Title: General Relativity for Cosmology - Lecture 11

Date: Oct 26, 2009 04:00 PM

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Abstract:

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GR for Cosmology, Achim Kempf, Fall 2009, Lecture 11

Recall:

- The curvature map, R, is defined through:

$$R:\ \xi_1,\xi_2,\xi_3 \to R(\xi_1,\xi_2)\xi_3 = \underbrace{\left(\nabla_{\xi_1}\nabla_{\xi_2} - \nabla_{\xi_2}\nabla_{\xi_1} - \nabla_{[\xi_1,\xi_2]}\right)}_{R(\xi_1,\xi_2)}\xi_3$$

So, "R" can stand for the tensor, the map and this R!

- 1st Bianchi Identity:

$$\sum_{\text{cyclic}} R(\xi,\eta)\nu = \sum_{\text{cyclic}}\Big(T(T(\xi,\eta),\nu) + (\nabla_\xi T)(\eta,\nu)\Big)$$

- 2nd Bianchi Identity:

$$\sum\Big((\nabla_\xi R)(\eta,\nu) + R(T(\xi,\eta),\nu)\Big) = 0$$

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$$\sum_{\text{cyclic}} \Big((\nabla_\xi R)(\eta, \nu) + R(T(\xi,\eta), \nu) \Big) = 0$$

In a chart?

(Assuming no torsion, and using $\frac{\partial}{\partial x^i}$, dx^i bases)

1st Bianchi: $\sum_{(jkl)} R^i{}_{jkl} = 0$

cyclic sum

2nd Bianchi: $\sum_{(klm)} R^i{}_{jkl;m} = 0$

Other useful properties:

(Note: This antisymmetry will be useful

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Other useful properties:

(Note: This antisymmetry will be useful because it allows one to view R as a 2 form, which is (1,1) tensor-valued)

- $R^i{}_{jkl} = -R^i{}_{jlk}$
- $R_{ijkl} = -R_{jikl}$ $\quad$ $\langle R(\xi,\eta)v,\zeta\rangle = \langle R(\xi,\eta)\zeta, v\rangle$

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Note Title 10/21/2009

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$$\langle R(\xi,\eta)v,\zeta\rangle = \langle R(\xi,\eta)\zeta, v\rangle$$

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Contractions of R:

The Ricci Tensor: $R_{jl} := R^i{}_{jil}$

clearly: $R_{jl}\, dx^j \otimes dx^l \in T_p(M)_2$

The Curvature Scalar: $R := g^{jl} R_{jl}$

Then, 2nd Bianchi identity implies:

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$$\left(R_i{}^k - \frac{1}{2}\delta^k_i R\right)_{;k} = 0$$

$\Rightarrow$ The so-called "Einstein tensor" $G_i{}^k := R_i{}^k - \frac{1}{2}\delta_i^k R$ obeys:

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(this property was crucial guidance for Einstein, as we will see

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Recall strategy:

- Specified g $\Rightarrow$ specified distances in M
  $\Rightarrow$ implicitly specified "shape" of M

Then, alternatively:

- Specified ∇ $\Rightarrow$ specified parallel transport in M
  $\Rightarrow$ specified "shape" of M, namely:
  ∇ specifies Torsion T and Curvature R.

Now assume a manifold is specified by giving a metric g.

There ought to exist a ∇ which describes the same manifold.

How does g determine Γ?

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Idea: The parallel transport of vectors η, ν must be such that their inner product (i.e. their lengths and relative angles) stays constant:

Consider any path γ and any two vector fields η, ν that are $\underbrace{\text{parallel transported}}_{\text{(i.e., autoparallel to } \gamma)}$ along γ, i.e., for which:

$$\nabla_{\dot\gamma}\, \eta(\gamma(t)) = 0, \quad \nabla_{\dot\gamma}\, \nu(\gamma(t)) = 0 \quad \text{for all } t.$$

Then, require:

$$\frac{d}{dt}\Big(g(\gamma(t))_{bc}\, \eta^b(\gamma(t))\, \nu^c(\gamma(t)) \Big) = 0$$

i.e.:

$$0 = \underbrace{\dot\gamma^a \left(g_{bc}\,\eta^b \nu^c\right)_{;a}}_{= \nabla_{\dot\gamma}\langle g, \eta \otimes \nu\rangle} \overset{\text{by } \nabla \text{ obeying Leibniz rule}}{=} \dot\gamma^a \Big(g_{bc;a}\,\eta^b \nu^c + g_{bc}\,\underbrace{\eta^b{}_{;a}}_{\substack{0 \\ \| \\ \text{because } \nabla_{\dot\gamma}\eta = 0}} \nu^c + g_{bc}\,\eta^b \underbrace{\nu^c{}_{;a}}_{\substack{0 \\ \| \\ \text{because } \nabla_{\dot\gamma}\nu = 0}} \Big)$$

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by ∇ obeying Leibniz rule

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i.e.: $0 = g_{bc;a}\,\xi^a \eta^b v^c$ for all arbitrary ξ, η, v !

