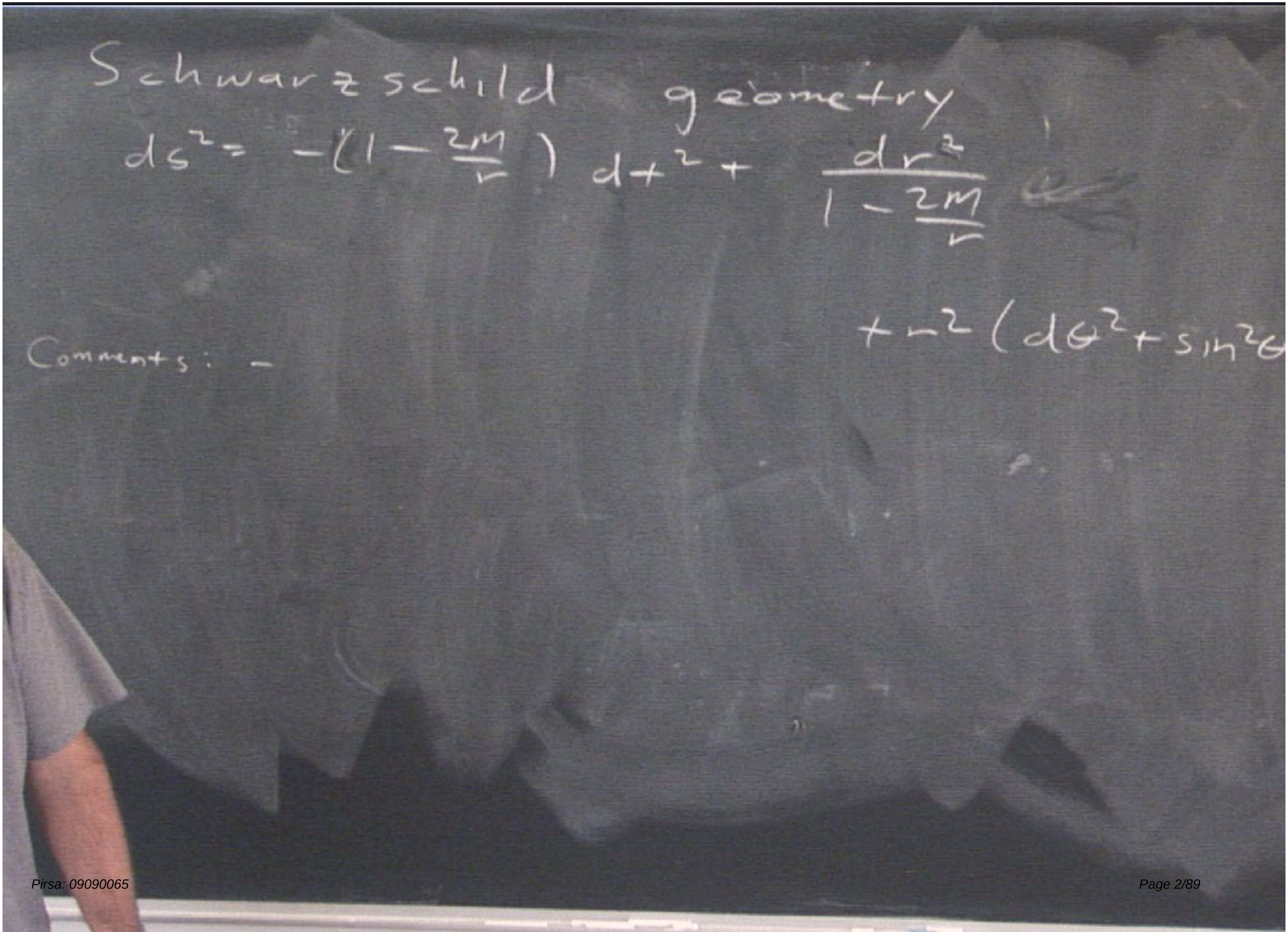
Title: PSI - Relativity (PHYS 604) - 8

Date: Sep 14, 2009 10:30 AM

URL: http://pirsa.org/09090065

Abstract:

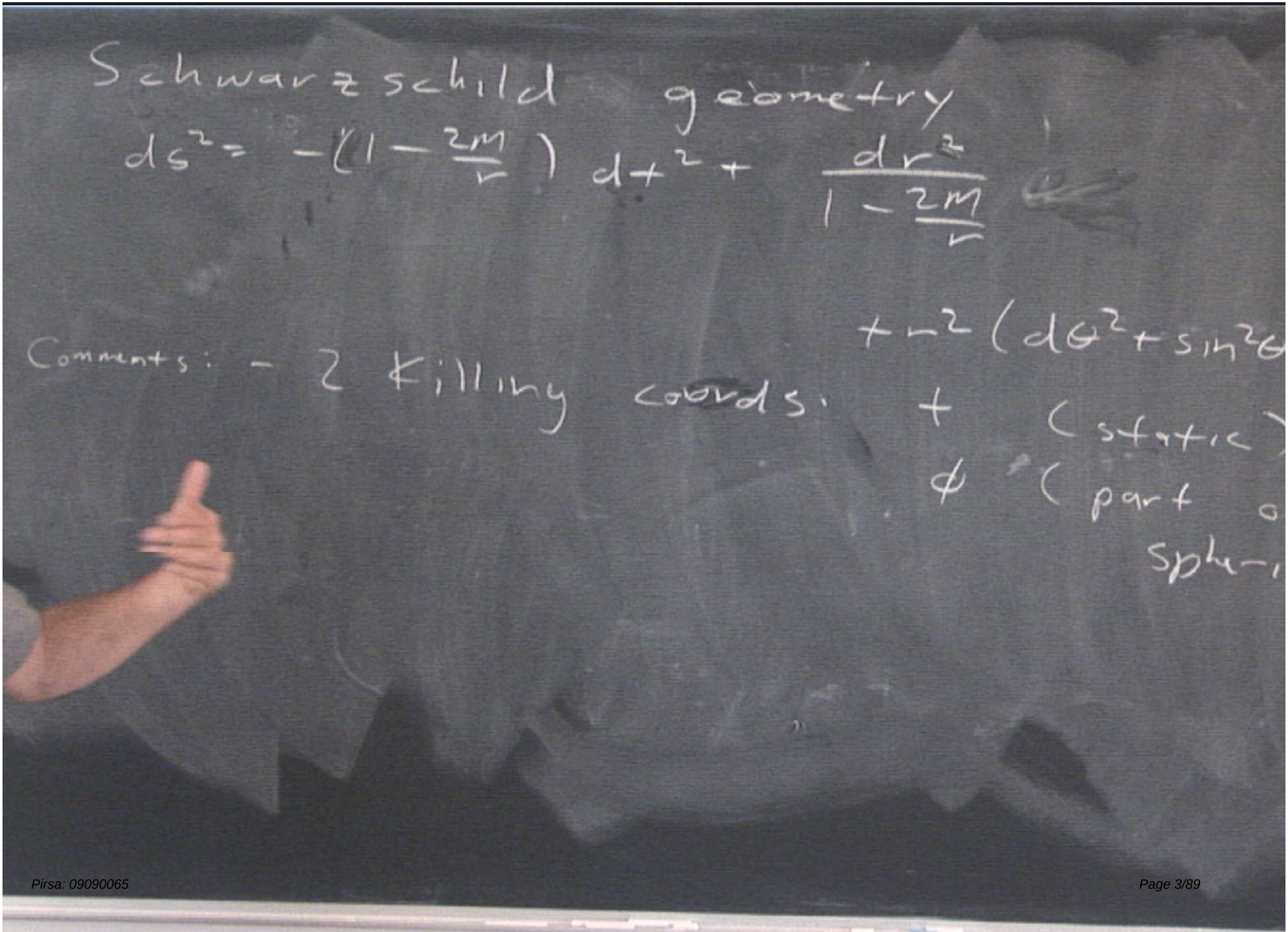


Schwarzschild geometry

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \frac{dr^2}{1-\frac{2M}{r}}$$

$$+ r^2 (d\theta^2 + \sin^2\theta$$

Comments: -



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Comments: - 2 Killing coords: t (static)

