Title: Relativity - Core (PHYS 604) - Lecture 13

Date: Sep 21, 2009 10:30 AM

URL: http://pirsa.org/09090072

Abstract:

Einstein's eqs

What replaces Newton's eq for the gravitational field?

$$\nabla^2 \underline{\Phi} = 4\pi G\rho$$

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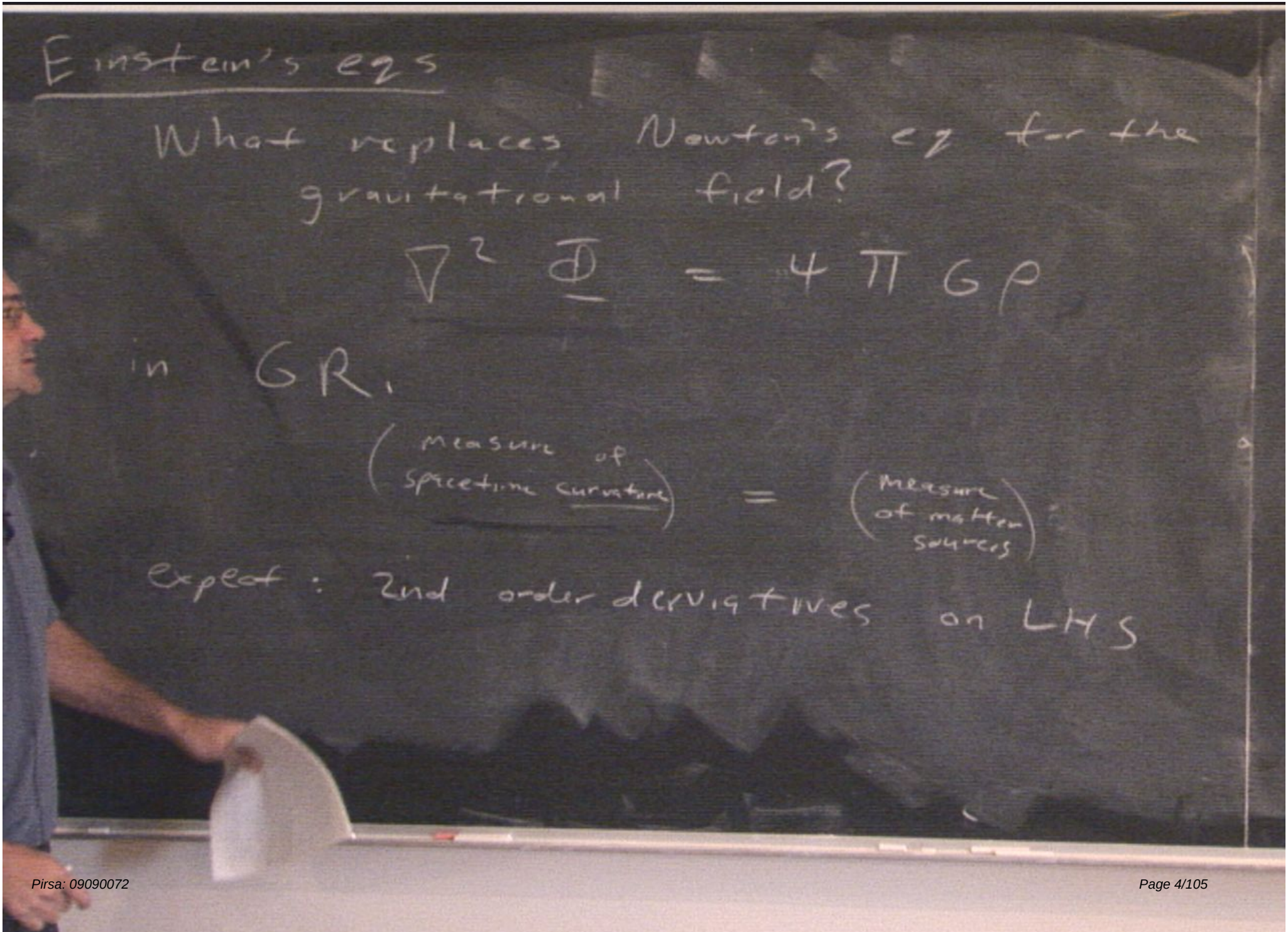
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(Measure of spacetime curvature) = (Measure of matter sources)
Except: 2nd order derivatives on LHS

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Expect: 2nd order derivatives on LHS both time and space derv's

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Except: 2nd order derivatives on LHS both time and space deriv's

$\Phi \rightarrow g_{\alpha\beta}$ so need a tensor's worth of eq's

→ also need tensor of matter sources

begin with LHS –

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begin with LHS – think about exp's which might be sensitive to curvature

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recall equivalence principle

→ also need tensor of matter sources

begin with LHS – think about exp's which might be sensitive to curvature

recall equivalence principle – physics same in inertial frames or freely falling frames in a uniform gravitational field

in math. terms:

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$$g_{\alpha\beta} = \eta_{\alpha\beta} \qquad g_{\alpha\beta,\lambda} = 0$$

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⟶ suggests we consider slightly larger exp's where non-uniformity of grav. field becomes evident

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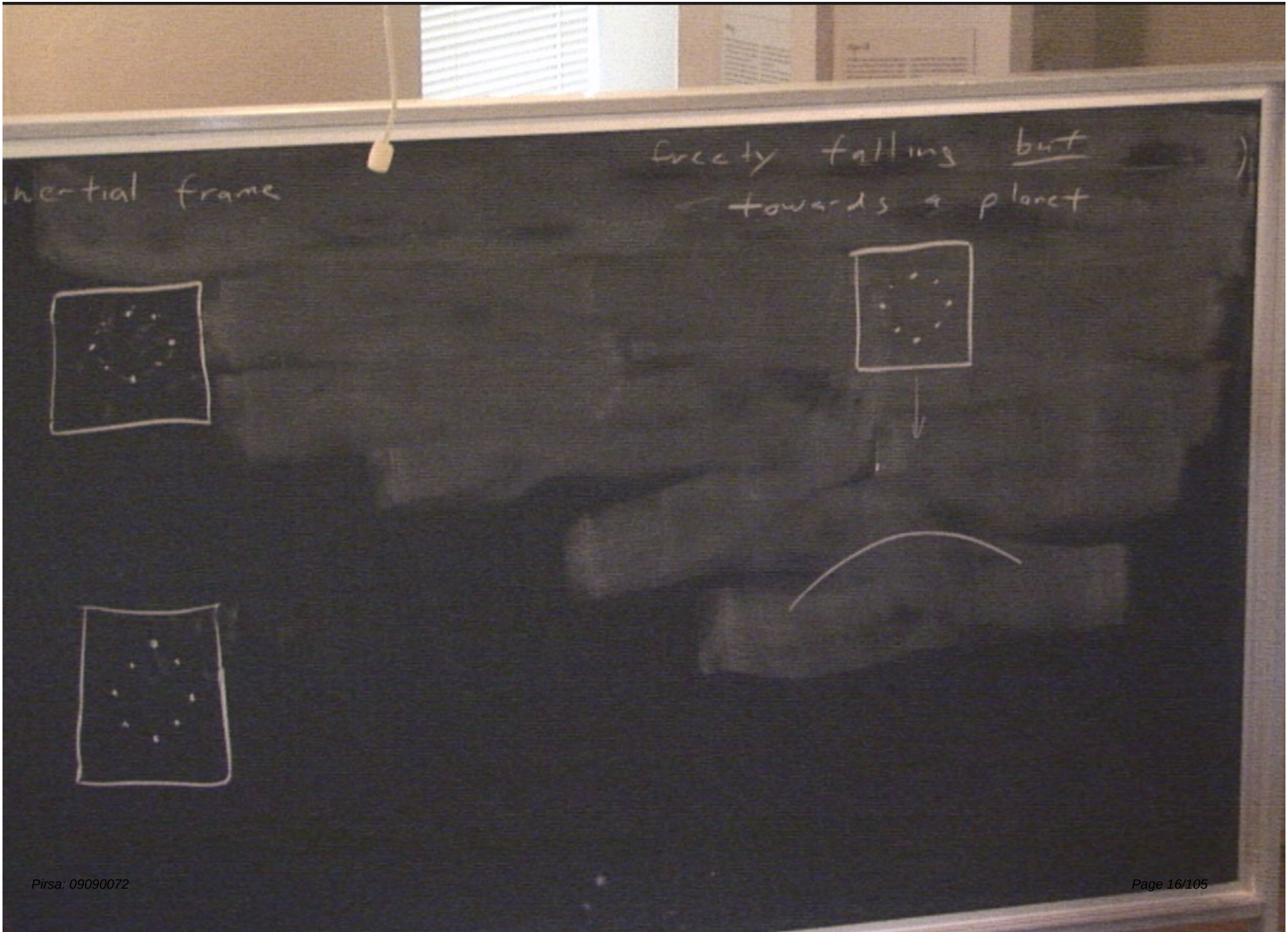
ertial frame

freely falling but
towards a planet


inertial frame
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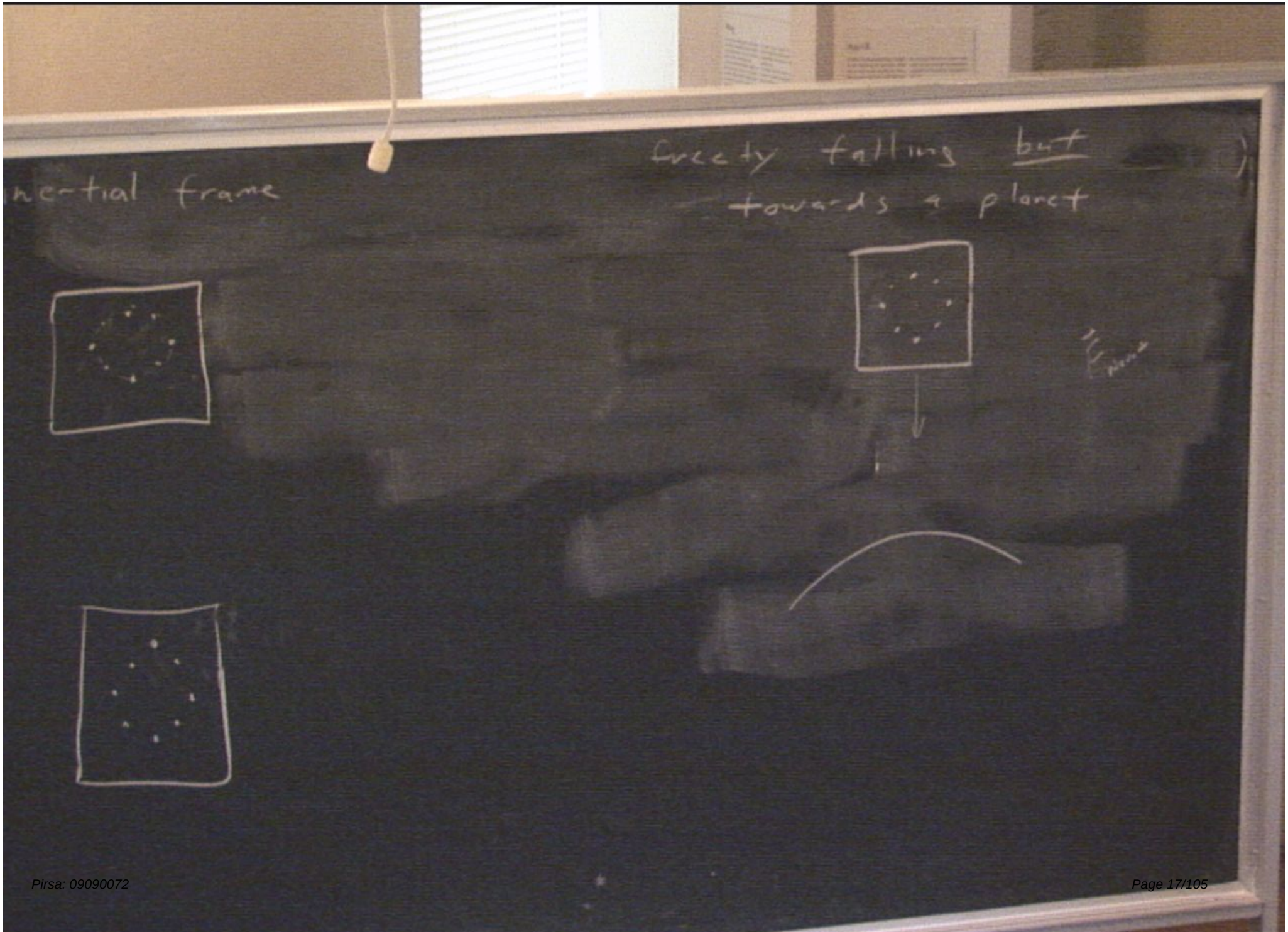
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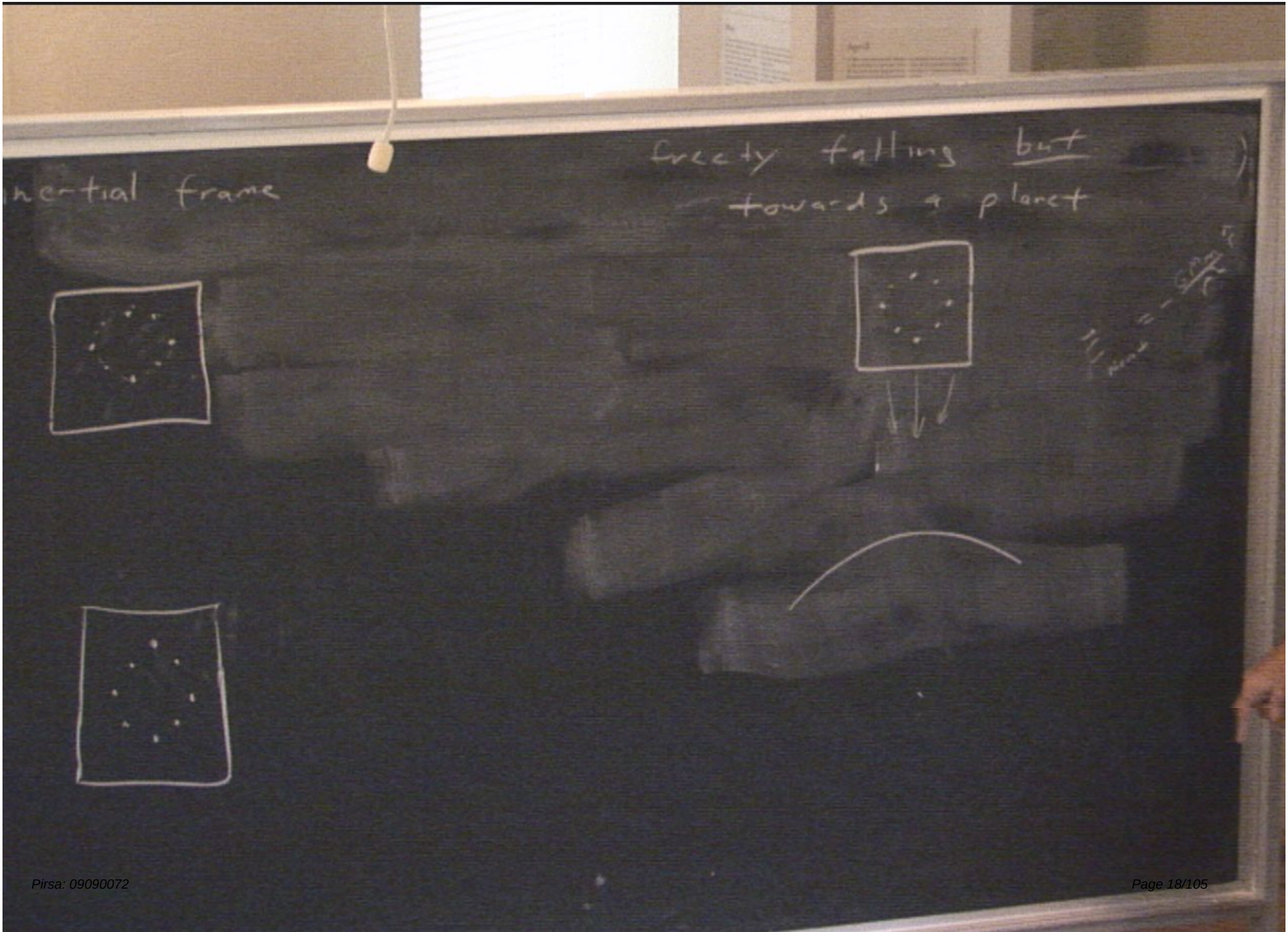
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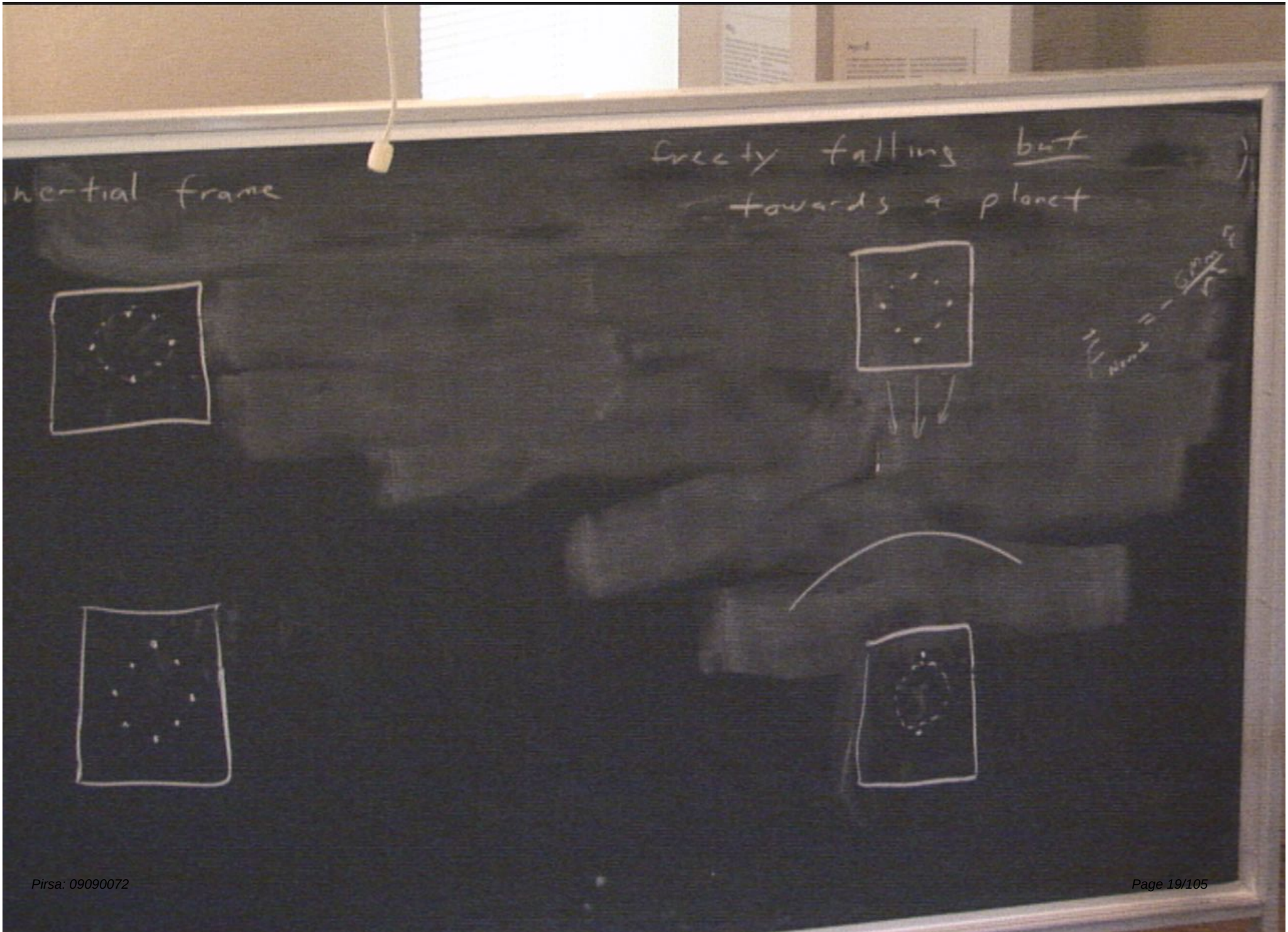
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→ suggests we consider slightly large- exp's where non-uniformity of g av. field becomes evident

