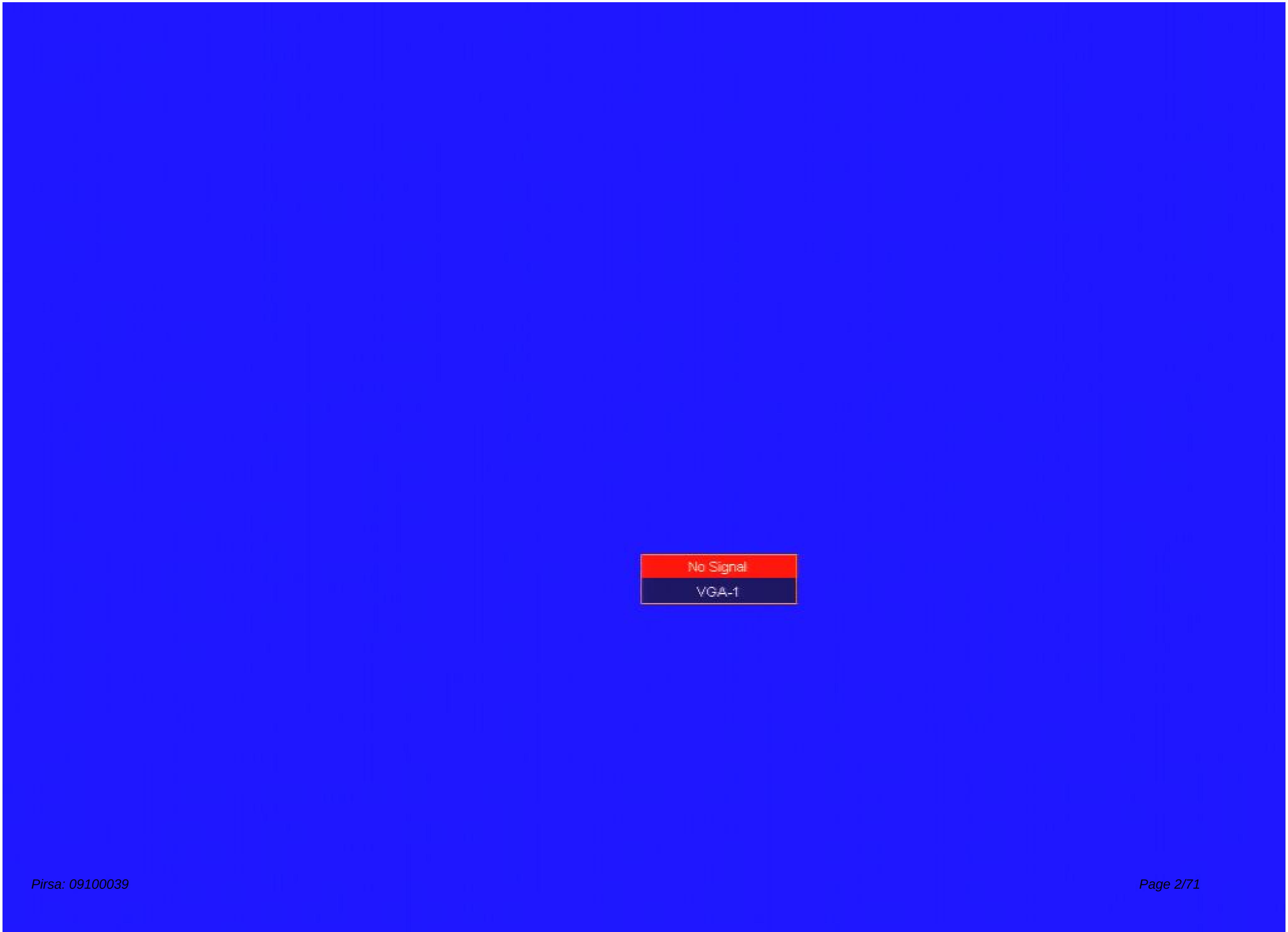
Title: Local Conformal Symmetry and its Violation at Quantum Level

Date: Oct 02, 2009 02:00 PM

URL: http://pirsa.org/09100039

Abstract: We present a short review of the local conformal symmetry and its anomalous violation in curved $4d$ space-time. Furthermore we discuss the ambiguities of conformal anomaly and the anomaly-induced effective actions. Despite the conformal symmetry is always broken at quantum level, it is useful for constructing the best known approximations for investigating quantum corrections to the classical action of gravity. These quantum corrections represent an appropriate basis for a number of applications in cosmology and black hole physics.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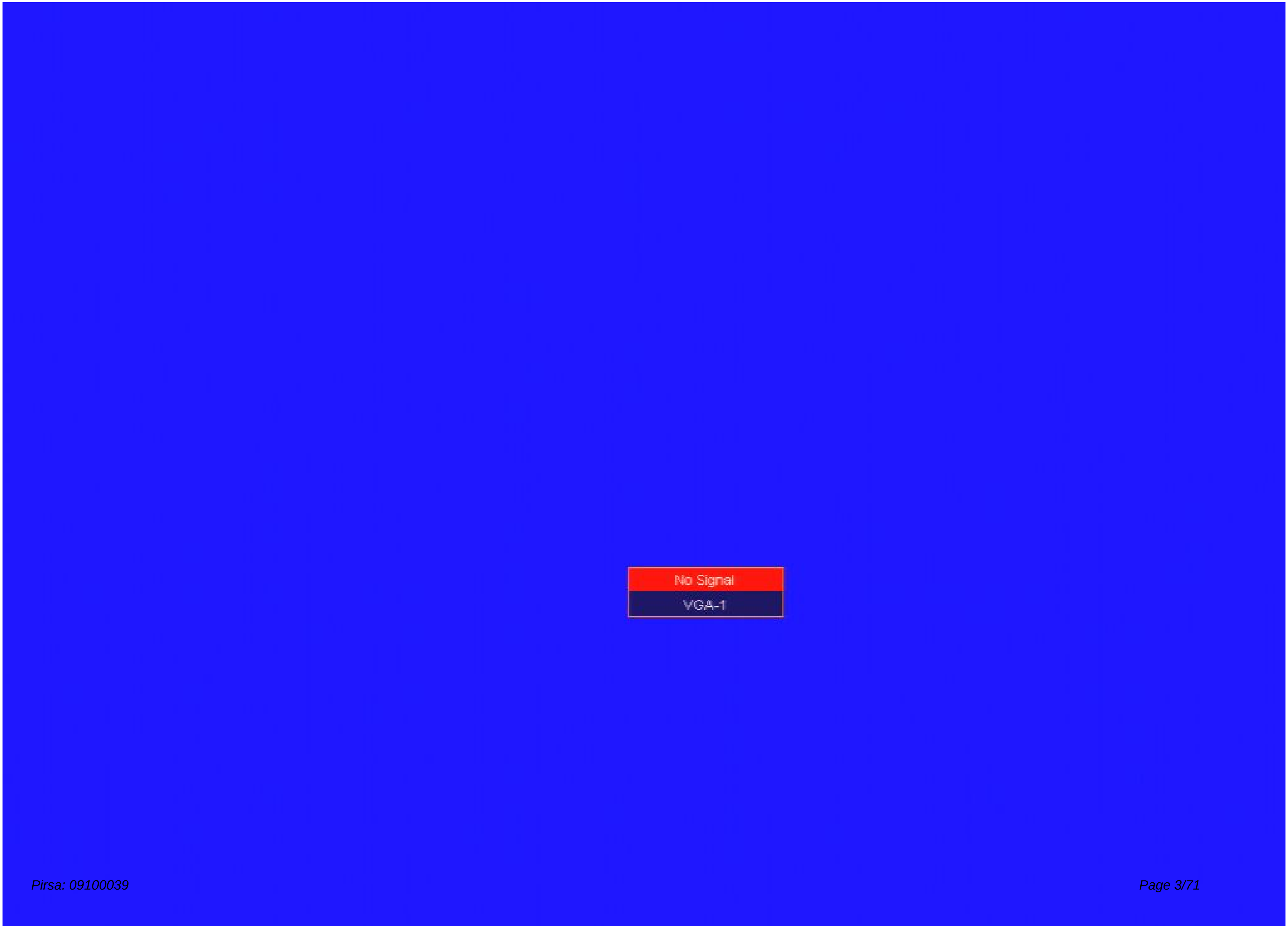
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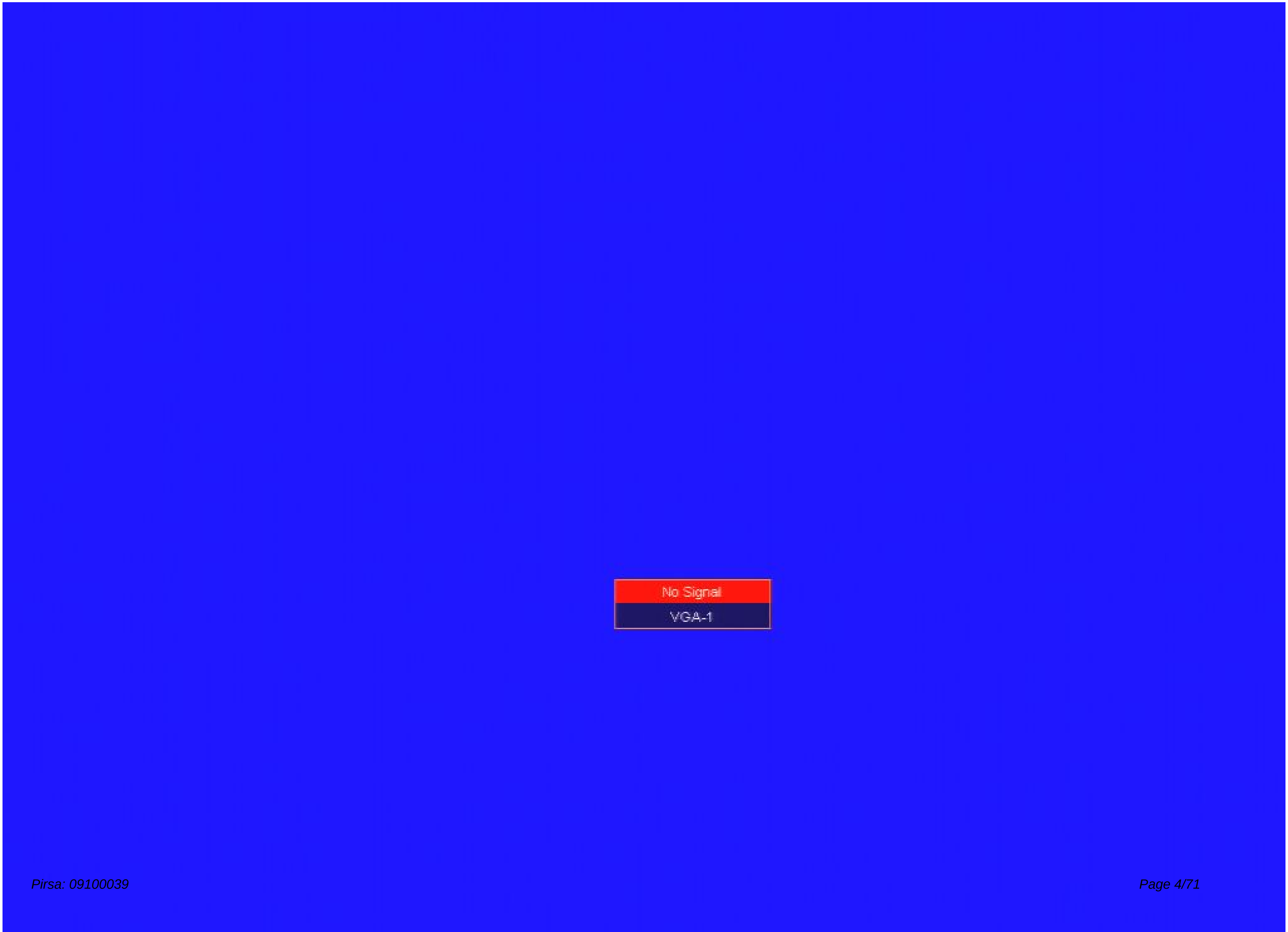
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Contents:

- Examples of $4d$ conformal theories.
- Conformal anomaly and its ambiguities.
- Anomaly induced effective action.
- Quantum corrections to the photon action.
- Light massive fields case.
- Applications: vacuum states of black holes and Starobinsky model.
- The case of conformal quantum gravity.

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Examples of $4d$ conformal theories

- **General metric-scalar theory**

$$S = \int d^4x\sqrt{-g}\,\{A(\phi)\,(\nabla\phi)^2 + B(\phi)R + C(\phi)\}$$

Consider conformal transformation of $g_{\mu\nu}$ plus scalar reparametrizations in the action

$$S = \int d^4x\sqrt{g'}\,\{R'\Phi + V(\Phi)\}$$

$$g'_{\mu\nu} = g_{\mu\nu}e^{2\sigma(\phi)}, \qquad \Phi = \Phi(\phi)\,.$$

O'Hanlon, PRL 29 (1972)

Simple calculation gives

$$A(\phi) = 6e^{2\sigma(\phi)}[\Phi\sigma_1 + \Phi_1]\sigma_1, \qquad B(\phi) = \Phi(\phi)e^{2\sigma},$$

where $B_1 = dB/d\phi$ etc.

The conformal symmetry corresponds to **GR**

I.Sh. & H. Takata, PRD 52 (1995) & PLB 361 (1995).

$$\Phi = const \implies A = \frac{3B_1^2}{2B}, \quad C = \lambda B^2.$$

The well-known particular case is

$$S = \frac{1}{2}\int d^4x\sqrt{g}\left\{\phi\,\Delta_2\,\phi + \frac{\lambda}{12}\,\phi^4\right\}, \quad \text{where} \quad \Delta_2 = \Box + R/6.$$

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More general action with ξ

$$S_{scal} = \int d^4x \sqrt{g} \left\{ \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \frac{1}{2} \xi R \phi^2 - \frac{\lambda}{4!} \phi^4 \right\}$$

is invariant under **global** but **not local** conformal transformation.

$$g_{\mu\nu} \;\to\; g'_{\mu\nu} = g_{\mu\nu}\, e^{2\lambda}, \qquad \phi \to \phi' = \phi\, e^{-\lambda},$$

$$\lambda = \textbf{const}\,.$$

Only in case $\quad \xi = \dfrac{1}{6}$

one meets **local** conformal symmetry

$$g_{\mu\nu} \;\to\; g'_{\mu\nu} = g_{\mu\nu}\, e^{2\sigma}, \qquad \phi \to \phi' = \phi\, e^{-\sigma},$$

$$\sigma = \sigma(x)\,.$$

- **Massles spinor and vector fields**

$$S_{1/2} = \frac{i}{2} \int d^4x \sqrt{g} \left\{ \bar{\psi} \gamma^\mu \nabla_\mu \psi - \nabla_\mu \bar{\psi} \gamma^\mu \psi \right\}$$

and

$$S_1 = -\frac{1}{4} \int d^4x \sqrt{g} \left\{ F_{\mu\nu} F^{\mu\nu} \right\}. \qquad \textbf{!!!}$$

The transformation rules are

$$\psi \to \psi' = \psi \, e^{-3\sigma/2}, \quad \bar{\psi} \to \bar{\psi}' = \bar{\psi} \, e^{-3\sigma/2}, \quad A_\mu \to A'_\mu = A_\mu ,$$

$$g_{\mu\nu} \to g'_{\mu\nu} = g_{\mu\nu} \, e^{2\sigma} \qquad \sigma = \sigma(x) .$$

Note: the difference between conformal weight and dimension for the vector field is due to

$$A_\mu = A_b \, e^b_\mu , \quad e^b_\mu \, e^a_\nu \, \eta_{ab} = g_{\mu\nu} .$$

There is direct relation between local & global conformal symmetries.

- **The conformal (Weyl) gravity** in the dimension $n = 4$

$$S_W = \int d^4x \sqrt{g}\, C^2 \,,$$

$$C^2 = C^{\mu\nu\alpha\beta}\, C_{\mu\nu\alpha\beta} = R^2_{\mu\nu\alpha\beta} - 2R^2_{\mu\nu} + \frac{1}{3} R^2 \,.$$

It can be easily generalized to an arbitrary dimension

$$C^2(n) = R^2_{\mu\nu\alpha\beta} - \frac{4}{n-2} R^2_{\mu\nu} + \frac{1}{(n-1)(n-2)} R^2 \,.$$

- **Fourth derivative scalars of the first kind** (2-nd kind exists, too).

$$S_4 = \int d^4x \sqrt{g}\, \varphi \Delta_4 \varphi \,,$$

where $\Delta_4 = \Box^2 + 2R^{\mu\nu}\nabla_\mu \nabla_\nu - \frac{2}{3} R\Box + \frac{1}{3} R_{;\mu} \nabla^\mu \,.$

The transformation law is $\varphi \to \varphi' \,.$

S. Paneitz, MIT preprint - 1983 ;

R.J. Riegert; E.S. Fradkin & A. Tseytlin, PLB - 1984.

