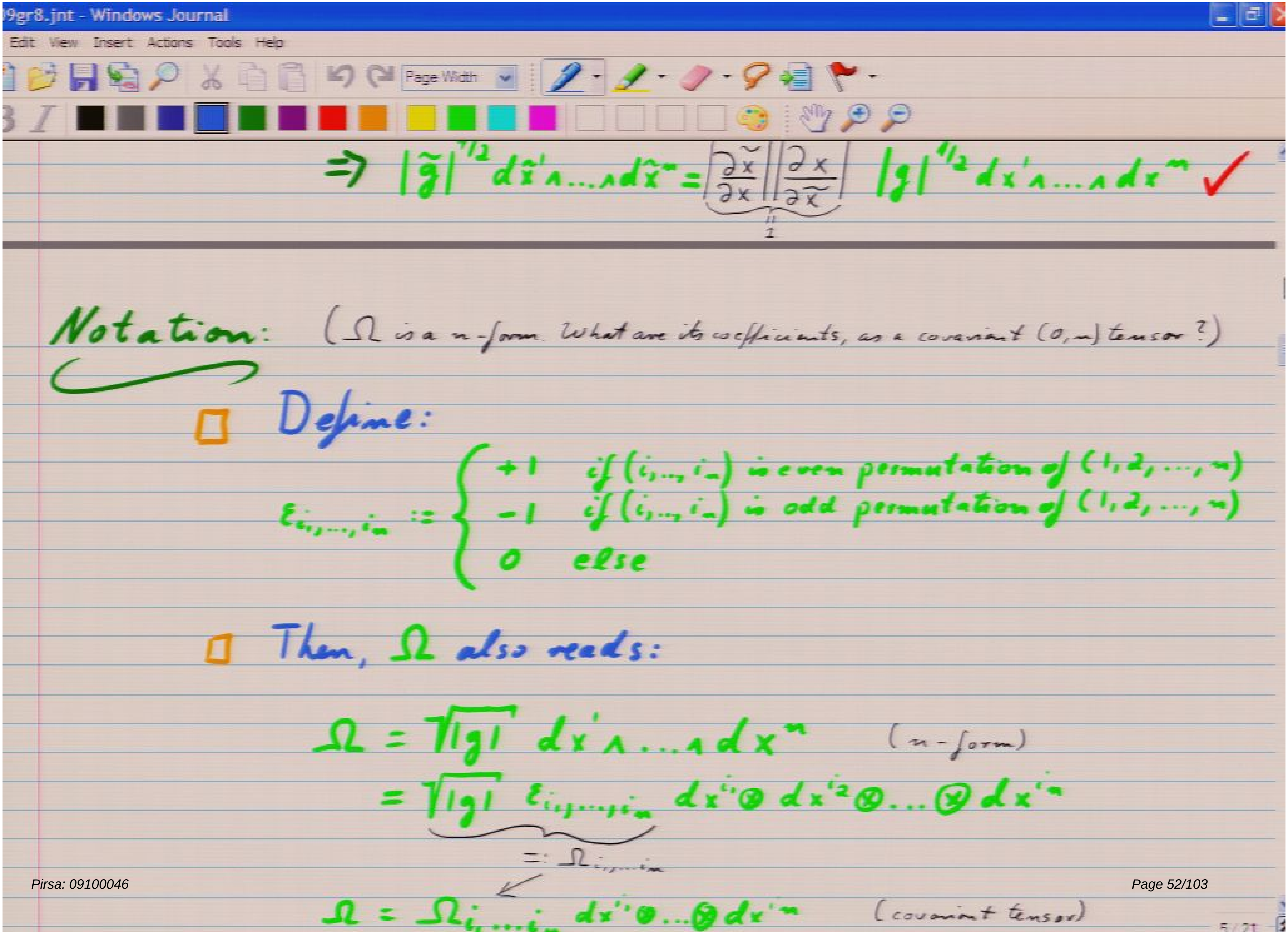

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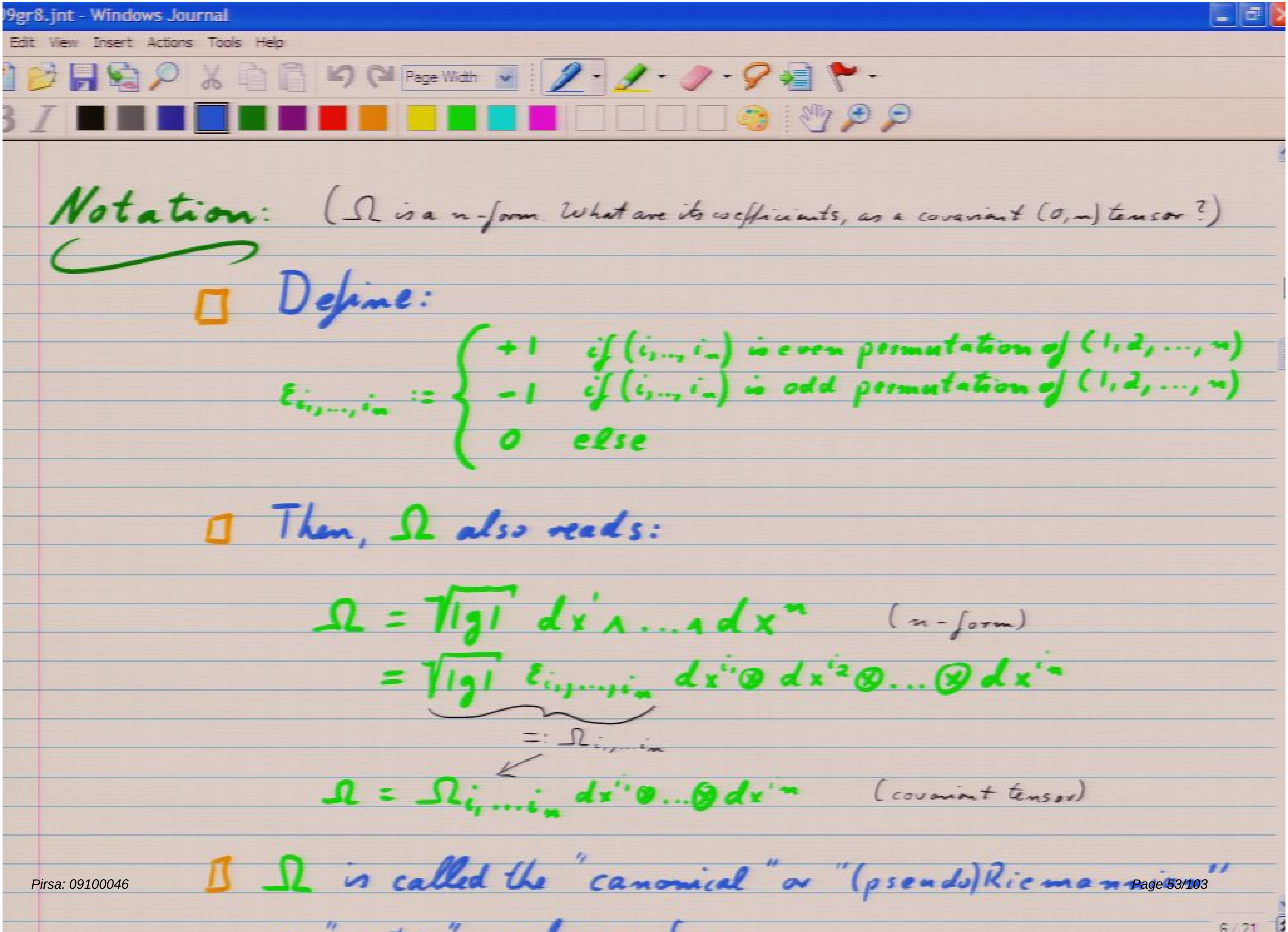

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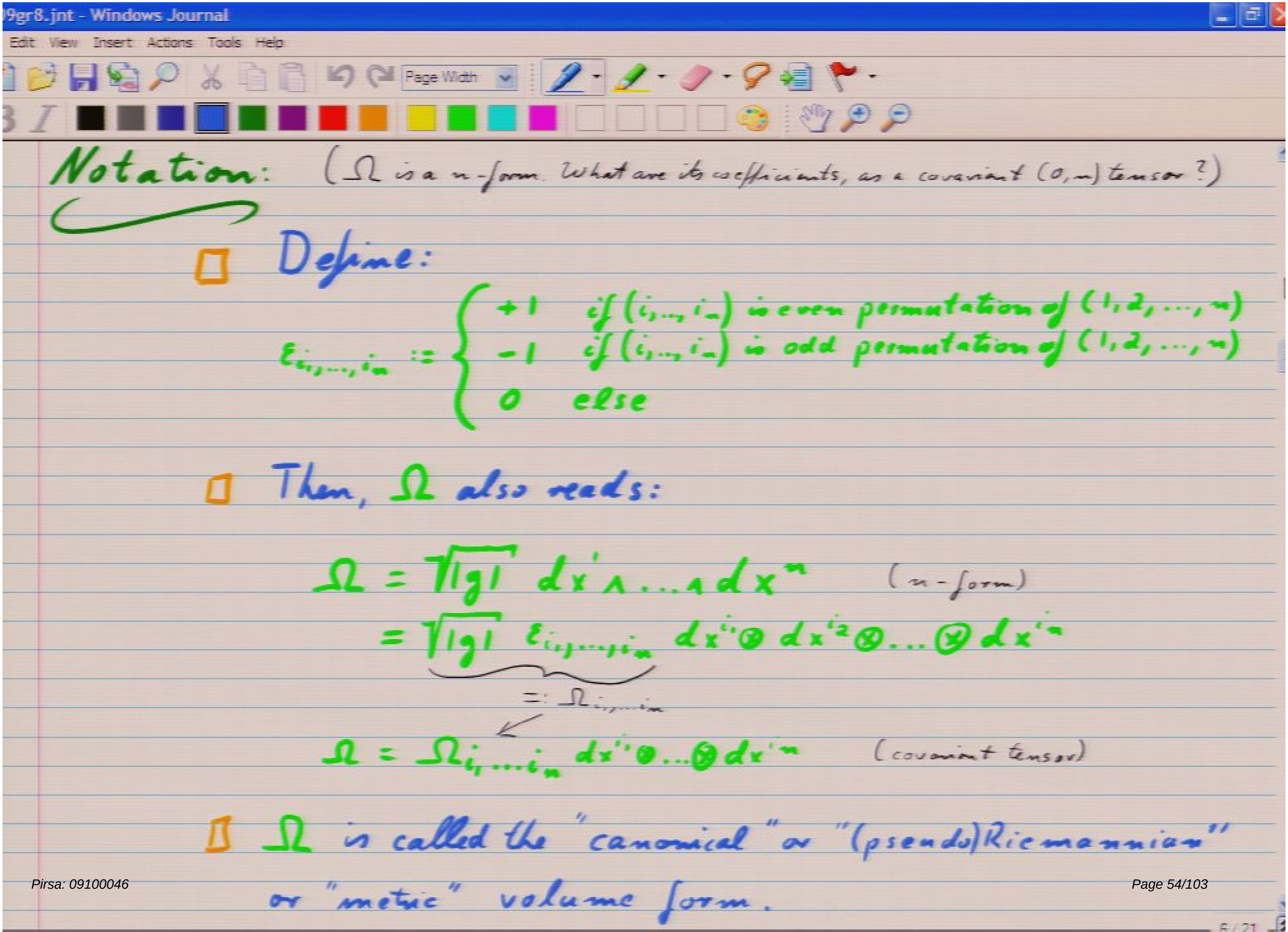

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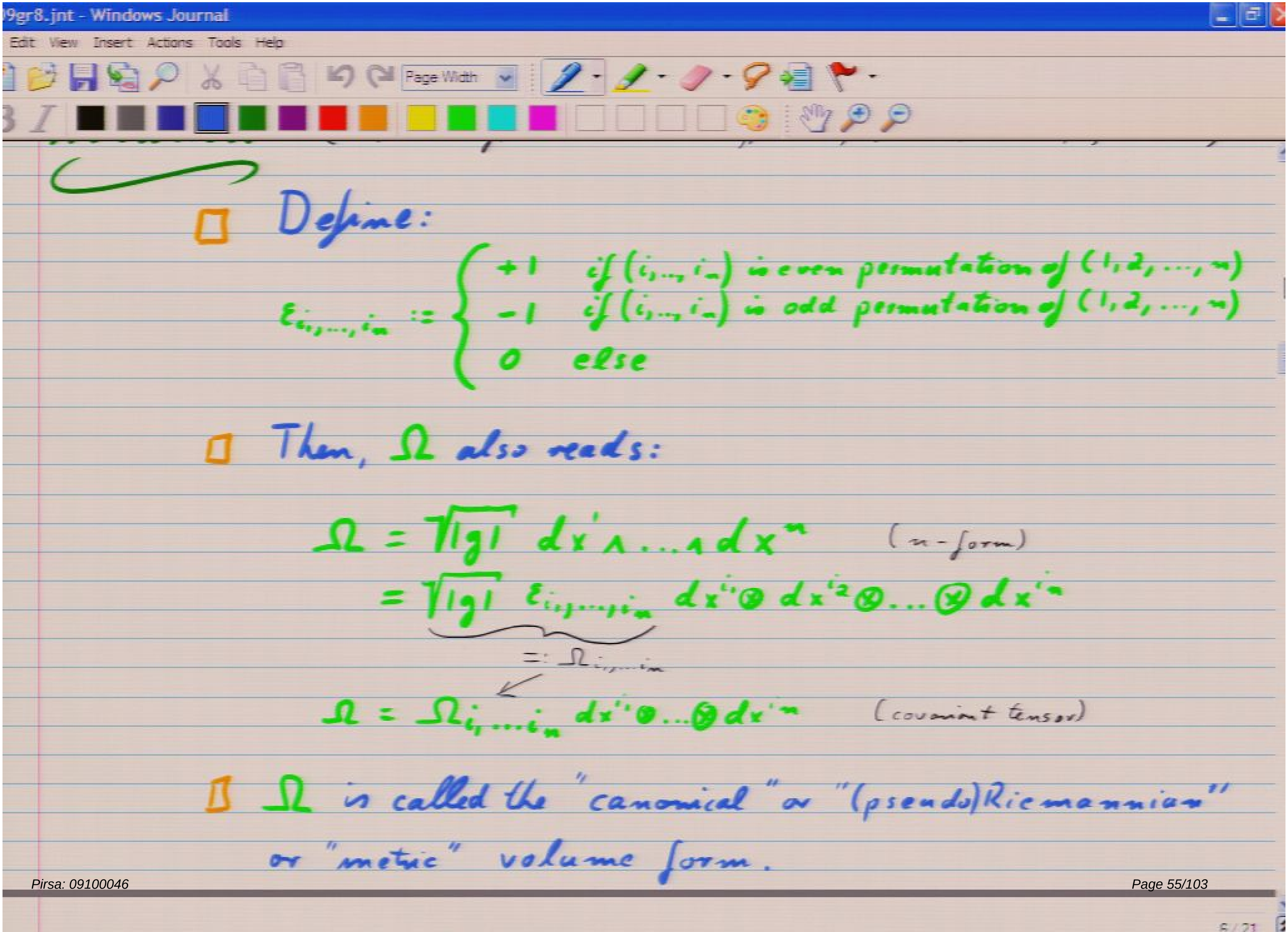

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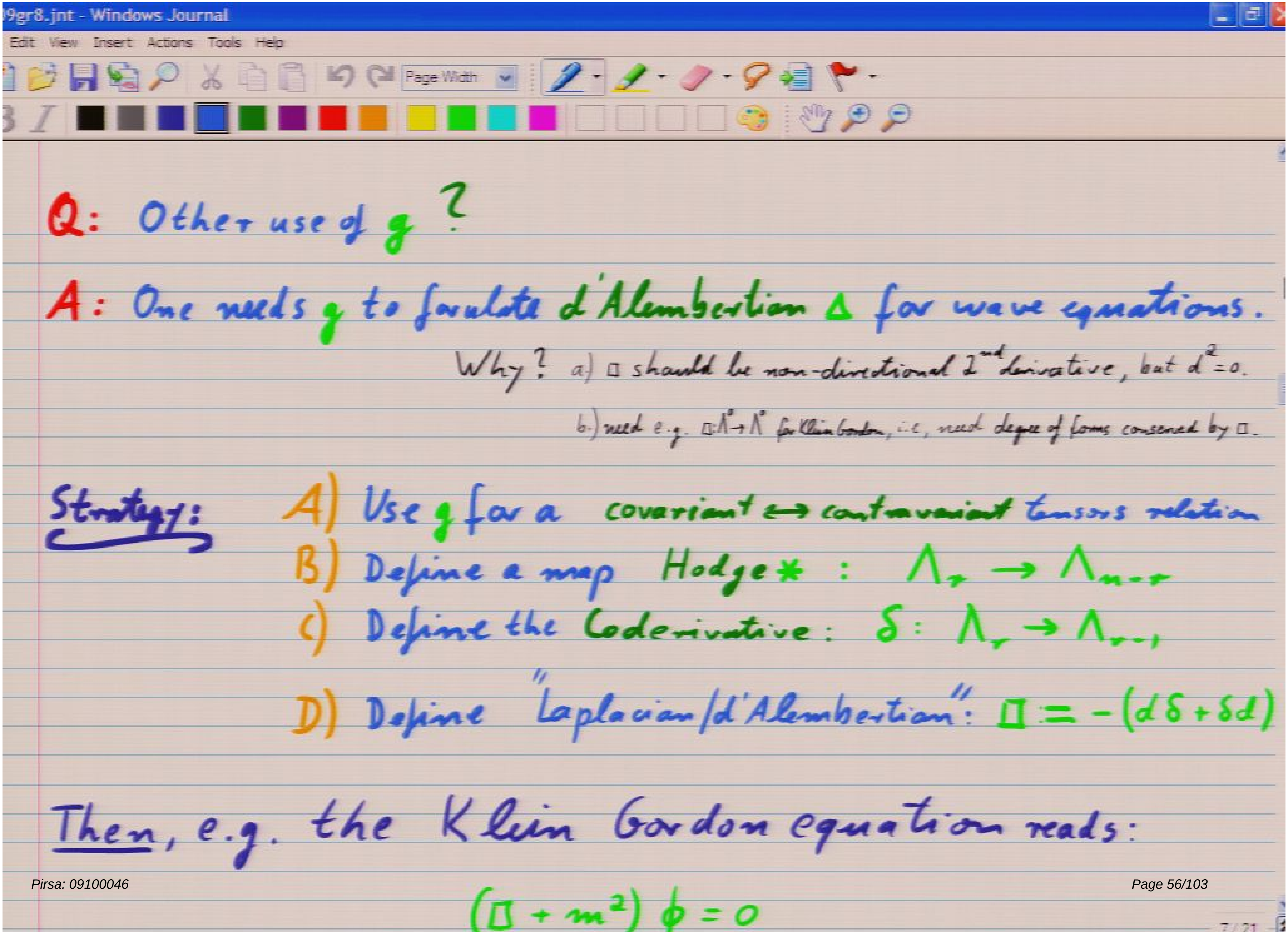

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A: One needs g to formulate d'Alembertian Δ for wave equations.

Why? a) $\Box$ should be non-directional 2^{nd} derivative, but $d^2 = 0$.

b.) need e.g. $\Box : \Lambda^0 \to \Lambda^0$ for Klein Gordon, i.e, need degree of forms conserved by $\Box$.

Strategy:

A) Use g for a covariant $\leftrightarrow$ contravariant tensors relation

B) Define a map Hodge $*$: $\Lambda_r \to \Lambda_{n-r}$

C) Define the Coderivative : $\delta : \Lambda_r \to \Lambda_{r-1}$

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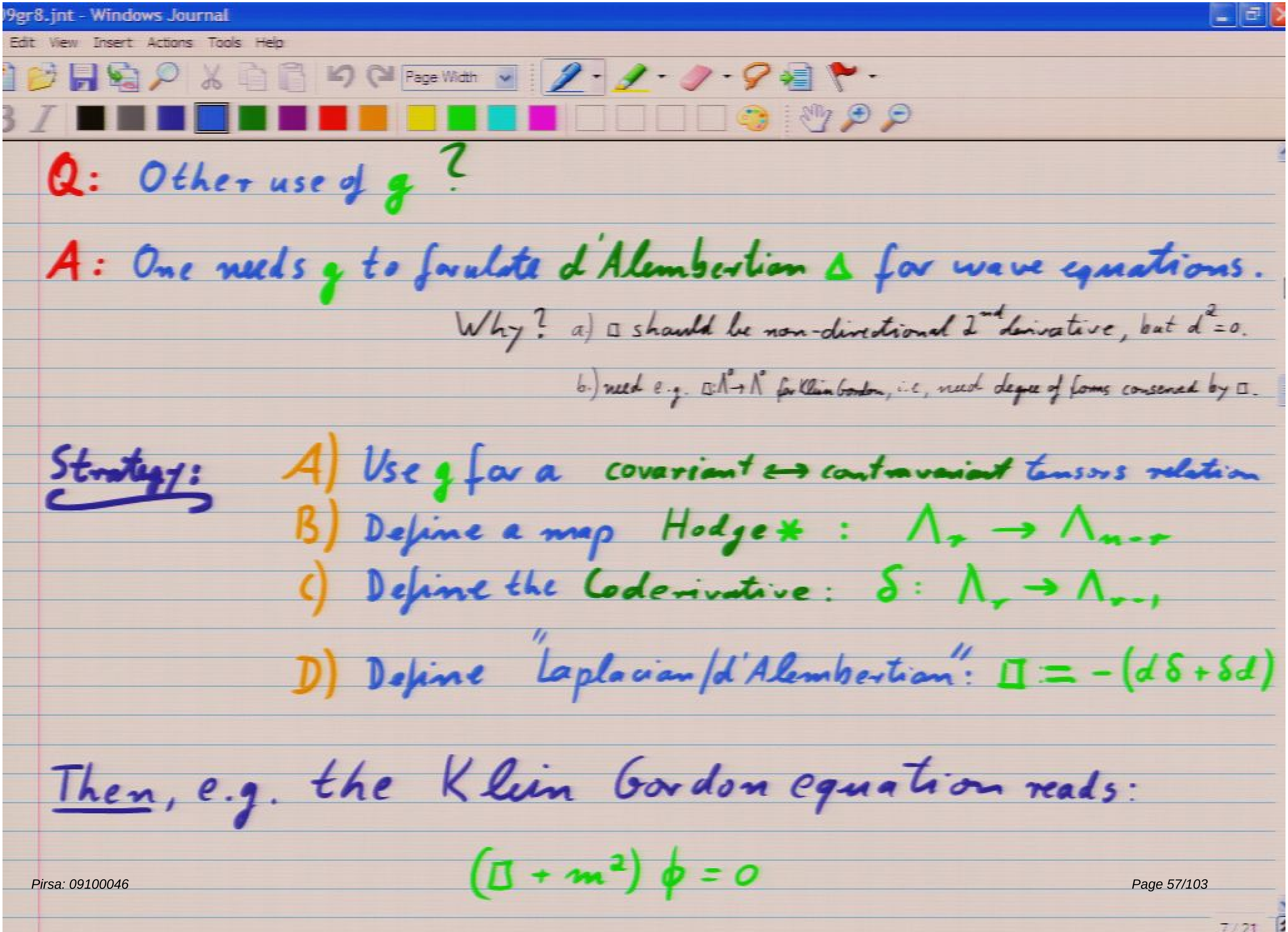

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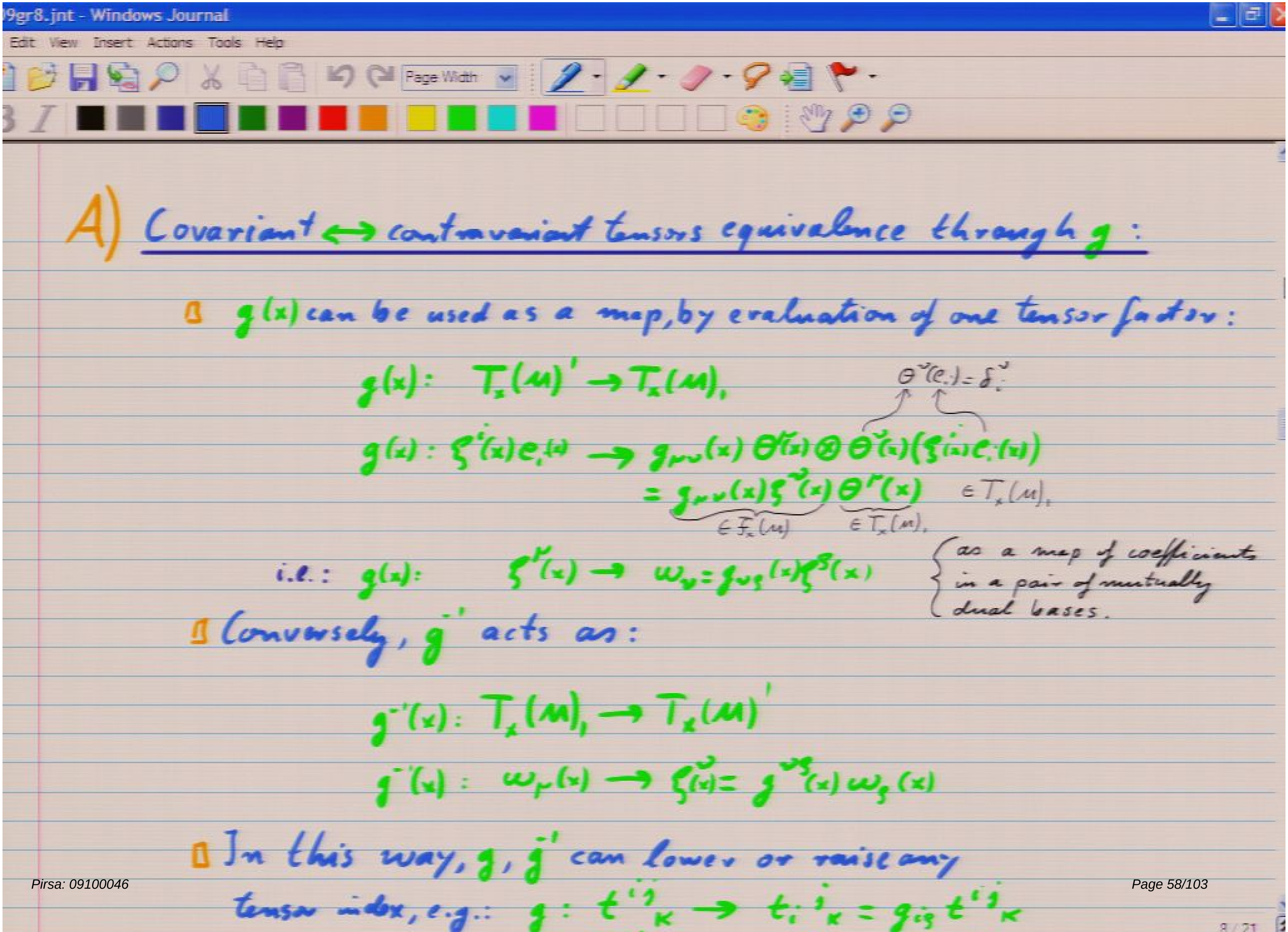

