Title: General Relativity for Cosmology - Lecture 5

Date: Oct 05, 2009 04:00 PM

URL: http://pirsa.org/09100041

Abstract:

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Questions:

Since L_ξ is the directional derivative on $\Lambda(M)$:

- Can L_ξ be extended to a directional derivative for all tensor fields? Yes!

- Can L_ξ be expressed as a Newton-Leibniz limit similar to

need an analog: a shift on a manifold, in the direction given by ξ.

$$f'(x) = \lim_{\varepsilon \to 0} \frac{f(\underbrace{x+\varepsilon}) - f(x)}{\varepsilon}$$? Yes!

To this end:

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The geometric definition of L_ξ:

- Recall that for any path

$$\gamma : \mathbb{R} \supset J \to M$$

an open interval of $\mathbb{R}$

$$\gamma : \quad t \to \gamma(t)$$

we have a tangent vector $\bar{\gamma}(t_0) \in T_{\gamma(t_0)}(M)$ at each point $\gamma(t_0)$ of the path:

$$\bar{\gamma}(t_0) : f \to \bar{\gamma}(t_0)(f) = \frac{d}{dt} f(\gamma(t))\Big|_{t=t_0}$$

(the geom. definition of the tangent space)

Definition: For a given vector field, ξ, a path γ is called an integral curve of ξ, if

$$\dot{\gamma}(t) = \xi(\gamma(t))$$

path's velocity vector at $\gamma(t)$

vector of field ξ at $\gamma(t) \in \mathcal{M}$.

From theory of ODEs:

For every $p \in \mathcal{M}$ there exists a maximal (i.e. inextendible) C^∞ integral curve through p.

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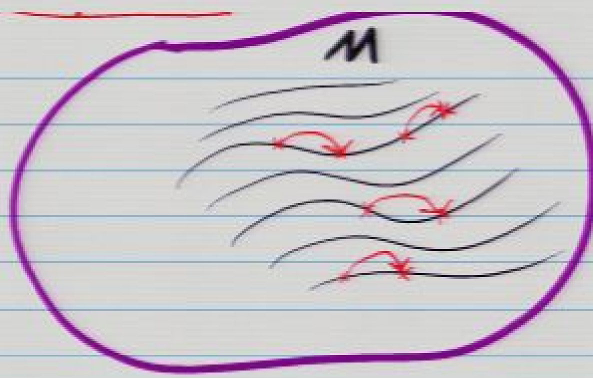
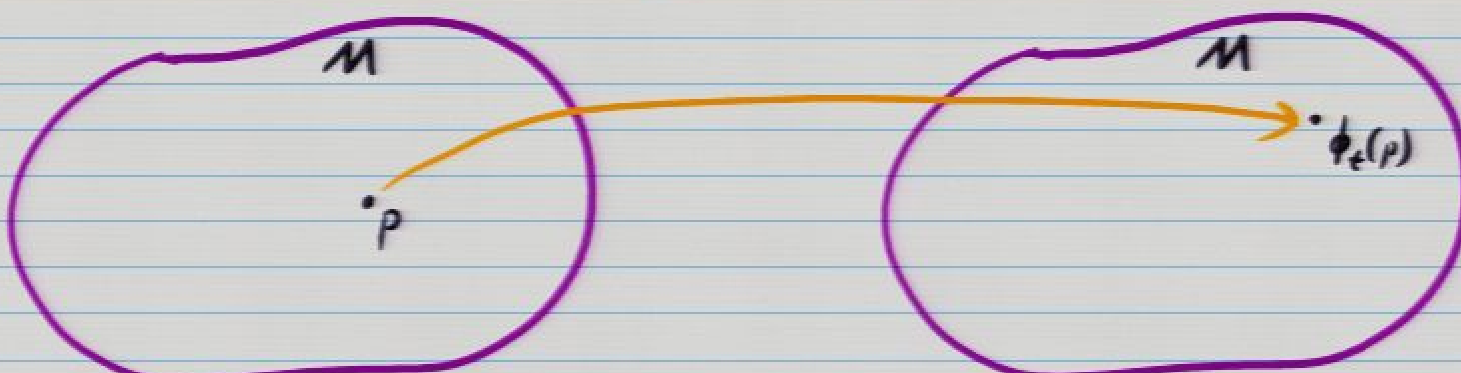


i.e., for any fixed value of the flow parameter t each point of M is mapped into another point of M.

The flow is a diffeomorphism "$\phi_t : M \to M$":



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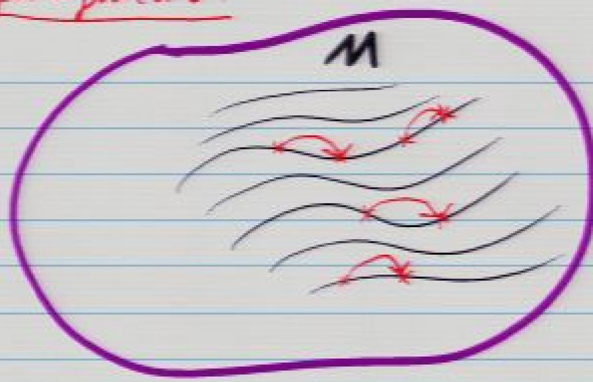
for a fixed t:



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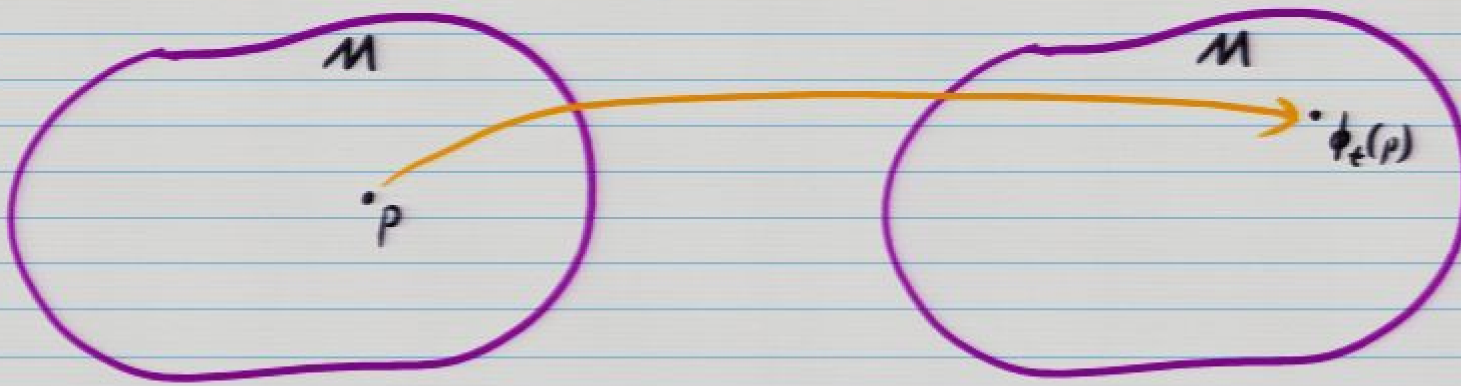

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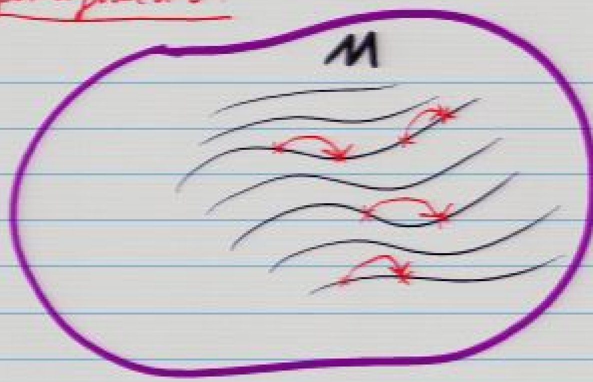

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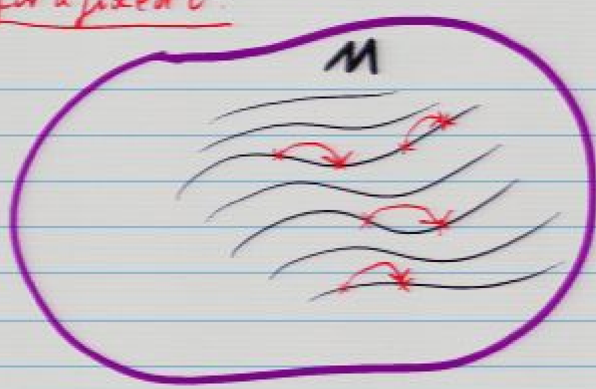

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corresponding isomorphisms of the tangent, cotangent and all tensor spaces at p and at $\phi_t(p)$ respectively:

$$\phi_t^* : T_p(M)^r_s \to T_{\phi_t(p)}(M)^r_s$$

Recall: A tensor field τ assigns to each $p \in M$ a tensor $\tau(p) \in T_p(M)^r_s$.

