Title: Quantum Field Theory (PHYS 601) - Lecture 2

Date: Sep 29, 2009 09:00 AM

URL: http://pirsa.org/09090109

Abstract:

Recall: $\phi_a(\underset{\sim}{x}, t)$.

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Lorentz Transformations: We want to construct relativistic field theories, with time and space placed on equal footing

The theory should be invariant under Lorentz transformations:

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e.g. $\Lambda^\mu{}_\sigma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ rotates by θ about z-axis

or $\Lambda^{\mu}{}_{\sigma} = \begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ is a boost by v along the x-axis, with $\gamma = \frac{1}{\sqrt{1-v^2}}$

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The Lorentz transformations have a representation on fields. For a scalar field, this is

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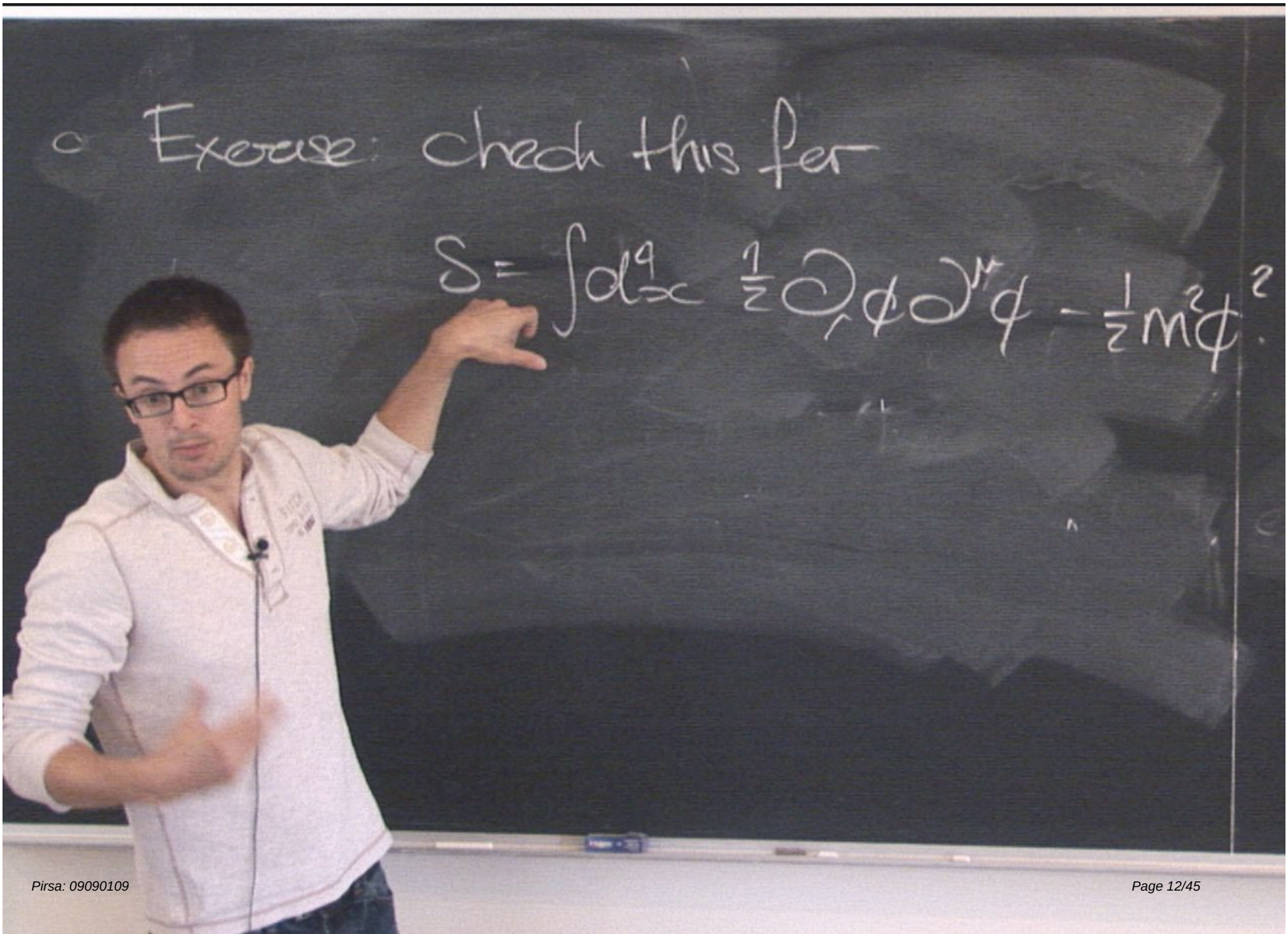
We can ensure that this is true by requiring that S is invariant.


