Title: PSI - Relativity (PHYS 604) - 17

Date: Sep 25, 2009 10:30 AM

URL: http://pirsa.org/09090076

Abstract:

	Newton	Einstein
vacuum	$\vec{\nabla}^2 \Phi = 0$	$R_{\alpha\beta} = 0$
with matter	$\vec{\nabla}^2 \Phi = 4\pi G \rho$	$R_{\alpha\beta} + \lambda g_{\alpha\beta} R = \kappa T_{\alpha\beta}$

$R = g^{\alpha\beta} R_{\alpha\beta}$

$\lambda = ?$

$\kappa = ?$

Some symmetric tensor

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$R = g^{\alpha\beta} R_{\alpha\beta}$

some symmetric tensor

$\lambda = ?$

$\kappa = ?$

"dust" – cloud of non-interacting particles

→ mass: m

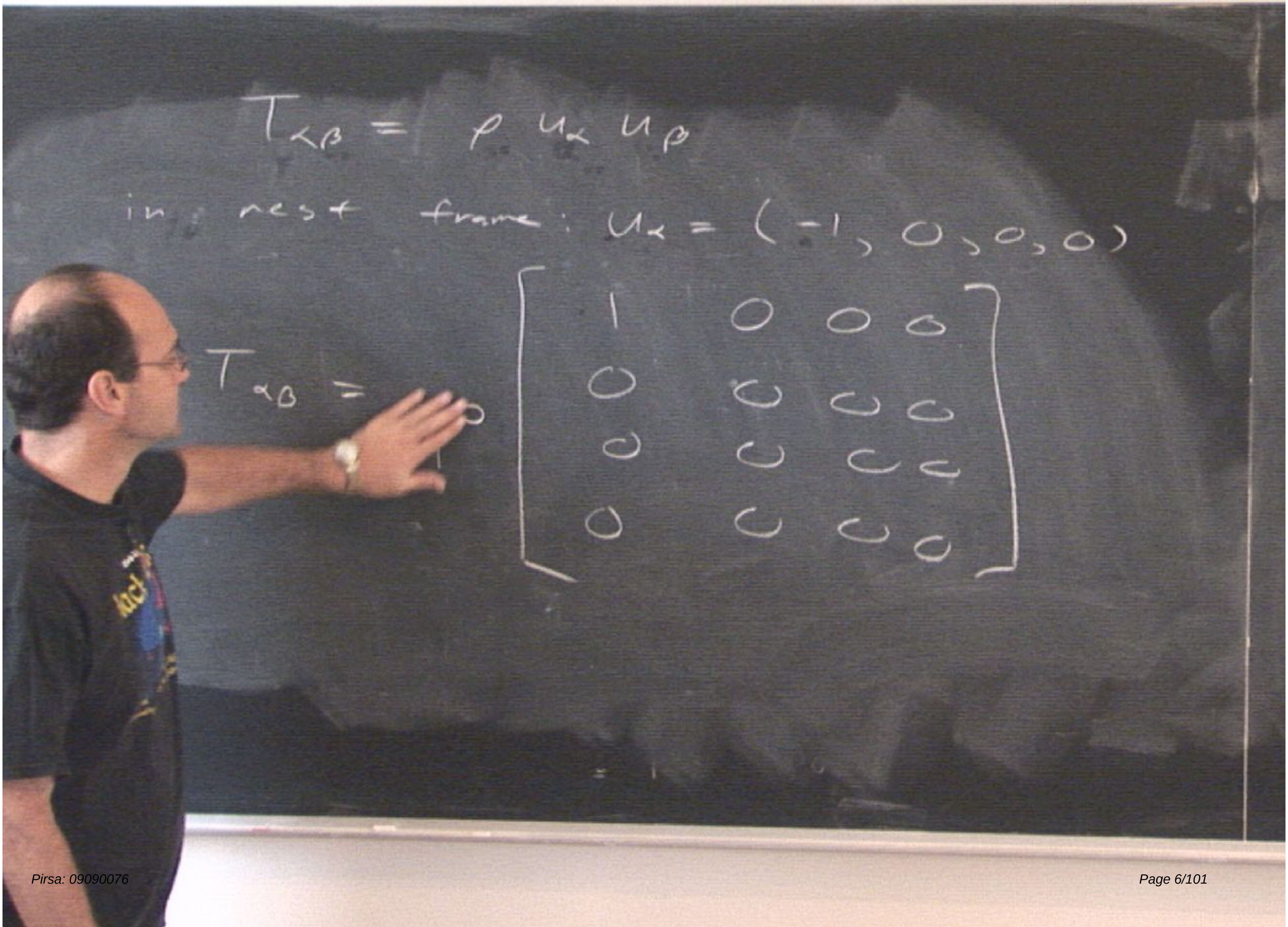
→ 4-velocity describes flow: $\underset{\sim}{u}(x)$

"proper" particle density: $N_0(x)$

for observer with $u^\alpha(x)$

"proper" energy/mass density: $\rho = m\, N_0(x)$

$$T_{\alpha\beta} = \rho\, u_\alpha u_\beta$$


Tαβ = ρ uα uβ
in rest frame: uα = (-1, 0, 0, 0)
Tαβ =
1 0 0 0
0 0 0 0
0 0 0 0
0 0 0 0

$$T_{\alpha\beta} = \rho \, u_\alpha u_\beta$$

in rest frame: $u_\alpha = (-1, 0, 0, 0)$

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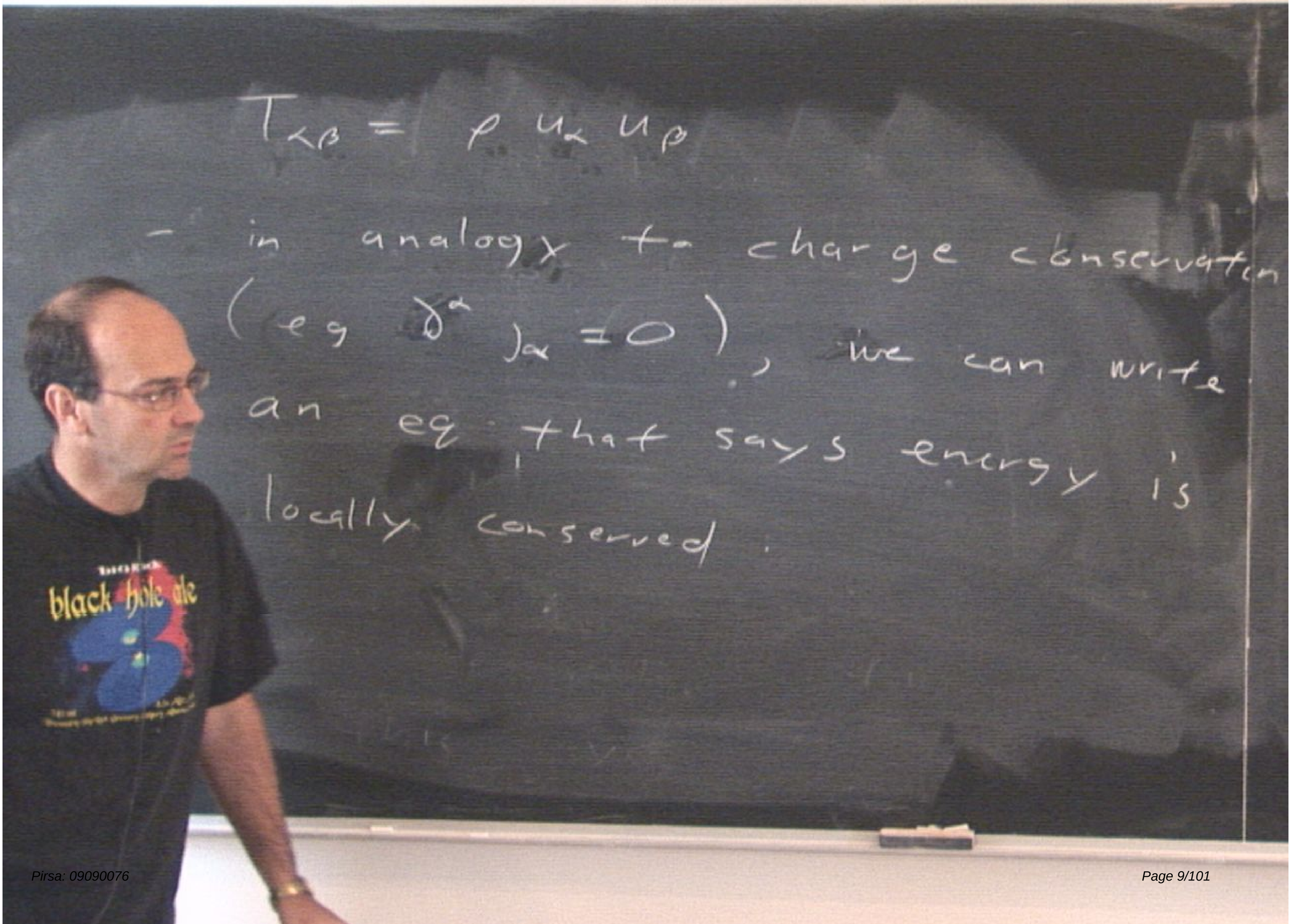
$\longrightarrow$ combined with Newtonian limit this says $\lambda \neq 0$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– in analogy to charge conservation
(eg $\partial^\alpha J_\alpha = 0$), we can write
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black hole ale