Definition:

We say that a tensor field τ is invariant under the flow induced by the vector field ξ if:

$$\phi_t^*(\tau(p)) = \tau(\phi_t(p)) \qquad \forall t \, \forall p$$

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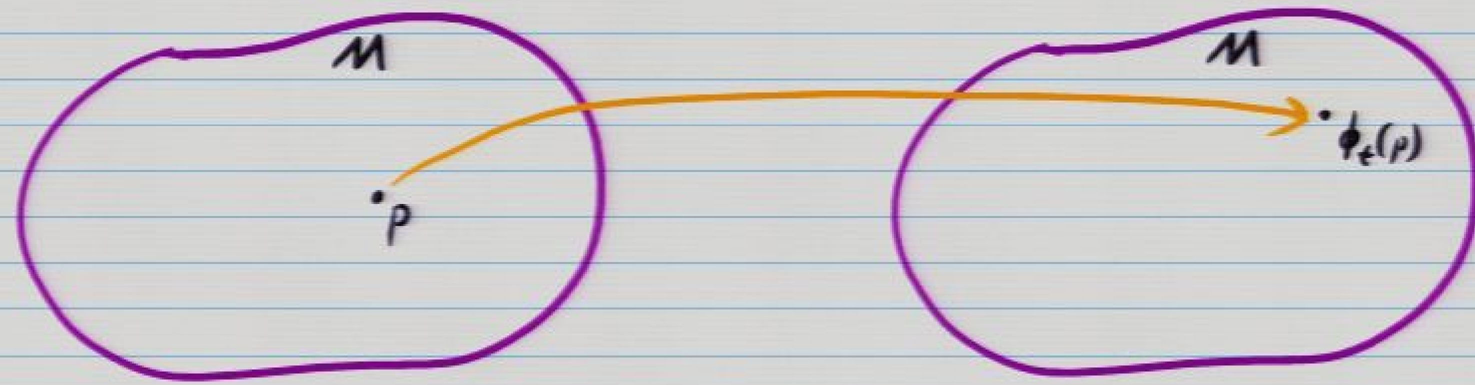

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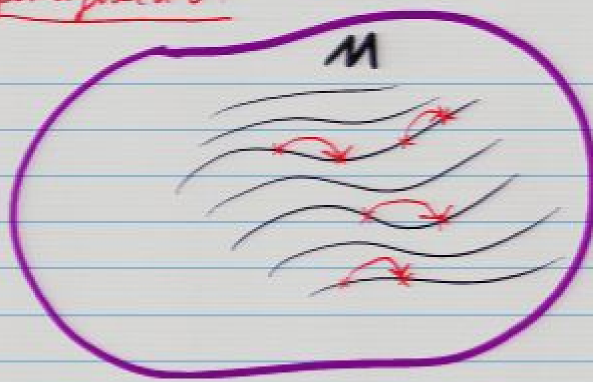

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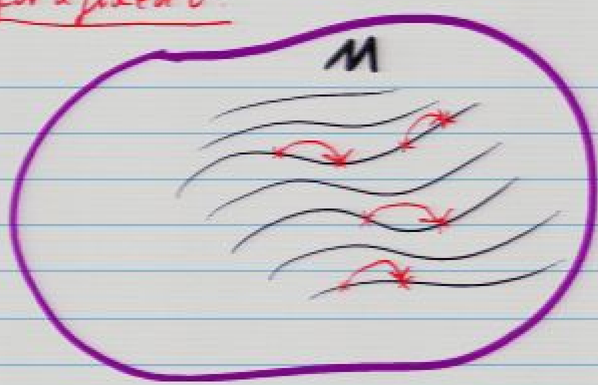

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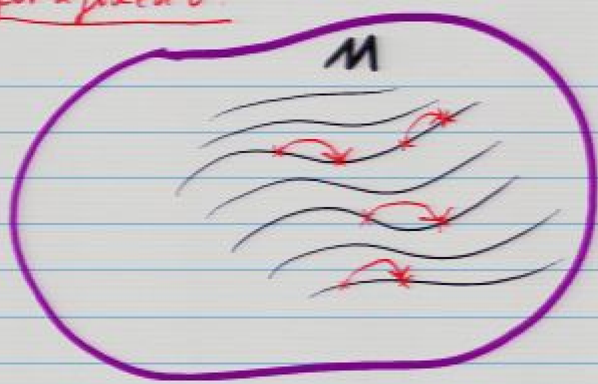

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(The flow produces an image of M in M:)

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The Lie derivative of any tensor field τ at the point $p = \gamma(0) \in M$ with respect to the flow induced by a vector field ξ is defined through:

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$$\mathcal{L}_\xi \tau := \lim_{t \to 0} \frac{1}{t} \left(\phi_t^{*\,-1}(\tau) - \tau \right)$$

tensor field value at image of p, i.e. $\in T_{\gamma(t)}(M)^r_s$

$$\text{i.e. } \mathcal{L}_\xi(\tau)(p) = \lim_{t \to 0} \frac{1}{t} \Big[\underbrace{(\phi^*)^{-1}\big(\overbrace{\tau(\gamma(t))}\big)}_{\in T_p(M)^r_s} - \tau(\underset{=\gamma(0)}{p}) \Big]$$

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Explicitly, in a chart:

- $\phi: x \to \tilde{x}$ with infinitesimal flow: $\tilde{x}^i(x) = x^i + t\,\xi^i(x) + O(t^2)$

- Jacobian matrix: $\dfrac{\partial \tilde{x}^i}{\partial x^j} = \delta^i_j + t\,\dfrac{\partial \xi^i(x)}{\partial x^j} + O(t^2)$

  we write $= \xi^i_{,j}$

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- Image of tensor at $\tau(\tilde{x})^{i_1 \dots i_r}_{j_1 \dots j_s}$ under flow, backwards, $\tilde{x} \to x$, has the components:

From now, we will omit writing Σ: Twice occurring indices are always to be summed over (Einstein convention)

$$\phi^*(\tau(x))^{i_1 \dots i_r}_{j_1 \dots j_s} = \tau^{\bar{i}_1 \dots \bar{i}_r}_{\bar{j}_1 \dots \bar{j}_s}(\tilde{x}) \frac{\partial x^{i_1}}{\partial \tilde{x}^{\bar{i}_1}} \cdots \frac{\partial x^{i_r}}{\partial \tilde{x}^{\bar{i}_r}} \frac{\partial \tilde{x}^{\bar{j}_1}}{\partial x^{j_1}} \cdots \frac{\partial \tilde{x}^{\bar{j}_s}}{\partial x^{j_s}}$$

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▫ Jacobian matrix: $\frac{\partial \tilde{x}^i}{\partial x^j} = \delta^i_j + t\,\frac{\partial \xi^i(x)}{\partial x^j} + O(t^2)$

