Title: PSI - Relativity (PHYS 604) - 16

Date: Sep 24, 2009 10:30 AM

URL: http://pirsa.org/09090075

Abstract:

Linearized Einstein Eq's (in vacuum)

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$$

↑ each $|h_{\alpha\beta}| \ll 1$

$$\Rightarrow R_{\alpha\beta} = 0 = \tfrac{1}{2}\left[-\Box h_{\alpha\beta} + \partial_\alpha V_\beta + \partial_\beta V_\alpha \right]$$

$\Box \equiv (-\partial_t^2 + \vec{\nabla}^2)$

with $V_\alpha \equiv \partial^\beta h_{\beta\alpha} - \tfrac{1}{2}\partial_\alpha h_\beta{}^\beta$

"Improved" Newtonian metric:

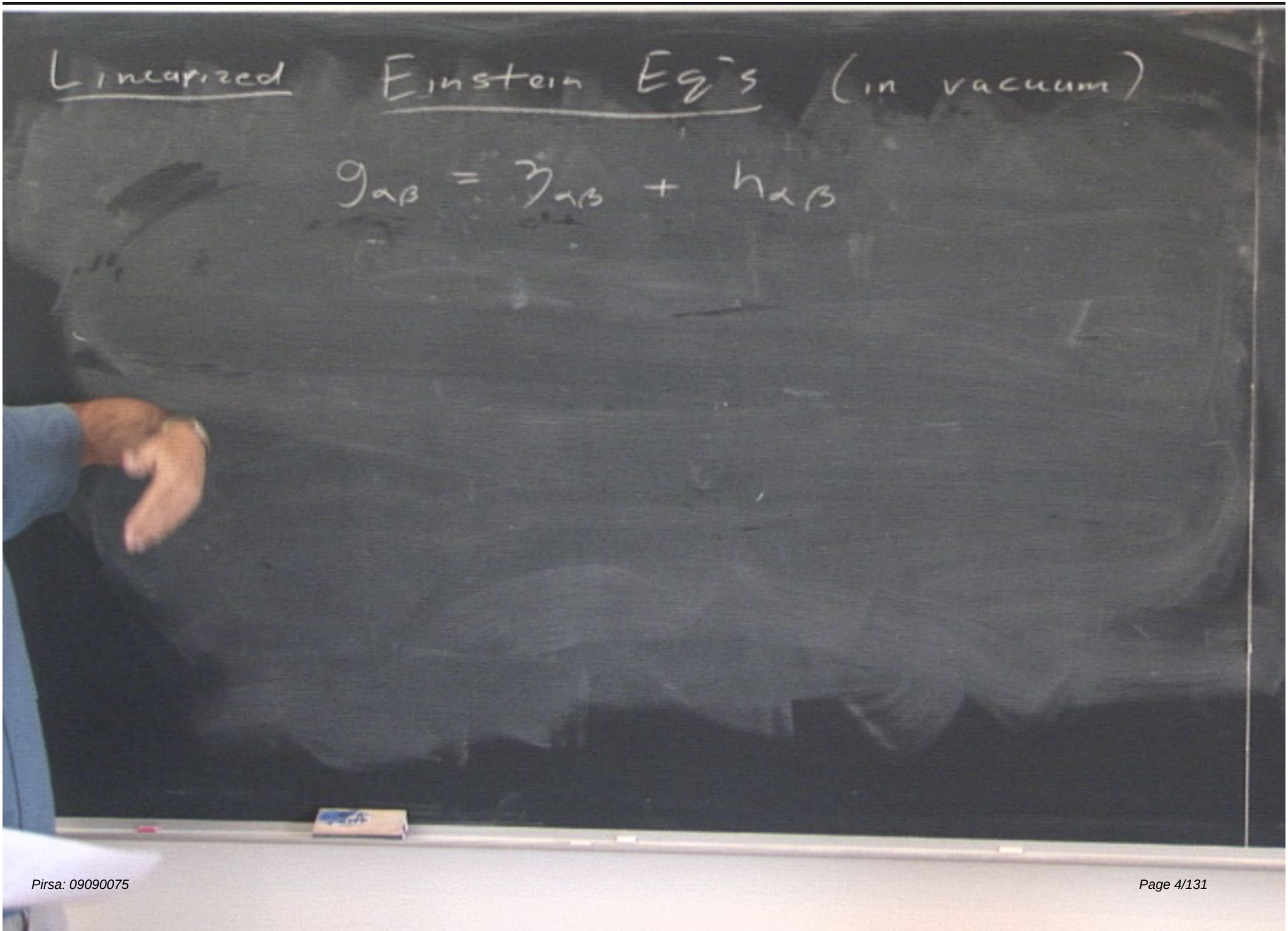
$$ds^2 = -(1+2\bar{\Phi})dt^2 + (1-2\bar{\Phi})(dx^2+dy^2+dz^2)$$

solves $\delta R_{\alpha\beta}=0 \leftarrow \Box h_{\alpha\beta}=0 \leftarrow \bar{\nabla}^2\bar{\Phi}=0$

$\leftarrow V_\alpha = 0$

- matches expansion of Schwarz. metric

in <u>isotropic coord's</u>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only work to linear order

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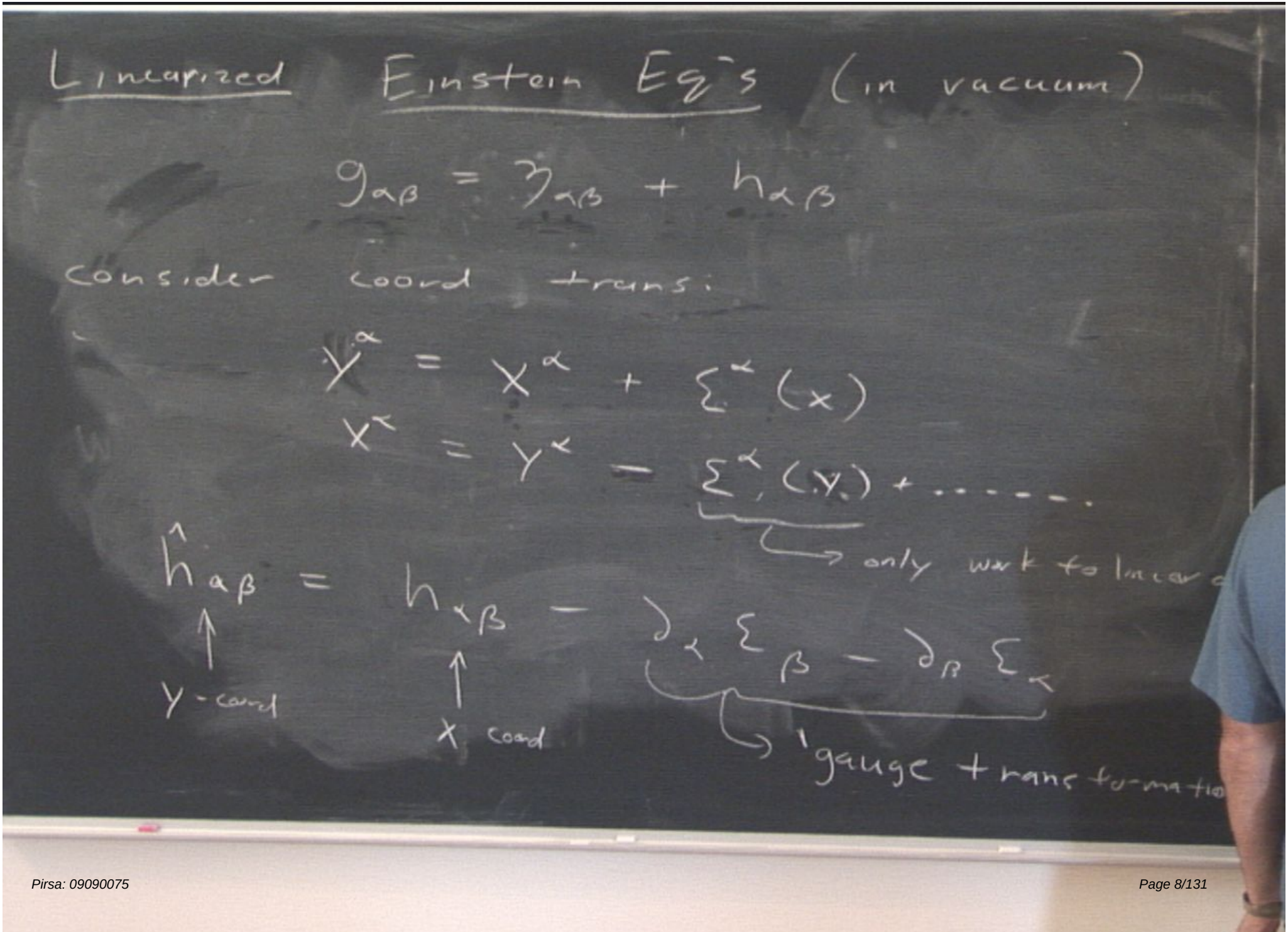
$$\hat{h}_{\alpha\beta} = h_{\alpha\beta} - \partial_\alpha \xi_\beta - \partial_\beta \xi_\alpha$$

$\hat{h}_{\alpha\beta}$: y-coord

$h_{\alpha\beta}$: x coord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$Y^\alpha = X^\alpha + \xi^\alpha(x)$
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only work to linear
$\hat{h}_{\alpha\beta} = h_{\alpha\beta} - \partial_\alpha \xi_\beta - \partial_\beta \xi_\alpha$
Y-coord
X coord
gauge transformation

"gauge fixing"

⟶ free to choose 4 arbitrary functions $\Sigma^{\tau}(x)$

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→ convenient to choose them to set

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$$\hat{V}_\alpha = \partial^\beta \hat{h}_{\beta\alpha} - \tfrac{1}{2}\partial_\alpha \hat{h}_\beta{}^\beta$$

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$$= \underline{\partial^\beta h_{\beta\alpha} - \tfrac{1}{2}\partial_\alpha h_\beta{}^\beta}$$

$$= V_\alpha$$

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$$= \underbrace{\partial^\beta h_{\beta\alpha} - \tfrac{1}{2}\partial_\alpha h_\beta{}^\beta}_{= V_\alpha} - \Box \Sigma_\alpha - \partial_\alpha(\nabla\cdot\Sigma)$$

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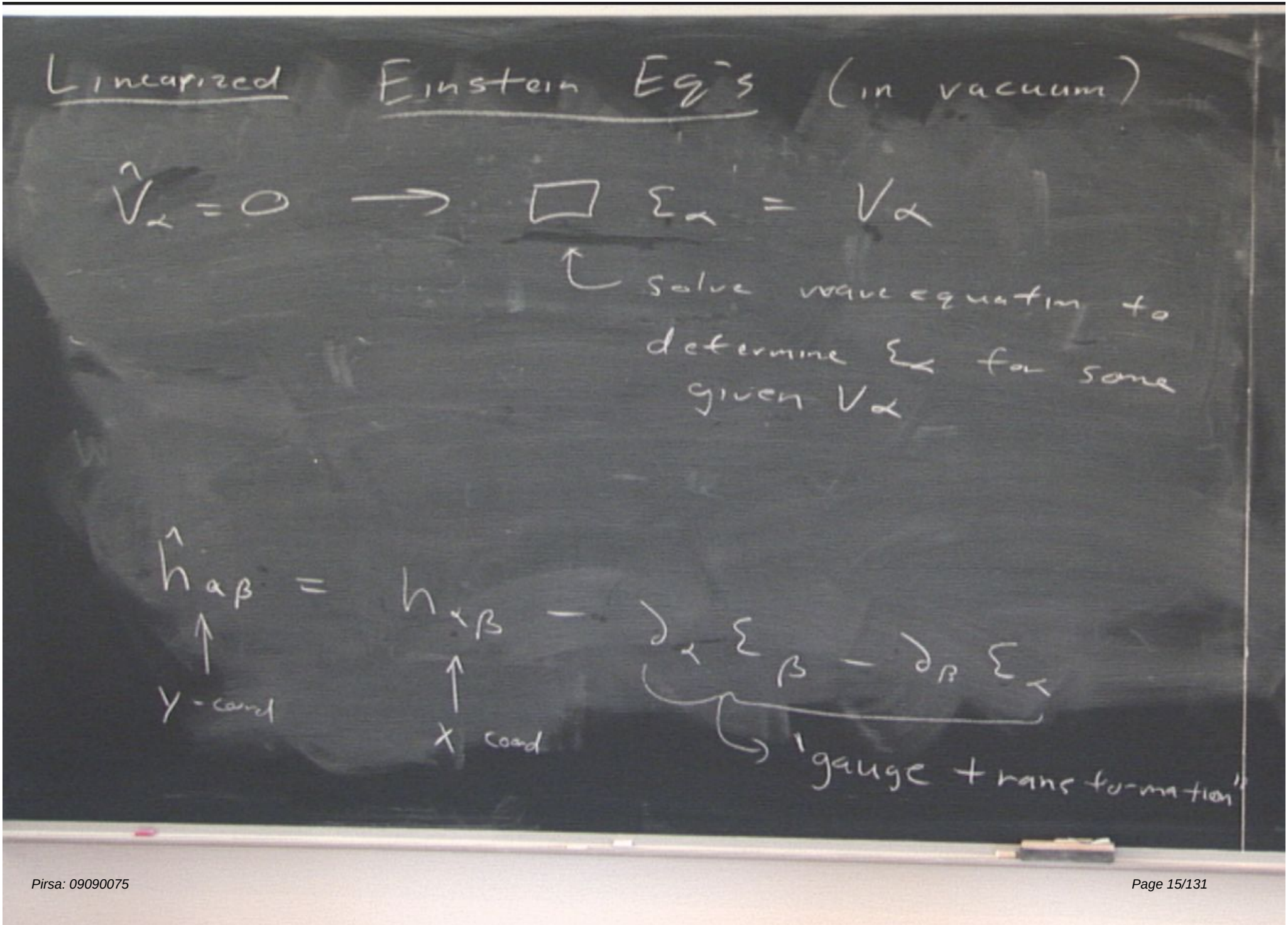
$$= \underbrace{\partial^\beta h_{\beta\alpha} - \tfrac{1}{2}\partial_\alpha h_\beta{}^\beta} - \Box\,\Sigma_\alpha - \partial_\alpha(\nabla\cdot\Sigma)$$

$$= V_\alpha \qquad -\tfrac{1}{2}\left(\partial_\alpha(-2\,\nabla\cdot\Sigma)\right)$$