- **Third derivative spinor field**

Fradkin & Tseytlin, Phys. Repts. - 1985;
G. de Berredo Peixoto & I.Sh. PLB - 2001.

$$S_3 = \frac{i}{2}\int d^4x\sqrt{-g}\,\{\bar{\psi}\gamma^\mu\mathcal{D}_\mu\psi - \mathcal{D}_\mu\bar{\psi}\gamma^\mu\psi\}\,,$$

where $$\mathcal{D}_\mu = \nabla_\mu\Box + R_{\mu\nu}\nabla^\nu - \frac{5}{12}R\nabla_\mu - \frac{1}{12}(\nabla_\mu R)\,.$$

The transformation law is $\psi \to \psi' = \psi\, e^{-\sigma/2}\,, \qquad \bar{\psi} \to \bar{\psi}' = \bar{\psi}\, e^{-\sigma/2}\,.$

- **Is it possible to construct more examples of conformal fields?**

Vectors? Scalars? Spinors? Spin $3/2$? Perhaps yes.
Erdmenger (CQG-1997); Erdmenger & Osborn (CQG-1998); ...

General Review: *V. Faraoni, E. Gunzig and P. Nardone, Fund. Cosmic Phys.* **20** *(1999); [gr-qc/9811047].*

Quantum (Semiclassical) Theory

Recent reviews (IOPP):
I.Sh. - hep-th/0610168, gr-qc/0801.0216.

The first step is to consistently formulate the action of quantum matter fields on classical curved background.

The criteria for choosing the action of the theory are as follows

- **Locality of the classical action.**
- **Renormalizability.**
- **Simplicity, e.g. no $[m^{-1}]$ parameters.**

The vacuum action satisfying these principles has the form

$$S_{vac} = S_{EH} + S_{HD}\,,$$

$$S_{HD} = \int d^4x\sqrt{-g}\Big\{\alpha C^2 + \beta E + \gamma\Box R + \delta R^2\Big\}\,,$$

where

$$E = R^2_{\mu\nu\alpha\beta} - 4R^2_{\alpha\beta} + R^2\,.$$

In the conformal theory, at the one-loop level it is sufficient to consider

$$S_{conf.\ vac} = \int d^4x\sqrt{-g}\,\{a_1 C^2 + a_2 E + a_3\Box R\}\,.$$

Conformal anomaly

The theory includes $g_{\mu\nu}$ and matter fields Φ.
k_Φ is the conformal weight of the field.

The Noether identity for the local conformal symmetry

$$\left[-2\, g_{\mu\nu}\,\frac{\delta}{\delta g_{\mu\nu}} + k_\Phi\,\Phi\,\frac{\delta}{\delta\Phi}\right]\, S(g_{\mu\nu},\,\Phi) \,=\, 0$$

produces $T^\mu_\mu = 0$ on shell

$$-\,\frac{2}{\sqrt{g}}\,g_{\mu\nu}\,\frac{\delta S_{vac}(g_{\mu\nu})}{\delta g_{\mu\nu}} \,=\, T^\mu_\mu \,=\, 0\,.$$

At quantum level $S_{vac}(g_{\mu\nu})$ is replaced by the effective action (EA)

$$\Gamma_{vac}(g_{\mu\nu})\,.$$

One can derive the conformal anomaly in many ways, depending on the choice of a regularization and renormalization schemes.

Consider, as an example, the derivation of anomaly through the most explicit method of dimensional regularization
M.J. Duff, NPB 1977.

The theory of matter includes massless fields:

N_0 scalars, $N_{1/2}$ spinors (Dirac, spin-1/2) N_1 vectors.

All N_s indicate a number of fields in curved space-time, taking conformal version for scalars.

We are interested in the 1-loop vacuum effects and therefore can restrict consideration by the free fields case.

The expression for divergences of the free fields

$$\Gamma_{div} = \frac{1}{\varepsilon} \int d^4x \sqrt{g} \left\{ \beta_1 C^2 + \beta_2 E + \beta_3 \Box R \right\} .$$

where

$$\begin{pmatrix} \beta_1 \\ -\beta_2 \\ \beta_3 \end{pmatrix} = \frac{1}{360(4\pi)^2} \begin{pmatrix} 3N_0 + 18N_{1/2} + 36N_1 \\ N_0 + 11N_{1/2} + 62N_1 \\ 2N_0 + 12N_{1/2} - 36N_1 \end{pmatrix}$$

The renormalized one-loop effective action has the form

$$\Gamma_R = S + \bar{\Gamma} + \Delta S , \qquad (1)$$

where $\bar{\Gamma} = \bar{\Gamma}_{div} + \bar{\Gamma}_{fin}$ is the naive quantum correction to the classical action and ΔS is a counterterm.

Remember that $S_{vac} = S_{EH} + S_{conf}$, but only conformal part of the vacuum action must be used in (1).

ΔS is an infinite local counterterm which is called to cancel the divergent part of $\bar{\Gamma}$.

ΔS **is the only source of noninvariance** of the EA, since naive (despite divergent) contributions of quantum fields are conformal.

The anomalous trace is

$$T = < T^{\mu}_{\mu} > = -\frac{2}{\sqrt{-g}}\, g_{\mu\nu}\, \frac{\delta\, \Gamma_R}{\delta\, g_{\mu\nu}}\bigg|_{n=4} = -\frac{2}{\sqrt{-g}}\, g_{\mu\nu}\, \frac{\delta\, \Delta S}{\delta\, g_{\mu\nu}}\bigg|_{n=4} .$$

The calculation of this expression can be easily done if we change parametrization of the metric,

$$g_{\mu\nu} = \bar{g}_{\mu\nu} \cdot e^{2\sigma} , \qquad \sigma = \sigma(x)$$

where $\bar{g}_{\mu\nu}$ is the fiducial metric with fixed determinant.

There is a useful relation

$$-\frac{2}{\sqrt{-g}}\, g_{\mu\nu} \frac{\delta\, A[g_{\mu\nu}]}{\delta\, g_{\mu\nu}} = -\frac{1}{\sqrt{-\bar{g}}}\, e^{-4\sigma}\, \frac{\delta\, A[\bar{g}_{\mu\nu}\, e^{2\sigma}]}{\delta\sigma}\Big|_{\bar{g}_{\mu\nu}\to g_{\mu\nu},\, \sigma\to 0,\, n\to 4} \qquad (*)$$

At that point we need a transformation laws for the structures presented in ΔS. They can be found, e.g., in gr-qc/0412113. E.g.,

$$\int d^n x\, \sqrt{-g}\, C^2(n) = \int d^n x \sqrt{-\bar{g}}\, e^{(n-4)\sigma}\, \bar{C}^2(n)\,.$$

Then

$$\frac{\delta}{\delta\sigma} \int \frac{d^4 x \sqrt{-\bar{g}}}{n-4}\, e^{(n-4)\sigma}\, \bar{C}^2(n)\Big|_{n\to 4} = \sqrt{-g}\, C^2\,.$$

All other expressions of our interest have the same factor $e^{(n-4)\sigma}$ and, on the top of that, some extra terms with derivatives of $\sigma(x)$. However, these terms are irrelevant due to the procedure in eq. (*).

In the simplest case $\sigma = \lambda = const$ we immediately arrive at the expression for T with $a' = \beta_3$. E.g.

$$\frac{\delta}{\delta\sigma} \int \frac{d^4x\sqrt{-\bar{g}}}{n-4}\, e^{(n-4)\sigma}\, \bar{C}^2(n) \Big|_{n\to 4} = C^2(4)\,.$$

For global conformal transform the same procedure always works,

$$< T^\mu_\mu >= \frac{1}{(4\pi)^2}\,\left(\omega C^2 + bE + c\Box R\right)\,, \qquad (\omega, b, c) = (\beta_1,\, \beta_2,\, \beta_3)$$

However in the local case $\sigma(x)$ things are more complicated, e.g.

$$\frac{\delta}{\delta g_{\mu\nu}} \int \sqrt{-g}\Box R \equiv 0\,.$$

We have a conflict between global and local conf. anomalies.