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▫ Inverse Jacobian: $\frac{\partial x^i}{\partial \tilde{x}^j} = \delta^i_j - t\,\frac{\partial \xi^i(x)}{\partial x^j} + O(t^2)$

▫ Image of tensor at $\tau(\tilde{x})^{i_1 \dots i_r}_{j_1 \dots j_s}$ under flow, backwards, $\tilde{x} \to x$, has the

From now, we will omit writing Σ: Twice occurring indices are always to be summed over (Einstein convention)

components:

$$\phi^*(\tau(x))^{i_1 \dots i_r}_{j_1 \dots j_s} = \tau^{\bar{i}_1 \dots \bar{i}_r}_{\bar{j}_1 \dots \bar{j}_s}(\tilde{x})\, \frac{\partial x^{i_1}}{\partial \tilde{x}^{\bar{i}_1}} \cdots \frac{\partial x^{i_r}}{\partial \tilde{x}^{\bar{i}_r}}\, \frac{\partial \tilde{x}^{\bar{j}_1}}{\partial x^{j_1}} \cdots \frac{\partial \tilde{x}^{\bar{j}_s}}{\partial x^{j_s}}$$

$$= \tau^{\bar{i}_1 \dots \bar{i}_r}_{\bar{j}_1 \dots \bar{j}_s}(x + t\xi)\left(\delta^{i_1}_{\bar{i}_1} - t\,\xi^{i_1}_{,\bar{i}_1}\right) \cdots \left(\delta^{i_r}_{\bar{i}_r} - t\,\xi^{i_r}_{,\bar{i}_r}\right)$$

$$\cdot\left(\delta^{\bar{j}_1}_{j_1} + t\,\xi^{\bar{j}_1}_{,j_1}\right) \cdots \left(\delta^{\bar{j}_s}_{j_s} + t\,\xi^{\bar{j}_s}_{,j_s}\right) + O(t^2)$$

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- Inverse Jacobian: $\frac{\partial x^i}{\partial \tilde{x}^j} = \delta^i_j - t\frac{\partial \xi^i(x)}{\partial x^j} + O(t^2)$

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$$\cdot\left(\delta^{\bar{j}_1}_{j_1} + t\xi^{\bar{j}_1}_{,j_1}\right)\cdots\left(\delta^{\bar{j}_s}_{j_s} + t\xi^{\bar{j}_s}_{,j_s}\right) + O(t^2)$$

$$= \tau^{i_1\dots i_r}_{j_1\dots j_s}(x) + t\,\tau^{i_1\dots i_r}_{j_1\dots j_s,k}(x)\,\xi^k(x)$$

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From now, we will omit writing $\sum$: twice occurring indices are always to be summed over (Einstein convention)

components:

$$\phi^{*}(\tau(x))^{i_1\ldots i_r}_{j_1\ldots j_s} = \tau^{\bar{i}_1\ldots\bar{i}_r}_{\bar{j}_1\ldots\bar{j}_s}(\tilde{x})\,\frac{\partial x^{i_1}}{\partial\tilde{x}^{\bar{i}_1}}\cdots\frac{\partial x^{i_r}}{\partial\tilde{x}^{\bar{i}_r}}\,\frac{\partial\tilde{x}^{\bar{j}_1}}{\partial x^{j_1}}\cdots\frac{\partial\tilde{x}^{\bar{j}_s}}{\partial x^{j_s}}$$

$$= \tau^{\bar{i}_1\ldots\bar{i}_r}_{\bar{j}_1\ldots\bar{j}_s}(x+t\xi)\left(\delta^{i_1}_{\bar{i}_1}-t\xi^{i_1}_{,\bar{i}_1}\right)\cdots\left(\delta^{i_r}_{\bar{i}_r}-t\xi^{i_r}_{,\bar{i}_r}\right)$$

$$\cdot\left(\delta^{\bar{j}_1}_{j_1}+t\xi^{\bar{j}_1}_{,j_1}\right)\cdots\left(\delta^{\bar{j}_s}_{j_s}+t\xi^{\bar{j}_s}_{,j_s}\right)+O(t^2)$$

$$= \tau^{i_1\ldots i_r}_{j_1\ldots j_s}(x) + t\,\tau^{i_1\ldots i_r}_{j_1\ldots j_s,k}(x)\,\xi^{k}(x)$$

$$- t\,\tau^{\bar{i}_1\ldots i_r}_{j_1\ldots j_s}(x)\,\xi^{i_1}_{,\bar{i}_1}(x) - \ldots - t\,\tau^{i_1\ldots\bar{i}_r}_{j_1\ldots j_s}(x)\,\xi^{i_r}_{,\bar{i}_r}(x)$$

$$+ t\,\tau^{i_1\ldots i_r}_{\bar{j}_1\ldots j_s}(x)\,\xi^{\bar{j}_1}_{,j_1}(x) + \ldots + t\,\tau^{i_1\ldots i_r}_{j_1\ldots\bar{j}_s}(x)\,\xi^{\bar{j}_s}_{,j_s}(x)$$

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$$\phi^{*-1}(\tau(x))^{i_1 \dots i_r}_{j_1 \dots j_s} = \tau^{\bar{i}_1 \dots \bar{i}_r}_{\bar{j}_1 \dots \bar{j}_s}(\tilde{x}) \frac{\partial x^{i_1}}{\partial \tilde{x}^{\bar{i}_1}} \cdots \frac{\partial x^{i_r}}{\partial \tilde{x}^{\bar{i}_r}} \frac{\partial \tilde{x}^{\bar{j}_1}}{\partial x^{j_1}} \cdots \frac{\partial \tilde{x}^{\bar{j}_s}}{\partial x^{j_s}}$$

$$= \tau^{\bar{i}_1 \dots \bar{i}_r}_{\bar{j}_1 \dots \bar{j}_s}(x + t\xi)\left(\delta^{i_1}_{\bar{i}_1} - t\xi^{i_1}_{,\bar{i}_1}\right) \cdots \left(\delta^{i_r}_{\bar{i}_r} - t\xi^{i_r}_{,\bar{i}_r}\right)$$

$$\cdot\left(\delta^{\bar{j}_1}_{j_1} + t\xi^{\bar{j}_1}_{,j_1}\right) \cdots \left(\delta^{\bar{j}_s}_{j_s} + t\xi^{\bar{j}_s}_{,j_s}\right) + O(t^2)$$

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From now, we will omit writing $\sum$: Twice occurring indices are always to be summed over (Einstein convention)

components:

$$\phi_t^{-1}(\tau(x))^{i_1\ldots i_r}_{j_1\ldots j_s} = \tau^{\bar{i}_1\ldots\bar{i}_r}_{\bar{j}_1\ldots\bar{j}_s}(\tilde{x})\,\frac{\partial x^{i_1}}{\partial\tilde{x}^{\bar{i}_1}}\cdots\frac{\partial x^{i_r}}{\partial\tilde{x}^{\bar{i}_r}}\,\frac{\partial\tilde{x}^{\bar{j}_1}}{\partial x^{j_1}}\cdots\frac{\partial\tilde{x}^{\bar{j}_s}}{\partial x^{j_s}}$$

$$= \tau^{\bar{i}_1\ldots\bar{i}_r}_{\bar{j}_1\ldots\bar{j}_s}(x+t\xi)\left(\delta^{i_1}_{\bar{i}_1} - t\xi^{i_1}_{,\bar{i}_1}\right)\cdots\left(\delta^{i_r}_{\bar{i}_r} - t\xi^{i_r}_{,\bar{i}_r}\right)$$

$$\cdot\left(\delta^{\bar{j}_1}_{j_1} + t\xi^{\bar{j}_1}_{,j_1}\right)\cdots\left(\delta^{\bar{j}_s}_{j_s} + t\xi^{\bar{j}_s}_{,j_s}\right) + O(t^2)$$

$$= \tau^{i_1\ldots i_r}_{j_1\ldots j_s}(x) + t\,\tau^{i_1\ldots i_r}_{j_1\ldots j_s,k}(x)\,\xi^k(x)$$

$$- t\,\tau^{\bar{i}_1\ldots i_r}_{j_1\ldots j_s}(x)\,\xi^{i_1}_{,\bar{i}_1}(x) - \ldots - t\,\tau^{i_1\ldots\bar{i}_r}_{j_1\ldots j_s}(x)\,\xi^{i_r}_{,\bar{i}_r}(x)$$

$$+ t\,\tau^{i_1\ldots i_r}_{\bar{j}_1\ldots j_s}(x)\,\xi^{\bar{j}_1}_{,j_1}(x) + \ldots + t\,\tau^{i_1\ldots i_r}_{j_1\ldots\bar{j}_s}(x)\,\xi^{\bar{j}_s}_{,j_s}(x)$$