(cancel)


Linearized Einstein Eq's (in vacuum)
$\hat{V}_\alpha = 0 \longrightarrow \Box \Sigma_\alpha = V_\alpha$
solve wave equation to determine Σ_α for some given V_α
$\hat{h}_{\alpha\beta} = h_{\alpha\beta} - \partial_\alpha \Sigma_\beta - \partial_\beta \Sigma_\alpha$
y-coord
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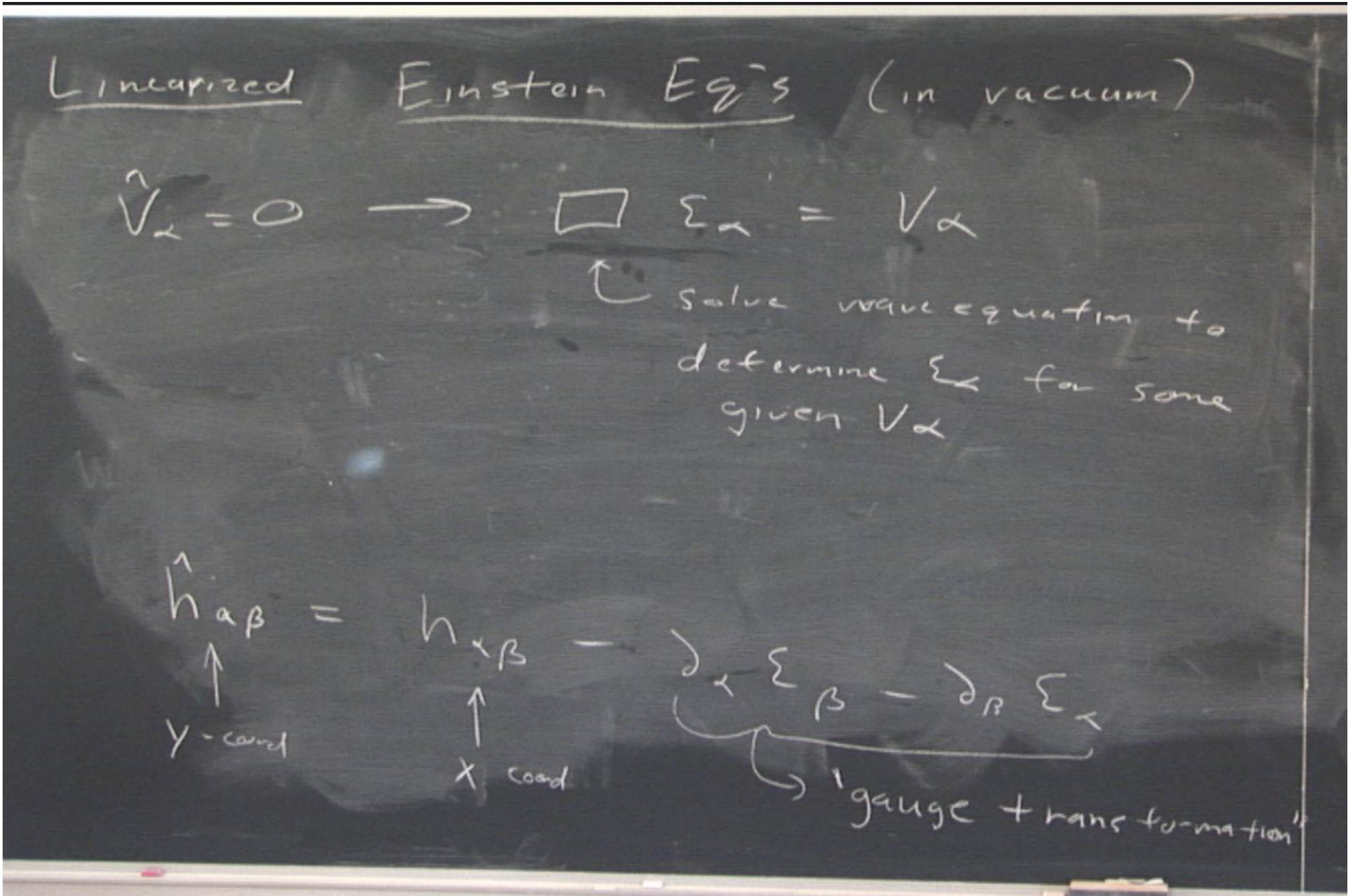
"gauge transformation"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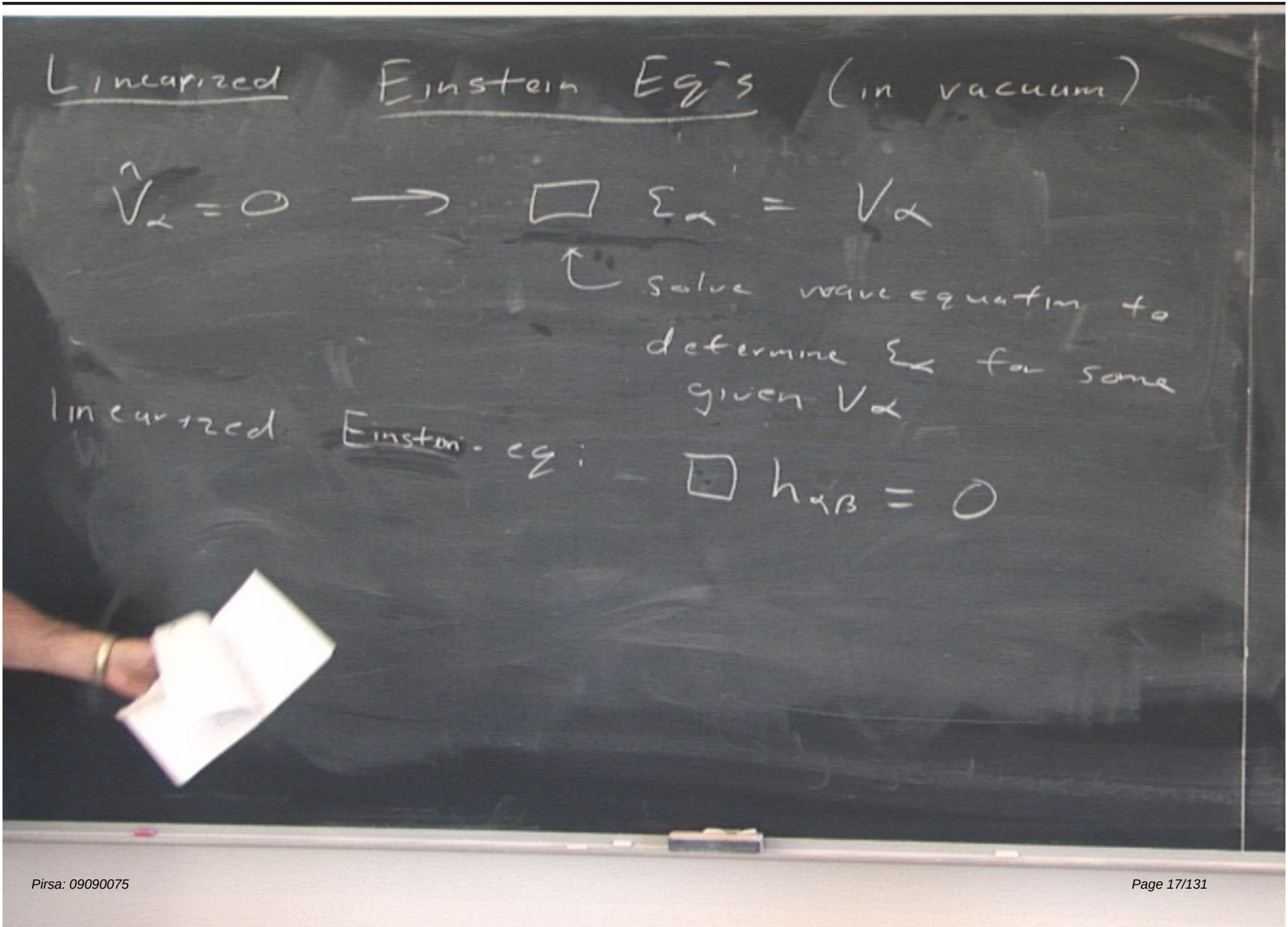
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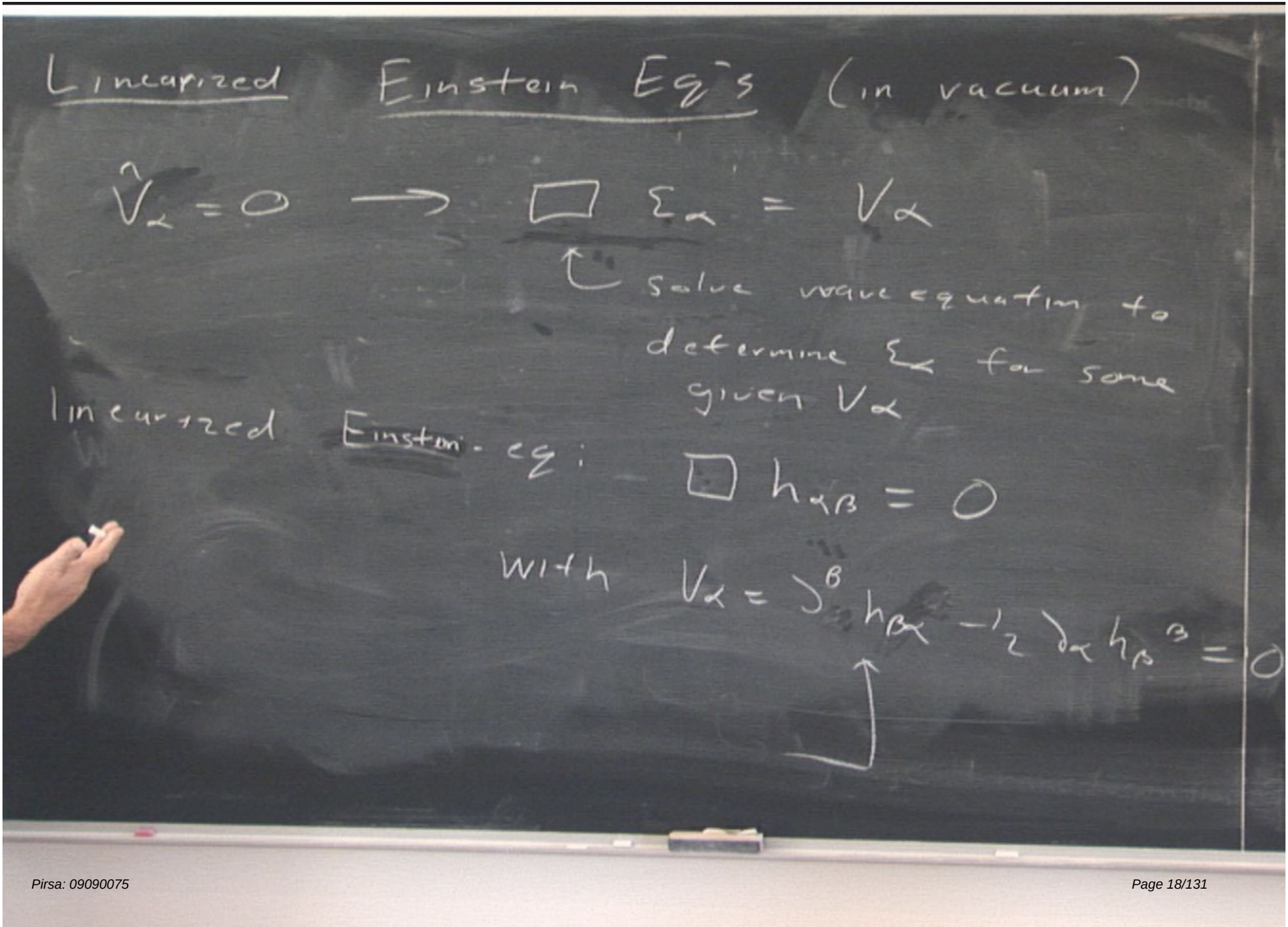
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Linearized Einstein Eq's (in vacuum)

$\hat{V}_\alpha = 0 \longrightarrow \Box \Sigma_\alpha = V_\alpha$

↳ solve wave equation to determine Σ_α for some given V_α

Linearized Einstein eq: $\Box h_{\alpha\beta} = 0$

with $V_\alpha = \partial^\beta h_{\beta\alpha} - \frac{1}{2}\partial_\alpha h_\beta{}^\beta = 0$

Lorentz gauge condition

→ have many solutions, eg plane waves

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$$h_{\alpha\beta} = q_{\alpha\beta} e^{i \underset{\sim}{k} \cdot \underset{\sim}{x}}$$

→ have many solutions, eg plane waves

$$h_{\alpha\beta} = q_{\alpha\beta}\, e^{i \underset{\sim}{k} \cdot \underset{\sim}{x}} \ldots$$

$$\underset{\sim}{k} \cdot \underset{\sim}{k} = 0 \longrightarrow k^{\alpha} = (\omega, \vec{k})$$

$$\omega = |\vec{k}|$$

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gauge cond. → $k^\alpha a_{\alpha\beta} - \frac{1}{2} k_\beta a_\alpha{}^\alpha = 0$

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$$\omega = |\vec{k}|$$

gauge cond. $\rightarrow$ $k^\alpha a_{\alpha\beta} - \frac{1}{2} k_\beta a_\alpha{}^\alpha = 0$

$\Longrightarrow$ Einstein's eq's supports waves in space-time geometry (even in the absence of matter