Or a conflict between formulas and intuitive expectations.

M.J. Duff, Class. Quantum. Grav. (1994)

- **Anomaly-induced Effective Action (EA) of vacuum**

One can use $< T^\mu_\mu >$ to obtain equation for the finite 1-loop EA

$$\frac{2}{\sqrt{-g}}\, g_{\mu\nu} \frac{\delta\,\bar{\Gamma}_{ind}}{\delta g_{\mu\nu}} = < T^\mu_\mu > = \frac{1}{(4\pi)^2}\,\big(\,\omega C^2 + bE + c\Box R\,\big)\ .$$

The solution is straightforward

Riegert; Fradkin & Tseytlin, PLB-1984.

It can be generalized for the theory with torsion or scalar field.

Buchbinder, Odintsov & I.Sh. Phys.Lett. B (1985).
Helayel-Neto, Penna-Firme & I.Sh. Phys.Lett. B (1998).
I.Sh., J. Solà, Phys.Lett. B (2002).

The simplest possibility is to parametrize metric

$$g_{\mu\nu} = \bar{g}_{\mu\nu} \cdot e^{2\sigma}\,, \qquad \sigma = \sigma(x)\,.$$

The solution for the effective action is

$$\bar{\Gamma}_{ind} = S_c[\bar{g}_{\mu\nu}] + \frac{1}{(4\pi)^2}\int d^4x\sqrt{-\bar{g}}\,\{\omega\sigma\bar{C}^2 \qquad (1)$$

$$+ b\sigma(\bar{E} - \frac{2}{3}\bar{\Box}\bar{R}) + 2b\sigma\bar{\Delta}_4\sigma - \frac{1}{12}(c + \frac{2}{3}b)[\bar{R} - 6(\bar{\nabla}\sigma)^2 - (\bar{\Box}\sigma)]^2)\}\,,$$

where $S_c[\bar{g}_{\mu\nu}] = S_c[g_{\mu\nu}]$ is an **unknown conformal functional** of the metric, which serves as an **integration constant** in eq. for Γ_{ind}.

The solution (1) has great merits:
1) Being simple, 2) Being exact in case $S_c[\bar{g}_{\mu\nu}]$ is irrelevant.
Example: FRW metrics.

An important disadvantage is that it is not covariant or, in other words, it is not expressed in terms of original metric $g_{\mu\nu}$.

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Now we obtain the **non-local covariant** solution and after represent it in the local form using **auxiliary fields.**
First one has to establish the relations

$$\sqrt{-g}C^2 = \sqrt{-\bar{g}}\bar{C}^2 , \qquad \sqrt{-\bar{g}}\,\bar{\Delta}_4 = \sqrt{-g}\,\Delta_4 .$$

$$\sqrt{-g}(E - \frac{2}{3}\Box R) = \sqrt{-\bar{g}}(\bar{E} - \frac{2}{3}\bar{\Box}\bar{R} + 4\bar{\Delta}_4\sigma)$$

and also introduce the Green function

$$\Delta_4\, G(x,y) = \delta(x,y) .$$

Using these formulas we find, for any $A(g_{\mu\nu}) = A(\bar{g}_{\mu\nu})$, the relation

$$\frac{\delta}{\delta\sigma}\int_x A\;(E - \frac{2}{3}\Box R)\Big| = 4\sqrt{-g}\Delta_4 A .$$

where $\int_x = \int d^4x\sqrt{-g(x)} , \qquad \Big| = \Big|_{\bar{g}_{\mu\nu}\to g_{\mu\nu}}$

As a consequence, we obtain

$$\frac{\delta}{\delta\sigma(y)} \iint_{x\,y} \frac{1}{4} C^2(x)\, G(x,y)\, \left(E - \frac{2}{3}\Box R\right)_y \Big|$$

$$= \int d^4x \sqrt{-\bar{g}(x)}\, \bar{\Delta}_4(x)\, \bar{G}(x,y)\, \bar{C}^2(x)$$

$$= \sqrt{-g}\, C^2(y)\,.$$

Hence, the part of Γ_{ind} which is responsible for $T_\omega = -\omega C^2$, is

$$\Gamma_\omega = \frac{\omega}{4} \iint_{x\,y} C^2(x)\, G(x,y)\, (E - \frac{2}{3}\Box R)_y$$

Similarly one can check that the variation $T_b = b\,(E - \frac{2}{3}\Box R)$ is produced by

$$\Gamma_b = \frac{b}{8} \iint_{x\,y} (E - \frac{2}{3}\Box R)_x\, G(x,y)\, (E - \frac{2}{3}\Box R)_y\,.$$

Finally, we can use simple relation

$$g_{\mu\nu}\,\frac{\delta}{\delta g_{\mu\nu}}\int_x R^2(x) \;=\; -6\,\sqrt{-g}\,\Box R\,.$$

to establish the remaining **local** constituent of Γ_{ind}

$$\Gamma_c = -\frac{3c+2b}{36(4\pi)^2}\int d^4x\sqrt{-g(x)}\,R^2(x)\,.$$

The **general covariant solution** for Γ_{ind} is the sum,

$$\Gamma_{ind} = S_c[g_{\mu\nu}] - \frac{3c+2b}{36(4\pi)^2}\int d^4x\sqrt{-g(x)}\,R^2(x)$$

$$+\frac{\omega}{4}\int_x\int_y C^2(x)\,G(x,y)\,(E-\frac{2}{3}\Box R)_y$$

$$+\frac{b}{8}\int_x\int_y (E-\frac{2}{3}\Box R)_x\,G(x,y)\,(E-\frac{2}{3}\Box R)_y\,.$$

One can rewrite this expression using **auxiliary scalars**.

The nonlocal terms can be rewritten in a **symmetric form**

$$\left(E - \frac{2}{3}\Box R\right)_x G(x,y)\left[\frac{a}{4}C^2 - \frac{b}{8}(E - \frac{2}{3}\Box R)\right]_y$$

$$= -\frac{b}{8}\int_x\int_y \left(E - \frac{2}{3}\Box R - \frac{\omega}{b}C^2\right)_x G(x,y)\left(E - \frac{2}{3}\Box R - \frac{a}{b}C^2\right)_y$$

$$- \frac{\omega^2}{8b}\int_x\int_y C^2_x\, G(x,y)\, C^2_y\,.$$

The last two terms are appropriate objects for rewriting them via the auxiliary fields! **We arrive at the local covariant expression for EA**

$$\Gamma_{ind} = S_c[g_{\mu\nu}] - \frac{3c+2b}{36(4\pi)^2}\int_x R^2(x) + \int_x \left\{\frac{1}{2}\varphi\Delta_4\varphi - \frac{1}{2}\psi\Delta_4\psi\right.$$

$$\left. + \frac{a}{8\pi\sqrt{-b}}\psi\, C^2 + \varphi\left[\frac{\sqrt{-b}}{8\pi}(E - \frac{2}{3}\Box R) - \frac{a}{8\pi\sqrt{-b}}C^2\right]\right\}.$$

The above form of EA is **the best one** for Γ_{ind}.
Jacksenaev, Sh., Phys. Lett. B (1994)

Similar expression has been independently introduced by
Mazur & Mottola, 1997-1998.

Some remarks are in order.
1) The local covariant form is dynamically equivalent to the non-local covariant form. The definition of the Cauchy problem for a non-local action requires boundary conditions for the Green functions $G(x,y)$ in the two terms Γ_a and Γ_b. The same can be achieved by imposing the boundary conditions on the two auxiliary fields φ and ψ.

2) Introducing the new term $\int C_x^2 G(x,y) C_y^2$ into the action may be viewed as redefinition of the conformal invariant functional $S_c[g_{\mu\nu}]$. However, writing the non-conformal terms in the symmetric form, we have modified the four point function in an essential way. Using ψ we restore the structure of the terms generated by anomaly.

3) **Importance for applications.** E.g., the **vacuum states of the black hole** (Boulware, Hartle-Hawking and Unruh) **can be classified by the initial conditions for the auxiliary fields**

Balbinot, Fabbri, Sh., 1999.