A) Covariant ↔ contravariant tensors equivalence through g :

- $g(x)$ can be used as a map, by evaluation of one tensor factor:

$$g(x): \; T_x(M)' \to T_x(M),$$

$$\theta^\nu(e_i) = \delta^\nu_i$$

$$g(x): \xi^i(x) e_i(x) \to g_{\mu\nu}(x)\, \theta^\mu(x) \otimes \theta^\nu(x) (\xi^i(x) e_i(x))$$

$$= \underbrace{g_{\mu\nu}(x) \xi^\nu(x)}_{\in \mathcal{F}_x(M)} \underbrace{\theta^\mu(x)}_{\in T_x(M)'} \quad \in T_x(M)'$$

i.e.: $g(x): \quad \xi^\mu(x) \to \omega_\nu = g_{\nu\beta}(x) \xi^\beta(x)$ $\left\{ \begin{array}{l} \text{as a map of coefficients} \\ \text{in a pair of mutually} \\ \text{dual bases.} \end{array} \right.$

- Conversely, g^{-1} acts as:

$$g^{-1}(x): \; T_x(M)' \to T_x(M)'$$

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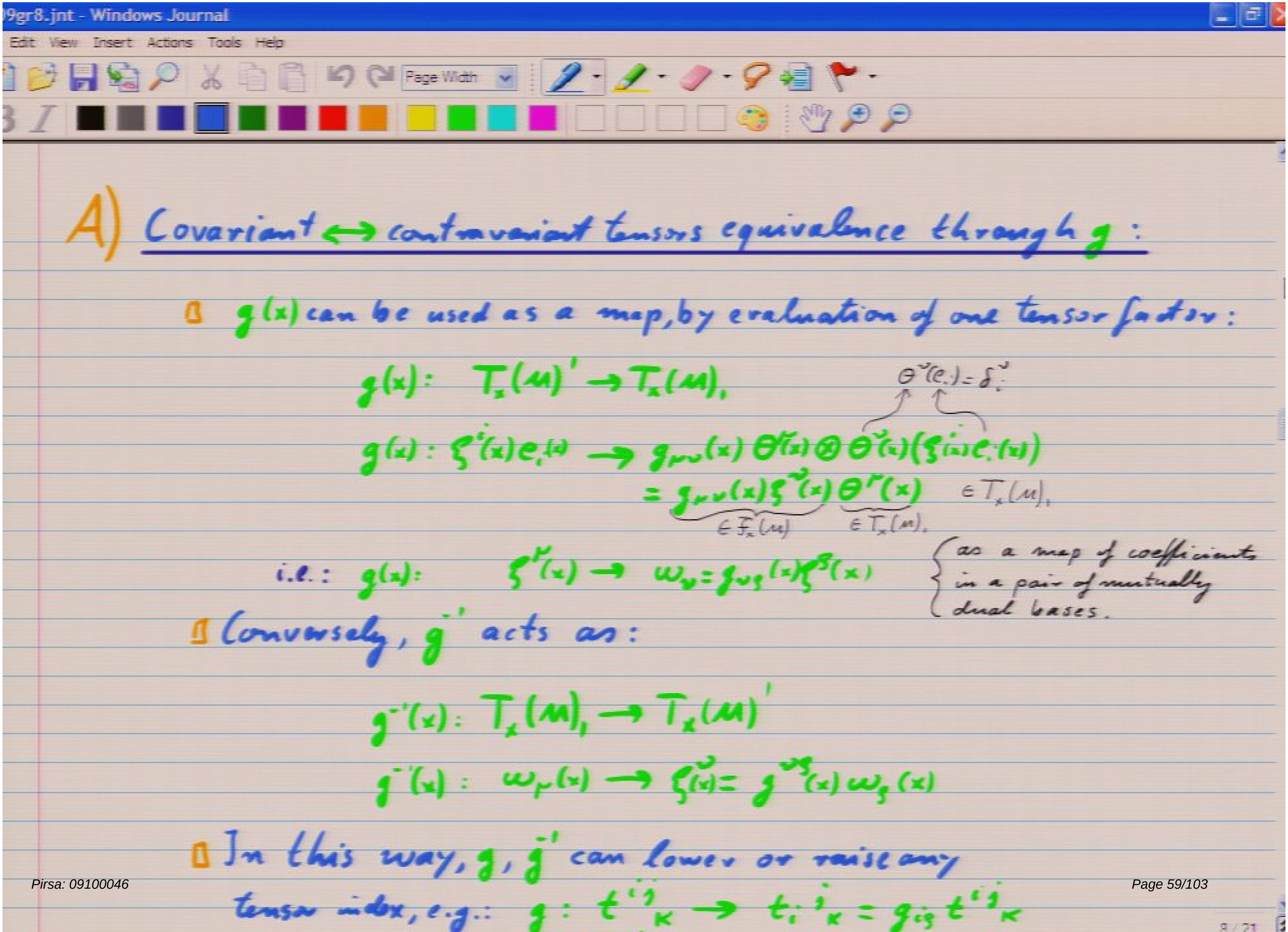

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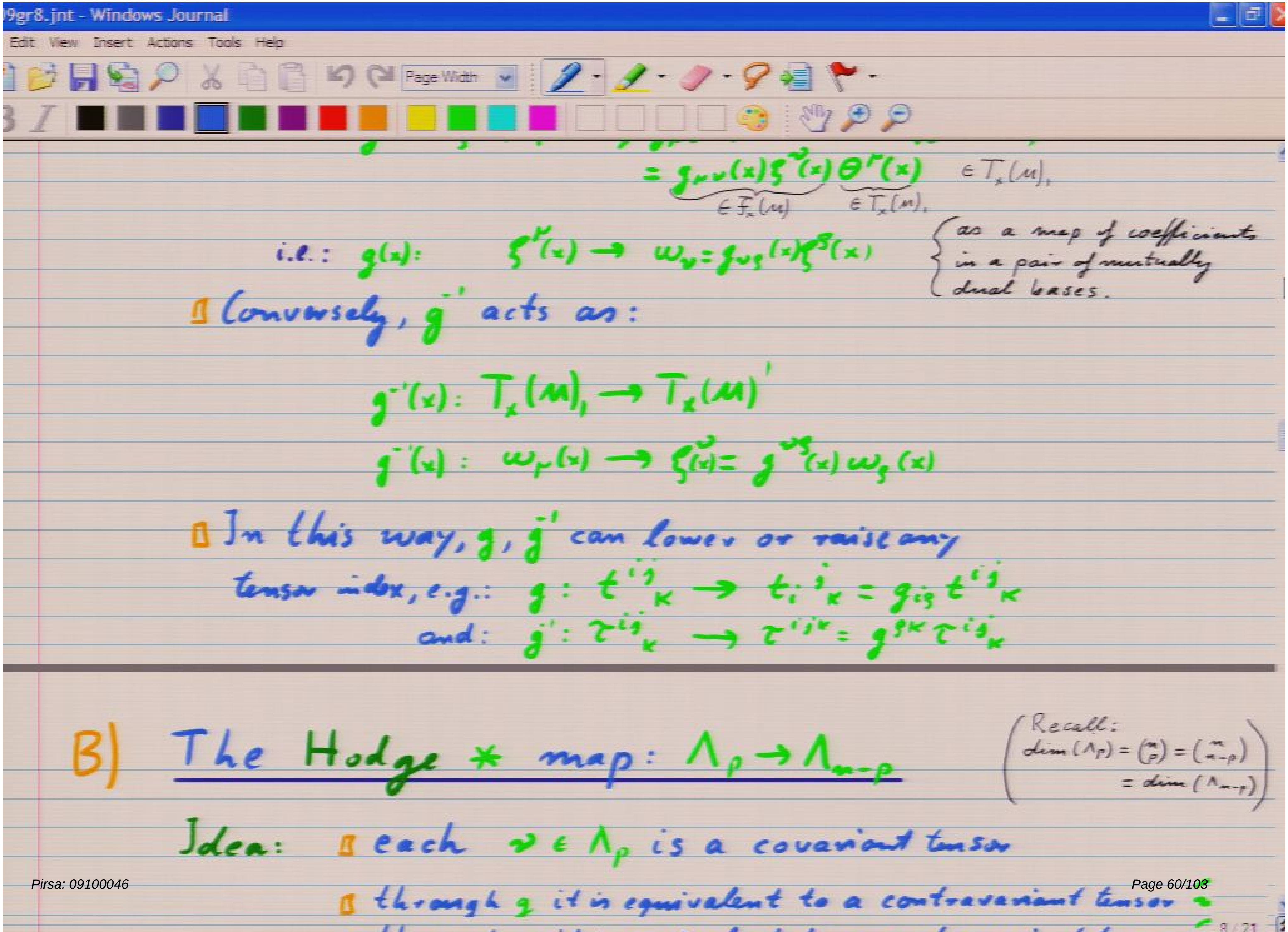

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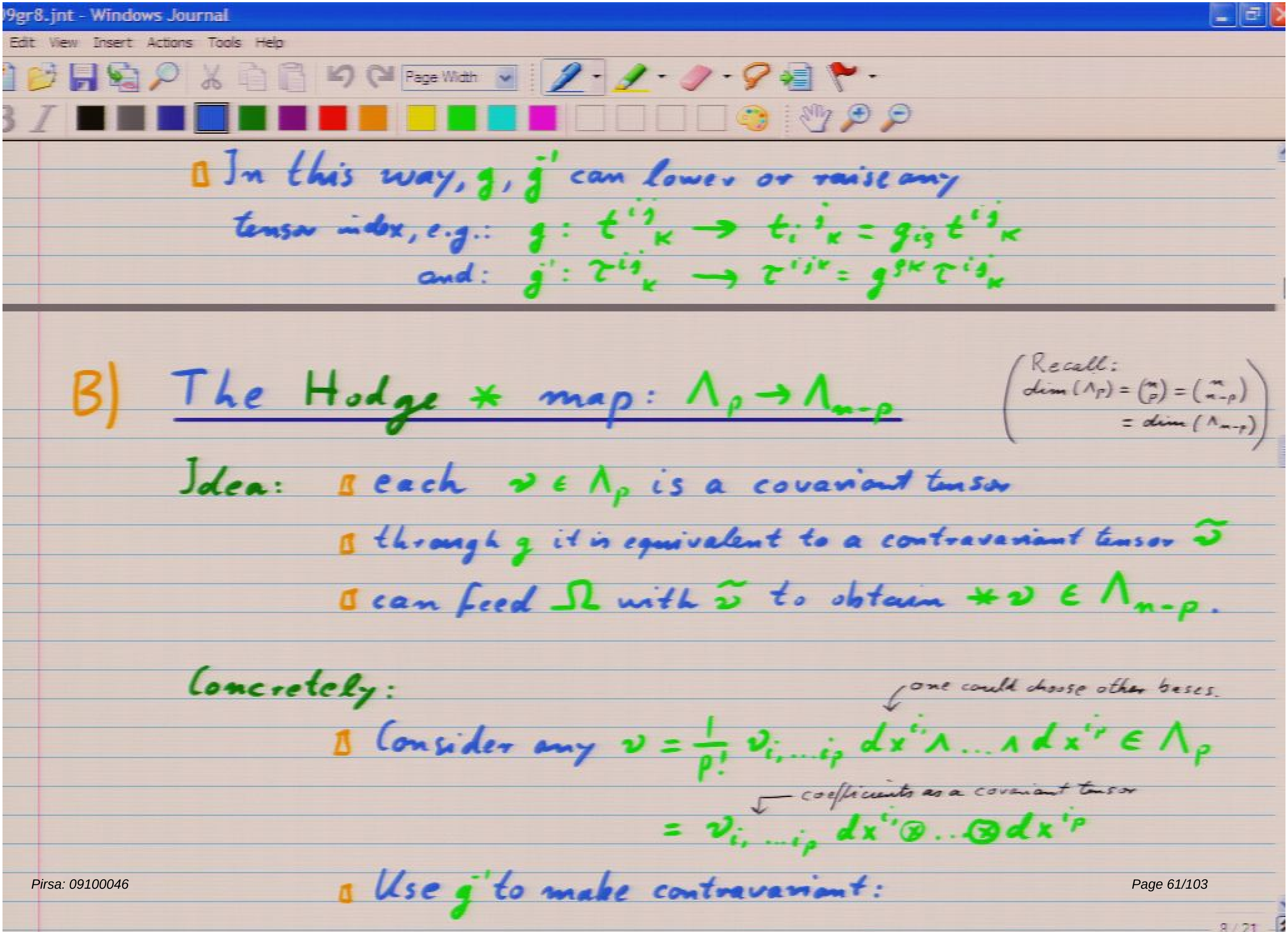
and: $g^{-1}: \tau^{ij}{}_{k} \to \tau^{ijr} = g^{rk}\tau^{ij}{}_{k}$

B) The Hodge * map: $\Lambda_p \to \Lambda_{n-p}$

(Recall: $\dim(\Lambda_p) = \binom{n}{p} = \binom{n}{n-p} = \dim(\Lambda_{n-p})$)

Idea: each $\nu \in \Lambda_p$ is a covariant tensor

through g it is equivalent to a contravariant tensor



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Concretely:

one could choose other bases.

- Consider any $v = \frac{1}{p!} v_{i_1 \ldots i_p} dx^{i_1} \wedge \ldots \wedge dx^{i_p} \in \Lambda_p$

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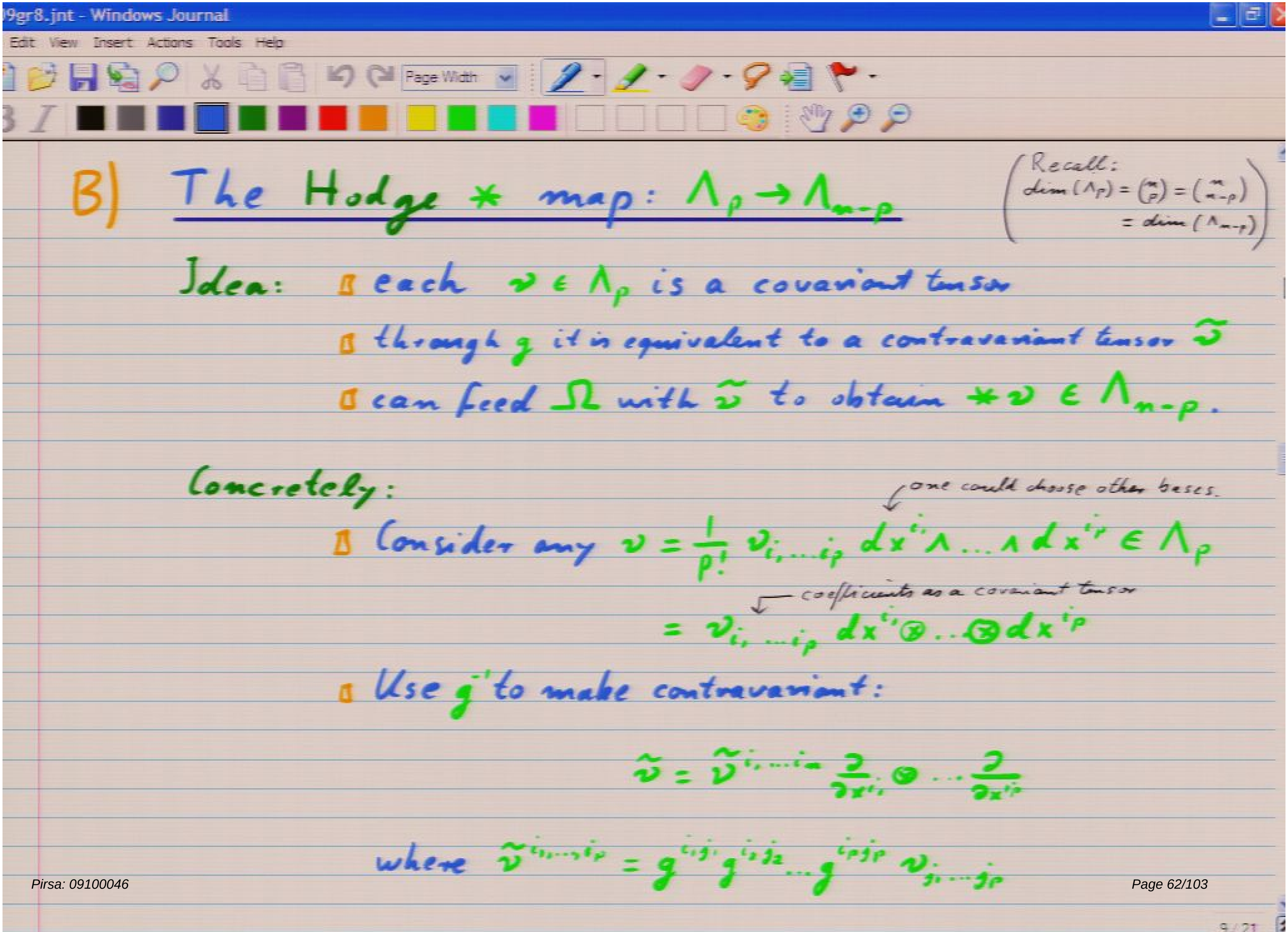

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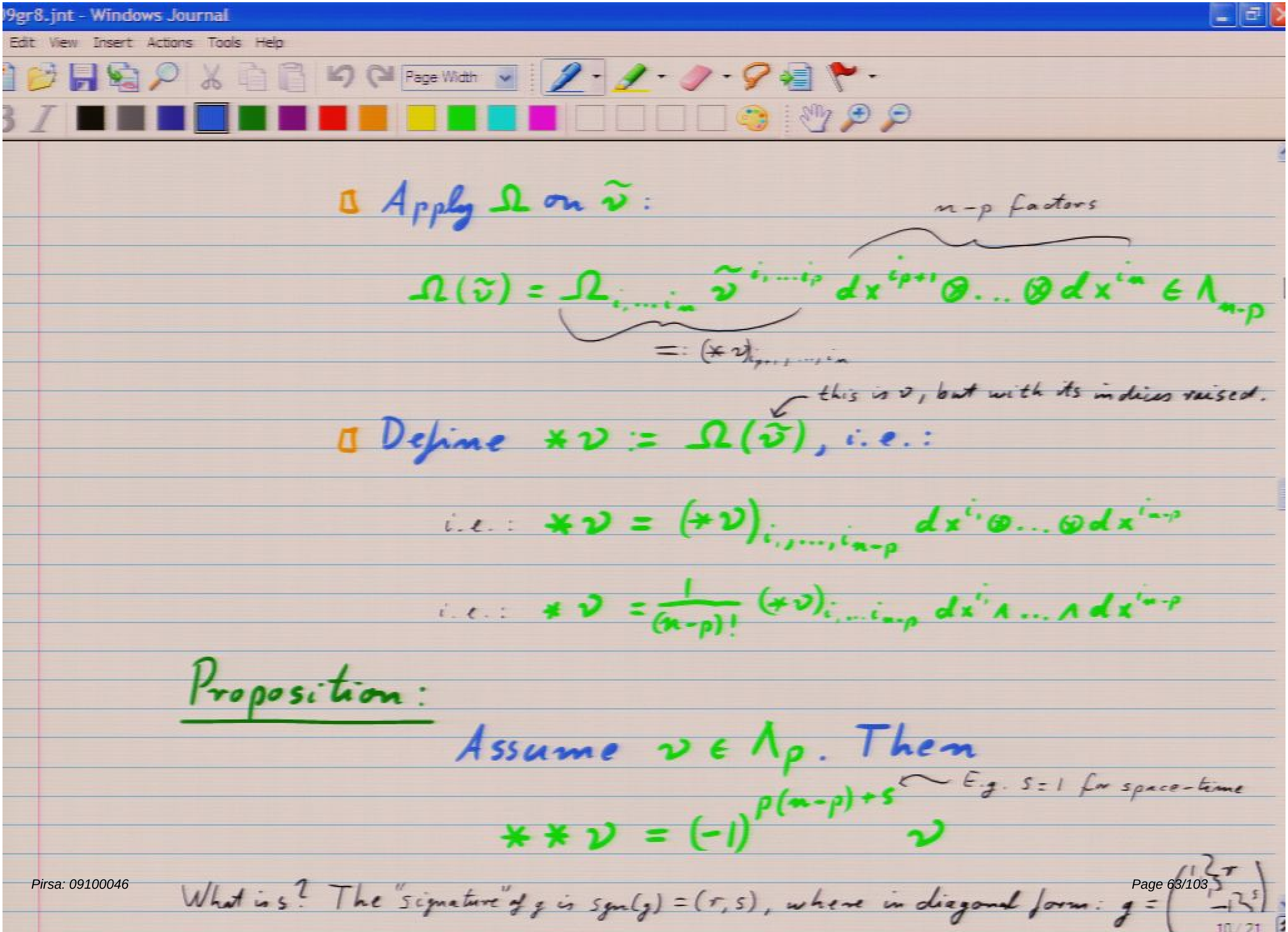
where $\tilde{v}^{i_1, \ldots, i_p} = g^{i_1 j_1} g^{i_2 j_2} \ldots g^{i_p j_p} v_{j_1 \ldots j_p}$



- Apply Ω on $\tilde{v}$:

$$\Omega(\tilde{v}) = \underbrace{\Omega_{i_1 \dots i_n} \tilde{v}^{i_1 \dots i_p}}_{=: (* v)_{i_{p+1}, \dots, i_n}} \overbrace{dx^{i_{p+1}} \otimes \dots \otimes dx^{i_n}}^{n-p \text{ factors}} \in \Lambda_{n-p}$$

- Define $* v := \Omega(\tilde{v})$, i.e.: ← this is v, but with its indices raised.