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$$\rho' = \gamma^2 \rho$$
$$= \frac{\rho}{1-v^2}$$

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recall $\rho' = \gamma^2 \rho$
$= \frac{\rho}{1-v^2}$

$\frac{\partial \rho'}{\partial t}$

$$T_{\alpha\beta} = \rho\, u_\alpha u_\beta$$

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recall $\quad = \gamma^2 \rho = \dfrac{\rho}{1-v^2}$

$$\frac{\partial \rho'}{\partial t} + \vec{\partial} \cdot (\rho' \vec{v})$$

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recall $\rho' = \gamma^2 \rho = \frac{\rho}{1-v^2}$

$$\frac{\partial \rho'}{\partial t} + \vec{\partial} \cdot (\rho' \vec{v}) = 0$$

can be expressed in terms of $T_{\alpha\beta}$

$$\partial_\alpha (\rho u^0 u^\alpha)$$

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in flat space (or local inertial frame)

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in flat space (or local inertial frame)

$\longrightarrow$ generalises to curved space

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in flat space (or local inertial frame)

→ generalises to curved space

$$\nabla_\alpha T^{t\alpha} = 0$$

→ in fact there is a vector's worth of conservation eq's

$$\nabla_\alpha T^{\beta\alpha} = 0$$

$$T_{\alpha\beta} = \rho\, u_\alpha u_\beta$$

$\longrightarrow$ spatial components

$$\partial_\alpha (\rho\, u^i u^\alpha) = 0$$

$$\Rightarrow \frac{\partial}{\partial t}(\rho' v^i) + \frac{\partial}{\partial x^j}(\rho' v^i v^j) = 0$$

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$$\frac{\partial \rho'}{\partial t}$$

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Navier-Stokes eq

$$T_{\alpha\beta} = \rho\, u_\alpha u_\beta \qquad \partial_\alpha(\rho\, u^0 u^\alpha) = 0$$

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Navier-Stokes eq

$$T^{\alpha\beta} = \rho u^\alpha u^\beta =$$

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$$T^{++} =$$

$$T^{\alpha\beta} = \rho u^\alpha u^\beta = \begin{bmatrix} \rho' & \rho' v^i \\ \rho' v^j & \rho' v^i v^j \end{bmatrix}$$

$$T^{tt} = \text{energy density}$$

$$T^{ti} = T^{it} = \text{momentum density}$$

$$T^{\alpha\beta} = \rho u^\alpha u^\beta = \begin{bmatrix} \rho' & \rho' v^i \\ \rho' v^j & \rho' v^i v^j \end{bmatrix}$$

T^{tt} = energy density

$T^{ti} = T^{it}$ = momentum density = flux of energy density

$T^{ij} = T^{ji} =$

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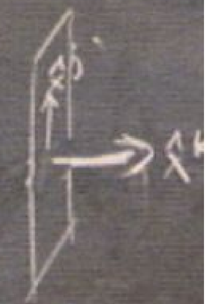
= "stress"

$$T^{\alpha\beta} = \rho u^\alpha u^\beta = \begin{bmatrix} \rho' & \rho' v^i \\ \rho' v^j & \rho' v^i v^j \end{bmatrix}$$

Stress



$T^{jk} \sim$ jth comp of force at surface element with normal $\hat{x}^k$

T^{tt} = energy density

$T^{ti} = T^{it}$ = momentum density = flux of energy density

$T^{ij} = T^{ji}$ = flux of momentum density = "stress"

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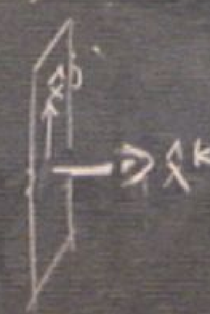

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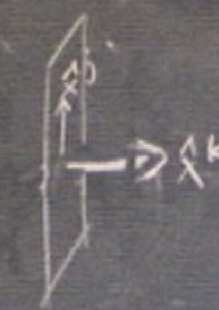

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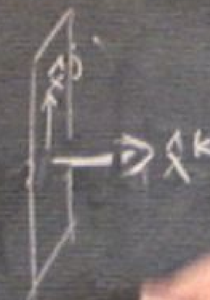

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$T^{ti} = T^{it}$ = momentum density = flux of energy density

$T^{ij} = T^{ji}$ = flux of momentum density = "stress"

$T_{\alpha\beta}$ = stress - energy tensor

or

energy - momentum tensor

$$\boxed{\nabla_\alpha T^{\alpha\beta} = 0}$$

$T_{\alpha\beta}$ = stress-energy tensor

or

energy-momentum tensor

$$\boxed{\nabla_\alpha T^{\alpha\beta} = 0} \longleftarrow \text{applies in general}$$