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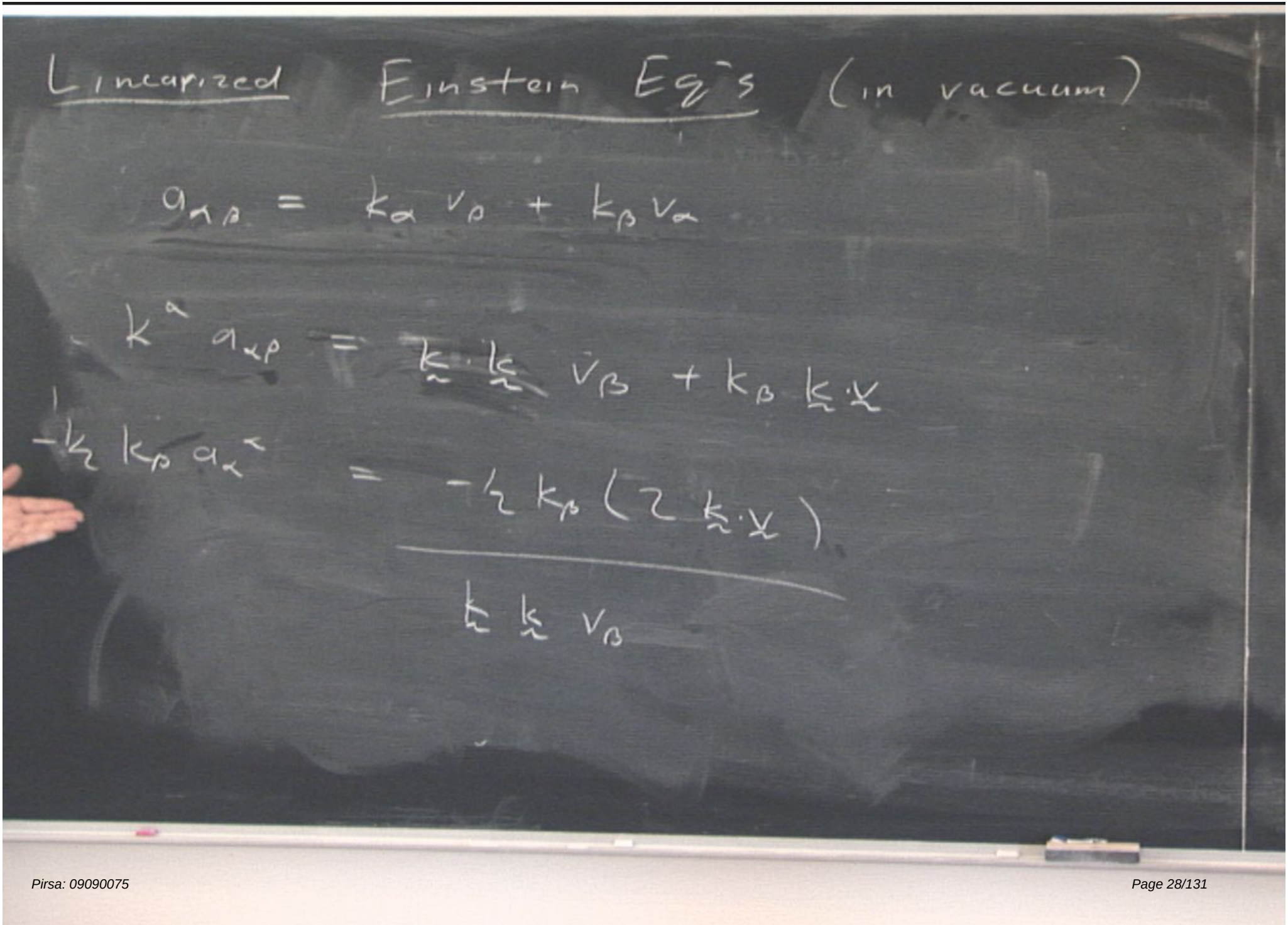
$$k^\alpha g_{\alpha\beta} = \underset{\sim}{k}\cdot\underset{\sim}{k}\, v_\beta + k_\beta\, \underset{\sim}{k}\cdot\underset{\sim}{v}$$

$$-\tfrac{1}{2} k_\beta g_\alpha{}^\alpha$$



Linearized Einstein Eq's (in vacuum)

$$a_{\alpha\beta} = k_\alpha v_\beta + k_\beta v_\alpha$$

$$k^\alpha a_{\alpha\beta} = \underset{\sim}{k} \cdot \underset{\sim}{k}\, v_\beta + k_\beta\, \underset{\sim}{k} \cdot \underset{\sim}{v}$$

$$-\tfrac{1}{2} k_\beta a_\alpha{}^\alpha = -\tfrac{1}{2} k_\beta \left(2\, \underset{\sim}{k} \cdot \underset{\sim}{v}\right)$$

$$\underset{\sim}{k}\, \underset{\sim}{k}\, v_\beta$$

Linearized Einstein Eq's (in vacuum)

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Linearized Einstein Eq's (in vacuum)

$$q_{\alpha\beta} = k_\alpha v_\beta + k_\beta v_\alpha \quad \leftarrow \text{satisfies gauge condition}$$

$$h_{\alpha\beta} = \partial_\alpha \left(-i v_\beta e^{ikx}\right) + \partial_\beta \left(-i v_\alpha e^{ikx}\right)$$

$$= \partial_\alpha \Sigma_\beta + \partial_\beta \Sigma_\alpha$$

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→ this solution in fact represents a coord shift

→ trivial

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$\longrightarrow$ this solution just represents a coord shift

$\longrightarrow$ trivial

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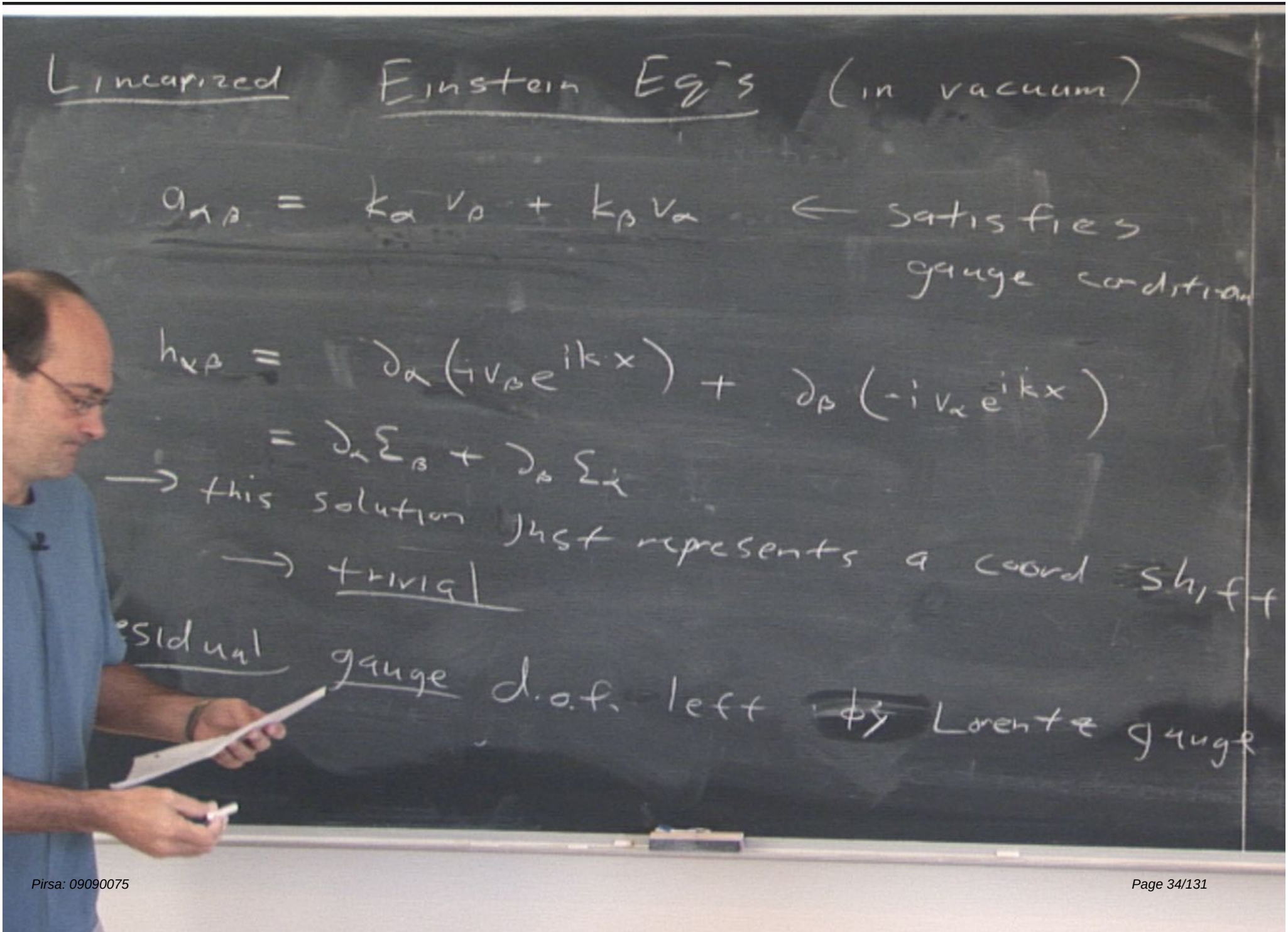
residual gauge d.o.f.: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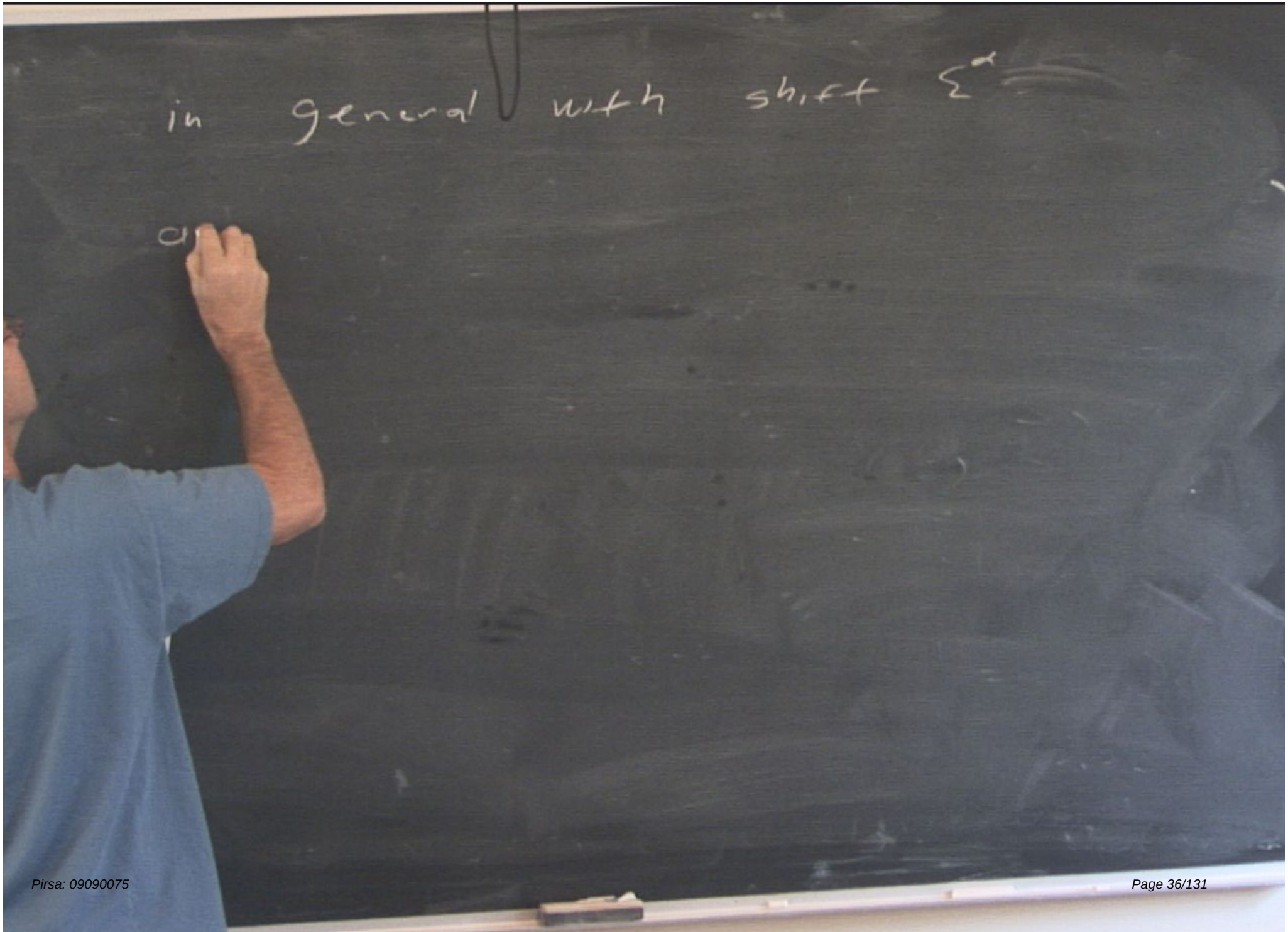
residual gauge d.o.f. left by Lorentz gauge


in general with shift Σ^a

in general with shift Σ^α

and $\Box \Sigma^\alpha = 0$

$\rightarrow$ doesn't effect gauge condition

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but still changes $h_{\alpha\beta}$

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→ doesn't effect gauge condition

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→ can simplify our sdutions further

again have four functions and so

make four choices!

in general with shift Σ^α

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again have four functions and so

make four choices: eg $h_{ti} = 0$

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go back to plane wave:

in general with shift Σ^α

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make four choices: eg $h_{ti} = 0 = h_\alpha{}^\alpha$

go back to plane wave: $a_{ti} = 0 = a_\alpha{}^\alpha$

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consider a special choice:

$$k^{\mu} = (\omega = k, 0, 0, k)$$

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Linearized Einstein Eq's (in vacuum)

consider a special choice:

$$k^\mu = (\omega = k, 0, 0, k)$$

$V_\alpha = 0 \rightarrow \alpha = +$:

$$\partial^\beta h_{\beta +} - \tfrac{1}{2} \partial_+ h^\gamma{}_\gamma = 0$$

$$- \partial_+ h_{++} = 0$$

$$\hookrightarrow -ik\, q_{++} = 0$$

Linearized Einstein Eq's (in vacuum)

consider a special choice:

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$-\partial_t h_{tt} = 0$

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$\alpha = x$ $\quad \partial_z h_{zx} = 0 \rightarrow ik \boxed{q_{zx} = 0}$

Linearized Einstein Eq's (in vacuum)