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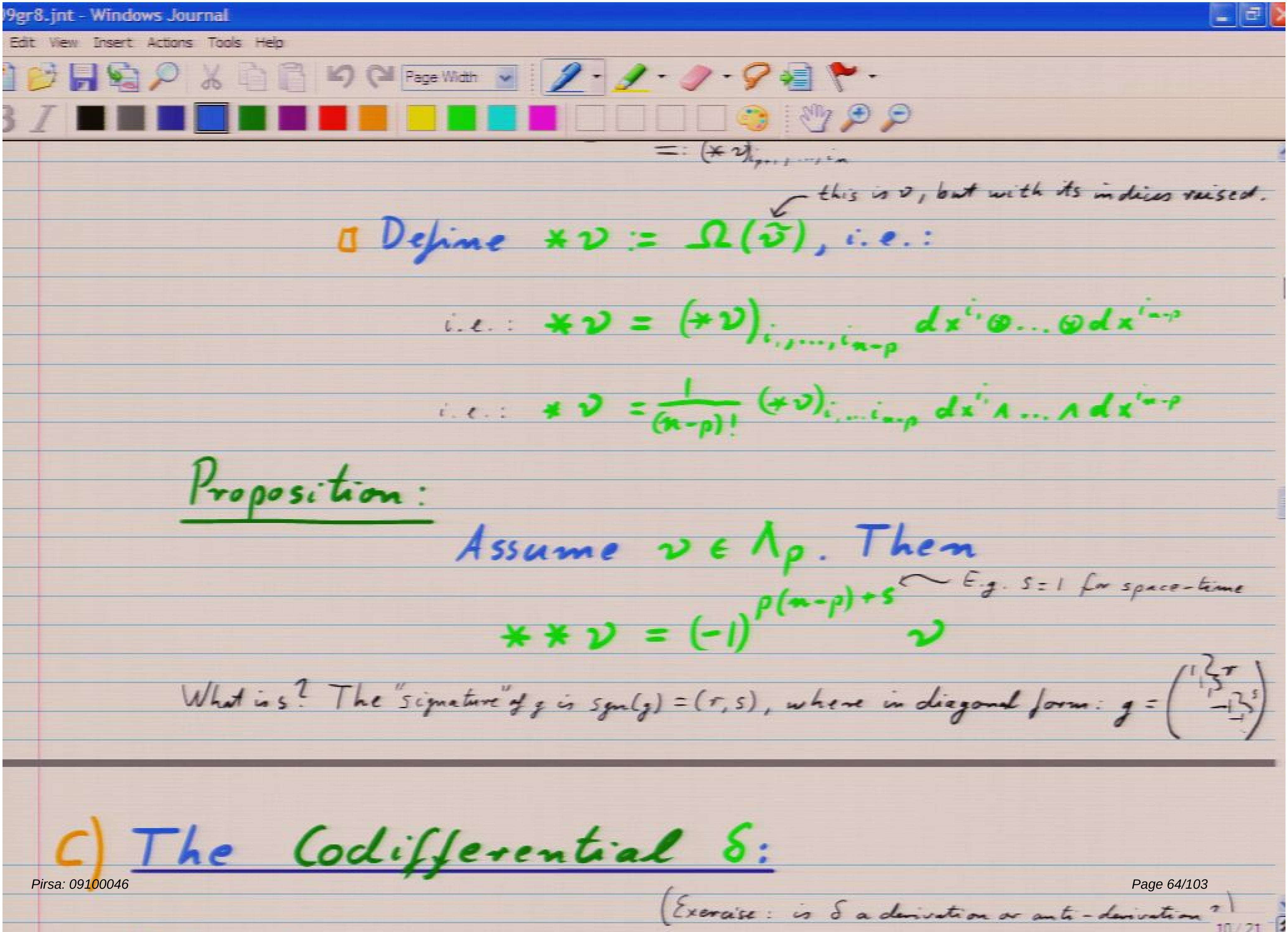
Proposition:

Assume $v \in \Lambda_p$. Then

$$* * v = (-1)^{p(n-p)+s} v$$

← E.g. $s = 1$ for space-time

What is s? The "signature" of g is $\text{sgn}(g) = (r, s)$, where in diagonal form: $g = \begin{pmatrix} 1 \,\}r & \\ & -1 \,\}s \end{pmatrix}$



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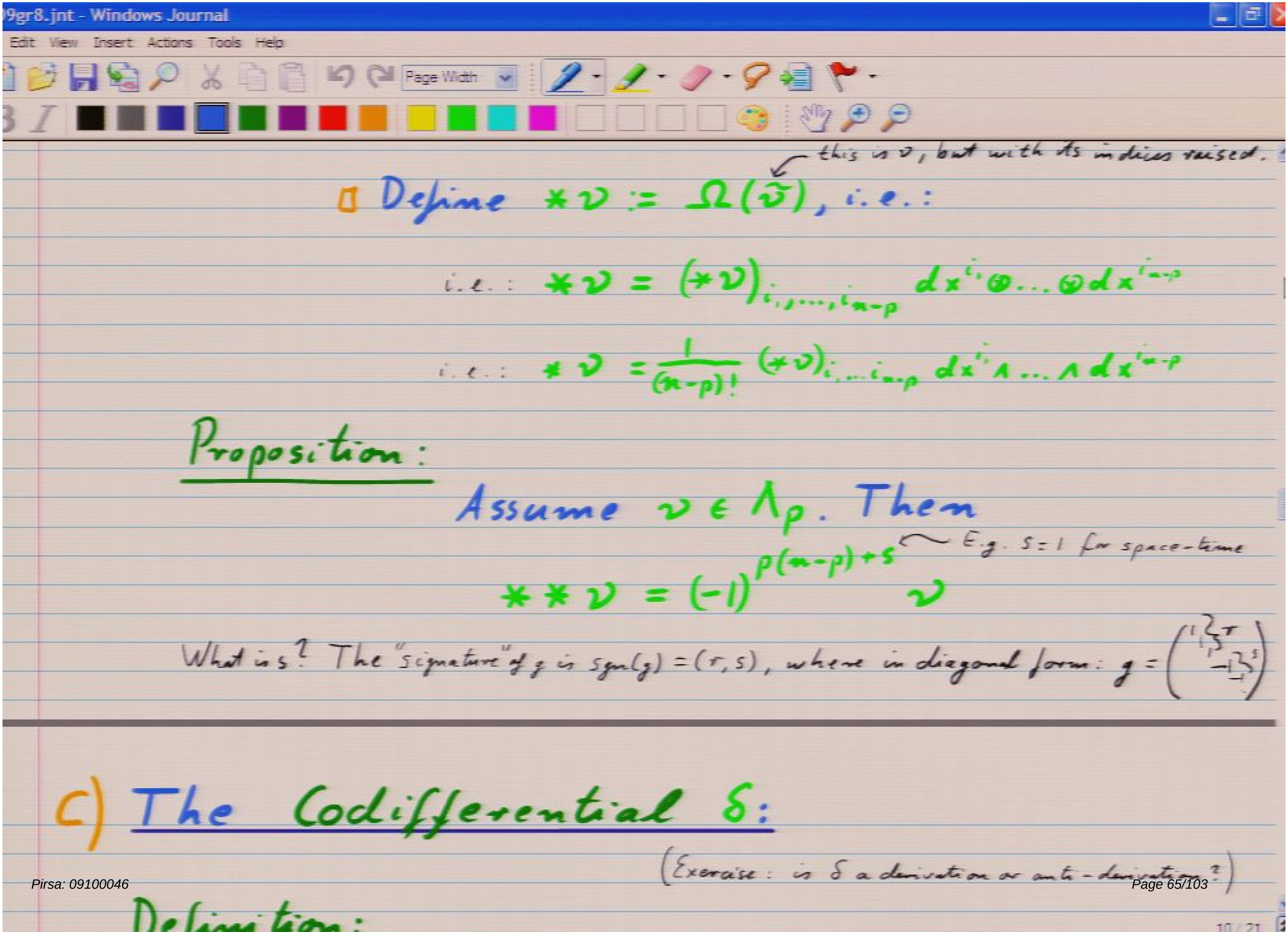
(Exercise: is δ a derivation or anti-derivation?)



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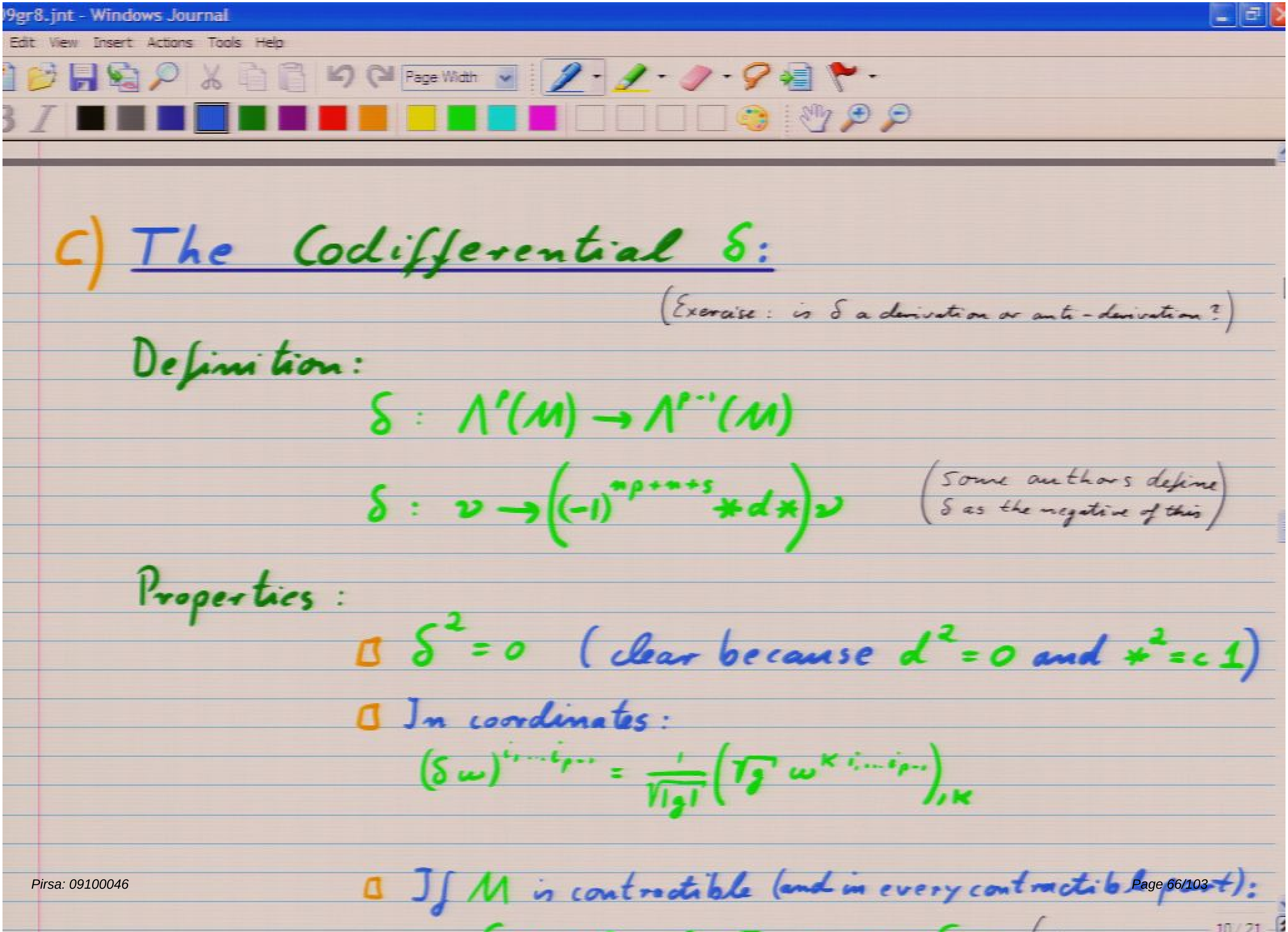

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$$\delta : \Lambda^p(M) \to \Lambda^{p-1}(M)$$

$$\delta : v \to \left((-1)^{np+n+s} * d *\right) v$$

(Some authors define δ as the negative of this)

Properties:

- $\delta^2 = 0$ (clear because $d^2 = 0$ and $*^2 = c\,1$)
- In coordinates:

$$(\delta\omega)^{i_1 \dots i_{p-1}} = \frac{1}{\sqrt{|g|}} \left(\sqrt{g}\,\omega^{k\, i_1 \dots i_{p-1}}\right)_{,k}$$

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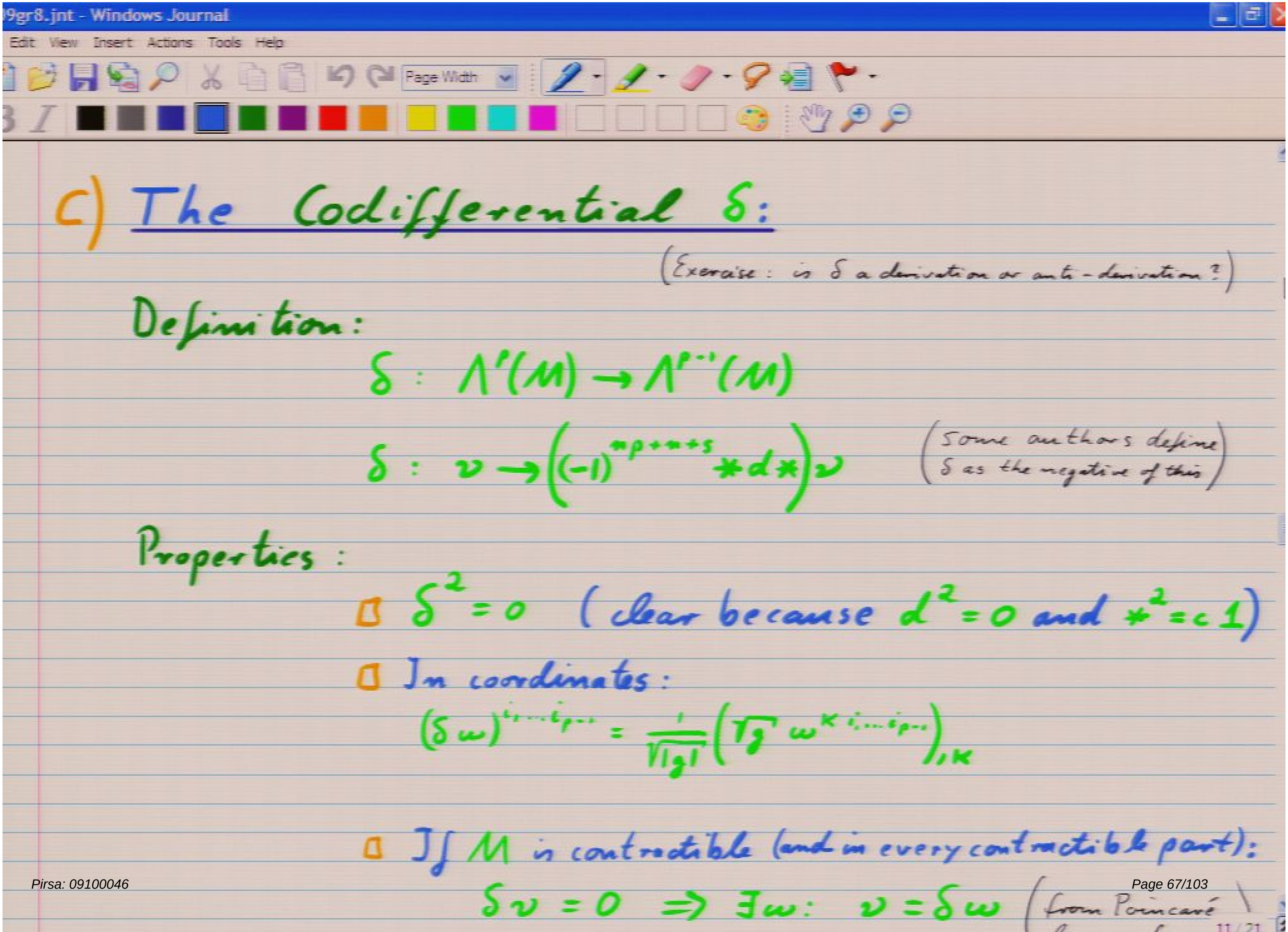

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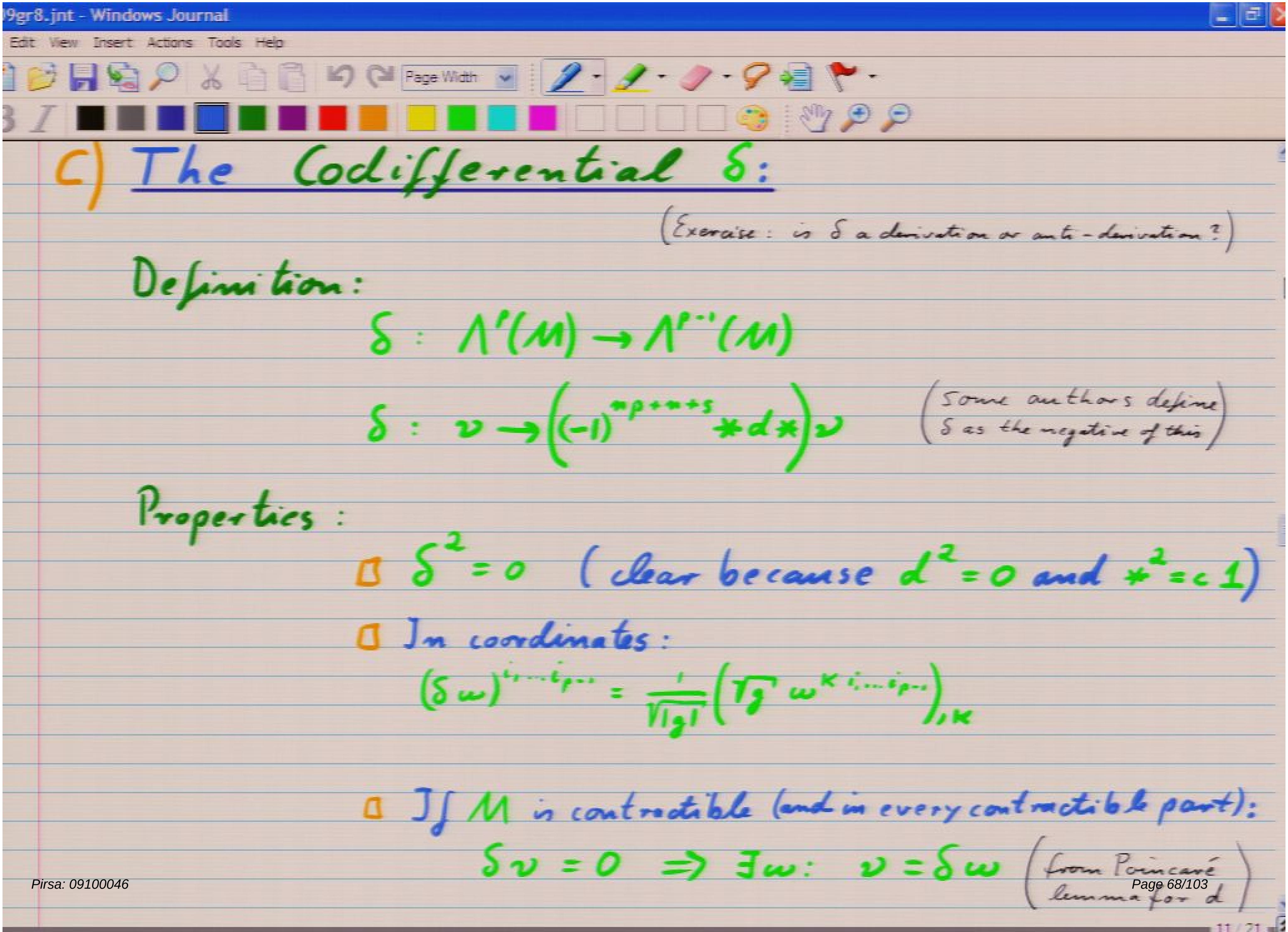
$$\delta v = 0 \Rightarrow \exists \omega : \; v = \delta\omega$$

(from Poincaré



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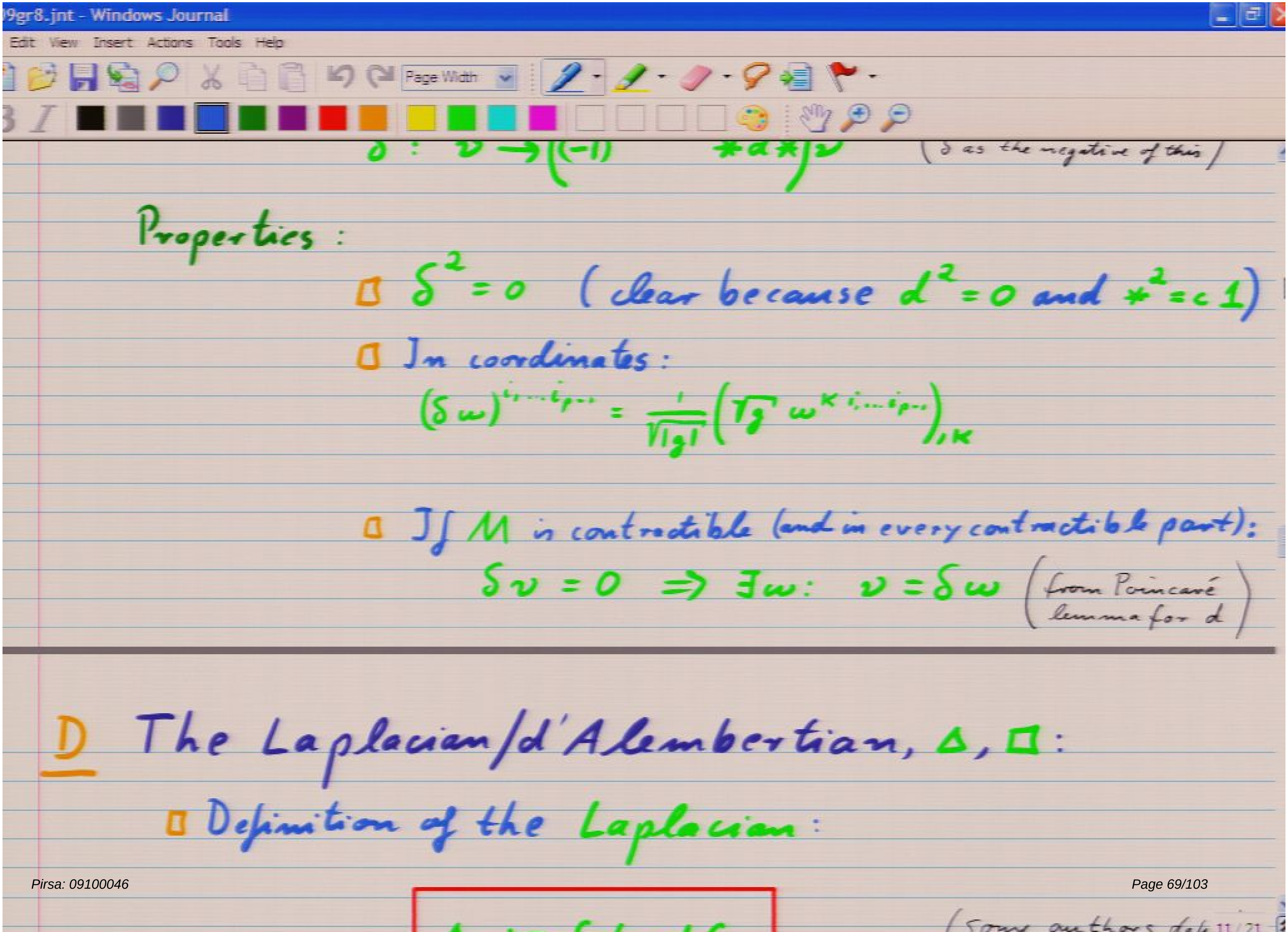

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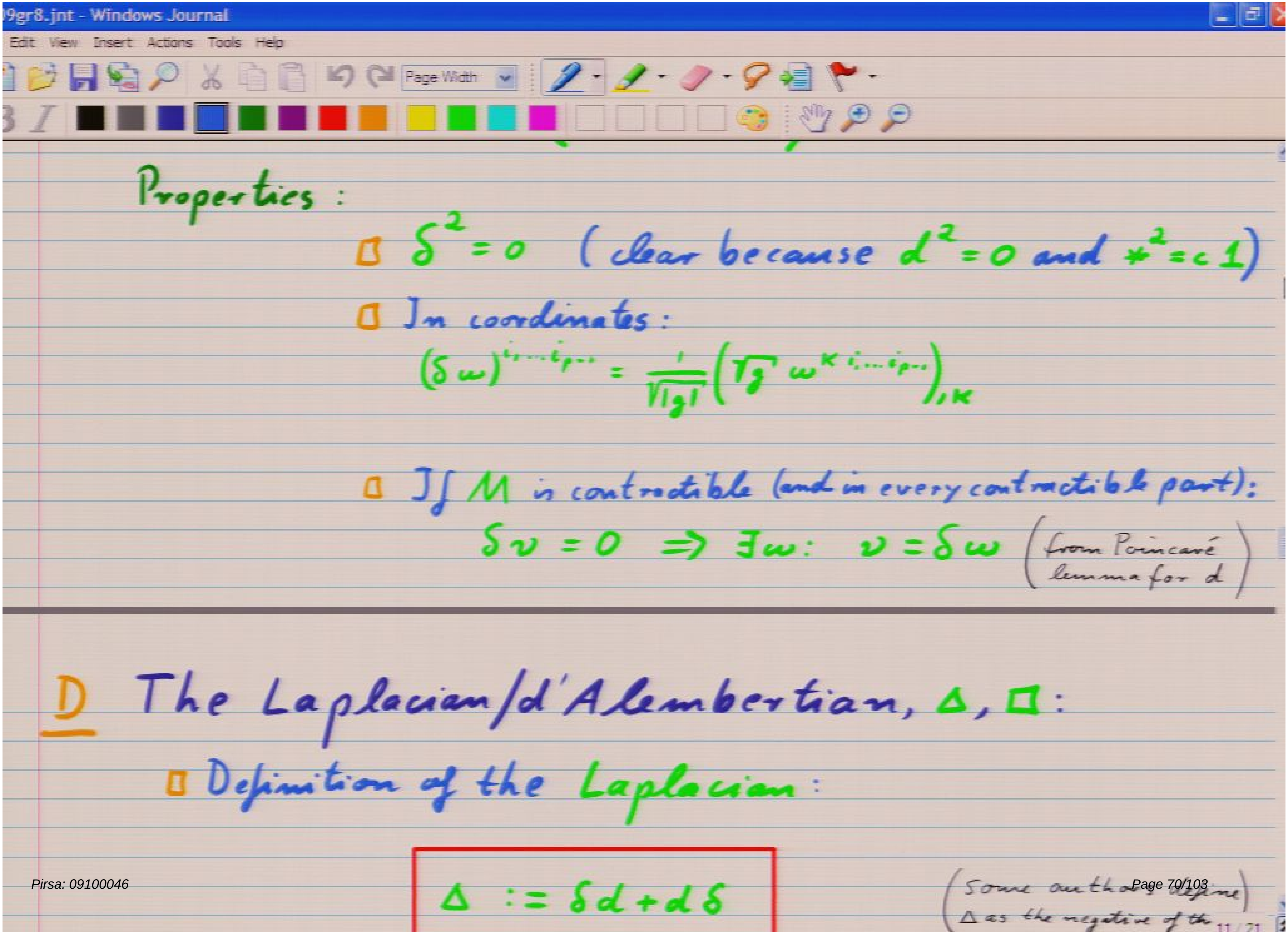

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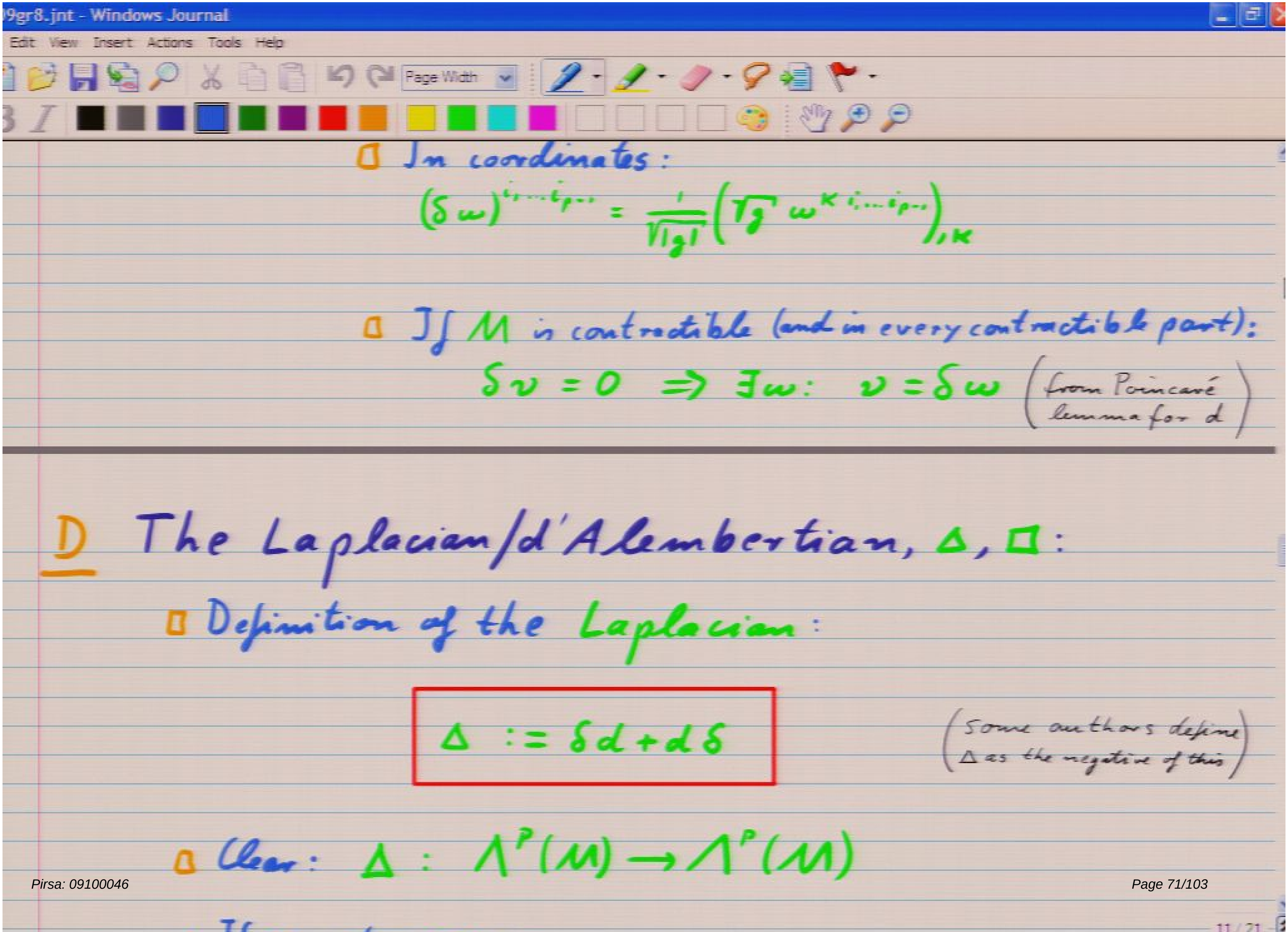