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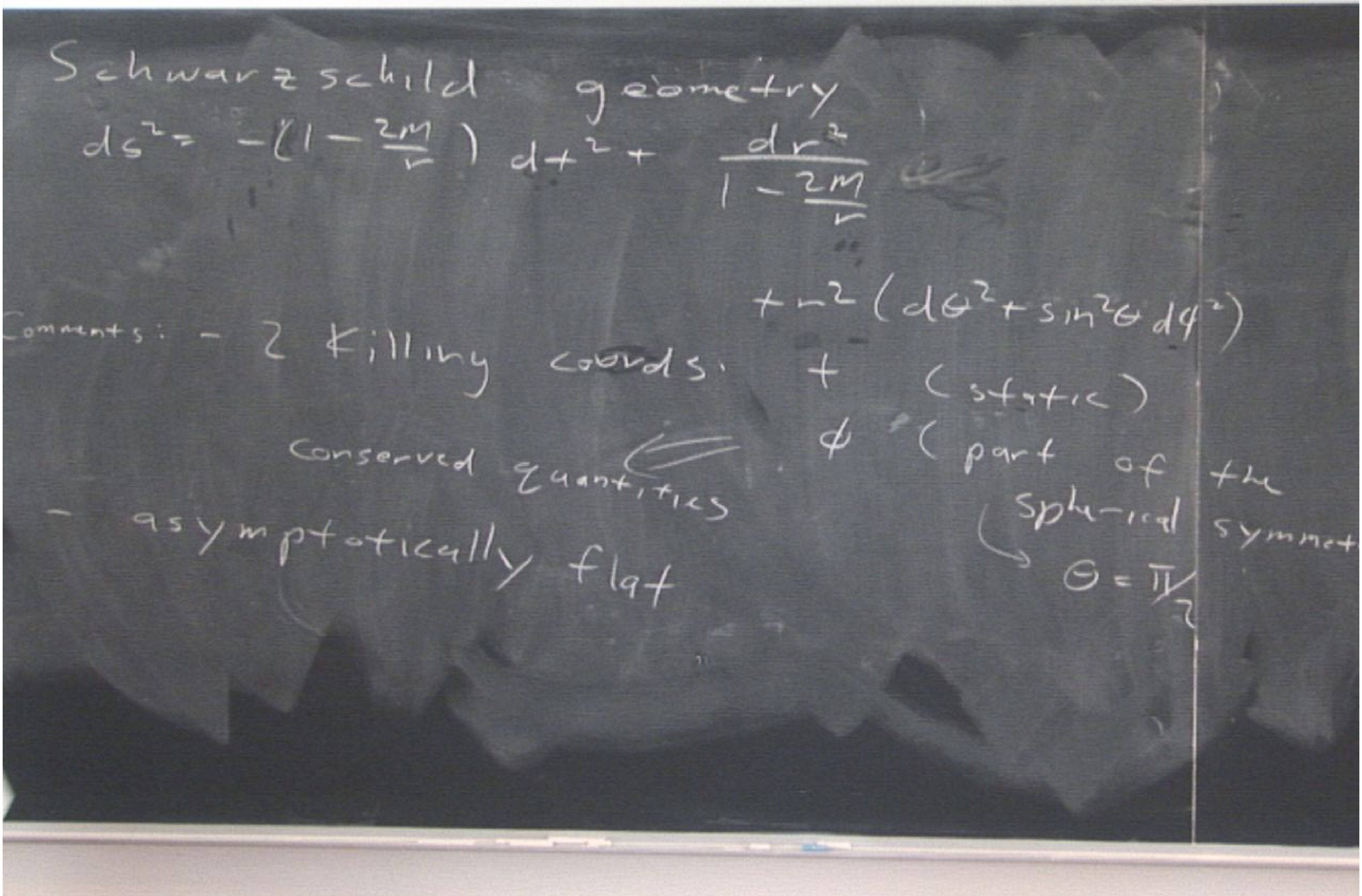