$\Longrightarrow$ Compatibility of ∇ with g means:

$$\nabla_\xi g = 0 \quad \text{for all } \xi$$

Is there a ∇ for each choice of g? Indeed:

Fund. theorem of (pseudo) Riemannian geometry:

For each (pseudo)Riemannian manifold (M, g) there exists a unique ∇ that is torsionless and compatible with g, i.e., which obeys $\nabla g = 0$, the Levi-Civita connecti

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take: $\frac{1}{2}\left(-\mathrm{I} + \mathrm{II} + \mathrm{III}\right)$

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take: $\frac{1}{2}\left(-\underline{\mathrm{I}} + \underline{\mathrm{II}} + \underline{\mathrm{III}}\right)$

$$\Rightarrow \quad \frac{1}{2}\left(g_{\alpha\mu,\nu} + g_{\nu\alpha,\mu} - g_{\mu\nu,\alpha}\right) = g_{\alpha\beta}\Gamma^{\beta}{}_{\nu\mu}$$

Thus:

$$\Gamma^{\beta}{}_{\nu\mu} = \frac{1}{2} g^{\alpha\beta}\left(g_{\alpha\mu,\nu} + g_{\nu\alpha,\mu} - g_{\mu\nu,\alpha}\right)$$

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Finally:

Let us now reformulate these results on a more abstract level:

- Allow arbitrary bases e_i, θ^i in (co-) tangent spaces: frames

- Allow forms to be tensor-valued: obtain, e.g., torsion and curvature forms. Also: connection forms.

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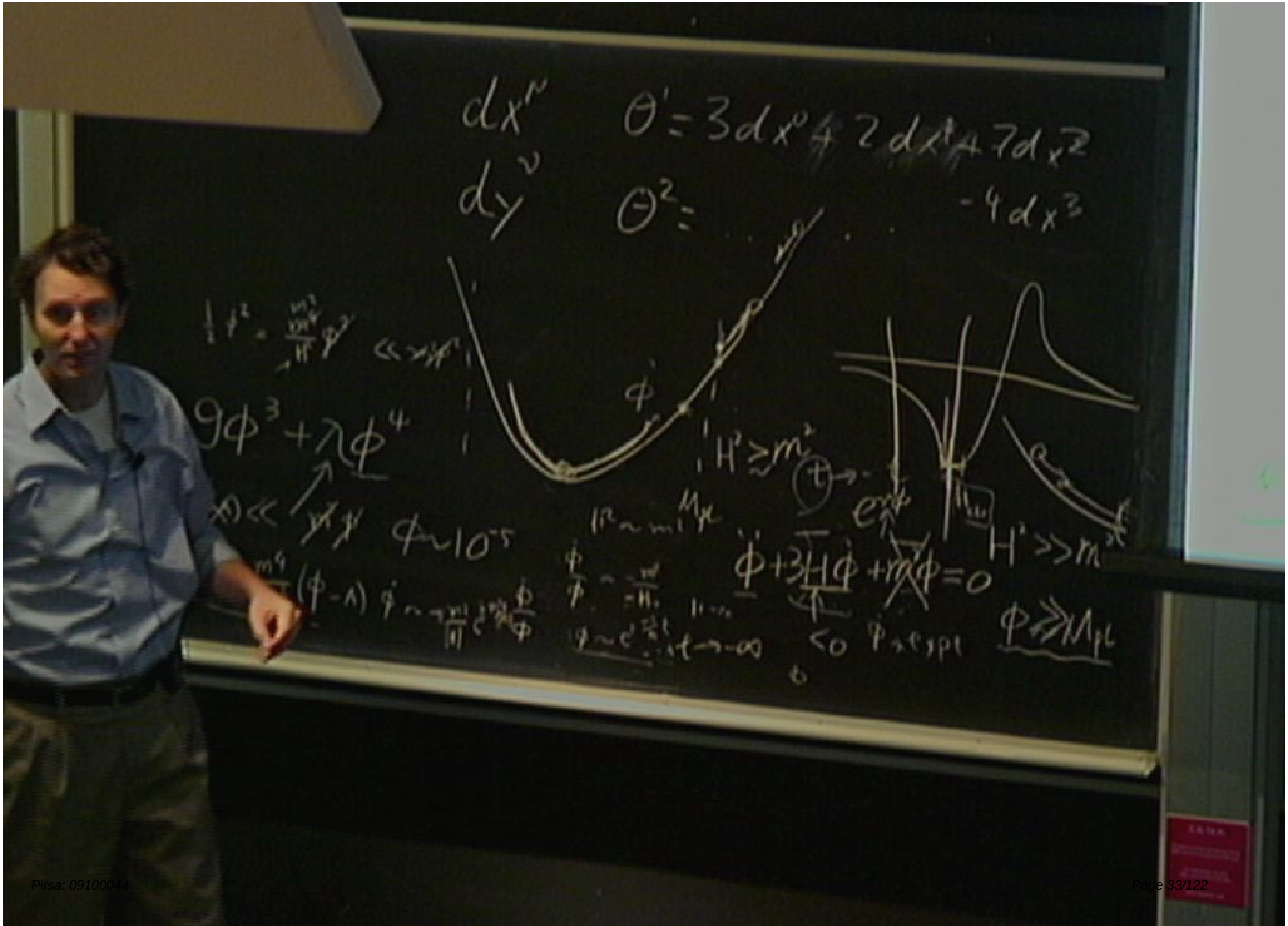
- Allow arbitrary bases e_i, θ^i in (co-) tangent spaces: frames
- Allow forms to be tensor-valued: obtain, e.g., torsion and curvature forms. Also: connection forms.