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$\alpha = x$ $\quad \partial_z h_{zx} = 0 \rightarrow ik\,\boxed{q_{zx} = 0}$

$q_{zy} = 0$

$q_{zz} = 0$

left with

$$h_{\alpha\beta} = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & & b & 0 \\ 0 & b & & 0 \\ 0 & 0 & 0 & c \end{bmatrix}}_{q_{\alpha\beta}} e^{ikx}$$

left with

$$h_{\alpha\beta} = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & a & b & 0 \\ 0 & b & -a & 0 \\ 0 & 0 & 0 & c \end{bmatrix}}_{a_{\alpha\beta}} e^{ik(z-t)}$$

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$\longrightarrow$ for each $\vec{k}$, precisely two physical polarizations for gravitational wave

differential identity of Ramis

differential identity of $R_{\alpha\beta\gamma\delta}$

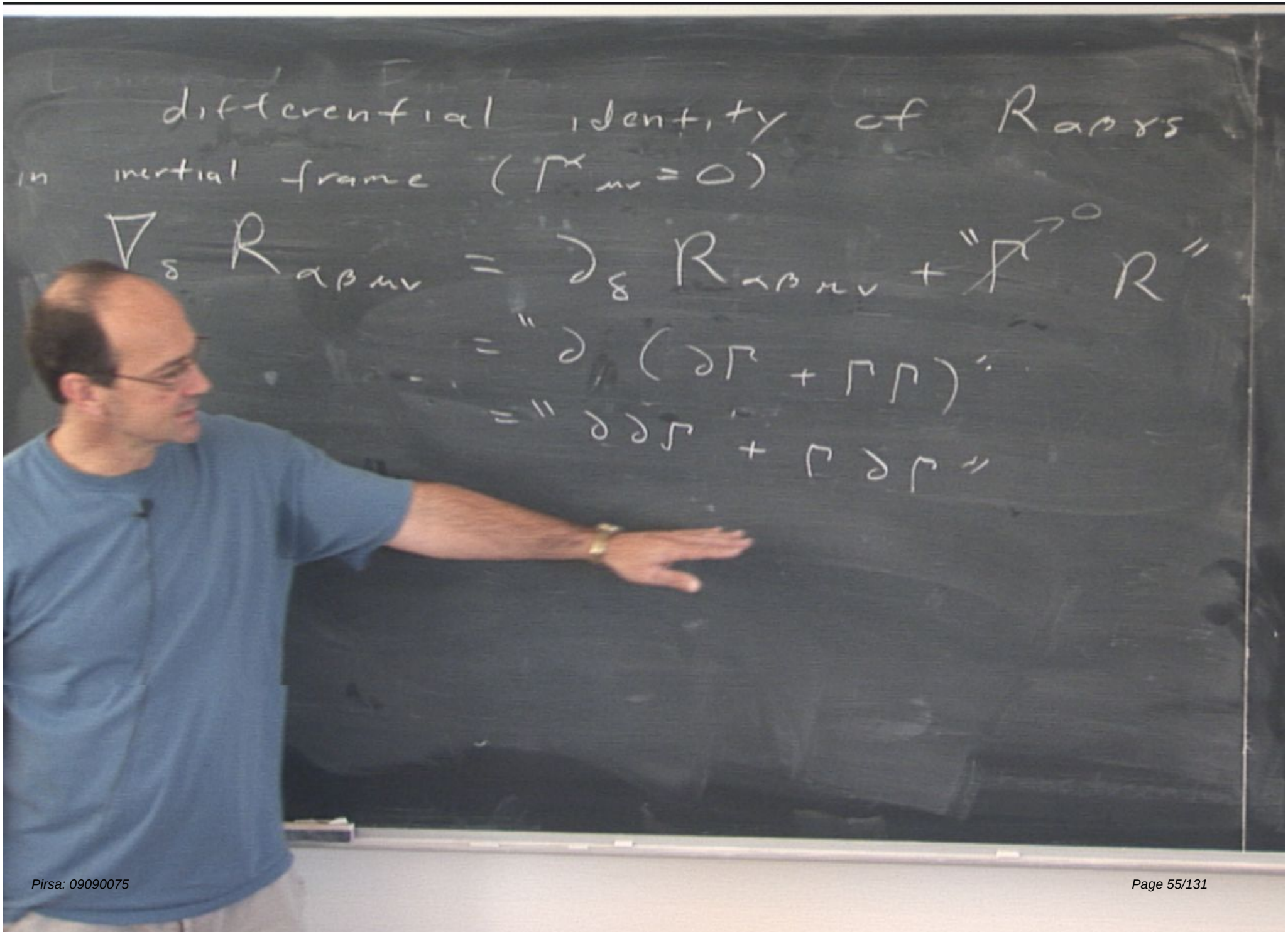
in inertial frame ($\Gamma^{\kappa}_{\mu\nu} = 0$)

$$\nabla_\delta R_{\alpha\beta\mu\nu}$$

differential identity of $R_{\alpha\beta\gamma\delta}$

in inertial frame ($\Gamma^{\kappa}_{\mu\nu} = 0$)

$$\nabla_\delta R_{\alpha\beta\mu\nu} = \partial_\delta R_{\alpha\beta\mu\nu} + \text{"}\Gamma\ R\text{"}$$


differential identity of $R_{\alpha\beta\gamma\delta}$
in inertial frame ($\Gamma^{\kappa}_{\mu\nu} = 0$)
$\nabla_\delta R_{\alpha\beta\mu\nu} = \partial_\delta R_{\alpha\beta\mu\nu} + \Gamma^{0} R$
$= \partial (\partial\Gamma + \Gamma\Gamma)$
$= \partial\partial\Gamma + \Gamma\partial\Gamma$

differential identity of $R_{\alpha\beta\gamma\delta}$

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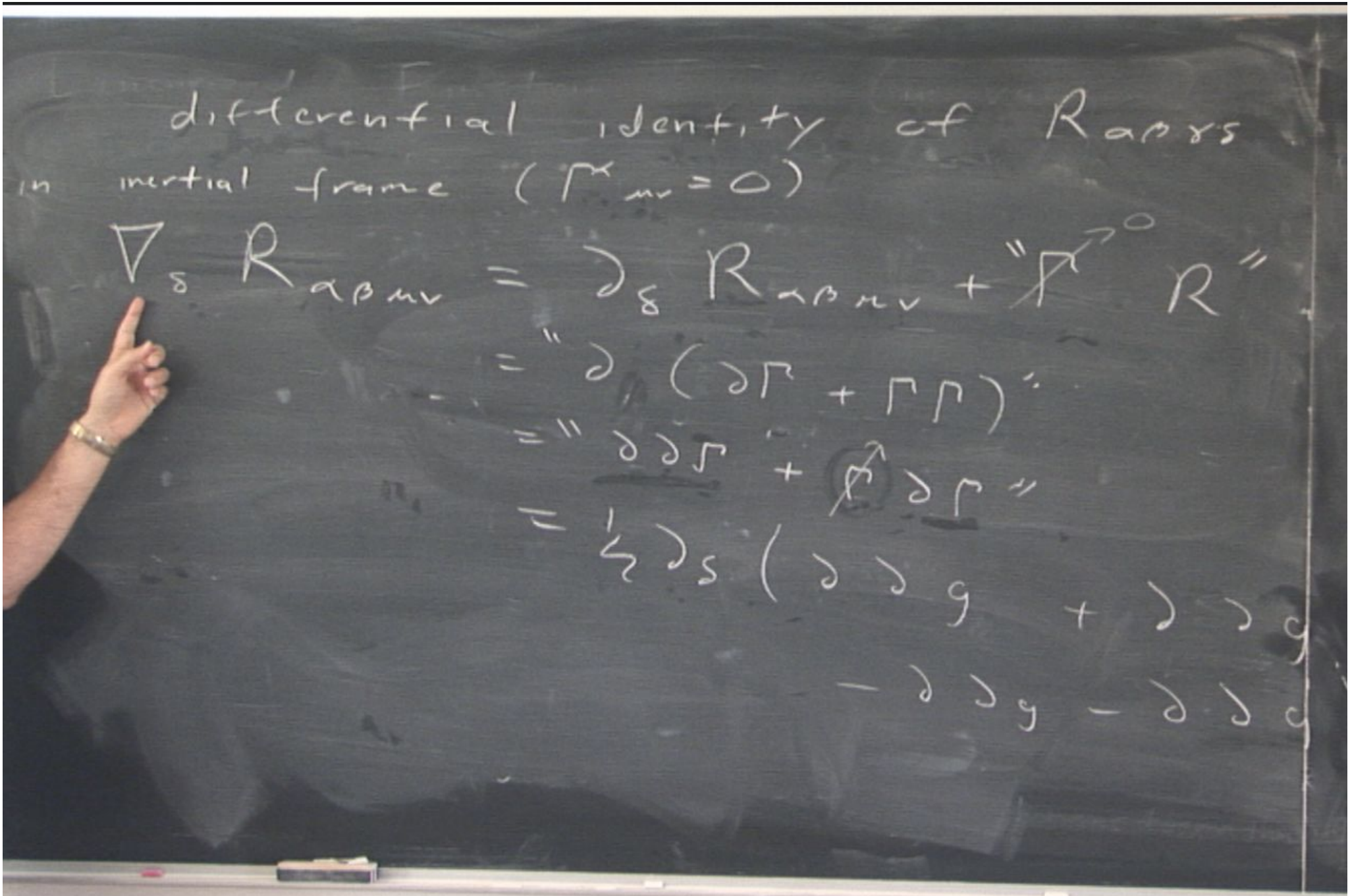
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in inertial frame ($\Gamma^{\kappa}_{\mu\nu} = 0$)
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$= "\partial(\partial\Gamma + \Gamma\Gamma)"$
$= "\partial\partial\Gamma + \Gamma\partial\Gamma"$
$= \frac{1}{2}\partial_\delta(\partial\partial g + \partial\partial g$
$- \partial\partial g - \partial\partial g$

differential identity of $R_{\alpha\beta\gamma\delta}$

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$$= \text{"}\partial(\partial\Gamma + \Gamma\Gamma)\text{"}$$

$$= \text{"}\partial\partial\Gamma + \Gamma\partial\Gamma\text{"}$$

$$= \tfrac{1}{2}\partial_\delta(\partial\partial g + \partial\partial g - \partial\partial g - \partial\partial g$$

$\longrightarrow$ cyclically permuting indices produces an interesting identity

$$\nabla_\sigma R_{\alpha\beta\mu\nu} + \nabla_\alpha R_{\beta\sigma\mu\nu} + \nabla_\beta R_{\sigma\alpha\mu\nu} = 0$$

Bianchi identity

implications for Ricci tensor

$$\nabla_\sigma R_{\alpha\beta\mu\nu} + \nabla_\alpha R_{\beta\sigma\mu\nu} + \nabla_\beta R_{\sigma\alpha\mu\nu} = 0$$

Bianchi identity

implications for Ricci tensor

$$R_{\alpha\beta} = g^{\mu\nu} R_{\mu\alpha\nu\beta}$$

$$\nabla_\sigma R_{\alpha\beta\mu\nu} + \nabla_\alpha R_{\beta\sigma\mu\nu} + \nabla_\beta R_{\sigma\alpha\mu\nu} = 0$$

Bianchi identity

implications for Ricci tensor

$$R_{\alpha\beta} = g^{\mu\nu} R_{\mu\alpha\nu\beta} \quad (= R_{\beta\alpha})$$

contract with $g^{\alpha\mu}$,

$$\nabla_\sigma R_{\alpha\beta\mu\nu} + \nabla_\alpha R_{\beta\sigma\mu\nu} + \nabla_\beta R_{\sigma\alpha\mu\nu} = 0$$

Bianchi identity

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contract with $g^{\alpha\mu}$,

$$\nabla_\sigma R_{\beta\nu} + \nabla^\mu R_{\beta\sigma\mu\nu} - \nabla_\beta R_{\sigma\nu} = 0$$

$$\nabla_\sigma R_{\alpha\beta\mu\nu} + \nabla_\alpha R_{\beta\sigma\mu\nu} + \nabla_\beta R_{\sigma\alpha\mu\nu} = 0$$