This can not be accomplished by using only one field φ.

Another important application is the anomaly-induced inflation, or the **Modified Starobinsky Model.**

Fabris, Pelinson, Solà, I. Sh. et al. (1998-2003)

4) **The** Δ_4**- based form of anomaly-induced effective action indicates a very complicated resummation of the loop and curvature expansions.**

S. Deser and A. Schwimmer, PLB 309 (1993);
S. Deser, PLB 479 (2000).

Ambiguity of local anomalous terms

The ambiguity of local anomalous term $\int \sqrt{-g} R^2$ in the effective action and the corresponding term $\Box R$ in the anomaly can be observed either in dimensional or covariant Pauli-Villars regularization.

Consider the dimensional regularization.

As we have already mentioned, the counterterm $\int \sqrt{-g} \Box R$ doesn't contribute to anomalous violation of local conformal symmetry.

According to Duff (1977), the anomaly comes from the $\int \sqrt{-g} C^2(d)$-type counterterm.

$$-\frac{2}{\sqrt{-g}}\, g_{\mu\nu}\, \frac{\delta}{\delta g_{\mu\nu}} \int d^n x \sqrt{-g}\, C^2(d) \Big|_{d=4,\, n\to 4} = C^2 - \frac{2}{3} \Box R\,.$$

However, the requirements of finiteness of renormalized effective and the locality leaves the freedom to choose the parameter d. If we take

$$d = n + \gamma \cdot [n - 4] \,,$$

where γ is an arbitrary parameter, we meet $a' \sim \gamma$.

The coefficient α' is arbitrary. Changing α' is equivalent to adding a $\int R^2$-term to the classical action.

Why we are allowed to add the $\int R^2$-term?

Because it would be a component of the action of **external fields**, which doesn't influence the dynamics of **quantum** fields.

In order to fix the arbitrariness in $\int R^2$ -term, one has to do the following:

- Introduce it into the classical action.
- Calculate quantum correction.
- Fix overall $\int R^2$-term by the renormalization condition.

Consider massive scalar field (Gorbar & I.Sh.) JHEP, 2003.

$$L = (1/2)\left\{(\nabla\varphi)^2 + m^2\varphi^2 + \left(\tilde{\xi} + 1/6\right) R\varphi^2\right\}.$$

In the second order in curvatures

$$\bar{\Gamma}^{(1)} = \int \frac{d^4x\sqrt{g}}{2(4\pi)^2}\, R\left\{\frac{\tilde{\xi}^2}{2\epsilon} + A\tilde{\xi}^2 + \frac{\tilde{\xi}A(4-a^2)}{6a^2}\right.$$

$$\left. + \frac{\tilde{\xi}}{18} + \frac{A\left(16 - 8a^2 + a^4\right)}{144a^4} + \frac{20 - 7a^2}{2160\,a^2}\right\} R + \dots,$$

$$A = 1 + \frac{1}{a}\ln\left|\frac{2-a}{2+a}\right| \quad \text{and} \quad a^2 = \frac{4\Box}{4m^2 - \Box}.$$

$$m = 0,\ \ \xi = 1/6 \quad \Rightarrow \quad -\frac{1}{12\cdot 180(4\pi)^2}\int d^4x\, g^{1/2}\, R^2,$$

fitting perfectly with the conformal anomaly obtained by point-splitting *Christensen, PRD 1978,* ζ-reg. *Cristley & Dowker, 76; Hawking, 77 ...*

$$< T^{\mu}_{\mu} >= \frac{1}{180(4\pi)^2}\Box R + \dots.$$

In the covariant Pauli-Villars regularization one has to introduce a set of "regulator" fields. E.g., in case of a massless conformal scalar φ we have to start from the action

$$S_{\rm reg} = \sum_{i=0}^{N} \int d^4x \sqrt{g}\, \{ (\nabla \varphi_i)^2 + (\xi_i R + m_i^2)\varphi_i^2 \} .$$

The physical scalar field $\varphi \equiv \varphi_0$ is conformal $\xi = 1/6$, $m_0 = 0$ and bosonic $s_0 = 1$, while PV regulators φ_i are massive $m_i = \mu_i M$ and can be bosonic $s_i = 1$ or fermionic $s_i = -2$.

The UV limit $M \to \infty$ produces the vacuum Eff. Action. The calculation is based on our result for the EA of the massive scalar. We assume that the Pauli-Villars regulators may have conformal $\xi_i = 1/6$ or non-conformal couplings $\xi_i \neq 1/6$.

Direct calculation shows that the $\int R^2$-term depends on the choice of ξ_i and hence is arbitrary
Asorey, Gorbar & I.Sh., CQG **21** *(2003).*

Even stronger arbitrariness

Consider interacting conformal scalar theory

$$S = \int d^4x\sqrt{g}\,\Big\{\frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi + \frac{1}{12}R\phi^2 - \frac{\lambda}{24}\phi^4\Big\}\,.$$

The Noether identity $\mathcal{T} = \frac{1}{\sqrt{g}}\Big(\phi\,\frac{\delta S}{\delta\phi} - 2\,g_{\mu\nu}\,\frac{\delta S}{\delta g_{\mu\nu}}\Big) = 0\,.$

At quantum level it is indeed violated $<\mathcal{T}>\neq 0$.

The ambigous part of anomaly $<\mathcal{T}> = \alpha_2'\Box\phi^2 + ...$ because the corresponding term in the EA is local

$$\Gamma_{ind} = \frac{\alpha'}{6}\int d^4x\sqrt{g}\,R\phi^2 + ...$$

Asorey, Berredo-Peixoto,Sh, PRD-2006.

Changing $R\varphi^2$-term in the classical action implies an essential change in the dynamics of quantum fields !!

Some examples of using the anomaly-induced EA

- **Classification of vacuum states in the vicinity of a black hole**

Anomaly is, in part, responsible for the Hawking radiation
S.M. Christensen, S.A. Fulling, PRD (1977).

The anomaly-induced effective action of gravity enables one to perform a kind of systematic classification of the vacuum states for the quantum fields on the black hole background.

We can distinguish the different vacuum states by choosing different boundary conditions for the auxiliary fields φ and ψ.

R. Balbinot, A. Fabbri & I.Sh., PRL **83**; *NPB* **559** *(1999).*

Generalization for the Reissner-Nordstrom black hole,
P.R. Anderson, E. Mottola & R. Vaulin, PRD **76** *(2007).*

At the classical level, the black hole (BH) does not emit radiation, but such emission can take place if we take quantum effects into account.

After the theoretical discovery by Hawking (1975), the same result has been obtained from analytical estimates of $\langle T_{\mu\nu} \rangle$ for matter fields propagating in a fixed Schwarzschild BH geometry.

Detailed analytical and numerical study, based on the analysis of $\langle T_{\mu\nu} \rangle$ in the classical black hole background:

P. Candelas, PRD **21** *(1980);*
D.N. Page, PRD **25** *(1982);*
M.R. Brown, A.C. Ottewill and D.N. Page, PRD **33** *(1986);*
V.P. Frolov and A.I. Zelnikov, PRD **35** *(1987);*
P.R. Anderson, W.A. Hiscock and D.A. Samuel, PRD **51** *(1995). ...*

A fundamental property is the existence of three different vacuum quantum states.

i) **The Boulware** $|B\rangle$ state reproduces the Minkowski vacuum $|M\rangle$ in the limit $r \to \infty$, where $\langle B|T_{\mu\nu}|B\rangle \sim r^{-6}$.
On the horizon this quantity is divergent in a free falling frame.

ii) **For Unruh vacuum** $|U\rangle$ the value $\langle U|T_{\mu\nu}|U\rangle$ is regular on the future event horizon but not on the past one. Asymptotically in the future $\langle U|T_{\mu\nu}|U\rangle$ has the form of a flux of radiation at the Hawking temperature $T_H = 1/8\pi M$.
This vacuum state is the most appropriate to discuss evaporation of black holes formed by gravitational collapse of matter.

iii) **The Israel-Hartle-Hawking** $|H\rangle$ state
$\langle H|T_{\mu\nu}|H\rangle$ for $r \to \infty$ describes a thermal bath of radiation at T_H.

The existence of three vacuum states reflects distinct positions of observers and the construction of different *in* and *out* modes with respect to the corresponding coordinates.