→ in freely frame the circular pattern is disorted — {closer / farther} masses experience {more / less} g

pulls test masses towards center of lab

→ suggests comparing trajectories of nearby test particles

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compare lets consider Newtonian problem

llection

pulls test masses towards center of lab

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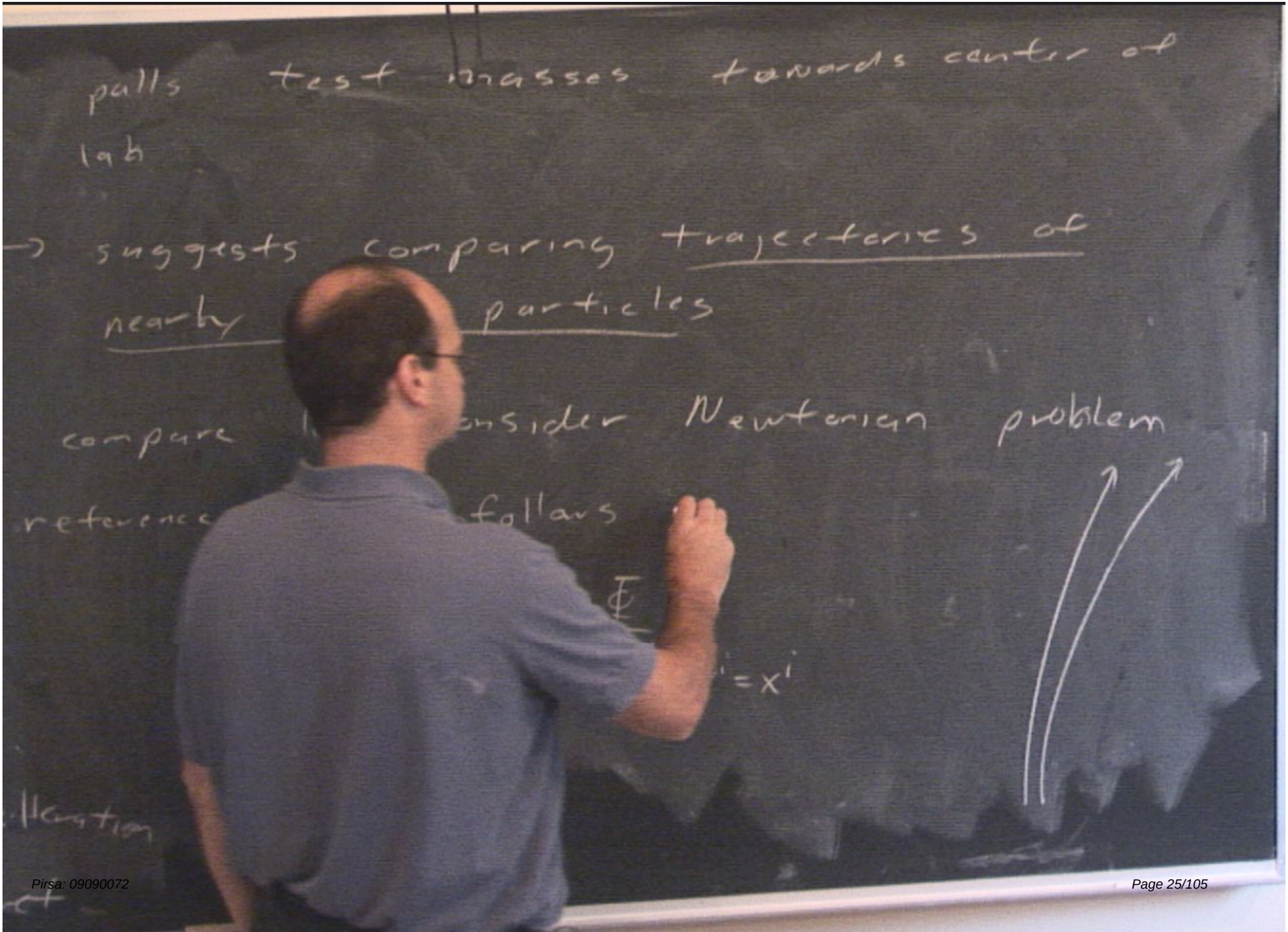
compare lets consider Newtonian problem


falls test masses towards center of
lab
→ suggests comparing trajectories of
nearby particles
compare
reference
$= x^i$

pulls test masses towards center of lab

→ suggests comparing trajectories of nearby test particles

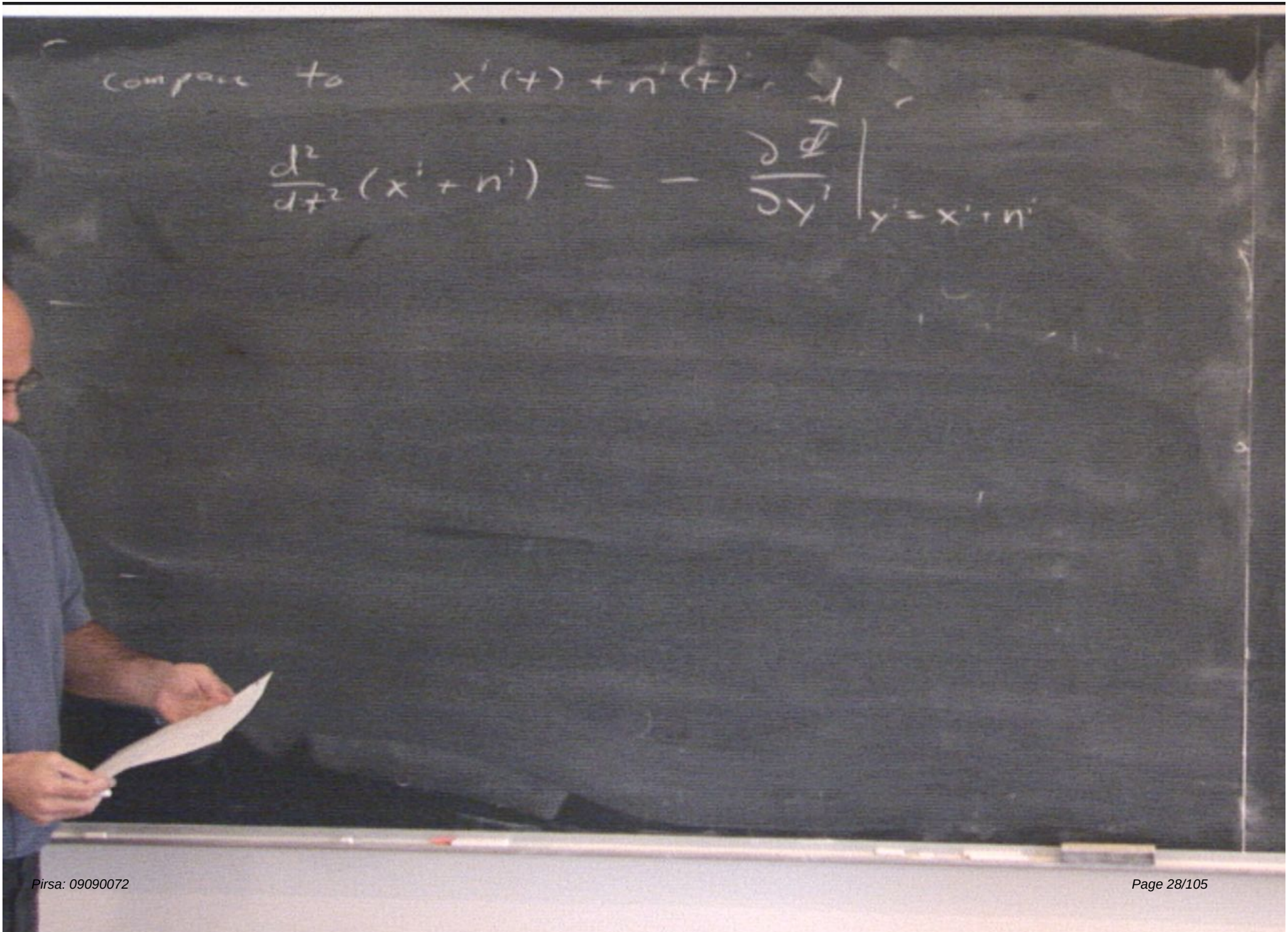
compare lets consider Newtonian problem

reference particle follows $x^i(t)$ set by

$$\frac{d^2x^i}{dt^2} = -\left.\frac{\partial \Phi}{\partial y^i}\right|_{y^i = x^i}$$

llection

compare to $x'(t) + n'(t) = 1$;


compare to $x^i(t) + n^i(t)$.
$$\frac{d^2}{dt^2}(x^i + n^i) = -\left.\frac{\partial \mathcal{V}}{\partial y^i}\right|_{y^i = x^i + n^i}$$

compare to $x^i(t) + n^i(t)$

$$\frac{d^2}{dt^2}(x^i + n^i) = -\left.\frac{\partial \bar{f}}{\partial y^i}\right|_{y^i = x^i + n^i}$$

subtract

$$\frac{d^2}{dt^2} n^i = -\left.\frac{\partial \bar{f}}{\partial y^i}\right|_{y^i = x^i + n^i} + \left.\frac{\partial f}{\partial y^i}\right|_{y^i = n^i}$$

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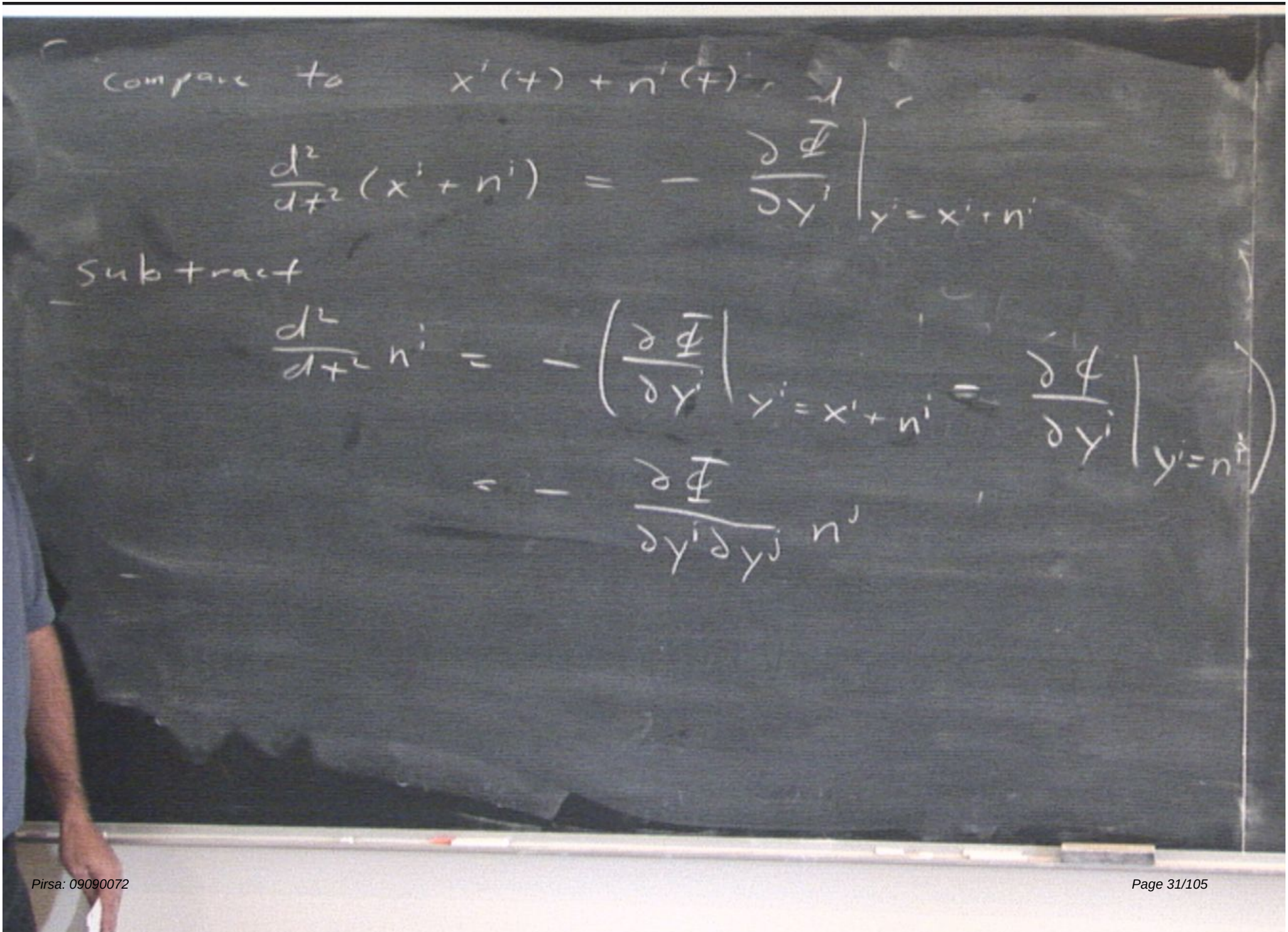
Subtract

$$\frac{d^2}{dt^2} n^i = -\left(\left.\frac{\partial \bar{\Phi}}{\partial y^i}\right|_{y^i = x^i + n^i} - \left.\frac{\partial \Phi}{\partial y^i}\right|_{y^i = n^i}\right)$$


compare to $x^i(t) + n^i(t)$
$\frac{d^2}{dt^2}(x^i + n^i) = -\frac{\partial \mathcal{F}}{\partial y^i}\Big|_{y^i = x^i + n^i}$
Subtract
$\frac{d^2}{dt^2} n^i = -\left(\frac{\partial \mathcal{F}}{\partial y^i}\Big|_{y^i = x^i + n^i} = \frac{\partial \mathcal{F}}{\partial y^i}\Big|_{y^i = n^i}\right)$
$= -\frac{\partial \mathcal{F}}{\partial y^i \partial y^j} n^j$

compare to $x^i(t) + n^i(t)$

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$$= -\frac{\partial \Phi}{\partial y^i \partial y^j} n^j + O(n^2)$$

$\frac{\partial \Phi}{\partial y^i \partial y^j}$ gives a measure of the forces that pull two masses closer or further apart

compare to $x^i(t) + n^i(t)$:

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$\longrightarrow$ "tidal forces"

Notice precisely this matrix enters Newton's grav. eq:

$$\nabla^2 \bar{\mathcal{E}} = 4\pi G \rho$$

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$$\nabla^2 \bar{\Phi} = 4\pi G \rho$$

$$\sum_i \frac{\partial \bar{\Phi}}{\partial x^i \partial x^i} \leftarrow \text{trace of previous matrix}$$

Notice precisely this matrix enters Newton's grav eq.

$$\nabla^2 \Phi = 4\pi G \rho$$

$$\sum_i \frac{\partial^2 \Phi}{\partial x^i \partial x^i} \leftarrow \text{trace of previous matrix}$$

eg, in the absence of matter (or in vacuum), the matrix is trace-free

- repeat for curved spacetime and nearby geodesics $\Longrightarrow$ "equation of geodesic deviation"

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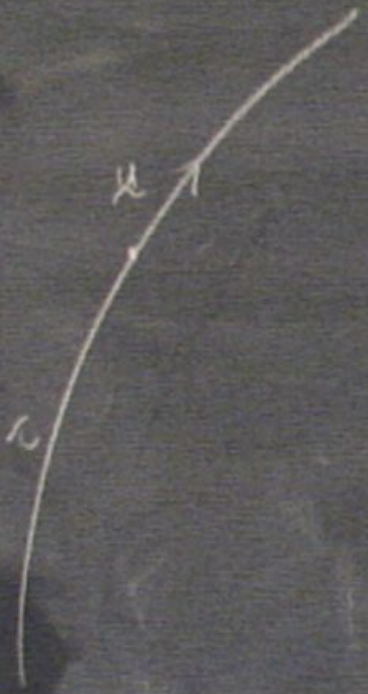
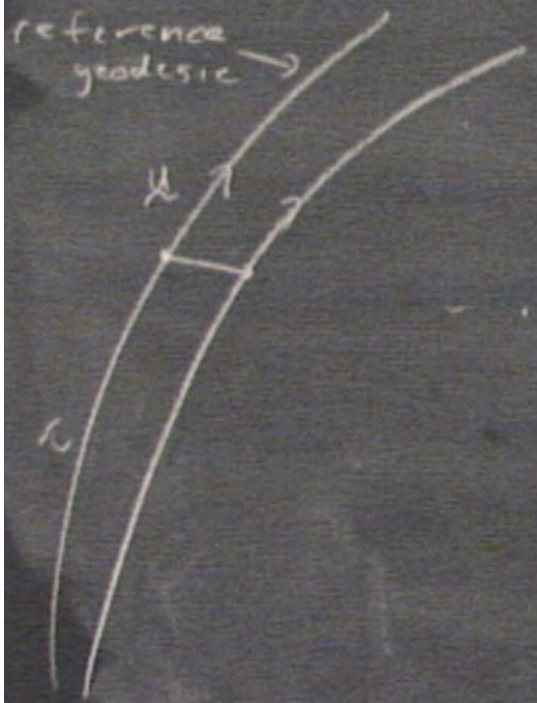


- repeat for curved spacetime and nearby geodesics ⟹ "equation of geodesic deviation"



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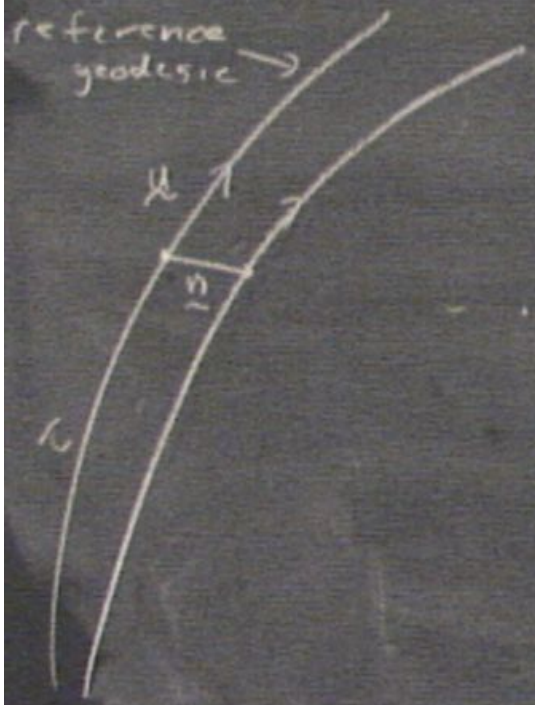

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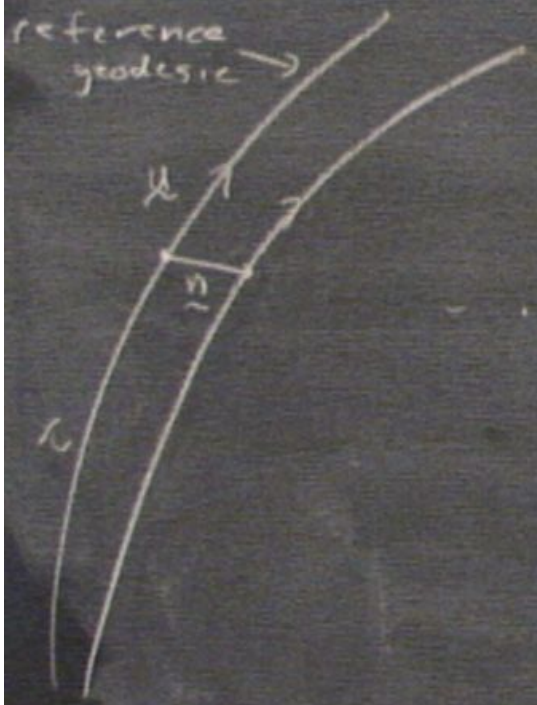

- separation is given by 4-vector – how do we chose n^α?

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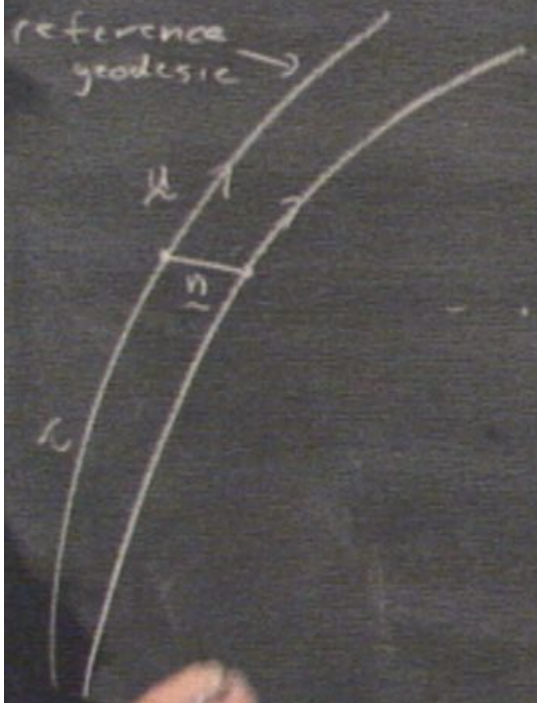

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→ options.