Exercise: check this for
$S = \int d^4x \, \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2$

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An example of a Lorentz transformation on a vector field, $A_\mu(x)$.

$$A_\mu(x) \to A'_\mu(x) = \Lambda_\mu{}^\nu A_\nu(\Lambda^{-1} x)$$

Symmetries + Noether's Theorem

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Noether's Thm: Every continuous symmetry of the Lagrangian gives rise to a conserved current $j^\mu(x)$, such that the equations of motion imply

$$\boxed{\partial_\mu j^\mu(x) = 0}$$

i.e. $$\frac{\partial}{\partial t} j^0 + \nabla \cdot \underline{j} = 0$$

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<u>ote</u>: A conserved current $\Rightarrow$ a conserved <u>charge</u>

$$Q = \int_{\mathbb{R}^3} d^3x \, j^0$$

which follows from $\dot{Q} = \int_{\mathbb{R}^3} \frac{\partial j^0}{\partial t} \, d^3x$

$$= -\int_{\mathbb{R}^3} \nabla \cdot j \, d^3x$$

$$= 0$$ assuming $j \to 0$ as $x \to \infty$.

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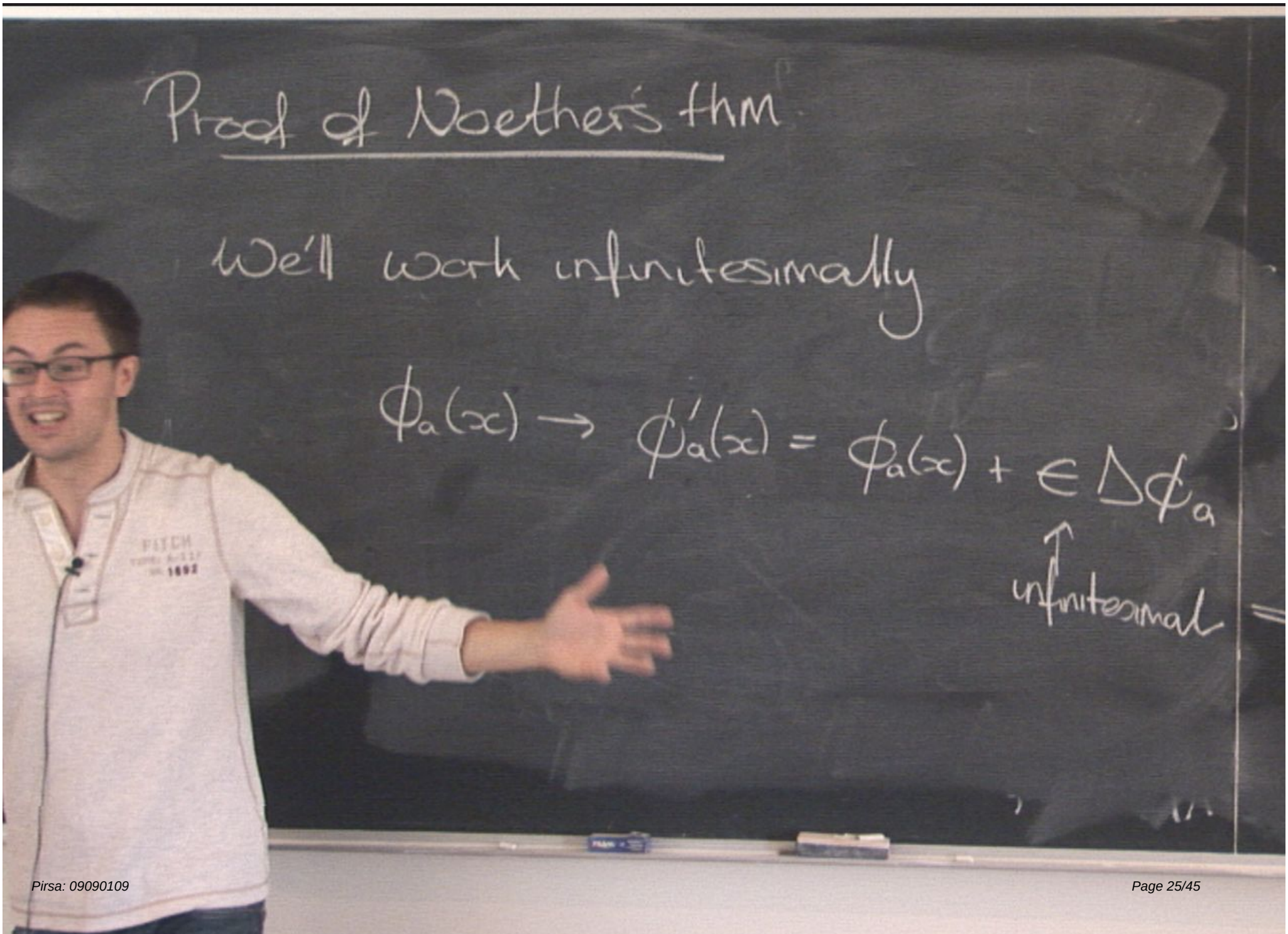
$$Q_V = \int_V j^0 \, d^3\underline{x}$$