The main difference between classical and quantum theories is that, in the first case we know how to transform the relevant quantities when we change the coordinate system.

The natural question is how to perform a transition between different vacuum states $|H\rangle$, $|B\rangle$ and $|U\rangle$?

The anomaly-induced effective action doesn't make any reference to a particular quantum state, but it includes the conformal invariant functional $S_c[g_{\mu\nu}]$ – a source of uncertainty.

Strategy: one has to fix the extended set of boundary conditions, including the ones for the auxiliary scalars φ and ψ.

The procedure for identifying the vacuum state is as follows:

1) Solving equations for φ and ψ.
The solutions always depend on the set of integration constants.

2) One has to find "appropriate" boundary conditions to identify $\langle V|T_{\mu\nu}|V\rangle$ for the given vacuum state $|V\rangle = (|B\rangle, |U\rangle, |H\rangle)$.

3) Use

$$\langle T_{\mu\nu}\rangle \longrightarrow \frac{2}{\sqrt{-g}}\frac{\delta\Gamma_{ind}}{\delta g_{\mu\nu}} = \langle S_{\mu\nu}\rangle$$

where of course $\langle S\rangle = \langle T\rangle$.

The general solution is $\phi(r,t) = d \cdot t + w(r)$, where $w(r)$ satisfies the equation

$$\frac{dw}{dr} = \frac{B}{3}r + \frac{2MB}{3} - \frac{A}{6} - \frac{\alpha}{72M} + \frac{1}{r-2M}\left(\frac{4}{3}BM^2 + \frac{C}{2M} - AM - \frac{\alpha}{24}\right) - \frac{C}{2Mr}$$

$$-\left[\frac{\alpha M}{r^3} + \frac{24AM-\alpha}{144\,M^2}\right]\frac{r^2\ln r}{r-2M} + \frac{(24AM-a)(r^3-8M^3)\ln(r-2M)}{3r(r-2M)48M^2}.$$

(d, A, B, C) are constants which specify the homogeneous solution $\Box^2\phi = 0$ and hence the quantum state.

For ψ we have a similar solution, but with (d', A', B', C').

In case of a Boulware state $|B\rangle$ we request

$$|B\rangle \to |M\rangle \quad \textbf{when} \quad r \to \infty \,.$$

In the Minkowski vacuum we can safely set $\varphi = \psi = 0$.

This asymptotic conditions enables one to arrive at the asymptotic expressions

$$\langle B|S_\mu^\nu|B\rangle \to \frac{1}{2}\frac{\alpha^2-\beta^2}{(24)^2}\frac{1}{(2M)^4\,(1-2M/r)^2} \times \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1/3 & 0 & 0 \\ 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 1/3 \end{pmatrix}$$

for $r \to 2M$ and

$$\langle B|S_\mu^\nu|B\rangle \propto \mathcal{O}(r^{-6}) \qquad \textbf{for} \qquad r \to \infty \,.$$

This behavior perfectly fits the ones observed within other methods.

Unruh vacuum case

Choosing another values of the integration constants we meet the following asymptotic behavior near the horizon $r \to 2M$:

$$\langle U|S_a{}^b|U\rangle \sim \frac{\alpha^2-\beta^2}{2(48M^2)^2}\begin{pmatrix} 1/f & -1 \\ 1/f^2 & -1/f \end{pmatrix},$$

regular on the future horizon, $a, b = r, t$. The asymptotic form

$$r \to \infty \qquad \langle U|S_\mu^{\ \nu}|U\rangle \to \frac{\alpha^2-\beta^2}{2r^2(24M)^2}\begin{pmatrix} -1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

These results are in exact agreement with the standard ones on Hawking radiation:

B.S. DeWitt, Phys. Rep. **C19** *(1975) 297.*

once the luminosity L of the radiating BH is identified with

$$\frac{L}{4\pi} = \frac{(\alpha^2-\beta^2)}{2(24M)^2}.$$

A bit more complicated situation takes place for the Hartle-Hawking vacuum.

One should not only properly choose initial conditions but also fine-tune the coefficient l_1 to order to achieve correspondence with the results achieved by other methods.

From the general perspective this situation looks somehow natural.

The unknown conformal invariant functional S_c is relevant for the spherically symmetric metric.

In case of the Hartle-Hawking vacuum, when the thermodynamic aspects play more important role, one can not really expect that the conformal anomaly gives complete description of the situation.

Cosmological Model based on the action

$$S_{total} = -\frac{M_P^2}{16\pi}\int d^4x\sqrt{-g}\,(R+2\Lambda) + S_{matter} + S_{vac} + \bar{\Gamma}_{ind}\,.$$

Equation of motion for $a(t)$, $dt = a(\eta)\,d\eta$

$$\frac{\dddot{a}}{a} + \frac{3\dot{a}\ddot{a}}{a^2} + \frac{\ddot{a}^2}{a^2} - \left(5+\frac{4b}{c}\right)\frac{\ddot{a}\dot{a}^2}{a^3} - 2k\left(1+\frac{2b}{c}\right)\frac{\ddot{a}}{a^3}$$

$$-\frac{M_P^2}{8\pi c}\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} - \frac{2\Lambda}{3}\right) = 0\,,$$

$k = 0, \pm 1$. Particular solutions (Starobinsky, PhLB-1980)

$$a(t) = a_0 e^{Ht}\,, \qquad k = 0\,,$$

where Hubble parameter $H = \dot{a}/a$ is

$$H^2 = -\frac{M_P^2}{32\pi b}\left(1 \pm \sqrt{1 + \frac{64\pi b}{3}\frac{\Lambda}{M_P^2}}\,\right).$$

For $0 < \Lambda \ll M_P^2$ there are two solutions:

$$H \approx \sqrt{\Lambda/3}\,; \qquad (IR)$$

$$H \approx \sqrt{-\frac{M_P^2}{16\pi b} - \frac{\Lambda}{3}} \approx \frac{M_P}{\sqrt{-16\pi b}}\,. \qquad (UV)$$

Perturbations of the conformal factor

$$\sigma(t) \to \sigma(t) + y(t).$$

The criterion for a **stable** (UV) **inflation**

$$c > 0 \iff N_1 < \frac{1}{3} N_{1/2} + \frac{1}{18} N_0\,,$$

in agreement with Starobinsky (1980).

The original Starobinsky model is **based on the unstable case** and involves **fine-tunings.** Our purpose is to avoid fine-tuning.

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iii) **The Israel-Hartle-Hawking** $|H\rangle$ state
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Consider massive scalar field (Gorbar & I.Sh.) JHEP, 2003.

$$L = (1/2)\left\{(\nabla\varphi)^2 + m^2\varphi^2 + \left(\tilde{\xi} + 1/6\right) R\varphi^2\right\}.$$

In the second order in curvatures

$$\bar{\Gamma}^{(1)} = \int \frac{d^4x\sqrt{g}}{2(4\pi)^2}\, R \left\{ \frac{\tilde{\xi}^2}{2\epsilon} + A\tilde{\xi}^2 + \frac{\tilde{\xi}A(4-a^2)}{6a^2}\right.$$

$$\left. + \frac{\tilde{\xi}}{18} + \frac{A\left(16 - 8a^2 + a^4\right)}{144a^4} + \frac{20 - 7a^2}{2160\, a^2} \right\} R + ...,$$

$$A = 1 + \frac{1}{a}\ln\left|\frac{2-a}{2+a}\right| \quad \text{and} \quad a^2 = \frac{4\Box}{4m^2 - \Box}.$$

$$m = 0,\ \xi = 1/6 \quad \Rightarrow \quad -\frac{1}{12\cdot 180(4\pi)^2}\int d^4x\, g^{1/2}\, R^2,$$

fitting perfectly with the conformal anomaly obtained by point-splitting *Christensen, PRD 1978,* ζ-reg. *Cristley & Dowker, 76; Hawking, 77 ...*

$$< T^\mu_\mu > = \frac{1}{180(4\pi)^2}\Box R + ...$$

The nonlocal terms can be rewritten in a **symmetric form**

$$\left(E - \frac{2}{3}\Box R\right)_x G(x,y)\left[\frac{a}{4}C^2 - \frac{b}{8}(E - \frac{2}{3}\Box R)\right]_y$$

$$= -\frac{b}{8}\int_x\int_y \left(E - \frac{2}{3}\Box R - \frac{\omega}{b}C^2\right)_x G(x,y)\left(E - \frac{2}{3}\Box R - \frac{a}{b}C^2\right)_y$$

$$- \frac{\omega^2}{8b}\int_x\int_y C_x^2\, G(x,y)\, C_y^2\,.$$

The last two terms are appropriate objects for rewriting them via the auxiliary fields! **We arrive at the local covariant expression for EA**

$$\Gamma_{ind} = S_c[g_{\mu\nu}] - \frac{3c+2b}{36(4\pi)^2}\int_x R^2(x) + \int_x \left\{\frac{1}{2}\varphi\Delta_4\varphi - \frac{1}{2}\psi\Delta_4\psi\right.$$

$$\left. + \frac{a}{8\pi\sqrt{-b}}\psi\, C^2 + \varphi\left[\frac{\sqrt{-b}}{8\pi}(E - \frac{2}{3}\Box R) - \frac{a}{8\pi\sqrt{-b}}C^2\right]\right\}.$$