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The **Lie derivative** of any tensor field τ at the point $p = \gamma(0) \in M$ with respect to the flow induced by a vector field ξ is defined through:

$$L_\xi \tau := \lim_{t \to 0} \frac{1}{t} \left(\phi_t^{*-1}(\tau) - \tau \right)$$

i.e. $$L_\xi(\tau)(p) = \lim_{t \to 0} \frac{1}{t} \Big[\underbrace{(\phi^*)^{-1}\big(\overbrace{\tau(\gamma(t))}^{\text{tensor field value at image of } p, \text{ i.e. } \in T_{\gamma(t)}(M)^r_s} \big)}_{\in T_p(M)^r_s} - \tau(\underbrace{p}_{= \gamma(0)}) \Big]$$

Explicitly, in a chart:

$\phi: x \to \tilde{x}$ with infinitesimal flow: $\tilde{x}^i(x) = x^i + t\,\xi^i(x) + O(t^2)$

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Recall: A tensor field τ assigns to each $p \in M$ a tensor $\tau(p) \in T_p(M)^r_s$.

Definition:

We say that a tensor field τ is invariant under the flow induced by the vector field ξ if:

$$\underbrace{\phi_t^*(\tau(p))}_{\text{image of the tensor field's value at } p} = \underbrace{\tau(\phi_t(p))}_{\text{tensor field's value at the image of } p} \qquad \forall t\, \forall p$$

(The flow produces an image of M in M:)

Definition:

geom. definition

Definition:

geom. definition

The Lie derivative of any tensor field τ at the point $p = \gamma(0) \in M$ with respect to the flow induced by a vector field ξ is defined through:

$$\mathcal{L}_\xi \tau := \lim_{t \to 0} \frac{1}{t} \left(\phi_t^{*-1}(\tau) - \tau \right)$$

i.e. $$\mathcal{L}_\xi(\tau)(p) = \lim_{t \to 0} \frac{1}{t} \Big[\underbrace{(\phi^*)^{-1}\big(\overbrace{\tau(\gamma(t))}^{\text{tensor field value at image of } p,\ \text{i.e. } \in T_{\gamma(t)}(M)^r_s}\big)}_{\in T_p(M)^r_s} - \tau(\underbrace{p}_{=\gamma(0)}) \Big]$$

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$$= \tau^{i_1\cdots i_r}_{j_1\cdots j_s}(x+t\xi)\left(\delta_{\bar{i}_1}^{i_1} - t\xi^{i_1}_{,\bar{i}_1}\right)\cdots\left(\delta_{\bar{i}_r}^{i_r} - t\xi^{i_r}_{,\bar{i}_r}\right)$$

$$\cdot\left(\delta^{\bar{j}_1}_{j_1} + t\xi^{\bar{j}_1}_{,j_1}\right)\cdots\left(\delta^{\bar{j}_s}_{j_s} + t\xi^{\bar{j}_s}_{,j_s}\right) + O(t^2)$$

$$= \tau^{i_1\cdots i_r}_{j_1\cdots j_s}(x) + t\,\tau^{i_1\cdots i_r}_{j_1\cdots j_s,k}(x)\,\xi^k(x)$$

$$- t\,\tau^{\bar{i}_1\cdots i_r}_{j_1\cdots j_s}(x)\,\xi^{i_1}_{,\bar{i}_1}(x) - \cdots - t\,\tau^{i_1\cdots \bar{i}_r}_{j_1\cdots j_s}(x)\,\xi^{i_r}_{,\bar{i}_r}(x)$$

$$+ t\,\tau^{i_1\cdots i_r}_{\bar{j}_1\cdots j_s}(x)\,\xi^{\bar{j}_1}_{,j_1}(x) + \cdots + t\,\tau^{i_1\cdots i_r}_{j_1\cdots \bar{j}_s}(x)\,\xi^{\bar{j}_s}_{,j_s}(x)$$

$$\Rightarrow (L_\xi\tau)^{i_1\cdots i_r}_{j_1\cdots j_s}(x) = \lim_{t\to 0}\frac{1}{t}\left(\phi^{*-1}_t(\tau(x))^{i_1\cdots i_r}_{j_1\cdots j_s} - \tau^{i_1\cdots i_r}_{j_1\cdots j_s}(\underbrace{x(0)}_{=x})\right)$$

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$$\cdot(\delta^{j_1}_{j_1}+t\xi^{j_1}_{j_1})\cdots(\delta^{j_s}_{j_s}+t\xi^{j_s}_{j_s})+O(t^2)$$

$$= \tau^{i_1\ldots i_r}_{j_1\ldots j_s}(x) + t\,\tau^{i_1\ldots i_r}_{j_1\ldots j_s,K}(x)\,\xi^K(x)$$

$$-t\,\tau^{\bar{i}_1\ldots i_r}_{j_1\ldots j_s}(x)\,\xi^{i_1}_{,\bar{i}_1}(x) - \ldots - t\,\tau^{i_1\ldots \bar{i}_r}_{j_1\ldots j_s}(x)\,\xi^{i_r}_{,\bar{i}_r}(x)$$

$$+t\,\tau^{i_1\ldots i_r}_{\bar{j}_1\ldots j_s}(x)\,\xi^{\bar{j}_1}_{,j_1}(x) + \ldots + t\,\tau^{i_1\ldots i_r}_{j_1\ldots \bar{j}_s}(x)\,\xi^{\bar{j}_s}_{,j_s}(x)$$

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$$= \tau^{i_1\ldots i_r}_{j_1\ldots j_s,K}(x)\,\xi^K(x) - \tau^{\bar{i}_1\ldots i_r}_{j_1\ldots j_s}(x)\,\xi^{i_1}_{,\bar{i}_1}(x) - \ldots$$

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$$\phi^{*-1}(\tau(\tilde x))^{i_1\dots i_r}_{j_1\dots j_s} = \tau^{\bar i_1\dots\bar i_r}_{\bar j_1\dots\bar j_s}(\tilde x)\,\frac{\partial x^{i_1}}{\partial\tilde x^{\bar i_1}}\cdots\frac{\partial x^{i_r}}{\partial \tilde x^{\bar i_r}}\,\frac{\partial\tilde x^{\bar j_1}}{\partial x^{j_1}}\cdots\frac{\partial \tilde x^{\bar j_s}}{\partial x^{j_s}}$$

$$= \tau^{\bar i_1\dots\bar i_r}_{\bar j_1\dots\bar j_s}(x+t\xi)\left(\delta^{i_1}_{\bar i_1} - t\,\xi^{i_1}_{,\bar i_1}\right)\cdots\left(\delta^{i_r}_{\bar i_r}-t\,\xi^{i_r}_{,\bar i_r}\right)$$

$$\cdot\left(\delta^{\bar j_1}_{j_1}+t\,\xi^{\bar j_1}_{,j_1}\right)\cdots\left(\delta^{\bar j_s}_{j_s}+t\,\xi^{\bar j_s}_{,j_s}\right)+O(t^2)$$