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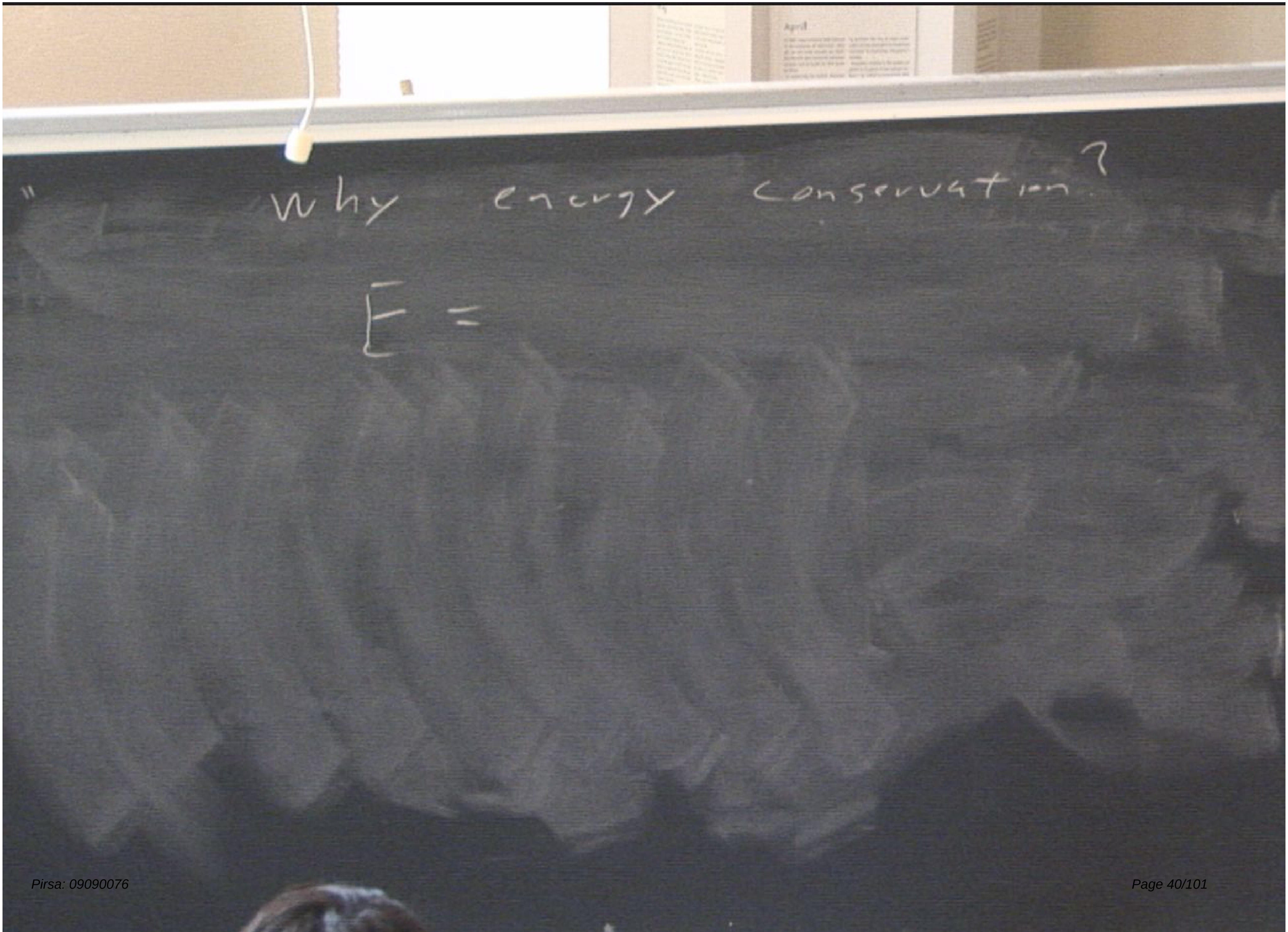
$\longrightarrow$ reflects the conservation of energy + momentum


" why energy conservation?
E =

"why energy conservation?

$$E = \int d^3x \; T^{tt}$$

$$\frac{\partial E}{\partial t} = \int d^3x \; \frac{\partial T^{tt}}{\partial t}$$

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$$E = \int d^3x \, T^{tt}$$

$$\frac{\partial E}{\partial t} = \int d^3x \, \frac{\partial T^{tt}}{\partial t}$$

$$= \int d^3x \left(- \frac{\partial}{\partial x^i} T^{it} \right)$$

$$= \oint d^2x \, n_i T^{it}$$

why energy conservation?

$$E = \int d^3x \; T^{tt}$$

$$\frac{\partial E}{\partial t} = \int d^3x \; \frac{\partial T^{tt}}{\partial t}$$

$$= \int d^3x \left(- \frac{\partial}{\partial x^i} T^{it} \right)$$

$$= \oint d^2x \; n_i T^{it} \qquad = 0$$

flux of energy escaping at ∞

recall

$$\nabla^{\alpha}\left(R_{\alpha\beta} + \lambda\, g_{\alpha\beta}\, R \right) = \kappa\, \cancelto{0}{\nabla^{\alpha} T}_{\alpha\beta}$$

recall

$$\nabla^{\alpha}\left(R_{\alpha\beta} + \lambda\, g_{\alpha\beta}\, R \right) = \kappa\, \cancelto{0}{\nabla^{\alpha} T_{\alpha\beta}}$$

$$\nabla^{\alpha} R_{\alpha\beta} = -\lambda\, \nabla_{\beta} R$$

recall

$$\nabla^\alpha \left(R_{\alpha\beta} + \lambda\, g_{\alpha\beta}\, R \right) = \kappa \cancelto{0}{\nabla^\alpha T_{\alpha\beta}}$$

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$\longrightarrow$ need this to be true always

recall

$$\nabla^{\alpha}\left(R_{\alpha\beta} + \lambda\, g_{\alpha\beta}\, R \right) = \kappa \cancel{\nabla^{\alpha} T_{\alpha\beta}}^{\,0}$$

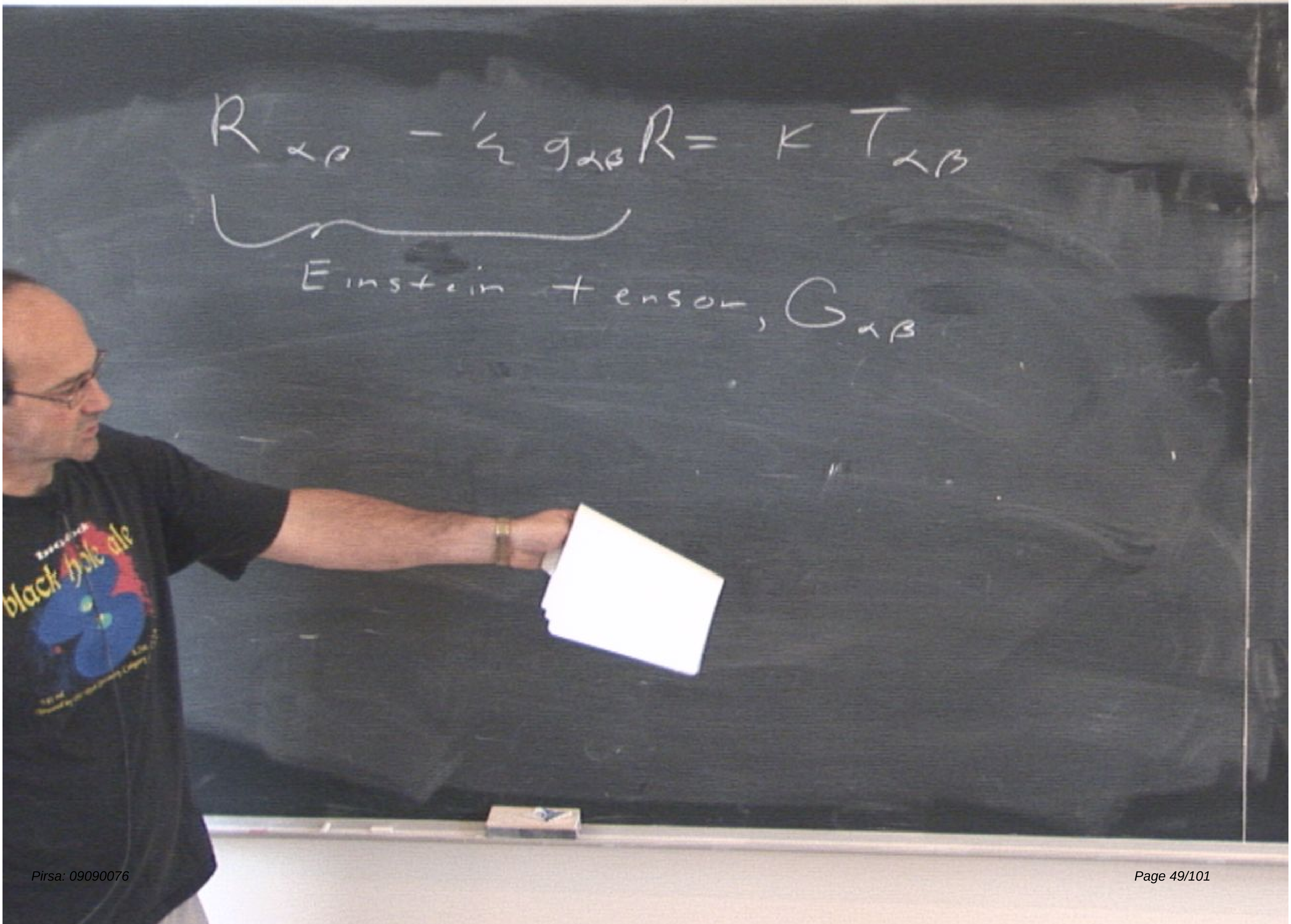
$$\nabla^{\alpha} R_{\alpha\beta} = -\lambda\, \nabla_{\beta} R$$