$$\dot{Q}_V = -\int_V \underline{\nabla} \cdot \underline{j} \, d^3\underline{x} = -\int_A \underline{j} \cdot d\underline{S}$$


Proof of Noether's thm
We'll work infinitesimally
$\phi_a(x) \to \phi'_a(x) = \phi_a(x) + \epsilon \Delta\phi_a$
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This is a symmetry if $\delta\mathcal{L} = 0$.

Consider making an arbitrary transformation

$$\phi_a \to \phi_a + \delta\phi_a$$

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi_a}\delta\phi_a + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta(\partial_\mu\phi_a)$$

$$= \left[\frac{\partial\mathcal{L}}{\partial\phi_a}\right.$$

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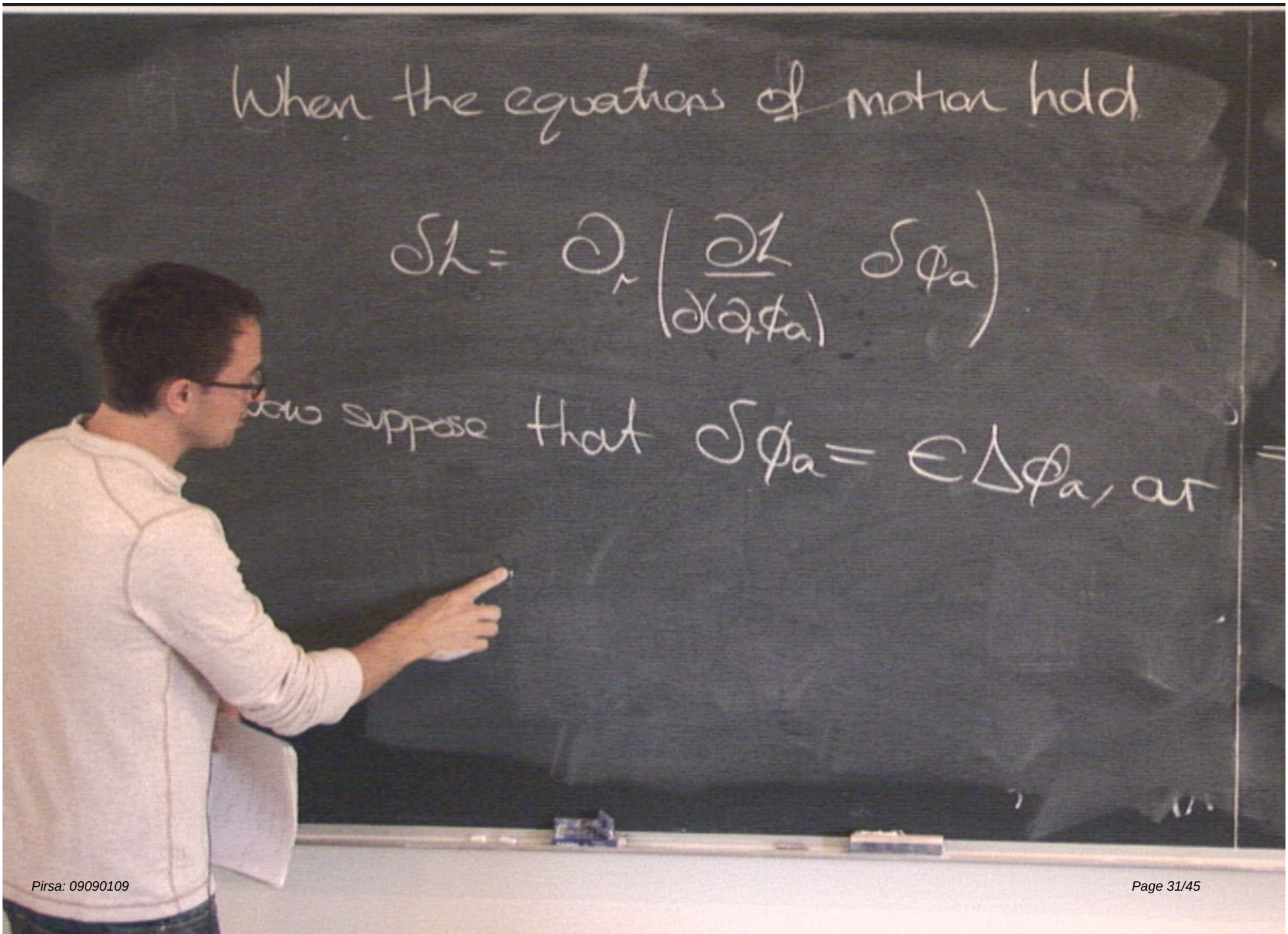
When the equations of motion hold