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$$+ t\,\tau^{i_1\dots i_r}_{\bar j_1\dots j_s}(x)\,\xi^{\bar j_1}_{,j_1}(x) + \dots + t\,\tau^{i_1\dots i_r}_{j_1\dots \bar j_s}(x)\,\xi^{\bar j_s}_{,j_s}(x)$$

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$$\Rightarrow\left(L_\xi\tau\right)^{i_1\ldots i_r}_{j_1\ldots j_s}(x)=\lim_{t\to 0}\frac{1}{t}\left(\phi^{*-1}(\tau(x))^{i_1\ldots i_r}_{j_1\ldots j_s}-\tau^{i_1\ldots i_r}_{j_1\ldots j_s}(\overbrace{x(0)}^{\overset{x}{\shortparallel}})\right)$$

$$=\tau^{i_1\ldots i_r}_{j_1\ldots j_s,k}(x)\,\xi^k(x)-\tau^{\bar{i}_1\ldots i_r}_{j_1\ldots j_s}(x)\,\xi^{i_1}{}_{,\bar{i}_1}(x)-\ldots$$

$$+\tau^{i_1\ldots i_r}_{\bar{j}_1\ldots j_s}(x)\,\xi^{\bar{j}_1}{}_{,j_1}(x)+\ldots\tau^{i_1\ldots i_r}_{j_1\ldots \bar{j}_s}(x)\,\xi^{\bar{j}_s}{}_{,j_s}(x)$$

□ Equivalent to algebraic definition of L_ξ?

Yes: Check e.g. that action on

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$$= \tau^{\bar i_1 \dots \bar i_r}_{\bar j_1 \dots \bar j_s}(x+t\xi)\left(\delta^{i_1}_{\bar i_1} - t\xi^{i_1}_{,\bar i_1}\right)\cdots\left(\delta^{i_r}_{\bar i_r} - t\xi^{i_r}_{,\bar i_r}\right)$$

$$\cdot\left(\delta^{\bar j_1}_{j_1} + t\xi^{\bar j_1}_{,j_1}\right)\cdots\left(\delta^{\bar j_s}_{j_s} + t\xi^{\bar j_s}_{,j_s}\right) + O(t^2)$$

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Yes: Check e.g. that action on $\Lambda_0(M)$ and $\Lambda_1(M)$ is the same:

- For $\tau \in \Lambda_0(M)$ we have $\tau = \tau(x)$

$$L_\xi \tau(x) = \xi^k \tau_{,k} = \xi^k \frac{\partial}{\partial x^k} \tau(x) \text{ is gradient } \checkmark$$

- Co-Vector field: $\tau = \tau_j(x)\, dx^j \in \Lambda_1(M)$

$$L_\xi \tau(x) = \left(\xi^k(x)\, \tau_{j,k}(x) + \tau_k(x)\, \xi^k_{\ ,j}(x) \right) dx^j$$

Exercise: verify that this agrees with the

$$+t\,\tau^{i_1\dots i_r}_{\bar{j}_1\dots j_s}(x)\,\xi^{\bar{j}_1}_{,j_1}(x)+\dots+t\,\tau^{i_1\dots i_r}_{j_1\dots \bar{j}_s}(x)\,\xi^{\bar{j}_s}_{,j_s}(x)$$

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$$= \tau^{i_1\dots i_r}_{j_1\dots j_s,k}(x)\,\xi^{k}(x) - \tau^{\bar{i}_1\dots i_r}_{j_1\dots j_s}(x)\,\xi^{i_1}_{,\bar{i}_1}(x) - \dots$$

$$+\,\tau^{i_1\dots i_r}_{\bar{j}_1\dots j_s}(x)\,\xi^{\bar{j}_1}_{,j_1}(x) + \dots\, \tau^{i_1\dots i_r}_{j_1\dots \bar{j}_s}(x)\,\xi^{\bar{j}_s}_{,j_s}(x)$$

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$$+\tau^{i_1\dots i_r}_{\bar{j}_1\dots j_s}(x)\,\xi^{\bar{j}_1}_{,j_1}(x) + \dots\,\tau^{i_1\dots i_r}_{j_1\dots \bar{j}_s}(x)\,\xi^{\bar{j}_s}_{,j_s}(x)$$

☐ Equivalent to algebraic definition of L_ξ?

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$$+t\,\tau^{i_1\cdots i_r}_{\bar{j}_1\cdots j_s}(x)\,\xi^{\bar{j}_1}_{,j_1}(x)+\ldots+t\,\tau^{i_1\cdots i_r}_{j_1\cdots \bar{j}_s}(x)\,\xi^{\bar{j}_s}_{,j_s}(x)$$

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$$= \tau^{i_1\cdots i_r}_{j_1\cdots j_s,k}(x)\,\xi^k(x) - \tau^{\bar{i}_1\cdots i_r}_{j_1\cdots j_s}(x)\,\xi^{i_1}_{,\bar{i}_1}(x) - \ldots$$

$$+\tau^{i_1\cdots i_r}_{\bar{j}_1\cdots j_s}(x)\,\xi^{\bar{j}_1}_{,j_1}(x)+\ldots\tau^{i_1\cdots i_r}_{j_1\cdots \bar{j}_s}(x)\,\xi^{\bar{j}_s}_{,j_s}(x)$$

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$$t\,\tau^{\,i_1\ldots i_r}_{j_1\ldots j_s}(x)\,\xi^{\bar{i}}_{,i_1}(x) \ldots - t\,\tau^{\,i_1\ldots \bar{i}}_{j_1\ldots j_s}(x)\,\xi^{i_r}_{,\bar{i}}(x)$$

$$+\,t\,\tau^{\,i_1\ldots i_r}_{\bar{j}\ldots j_s}(x)\,\xi^{\bar{j}}_{,j_1}(x)+\ldots+t\,\tau^{\,i_1\ldots i_r}_{j_1\ldots \bar{j}}(x)\,\xi^{\bar{j}}_{,j_s}(x)$$

$$\Rightarrow \left(L_\xi \tau\right)^{i_1\ldots i_r}_{j_1\ldots j_s}(x) = \lim_{t\to 0}\frac{1}{t}\left(\phi^{*-1}_t(\tau(x))^{i_1\ldots i_r}_{j_1\ldots j_s} - \tau^{i_1\ldots i_r}_{j_1\ldots j_s}(\overbrace{x(0)}^{x\,=})\right)$$

$$= \tau^{\,i_1\ldots i_r}_{j_1\ldots j_s,k}(x)\,\xi^k(x) - \tau^{\,\bar{i}\ldots i_r}_{j_1\ldots j_s}(x)\,\xi^{i_1}_{,\bar{i}}(x) - \ldots$$

$$+\,\tau^{\,i_1\ldots i_r}_{\bar{j}\ldots j_s}(x)\,\xi^{\bar{j}}_{,j_1}(x)+\ldots\,\tau^{\,i_1\ldots i_r}_{j_1\ldots \bar{j}}(x)\,\xi^{\bar{j}}_{,j_s}(x)$$

□ Equivalent to algebraic definition of L_ξ?

Yes: Check ...

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$\Lambda_0(M)$ and $\Lambda_1(M)$ is the same:

- For $\tau \in \Lambda_0(M)$ we have $\tau = \tau(x)$

$$L_\xi \tau(x) = \xi^k \tau_{,k} = \xi^k \frac{\partial}{\partial x^k} \tau(x) \text{ is gradient } \checkmark$$

- Co-Vector field: $\tau = \tau_j(x)\, dx^j \in \Lambda_1(M)$

$$L_\xi \tau(x) = \left(\xi^k(x)\, \tau_{j,k}(x) + \tau_k(x)\, \xi^k_{\ ,j}(x)\right) dx^j$$

Exercise: verify that this agrees with the algebraically defined action of L_ξ on $\Lambda_1(M)$.