Bianchi identity

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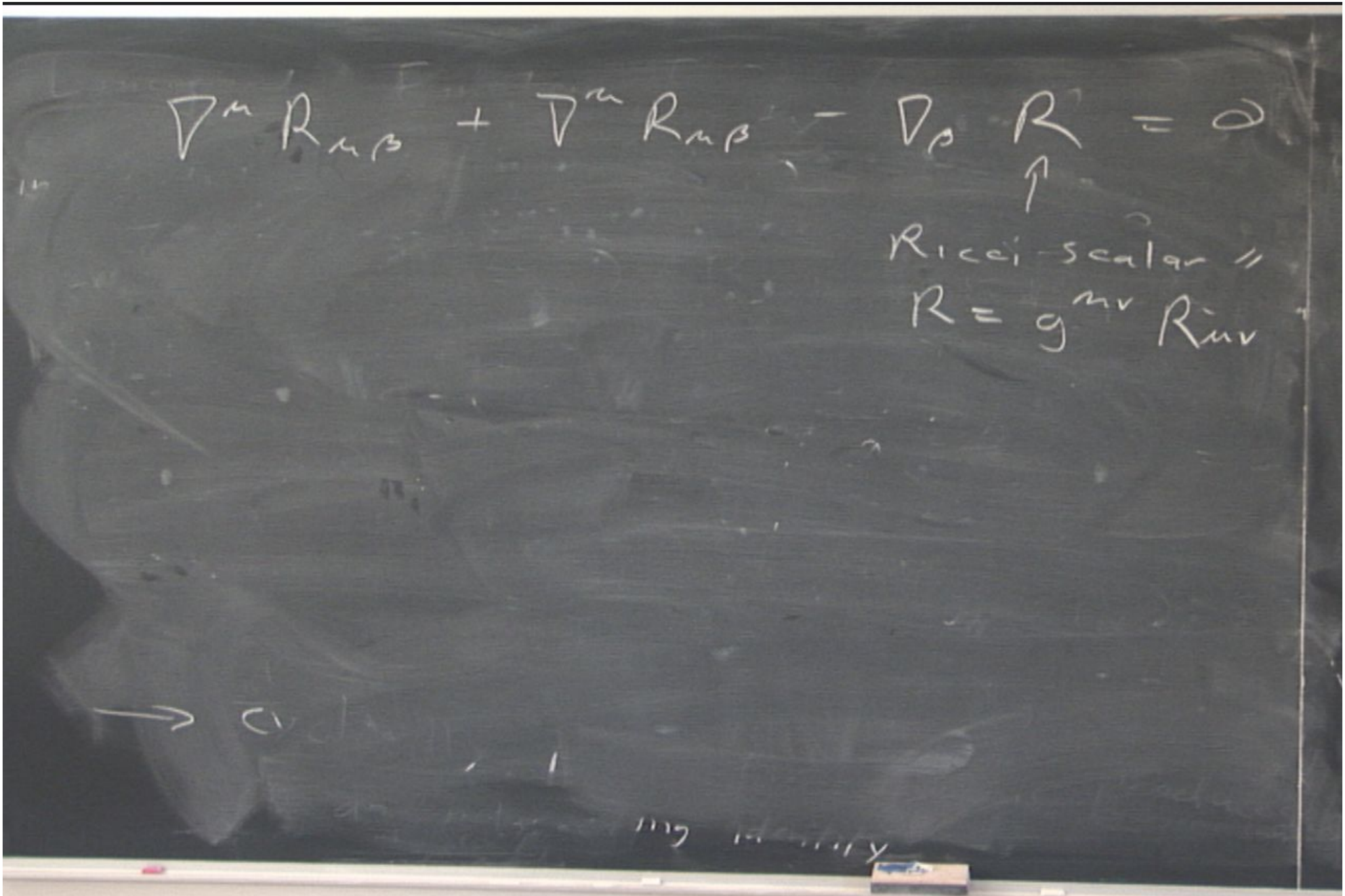
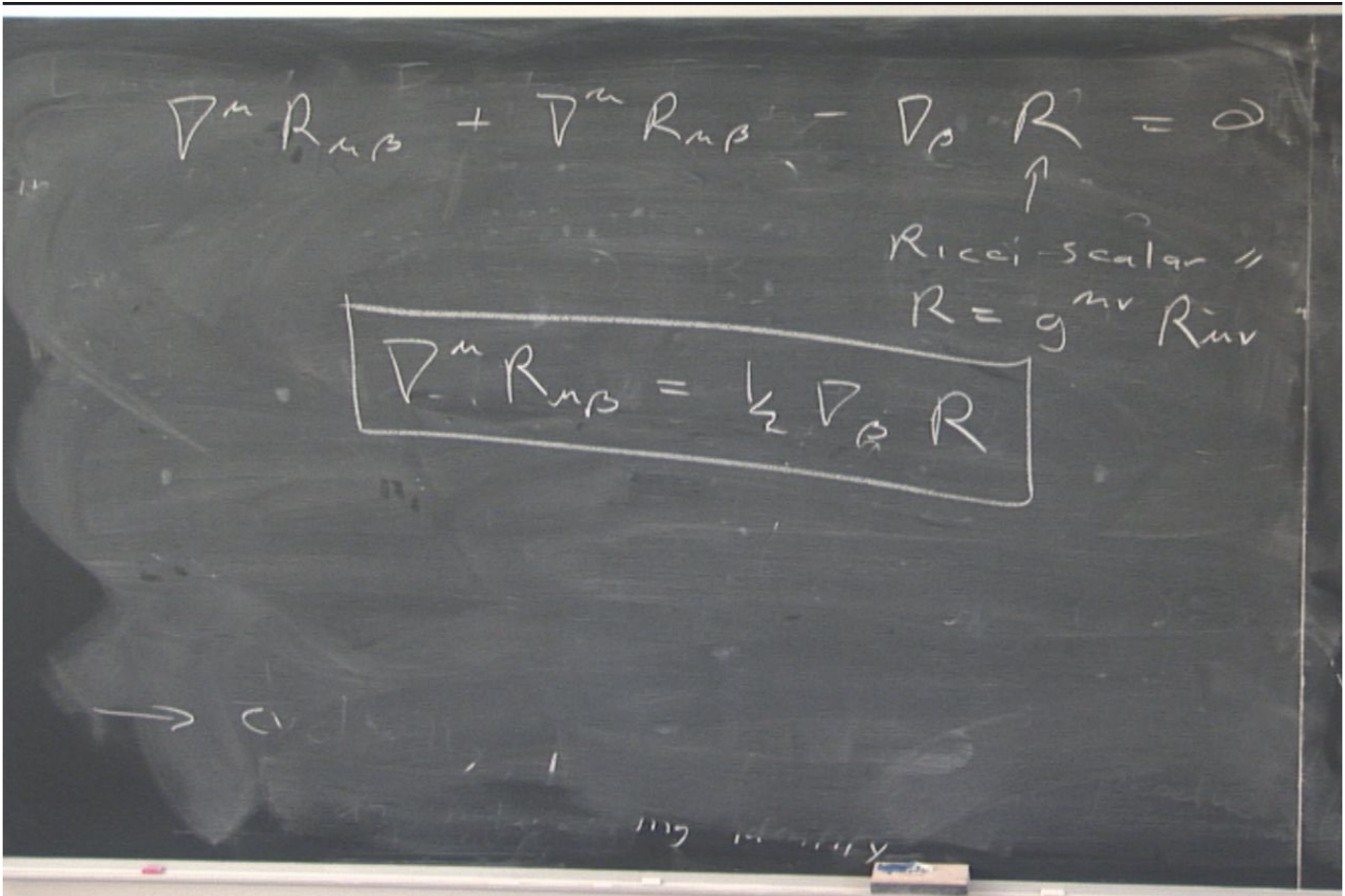
$$R_{\alpha\beta} = g^{\mu\nu} R_{\mu\alpha\nu\beta} \quad (= R_{\beta\alpha})$$

contract with $g^{\alpha\mu}$,

$$\nabla_\sigma R_{\beta\nu} + \nabla^\mu R_{\beta\sigma\mu\nu} - \nabla_\beta R_{\sigma\nu} = 0$$

contract $g^{\sigma\nu}$


$\nabla^\mu R_{\mu\beta} + \nabla^\mu R_{\mu\beta} - \nabla_\beta R = 0$
Ricci scalar
$R = g^{\mu\nu} R_{\mu\nu}$


$\nabla^\mu R_{\mu\beta} + \nabla^\mu R_{\mu\beta} - \nabla_\beta R = 0$
Ricci-scalar
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Einstein eqs

$$(\text{curvature})_{\alpha\beta} = (\text{matter})_{\alpha\beta}$$

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$T_{\alpha\beta}$: sym, some matrix describing mat

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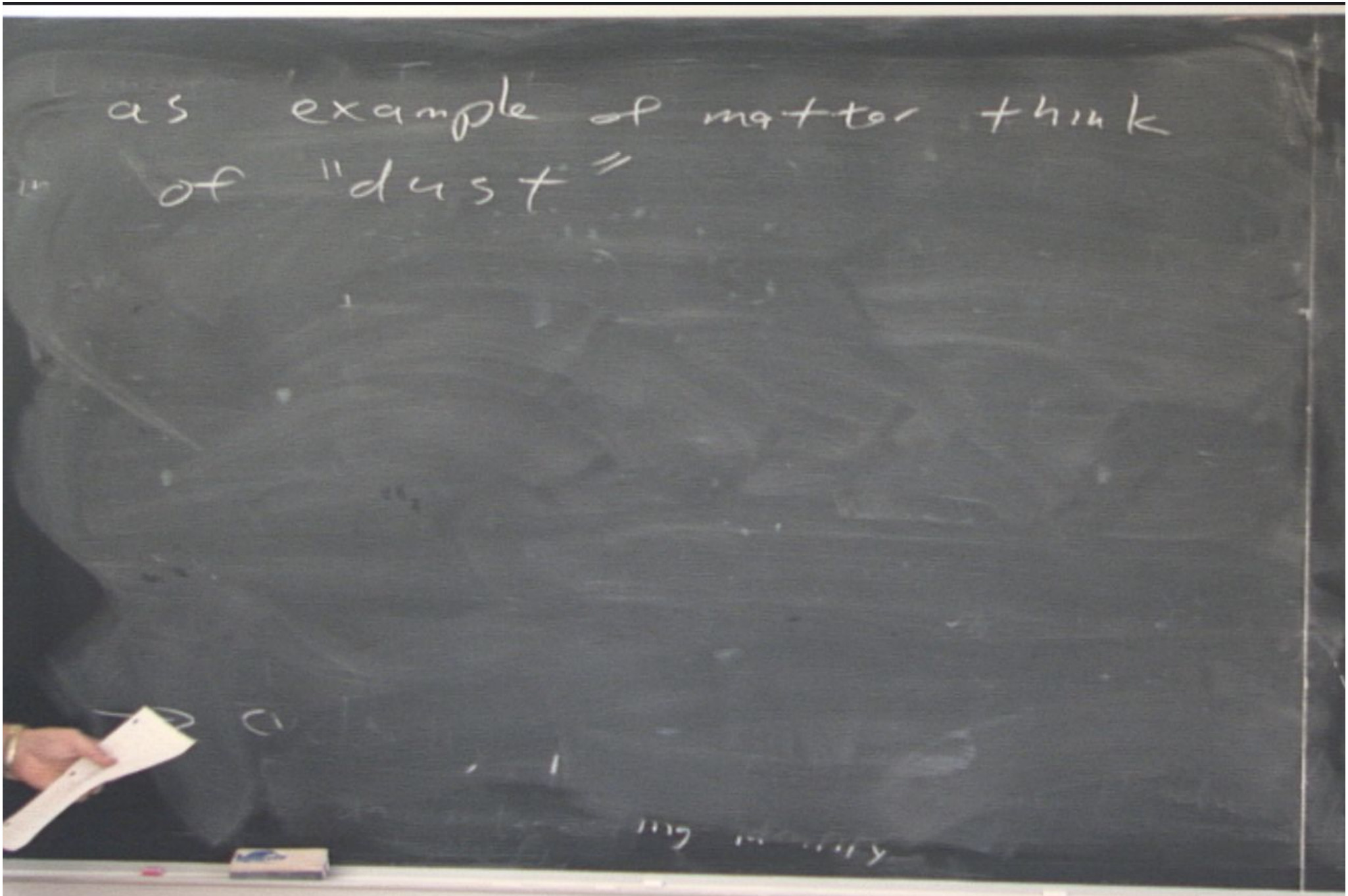
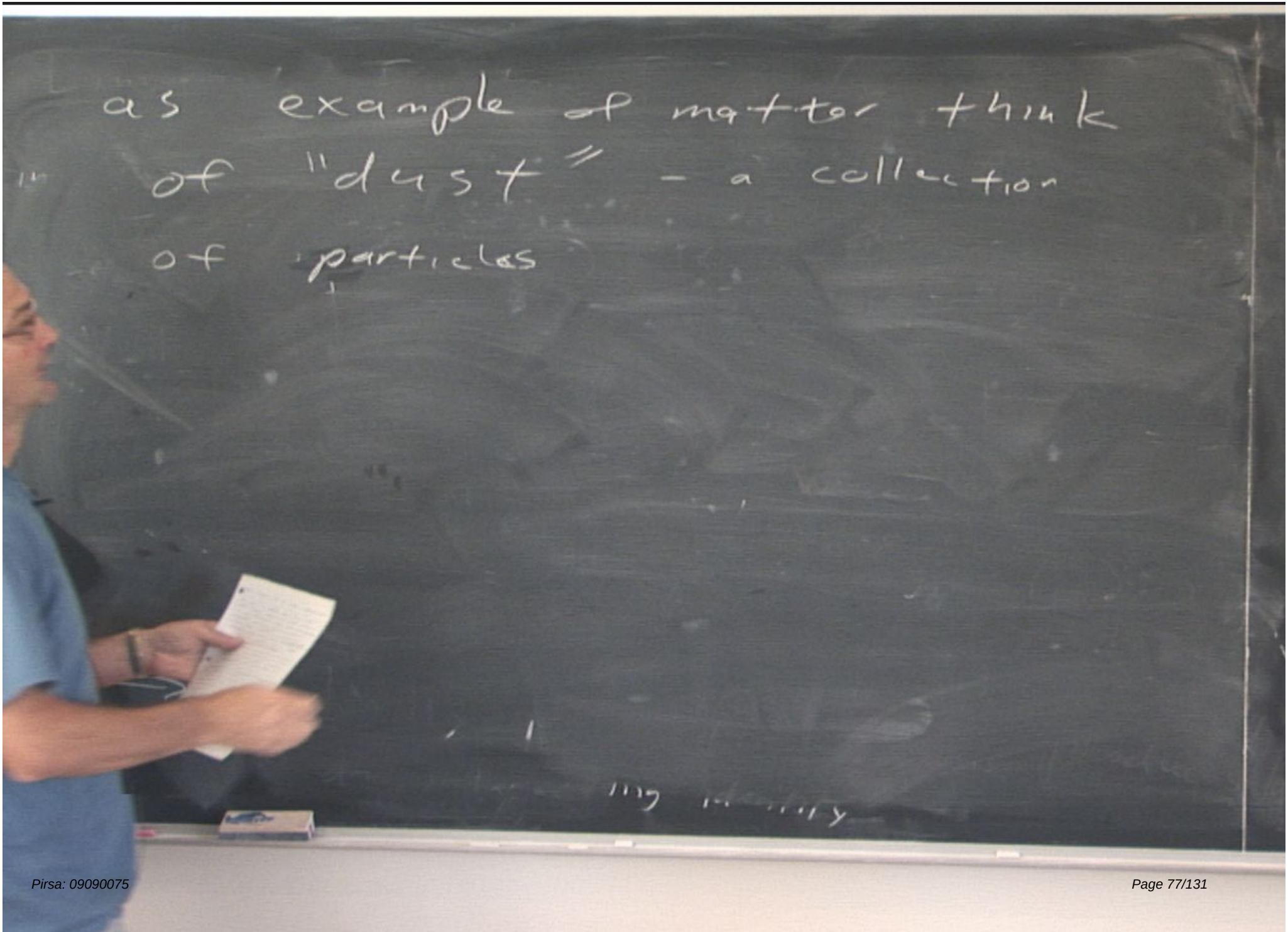