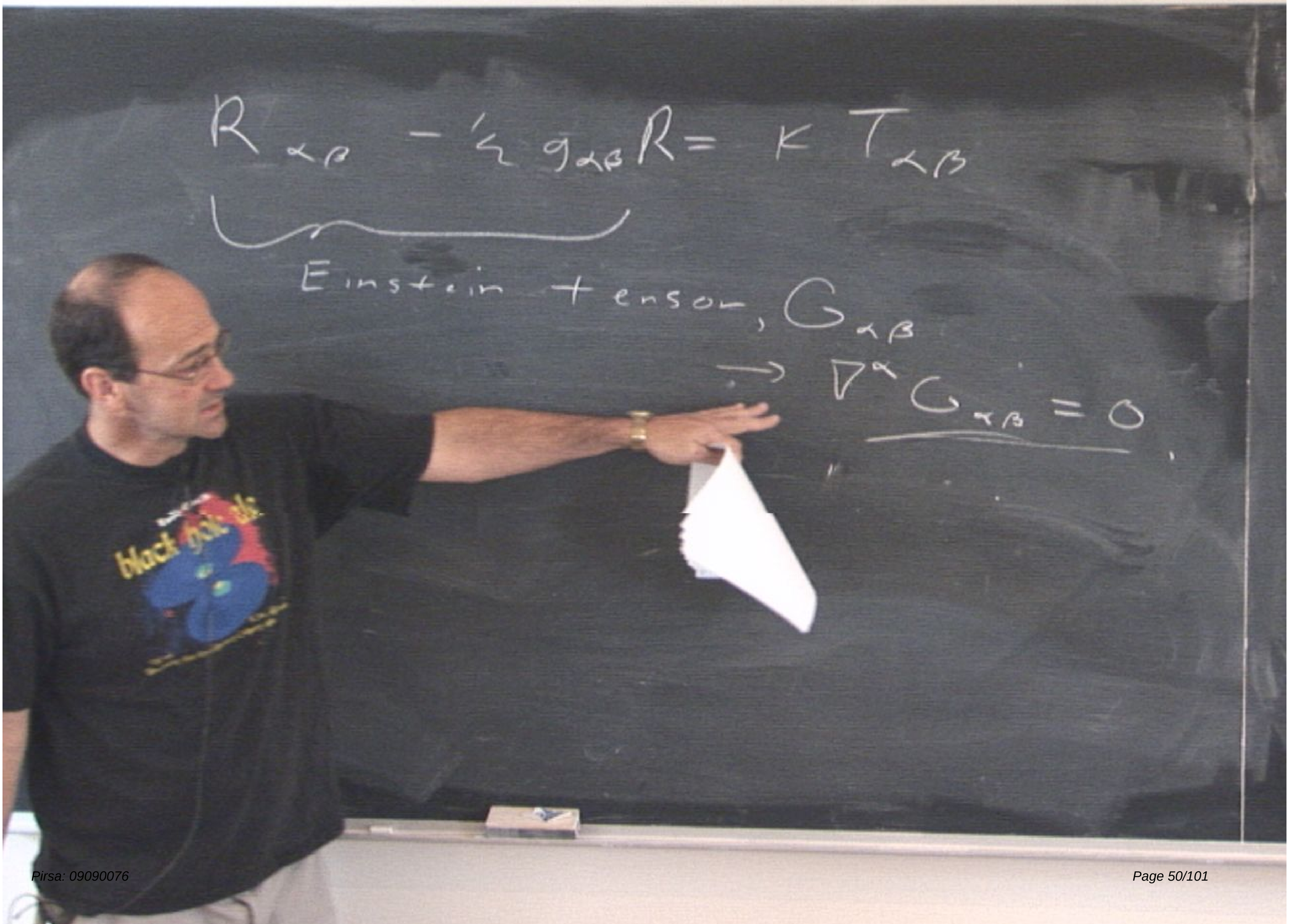
$\rightarrow$ need this to be true always

from Bianchi identity, we know

this is satisfied if $\lambda = -\frac{1}{2}$

$$\underbrace{R_{\alpha\beta} - \tfrac{1}{2} g_{\alpha\beta} R} = \kappa T_{\alpha\beta}$$


$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = \kappa T_{\alpha\beta}$
Einstein tensor, $G_{\alpha\beta}$


$\underbrace{R_{\alpha\beta} - \tfrac{1}{2} g_{\alpha\beta} R}_{\text{Einstein tensor, } G_{\alpha\beta}} = \kappa T_{\alpha\beta}$
$\longrightarrow \nabla^{\alpha} G_{\alpha\beta} = 0$
black

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$$\longrightarrow \underline{\nabla^{\alpha} G_{\alpha\beta} = 0}.$$

<u>to fix κ</u>:

consider Newtonian limit with the dust stress tensor

$$\underbrace{R_{\alpha\beta} - \tfrac{1}{2} g_{\alpha\beta} R}_{\text{Einstein tensor, } G_{\alpha\beta}} = \kappa\, T_{\alpha\beta}$$

$$\longrightarrow \nabla^{\alpha} G_{\alpha\beta} = 0$$

to fit κ:

- consider Newtonian limit with a dust stress tensor
-

$\longrightarrow$ compare to $\quad \nabla^2 \Phi = 4\pi G \rho$

$$\boxed{\kappa = \frac{8\pi G}{c^4}}$$

→ compare to $\nabla^2 \Phi = 4\pi G \rho$

$$\boxed{\kappa = \frac{8\pi G}{c^4}}$$

Action Principle

→ classical mechanics: $\int dt \, L(\dot{x}, x)$

→ compare to $\nabla^2 \Phi = 4\pi G \rho$

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Action Principle

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Action Principle

→ classical mechanics: $\int dt\, L(\dot{x}^a, x^a)$

→ fields : $\int$ $L(\dot{\phi}, \partial_i \phi, \phi)$

$\longrightarrow$ compare to $\nabla^2 \Phi = 4\pi G \rho$

$$\boxed{\kappa = \frac{8\pi G}{c^4}}$$

Action Principle

$\longrightarrow$ classical mechanics: $\int dt \, L(\dot{x}^a, x^a)$

$\longrightarrow$ fields : $\int dt \, d^3x \, L(\dot{\phi}, \partial_i \phi, \phi)$ (eg $\int d^4x \, (-\frac{1}{4} F$

$\times F_a$

$\longrightarrow$ compare to $\nabla^2 \Phi = 4\pi G \rho$

$$\boxed{\kappa = \frac{8\pi G}{c^4}}$$

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gravity? field $\longrightarrow g_{\alpha\beta}(x)$