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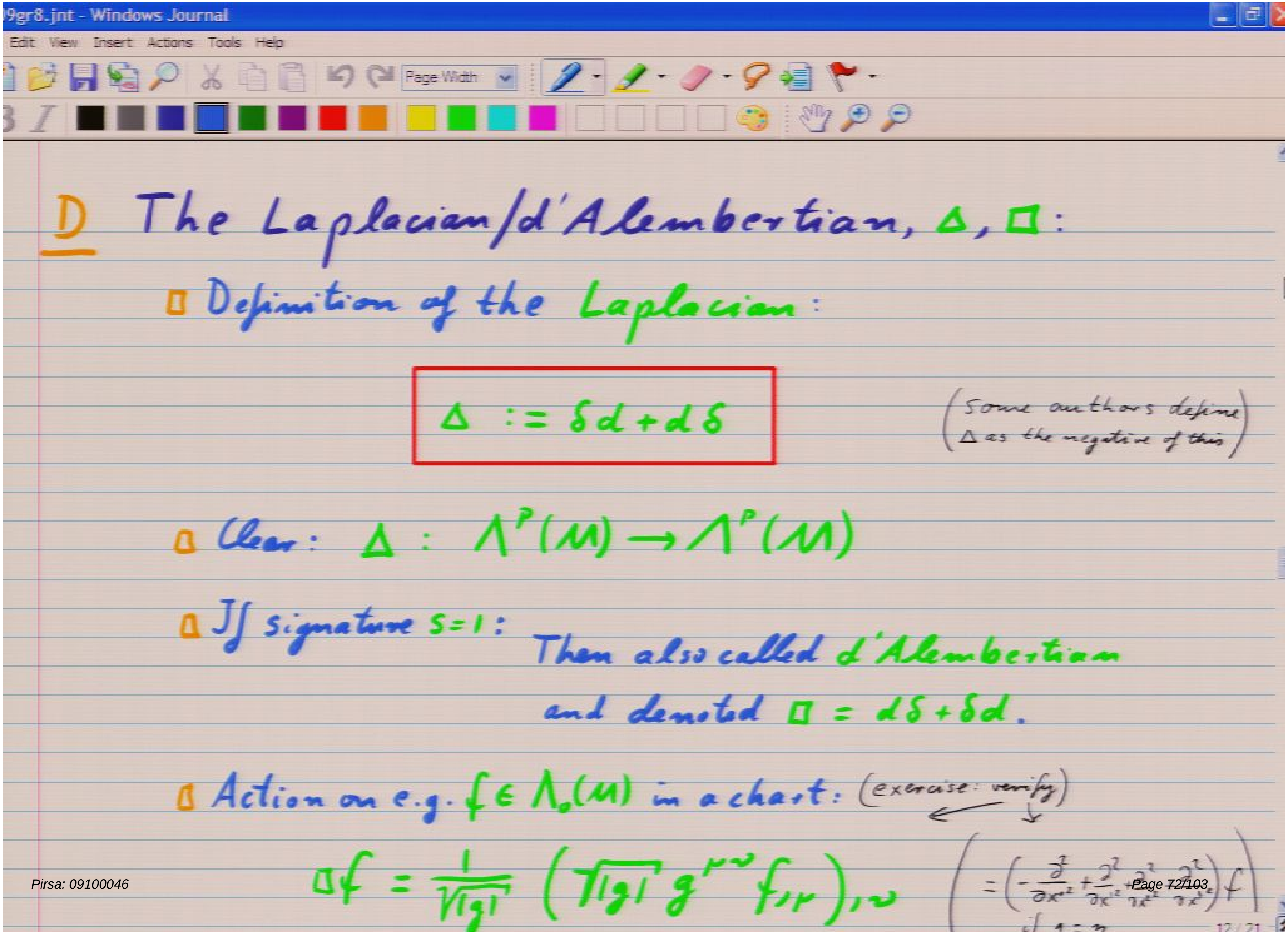

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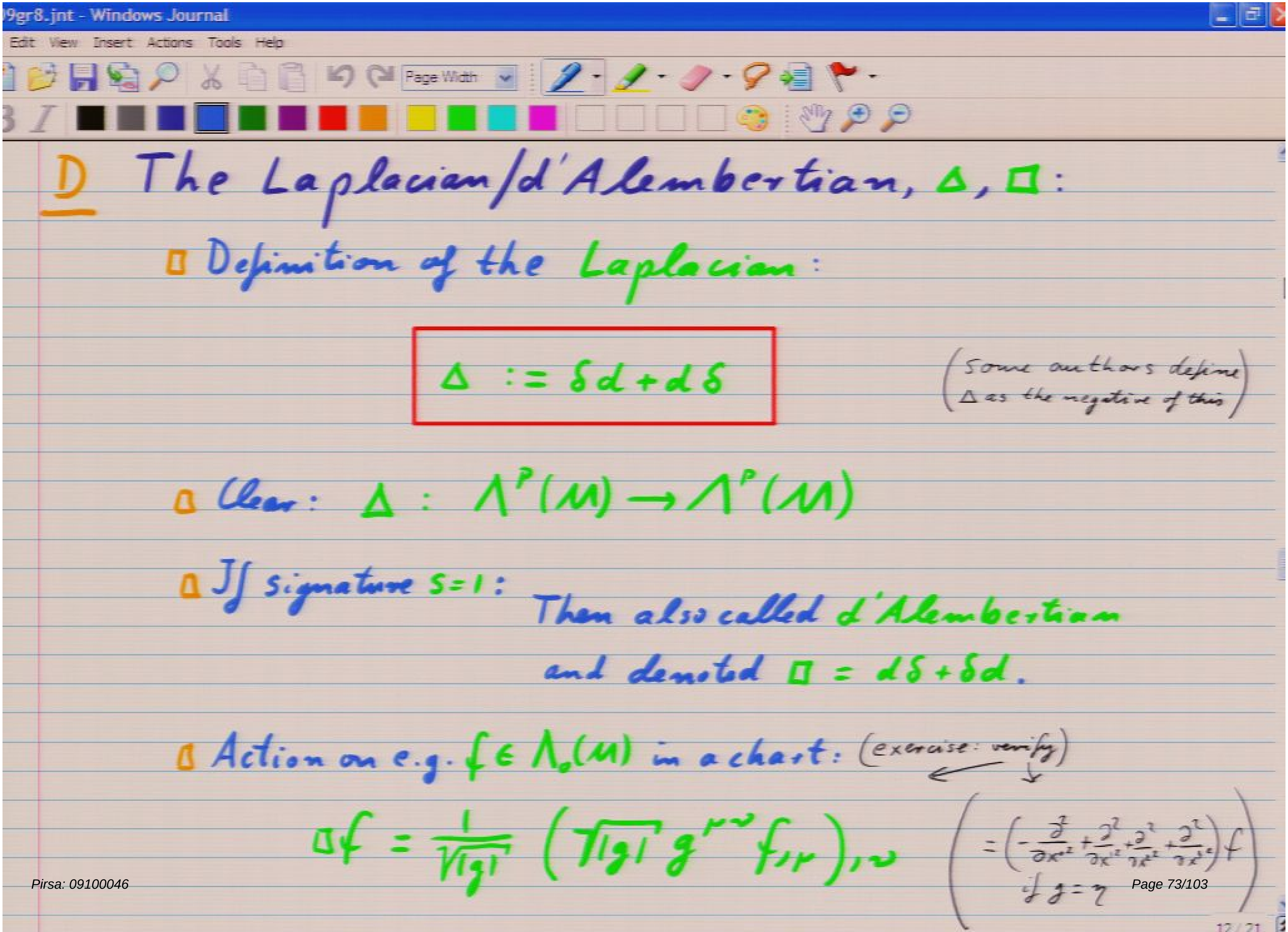

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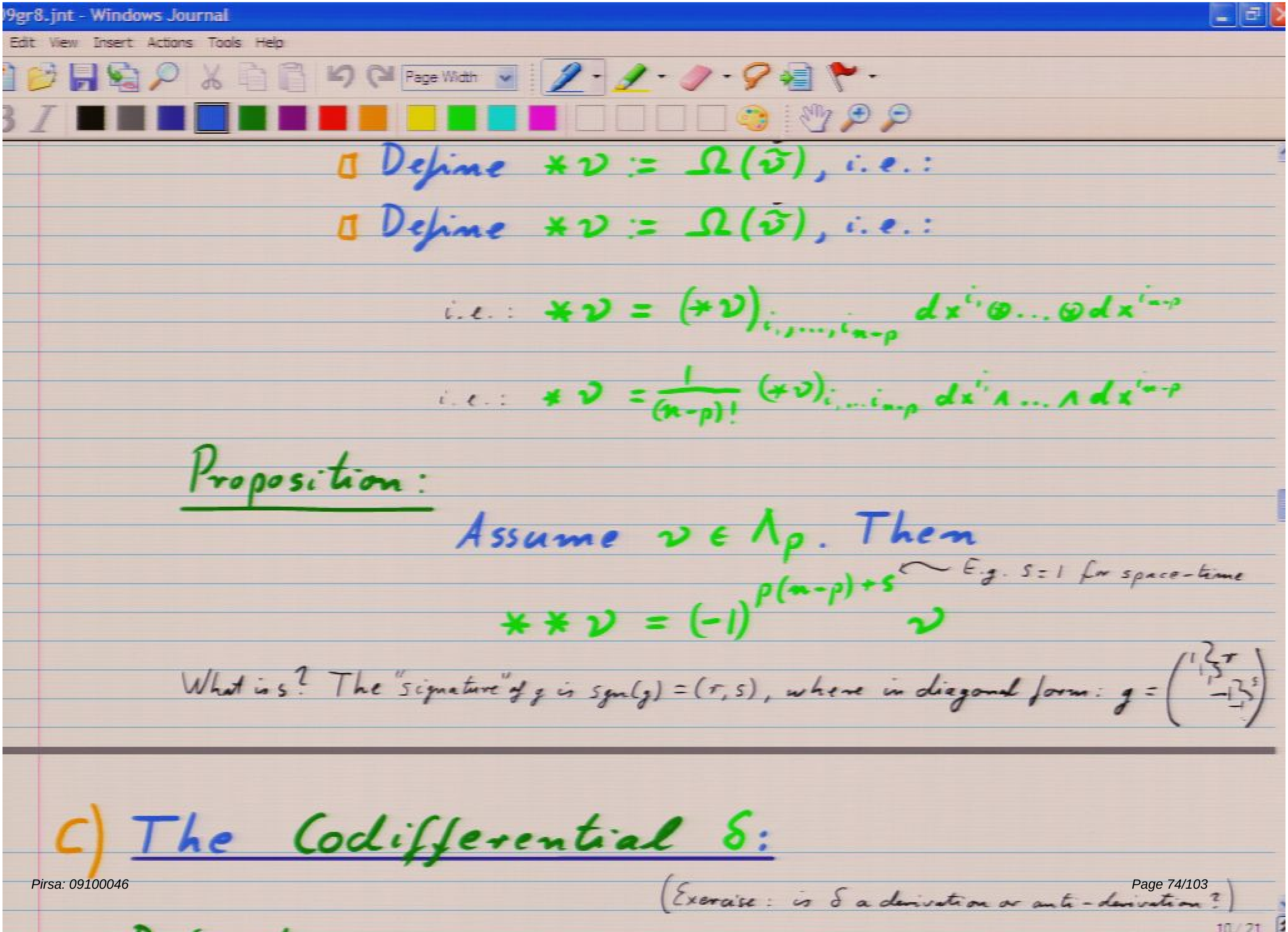

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Proposition:

Assume $\nu \in \Lambda_p$. Then

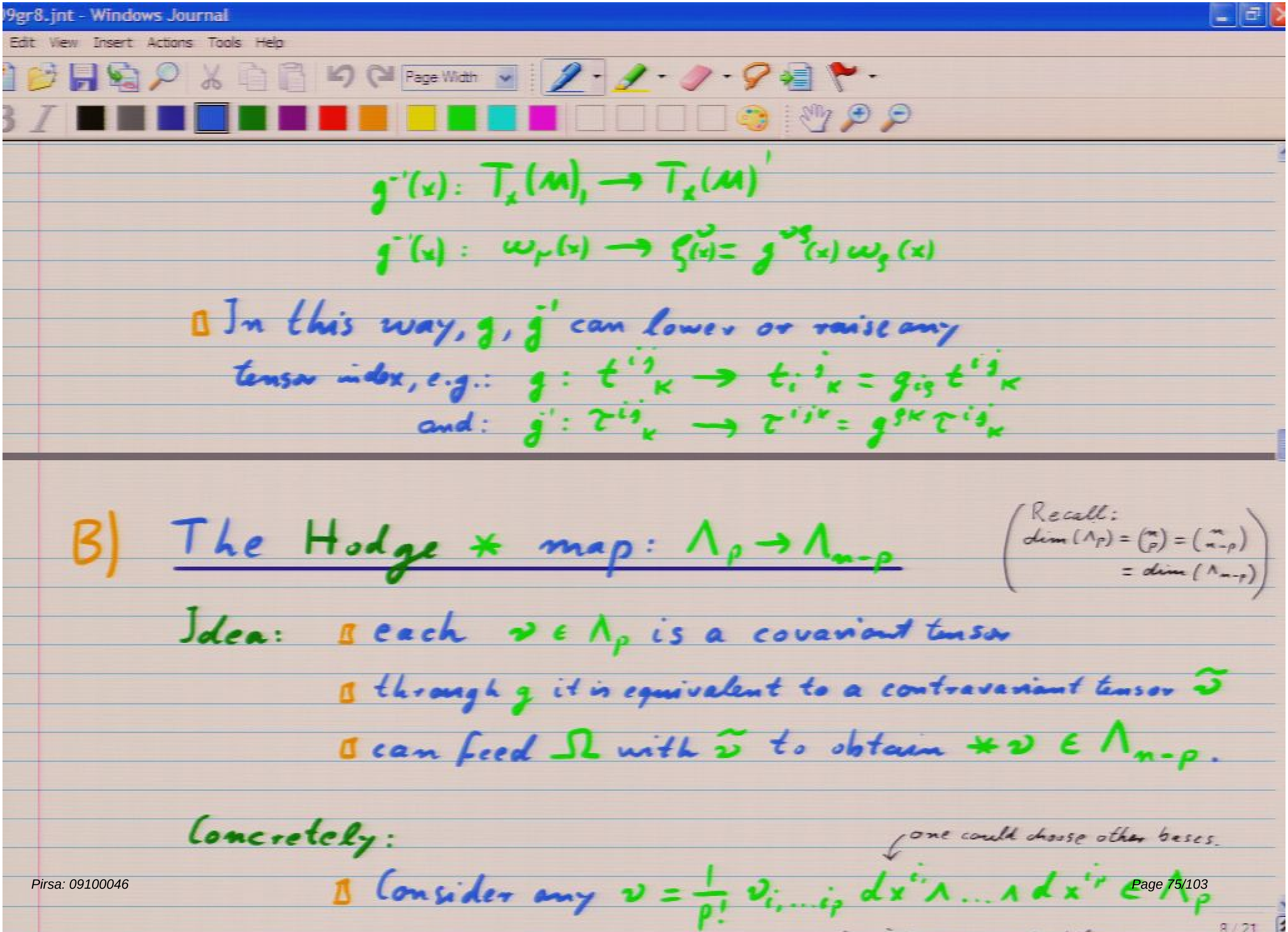
E.g. $s=1$ for space-time

$$**\nu = (-1)^{p(n-p)+s}\,\nu$$

What is s? The "signature" of g is $\mathrm{sgn}(g) = (r,s)$, where in diagonal form: $g = \begin{pmatrix} \left.\begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix}\right\} r & \\ & \left.\begin{matrix}-1 & & \\ & \ddots & \\ & & -1\end{matrix}\right\} s \end{pmatrix}$

c) The Codifferential δ:

(Exercise: is δ a derivation or anti-derivation?)



$$g^{-1}(x): T_x(M)_1 \to T_x(M)^1$$

$$g^{-1}(x): \omega_\mu(x) \to \xi^\nu(x) = g^{\nu\rho}(x)\,\omega_\rho(x)$$

- In this way, g, g^{-1} can lower or raise any tensor index, e.g.: $g: t^{ij}{}_k \to t_i{}^j{}_k = g_{ij}\,t^{ij}{}_k$
  and: $g^{-1}: \tau^{ij}{}_k \to \tau^{ijr} = g^{rk}\,\tau^{ij}{}_k$

B) The Hodge $*$ map: $\Lambda_p \to \Lambda_{n-p}$

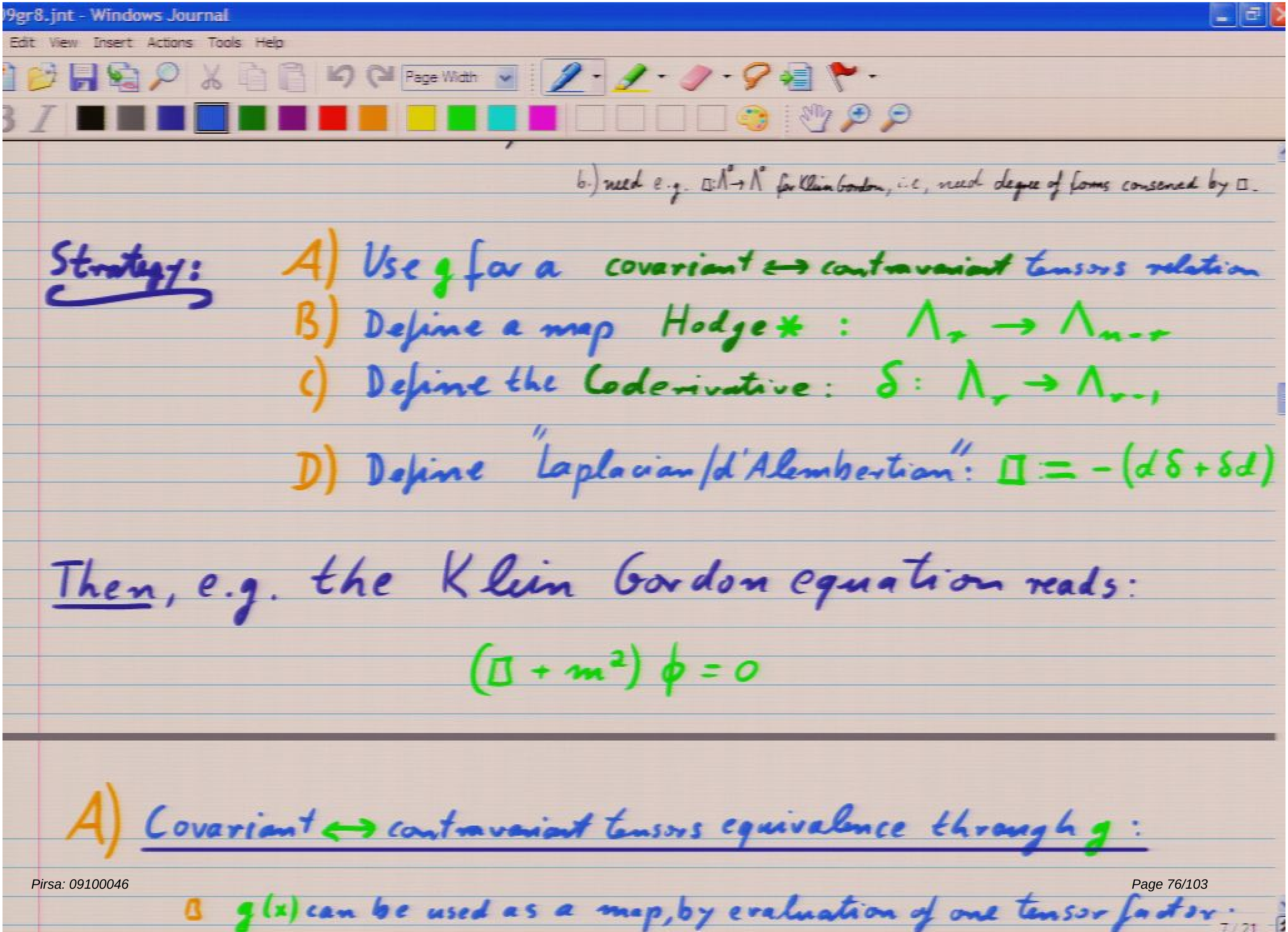
(Recall:
$\dim(\Lambda_p) = \binom{n}{p} = \binom{n}{n-p}$
$= \dim(\Lambda_{n-p})$)

Idea:
- each $v \in \Lambda_p$ is a covariant tensor
- through g it is equivalent to a contravariant tensor $\tilde{v}$
- can feed Ω with $\tilde{v}$ to obtain $*v \in \Lambda_{n-p}$.

Concretely:

one could choose other bases.

- Consider any $v = \frac{1}{p!} v_{i_1 \dots i_p}\, dx^{i_1} \wedge \dots \wedge dx^{i_p} \in \Lambda_p$



b.) need e.g. $\Box:\Lambda^0\to\Lambda^0$ for Klein Gordon, i.e, need degree of forms conserved by $\Box$.

Strategy:

A) Use g for a covariant ↔ contravariant tensors relation

B) Define a map Hodge $*$: $\Lambda_r \to \Lambda_{n-r}$

C) Define the Coderivative: $\delta : \Lambda_r \to \Lambda_{r-1}$

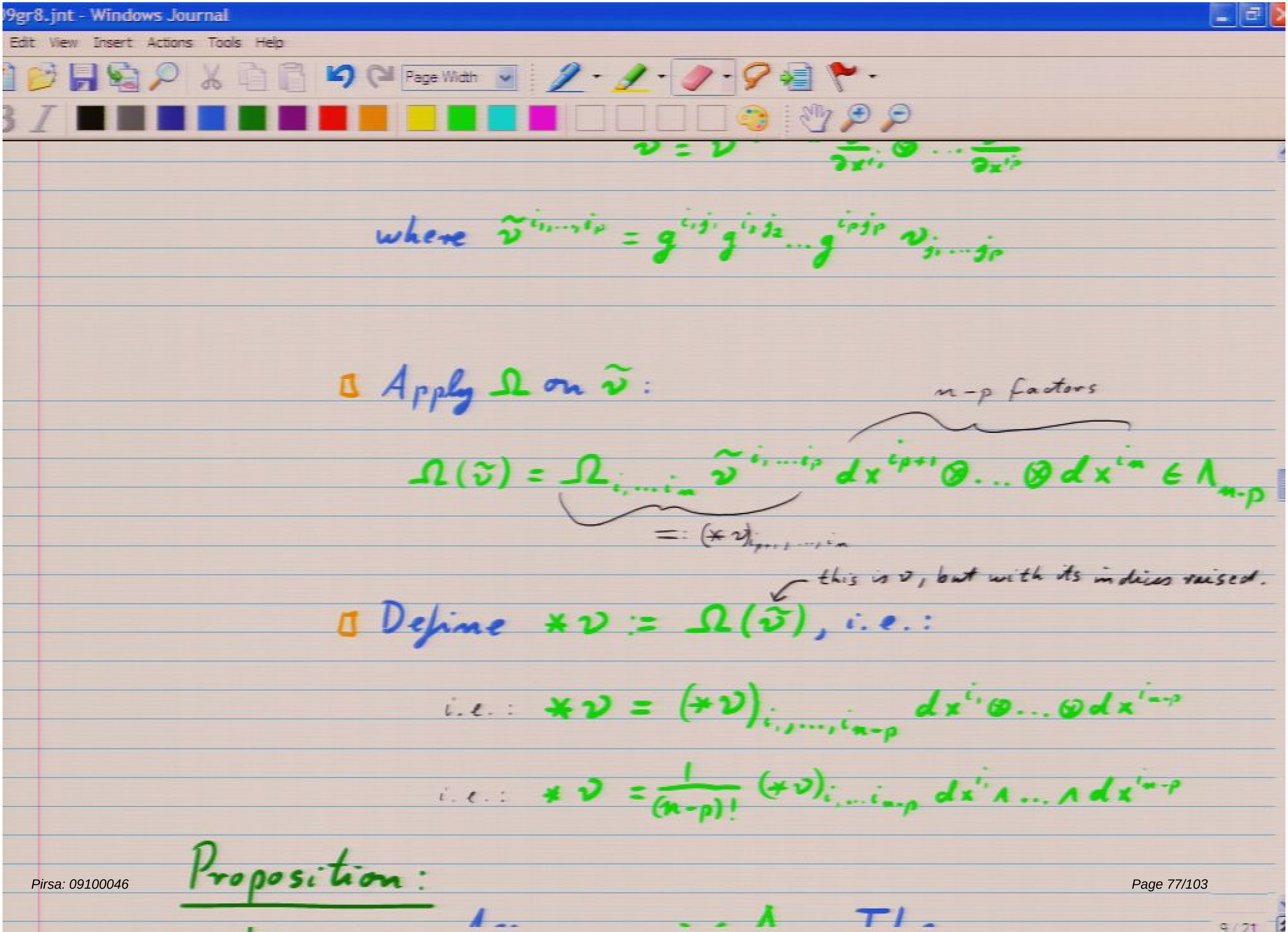
D) Define "Laplacian/d'Alembertian": $\Box := -(d\delta + \delta d)$

Then, e.g. the Klein Gordon equation reads:

$$(\Box + m^2)\,\phi = 0$$

A) Covariant ↔ contravariant tensors equivalence through g:

- $g(x)$ can be used as a map, by evaluation of one tensor factor:



$v = v \quad \frac{}{\partial x^{i_1}} \otimes \ldots \frac{}{\partial x^{i_p}}$

where $\tilde{v}^{i_1,\ldots,i_p} = g^{i_1 j_1} g^{i_2 j_2} \ldots g^{i_p j_p} v_{j_1 \ldots j_p}$

▯ Apply Ω on $\tilde{v}$:

$$\Omega(\tilde{v}) = \underbrace{\Omega_{i_1 \ldots i_n} \tilde{v}^{i_1 \ldots i_p}}_{=: (*v)_{i_{p+1},\ldots,i_n}} \overbrace{dx^{i_{p+1}} \otimes \ldots \otimes dx^{i_n}}^{n-p \text{ factors}} \in \Lambda_{n-p}$$

▯ Define $*v := \Omega(\tilde{v})$, i.e.:

this is v, but with its indices raised.