$\Rightarrow$ We will obtain powerful, simple equations that relate ∇, g, R, T. (Even the Bianchi identities will look simple)

Now: Assume again that ∇ and g are still unrelated and $T \neq 0$.



dx^μ
dy^ν
$\theta^1 = 3dx^0 + 2dx^1 + 7dx^2 - 4dx^3$
$\theta^2 =$
$g\phi^3 + \lambda\phi^4$
$\phi \sim 10^{-5}$
$H^2 \geq m^2$
$H^2 >> m^2$
$\ddot\phi + 3H\dot\phi + m^2\phi = 0$
$\phi >> M_{pl}$

"Moving frames":

Def: A moving frame is a set, $\{e_i\}_{i=1}^{n}$, of contravariant vector fields e_i which, together, at each point $p \in M$ form a basis of $T_p(M)$'.

Def: We denote the dual basis $\{\theta^i\}_{i=1}^{n}$.

Def: For $n=4$ it may be called vierbein (german: 4 legs.) or tetrad.

Def: Since θ^i is a 1-form, $d\theta^i$ is a 2-form. We denote the expansion coefficients by functions C^i_{jk}:

$$d\theta^i = \underbrace{-\tfrac{1}{2}}_{\text{convention}} C^i_{jk} \, \underbrace{\theta^j \wedge \theta^k}_{\text{basis for space of all 2 forms}} \quad \text{with} \quad \underbrace{C^i_{jk} = -C^i_{kj}}_{\text{the symm. part would drop out}}$$

coefficient functions depend on choice of frame

□ choice $\theta^i = dx^i$ would yield $d\theta^i = 0$ i.e. all $C^i_{jk} = 0$.

□ other choices, say $\theta^i = \ldots$

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$$\Theta^i(x) = \underbrace{\alpha^i{}_j(x)}\, dx^j$$

$$d\Theta^i(x) \neq 0$$

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□ choice $\theta^i = dx^i$ would yield $d\theta^i = 0$ i.e. all $C^i_{jk} = 0$.

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coefficient functions depend on choice of frame

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Def: A moving frame is a set, $\{e_i\}_{i=1}^{n}$, of contravariant vector fields e_i which, together, at each point $p \in M$ form a basis of $T_p(M)'$.