$\xi \cdot \eta = 0$

$\eta^t = 0$ in our favorite coordinate system

→

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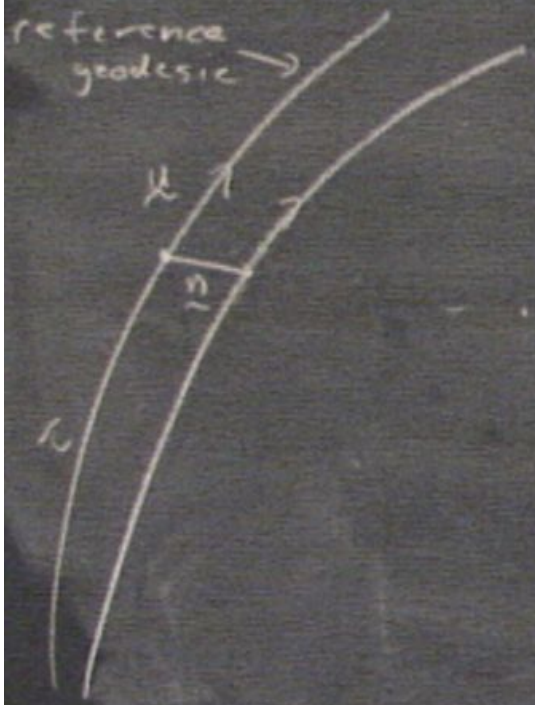

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→ options: $\Delta \cdot u = 0$

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→ details don't matter – so we make some choice

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- Some ambiguity in setting up basis along entire geodesic

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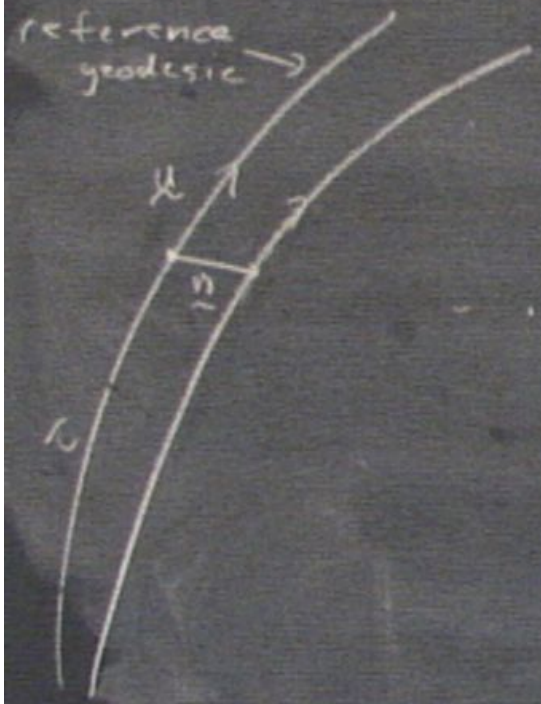
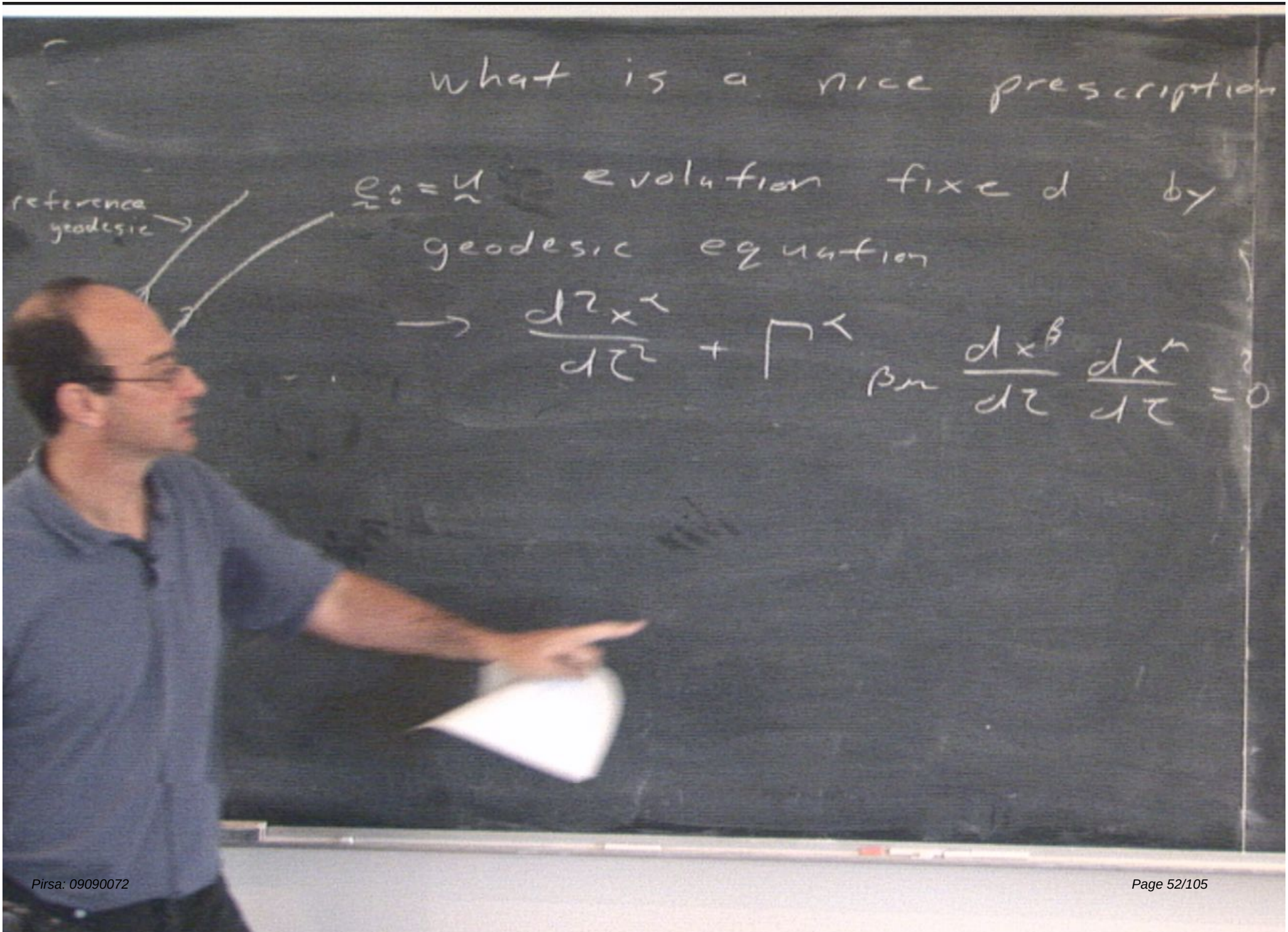
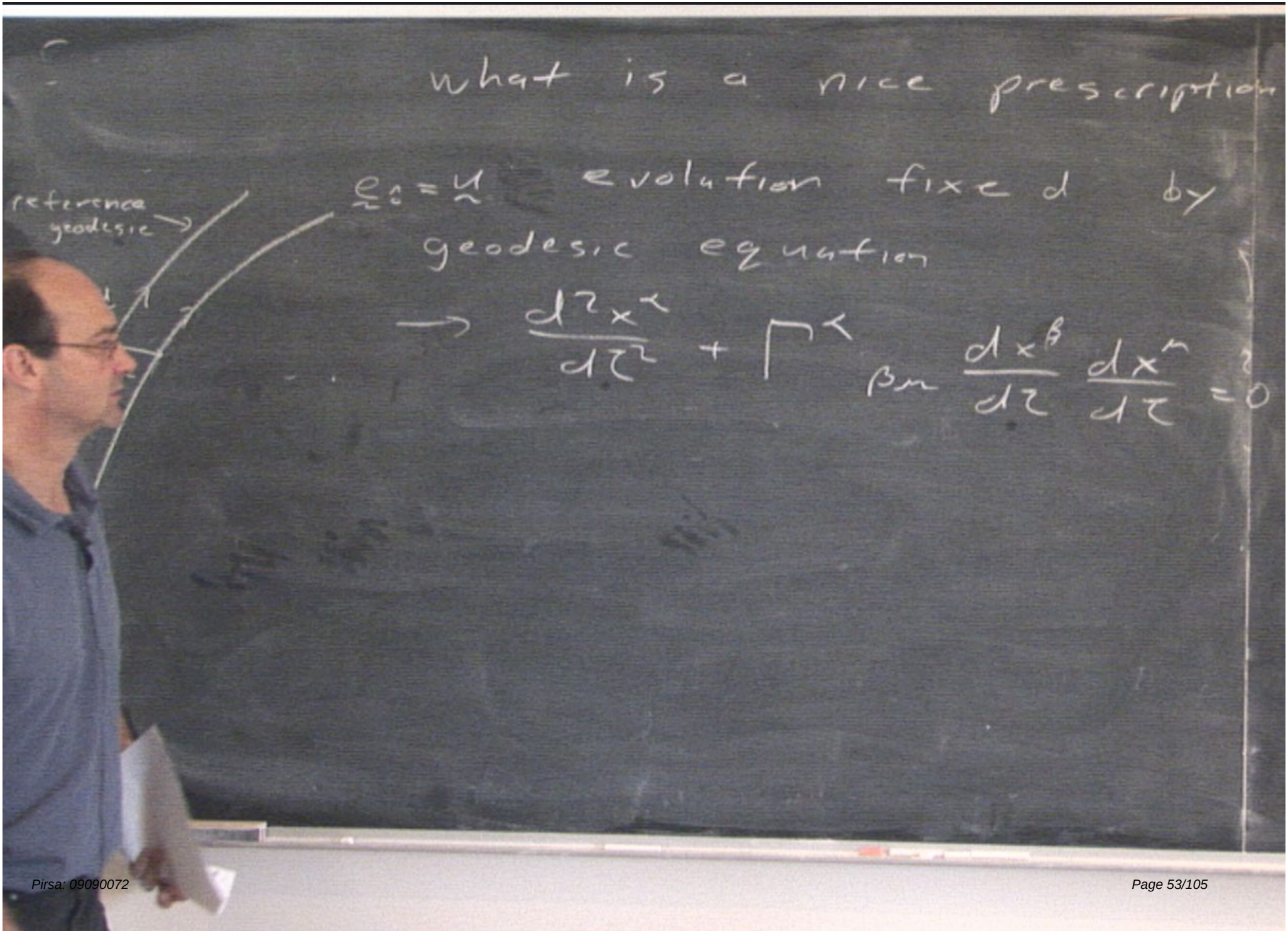



what is a nice prescription
reference geodesic
$\xi := u$ evolution fixed by
geodesic equation
$\rightarrow \frac{d^2 x^\alpha}{d\tau^2} + \Gamma^\alpha_{\beta\gamma} \frac{dx^\beta}{d\tau} \frac{dx^\gamma}{d\tau} = 0$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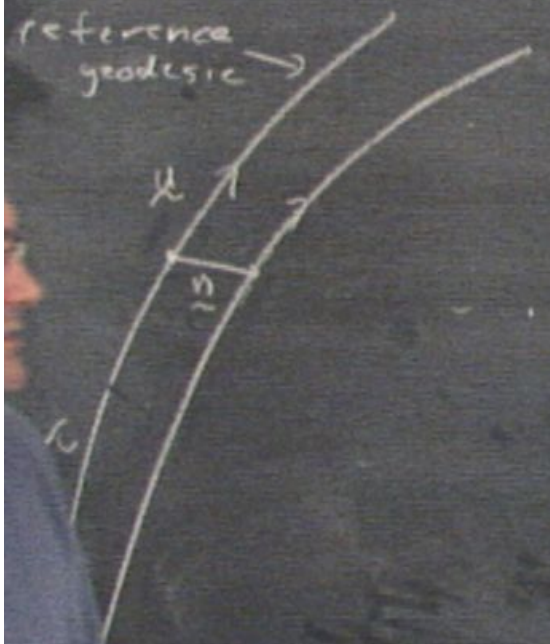

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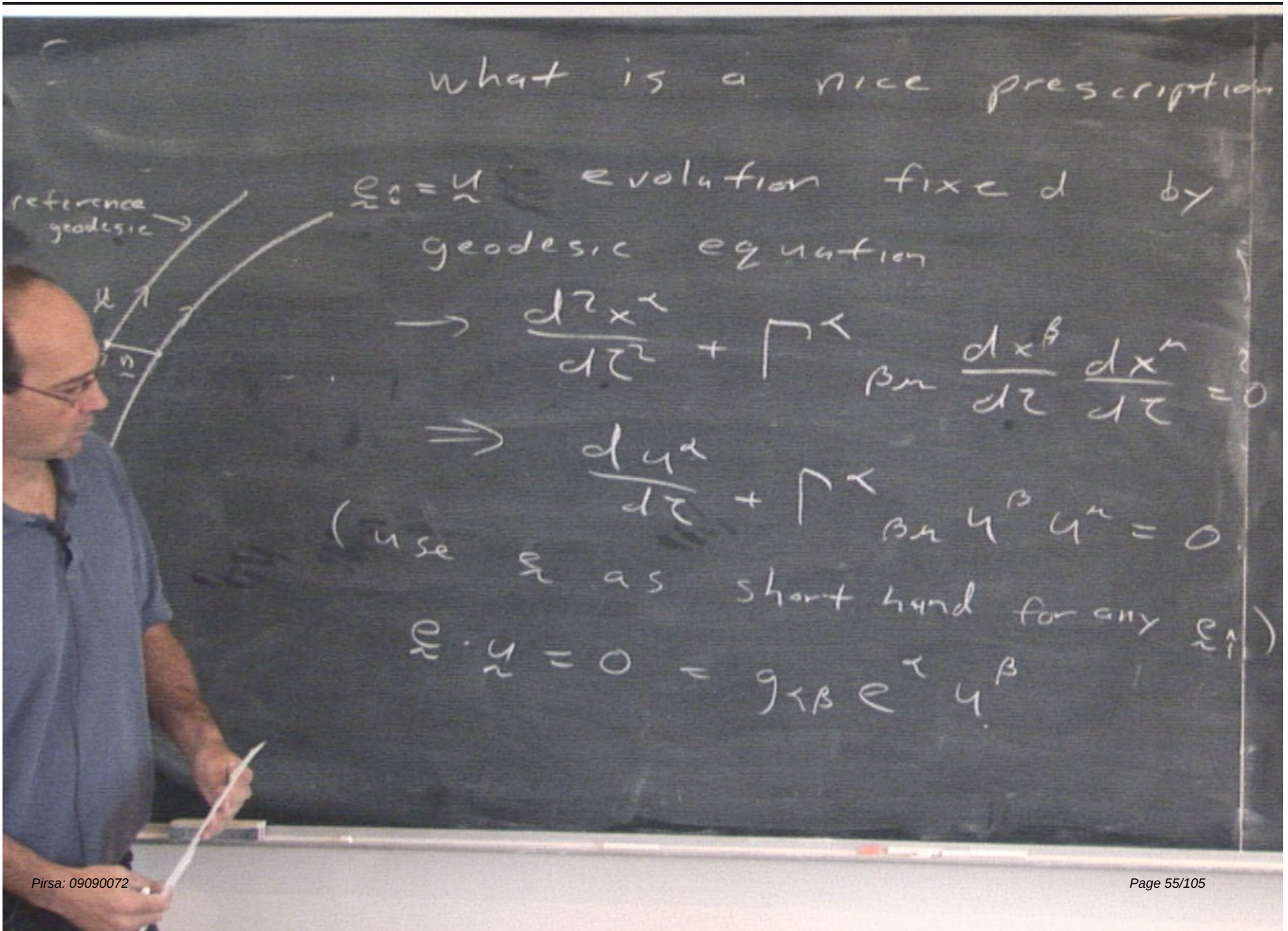
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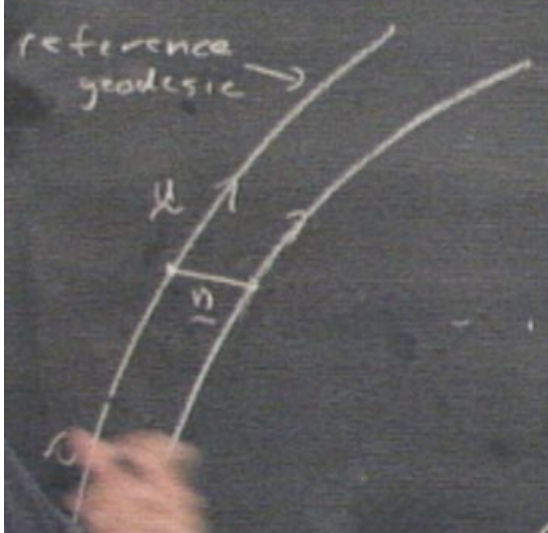
(use ξ as short hand for any $\xi \uparrow$)
$\xi \cdot u = 0 = g_{\tau\beta} \xi^\tau u^\beta$

what is a nice prescription



$\underset{\sim}{\xi}(\tau) = \underset{\sim}{\eta}$ evolution fixed by geodesic equation

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$$\Rightarrow \frac{d\eta^\alpha}{d\tau} + \Gamma^\alpha_{\beta\gamma} \eta^\beta \eta^\gamma = 0$$

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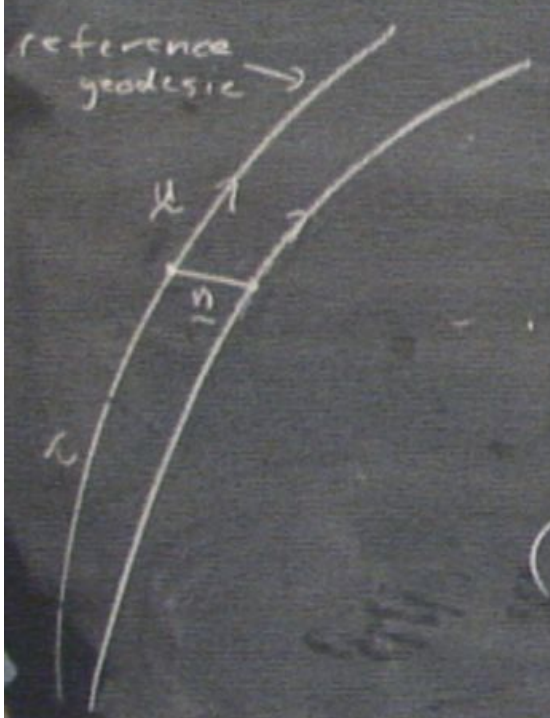