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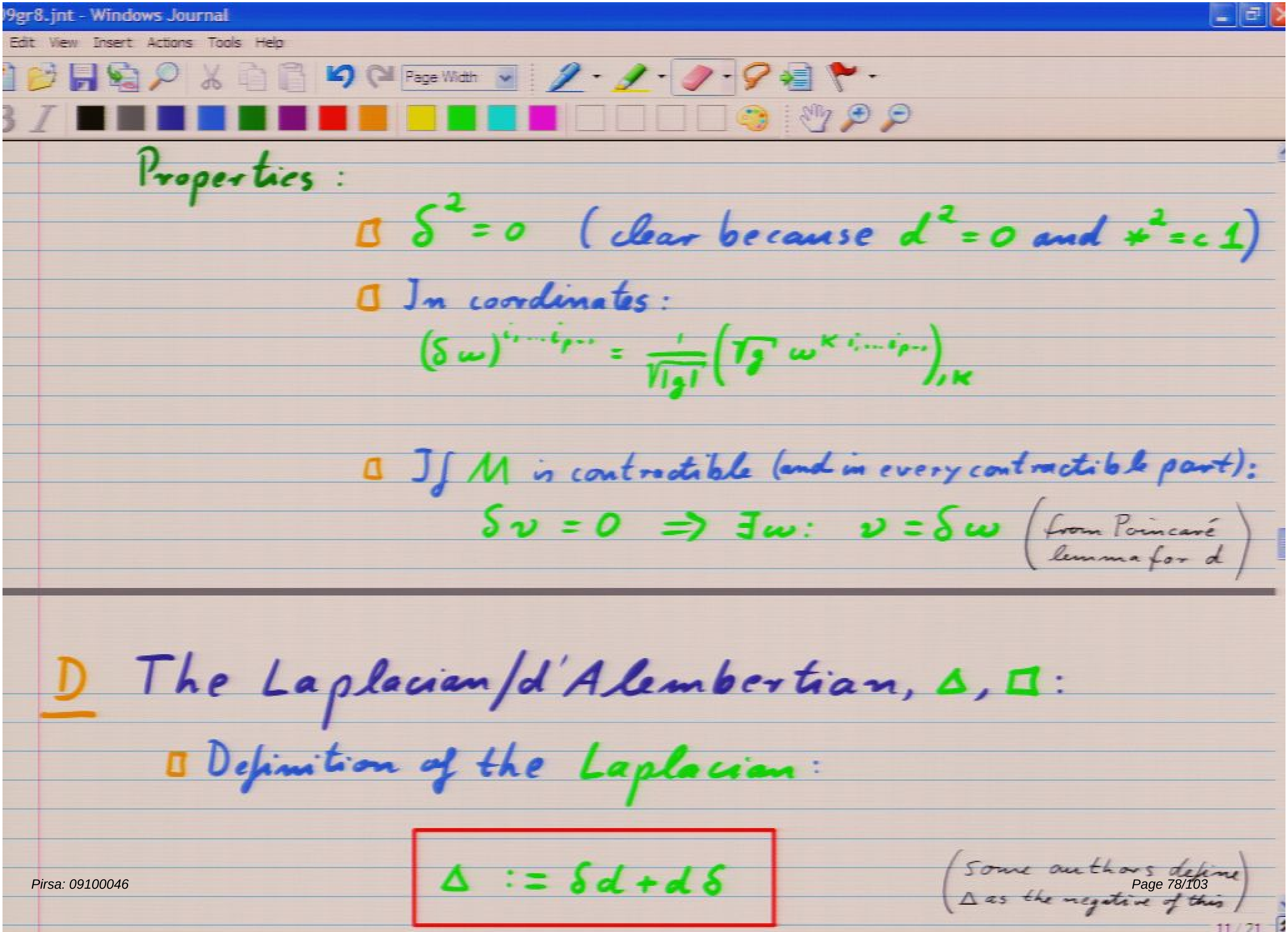

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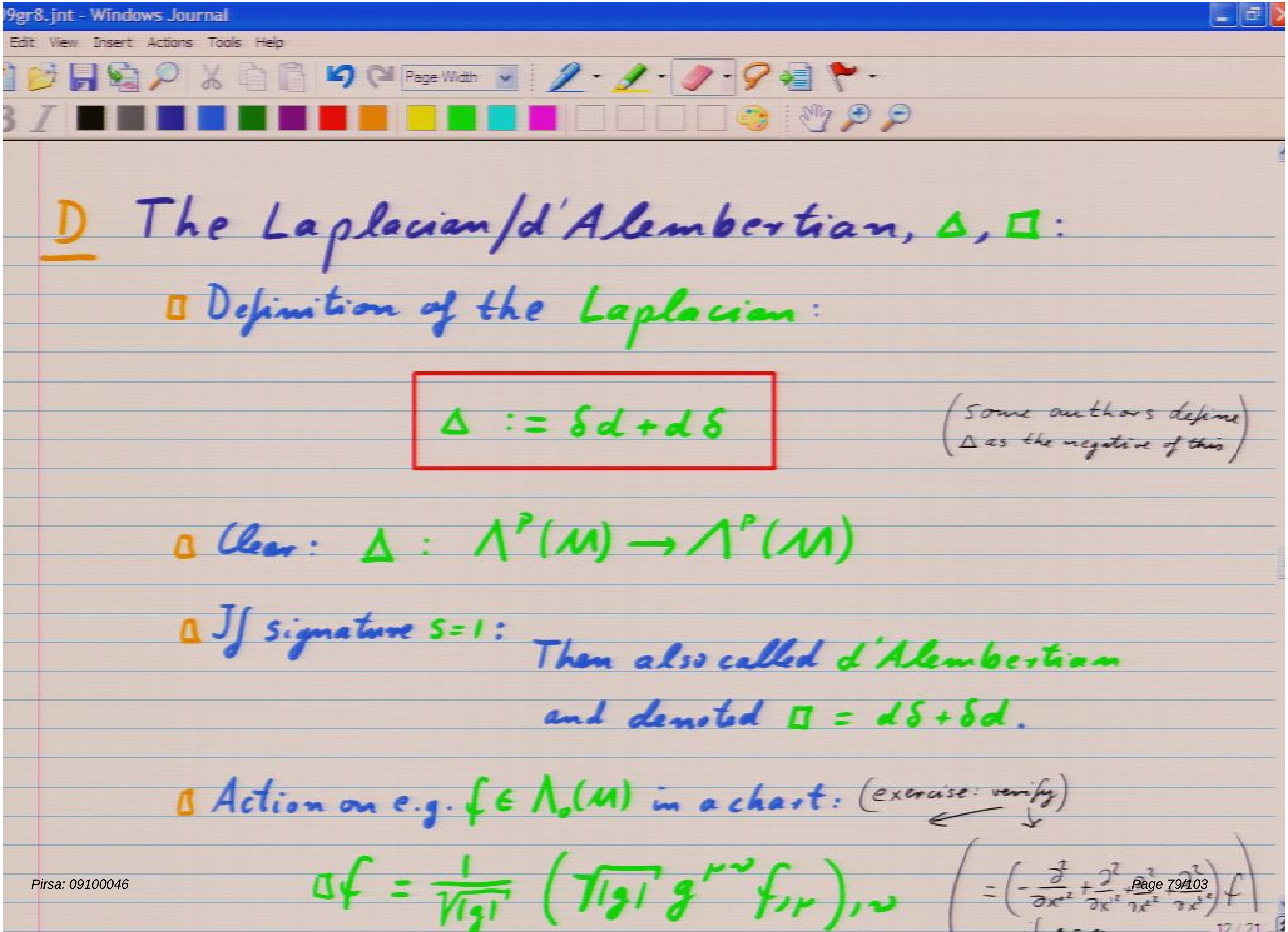

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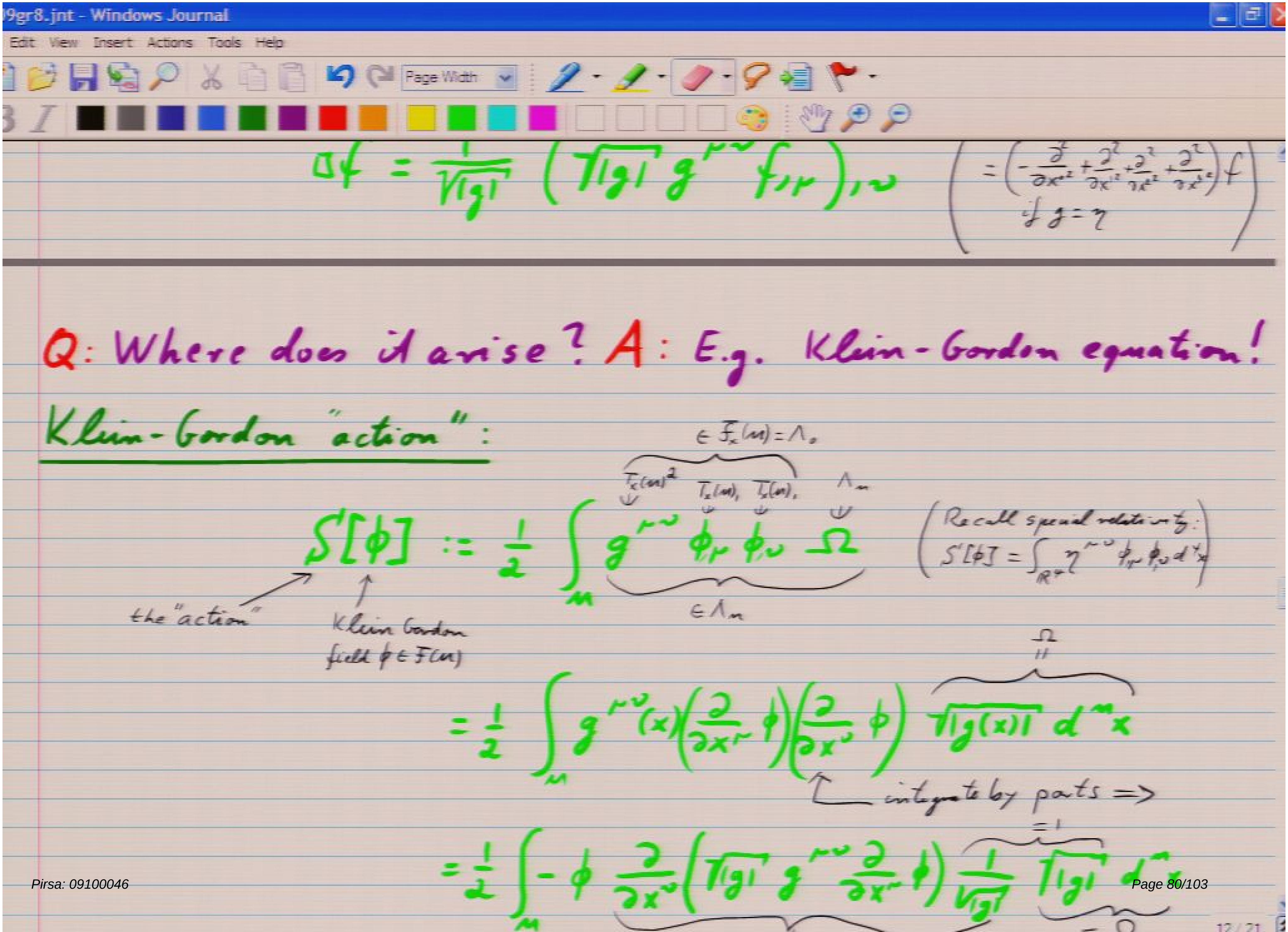

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Q: Where does it arise? A: E.g. Klein-Gordon equation!

Klein-Gordon "action":

$$S[\phi] := \frac{1}{2}\int_M \underbrace{\overbrace{\overset{T_x(M)^2}{\overset{\downarrow}{g^{\mu\nu}}}\ \overset{T_x(M)_*}{\overset{\downarrow}{\phi_{,\mu}}}\ \overset{T_x(M)_*}{\overset{\downarrow}{\phi_{,\nu}}}}^{\in \mathcal{F}_x(M) = \Lambda_0}\ \overset{\Lambda_m}{\overset{\downarrow}{\Omega}}}_{\in \Lambda_m}$$

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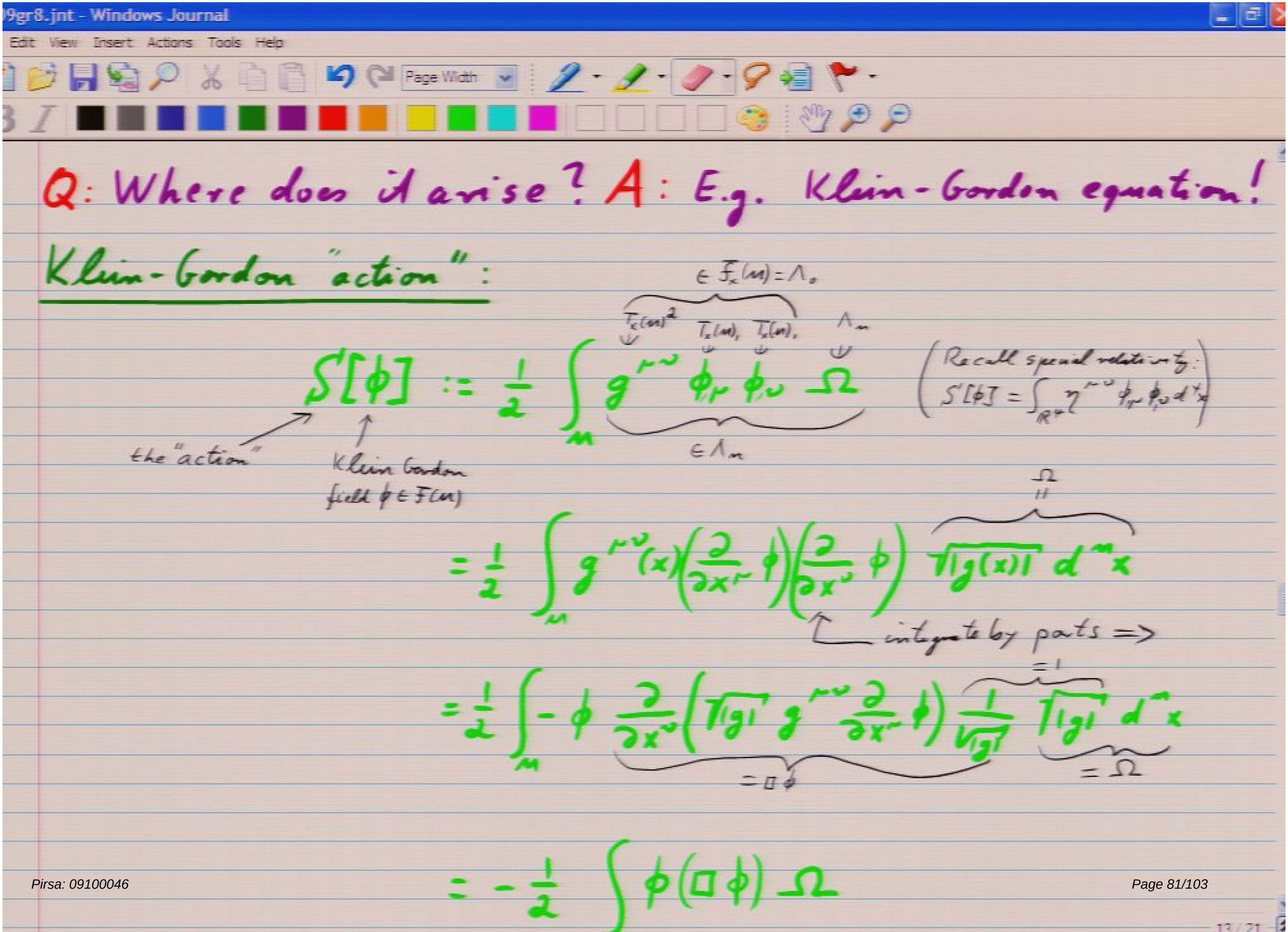
Klein Gordon field $\phi \in \mathcal{F}(M)$

$$\left(\text{Recall special relativity:} \quad S[\phi] = \int_{\mathbb{R}^4} \eta^{\mu\nu}\phi_{,\mu}\phi_{,\nu}\, d^4x \right)$$

$$= \frac{1}{2}\int_M g^{\mu\nu}(x)\left(\frac{\partial}{\partial x^\mu}\phi\right)\left(\frac{\partial}{\partial x^\nu}\phi\right) \overbrace{\sqrt{|g(x)|}\, d^m x}^{\overset{\Omega}{\parallel}}$$

integrate by parts $\Rightarrow$

$$= \frac{1}{2}\int_M \underbrace{-\phi\, \frac{\partial}{\partial x^\nu}\left(\sqrt{|g|}\, g^{\mu\nu}\frac{\partial}{\partial x^\mu}\phi\right)} \overbrace{\frac{1}{\sqrt{|g|}}\sqrt{|g|}}^{=1}\, d^m x$$



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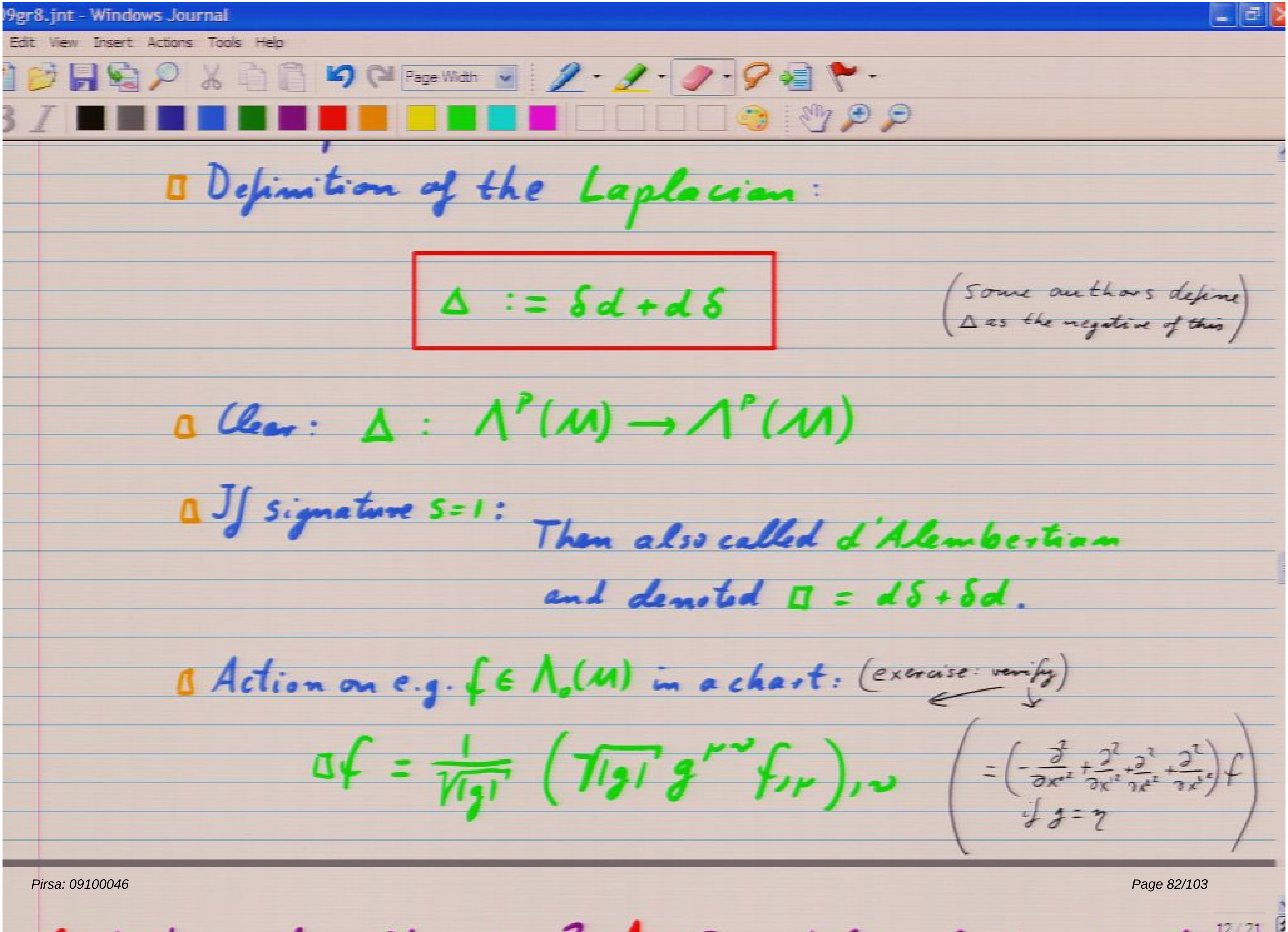

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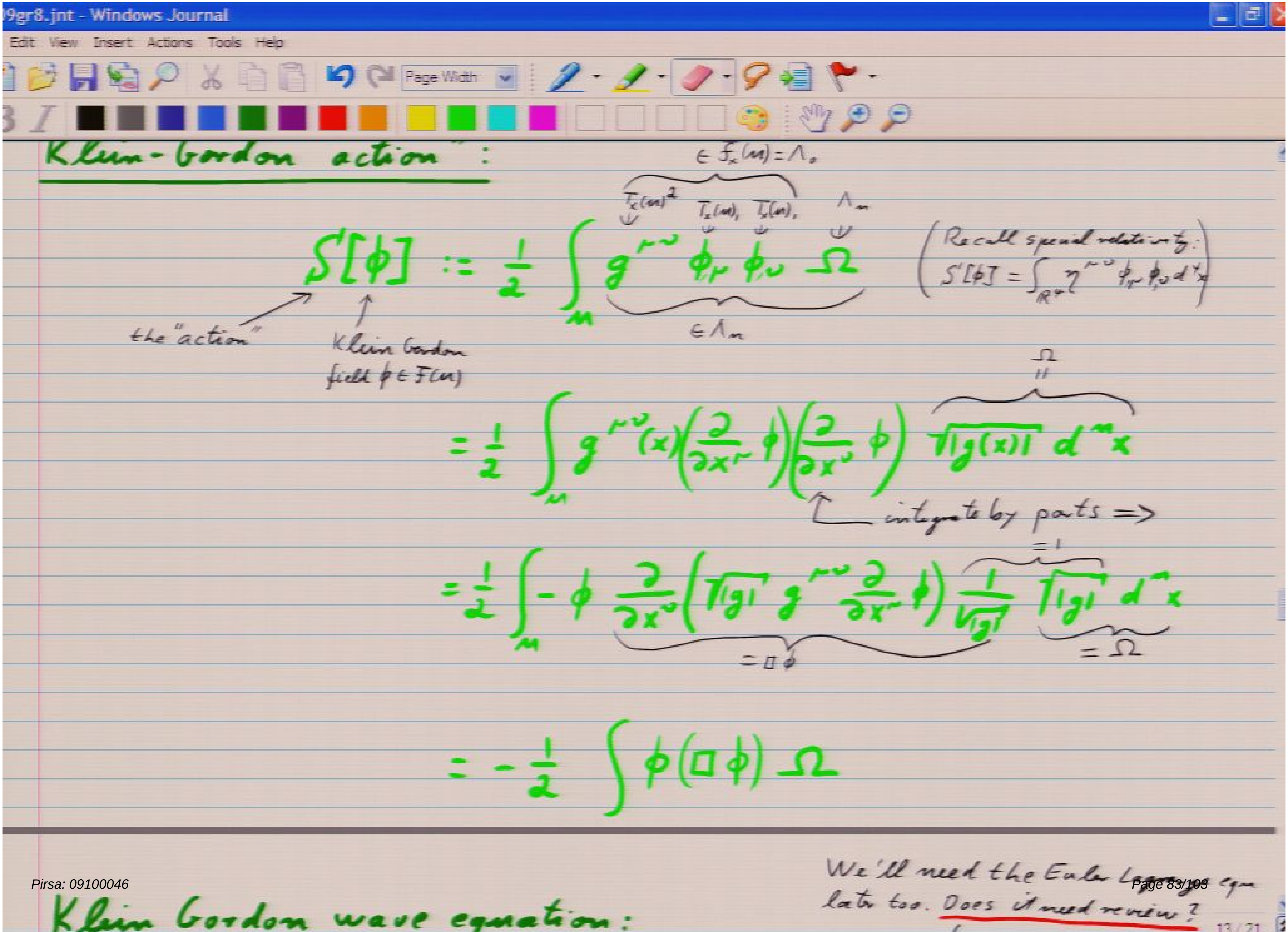