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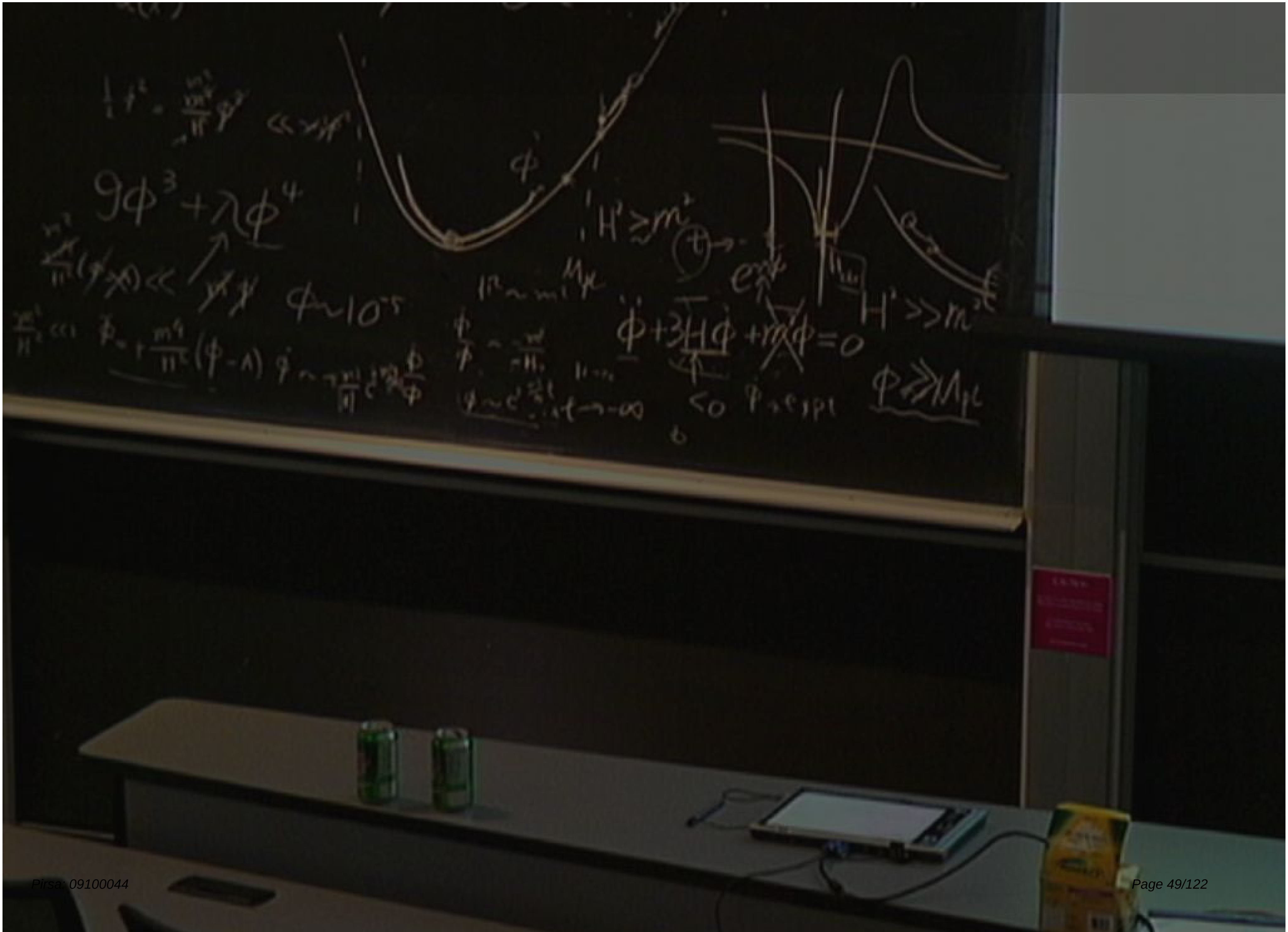
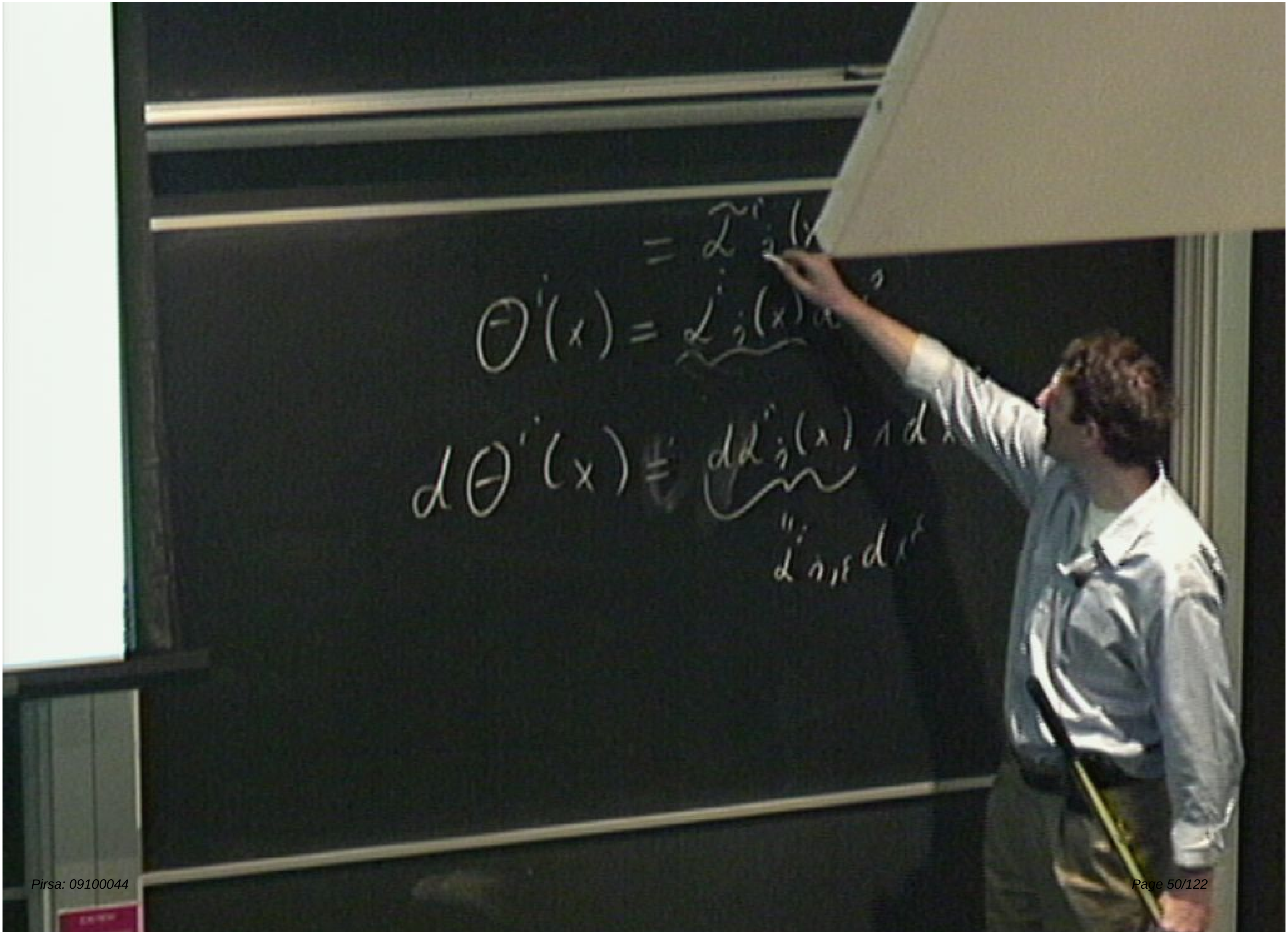