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$\int d(\qquad)$

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$\int d($ $)$ "scalar" $($

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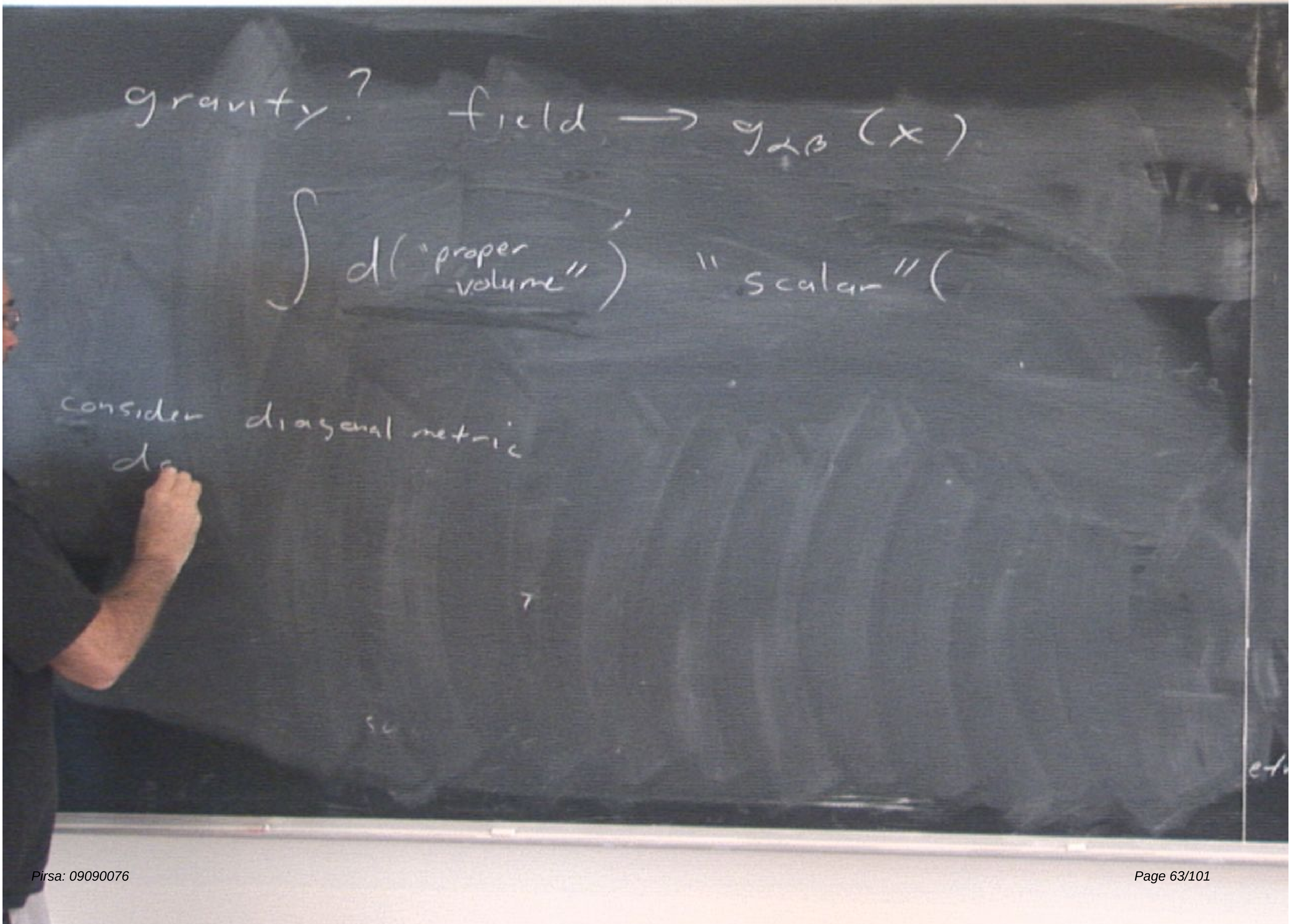
$\int d($"proper volume"$)$ "scalar"(


gravity? field → $g_{\alpha\beta}(x)$
$\int d($"proper volume"$)$ "scalar"(
consider diagonal metric

gravity? field $\longrightarrow g_{\alpha\beta}(x)$

$$\int d(\text{"proper volume"}) \ \text{"scalar"}($$

consider diagonal metric

$$ds^2 = \underbrace{g_{tt}}_{\text{negative}} dt^2 + g_{xx}\,dx^2 + \ldots$$

$$dV_{proper} = \sqrt{-g_{tt}\, g_{xx}\, g_{yy}\, g_{zz}}\ dt\,dx\,dy\,dz$$

gravity? field $\longrightarrow g_{\alpha\beta}(x)$

$$\int d(\text{"proper volume"}) \quad \text{"scalar"}($$

$\hookrightarrow \sqrt{-g}\, d^4x$

$\uparrow$ det $g_{\alpha\beta}$

consider diagonal metric

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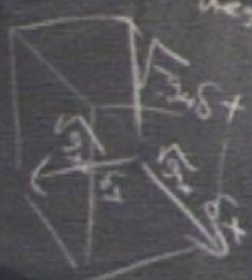

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general result $\longrightarrow \sqrt{-g}\; d^4x$

det $g_{\alpha\beta}$

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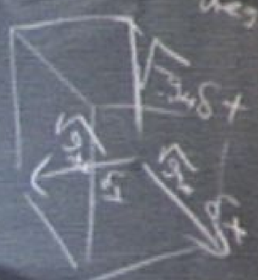

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gravity? field $\longrightarrow g_{\alpha\beta}(x)$

$$\int d(\text{"proper volume"}) \ \text{"scalar"}(g, \partial g$$

general result $\longrightarrow \sqrt{-g}\, d^4x$ ← det $g_{\alpha\beta}$

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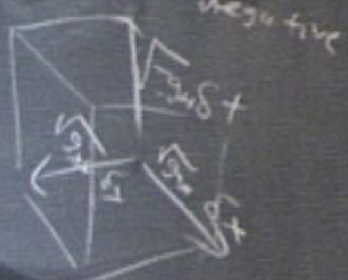

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gravity? field $\longrightarrow g_{\alpha\beta}(x)$

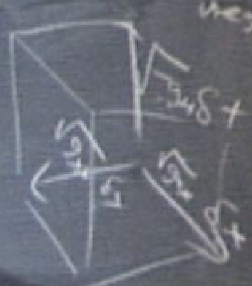
$$\int d(\text{"proper volume"}) \; \text{"scalar"}(g, \partial g)$$

general result $\longrightarrow \sqrt{-g}\, d^4x \longrightarrow$ none (constant)

$\sqrt{-g} \leftarrow \det g_{\alpha\beta}$

consider diagonal metric

$$ds^2 = \underbrace{g_{tt}}_{\text{negative}}\, dt^2 + g_{xx}\, dx^2 + \dots$$



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gravity? field $\longrightarrow g_{\alpha\beta}(x)$

$$\int d(\text{"proper volume"}) \; \text{"scalar"}(g, \partial g)$$

general result $\longrightarrow \sqrt{-g}\, d^4x$ $\longleftarrow \det g_{\alpha\beta}$

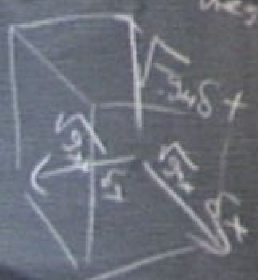
$\longrightarrow$ none (constant)

"scalar" $(g, \partial g, \partial\partial g)$

linear

consider diagonal metric

$$ds^2 = \underbrace{g_{tt}}_{\text{negative}}\, dt^2 + g_{xx}\, dx^2 + \cdots$$



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general result $\longrightarrow \sqrt{-g}\; d^4x \longrightarrow$ none (constant)

$\sqrt{-g}$ ← det $g_{\alpha\beta}$

"scalar"$(g, \partial g, \partial\partial g)$ $\longrightarrow$ one $\quad R = g^{\alpha\beta} R_{\alpha\beta}$