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Klein Gordon wave equation:

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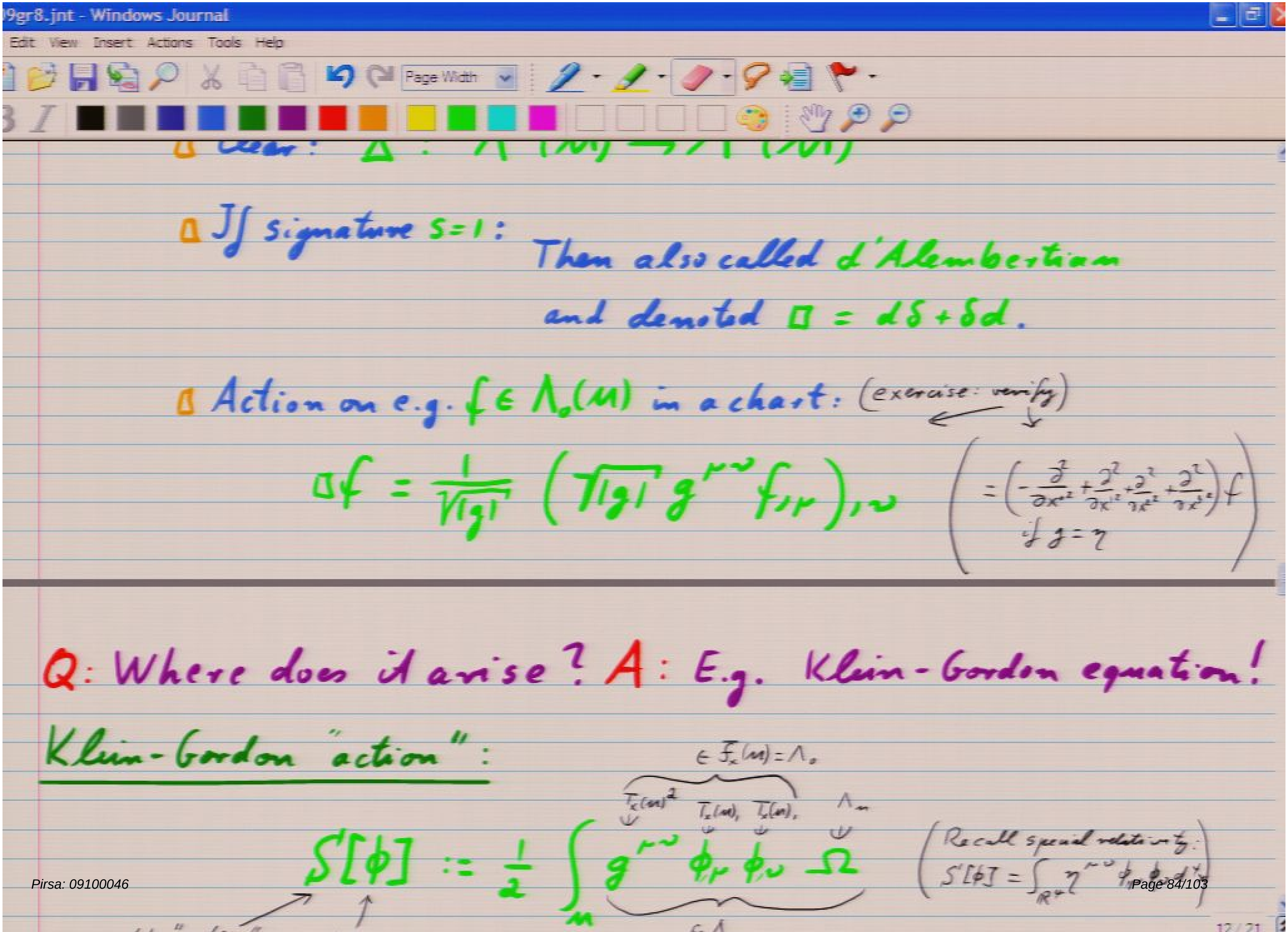

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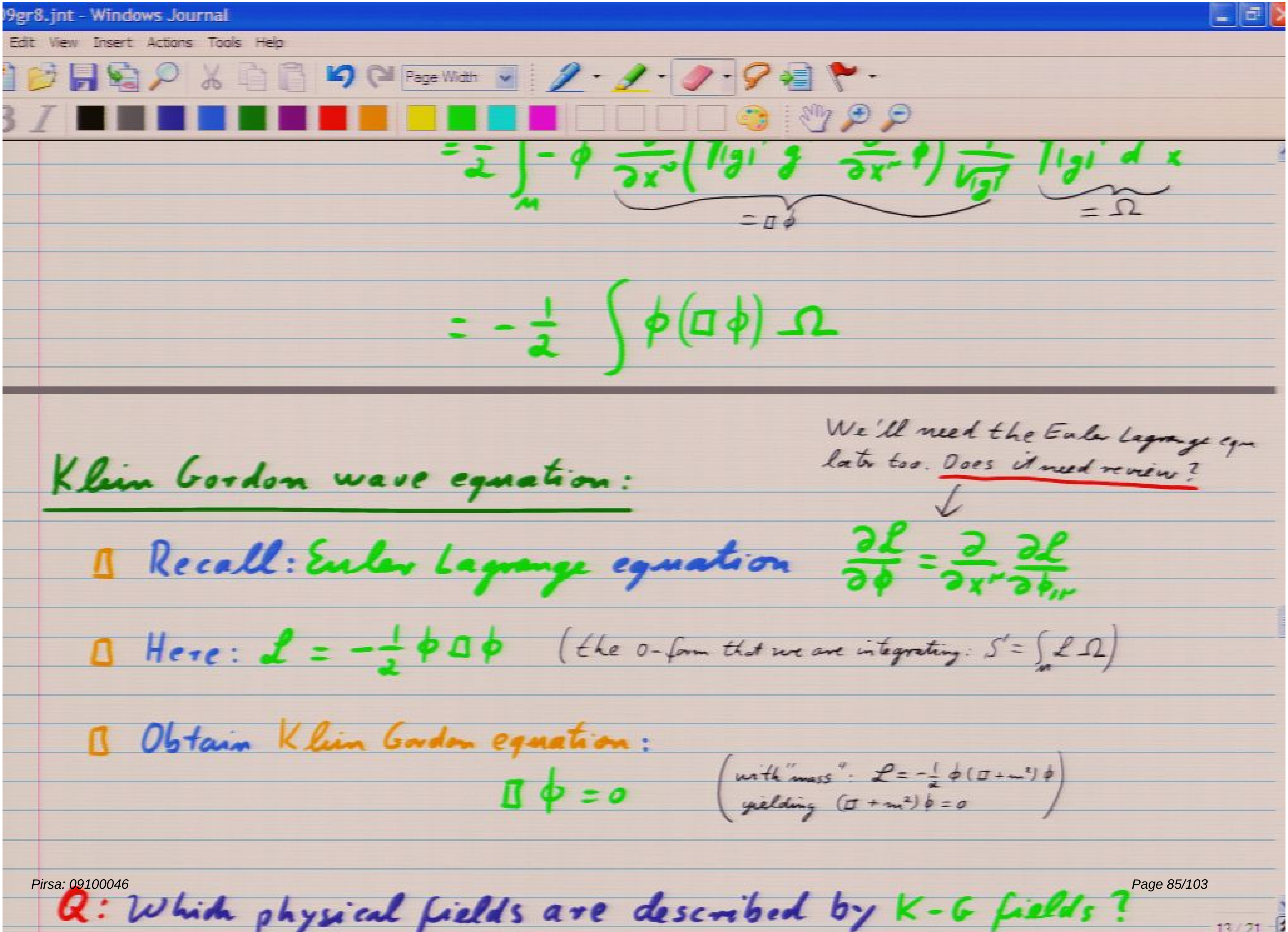

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- Recall: Euler Lagrange equation $\frac{\partial \mathcal{L}}{\partial \phi} = \frac{\partial}{\partial x^\mu}\frac{\partial \mathcal{L}}{\partial \phi_{,\mu}}$
- Here: $\mathcal{L} = -\frac{1}{2}\phi\Box\phi$ (the 0-form that we are integrating: $S = \int_M \mathcal{L}\,\Omega$)
- Obtain Klein Gordon equation:

$$\Box\phi = 0$$

(with "mass": $\mathcal{L} = -\frac{1}{2}\phi(\Box + m^2)\phi$
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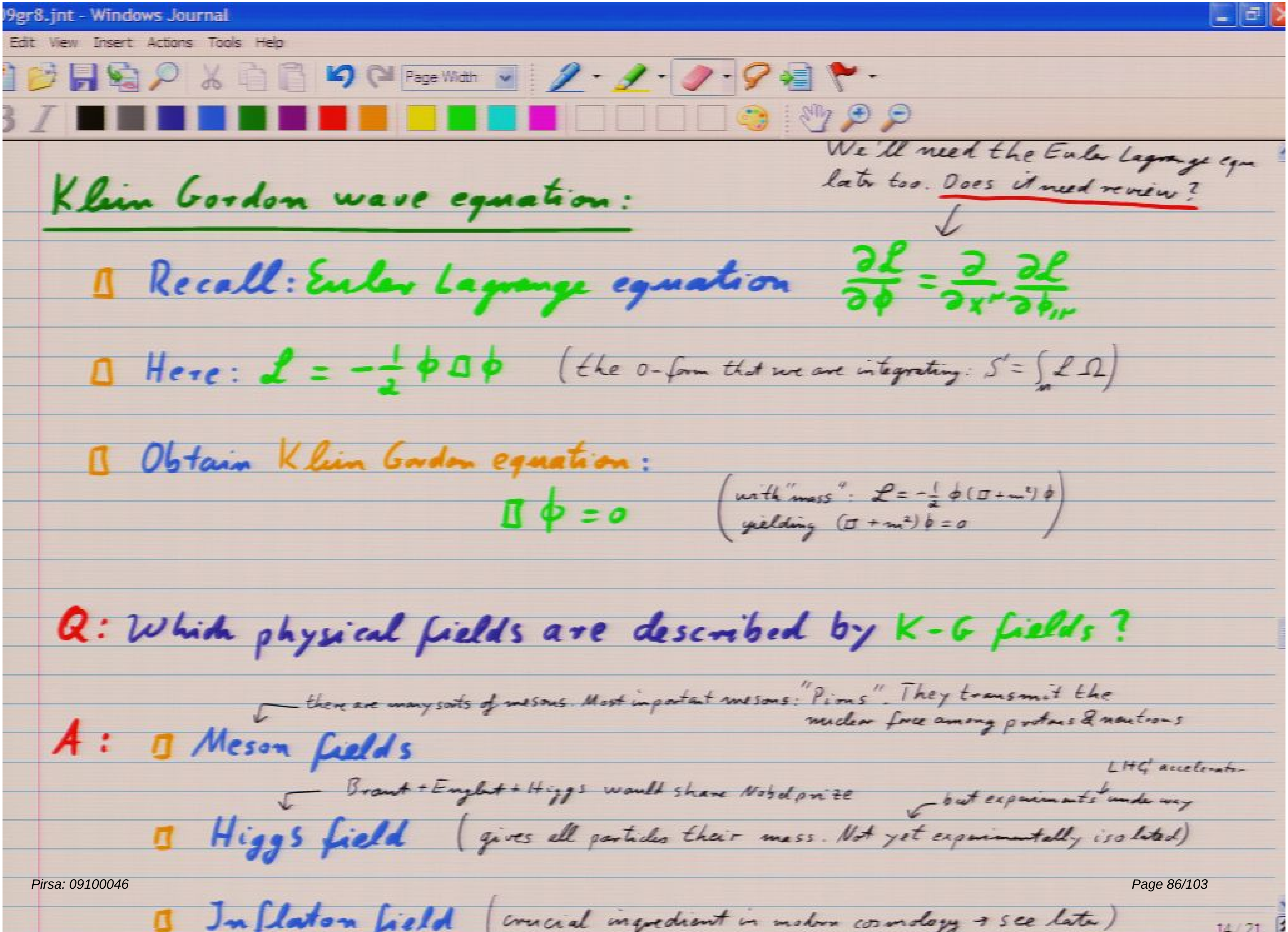
Q: Which physical fields are described by K-G fields?



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there are many sorts of mesons. Most important mesons: "Pions". They transmit the nuclear force among protons & neutrons

A: ☐ Meson fields

Brout + Englert + Higgs would share Nobel prize

LHC accelerator

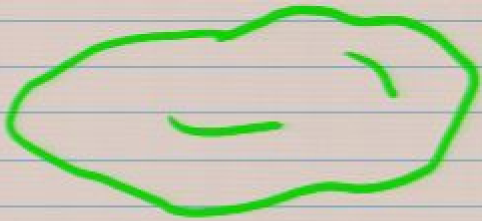
but experiments under way

☐ Higgs field (gives all particles their mass. Not yet experimentally isolated)

☐ Inflaton field (crucial ingredient in modern cosmology → see later)

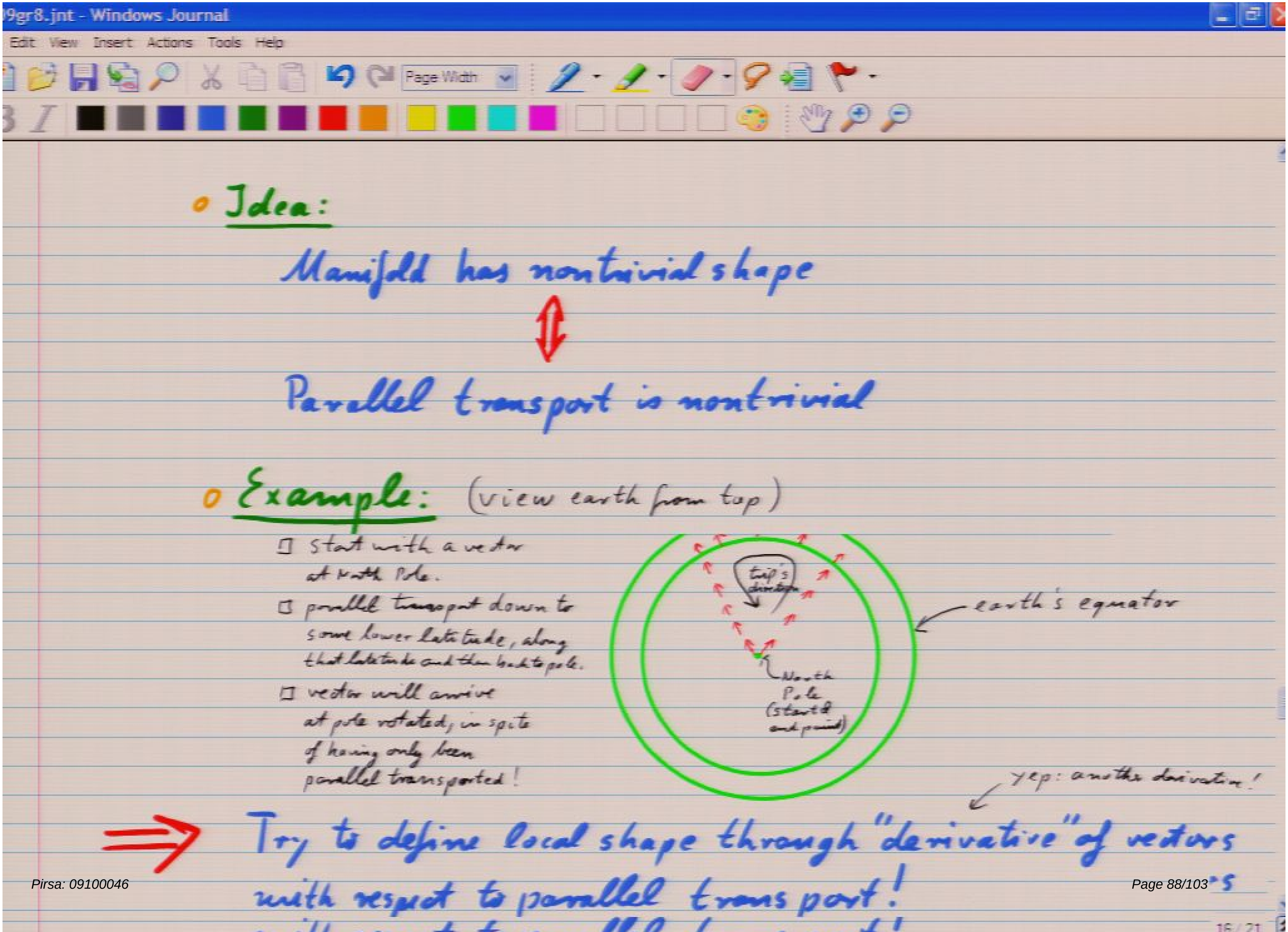
Need new intuition:

- Having defined g allows us to calculate distances, and so g defines the shape of the manifold:



E.g. on a potato-shaped surface the distance relations are non-Pythagorean, and this shows through $g_{ij}(x) \neq \eta_{ij} = \begin{pmatrix}1&0\\0&1\end{pmatrix} \; \forall x$ in any chart.

- But there is also an alternative way to define the shape of a manifold, that is equivalent but often more convenient!



- o **Idea:**

Manifold has nontrivial shape

⇕

Parallel transport is nontrivial

- o **Example:** (view earth from top)

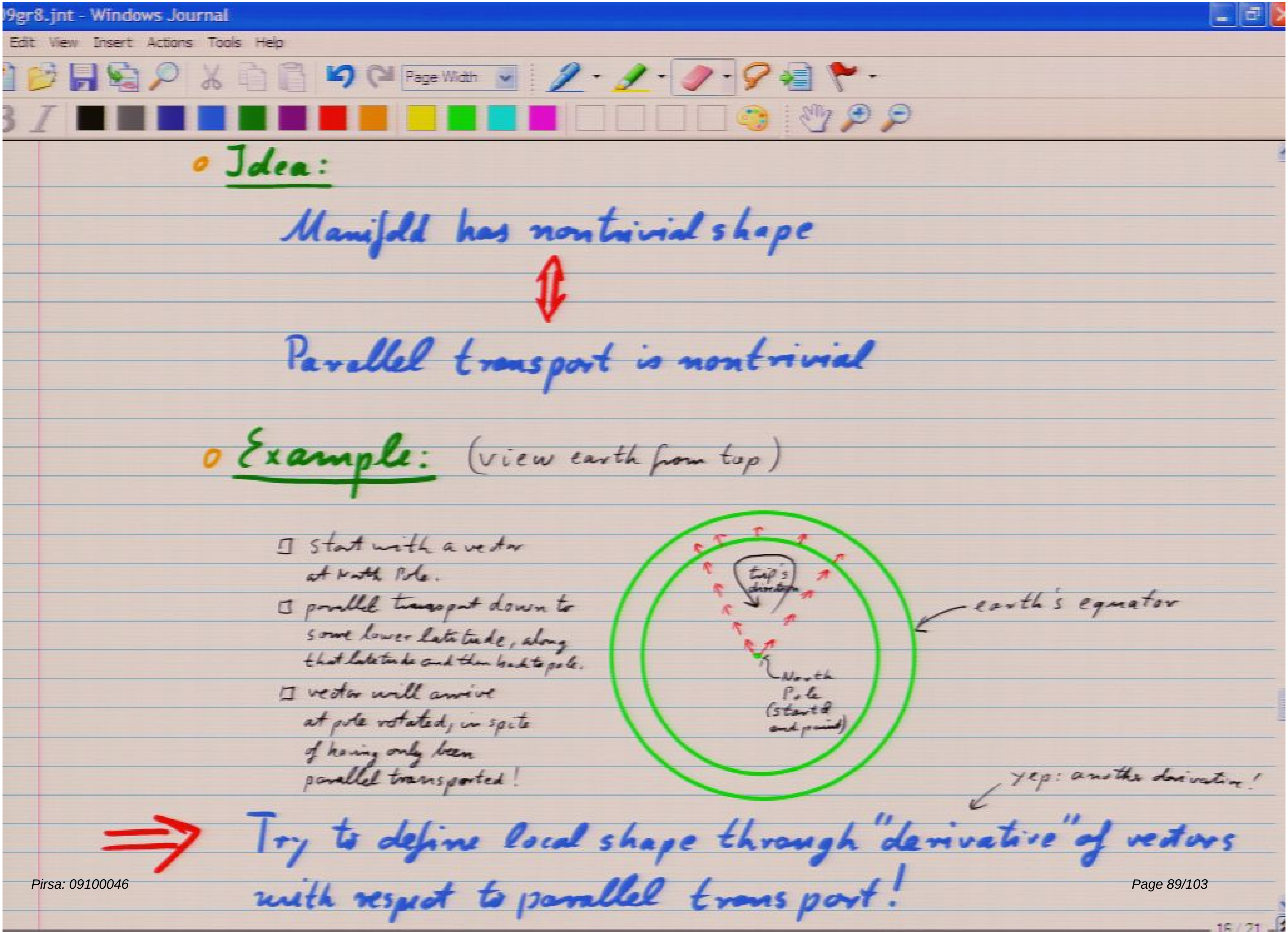
☐ start with a vector at North Pole.

☐ parallel transport down to some lower latitude, along that latitude and then back to pole.

☐ vector will arrive at pole rotated, in spite of having only been parallel transported!

yep: another derivation!

⇒ Try to define local shape through "derivative" of vectors with respect to parallel transport!



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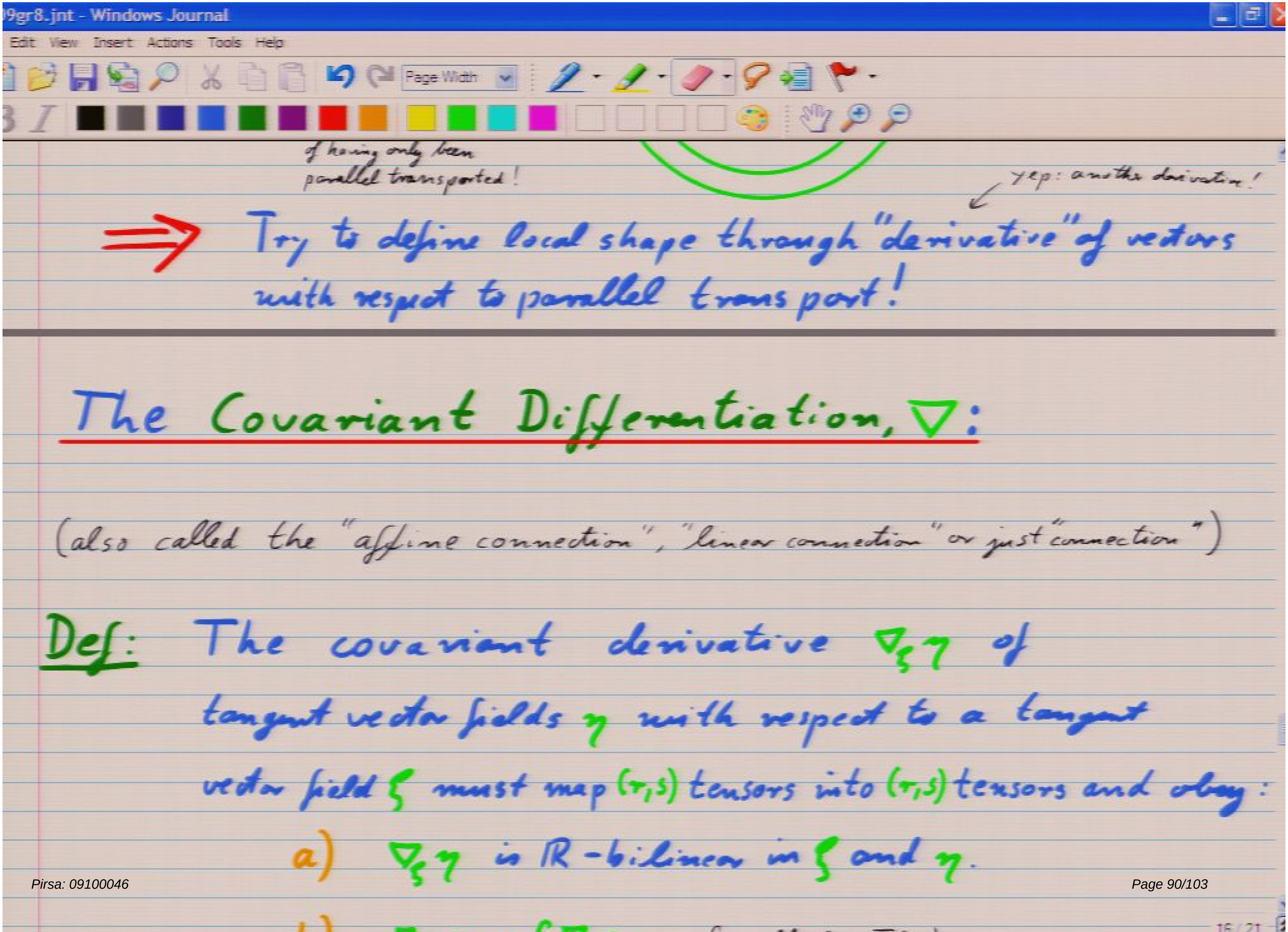
⇒ Try to define local shape through "derivative" of vectors with respect to parallel transport!