$\underset{\sim}{e}_0 = \underset{\sim}{u}$ evolution fixed by geodesic equation

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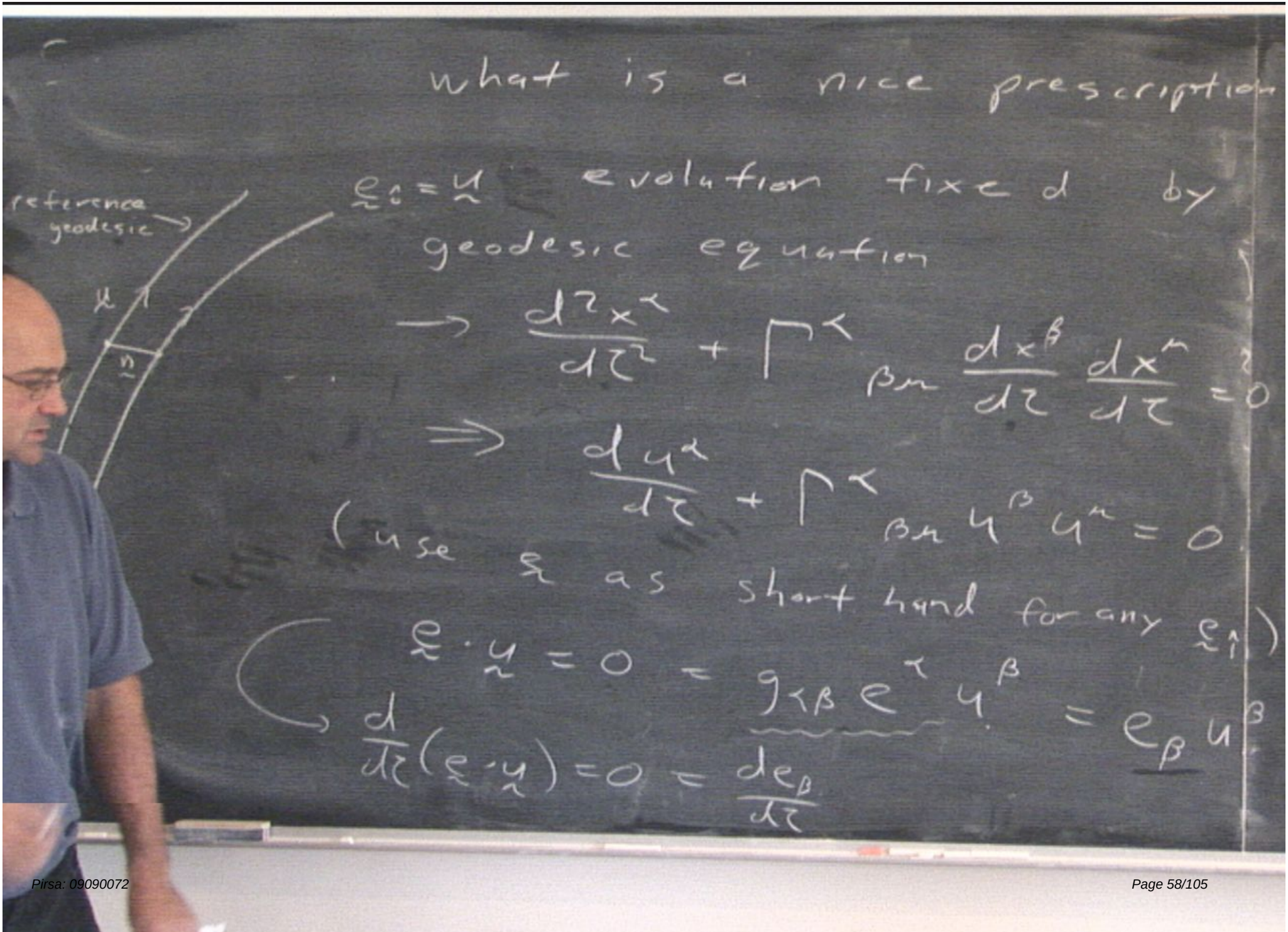
$$\Rightarrow \frac{du^\alpha}{d\tau} + \Gamma^\alpha{}_{\beta\gamma} u^\beta u^\gamma = 0$$

(use $\underset{\sim}{e}$ as shorthand for any $\underset{\sim}{e}_\mu$)

$$\underset{\sim}{e} \cdot \underset{\sim}{u} = 0 = g_{\alpha\beta} e^\alpha u^\beta = e_\beta u^\beta$$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$\Rightarrow \frac{du^\alpha}{d\tau} + \Gamma^\alpha{}_{\beta\mu} u^\beta u^\mu = 0$
(use $\underset{\sim}{e}$ as short hand for any $\underset{\sim}{e}_{\hat{1}}$)
$\underset{\sim}{e} \cdot \underset{\sim}{u} = 0 = g_{\alpha\beta} e^\alpha u^\beta = e_\beta u^\beta$
$\frac{d}{d\tau}(\underset{\sim}{e} \cdot \underset{\sim}{u}) = 0 = \frac{de_\beta}{d\tau}$

$$u^\beta \left(\frac{de_\beta}{d\tau} - \Gamma^\alpha{}_{\beta\mu} e_\alpha u^\mu \right) = 0$$

$$u^\beta \left(\frac{de_\beta}{d\tau} - \Gamma^\alpha{}_{\beta\mu} e_\alpha u^\mu \right) = 0$$

$= 0$ is a simple choice to evolve the basis vectors forward

(5.1)

$$u^\beta \left(\underbrace{\frac{de_\beta}{d\tau} - \Gamma^\alpha{}_{\beta\mu} e_\alpha u^\mu}_{=0} \right) = 0$$

is a simple choice to evolve the basis vectors forward

propose: $\frac{d(e_\alpha)_\beta}{d\tau}$

$$u^\beta \left(\frac{de_\beta}{d\tau} - \Gamma^\alpha{}_{\beta\mu} e_\alpha u^\mu \right) = 0$$

$= 0$ is a simple choice to evolve the basis vectors forward

propose: $\frac{d(e_a)_\beta}{d\tau} = \Gamma^\alpha{}_{\beta\mu} (e_a)_\alpha u^\mu$

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propose: $\frac{d(e_a)_\beta}{d\tau} = \Gamma^\alpha_{\beta\mu} (e_a)_\alpha u^\mu$

→ can check that applies for $u_\alpha = (e_a)_\alpha$

→ can check

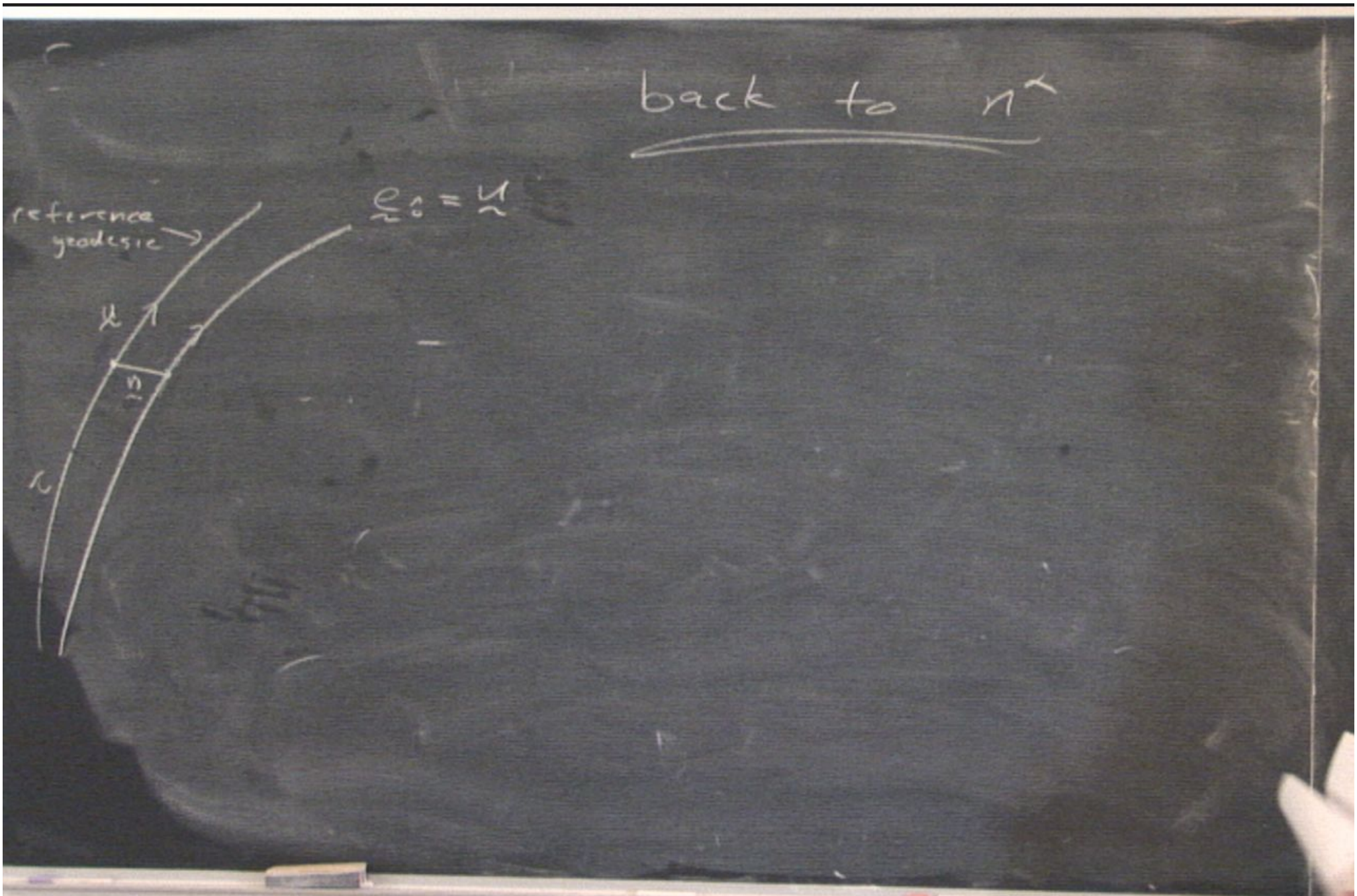
$$u^\beta \left(\frac{de_\beta}{d\tau} - \Gamma^\alpha_{\beta\mu} e_\alpha u^\mu \right) = 0$$

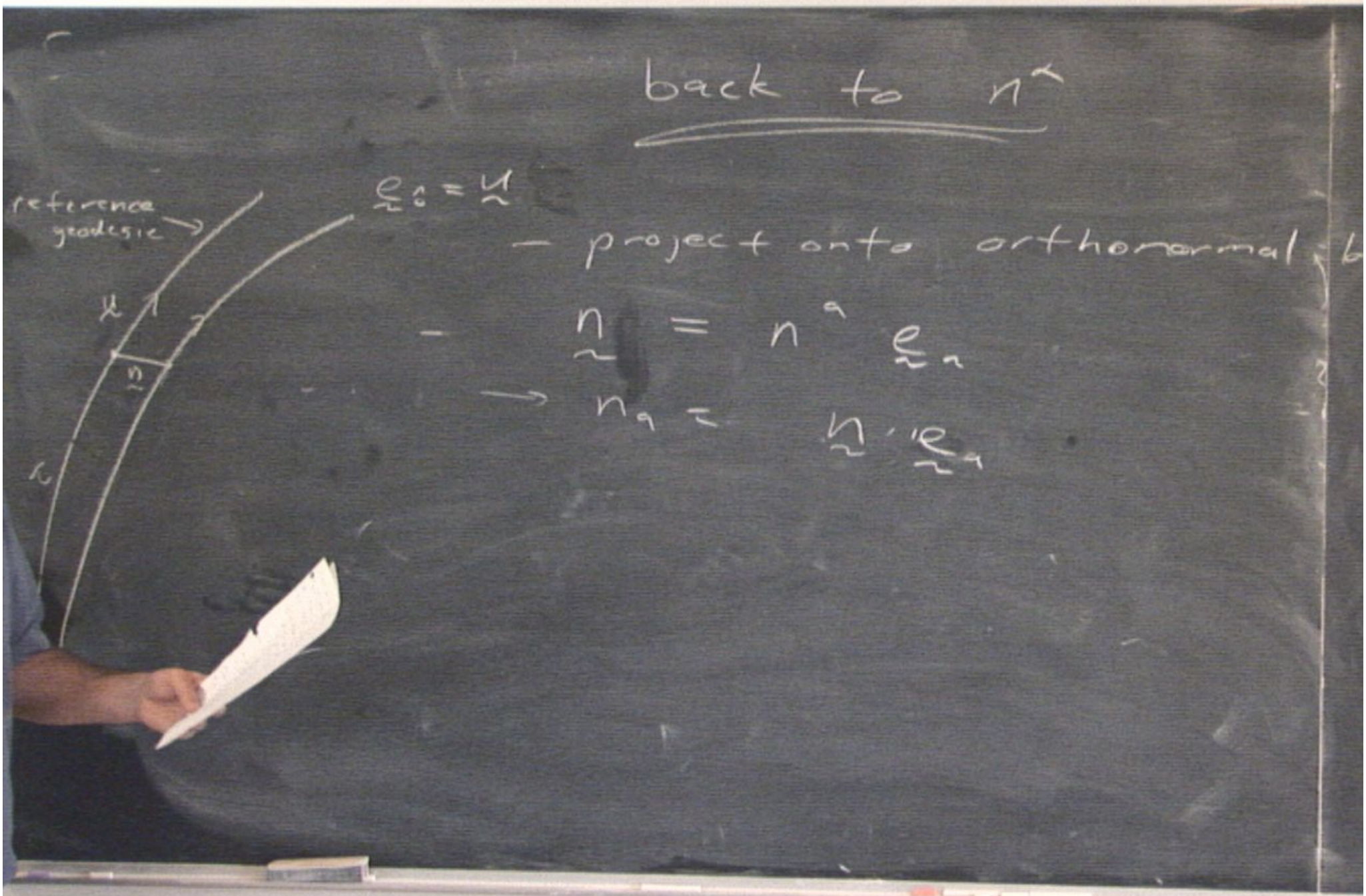
$= 0$ is a simple choice to evolve the basis vectors forward

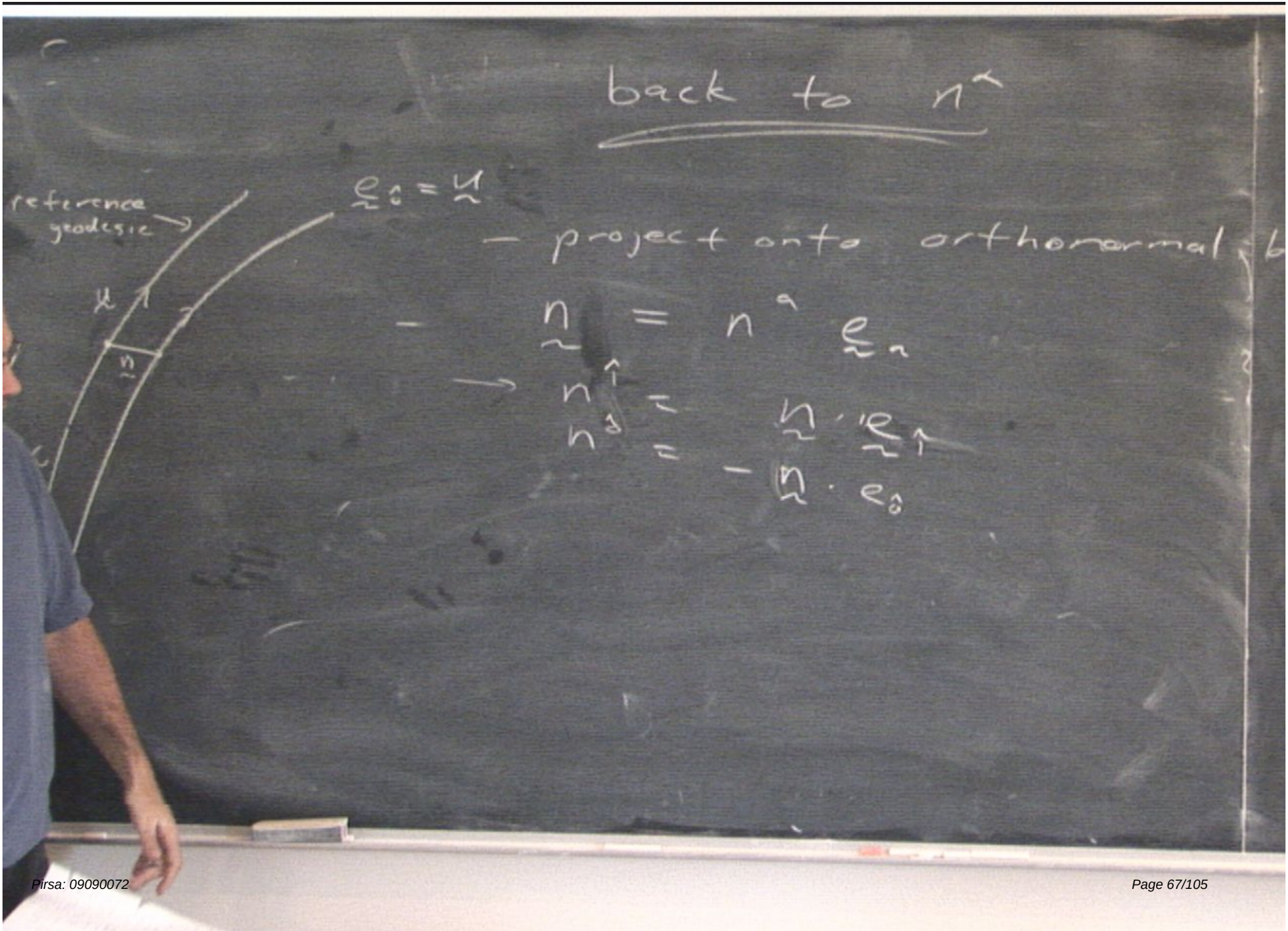
propose: $\frac{d(e_a)_\beta}{d\tau} = \Gamma^\alpha_{\beta\mu} (e_a)_\alpha u^\mu$

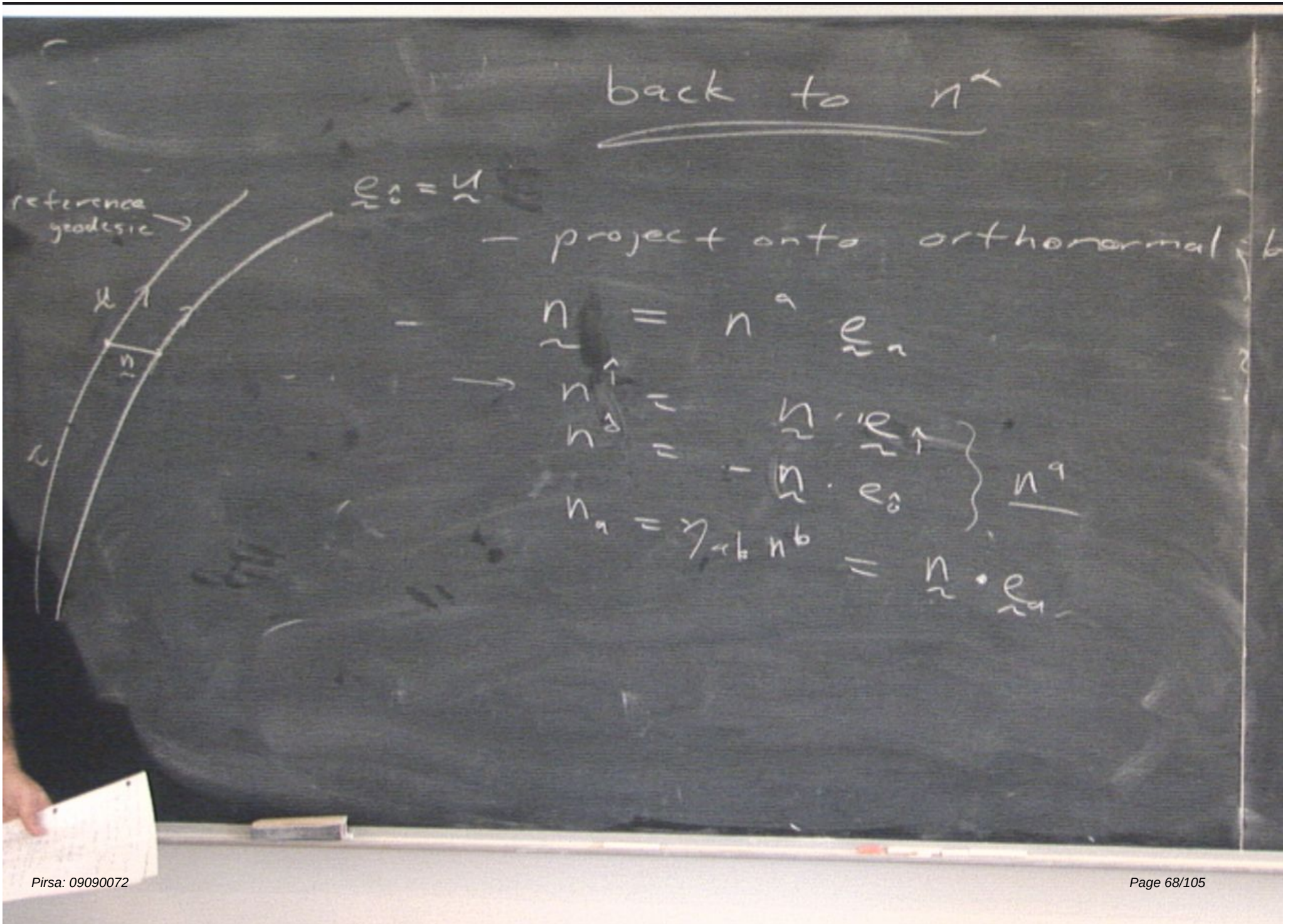
→ can check that applies for $u_\alpha = (e_0)_\alpha$