- Collected properties: (without proof)

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$$- t\, \tau^{\cdot\, \dots\, i_r}_{j_1 \dots j_s}(x)\, \xi^{i_1}_{,\cdot}(x) - \dots - t\, \tau^{i_1 \dots\, \cdot}_{j_1 \dots j_s}(x)\, \xi^{i_r}_{,\cdot}(x)$$

$$+ t\, \tau^{i_1 \dots i_r}_{\cdot\, \dots j_s}(x)\, \xi^{\cdot}_{,j_1}(x) + \dots + t\, \tau^{i_1 \dots i_r}_{j_1 \dots\, \cdot}(x)\, \xi^{\cdot}_{,j_s}(x)$$

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$$+ \tau^{i_1 \dots i_r}_{\cdot\, \dots j_s}(x)\, \xi^{\cdot}_{,j_1}(x) + \dots\, \tau^{i_1 \dots i_r}_{j_1 \dots\, \cdot}(x)\, \xi^{\cdot}_{,j_s}(x)$$

□ Equivalent to algebraic definition of L_ξ?

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equivalent to algebraic definition of L_ξ.

Yes: Check e.g. that action on $\Lambda_0(M)$ and $\Lambda_1(M)$ is the same:

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- Collected properties: (without proof)

- $L_\xi : T_p(M)^r_s \rightarrow T_p(M)^r_s$ (i.e. not just $\Lambda_s \rightarrow \Lambda_s$)

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Collected properties: (without proof)

- $L_\xi : T_p(M)^r_s \longrightarrow T_p(M)^r_s$ (i.e. not just $\Lambda_s \to \Lambda_s$)

- In particular, the Lie derivative of a vector field η is:

$$L_\xi : \eta \to L_\xi(\eta) = [\xi, \eta]$$

- One also finds:

$$L_{\xi+\eta} = L_\xi + L_\eta$$

$$L_{[\xi,\eta]} = [L_\xi, L_\eta] \quad \left(= L_\xi \circ L_\eta - L_\eta \circ L_\xi \right)$$

- Does it still obey a Leibniz rule?

Yes: $L_\xi(\tau \otimes \sigma) = L_\xi(\tau) \otimes \sigma + \tau \otimes L_\xi(\sigma)$

(tensors form an algebra w. respect to multiplication $\otimes$)

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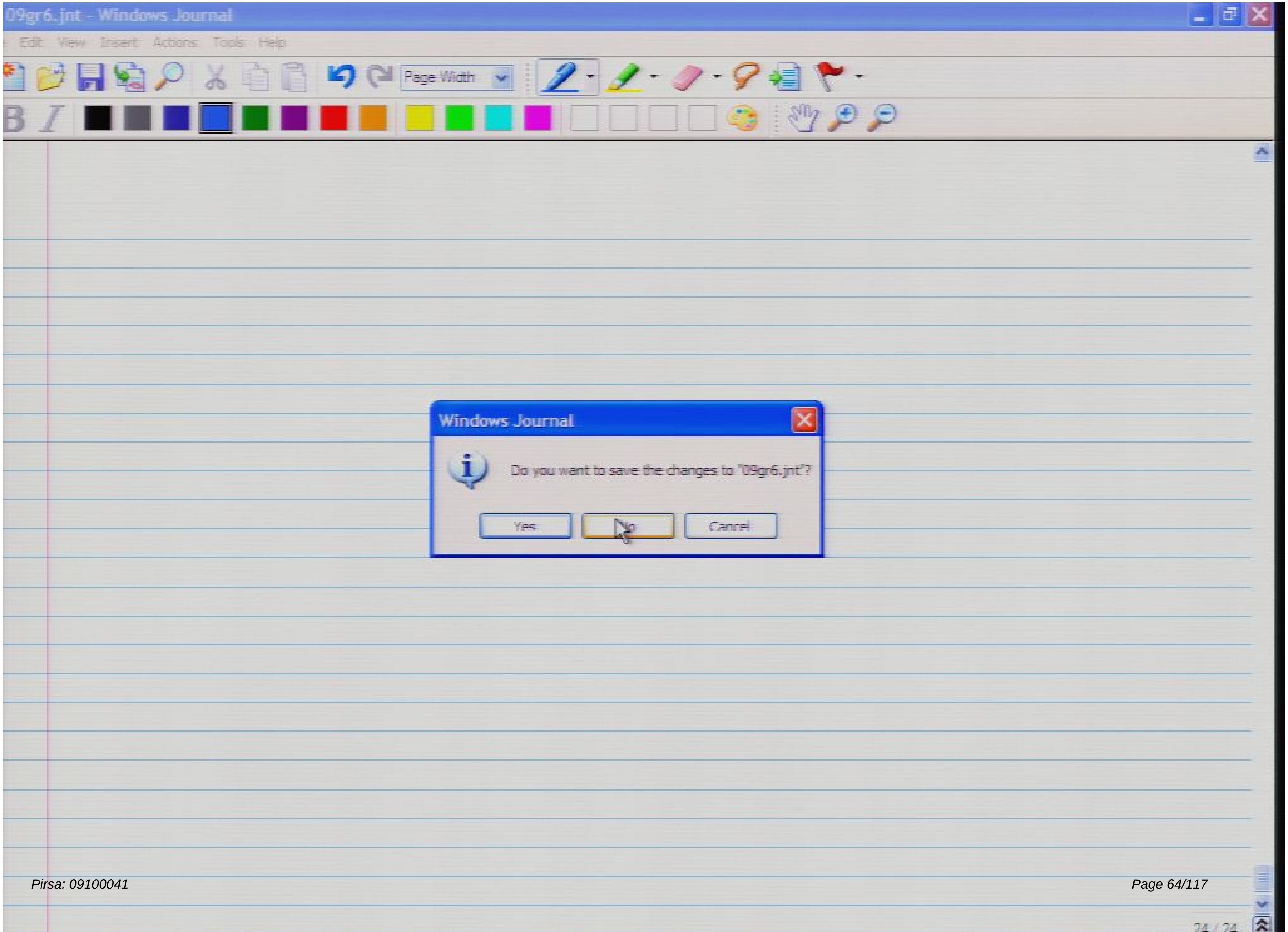
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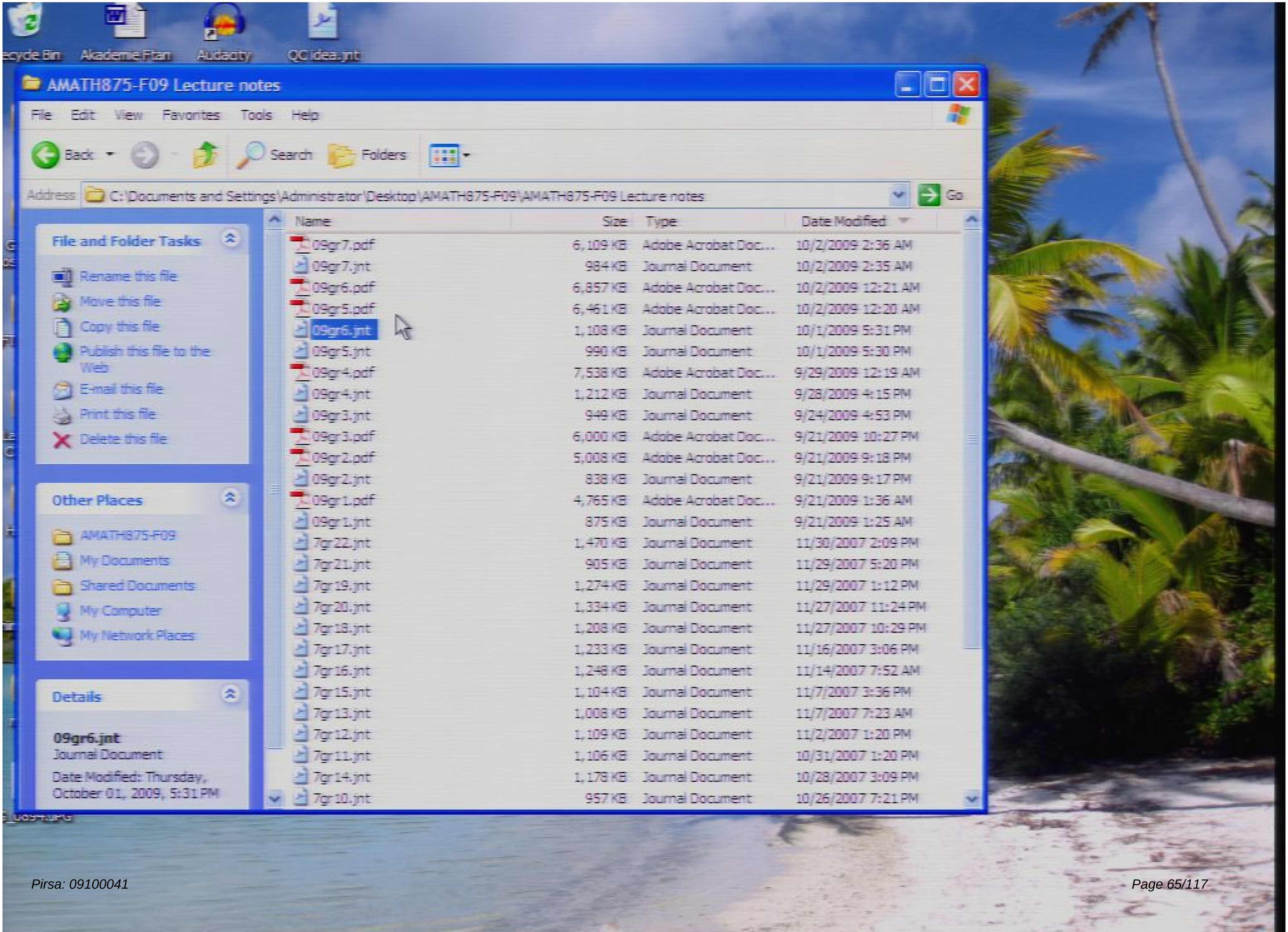
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GR for Cosmology, Achim Kempf, Fall 09, Lecture 7