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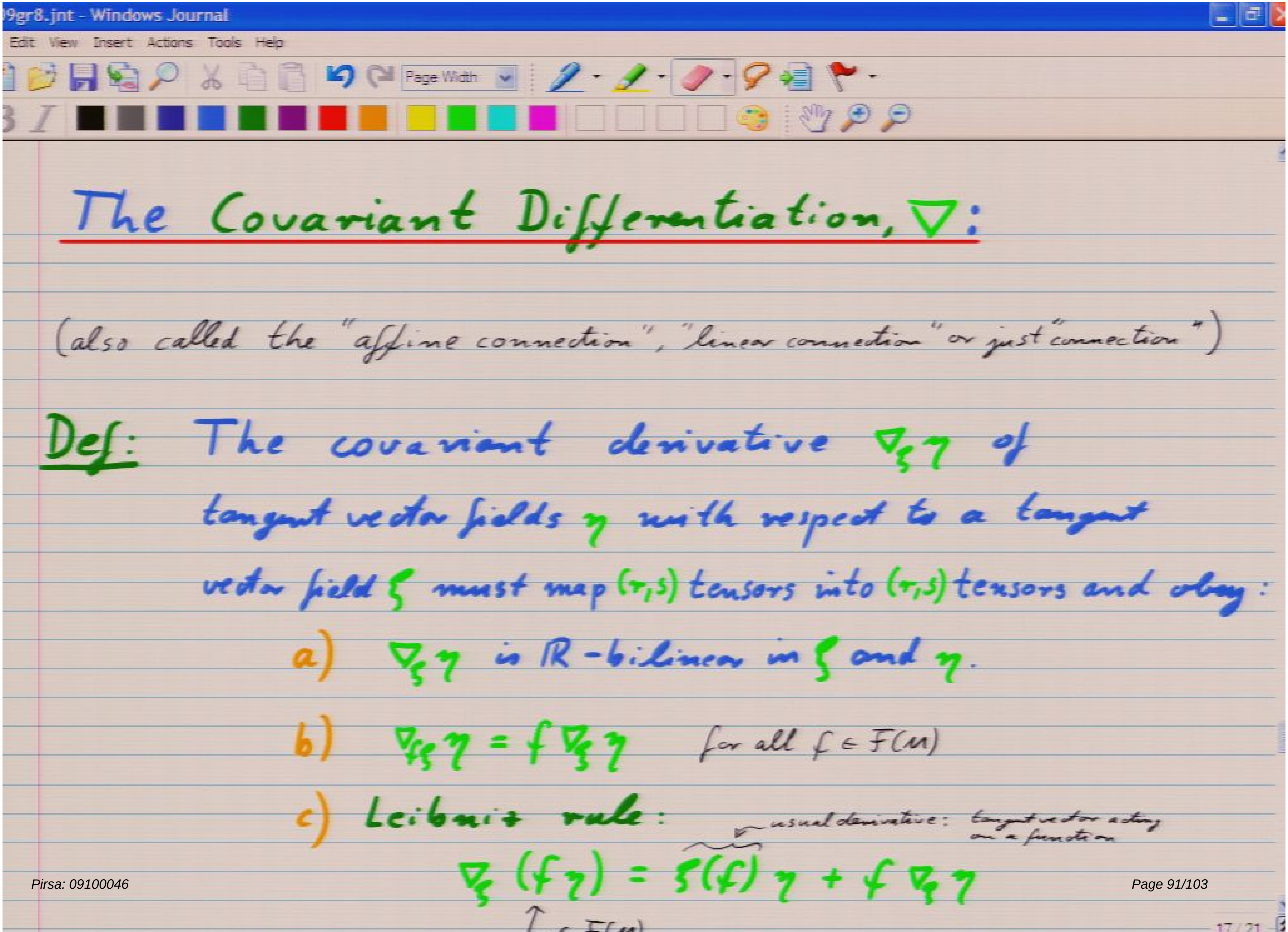
The Covariant Differentiation, ∇:

(also called the "affine connection", "linear connection" or just "connection")

Def: The covariant derivative $\nabla_\xi \eta$ of tangent vector fields η with respect to a tangent vector field ξ must map (r,s) tensors into (r,s) tensors and obey:

a) $\nabla_\xi \eta$ is $\mathbb{R}$-bilinear in ξ and η.



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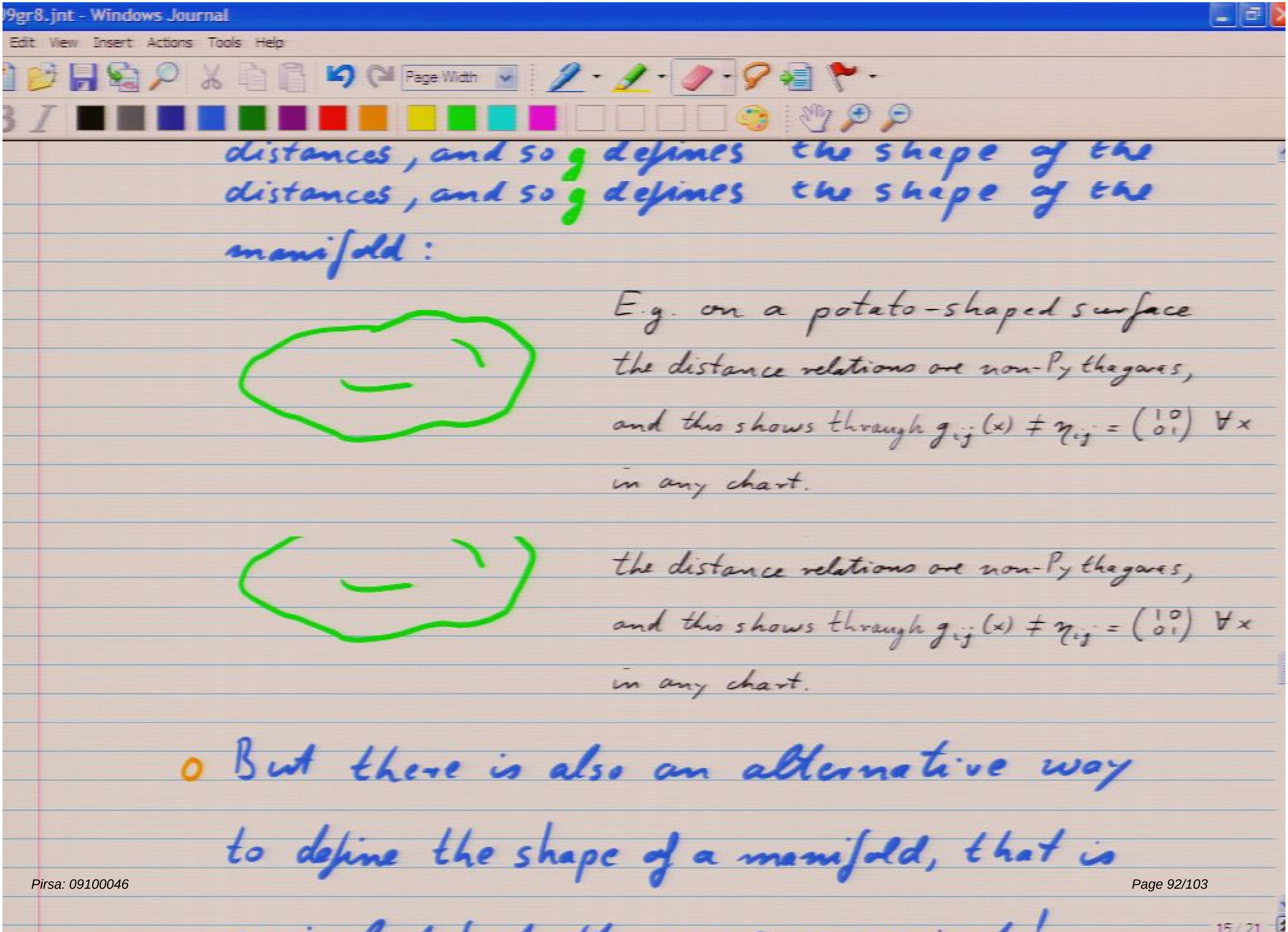
a) $\nabla_\xi \eta$ is $\mathbb{R}$-bilinear in ξ and η.

b) $\nabla_{f\xi} \eta = f \nabla_\xi \eta$ for all $f \in \mathcal{F}(M)$

c) **Leibniz rule:**

$$\nabla_\xi (\underset{\uparrow \in \mathcal{F}(M)}{f} \eta) = \overbrace{\xi(f)}^{\text{usual derivative: tangent vector acting on a function}} \eta + f \nabla_\xi \eta$$



distances, and so g defines the shape of the

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manifold:

E.g. on a potato-shaped surface the distance relations are non-Pythagoras, and this shows through $g_{ij}(x) \neq \eta_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \; \forall x$ in any chart.

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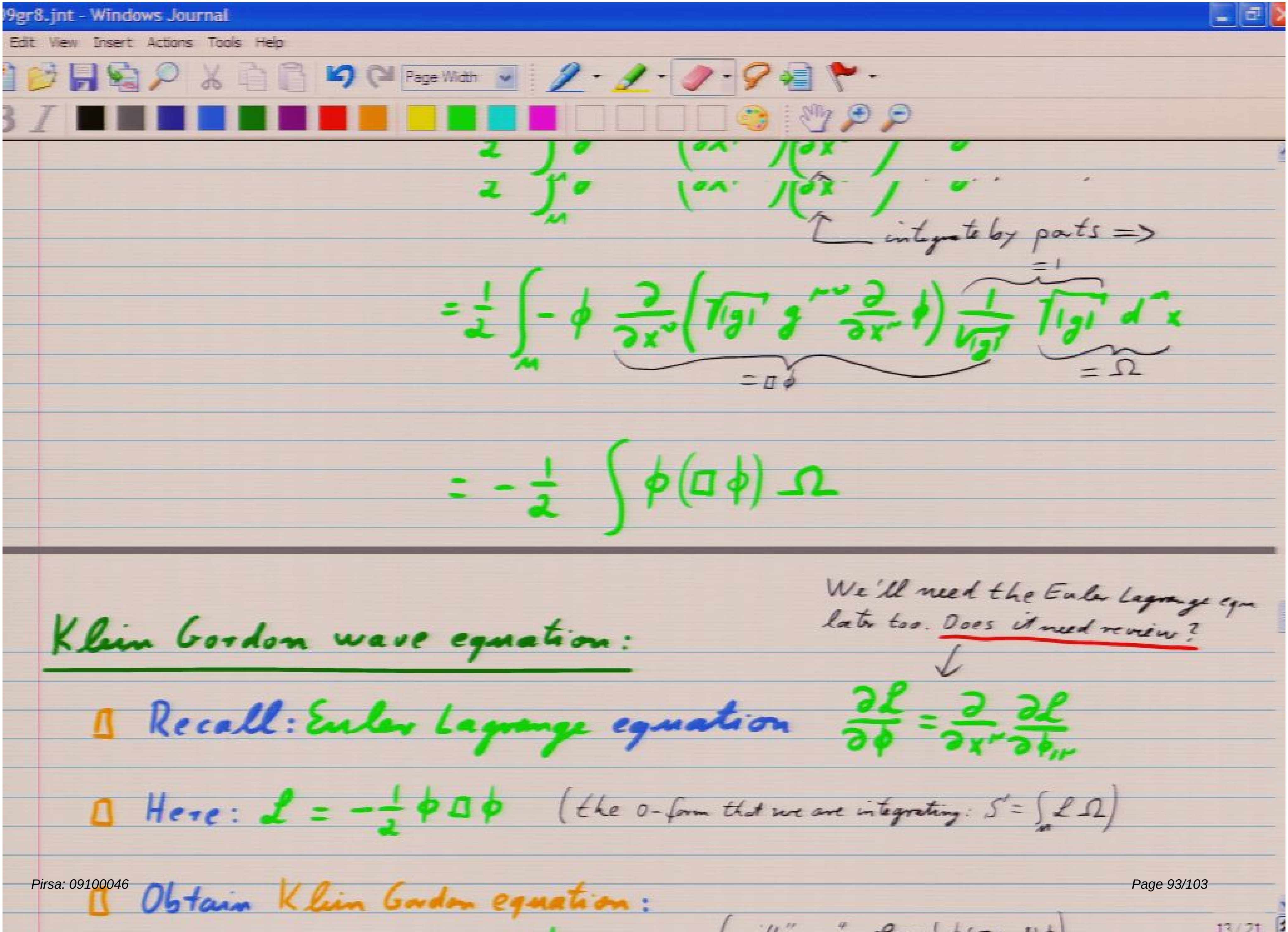
- But there is also an alternative way to define the shape of a manifold, that is



integrate by parts =>

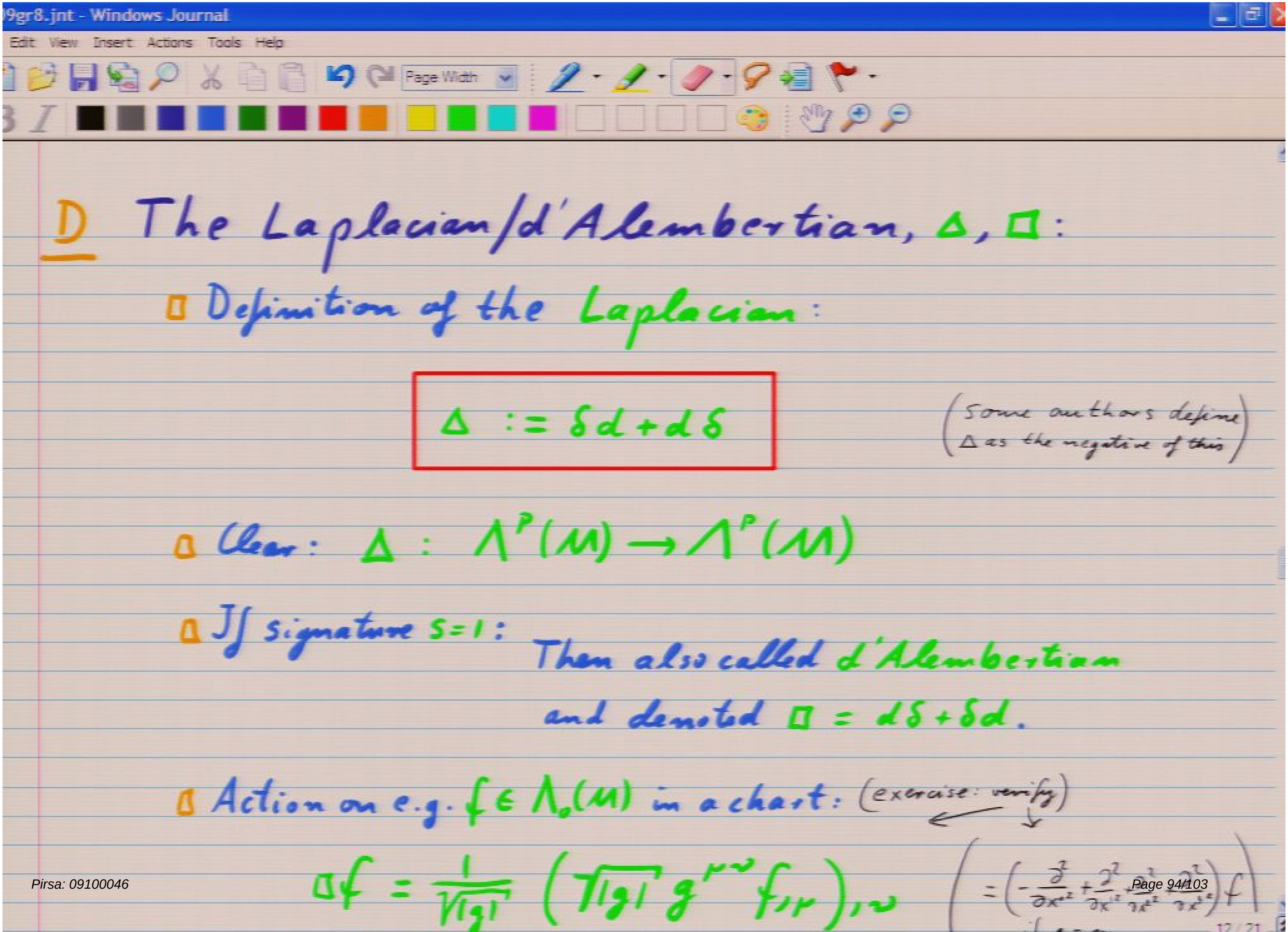
$$=\frac{1}{2}\int_M -\phi\,\underbrace{\frac{\partial}{\partial x^\nu}\left(\sqrt{|g|}\,g^{\mu\nu}\frac{\partial}{\partial x^\mu}\phi\right)\frac{1}{\sqrt{|g|}}}_{=\Box\phi}\overbrace{}^{=1}\underbrace{\sqrt{|g|}\,d^nx}_{=\Omega}$$

$$=-\frac{1}{2}\int\phi(\Box\phi)\,\Omega$$

Klein Gordon wave equation:

We'll need the Euler Lagrange eqn later too. Does it need review?

- Recall: Euler Lagrange equation $\frac{\partial\mathcal{L}}{\partial\phi}=\frac{\partial}{\partial x^\mu}\frac{\partial\mathcal{L}}{\partial\phi_{,\mu}}$
- Here: $\mathcal{L}=-\frac{1}{2}\phi\Box\phi$ (the 0-form that we are integrating: $S=\int_M\mathcal{L}\,\Omega$)
- Obtain Klein Gordon equation:



D The Laplacian/d'Alembertian, Δ, $\Box$:

- Definition of the Laplacian:

$$\boxed{\Delta := \delta d + d\delta}$$

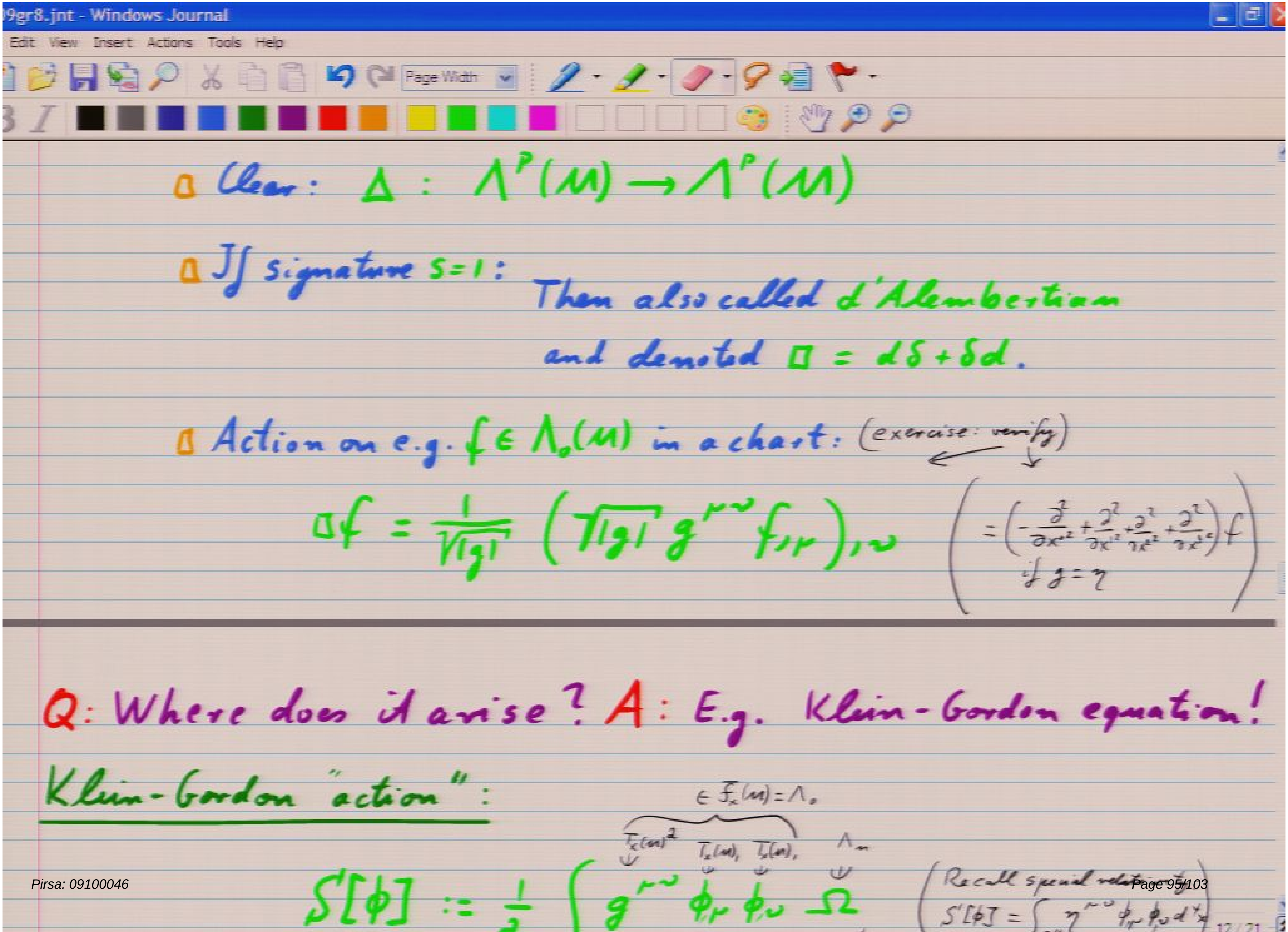
(Some authors define Δ as the negative of this)

- Clear: $\Delta : \Lambda^p(M) \to \Lambda^p(M)$

- If signature $s=1$: Then also called d'Alembertian and denoted $\Box = d\delta + \delta d$.

- Action on e.g. $f \in \Lambda_0(M)$ in a chart: (exercise: verify)

$$\Box f = \frac{1}{\sqrt{|g|}} \left(\sqrt{|g|}\, g^{\mu\nu} f_{,\mu}\right)_{,\nu} \qquad \left(= \left(-\frac{\partial^2}{\partial x^{0\,2}} + \frac{\partial^2}{\partial x^{1\,2}} + \frac{\partial^2}{\partial x^{2\,2}} + \frac{\partial^2}{\partial x^{3\,2}} \right) f \right.$$



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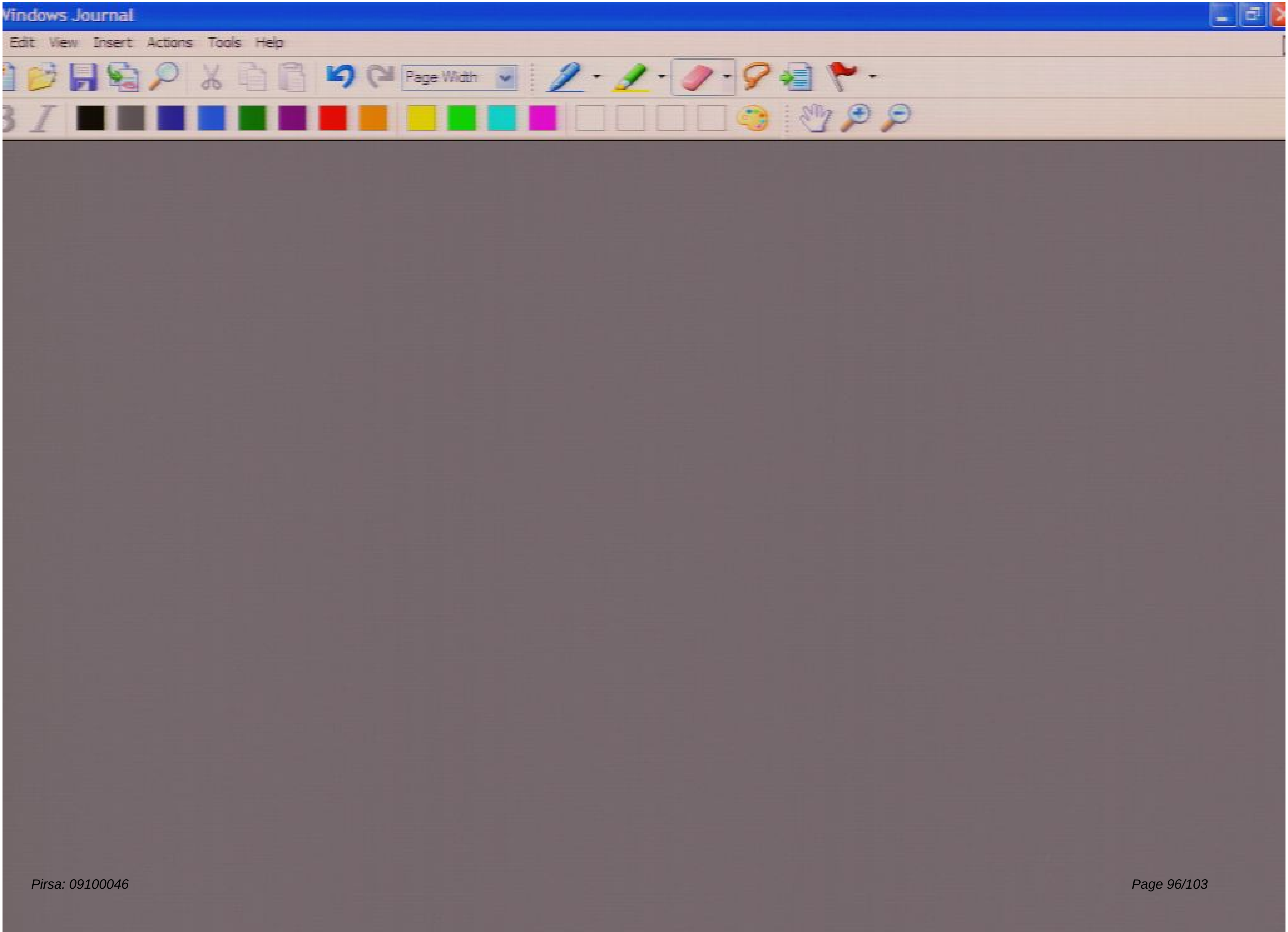
Q: Where does it arise? A: E.g. Klein-Gordon equation!