$$\delta \mathcal{L} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \delta \phi_a \right)$$


When the equations of motion hold
$\delta L = \partial_\mu \left(\frac{\partial L}{\partial(\partial_\mu \phi_a)} \delta \phi_a \right)$
now suppose that $\delta \phi_a = \epsilon \Delta \phi_a$, as

When the equations of motion hold

$$\delta \mathcal{L} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \delta \phi_a \right)$$

Now suppose that $\delta \phi_a = \epsilon \Delta \phi_a$, as a symmetry, then $\delta \mathcal{L} = 0$

$$\Rightarrow \partial_\mu j^\mu = 0 \quad \text{where} \quad \boxed{j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \Delta \phi_a}$$

Slight Generalization

We can weaken our requirement of a symmetry to

$$\delta \mathcal{L} = \partial_\mu F^\mu \quad \text{for some } F^\mu.$$

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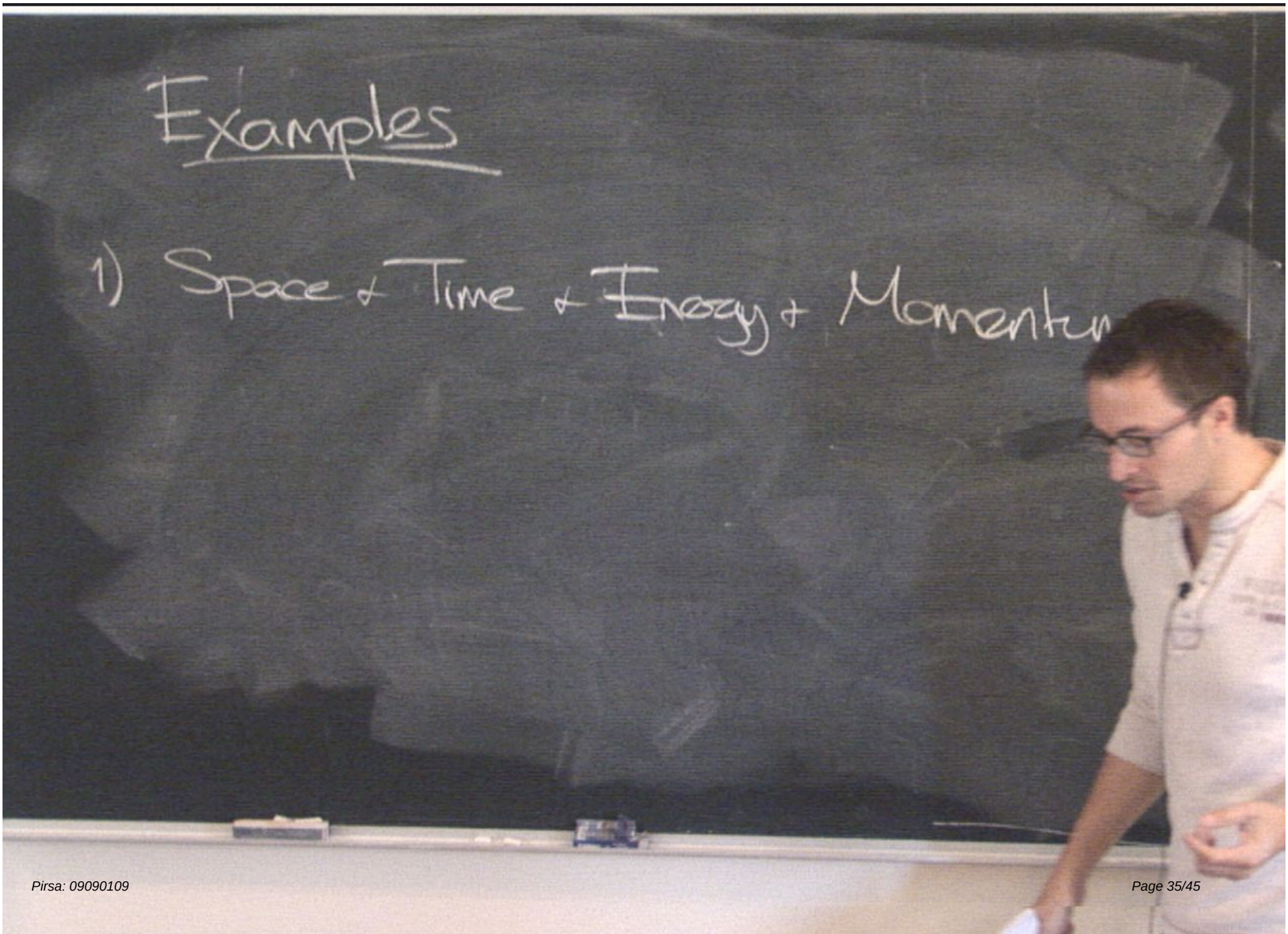
$$\Rightarrow \quad j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \Delta\phi_a - F^\mu \quad \text{is conserved}$$


Examples
1) Space & Time & Energy & Momentum

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1) Space + Time + Energy + Momentum

Consider $x^\mu \to x^\mu + \epsilon^\mu$