Integration!

(leading up to the famous Stokes' theorem)

Q: What is special about totally antisymmetric covariant tensors, i.e., about differential forms?

A: Antisymmetry $\Rightarrow$ special transformation property under chart changes:

$\sim$ det(Jacobian)

$\Rightarrow$ suitable for integration: s-forms have natural integrals in s-dimensional manifolds

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Q: What is special about totally antisymmetric covariant tensors, i.e., about differential forms?

A: Antisymmetry ⇒ special transformation property under chart changes:
~ det (Jacobian)
⇒ suitable for integration:
s-forms have natural integrals in s-dimensional manifolds

Except: Depending on charts, sign of Jacobian may be wrong!

Thus: Before defining integration on mflds, must study notion of "Orientation" of the mfld.

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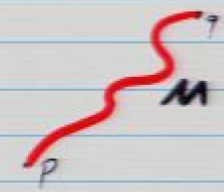
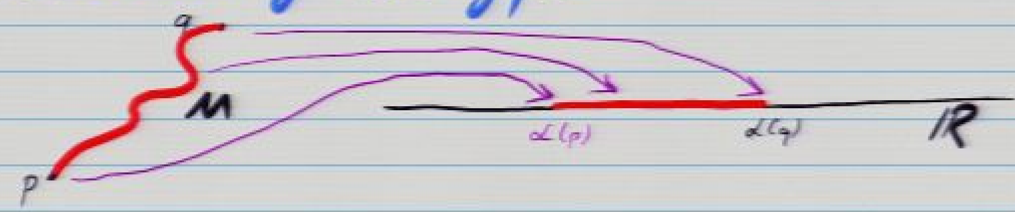
Thus: Before defining integration on mflds, must study notion of "Orientation" of the mfld.

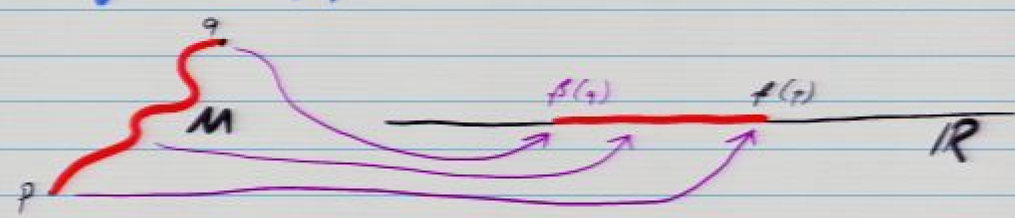
Namely: Consider e.g. 1-dim mfld:



- Could have charts of the type



- or charts of the type



- But, since $\int_a^b f(t)\,dt = -\int_b^a f(t')\,dt'$ one needs to decide!

because $\frac{dt}{dt'} = -1$ (which is det[Jacobian ...

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For n-dim mflds, may need several charts.

Definitions:

- A complete collection of charts, i.e., an Atlas, A, is called oriented if among all overlapping charts with coordinates say $x, \tilde{x}$ the Jacobi determinants are positive:

$$\det\left(\frac{\partial \tilde{x}^i}{\partial x^j}\right) > 0$$

- A mfld M is called orientable

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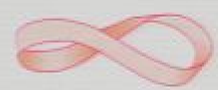
Example: Möbius strips



are not orientable.

- A mfld, M, together with a choice of oriented atlas, A, is called an oriented manifold.

- Then, an arbitrary chart is called positive (or negative) if its Jacobian determinant with charts of the atlas A is positive (or negative).

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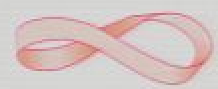

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Integration:

- Recall change of cds in integration in $\mathbb{R}^n$:

For $(x^1, \dots, x^n) \to (\tilde{x}^1, \dots, \tilde{x}^n)$:

$g: \mathbb{R}^n \to \mathbb{R}$

Riemann or Lebesgue integrals

$$\int_{\mathbb{R}^n} g(x^1, \dots, x^n)\, dx^1 \dots dx^n = \int_{\mathbb{R}^n} g(x(\tilde{x}))\, \det\left(\frac{\partial x^i}{\partial \tilde{x}^j}\right) d\tilde{x}^1 \dots d\tilde{x}^n \quad (*)$$

Jacobian determinant is negative if coordinate systems change handedness.

- Now for a general diffable mfld M,

consider an n-form ω in a chart:

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Then what is ω in an overlapping, second chart?

$$\omega = f(x(\tilde{x}))\,\frac{\partial x^1}{\partial \tilde{x}^{i_1}}\frac{\partial x^2}{\partial \tilde{x}^{i_2}}\dots\frac{\partial x^n}{\partial \tilde{x}^{i_n}}\,d\tilde{x}^{i_1}\wedge d\tilde{x}^{i_2}\wedge\dots\wedge d\tilde{x}^{i_n}$$

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- terms are nonzero only if contain each number $1, \ldots, n$ exactly once, e.g. $d\tilde{x}^1 \wedge d\tilde{x}^3 \wedge d\tilde{x}^2 \wedge d\tilde{x}^4 \wedge d\tilde{x}^5 \wedge \ldots \wedge d\tilde{x}^n$.
- Reorder those terms - they are all

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$\Rightarrow$

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Compare with equation (*) above $\Rightarrow$

The following definition of the integral of n-forms in an n-dim

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The following definition of the integral of n-forms in an n-dim. diffable mfld is chart-independent, i.e., is well-defined:

Definition:

Assume M is an oriented n-dim mfld and $\omega \in \Lambda_n(M)$ reads in a chart d: $\omega = f(x)\, dx^1 \wedge \ldots \wedge dx^n$.