as example of matter think
of "dust"


as example of matter think
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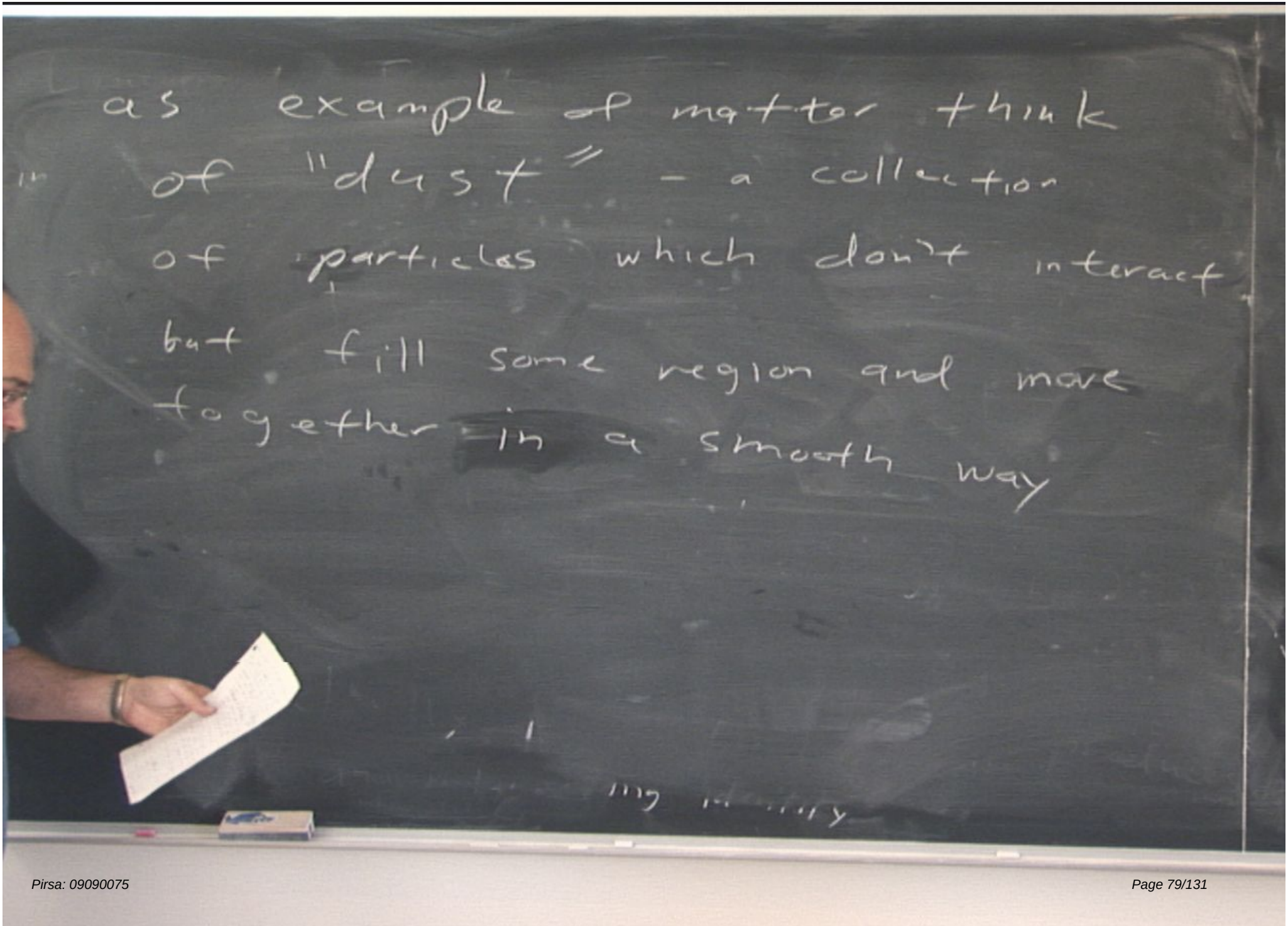
as example of matter think of "dust" – a collection of particles which don't interact.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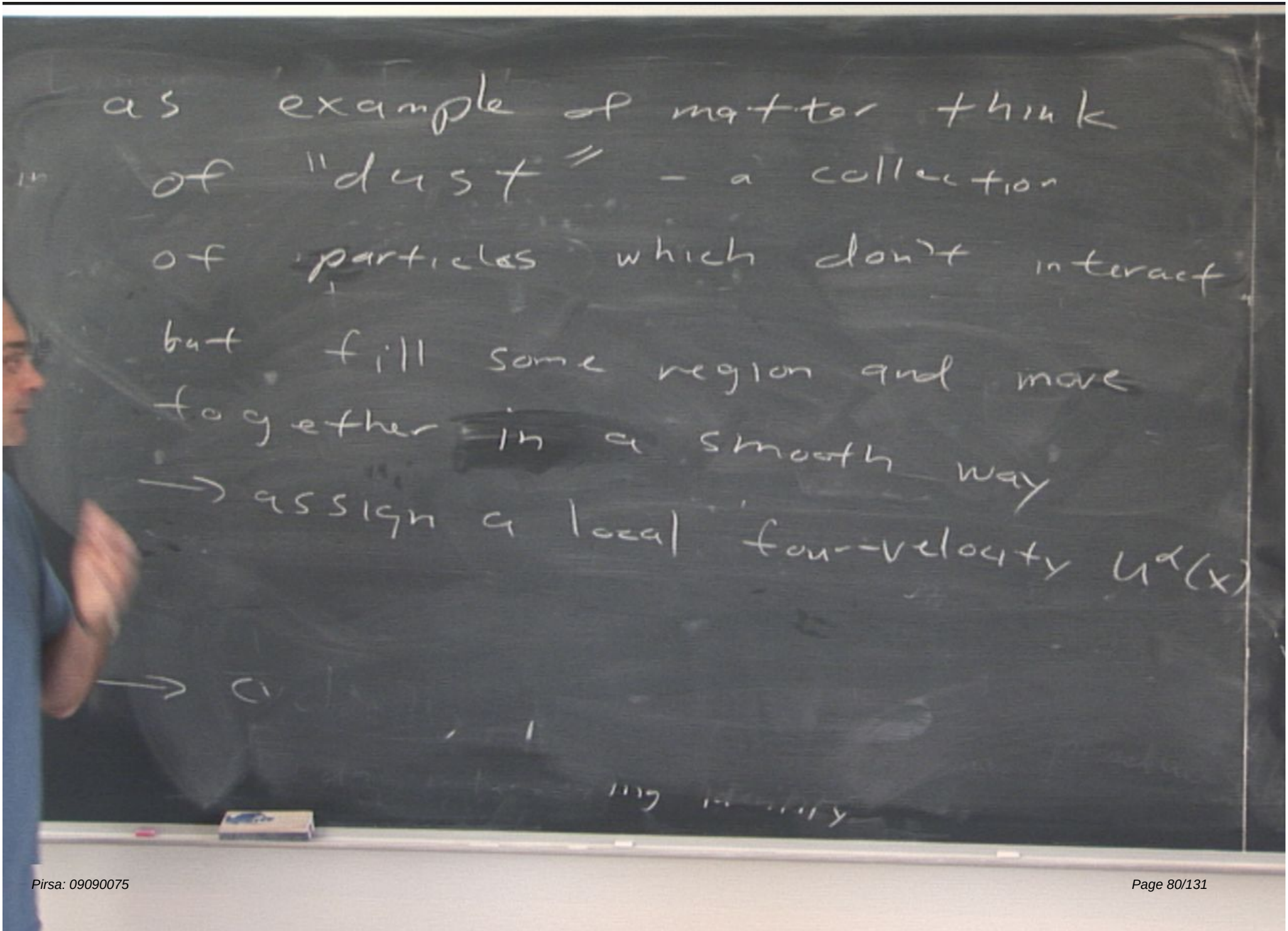
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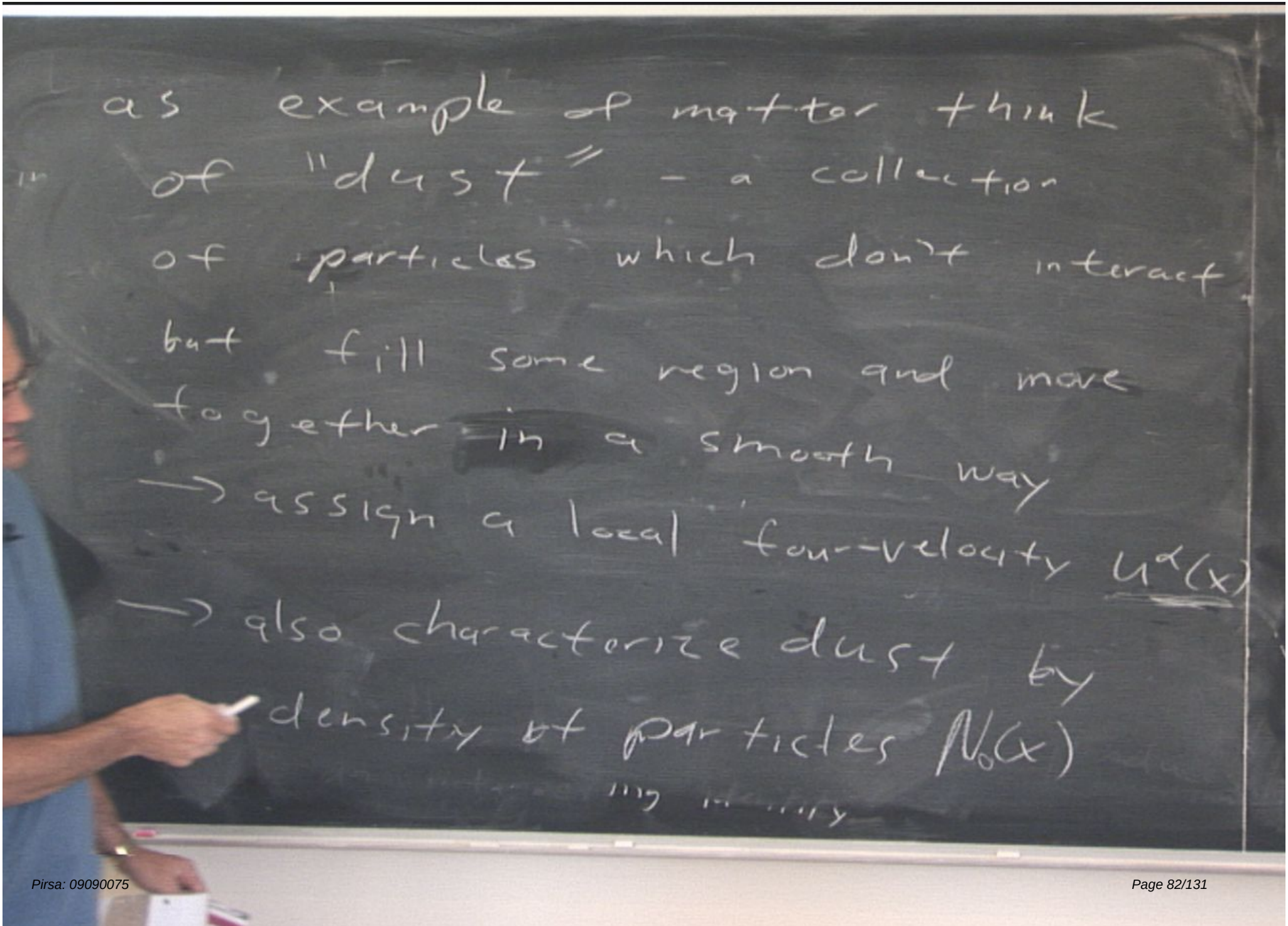
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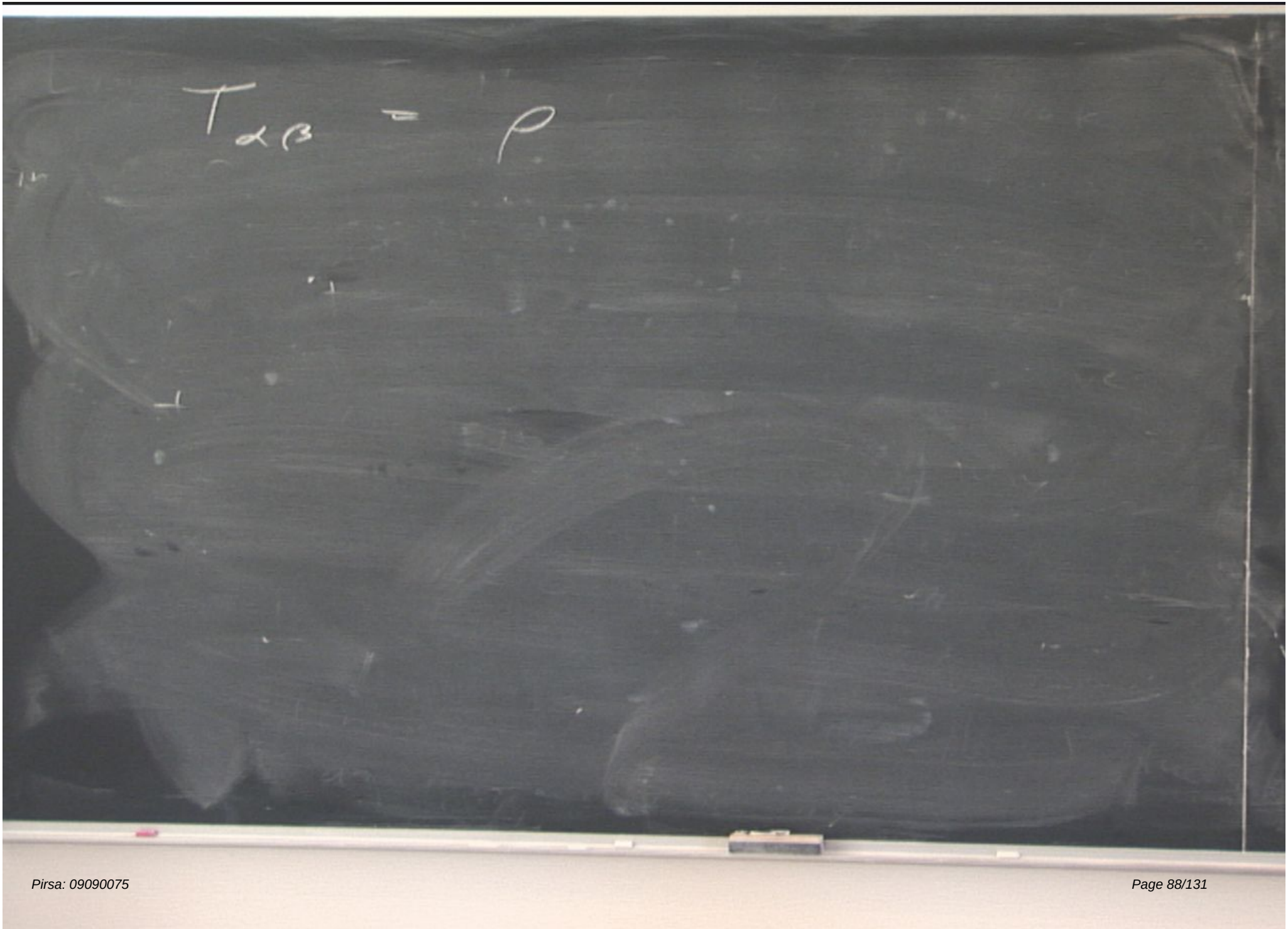
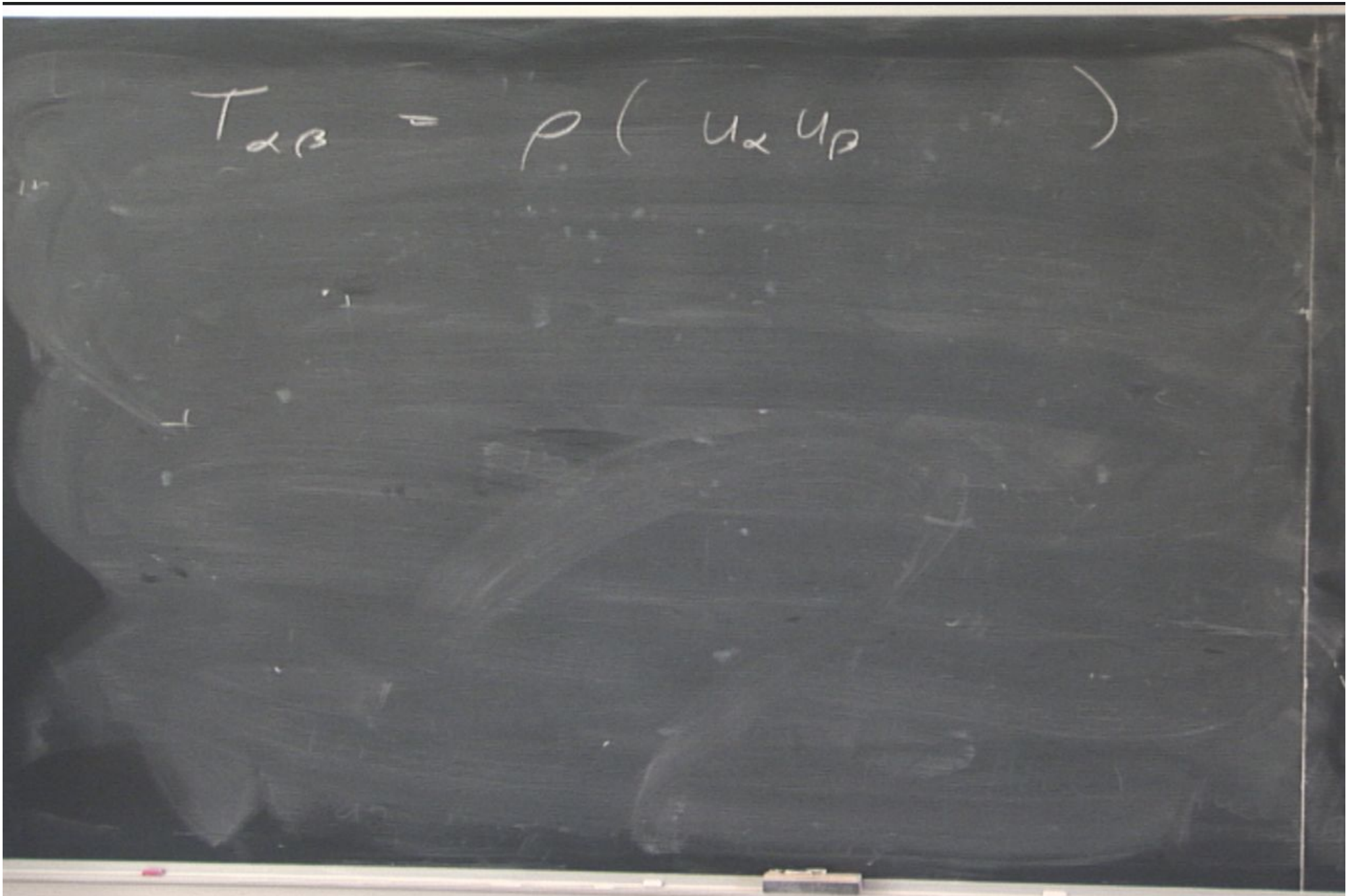