$g\phi^3 + \lambda\phi^4$
$\phi \sim 10^{-5}$
$H^2 > m^2$
$H^2 >> m^2$
$\ddot{\phi} + 3H\dot{\phi} + m^2\phi = 0$
$\phi >> M_{pl}$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$$= \tilde{\alpha}^i{}_j(y)\, dy^j$$

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Another step towards more abstract formulation:

Tensor-valued p-forms:

Def: A (r,s)-tensor-valued p-form ϕ is an anti-symmetric p-multilinear mapping at each $q \in M$:

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Special cases:

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- We recall that $T(\xi,\eta) = -T(\eta,\xi) \Rightarrow$ can define the torsion's (1,0) tensor-valued 2-form through its action on 2 vector fields ξ,η:

"torsion 2-form"

$$\underbrace{\Theta^i(\xi,\eta)\,e_i}_{\text{the 2 form } \Theta^i \text{ fed 2 vectors to yield a vector}} := T(\xi,\eta)$$

- Given a frame:

using their antisymmetry

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Curvature 2-form:

recall that in canonical basis:

$R^i{}_{jkl} = - R^i{}_{jlk}$

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Recall: $R: \xi \eta e_j \to \nabla_\xi \nabla_\eta e_j - \nabla_\eta \nabla_\xi e_j - \nabla_{[\eta,\xi]} e_j$