Then, if one chart suffices:

$$\int_M \omega := \overbrace{\int_{d(M)} f(x)\, dx^1 dx^2 \ldots dx^n}^{\text{usual Riemann or Lebesgue integral}}$$

image of M in $\mathbb{R}^n$

Else: Piece right hand side together from several charts

Note: how to piece together does not matter as long as charts are from the

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up to a possible factor -1 because $dx^i \wedge dx^j = -dx^j \wedge dx^i$

$\Rightarrow$

$$\omega = f(x(\tilde{x}))\, \det\left(\frac{\partial x^i}{\partial \tilde{x}^j}\right) d\tilde{x}^1 \wedge d\tilde{x}^2 \wedge \ldots \wedge d\tilde{x}^n$$

Compare with equation (*) above $\Rightarrow$

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o Reorder those terms - they are all

$d\tilde{x}^1 \wedge d\tilde{x}^2 \wedge \ldots \wedge d\tilde{x}^n$

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$d(M)$: image of M in $\mathbb{R}^n$

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An n-form $\Omega \in \Lambda_n(M)$ is called a volume form if it nowhere vanishes. (We will later find a preferred volume form for space-time)

Proposition:

M possesses a volume form

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M is orientable

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Definition:

- Assume $G \subset M$ is a region (i.e., open and connected subset) of the n-dim mfld M.

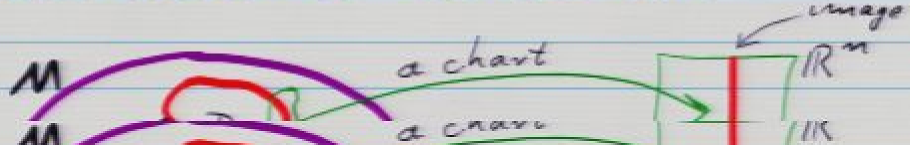
We denote the $(n-1)$ dim. boundary manifold of G by ∂G:

the boundary operator

$$\partial G := \text{boundary}(G)$$

- We say that ∂G is smooth if locally there exist charts so that:

image of boundary surface is $(n-1)$ dim. plane in $\mathbb{R}^n$.



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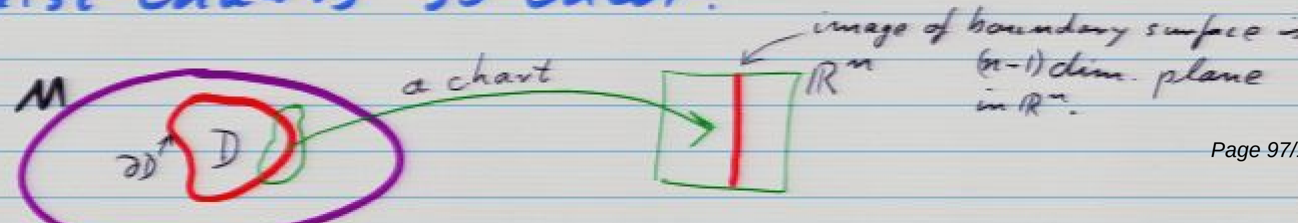

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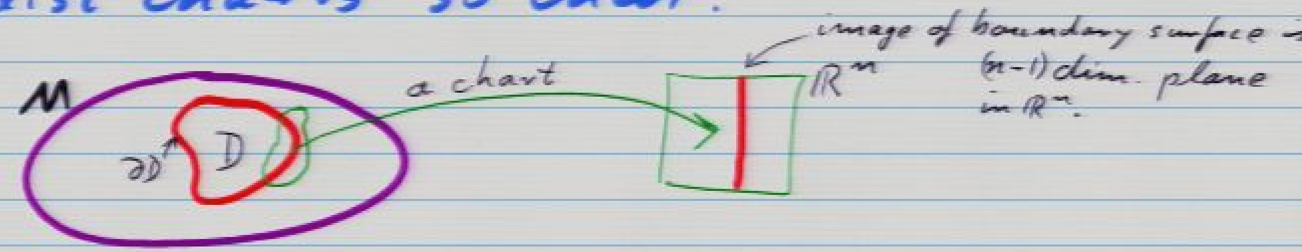

Proposition: If M is orientable, then so is G. Also, the orientation of G induces an orientation of ∂G.

We finally have all ingredients for one of Math's most important theorems:

Stokes' theorem: If closure $\bar{G}$ of G is a compact n-dim region, then:

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dd: $= 0$ always for algebraic reasons.

$\partial\partial H = 0$ for geometric reasons because, indeed, boundaries don't possess boundaries:

E.g.

G = half sphere
∂G = equator
$\partial\partial G = \emptyset$

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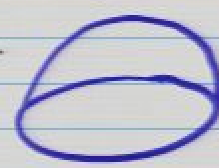
E.g. 

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Special case I:

Assume: $M = \mathbb{R}$, $G = (a,b)$

Therefore: $\partial G = \{a, b\}$

Then, Stokes' theorem is $\int_G df = \int_{\partial G} f$, namely:

$$\int_a^b df = f\Big|_a^b$$

(fund. thm of calculus)

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Special case II:

- $M = \mathbb{R}^2$, $G \subset \mathbb{R}^2$ a region with (closed) boundary curve ∂G.

  recall: this is automatic because $\partial \circ \partial = 0$

- Consider a 1-form $\omega \in \Lambda_1(M)$:

$$\omega = \omega_1(x)\, dx^1 + \omega_2(x)\, dx^2$$

$$\Rightarrow d\omega = \left(\frac{\partial \omega_2}{\partial x^1} - \frac{\partial \omega_1}{\partial x^2}\right) dx^1 \wedge dx^2$$

Now Stokes' theorem:

$$\int_G d\omega = \int_{\partial G} \omega \Rightarrow \iint_G \left(\frac{\partial \omega_2}{\partial x^1} - \frac{\partial \omega_1}{\partial x^2}\right) dx^1 dx^2 = \int_{\partial G} (\omega_1\, dx^1 + \omega_2\, dx^2)$$

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Special case III: (exercise)

Similarly, one can show that what is often called the Stokes theorem for $M = \mathbb{R}^3$, namely

"cross product": $\vec{a} \times \vec{b} = (a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1)$

$\left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\right)$

$$\int_A \vec{\nabla} \times \vec{w} \, dA = \int_{\partial A} \vec{w} \cdot \vec{ds}$$

vector field

a 2 dim submanifold of M

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A: a 2dim submanifold of M

$\vec{w}$: vector field

∂A: 1dim boundary of A.

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$G = A$ and $\omega \in \Lambda_1(A)$ with $\vec{\nabla} \times \vec{\omega} = d\omega \in \Lambda_2(A)$

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Special case IV:

The very important special case of the Gauß' theorem can be obtained only after a volume form $\Omega \in \Lambda_n(M)$ is used to define yet a new derivative:

Def: The Divergence of a vector field ξ with respect to a volume form Ω is defined to be:

$$\boxed{\mathrm{div}_\Omega \xi := L_\xi(\Omega)}$$

Lie derivative

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Thus:

$\square \quad div_\Omega \xi \in \Lambda_n(M)$

$\square \quad div_\Omega \xi = (d \circ i_\xi + i_\xi \circ d)\, \Omega$

Recall: $d\Omega = 0$ because anti-symmetry doesn't allow $(n+1)$ forms.

$$\Rightarrow \boxed{div_\Omega \xi = d \circ i_\xi(\Omega)}$$

From Stokes $\Rightarrow$

$$\int_G d\, i_\xi(\Omega) = \int_{\partial G} i_\xi(\Omega)$$

i.e.: $\int \overbrace{div_\xi(\Omega)}^{n\text{-form}} = \int \overbrace{i_\xi(\Omega)}^{(n-1)\text{ form}}$ "Gauß' theorem"

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