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Now we obtain the **non-local covariant** solution and after represent it in the local form using **auxiliary fields.**
First one has to establish the relations

$$\sqrt{-g}C^2 = \sqrt{-\bar{g}}\bar{C}^2 \,, \qquad \sqrt{-\bar{g}}\,\bar{\Delta}_4 = \sqrt{-g}\,\Delta_4 \,.$$

$$\sqrt{-g}(E - \frac{2}{3}\Box R) = \sqrt{-\bar{g}}(\bar{E} - \frac{2}{3}\bar{\Box}\bar{R} + 4\bar{\Delta}_4\sigma)$$

and also introduce the Green function

$$\Delta_4\, G(x,y) = \delta(x,y)\,.$$

Using these formulas we find, for any $A(g_{\mu\nu}) = A(\bar{g}_{\mu\nu})$, the relation

$$\frac{\delta}{\delta\sigma}\int_x A\;(E - \frac{2}{3}\Box R)\Big| = 4\sqrt{-g}\Delta_4\, A\,.$$

where $\int_x = \int d^4x\sqrt{-g(x)}\,, \qquad \Big| = \Big|_{\bar{g}_{\mu\nu}\to g_{\mu\nu}}$

The simplest possibility is to parametrize metric

$$g_{\mu\nu} = \bar{g}_{\mu\nu} \cdot e^{2\sigma}\,, \qquad \sigma = \sigma(x)\,.$$

The solution for the effective action is

$$\bar{\Gamma}_{ind} = S_c[\bar{g}_{\mu\nu}] + \frac{1}{(4\pi)^2}\int d^4x\sqrt{-\bar{g}}\,\{\omega\sigma\bar{C}^2 \qquad (1)$$

$$+\, b\sigma(\bar{E} - \frac{2}{3}\bar{\Box}\bar{R}) + 2b\sigma\bar{\Delta}_4\sigma - \frac{1}{12}(c + \frac{2}{3}b)[\bar{R} - 6(\bar{\nabla}\sigma)^2 - (\bar{\Box}\sigma)]^2)\}\,,$$

where $S_c[\bar{g}_{\mu\nu}] = S_c[g_{\mu\nu}]$ is an **unknown conformal functional** of the metric, which serves as an **integration constant** in eq. for Γ_{ind}.

The solution (1) has great merits:
1) Being simple, 2) Being exact in case $S_c[\bar{g}_{\mu\nu}]$ is irrelevant.
Example: FRW metrics.

An important disadvantage is that it is not covariant or, in other words, it is not expressed in terms of original metric $g_{\mu\nu}$.

In case of a Boulware state $|B\rangle$ we request

$$|B\rangle \rightarrow |M\rangle \quad \textbf{when} \quad r \rightarrow \infty \, .$$

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for $r \rightarrow 2M$ and

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in agreement with Starobinsky (1980).

The original Starobinsky model is **based on the unstable case** and involves **fine-tunings.** Our purpose is to avoid fine-tuning.

Suppose at **UV** $(H \gg M_F)$ there is **SUSY**, e.g. **MSSM** with a particle content

$$N_1 = 12\,, \qquad N_{1/2} = 32\,, \qquad N_0 = 104\,.$$

This provides **stable inflation**, because $c > 0$

$$N_1 \, < \, \frac{1}{3}\, N_{1/2} + \frac{1}{18}\, N_0\,.$$

Similar for any realistic SUSY model. Inflation is **independent on initial data.**

Fine!

But why should inflation end?
Already for MSM $(\,N_{1,\,1/2,\,0} = 12, 24, 4)$, $\;c < 0$, **inflation is unstable.**

Natural interpretation: I.Sh. Int.J.Mod.Ph. *11D* (2002)

All **sparticles** are heavy $\Rightarrow$ **decouple**, when H becomes smaller than their masses.

According to our calculations (Gorbar, I.Sh. JHEP,2003) the transition $c > 0 \Longrightarrow c < 0$ is smooth, giving a hope for a graceful exit.

Simple test of the model. Late Universe, $k = 0$, $H_0 = \sqrt{\Lambda/3}$.
Only photon is active

$$N_0 = 0\,, \qquad N_{1/2} = 0\,, \qquad N_1 = 1\,.$$

Graviton typical energy is $H_0 \approx 10^{-42}\, GeV$, $\Longrightarrow$ all massive particles (even neutrino) $m_\nu \geq 10^{-12}\, GeV$ **decouple** from gravity. $c < 0 \quad \Longrightarrow$ **today inflation is unstable.**

Stability for the small $H = H_0$ **case:** $H \to H_0 + const \cdot e^{\lambda t} \quad \Longrightarrow$

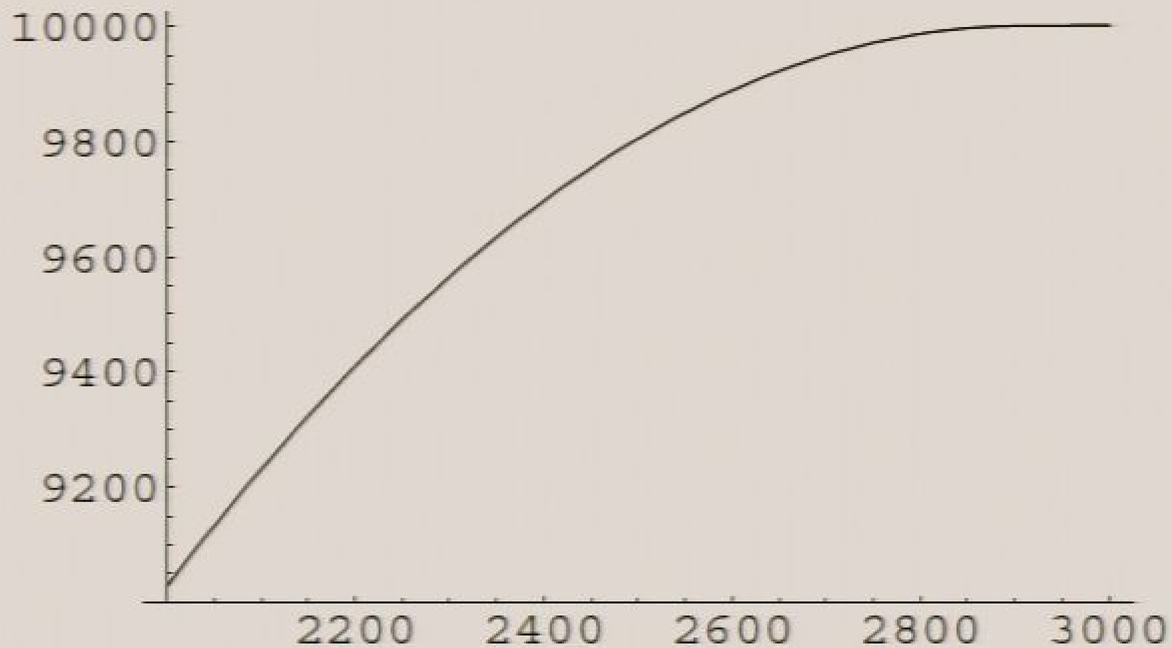
$$\lambda^3 + 7H_0\lambda^2 + \left[\frac{(3c-b)4{H_0}^2}{c} - \frac{M_P^2}{8\pi c}\right]\lambda - \frac{32\pi b{H_0}^3 + M_P^2 H_0}{2\pi c} = 0\,.$$

The solutions are $\lambda_1 = -4H_0\,, \quad \lambda_{2/3} = -\frac{3}{2}H_0 \pm \frac{M_P}{\sqrt{8\pi|c|}}\,i\,.$

$\Lambda > 0$ **protects our world from quantum corrections!**

Anomaly-induced inflation slows down if taking masses of quantum fields into account.