linear

consider diagonal metric

$$ds^2 = \underbrace{g_{tt}}_{\text{negative}} dt^2 + g_{xx}\, dx^2 + \cdots$$

$$dV_{proper} = \sqrt{-g_{tt}\, g_{xx}\, g_{yy}\, g_{zz}} \; dt\, dx\, dy\, dz$$

etric

variations

$$\delta I = \int d^4x \; \delta g_{\mu\nu} \, (EOM)^{\mu\nu}$$

instead vary inverse metric

$$- \delta I = \int d^4x \; \delta g^{\mu\nu} \, (\widetilde{EOM})_{\mu\nu}$$

variations

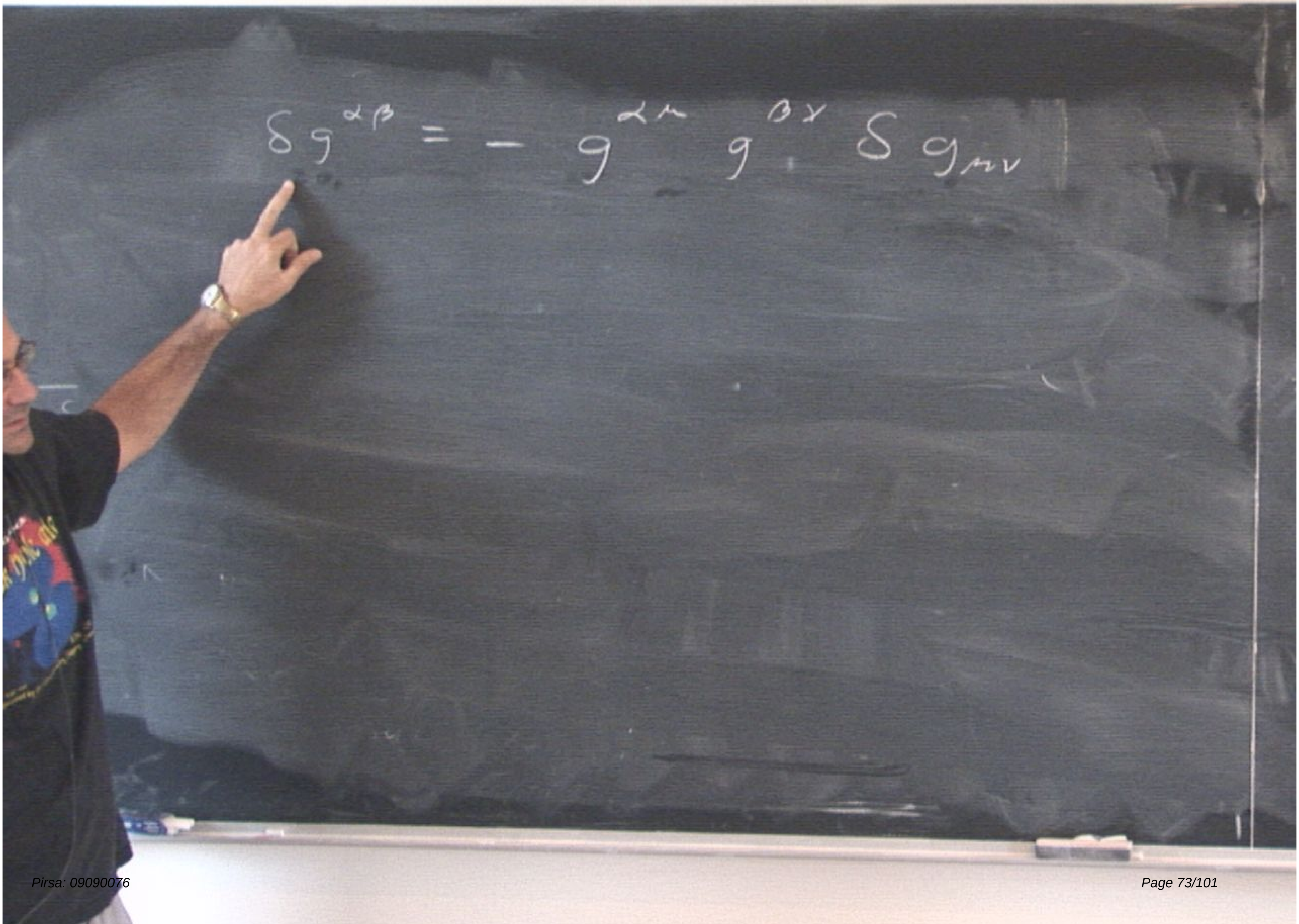
$$\delta I = \int d^4x \; \delta g_{\mu\nu} \, (EOM)^{\mu\nu}$$

instead vary inverse metric

$$-\delta I = \int d^4x \; \delta g^{\mu\nu} \, (\widetilde{EOM})_{\mu\nu}$$

Note $g^{\mu\nu} g_{\nu\alpha} = \delta^{\mu}{}_{\alpha}$

$$\delta g^{\mu\nu} g_{\nu\alpha} + g^{\mu\nu} \delta g_{\nu\alpha} = 0$$


$\delta g^{\alpha\beta} = - g^{\alpha\mu} g^{\beta\nu} \delta g_{\mu\nu}$

$$\delta g^{\alpha\beta} = - g^{\alpha\mu} g^{\beta\nu} \delta g_{\mu\nu}$$

$$I = \int d^4x \sqrt{-g}\; g^{\mu\nu} R_{\mu\nu}$$

$$\delta g^{\alpha\beta} = - g^{\alpha\mu} g^{\beta\nu} \delta g_{\mu\nu}$$

$$I = \int d^4x \sqrt{-g}\, g^{\mu\nu} R_{\mu\nu}$$

$$\delta I = \delta I_1 + \delta I_2 + \delta I_3$$

$$\delta I_2 = \int d^4x$$

$$\delta g^{\alpha\beta} = - g^{\alpha\mu} g^{\beta\nu} \delta g_{\mu\nu}$$

$$I = \int d^4x \sqrt{-g}\; g^{\mu\nu} R_{\mu\nu}$$

$$\delta I = \delta I_1 + \delta I_2 + \delta I_3$$

$$\delta I_2 = \int d^4x \sqrt{-g}\; \delta g^{\mu\nu} R_{\mu\nu}$$

$\delta I_1 = ?$

$\delta \det(-g_{\alpha\beta}) =$

$$\delta Z_1 = ?$$

$$\delta \det(-g_{\alpha\beta}) = \delta\left(\frac{1}{\det(-g^{\alpha\beta})}\right)$$

$$= \frac{1}{\det(g^{\alpha\beta} + \delta g^{\alpha\beta})}$$

$$= \frac{1}{\det(g^{\alpha\beta})\det(\delta_\gamma^\rho + g_{\gamma\delta}\delta g^{\delta\rho})}$$

$$\delta Z_1 = ?$$

$$\delta \det(g_{\alpha\beta}) = \delta\left(\frac{1}{\det(g^{\alpha\beta})}\right)$$

$$= \frac{1}{\det(g^{\alpha\beta} + \delta g^{\alpha\beta})}$$

$$= \frac{1}{\det(g^{\alpha\beta})\det(\delta_\gamma^\rho + g_{\gamma\delta}\delta g^{\delta\rho})}$$