$T_{\alpha\beta} = \rho$


$T_{\alpha\beta} = \rho (u_\alpha u_\beta$)

$$T_{\alpha\beta} = \rho\left(a\, u_\alpha u_\beta + b\, g_{\alpha\beta}\right)$$

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What is the energy density measured by observer at rest in the cloud

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recall derivation of vacuum Einstein eq

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consider observers moving at fixed velocity wrt dust with V^α

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$$T_{\alpha\beta} V^\alpha V^\beta = \rho \left(a (\underset{\sim}{u} \cdot \underset{\sim}{v})^2 - b \right)$$

pick explicit frame

$$U^\alpha = (1, 0, 0, 0)$$

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particles are moving wrt "V-observers"

and so rel. energy of each particle

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$\sqrt{\quad}$ 1

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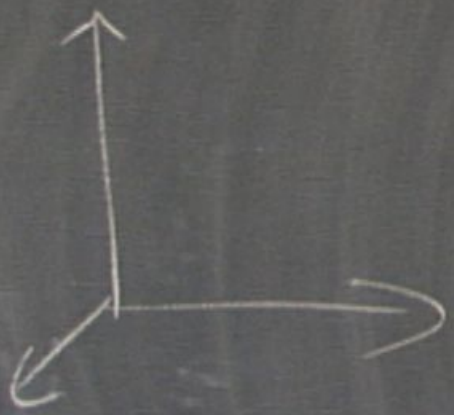

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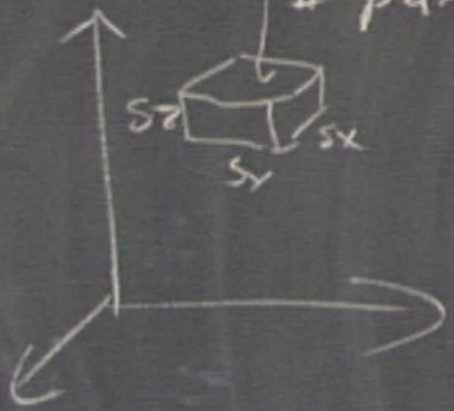

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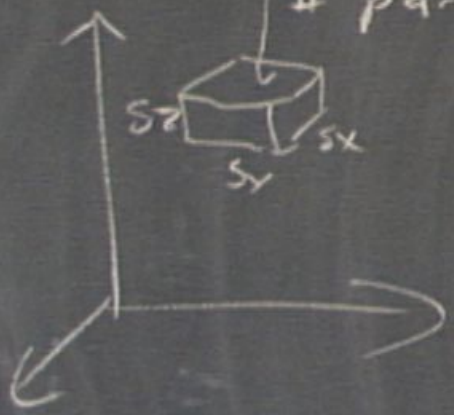

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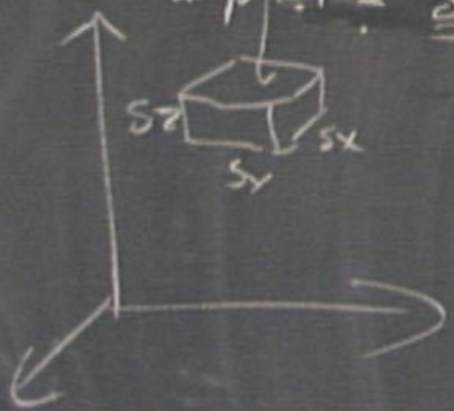

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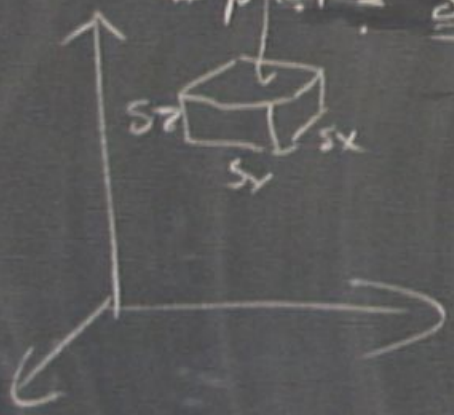

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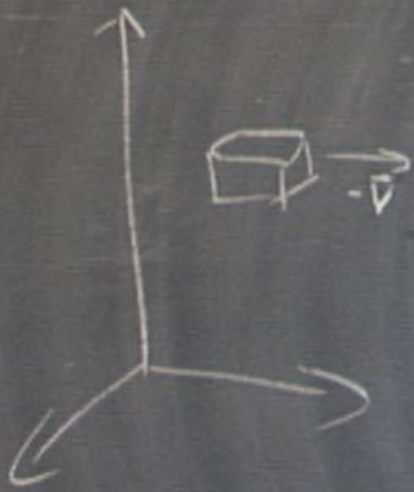

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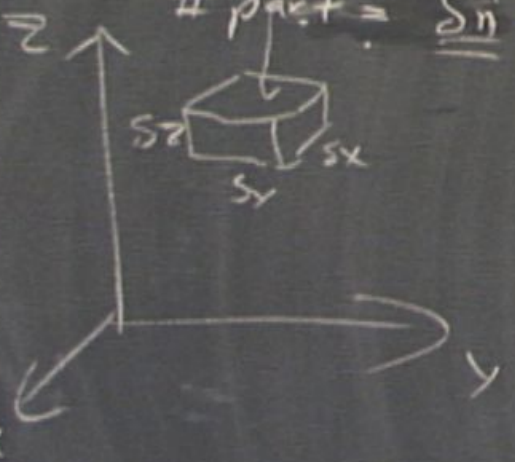

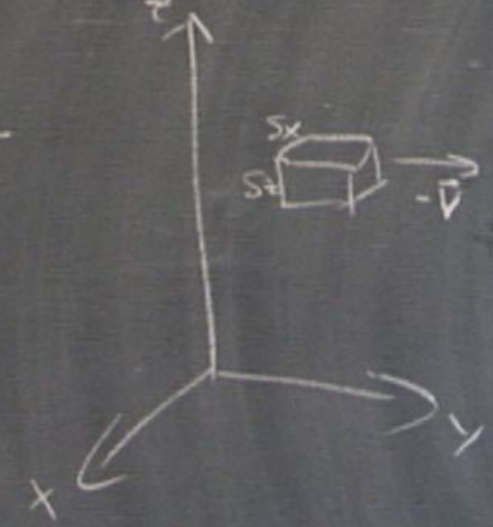

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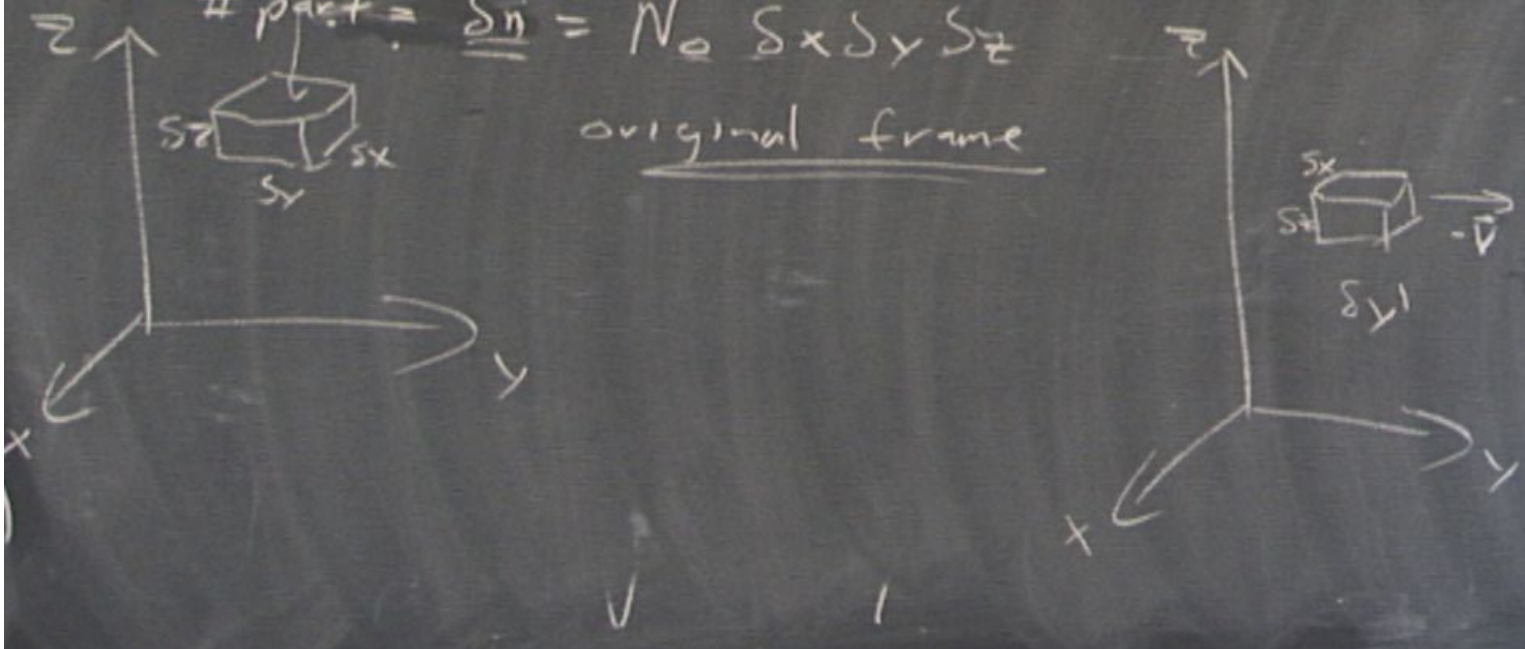