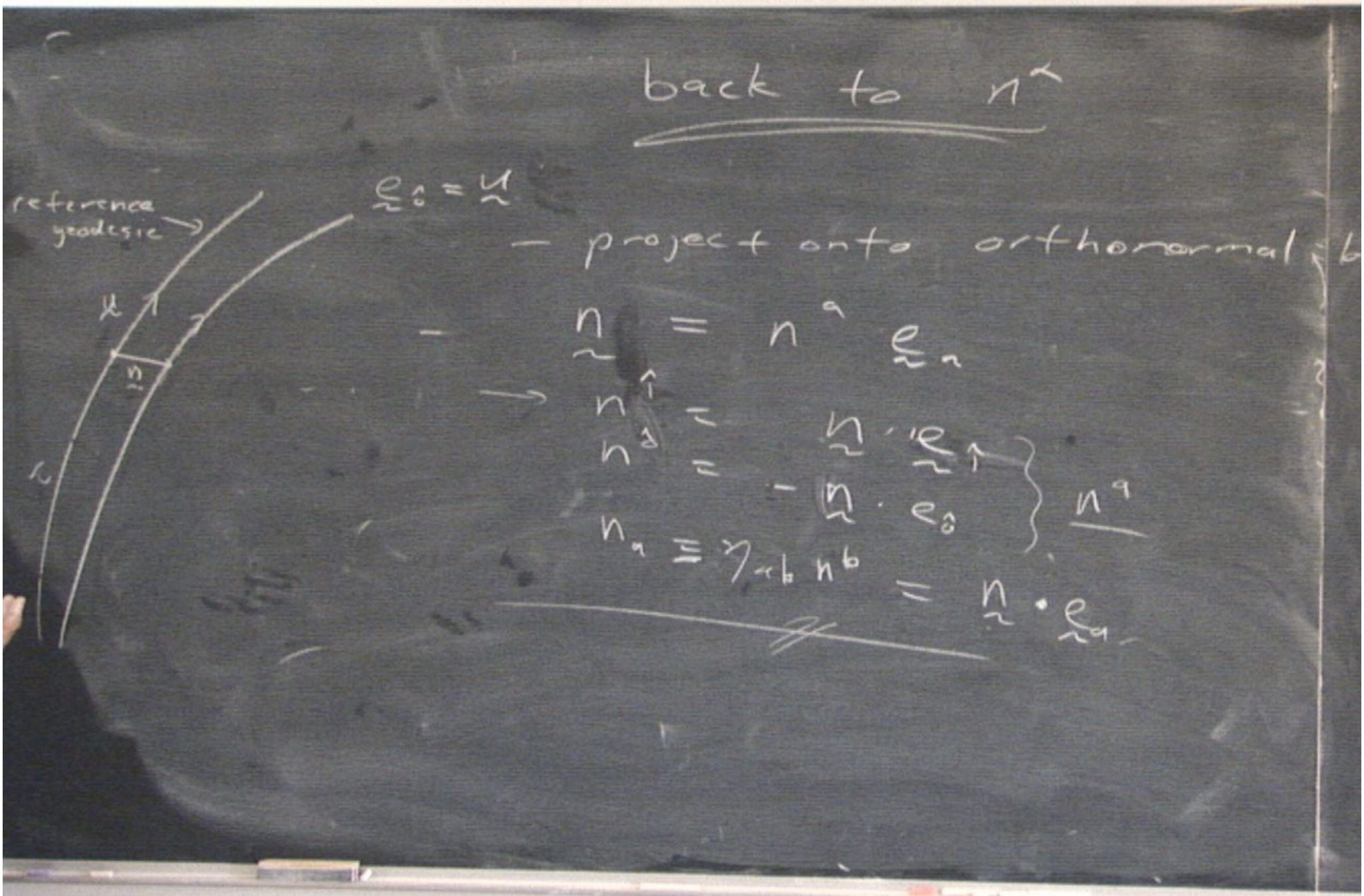
→ can check that compatible with $e_i \cdot e_j = \delta_{ij}$

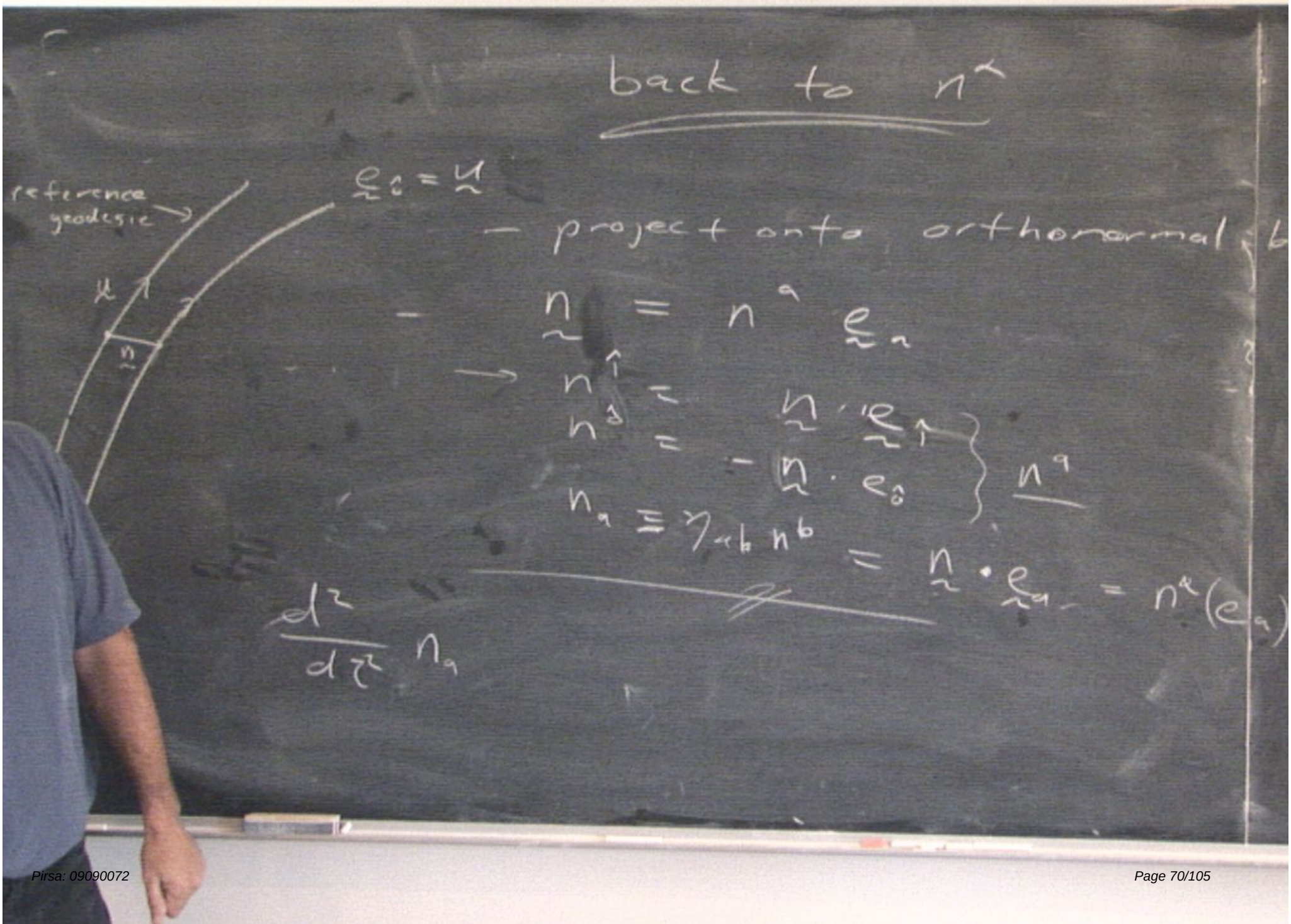

back to n^α
reference geodesic
$\underset{\sim}{e}_0 = \underset{\sim}{u}$
$\underset{\sim}{u}$
$\underset{\sim}{n}$
$\underset{\sim}{n}$

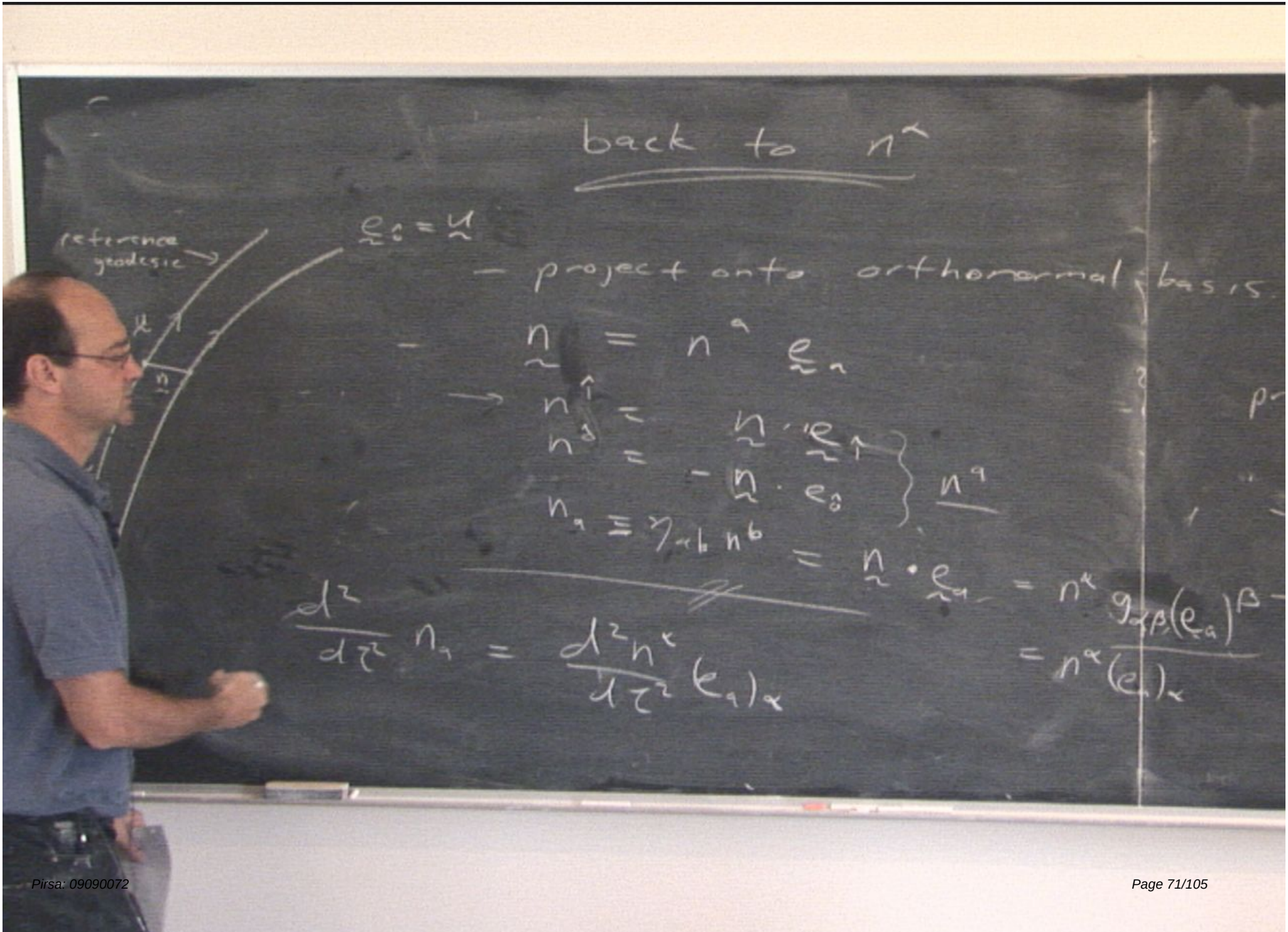

back to n^α
reference geodesic
$\underset{\sim}{e}_0 = \underset{\sim}{u}$
– project onto orthonormal b
– $\underset{\sim}{n} = n^a \underset{\sim}{e}_a$
$\rightarrow n_a = \underset{\sim}{n} \cdot \underset{\sim}{e}_a$


back to n̂
reference geodesic
$\underset{\sim}{e}_0 = \underset{\sim}{u}$
– project onto orthonormal
$\underset{\sim}{n} = n^{\hat{a}}\,\underset{\sim}{e}_{\hat{a}}$
$n^{\hat{1}} = \underset{\sim}{n}\cdot\underset{\sim}{e}_{\hat{1}}$
$n^{\hat{0}} = -\underset{\sim}{n}\cdot\underset{\sim}{e}_0$


back to $n^\hat{a}$
reference geodesic
$\underset{\sim}{e}_{\hat{0}} = \underset{\sim}{u}$
– project onto orthonormal
$\underset{\sim}{n} = n^{\hat{a}}\,\underset{\sim}{e}_{\hat{a}}$
$n^{\hat{i}} = \underset{\sim}{n}\cdot\underset{\sim}{e}_{\hat{i}}$
$n^{\hat{0}} = -\underset{\sim}{n}\cdot\underset{\sim}{e}_{\hat{0}}$
$n_{\hat{a}} = \eta_{\hat{a}\hat{b}}\,n^{\hat{b}} = \underset{\sim}{n}\cdot\underset{\sim}{e}_{\hat{a}}$
$\underline{n^{\hat{a}}}$


back to $n^{\hat{a}}$
reference geodesic
$\underset{\sim}{e}_{\hat{0}} = \underset{\sim}{u}$
– project onto orthonormal
$\underset{\sim}{n} = n^{\hat{a}}\,\underset{\sim}{e}_{\hat{a}}$
$n^{\hat{0}} = \underset{\sim}{n}\cdot\underset{\sim}{e}_{\hat{0}} = -\underset{\sim}{n}\cdot\underset{\sim}{e}_{0}$
$n^{\hat{i}} = \underset{\sim}{n}\cdot\underset{\sim}{e}_{\hat{i}}$
$\underline{n}^{a}$
$n_a \equiv \eta_{ab}\, n^b = \underset{\sim}{n}\cdot\underset{\sim}{e}_{a}$


back to n^α
reference geodesic
$\underset{\sim}{e}_0 = \underset{\sim}{u}$
$\underset{\sim}{u}$
$\underset{\sim}{n}$
– project onto orthonormal
$\underset{\sim}{n} = n^a \underset{\sim}{e}_a$
$n^\hat{1} = \underset{\sim}{n} \cdot \underset{\sim}{e}_\hat{1}$
$n^\hat{0} = -\underset{\sim}{n} \cdot \underset{\sim}{e}_0$
n^a
$n_a \equiv \eta_{ab} n^b = \underset{\sim}{n} \cdot \underset{\sim}{e}_a = n^\alpha (e_a)_\alpha$
$\frac{d^2}{d\tau^2} n_a$


back to n^α
reference geodesic
$\underset{\sim}{e}_0 = \underset{\sim}{u}$
– project onto orthonormal basis.
$\underset{\sim}{n} = n^a \underset{\sim}{e}_a$
$n^0 = \underset{\sim}{n} \cdot \underset{\sim}{e}_0$
$n^i = -\underset{\sim}{n} \cdot \underset{\sim}{e}_i$
n^a
$n_a \equiv \eta_{ab} n^b = \underset{\sim}{n} \cdot \underset{\sim}{e}_a = n^\alpha g_{\alpha\beta} (e_a)^\beta$
$= n^\alpha (e_a)_\alpha$
$\frac{d^2}{d\tau^2} n_a = \frac{d^2 n^\alpha}{d\tau^2} (e_a)_\alpha$

back to n^α

$\underset{\sim}{e}_0 = \underset{\sim}{u}$

– project onto orthonormal basis.

– $\underset{\sim}{n} = n^a \underset{\sim}{e}_a$

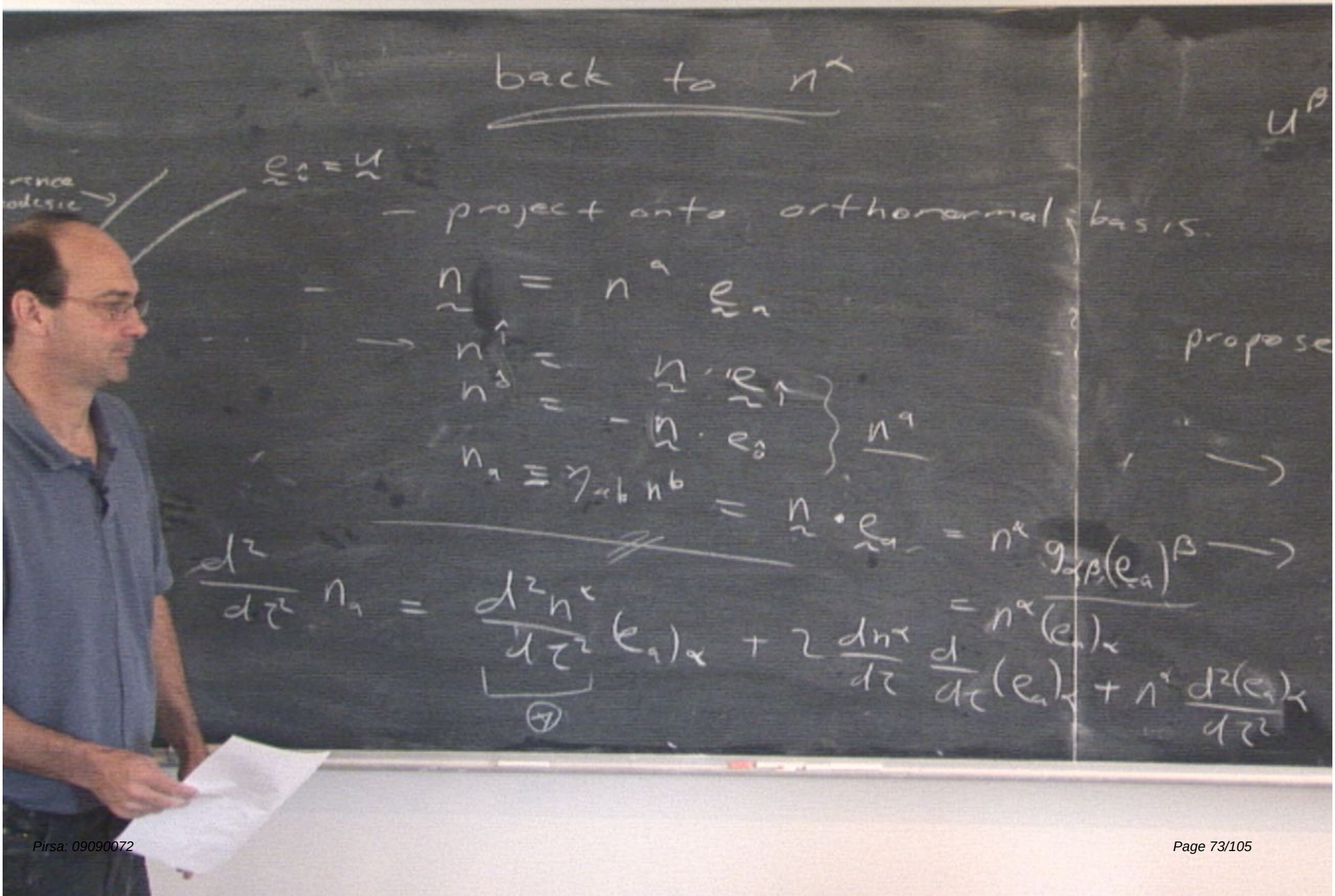
$\longrightarrow$ $n^\hat{1} = \underset{\sim}{n} \cdot \underset{\sim}{e}_\hat{1}$