$$\delta Z_1 = ?$$

$$\delta g = g \cdot g^{\mu\nu} \, \delta g_{\mu\nu}$$

$$= - g \cdot g_{\mu\nu} \, \delta g^{\mu\nu}$$

$$\delta Z_1 = ?$$

$$\delta g = g \cdot g^{\mu\nu} \, \delta g_{\mu\nu}$$

$$= - g \cdot g_{\mu\nu} \, \delta g^{\mu\nu}$$

$$\delta Z_1 = ?$$

$$\delta g = g \, g^{\mu\nu} \, \delta g_{\mu\nu}$$

$$= - g \, g_{\mu\nu} \, \delta g^{\mu\nu}$$

$$\delta(-g$$

$$\delta Z_1 = ?$$

$$\delta g = g \, g^{\mu\nu} \, \delta g_{\mu\nu}$$

$$= - g \, g_{\mu\nu} \, \delta g^{\mu\nu}$$

$$\delta \sqrt{-g} = + \frac{1}{2} \frac{1}{\sqrt{-g}} \, \delta g$$

$$= - \frac{1}{2} \sqrt{-g} \, g_{\mu\nu} \, \delta g^{\mu\nu}$$

$$\delta I_1 = ?$$

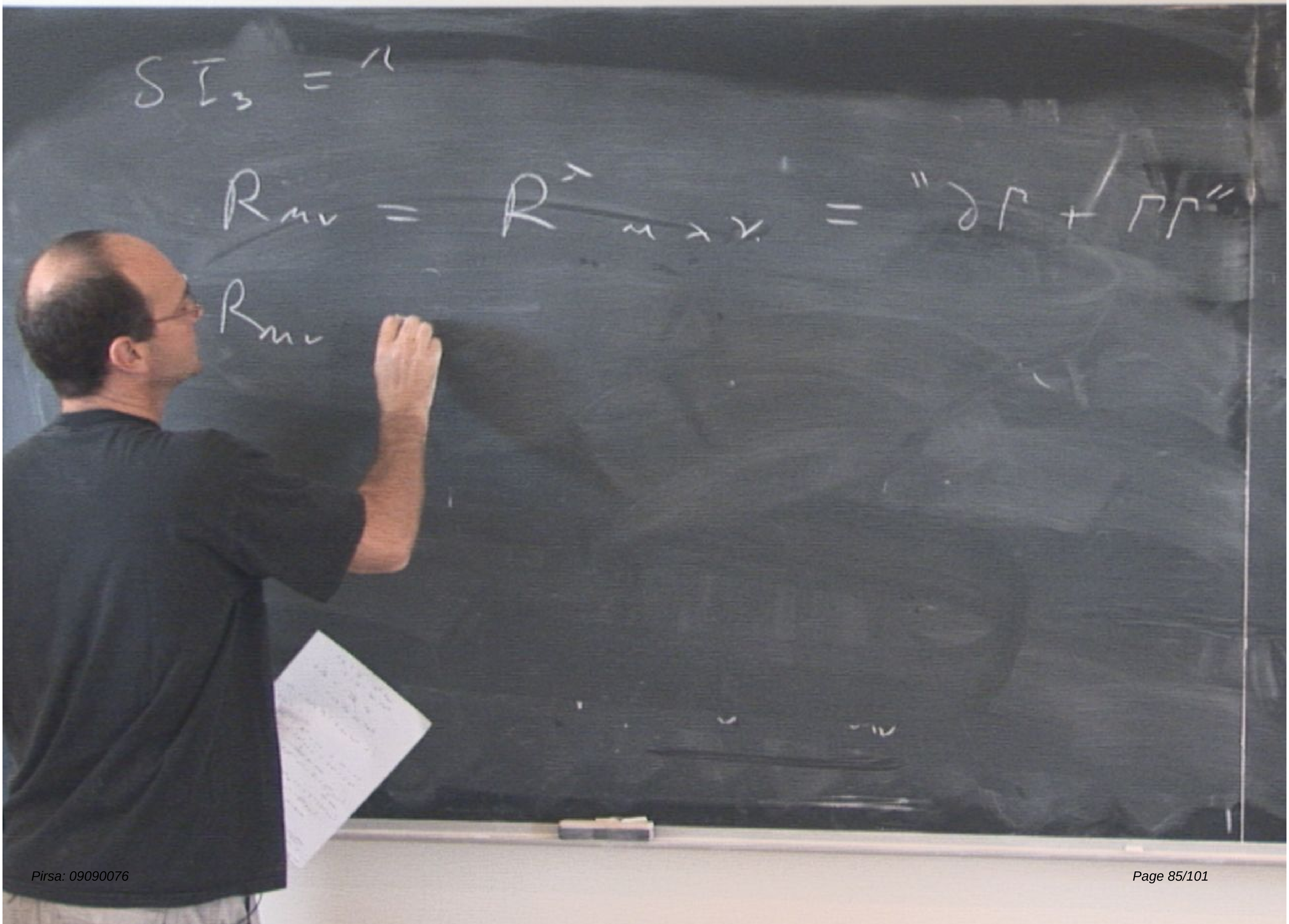
$$\delta g = g \cdot g^{\mu\nu} \, \delta g_{\mu\nu}$$

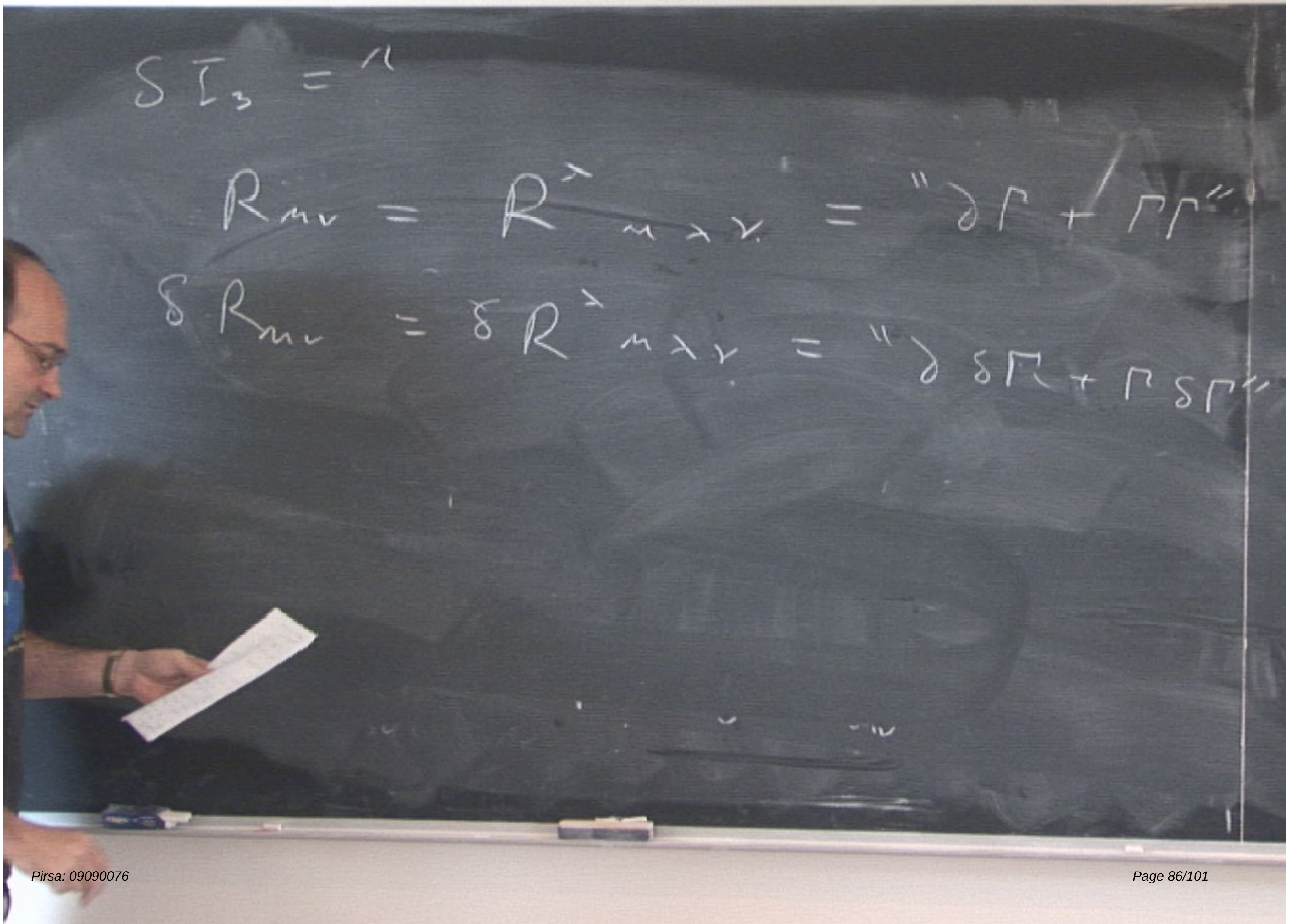
$$= - g \cdot g_{\mu\nu} \, \delta g^{\mu\nu}$$

$$\delta \sqrt{-g} = + \frac{1}{2} \frac{1}{\sqrt{-g}} \, \delta g$$

$$= - \frac{1}{2} \sqrt{-g} \, g_{\mu\nu} \, \delta g^{\mu\nu}$$

$$\delta I_1 = \int d^4x \sqrt{-g} \left(- \frac{1}{2} g_{\mu\nu} \, \delta g^{\mu\nu} \right) R$$


$S I_3 =$
$R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu} =$ "$\partial\Gamma + \Gamma\Gamma$"
$R_{\mu\nu}$