$$\Longrightarrow \quad \phi_a(x) \to \phi_a(x) + \epsilon^\nu \partial_\nu \phi_a(x)$$

If the Lagrangian has no explicit x^μ dependence

$$\mathcal{L}(x) \to \mathcal{L}(x) + \epsilon^\nu \partial_\nu \mathcal{L}$$

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$\Rightarrow$ we get four conserved currents

Examples

$$(j^\mu)_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \partial_\nu \phi_a - \delta^\mu_\nu \mathcal{L}$$

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$$\equiv T^\mu{}_\nu \,,$$

is called the <u>energy-momentum tensor</u>

This obeys $\partial_\mu T^\mu{}_\nu = 0$

Four conserved quantities

$$E = \int d^3x \, T^{00}$$ is total energy of field

$$P^i = \int d^3x \, T^{0i}$$ is total momentum of field.

An example:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2$$

$$\Rightarrow T^{\mu\nu} = \partial^\mu\phi\,\partial^\nu\phi - \eta^{\mu\nu}\mathcal{L}$$

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$$\Rightarrow T^{\mu\nu} = \partial^\mu\phi\,\partial^\nu\phi - \eta^{\mu\nu}\mathcal{L}$$

$$\Rightarrow E = \int d^3x\ \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2$$

$$P^i = \int d^3x\ \dot{\phi}\,(\partial^i\phi)$$