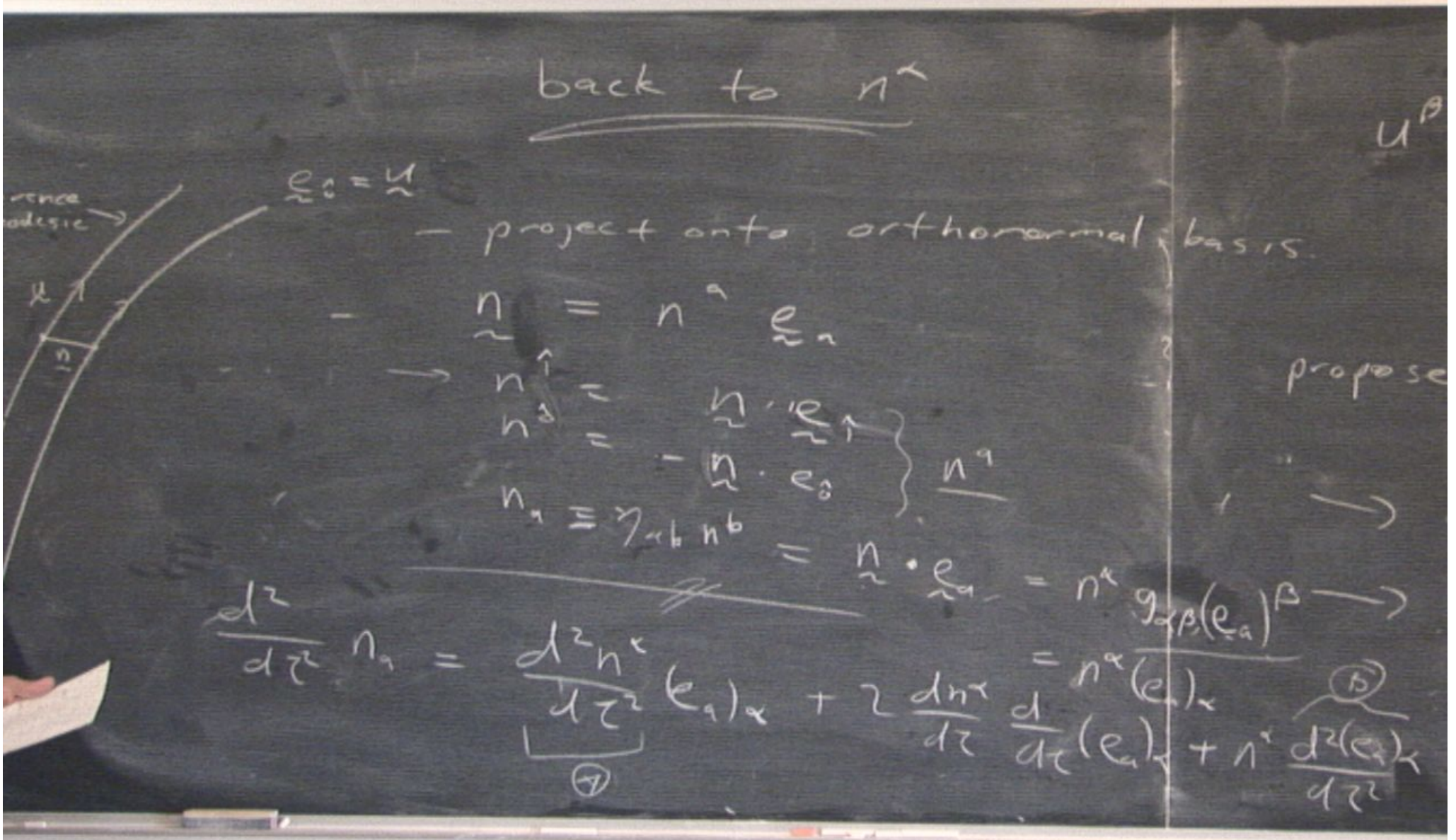
$n^\hat{0} = -\underset{\sim}{n} \cdot \underset{\sim}{e}_0$ } n^a

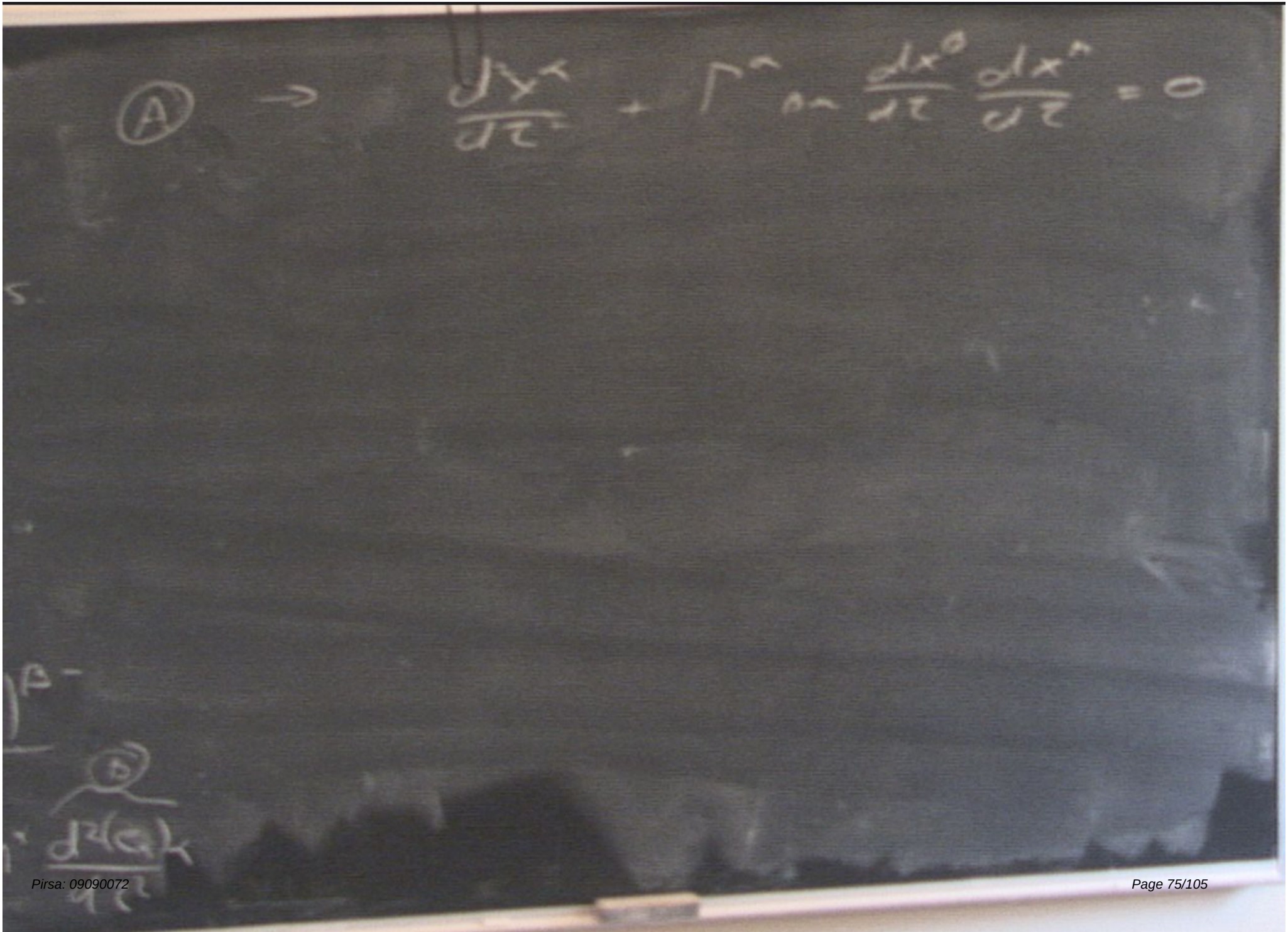
$n_a \equiv \eta_{ab} n^b = \underset{\sim}{n} \cdot \underset{\sim}{e}_a = n^\alpha g_{\alpha\beta} (e_a)^\beta$

$\frac{d^2}{d\tau^2} n_a = \frac{d^2 n^\alpha}{d\tau^2} (e_a)_\alpha + 2 \frac{dn^\alpha}{d\tau} \frac{d}{d\tau}(e_a)_\alpha + n^\alpha \frac{d^2}{d\tau^2}(e_a)_\alpha$

$= n^\alpha (e_a)_\alpha$


back to n^α
$\underset{\sim}{e}_0 = \underset{\sim}{u}$
– project onto orthonormal basis.
$\underset{\sim}{n} = n^a \underset{\sim}{e}_a$
$n^0 = \underset{\sim}{n} \cdot \underset{\sim}{e}_0$
$n^i = \underset{\sim}{n} \cdot \underset{\sim}{e}_i$
n^a
$n_a \equiv \gamma_{ab} n^b$
$= \underset{\sim}{n} \cdot \underset{\sim}{e}_a = n^\alpha g_{\alpha\beta} (e_a)^\beta$
$= n^\alpha (e_a)_\alpha$
$\frac{d^2}{d\tau^2} n_a = \frac{d^2 n^\alpha}{d\tau^2} (e_a)_\alpha + 2 \frac{dn^\alpha}{d\tau} \frac{d}{d\tau} (e_a)_\alpha + n^\alpha \frac{d^2 (e_a)_\alpha}{d\tau^2}$
u^β
propose


back to n^α
$e_0 = u$
– project onto orthonormal basis.
$\underset{\sim}{n} = n^a \underset{\sim}{e}_a$
$n^0 = \underset{\sim}{n} \cdot \underset{\sim}{e}_0$
$n^i = \underset{\sim}{n} \cdot \underset{\sim}{e}_i$
n^a
$n_a \equiv \eta_{ab} n^b = \underset{\sim}{n} \cdot \underset{\sim}{e}_a = n^\alpha g_{\alpha\beta} (e_a)^\beta$
$= n^\alpha (e_a)_\alpha$
$\frac{d^2}{d\tau^2} n_a = \frac{d^2 n^\alpha}{d\tau^2} (e_a)_\alpha + 2 \frac{dn^\alpha}{d\tau} \frac{d}{d\tau}(e_a)_\alpha + n^\alpha \frac{d^2 (e_a)_\alpha}{d\tau^2}$
a
b
propose



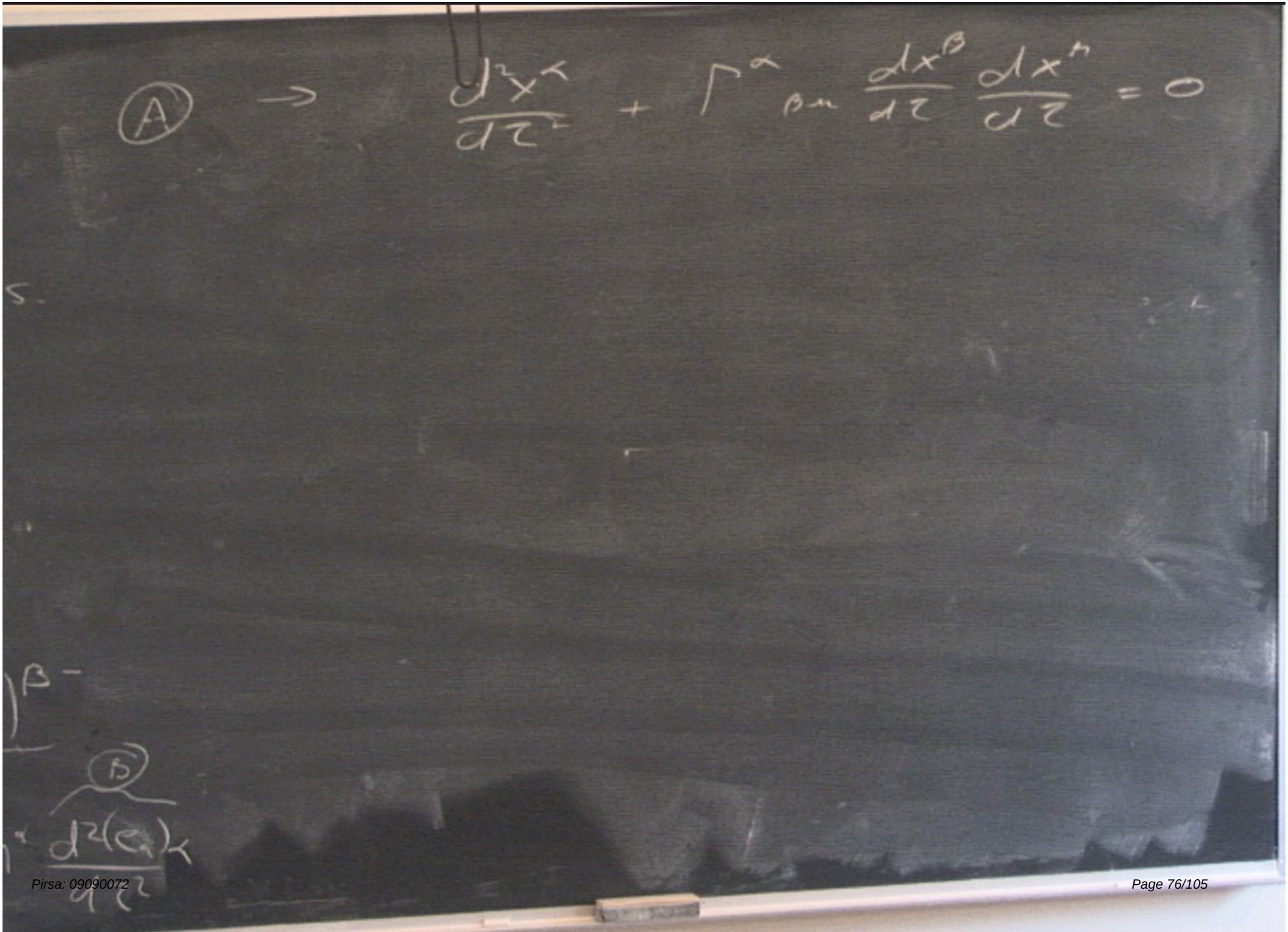
$$\text{(A)} \rightarrow \frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha{}_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau} = 0$$

s.

β -

(B)

$\frac{d^2(e_\alpha)_\alpha}{d\tau^2}$



$$\text{(A)} \rightarrow \frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha{}_{\beta\mu}\frac{dx^\beta}{d\tau}\frac{dx^\mu}{d\tau} = 0$$

s.

$)^{\beta -}$

$$\overbrace{\frac{d^2(e_\alpha)_\alpha}{d\tau^2}}^{\text{(B)}}$$

$$\text{(A)} \rightarrow \frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha_{\beta\mu}\frac{dx^\beta}{d\tau}\frac{dx^\mu}{d\tau} = 0$$

$$\frac{d^2(x^\alpha + \pi^\alpha)}{d\tau^2} + \Gamma^\alpha_{\beta\mu}($$

(B)

$$\frac{d^2(\quad)^\kappa}{d\tau^2}$$

(A) $\rightarrow$ $\frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha_{\beta\mu}\frac{dx^\beta}{d\tau}\frac{dx^\mu}{d\tau} = 0$

$\frac{d^2(x^\alpha+\eta^\alpha)}{d\tau^2} + \Gamma^\alpha_{\beta\mu}(x^\nu+\eta^\nu)\frac{d(x+\eta)^\beta}{d\tau}\frac{d(x+\eta)^\mu}{d\tau} = 0$

5.

(B)

$\frac{d^2(\eta)^\alpha}{d\tau^2}$

$$\text{(A)} \rightarrow \frac{d^2 x^\alpha}{d\tau^2} + \Gamma^\alpha{}_{\beta\mu} \frac{dx^\beta}{d\tau} \frac{dx^\mu}{d\tau} = 0$$

$$\frac{d^2 (x^\alpha + n^\alpha)}{d\tau^2} + \Gamma^\alpha{}_{\beta\mu} (x^\nu + n^\nu) \frac{d(x+n)^\beta}{d\tau} \frac{d(x+n)^\mu}{d\tau} = 0$$

s.

btract : $\frac{d^2 n^\alpha}{d\tau^2} =$

$)^{\beta -}$

$$\overbrace{n^\alpha \frac{d^2 (e_\alpha)_\alpha}{d\tau^2}}^{\text{(B)}}$$

(A) $\rightarrow \quad \frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha_{\beta\mu} \frac{dx^\beta}{d\tau} \frac{dx^\mu}{d\tau} = 0$

$$\frac{d^2(x^\alpha + \eta^\alpha)}{d\tau^2} + \Gamma^\alpha_{\beta\mu}(x^\nu + \eta^\nu) \frac{d(x+\eta)^\beta}{d\tau} \frac{d(x+\eta)^\mu}{d\tau} = 0$$

5.

btract: $\frac{d^2\eta^\alpha}{d\tau^2} = -2\Gamma^\alpha_{\beta\mu}(x) \frac{d\eta^\beta}{d\tau} \frac{dx^\mu}{d\tau}$

$]^\beta -$

(B)

$\eta^\alpha \frac{d^2(e_\eta)_\alpha}{d\tau^2}$

$$\text{(A)} \rightarrow \frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha{}_{\beta\mu}\frac{dx^\beta}{d\tau}\frac{dx^\mu}{d\tau} = 0$$

$$\frac{d^2(x^\alpha+n^\alpha)}{d\tau^2} + \Gamma^\alpha{}_{\beta\mu}(x^\nu+n^\nu)\frac{d(x+n)^\beta}{d\tau}\frac{d(x+n)^\mu}{d\tau} = 0$$

s.

btract: $$\frac{d^2n^\alpha}{d\tau^2} = -2\,\Gamma^\alpha{}_{\beta\mu}(x)\frac{dn^\beta}{d\tau}\frac{dx^\mu}{d\tau} - \frac{\partial}{\partial x^\delta}\Gamma^\alpha{}_{\beta\mu}\Big|_{x^\mu}\frac{dx^\beta}{d\tau}\frac{dx^\mu}{d\tau}$$

$)^{\beta -}$

$$\overbrace{n^\alpha \frac{d^2(e_\alpha)_\alpha}{d\tau^2}}^{\text{(B)}}$$

(A) $\rightarrow \quad \frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha_{\beta\mu} \frac{dx^\beta}{d\tau} \frac{dx^\mu}{d\tau} = 0$

$$\frac{d^2(x^\alpha + \eta^\alpha)}{d\tau^2} + \Gamma^\alpha_{\beta\mu}(x^\nu + \eta^\nu) \frac{d(x+\eta)^\beta}{d\tau} \frac{d(x+\eta)^\mu}{d\tau} = 0$$

5.

btract: $\frac{d^2\eta^\alpha}{d\tau^2} = -2\Gamma^\alpha_{\beta\mu}(x) \frac{d\eta^\beta}{d\tau} \frac{dx^\mu}{d\tau}$

$$- \frac{2}{\partial x^\delta} \Gamma^\alpha_{\beta\mu}\Big|_{x^\mu} \frac{dx^\beta}{d\tau} \frac{dx^\mu}{d\tau} \eta^\delta$$

$)^\beta -$

(B)

$\eta^\alpha \frac{d^2(e_\eta)_\alpha}{d\tau^2}$

(A) $\rightarrow \quad \frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha_{\beta\mu} \frac{dx^\beta}{d\tau} \frac{dx^\mu}{d\tau} = 0$

$$\frac{d^2(x^\alpha + \eta^\alpha)}{d\tau^2} + \Gamma^\alpha_{\beta\mu}(x^\nu + \eta^\nu) \frac{d(x+\eta)^\beta}{d\tau} \frac{d(x+\eta)^\mu}{d\tau} = 0$$

5.

btract: $\frac{d^2\eta^\alpha}{d\tau^2} = -2\Gamma^\alpha_{\beta\mu}(x) \frac{d\eta^\beta}{d\tau} \frac{dx^\mu}{d\tau}$

$$- \frac{\partial}{\partial x^\delta} \Gamma^\alpha_{\beta\mu}\Big|_{x^\mu} \frac{dx^\beta}{d\tau} \frac{dx^\mu}{d\tau} \eta^\delta$$

(B)

$\beta -$

(B)

$\alpha \frac{d^2(\eta)^\alpha}{d\tau^2}$

(A) $\rightarrow \quad \frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha_{\beta\mu} \frac{dx^\beta}{d\tau} \frac{dx^\mu}{d\tau} = 0$

$$\frac{d^2(x^\alpha + \eta^\alpha)}{d\tau^2} + \Gamma^\alpha_{\beta\mu}(x^\nu + \eta^\nu) \frac{d(x+\eta)^\beta}{d\tau} \frac{d(x+\eta)^\mu}{d\tau} = 0$$

5.

btract: $\frac{d^2\eta^\alpha}{d\tau^2} = -2\Gamma^\alpha_{\beta\mu}(x) \frac{d\eta^\beta}{d\tau} \frac{dx^\mu}{d\tau}$

$$- \frac{2}{\partial x^\delta} \Gamma^\alpha_{\beta\mu}\Big|_{x^\mu} \frac{dx^\beta}{d\tau} \frac{dx^\mu}{d\tau} \eta^\delta$$

(B) $\frac{d^2(e_a)_\alpha}{d\tau^2} = \frac{d}{d\tau}\left(\frac{d(e_a)_\alpha}{d\tau}\right)$

$]^\beta -$

(B)

$\eta^\alpha \frac{d^2(e_a)_\alpha}{d\tau^2}$

$$\text{(A)} \rightarrow \frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha_{\beta\mu}\frac{dx^\beta}{d\tau}\frac{dx^\mu}{d\tau} = 0$$

$$\frac{d^2(x^\alpha + \eta^\alpha)}{d\tau^2} + \Gamma^\alpha_{\beta\mu}(x^\nu + \eta^\nu)\frac{d(x+\eta)^\beta}{d\tau}\frac{d(x+\eta)^\mu}{d\tau} = 0$$

5.