$\delta I_3 =$
$R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu} = \text{"}\partial\Gamma + \Gamma\Gamma\text{"}$
$\delta R_{\mu\nu} = \delta R^{\lambda}{}_{\mu\lambda\nu} = \text{"}\partial\,\delta\Gamma + \Gamma\,\delta\Gamma\text{"}$

$$\delta I_3 =$$

$$R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu} = \text{``}\partial\Gamma + \Gamma\Gamma\text{''}$$

$$\delta R_{\mu\nu} = \delta R^{\lambda}{}_{\mu\lambda\nu} = \text{``}\partial\,\delta\Gamma + \Gamma\,\delta\Gamma\text{''}$$

tensor

$$= \nabla_{\lambda}(\delta\Gamma^{\lambda}{}_{\nu\mu}) - \nabla_{\nu}(\delta\Gamma^{\lambda}{}_{\lambda\mu})$$

$$\delta I_3 = {}^{\Lambda}$$

$$R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu} = \text{"}\partial\Gamma + \Gamma\Gamma\text{"}$$

$$\delta R_{\mu\nu} = \delta R^{\lambda}{}_{\mu\lambda\nu} = \text{"}\partial\,\delta\Gamma + \Gamma\,\delta\Gamma\text{"}$$

tensor

$$= \nabla_\lambda(\delta\Gamma^\lambda{}_{\nu\mu}) - \nabla_\nu(\delta\Gamma^\lambda{}_{\lambda\mu})$$

$$g^{\mu\nu}\,\delta R_{\mu\nu} = \nabla^\lambda(\delta\Gamma_{\lambda\mu}{}^{\mu}) - \nabla^\mu(\delta\Gamma^\lambda{}_{\lambda\mu})$$

$$\delta I_3 =$$

$$R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu} = \text{"}\partial\Gamma + \Gamma\Gamma\text{"}$$

$$\delta R_{\mu\nu} = \delta R^{\lambda}{}_{\mu\lambda\nu} = \text{"}\partial\,\delta\Gamma + \Gamma\,\delta\Gamma\text{"}$$

tensor

$$= \nabla_{\lambda}(\delta\Gamma^{\lambda}{}_{\nu\mu}) - \nabla_{\nu}(\delta\Gamma^{\lambda}{}_{\lambda\mu})$$

$$g^{\mu\nu}\,\delta R_{\mu\nu} = \nabla^{\lambda}(\delta\Gamma_{\lambda\mu}{}^{\mu}) - \nabla^{\mu}(\delta\Gamma^{\lambda}{}_{\lambda\mu})$$

$$= \nabla^{\mu}(\delta S_{\mu})$$

$$\delta I_3 = {}^{\wedge}$$

$$R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu} = \text{"}\partial\Gamma + \Gamma\Gamma\text{"}$$

$$\delta R_{\mu\nu} = \underbrace{\delta R^{\lambda}{}_{\mu\lambda\nu}}_{\text{tensor}} = \text{"}\partial\,\delta\Gamma + \Gamma\,\delta\Gamma\text{"}$$

$$= \nabla_{\lambda}(\delta\Gamma^{\lambda}{}_{\nu\mu}) - \nabla_{\nu}(\delta\Gamma^{\lambda}{}_{\lambda\mu})$$

$$g^{\mu\nu}\,\delta R_{\mu\nu} = \nabla^{\lambda}(\delta\Gamma_{\lambda\mu}{}^{\mu}) - \nabla^{\mu}(\delta\Gamma^{\lambda}{}_{\lambda\mu})$$

$$= \nabla^{\mu}(\delta S_{\mu})$$

$$\delta I_3 = \int d^4x \sqrt{-g} \; \nabla^\mu (\delta w_\mu)$$

$$= \oint d^4x \sqrt{-g} \; n^\mu \; \delta w_\mu$$

$$\delta I_3 = \int d^4x \sqrt{-g}\ \nabla^\mu (\delta w_\mu)$$

$$= \oint d^4x \sqrt{-g}\ n^\mu\ \delta w_\mu$$

use divergence theorm

$$\delta I_3 = \int d^4x \sqrt{-g}\; \nabla^\mu (\delta w_\mu)$$

$$= \oint d^4x \sqrt{-g}\; n^\mu \, \delta w_\mu$$

use divergence theorem

$$\delta I = \delta I_1 + \delta I_2 + \cancel{\delta I_3}$$

boundary term

$$= \int d^4x \sqrt{-g}\; \delta g^{\mu\nu} \Big(R_{\mu\nu} \quad \Lambda$$

$$\delta I_3 = \int d^4x \sqrt{-g}\, \nabla^\mu (\delta w_\mu)$$

$$= \oint d^4x \sqrt{-g}\, n^\mu\, \delta w_\mu$$

use divergence theorem

$$\delta I = \delta I_1 + \delta I_2 + \cancel{\delta I_3}$$

boundary term

$$= \int d^4x \sqrt{-g}\, \delta g^{\mu\nu} \left(R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R \right.$$

$$\delta I_3 = \int d^4x \sqrt{-g}\; \nabla^\mu (\delta w_\mu)$$

$$= \oint d^4x \sqrt{-g}\; n^\mu \, \delta w_\mu$$

use divergence theorm

$$\delta I = \delta I_1 + \delta I_2 + \cancel{\delta I_3}$$

boundary term

$$= \int d^4x \sqrt{-g}\; \delta g^{\mu\nu} \underbrace{\left(R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R \right)}_{G_{\mu\nu}}$$

EOM: $G_{\mu\nu} = 0$

in absence of any matter

$$g^{\mu\nu}\,\delta R_{\mu\nu} = \nabla_\lambda(\delta\Gamma^{\lambda}{}_{\nu\mu}) - \nabla_\nu(\delta\Gamma^{\lambda}{}_{\lambda\mu})$$

$$= \nabla^\lambda(\delta\Gamma_{\lambda\mu}{}^{\mu}) - \nabla^\mu(\delta\Gamma^{\lambda}{}_{\lambda\mu})$$

$$= \nabla^\mu(\delta W_\mu)$$

$$\delta I_3 = \int d^4x \sqrt{-g}\, \nabla^\mu (\delta w_\mu)$$

$$= \oint d^4x \sqrt{-g}\; n^\mu\, \delta w_\mu$$

use divergence theorem

$$\delta I = \delta I_1 + \delta I_2 + \cancel{\delta I_3}$$

boundary term

$$= \int d^4x \sqrt{-g}\; \underset{\substack{\| \\ (\nabla^\nu \xi^\mu + \nabla^\mu \xi^\nu)}}{\delta g^{\mu\nu}} \Big(\underbrace{R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R}_{G_{\mu\nu}} \Big)$$

Aside:

$$\delta I_3 = \int d^4x \sqrt{-g}\; \nabla^{\mu}(\delta w_{\mu})$$

$$= \oint d^4x \sqrt{-g}\; n^{\mu}\, \delta w_{\mu}$$

use divergence theorm

boundary term

$$\delta I = \delta I_1 + \delta I_2 + \cancel{\delta I_3}$$

$$= \int d^4x \sqrt{-g}\; \underset{\substack{\| \\ (\nabla^{\nu}\xi^{\mu} + \nabla^{\mu}\xi^{\nu})}}{\delta g^{\mu\nu}} \Big(\underbrace{R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R}_{G_{\mu\nu}} \Big)$$

Aside:

EOM: $G_{\mu\nu} = 0$

in absence of any matter

if $\delta g^{\mu\nu} = \nabla^\mu \xi^\nu + \nabla^\nu \xi^\mu$ ← infinitesimal coord change

$\delta I = 0$

EOM: $G_{\mu\nu} = 0$

in absence of any matter

if $\delta g^{\mu\nu} = \nabla^{\mu} \xi^{\nu} + \nabla^{\nu} \xi^{\mu}$ $\leftarrow$ infinitesimal coord change

$$\delta I = 0 \longrightarrow \nabla^{\mu} G_{\mu\nu} = 0$$

$$\delta I_3 = \int d^4x \sqrt{-g}\; \nabla^\mu (\delta w_\mu)$$

$$= \oint d^4x \sqrt{-g}\; n^\mu\, \delta w_\mu$$

use divergence theorem

$$\delta I = \delta I_1 + \delta I_2 + \cancel{\delta I_3}$$

boundary term

$$= \int d^4x \sqrt{-g}\; \delta g^{\mu\nu} \underbrace{\left(R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R\right)}_{G_{\mu\nu}}$$

Aside: $\delta g^{\mu\nu} = (\nabla^\nu \xi^\mu + \nabla^\mu \xi^\nu)$