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Proposition: cov. deriv. for covectors reads

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Thus: $\underbrace{\nabla_\xi e_j}_{\text{vector}} = \underbrace{\overbrace{\omega^i{}_j(\xi)}^{\text{numbers}}\, e_i}_{\text{vector}}$. (because $\nabla_{\xi^k e_k} e_j = \xi^k \nabla_{e_k} e_j$)

Proposition: cov. deriv. for covectors reads

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Leibniz rule and using that ∇ and contractions commute

Proof: $0 = \nabla_\xi \underbrace{\langle \theta^i, e_j \rangle}_{=\delta^i_j} = \langle \nabla_\xi \theta^i, e_j \rangle + \langle \theta^i, \nabla_\xi e_j \rangle$

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so can't be a 2-form (but can be 1-form):

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→ indeed:

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$$\nabla_\xi \theta^i = -\omega^i_{\ l}(\xi)\,\theta^l$$

(contract with $\langle\ ,e_j\rangle$ to verify that this is Eq. (*))

✓

Connection 1-forms are non-tensorial:

- Under change of frame $\bar\theta^i(x) = A^i_{\ j}(x)\,\theta^j(x)$ the transformation is:

$$\underbrace{\bar\omega^a_{\ b}}_{\text{1-form}} = \underbrace{A^a_{\ i}\,\overbrace{\omega^i_{\ j}}^{\text{1-form}}\,A^{-1\,j}_{\ \ \ b}}_{\text{1-form}} - \underbrace{\underbrace{(dA)^a_{\ i}}_{\text{1-form}}\,\underbrace{(A^{-1})^i_{\ b}}_{\text{functions}}}_{\text{1-form}}$$

functions   function   matrix inverse

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$$-\bar{\omega}(\xi)\bar{\theta} = \nabla_\xi \bar{\theta} = \nabla_\xi (A\theta) \overset{\text{Leibniz}}{=} \big(dA(\xi)\big)\theta + A\nabla_\xi \theta$$

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functions, 1 form, function, matrix inverse.

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Leibniz

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Back to tensor-valued p-forms:

The "absolute exterior differential" D:

(It generalizes both ∇ and d)

Proposition: (proof, see e.g. Straumann: check tensorial behavior under frame change)

For every (r,s) tensor-valued p-form ϕ there exists a unique (r,s) tensor-valued $(p+1)$ form $D\phi$ whose components relative to $\{\theta^i\}$ are:

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How are $\omega, g, \Theta, \Omega$ related now?

Proposition:

An affine connection ∇ is metric,
if and only if $Dg = 0$,

Exercise: check

(0,2) tensor-valued 0-form

i.e., iff: $\underbrace{dg_{ik}}_{(0,2)\text{ tensor-valued 1-form}} - \omega_{ik} - \omega_{ki} = 0$

Theorem: "The Cartan structure equations"

absolute exterior derivative.

$=0$ for metric connection

Torsion $\Theta = \Theta^i e_i$ is $(1,0)$ tensor-valued 2-form

special case of frame $\theta^i = dx^i$:

$T^i_{kj} = \Gamma^i_{kj} - \Gamma^i_{jk}$

1.) $\Theta^i = d\theta^i + \omega^i{}_j \wedge \theta^j$ i.e. $\Theta^i = D\theta^i$

(The frame, $\theta = \theta^i e_i$ is a $\underbrace{(1,0)\text{ tensor-valued } 1\text{-f}}$
notice the upper index

$$D = d$$

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1-form 1-form 1-form functions
1-form

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Proposition: D is an anti-derivation: degree of ϕ

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degree of ϕ

$$D(\phi \wedge \psi) = D\phi \wedge \psi + (-1)^p \phi \wedge D\psi$$

Special cases:

- An ordinary p-form is $(0,0)$ tensor-valued. In this case, clearly:

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Theorem: "The Cartan structure equations"

absolute exterior derivative

= 0 for metric connection

Torsion $\Theta = \theta e_i$ is (1,0) tensor-valued 2-form

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How are ω, g, Θ, Ω related now?