btract: $$\frac{d^2\eta^\alpha}{d\tau^2} = -2\Gamma^\alpha_{\beta\mu}(x)\frac{d\eta^\beta}{d\tau}\frac{dx^\mu}{d\tau}$$

$$- \frac{2}{\partial x^\delta}\Gamma^\alpha_{\beta\mu}\Big|_{x^\mu}\frac{dx^\sigma}{d\tau}\frac{dx^\mu}{d\tau}\eta^\delta$$

(B) $$\frac{d^2(e_a)_\alpha}{d\tau^2} = \frac{d}{d\tau}\left(\frac{d(e_a)_\alpha}{d\tau}\right) = \frac{d}{d\tau}\left(\Gamma^\beta_{\alpha\mu}u^\mu(e_a)_\beta\right)$$

β-

(B)

α $\frac{d^2(e_a)_\alpha}{d\tau^2}$

(A) $\rightarrow \quad \frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha_{\beta\mu} \frac{dx^\beta}{d\tau} \frac{dx^\mu}{d\tau} = 0$

$$\frac{d^2(x^\alpha + \eta^\alpha)}{d\tau^2} + \Gamma^\alpha_{\beta\mu}(x^\nu + \eta^\nu) \frac{d(x+\eta)^\beta}{d\tau} \frac{d(x+\eta)^\mu}{d\tau} = 0$$

5.

btract: $\frac{d^2\eta^\alpha}{d\tau^2} = -2\Gamma^\alpha_{\beta\mu}(x) \frac{d\eta^\beta}{d\tau} \frac{dx^\mu}{d\tau}$

$$- \frac{\partial}{\partial x^\delta} \Gamma^\alpha_{\beta\mu}\Big|_{x^\mu} \frac{dx^\beta}{d\tau} \frac{dx^\mu}{d\tau} \eta^\delta$$

(B) $\frac{d^2(e_a)_\alpha}{d\tau^2} = \frac{d}{d\tau}\left(\frac{d(e_a)_\alpha}{d\tau}\right) = \frac{d}{d\tau}\left(\Gamma^\beta_{\alpha\mu} u^\mu (e_a)_\beta\right)$

$$= \left(\frac{d}{d\tau}\Gamma^\beta_{\alpha\mu}\right) u^\mu (e_a)_\beta + \Gamma^\beta_{\alpha\mu} \frac{du^\mu}{d\tau}(e_a)_\beta + \Gamma^\beta_{\alpha\mu} u^\mu \frac{d(e_a)_\beta}{d\tau}$$

(B) $\eta^\alpha \frac{d^2(e_a)_\alpha}{d\tau^2}$

(A) $\rightarrow \quad \frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha_{\beta\mu}\frac{dx^\beta}{d\tau}\frac{dx^\mu}{d\tau} = 0$

$$\frac{d^2(x^\alpha+\eta^\alpha)}{d\tau^2} + \Gamma^\alpha_{\beta\mu}(x^\nu+\eta^\nu)\frac{d(x+\eta)^\beta}{d\tau}\frac{d(x+\eta)^\mu}{d\tau} = 0$$

btract: $\frac{d^2\eta^\alpha}{d\tau^2} = -2\Gamma^\alpha_{\beta\mu}(x)\frac{d\eta^\beta}{d\tau}\frac{dx^\mu}{d\tau}$

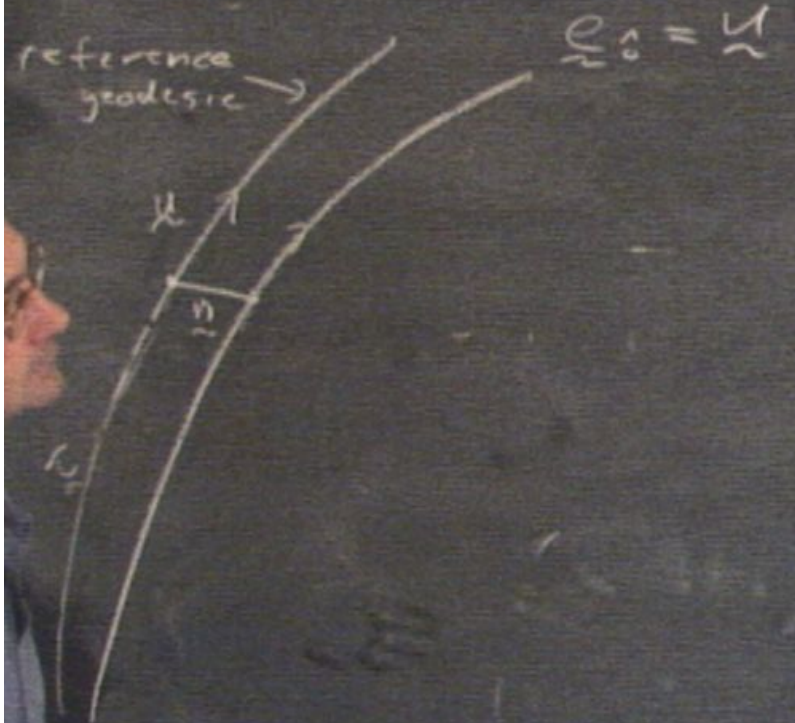
$$- \frac{2}{\partial x^\delta}\Gamma^\alpha_{\beta\mu}\Big|_{x^\mu}\frac{dx^\beta}{d\tau}\frac{dx^\mu}{d\tau}\eta^\delta$$

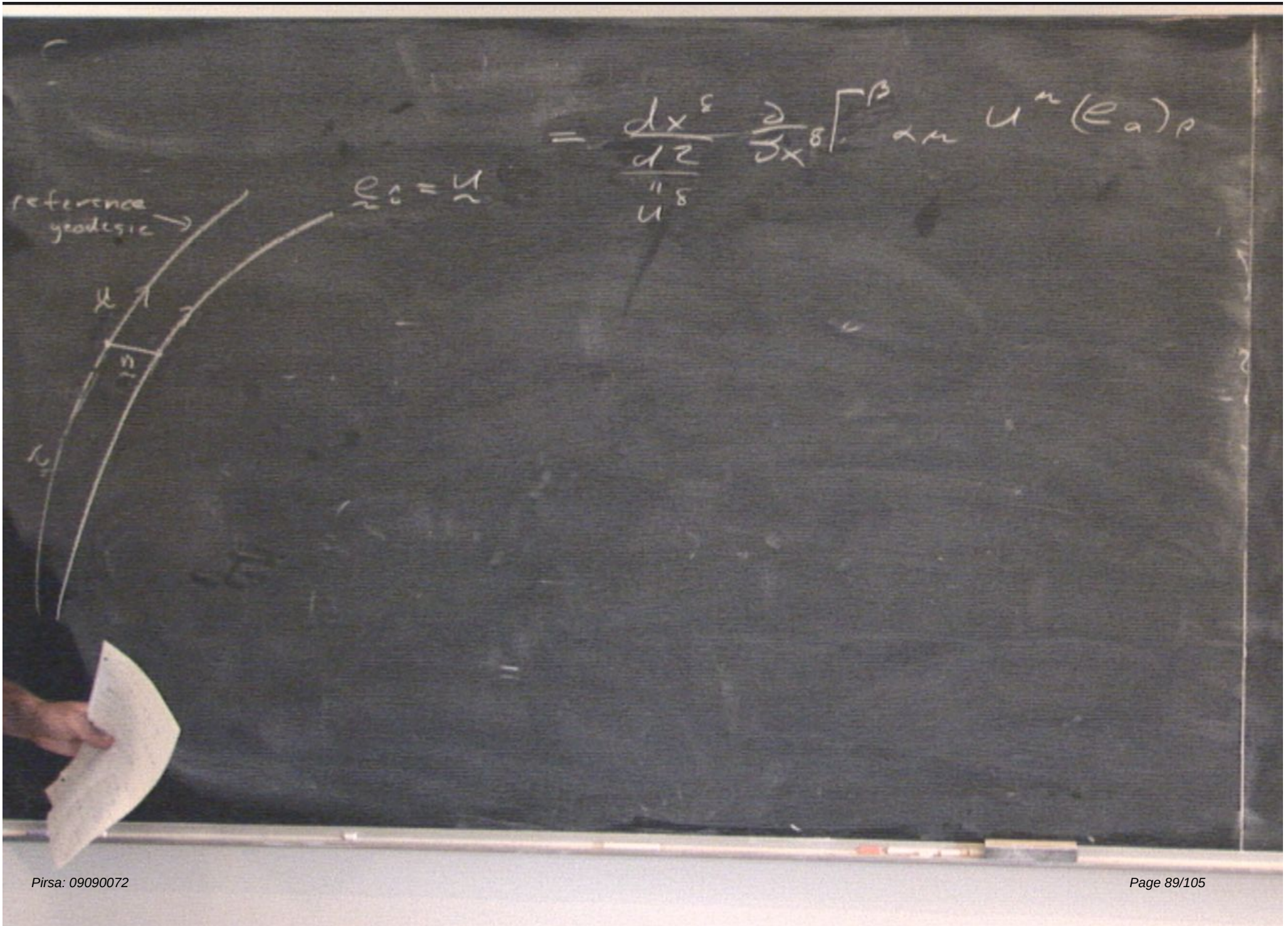
(B) $\frac{d^2(e_a)_\alpha}{d\tau^2} = \frac{d}{d\tau}\left(\frac{d(e_a)_\alpha}{d\tau}\right) = \frac{d}{d\tau}\left(\Gamma^\beta_{\alpha\mu}u^\mu(e_a)_\beta\right)$

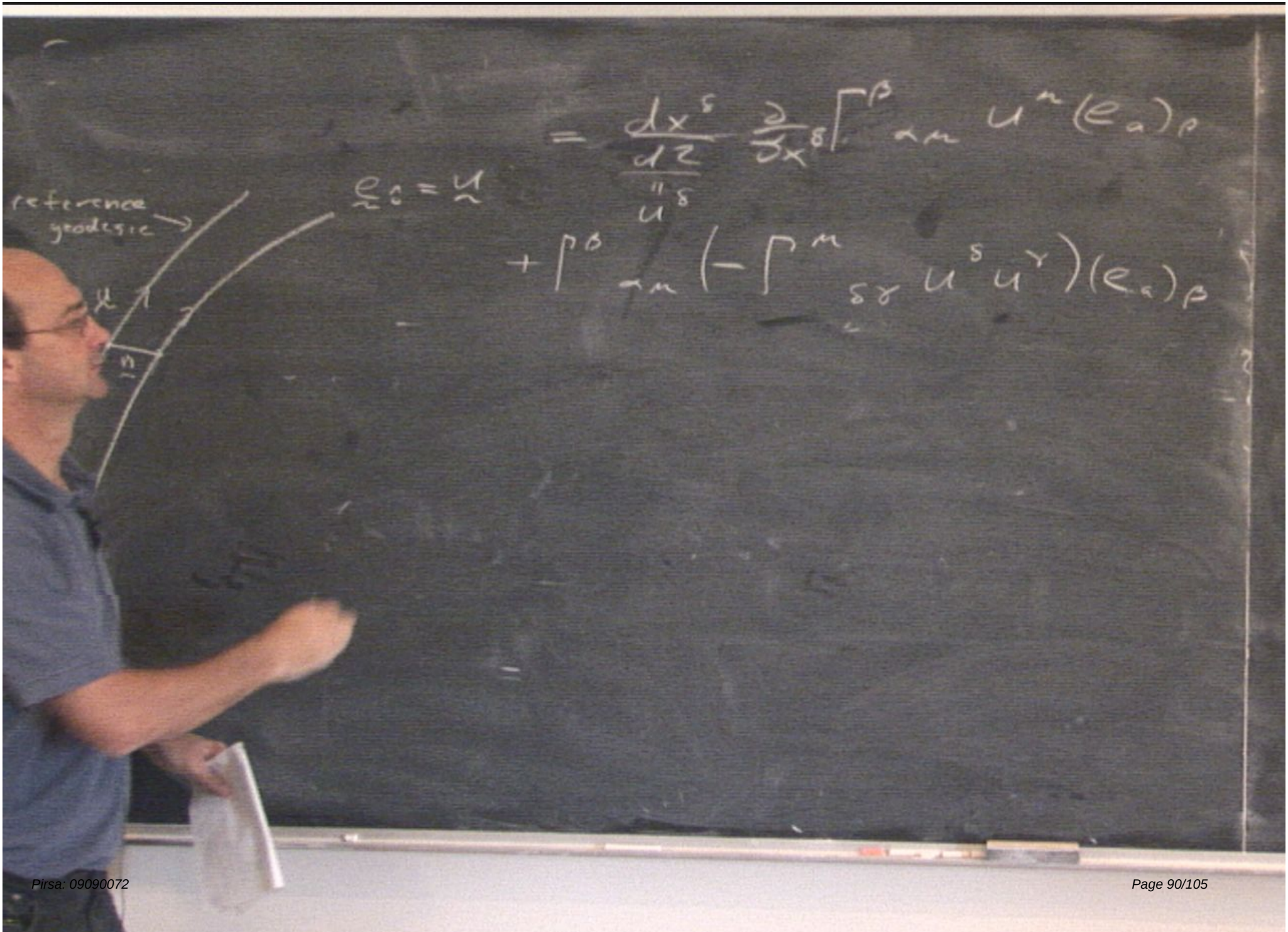
$$= \left(\frac{d}{d\tau}\Gamma^\beta_{\alpha\mu}\right)u^\mu(e_a)_\beta + \Gamma^\beta_{\alpha\mu}\frac{du^\mu}{d\tau}(e_a)_\beta + \Gamma^\beta_{\alpha\mu}u^\mu\frac{d(e_a)_\beta}{d\tau}$$

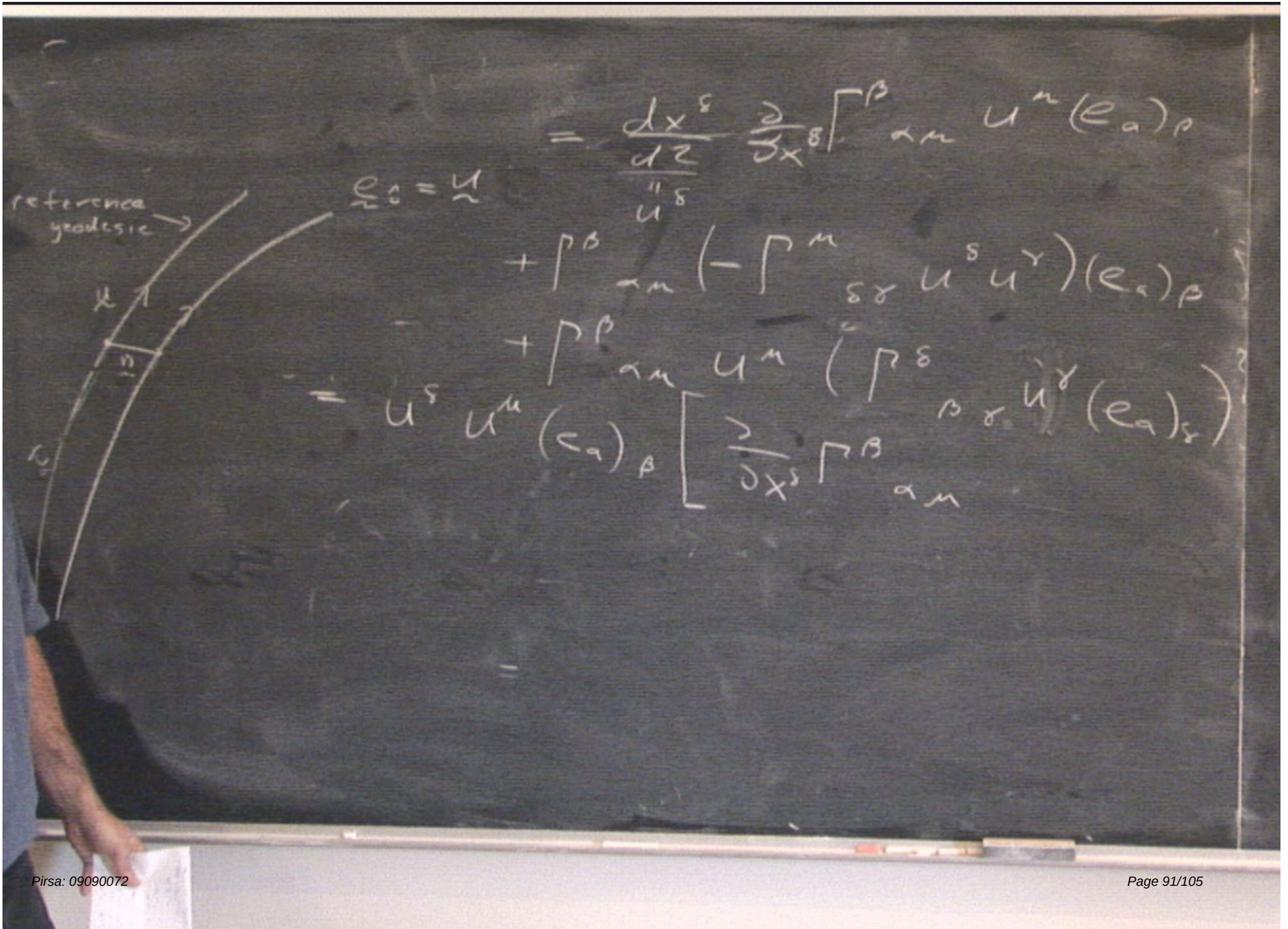
(B) $\frac{d^2(e_a)_\alpha}{d\tau^2}$

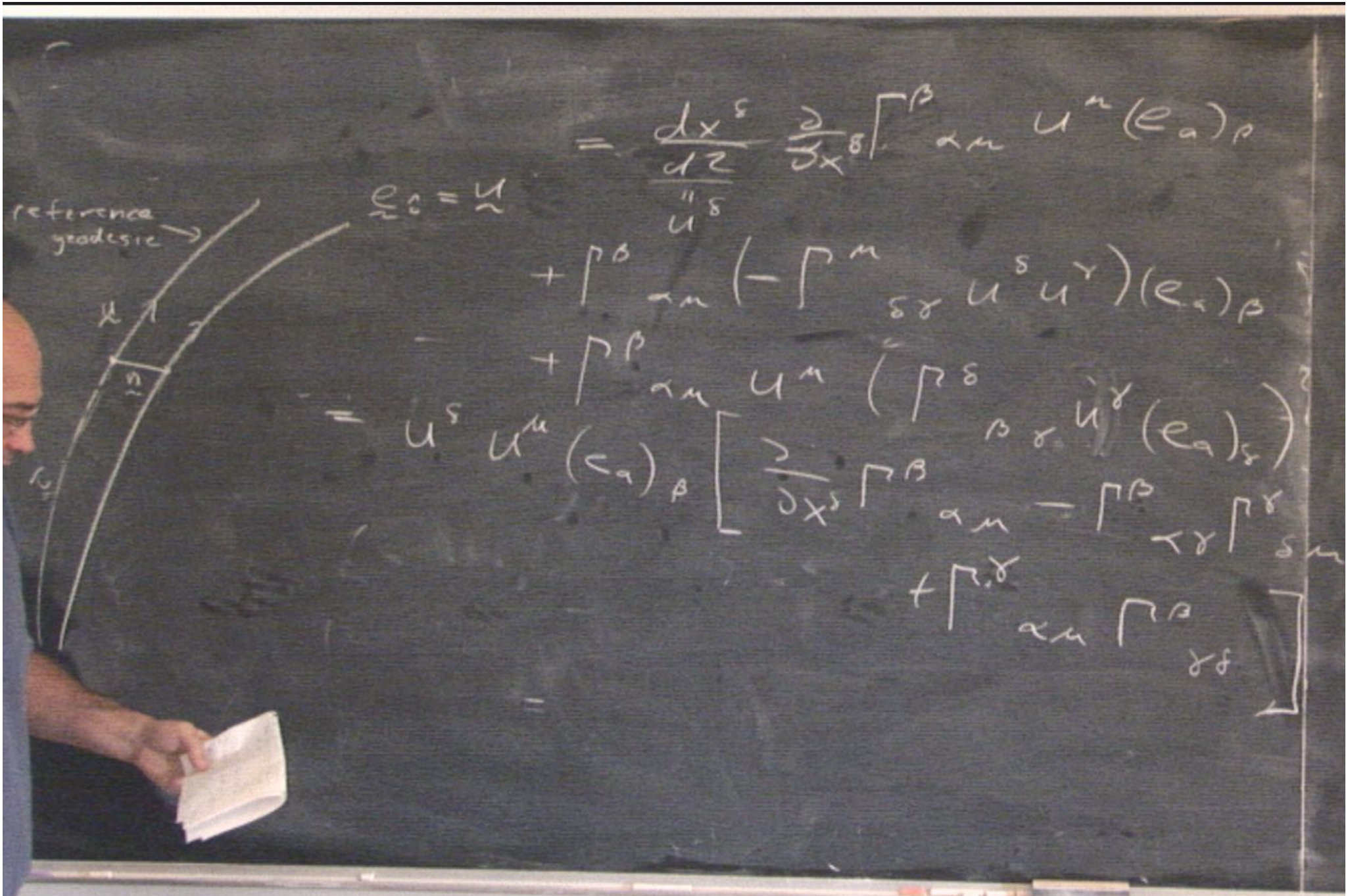
$$= \frac{dx^\delta}{d\tau} \frac{\partial}{\partial x^\delta} \Gamma^\beta{}_{\alpha\mu} u^\mu (e_a)_\beta$$