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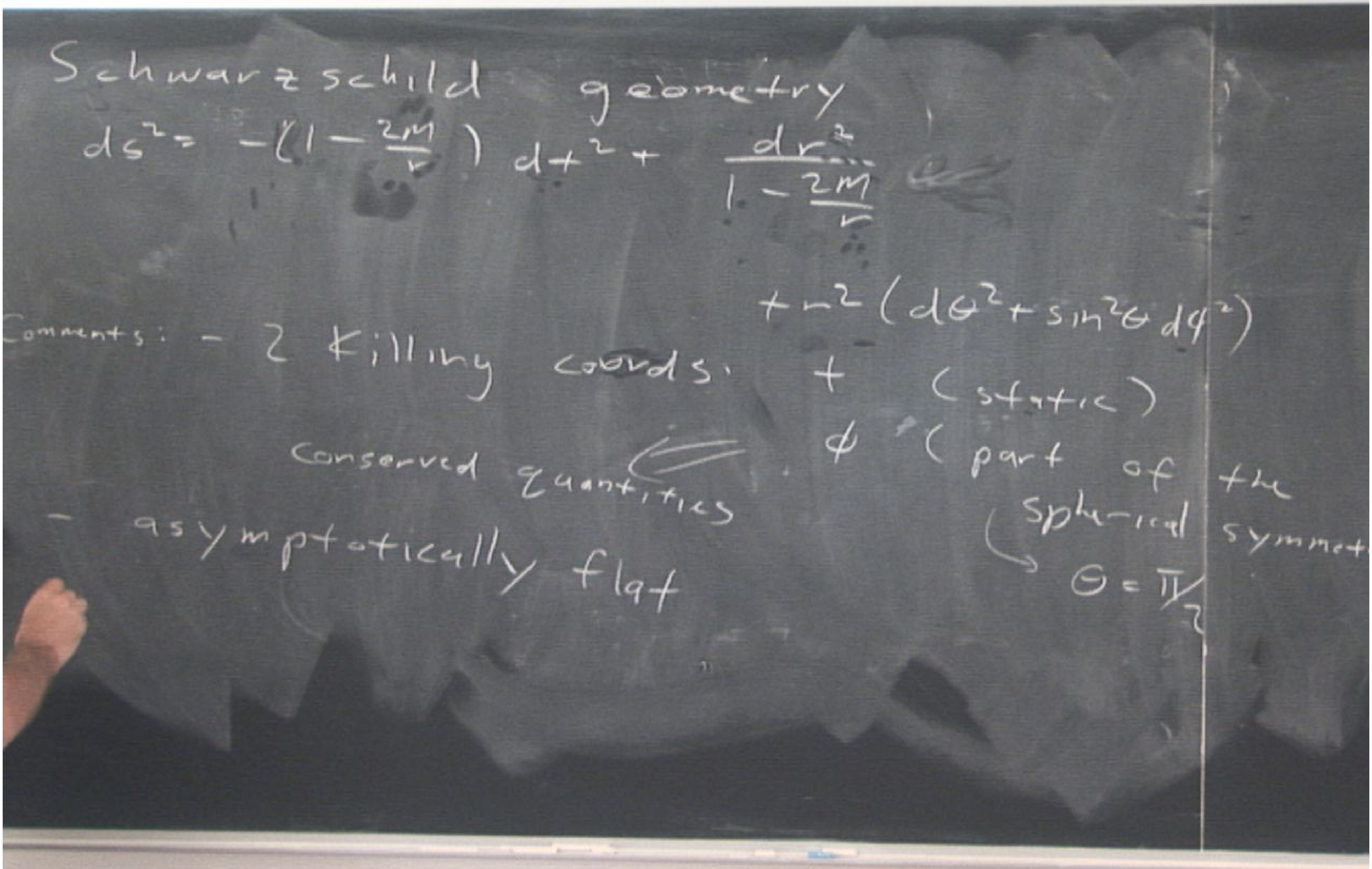
$\to \theta = \pi/2$