Proposition:

An affine connection ∇ is metric,

if and only if $Dg = 0$,

Exercise: check

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Proof of 2.:

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Theorem: The Cartan structure equations

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Proof of 2.:

$$\Omega^i{}_j(\xi,\eta)e_i = \nabla_\xi \nabla_\eta e_j - \nabla_\eta \nabla_\xi e_j - \nabla_{[\xi,\eta]} e_j$$

$$= \nabla_\xi\left(\omega^i{}_j(\eta)e_i\right) - \nabla_\eta\left(\omega^i{}_j(\xi)e_i\right) - \omega^i{}_j([\xi,\eta])e_i$$

Leibniz

$$= \left(\xi\left(\omega^i{}_j(\eta)\right) - \eta\left(\omega^i{}_j(\xi)\right) - \omega^i{}_j([\xi,\eta])\right)e_i$$

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Theorem: "The Cartan structure equations"

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exercise: check

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$R^i{}_{jkl} = \Gamma^i{}_{lj,k} - \Gamma^i{}_{kj,l} + \Gamma^s{}_{lj}\Gamma^i{}_{ks} - \Gamma^s{}_{kj}\Gamma^i{}_{ls}$

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$$\stackrel{\text{Leibnitz}}{=} \left(\xi\left(\omega^i{}_j(\eta)\right) - \eta\left(\omega^i{}_j(\xi)\right) - \omega^i{}_j([\xi,\eta])\right)e_i$$

$$+ \left(\omega^i{}_j(\eta)\,\omega^k{}_i(\xi) - \omega^i{}_j(\xi)\,\omega^k{}_i(\eta)\right)e_k$$

Exercise: Fill in all steps.

$$= d\omega^i{}_j(\xi,\eta)e_i + \left(\omega^i{}_k \wedge \omega^k{}_j\right)(\xi,\eta)e_i$$

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$$\Omega^i{}_j(\xi,\eta)e_i = \nabla_\xi \nabla_\eta e_j - \nabla_\eta \nabla_\xi e_j - \nabla_{[\xi,\eta]} e_j$$

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$$= d\omega^i{}_j(\xi,\eta)e_i + (\omega^i{}_k \wedge \omega^k{}_j)(\xi,\eta)e_i$$

true for all $\xi, \eta, e_i \Rightarrow$ ✓

Use of the Cartan Structure equations?

- Allow proof of simple formulation of the Bianchi identities:

  1st Bianchi: $$D\Theta^i = \Omega^i{}_j \wedge \theta^j$$

  2nd Bianchi: $$D\Omega^i{}_j = 0$$

- Thus, for metric connection, i.e. when ← i.e. "Riemannian", i.e. "Levi-Civita"

  $$dg_{ik} = \omega_{ik} + \omega_{ki} \qquad (\text{same as } \nabla g = 0)$$

  then:
  $$\boxed{\begin{aligned} \Omega^i{}_j \wedge \theta^j &= 0 \\ D\Omega^i{}_j &= 0 \end{aligned}}$$

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Proposition:

- In the case of metric connection, the Cartan equations yield for arbitrary bases:

$C^l_{ki} = 0$ in canonical frame $\{dx^i\}$

$$\Gamma^l_{\ ki} = \frac{1}{2}\left(C^l_{\ ki} - g_{is}\, g^{lj} C^s_{\ kj} - g_{ks}\, g^{lj} C^s_{\ ij}\right) + \frac{1}{2}\, g^{lj}\left(g_{ij,k} + g_{jk,i} - g_{ki,j}\right)$$

Recall:

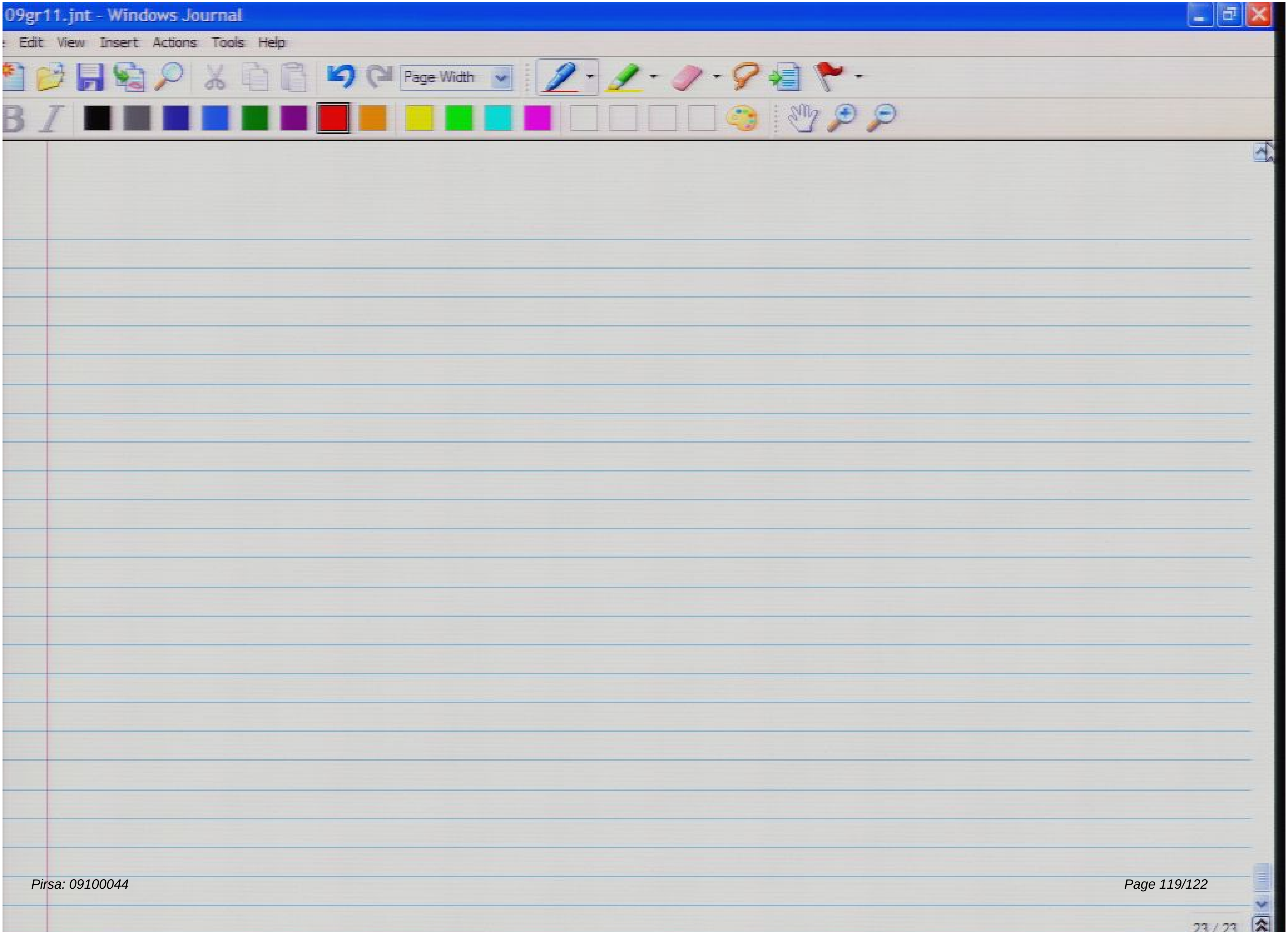
$$d\theta^i = \overbrace{-\frac{1}{2}}^{\text{convention}} C^i_{\ jk}\, \underbrace{\theta^j \wedge \theta^k}_{\text{basis for space of all 2 forms}}$$

coefficient functions depend on choice of frame

- In this case, also:

absent in canonical frame

$$R^i_{\ jkl} = \Gamma^i \ldots - \Gamma^i \ldots + \Gamma^i \Gamma^e \ldots - \Gamma^i \Gamma^e \ldots - \Gamma^i \ldots$$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Proposition:

- In the case of metric connection, the Cartan equations yield for arbitrary bases:

$$\Gamma^{l}{}_{ki} = \frac{1}{2}\Big(\overbrace{C^{l}{}_{ki} - g_{is}\, g^{lj} C^{s}{}_{kj} - g_{ks}\, g^{lj} C^{s}{}_{ij}}^{C^{l}_{ki} = 0 \text{ in canonical frame } \{dx^i\}}\Big)$$

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$$R^{i}{}_{jab} = \Gamma^{i}{}_{bj,a} - \Gamma^{i}{}_{aj,b} + \Gamma^{i}{}_{ae}\Gamma^{e}{}_{bj} - \Gamma^{i}{}_{be}\Gamma^{e}{}_{aj} - \Gamma^{i}{}_{ej}\, C^{e}{}_{\ldots}$$

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$$R^{i}_{\ jab} = \Gamma^{i}_{\ bj,a} - \Gamma^{i}_{\ aj,b} + \Gamma^{i}_{\ a\ell}\Gamma^{\ell}_{\ bj} - \Gamma^{i}_{\ b\ell}\Gamma^{\ell}_{\ aj} - \Gamma^{i}_{\ \ell j}\, C^{\ell}_{\ ab}$$