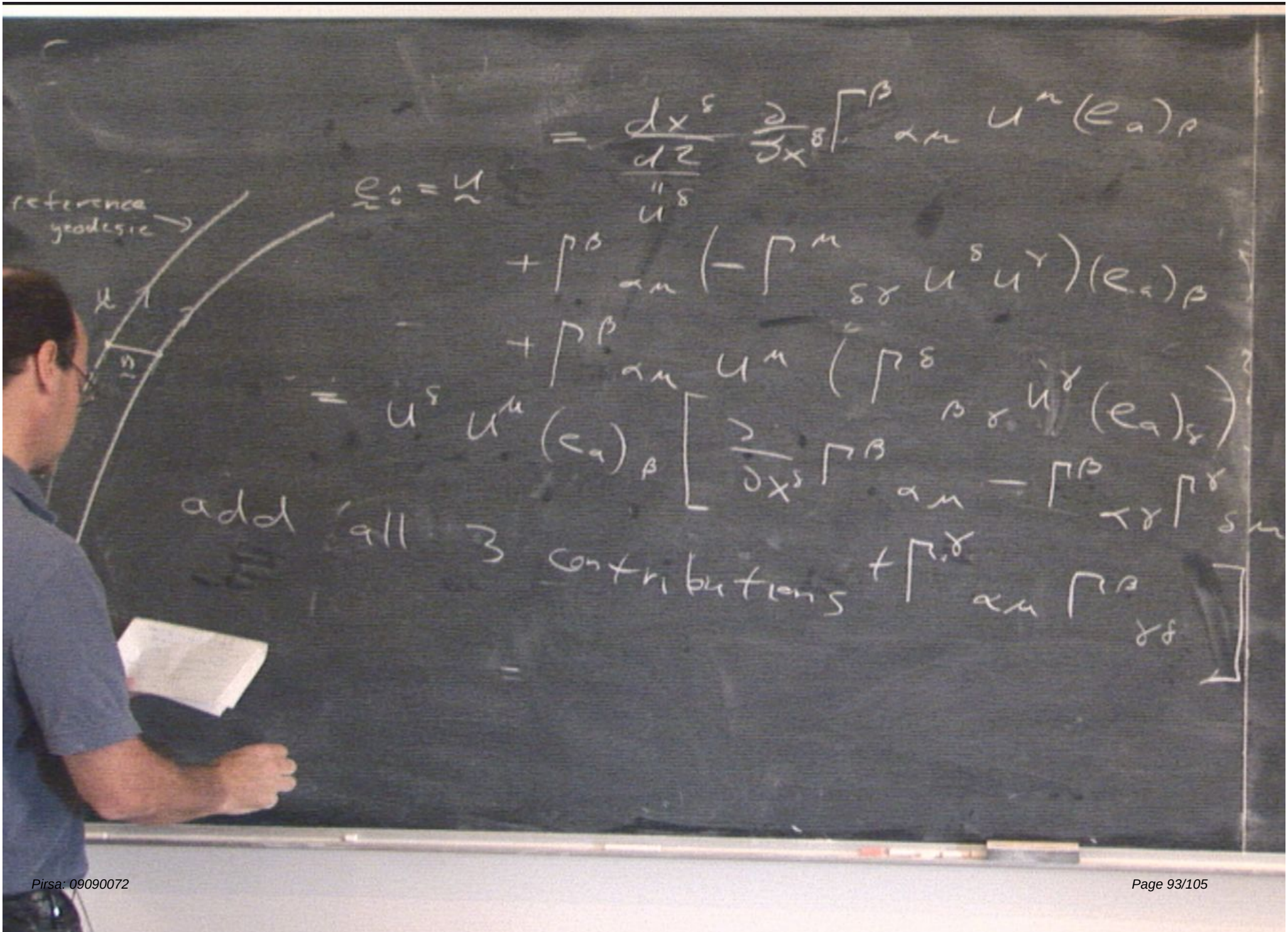


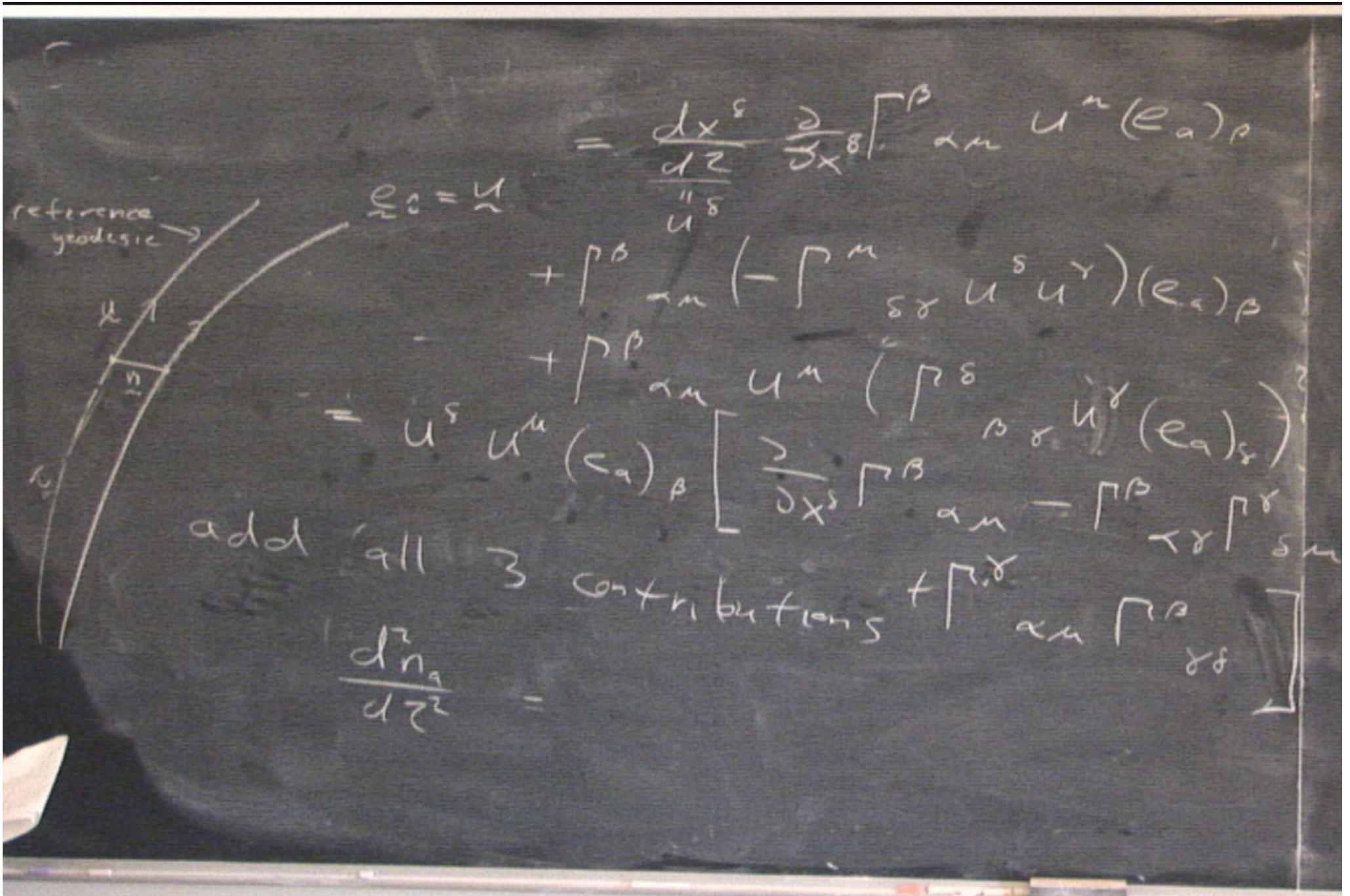

reference geodesic

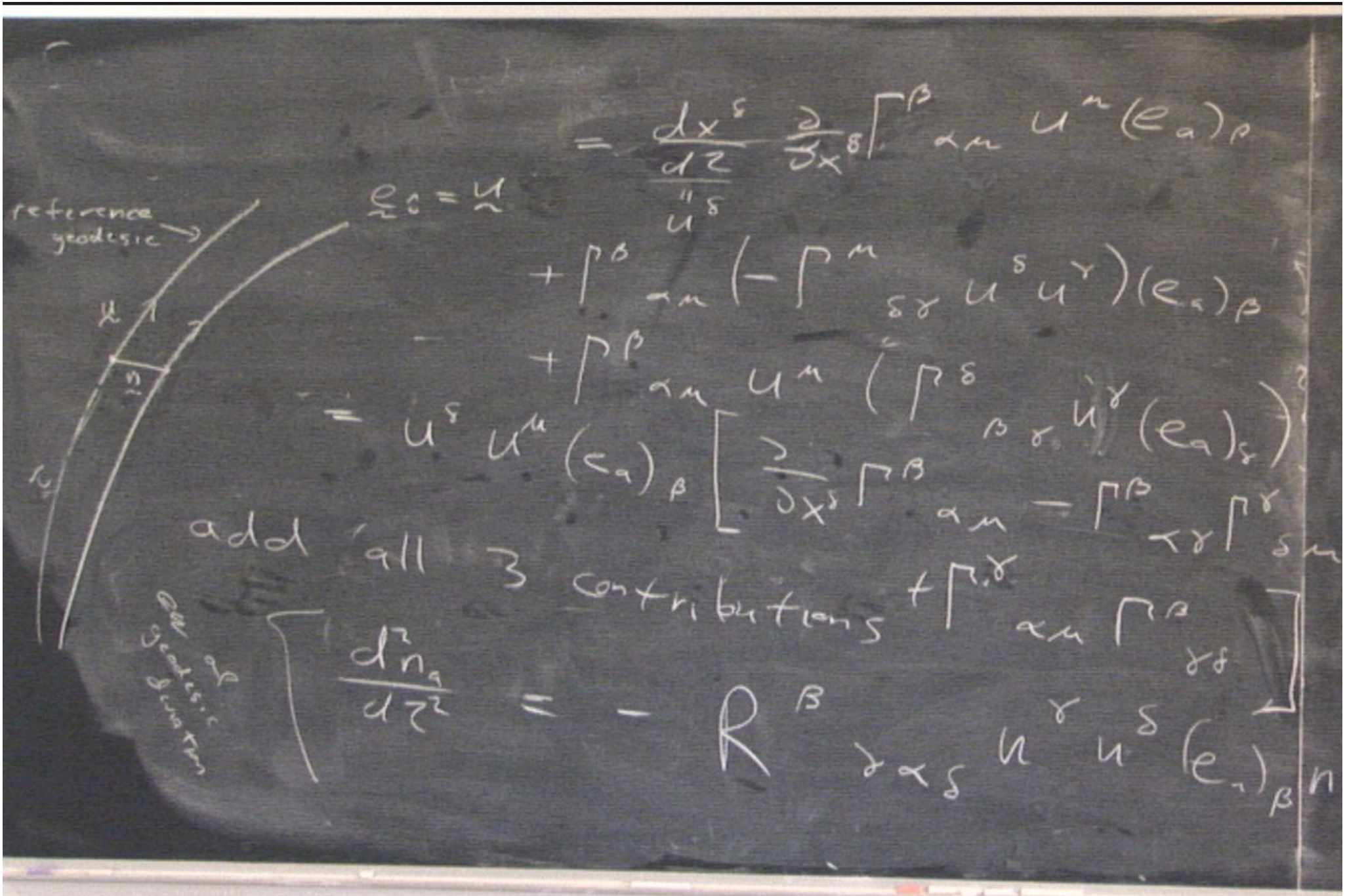

reference geodesic


reference geodesic


reference geodesic


reference geodesic
add all 3 contributions


reference geodesic
$\vec{e}_0 = \vec{u}$
$\vec{n}$
$\vec{u}$
$= \frac{dx^\beta}{d\tau} \frac{\partial}{\partial x^\beta} \Gamma^\alpha{}_{\mu\nu} u^\mu (e_a)^\nu$
$\frac{dx^\beta}{d\tau} = u^\beta$
$+ \Gamma^\alpha{}_{\mu\nu} \left(-\Gamma^\mu{}_{\beta\gamma} u^\beta u^\gamma\right)(e_a)^\nu$
$+ \Gamma^\alpha{}_{\mu\nu} u^\mu \left(\Gamma^\beta{}_{\nu\gamma} u^\gamma (e_a)_\beta\right)$
$= u^\beta u^\mu (e_a)_\nu \left[\frac{\partial}{\partial x^\beta} \Gamma^\alpha{}_{\mu\nu} - \Gamma^\alpha{}_{\tau\gamma} \Gamma^\gamma{}_{\beta\mu} + \Gamma^\gamma{}_{\mu\nu} \Gamma^\alpha{}_{\gamma\beta}\right]$
add 'all 3 contributions'
$\frac{d^2 n_a}{d\tau^2} =$


reference geodesic
$\underset{\sim}{e}_0 = \underset{\sim}{u}$
$= \frac{dx^\delta}{d\tau} \frac{\partial}{\partial x^\delta} \Gamma^\beta{}_{\alpha\mu} u^\mu (e_a)_\beta$
$\frac{dx^\delta}{d\tau} = u^\delta$
$+ \Gamma^\beta{}_{\alpha\mu} (-\Gamma^\mu{}_{\delta\gamma} u^\delta u^\gamma)(e_a)_\beta$
$+ \Gamma^\beta{}_{\alpha\mu} u^\mu (\Gamma^\delta{}_{\beta\gamma} u^\gamma (e_a)_\delta)$
$= u^\delta u^\mu (e_a)_\beta \left[\frac{\partial}{\partial x^\delta} \Gamma^\beta{}_{\alpha\mu} - \Gamma^\beta{}_{\alpha\gamma} \Gamma^\gamma{}_{\delta\mu} + \Gamma^\gamma{}_{\alpha\mu} \Gamma^\beta{}_{\gamma\delta} \right]$
add all 3 contributions
$\frac{d^2 n_a}{d\tau^2} = - R^\beta{}_{\gamma\alpha\delta} u^\gamma u^\delta (e_a)_\beta n$
geodesic deviation

$$R^{\rho}{}_{\gamma\alpha\delta} = \frac{\partial}{\partial x^{\alpha}}\Gamma^{\rho}{}_{\gamma\delta} - \frac{\partial}{\partial x^{\delta}}\Gamma^{\beta}{}_{\gamma\alpha}$$

$$R^{\beta}{}_{\gamma\alpha\delta} = \left[\frac{\partial}{\partial x^{\alpha}} \Gamma^{\beta}{}_{\gamma\delta} - \frac{\partial}{\partial x^{\delta}} \Gamma^{\beta}{}_{\gamma\alpha} + \Gamma^{\beta}{}_{\alpha\varepsilon} \Gamma^{\varepsilon}{}_{\gamma\delta} - \Gamma^{\beta}{}_{\delta\varepsilon} \Gamma^{\varepsilon}{}_{\gamma\alpha} \right]$$

$$R^{\beta}{}_{\gamma\alpha\delta} = \left[\frac{\partial}{\partial x^{\alpha}} \Gamma^{\beta}{}_{\gamma\delta} - \frac{\partial}{\partial x^{\delta}} \Gamma^{\beta}{}_{\gamma\alpha} + \Gamma^{\beta}{}_{\alpha\varepsilon} \Gamma^{\varepsilon}{}_{\delta\gamma} - \Gamma^{\beta}{}_{\delta\varepsilon} \Gamma^{\varepsilon}{}_{\alpha\delta} \right]$$

$\longrightarrow$ tensor describing spacetime curvature

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$\longrightarrow$ tells us about "tidal forces"

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$\longrightarrow$ tensor describing spacetime curvature

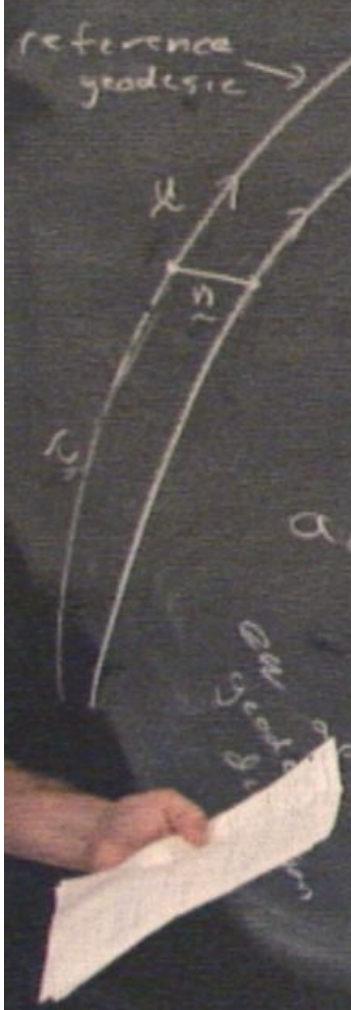
$\longrightarrow$ Riemann curvature

$\longrightarrow$ tells us about "tidal forces"

$\longrightarrow$ contains 2nd derivatives of $g_{\alpha\beta}$

$\longrightarrow$ symmetries eg $R^{\beta}{}_{\gamma\alpha\delta} = -R^{\beta}{}_{\gamma\delta\alpha}$

reduce the number of independent components from $4^4 = 256 \rightarrow 20$

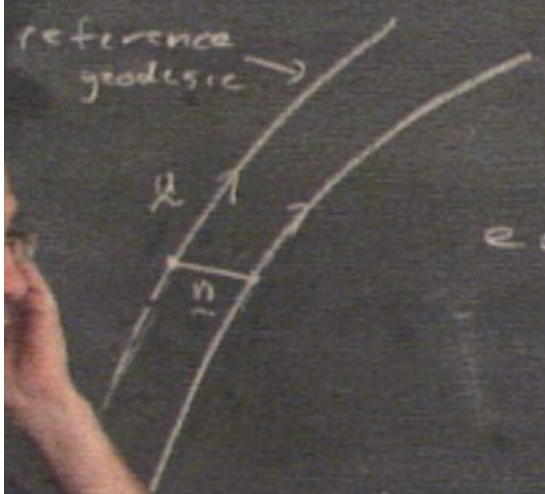


$$= u\,u\,(e_n)_\beta \left[\frac{\partial}{\partial x^\delta} \Gamma^\beta_{\alpha\mu} - \Gamma^\gamma_{\alpha\mu} \Gamma^\beta_{\gamma\delta} \right]$$

add all 3 contributions

$$\frac{d^2 n_\alpha}{d\tau^2} = - R^\beta{}_{\gamma\alpha\delta}\, u^\gamma u^\delta (e_n)_\beta \, n$$

reduce the number of independent components from $4^4 = 256 \to 20$



eq takes form $\dfrac{d^2 n_a}{d\tau^2} = -K^a{}_b\, n^b$

add all 3 contributions $+ \Gamma^{\mu\nu}{}_{\alpha\mu}\, \Gamma^{\beta}{}_{\gamma\delta}$

eq of geodesic deviation

$$\frac{d^2 n_a}{d\tau^2} = -R^{\beta}{}_{\gamma\alpha\delta}\, u^{\gamma} u^{\delta} (e_a)_{\beta}\, n$$

reduce the number of independent components from $4^4 = 256 \to 20$

$\underset{\sim}{e}_0 = \underset{\sim}{u}$

eq takes form $\frac{d^2 n_a}{d\tau^2} = -K^a{}_b n^b$

$\longrightarrow$ suggests a particular set of eq's for vacuum Einstein's eqs

add all 3 contributions $+ \Gamma^{\mu\nu}{}_{\alpha\mu} \Gamma^{\beta}{}_{\gamma\delta}$

$$\frac{d^2 n_a}{d\tau^2} = - R^{\beta}{}_{\gamma\alpha\delta} u^{\gamma} u^{\delta} (e_a)_{\beta} n^{\alpha}$$

R