Klein-Gordon "action":

$$S[\phi] := \frac{1}{2}\int \overbrace{\underset{\substack{\downarrow}}{\overset{T_x(M)^2}{g^{\mu\nu}}}\;\overset{T_x(M),}{\phi_{,\mu}}\;\overset{T_x(M),}{\phi_{,\nu}}}^{\in F_x(M) = \Lambda_0}\;\overset{\Lambda_n}{\Omega}$$

(Recall special relativity
$S[\phi] = \int \eta^{\mu\nu}\phi_{,\mu}\phi_{,\nu}\, d^4x$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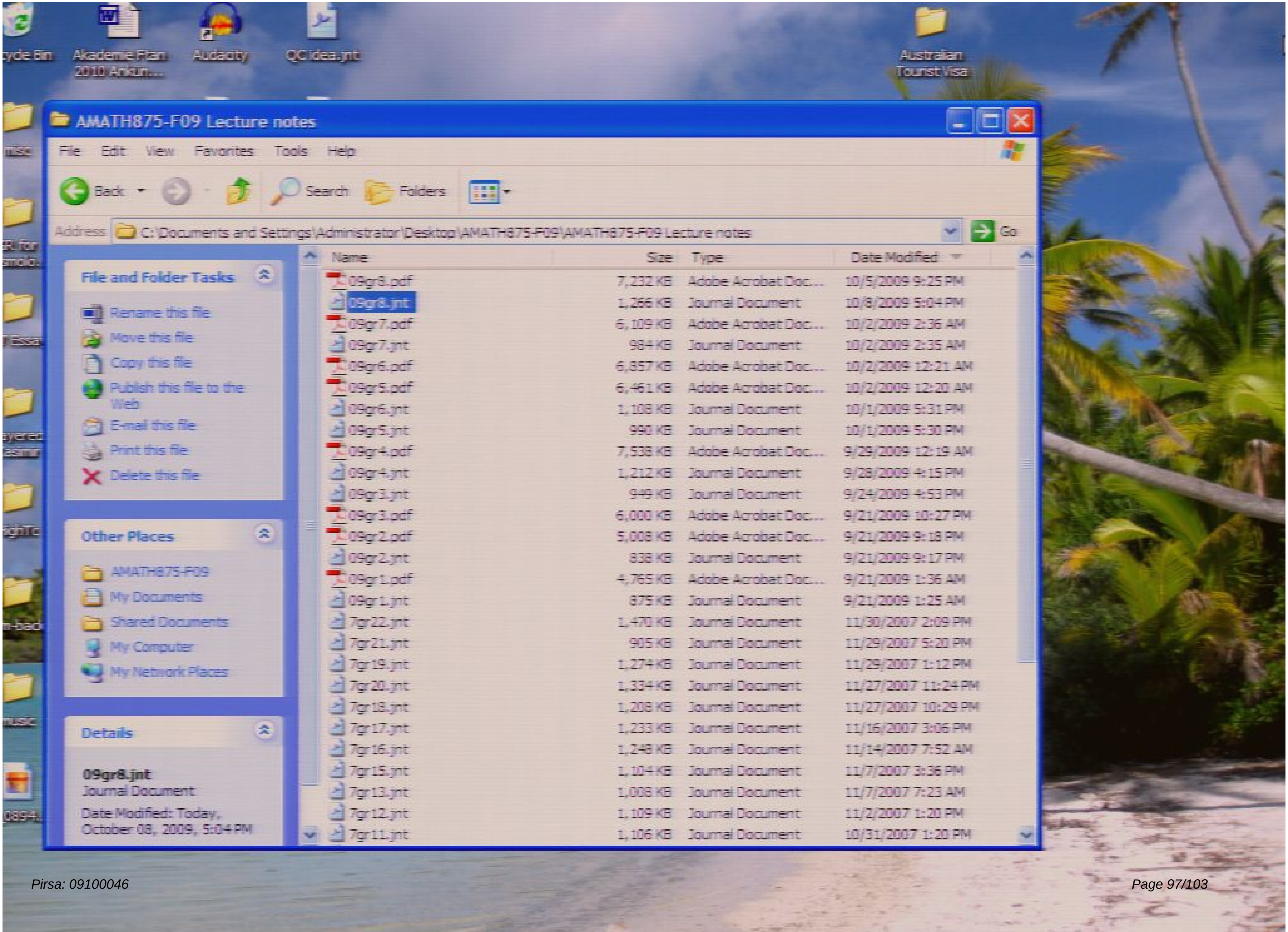

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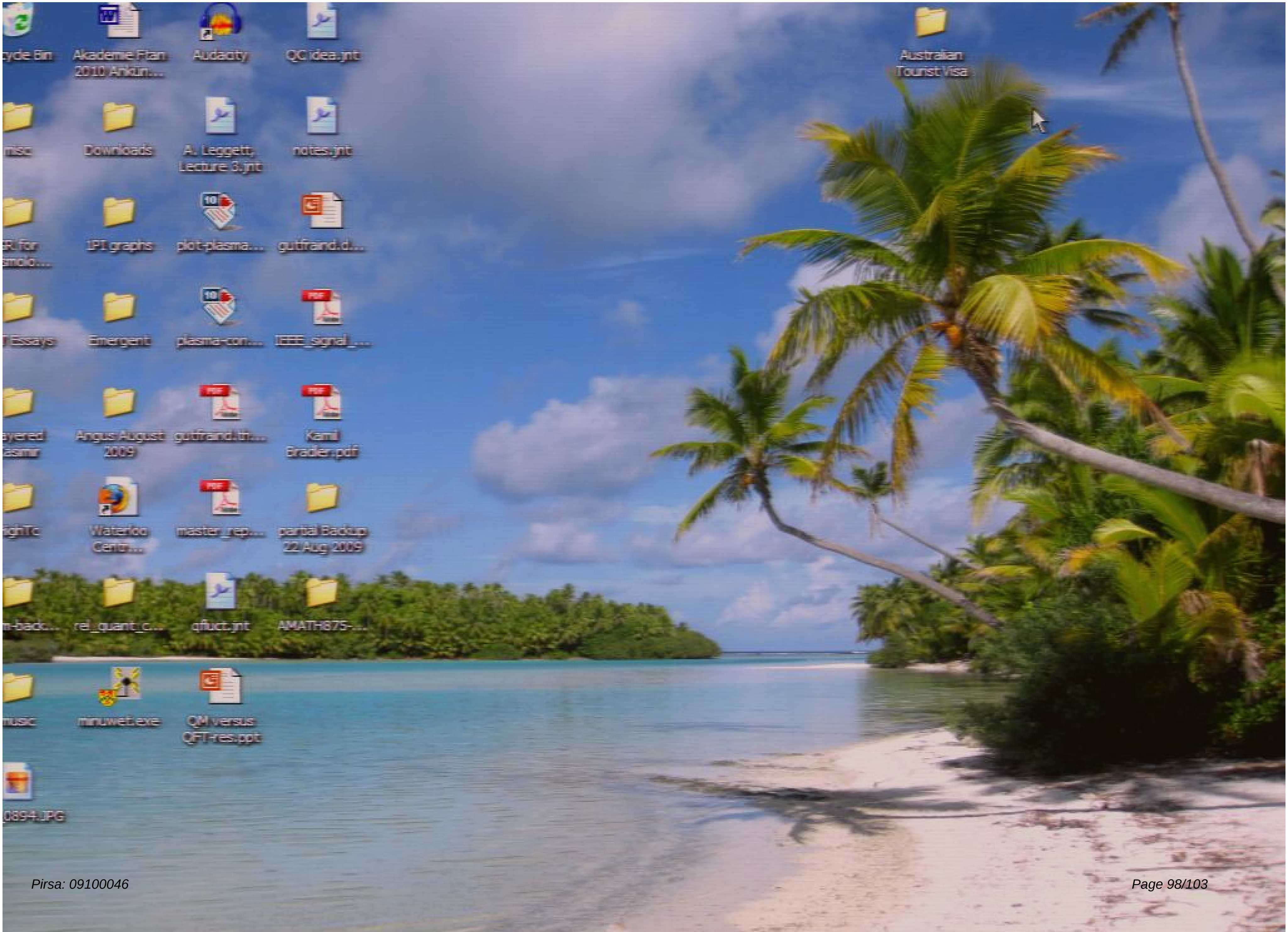
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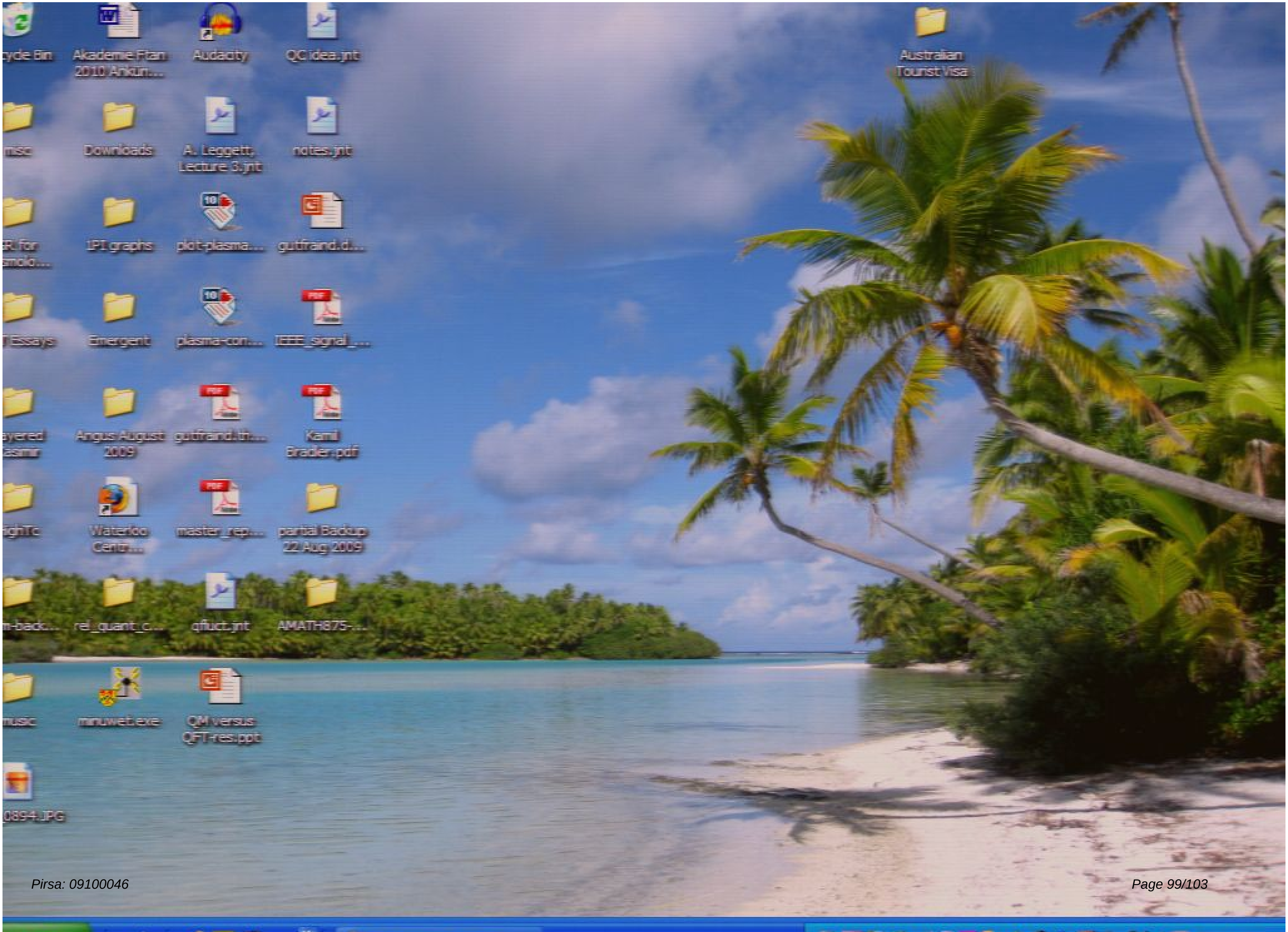

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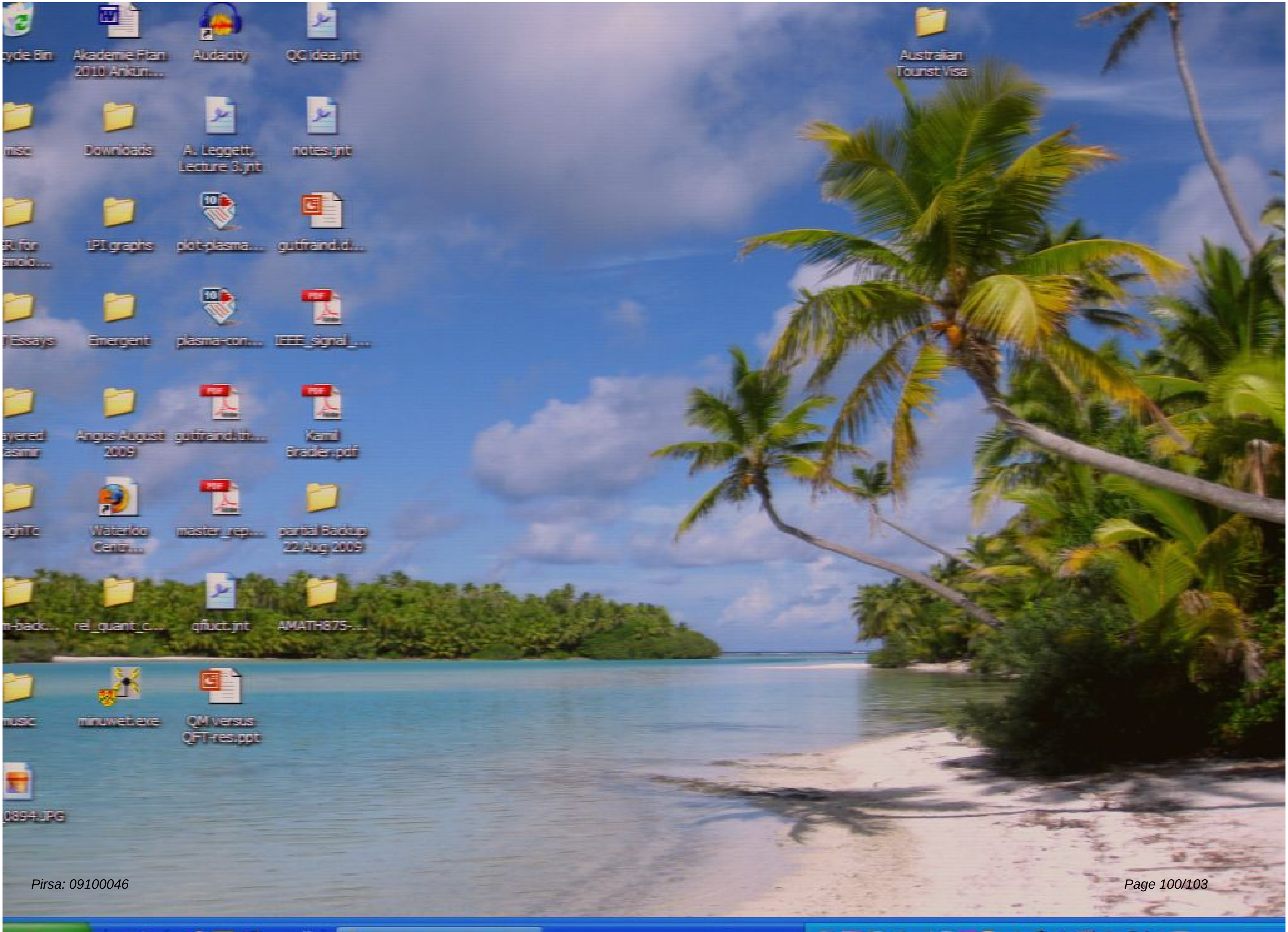

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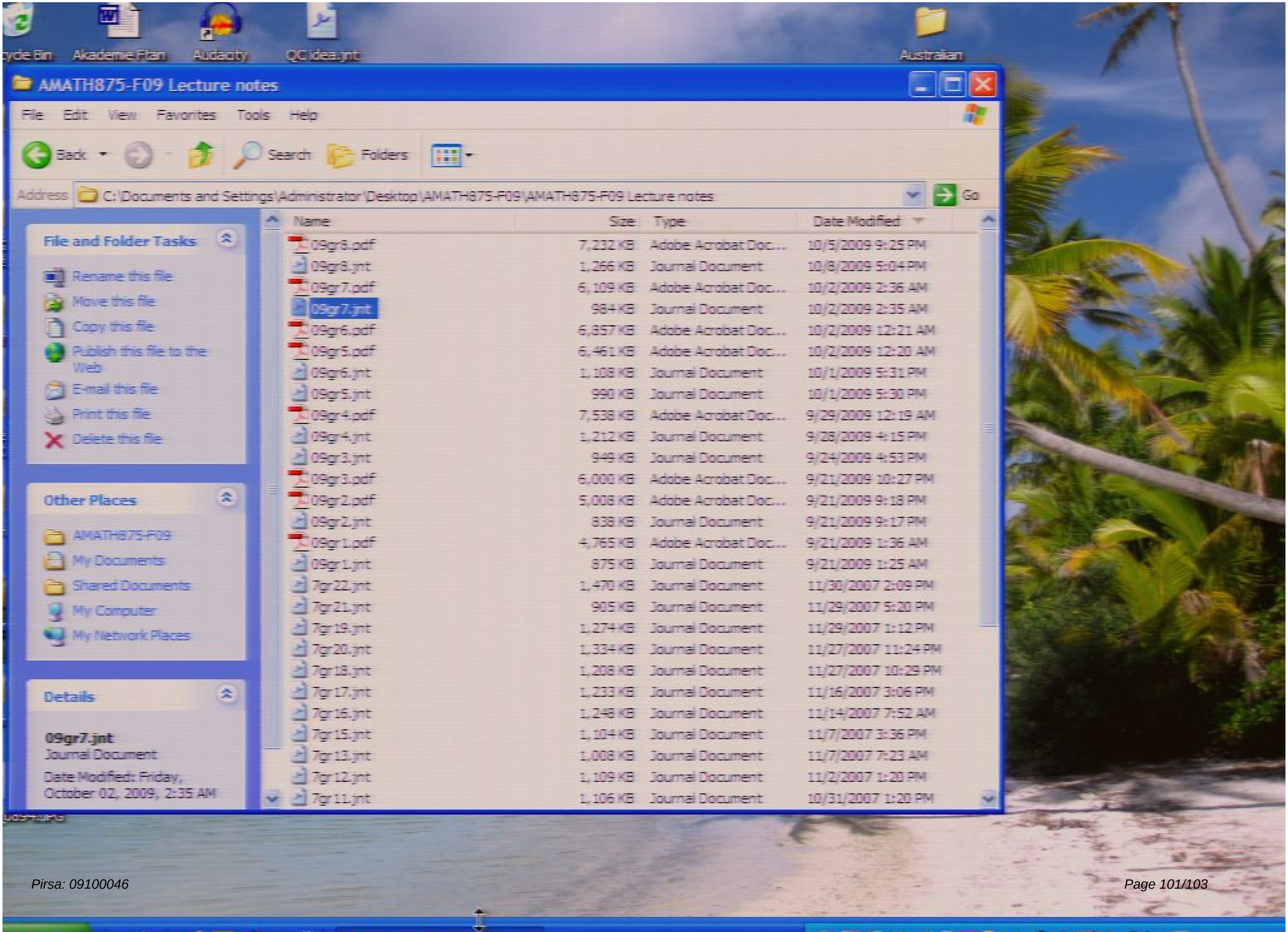

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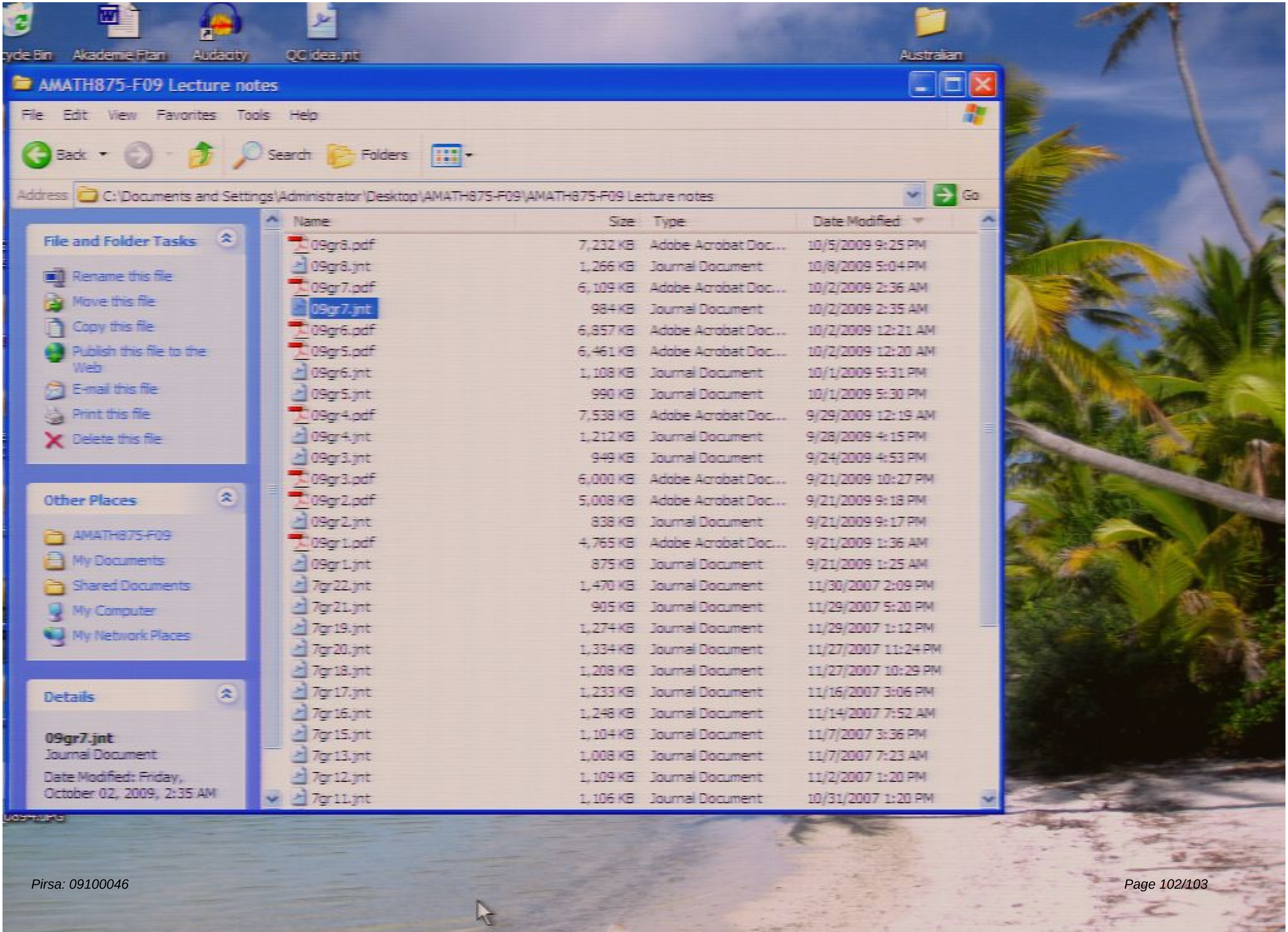

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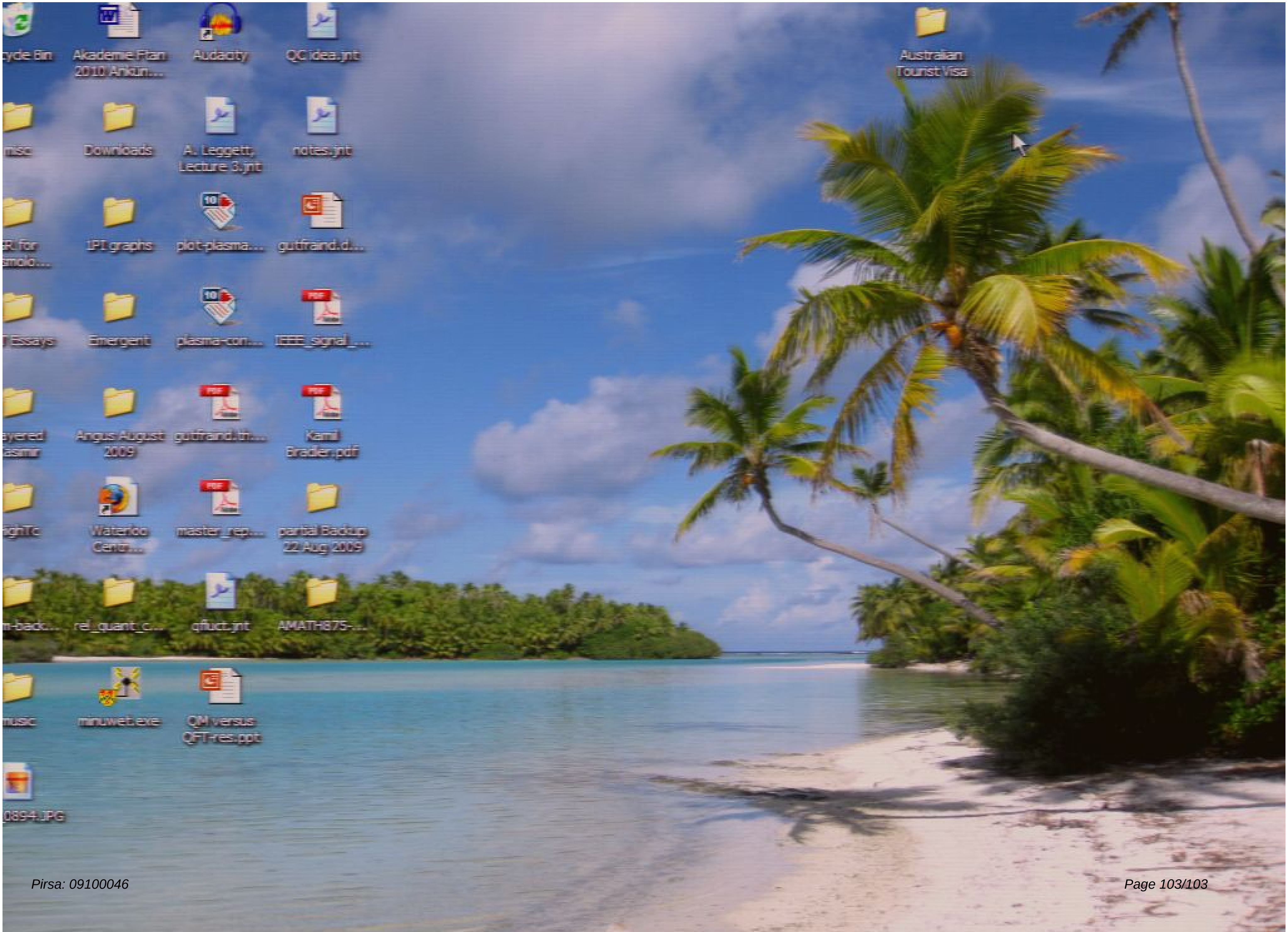
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