conserved quantities $\Leftarrow$

- asymptotically flat



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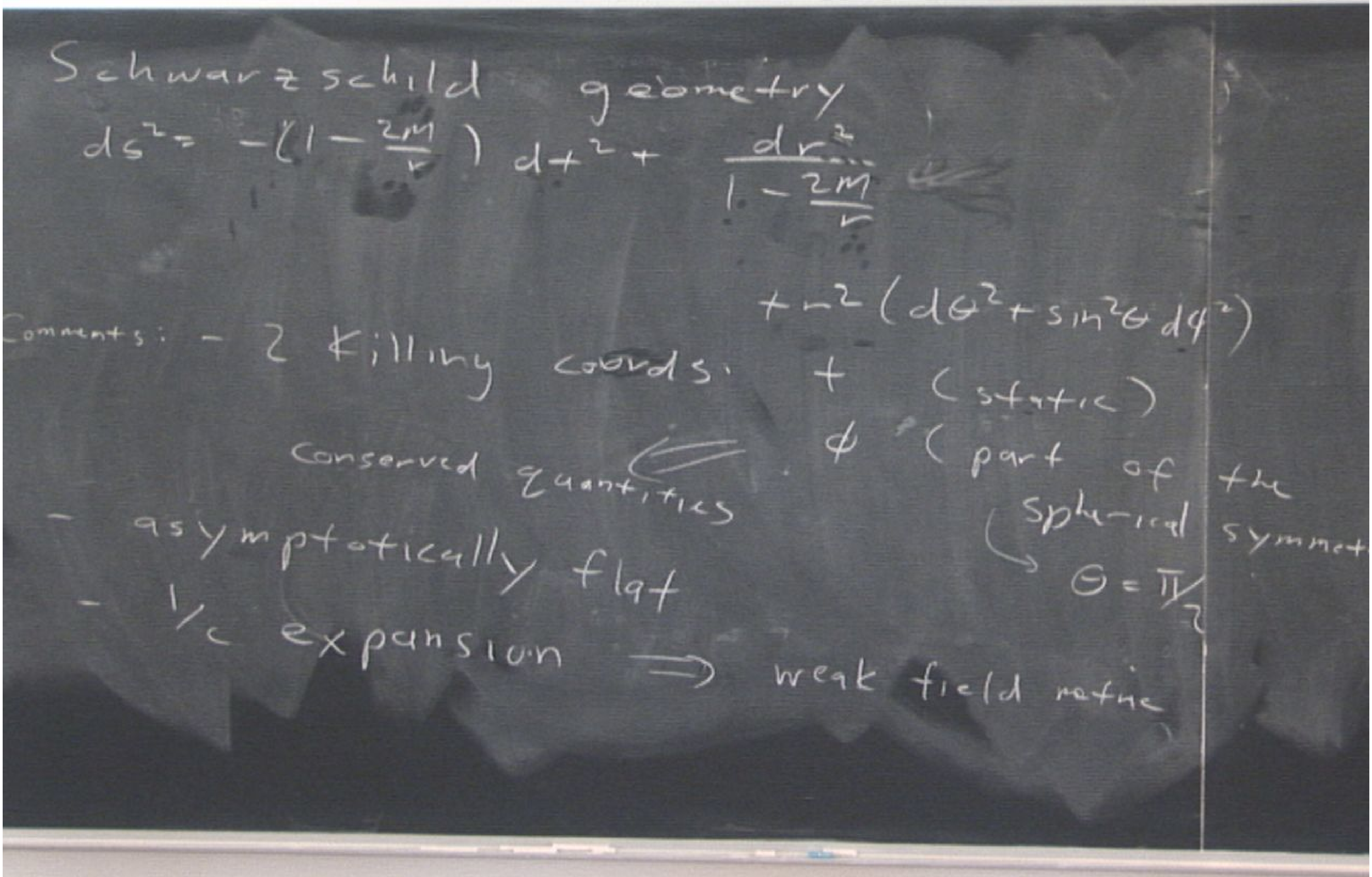