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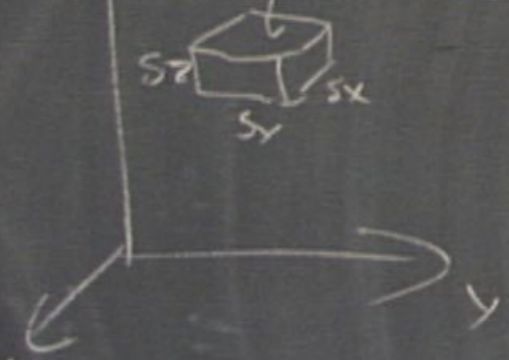

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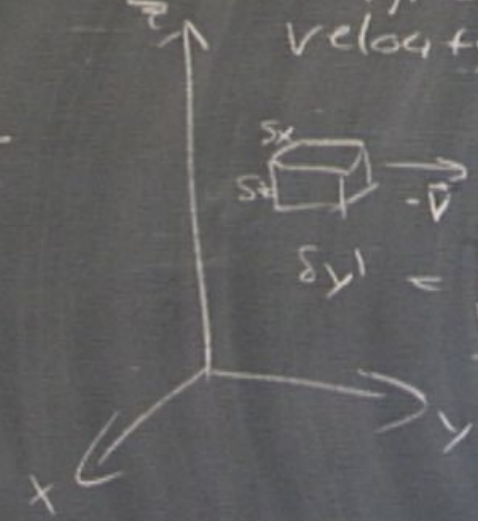
imagine duct moves with velocity v along y-axis

- δx / δz unchanged

$\delta y' = \sqrt{1-v^2}\,\delta y$



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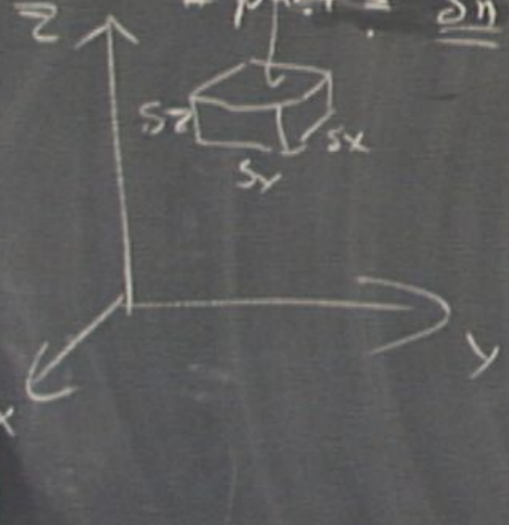

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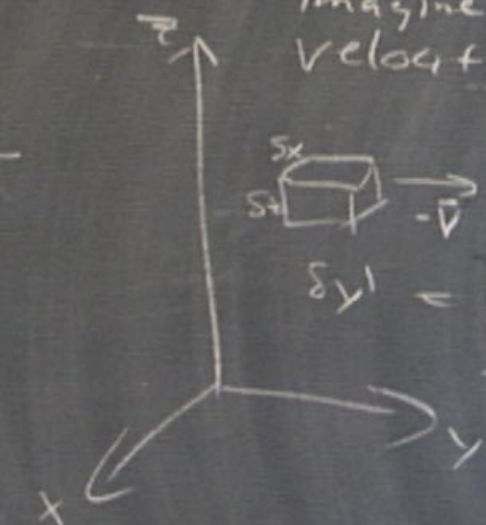

length contracted

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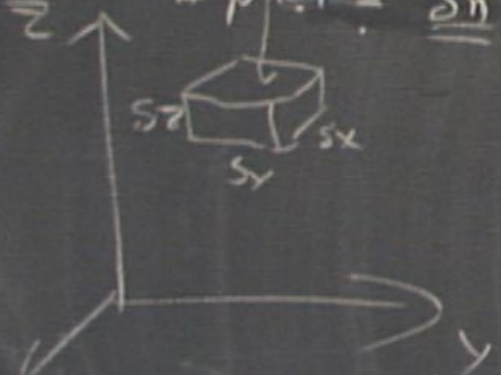

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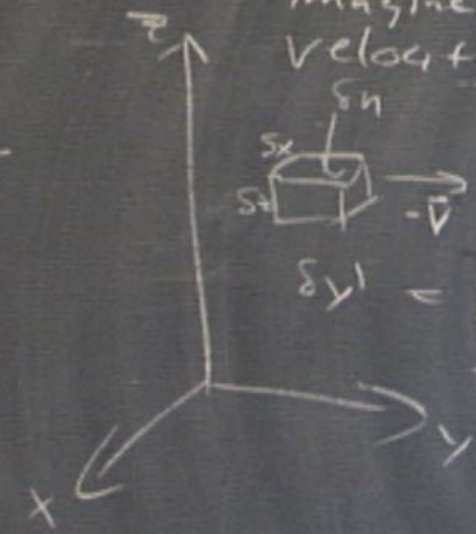

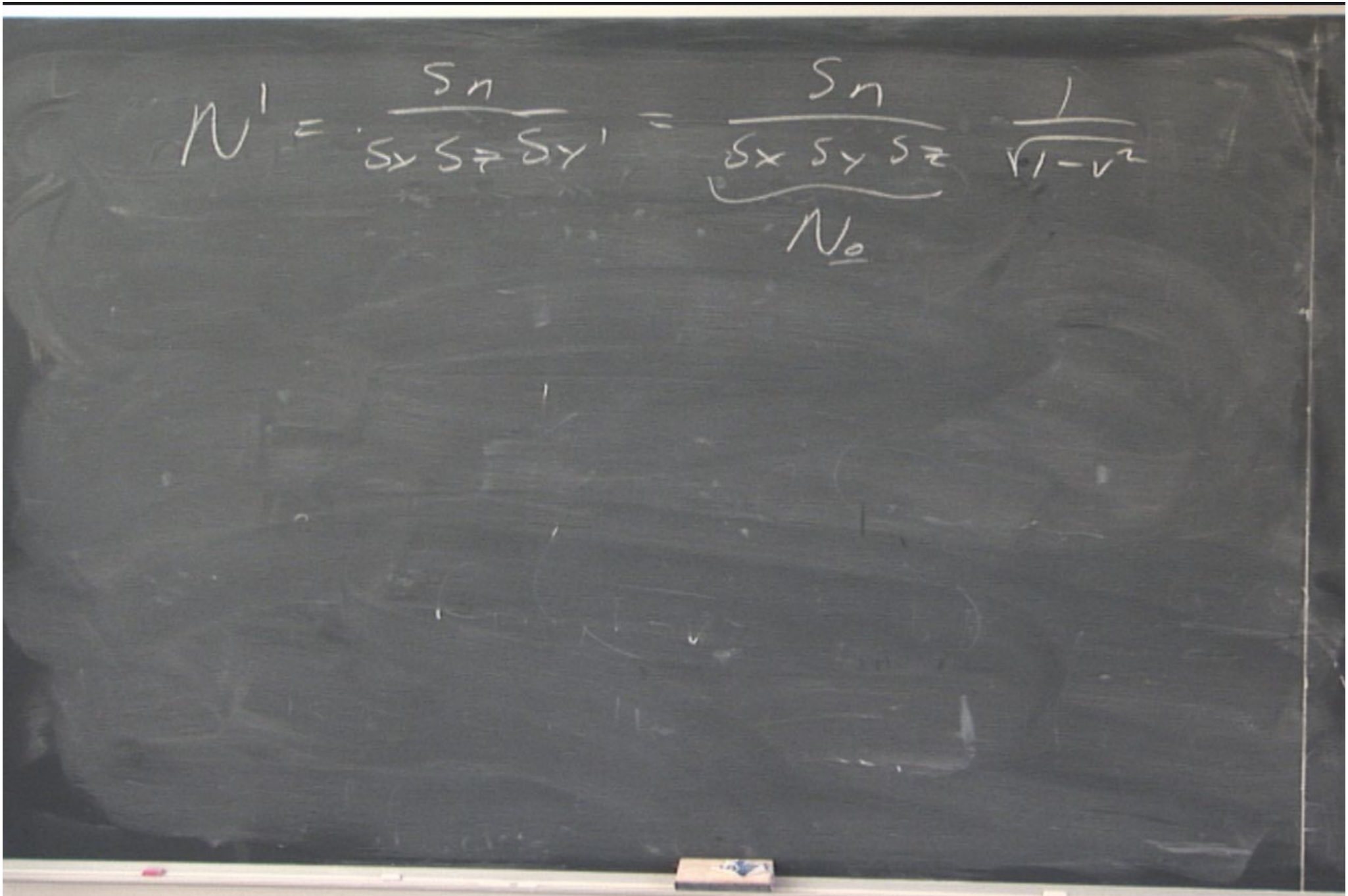
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$$\frac{1}{1-v^2} \longrightarrow$$

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$$\frac{1}{1-v^2} \longrightarrow b = 0$$

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$$\frac{1}{1-v^2} \longrightarrow \underline{b=0 \quad a=1}$$

$$T_{\alpha\beta} = \rho \, u_\alpha u_\beta \cdots$$

$$N' = \frac{\delta n}{\delta x' \delta z' \delta y'} = \frac{\delta n}{\underbrace{\delta x \delta y \delta z}_{N_0}} \frac{1}{\sqrt{1-v^2}}$$

$$\rho' = \frac{m}{\sqrt{1-v^2}} N' = \frac{m N_0}{1-v^2} = \frac{\rho}{1-v^2}$$

So new source should increase by precisely

$$\frac{1}{1-v^2} \longrightarrow \underline{b=0} \quad \underline{q=1}$$

$$T_{\alpha\beta} = \rho u_\alpha u_\beta \cdots$$

$$N' = \frac{\delta n}{\delta x \, \delta z \, \delta y'} = \underbrace{\frac{\delta n}{\delta x \, \delta y \, \delta z}}_{N_0} \frac{1}{\sqrt{1-v^2}}$$

$$\rho' = \frac{m}{\sqrt{1-v^2}} N' = \frac{m N_0}{1-v^2} = \frac{\rho}{1-v^2}$$

So new source should increase by precisely

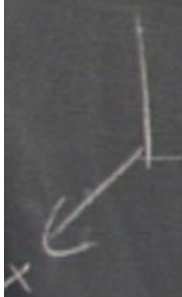
$$\frac{1}{1-v^2} \longrightarrow \underline{b=0} \quad \underline{a \neq 1}$$

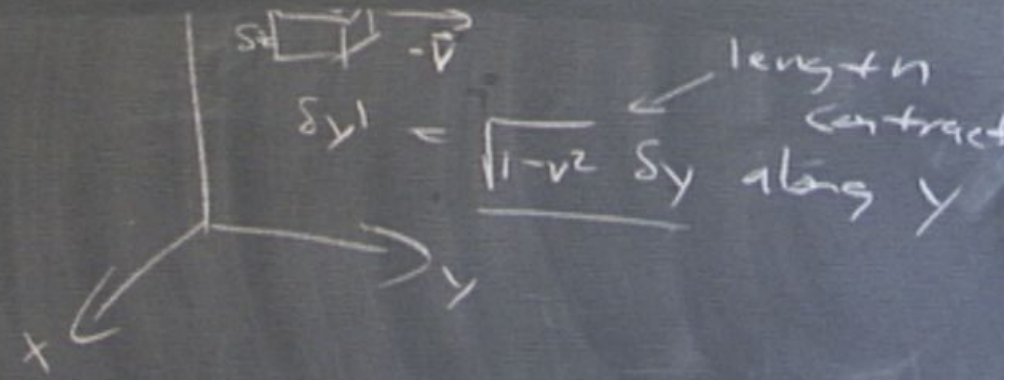
$$\boxed{T_{\alpha\beta} = \rho \, u_\alpha u_\beta}$$

" a miracle occurs " fixing

$\lambda = -1/2$

$$R_{\alpha\beta} - \tfrac{1}{2} g_{\alpha\beta} R = \kappa T_{\alpha\beta}$$





"a miracle occurs" fixing

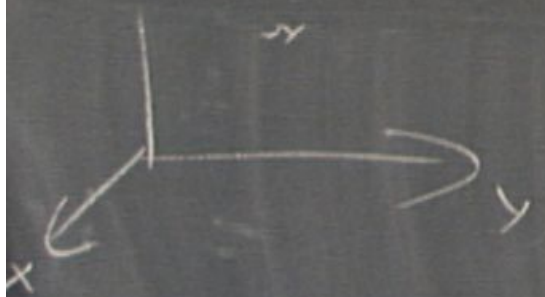
$\lambda = -1/2$

$$R_{\alpha\beta} - \tfrac{1}{2} g_{\alpha\beta} R = \kappa T_{\alpha\beta}$$

project with $u^\alpha u^\beta$

$$\longrightarrow \quad \nabla^2 \Phi = (\quad) \rho$$

$4\pi G$



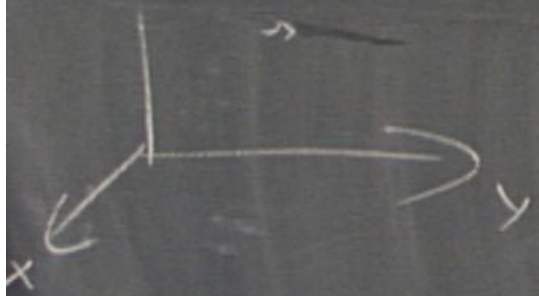
"a miracle occurs" fixing

$\lambda = -1/2$

$$R_{\alpha\beta} - \tfrac{1}{2} g_{\alpha\beta} R = \kappa T_{\alpha\beta}$$

project with $u^\alpha u^\beta$ + substitute

$$\longrightarrow \nabla^2 \Phi = (\quad) \rho$$

improved Newtonian metric

$4\pi G$