Sh., Solà, Phys.Lett. 530B (2002);
Pelinson, Sh. & Takakura, Nucl.Ph. 648B (2003).



$$\sigma(t) = \ln a(t) \approx H_0\, t - \frac{H_0^2}{4}\,\tilde{f}\, t^2\,, \qquad H_0 \propto M_P$$

The total amount of e-folds may be as large as 10^{32}, **but only** 65 **last ones**, where $H \propto M_*$ (SUSY breaking scale) **are relevant.**

Other features of the Modified Starobinsky Model.

In the last 65 *e*-folds the production of gravitational waves is restricted

$$H(t) \ll 10^{-5} M_P.$$

Furthermore, once created, in this model **gravitational waves do not amplify.**

Fabris, Pelinson, Sh., Nucl.Phys. *597B*(2001);
Pelinson, Sh., Takakura, Nucl.Phys. *648B* (2003);
Fabris, Pelinson, Sh, Takakura, NPB(PS) *127*(2004).

All in all, modified Starobinsky model is a **promising candidate** to describe inflation in a natural way. However, **small information is available about intermediate stage of inflation.**

In order to obtain this information one needs further development of QFT in curved space-time.

Quantum correction to photon sector

Classical action:

$$S_{em} = -\frac{1}{4}\int d^4x\,\sqrt{g}\,F_{\mu\nu}\,F^{\mu\nu}$$

possesses local conformal invariance,

$$g_{\mu\nu} \to g'_{\mu\nu} = g_{\mu\nu}\,e^{2\sigma(x)}\,, \quad A_\mu \to A'_\mu = A_\mu\,,$$

Why it is important that the local conformal symmetry is violated at quantum level?

1) Broken conformal symmetry changes the equation of state (EOS) of radiation. Modified EOS for the radiation may lead to interesting new cosmological models.
Modified EOS changes the expansion of the Universe, entropy production, affects the red-shift dependence of CMB, etc.

2) The broken conformal symmetry may affect the rate of photons creation in the reheating period, leading to potentially observable consequences.

3) The conformal symmetry violation can be important for creation of initial seeds of magnetic field of the galaxies.

4) Quantum corrections can move the pole in the photon propagator and make the speed different from the one in the classical case.

Similar effect occurs in curved space,

The two main questions are as follows:

1) What is the mechanism of violation of the local conformal symmetry at the quantum level?

2) To which extent the finite quantum corrections and, in particular, violation of local conformal symmetry, are universal?

In other words, do we have an ambiguity in the quantum terms?

- **Incorporating the electromagnetic field into the anomaly-induced effective action of $g_{\mu\nu}$ and A_μ is very simple.**

We only need to replace

$$wC^2 \quad \to \quad wC^2 + \beta F_{\mu\nu}^2$$

everywhere. The result is

$$\Gamma_{ind} \;=\; S_c[g_{\mu\nu}, A_\mu] - \frac{3c+2b}{36(4\pi)^2}\int_x R^2(x)$$

$$+ \int_x \left\{ \frac{1}{2}(\varphi\Delta_4\varphi - \psi\Delta_4\psi) + \frac{1}{8\pi\sqrt{-b}}(\psi - \varphi)\left(wC^2 + \beta F_{\mu\nu}^2\right)\right.$$

$$\left. + \frac{\sqrt{-b}}{8\pi}\varphi\left(E - \frac{2}{3}\Box R\right)\right\}.$$

What about quantum effects of massive fields?

The one-loop effective action (EA) in the case of $g_{\mu\nu}$ and A_μ background can be defined via the path integral

$$e^{i\Gamma^{(1)}[g_{\mu\nu},\, A_\mu]} = \int D\psi D\bar{\psi}\, e^{iS_{QED}}\,,$$

or

$$\bar{\Gamma}^{(1)} = -\frac{1}{2}\,\text{Log Det}\,\hat{H}\,,$$

where

$$\hat{H} = i\left(\gamma^\mu \nabla_\mu - iM - ie\gamma^\mu A_\mu\right)$$

We use heat-kernel method and the Schwinger-DeWitt technique in curved space QED.

Reducing the problem to the derivation of Log Det $\hat{\mathcal{O}}$,

$$\hat{\mathcal{O}} = \hat{\Box} + 2\hat{h}^\mu \nabla_\mu + \hat{\Pi} \, .$$

Multiply $\hat{H}$ by an appropriate conjugate $\hat{H}^*$,

$$\hat{\mathcal{O}} = \hat{H} \cdot \hat{H}^*$$

and use the relation Log Det $\hat{H}$ = Log Det $\hat{\mathcal{O}}$ − Log Det $\hat{H}^*$.

The simplest choice,

$$\hat{H}_1^* = -i \left(\gamma^\mu \nabla_\mu + iM - ie\gamma^\mu A_\mu \right) .$$

According to *G. De Berredo-Peixoto, M.Ph.L.A16 (2001)*

$$\text{Log Det } \hat{H} = \text{Log Det } \hat{H}_1^* \quad \Rightarrow \quad \text{Log Det } \hat{H} = \frac{1}{2} \text{Log Det } (\hat{H}\hat{H}_1^*) \, .$$

In 1989-1990 **Avramidy and Barvinsky & Vilkovisky derived a very important summation of the Schwinger-DeWitt series.**

Application to massive fields:

E.Gorbar & I.Sh., JHEP-2003,2004.

Our result for the one-loop Euclidean effective action is

$$\bar{\Gamma}^{(1)}_{\sim F^2} = \frac{e^2}{2(4\pi)^2} \int d^4x \sqrt{g}\, F_{\mu\nu} \left[\frac{2}{3\epsilon} + k_1^{FF} \right] F^{\mu\nu},$$

where $$k_1^{FF} = k_1^{FF}(a) = Y\left(2 - \frac{8}{3a^2}\right) - \frac{2}{9}.$$

$$Y = 1 - \frac{1}{a} \ln\left(\frac{2+a}{2-a}\right), \qquad a^2 = \frac{4\Box}{\Box - 4m^2}.$$

This expression represents a complete one-loop contribution.

Goncalves, de B.-Peixoto & I.Sh., hep-th/0904.4171

Conclusions

1. There is a certain variety of theories with local conformal symmetry in $D = 4$.

2. In the semiclassical theory local conformal symmetry is violated by the anomaly. Local terms in the anomaly-induced EA are plagued by ambiguities, indicating certain inconsistency.

3. For the interacting conformal scalar field (and also for the conformal quantum gravity) similar ambiguities produce an inconsistency of the theory beyond the one loop level.

4. In general, conformal invariant theories are not consistent at quantum level. The conformal symmetry may be only approximate, despite it is a very useful tool for calculating quantum corrections.

We use heat-kernel method and the Schwinger-DeWitt technique in curved space QED.

Reducing the problem to the derivation of Log Det $\hat{\mathcal{O}}$,

$$\hat{\mathcal{O}} = \hat{\Box} + 2\hat{h}^{\mu}\nabla_{\mu} + \hat{\Pi}\,.$$

Multiply $\hat{H}$ by an appropriate conjugate $\hat{H}^{*}$,

$$\hat{\mathcal{O}} = \hat{H} \cdot \hat{H}^{*}$$

and use the relation $\text{Log Det } \hat{H} = \text{Log Det } \hat{\mathcal{O}} - \text{Log Det } \hat{H}^{*}\,.$

The simplest choice,

$$\hat{H}_1^{*} = -i\left(\gamma^{\mu}\nabla_{\mu} + iM - ie\gamma^{\mu}A_{\mu}\right).$$

According to *G. De Berredo-Peixoto, M.Ph.L.A16 (2001)*

$$\text{Log Det } \hat{H} = \text{Log Det } \hat{H}_1^{*} \quad \Rightarrow \quad \text{Log Det } \hat{H} = \frac{1}{2}\,\text{Log Det } (\hat{H}\hat{H}_1^{*})\,.$$