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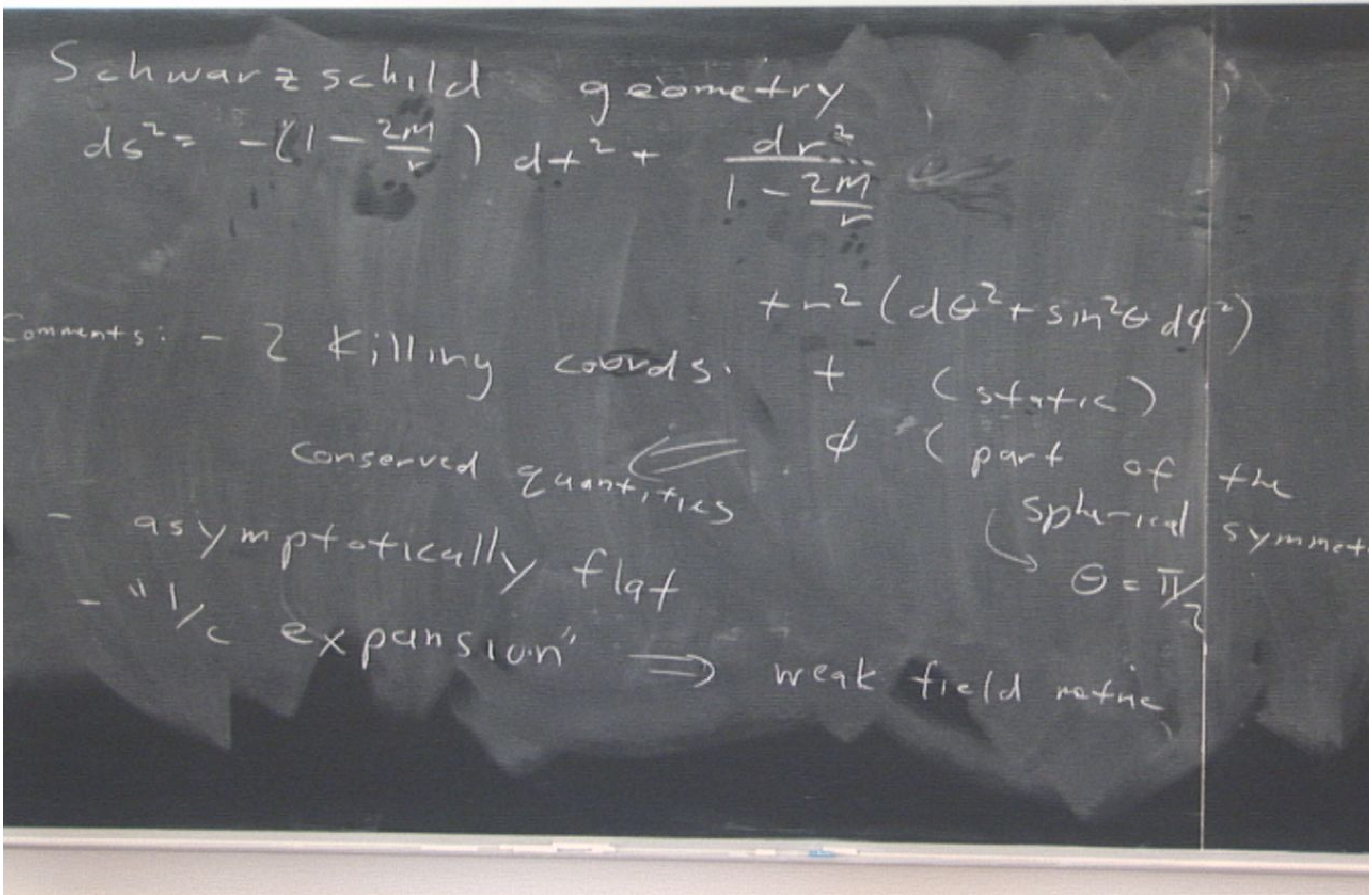

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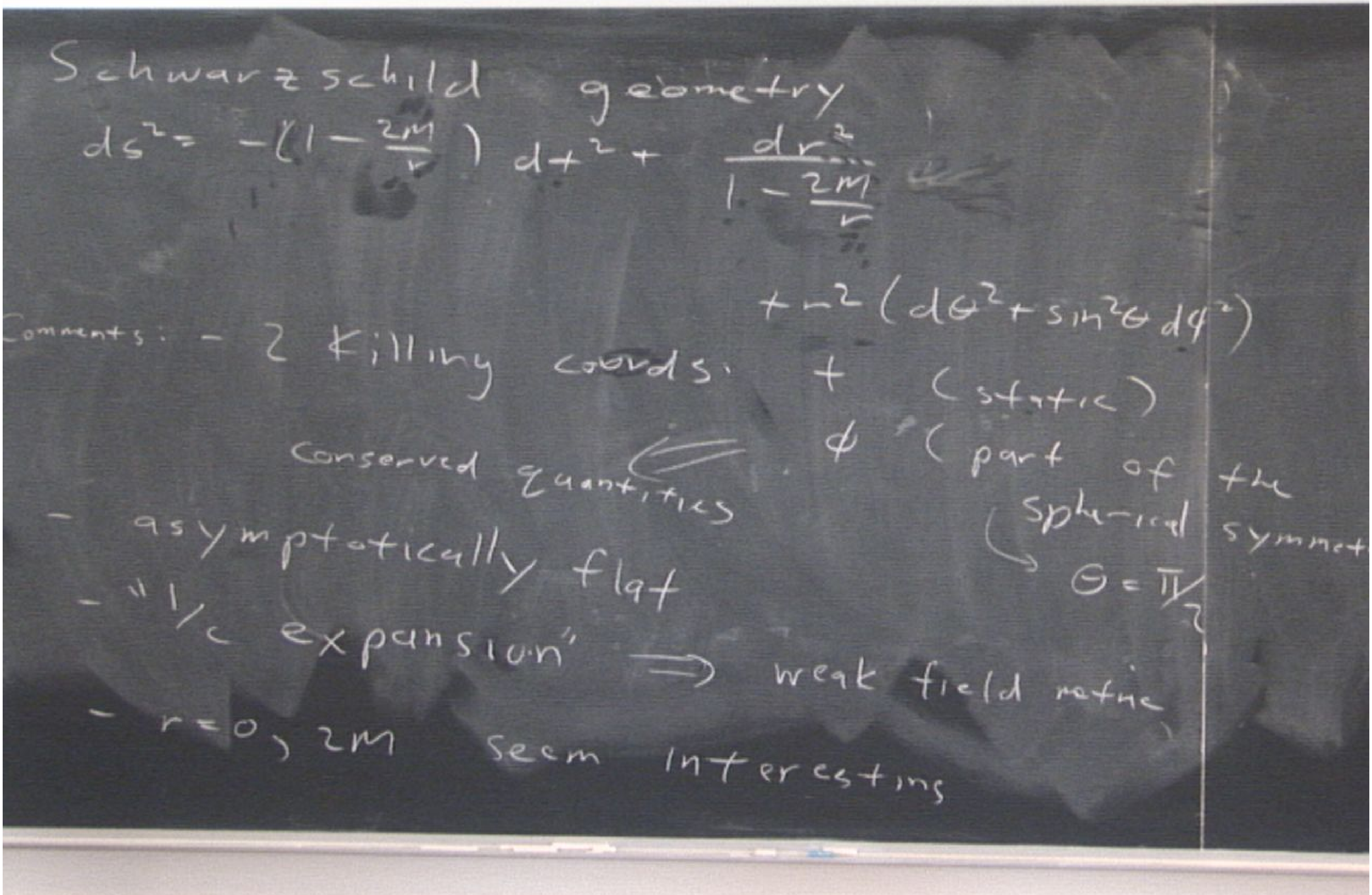

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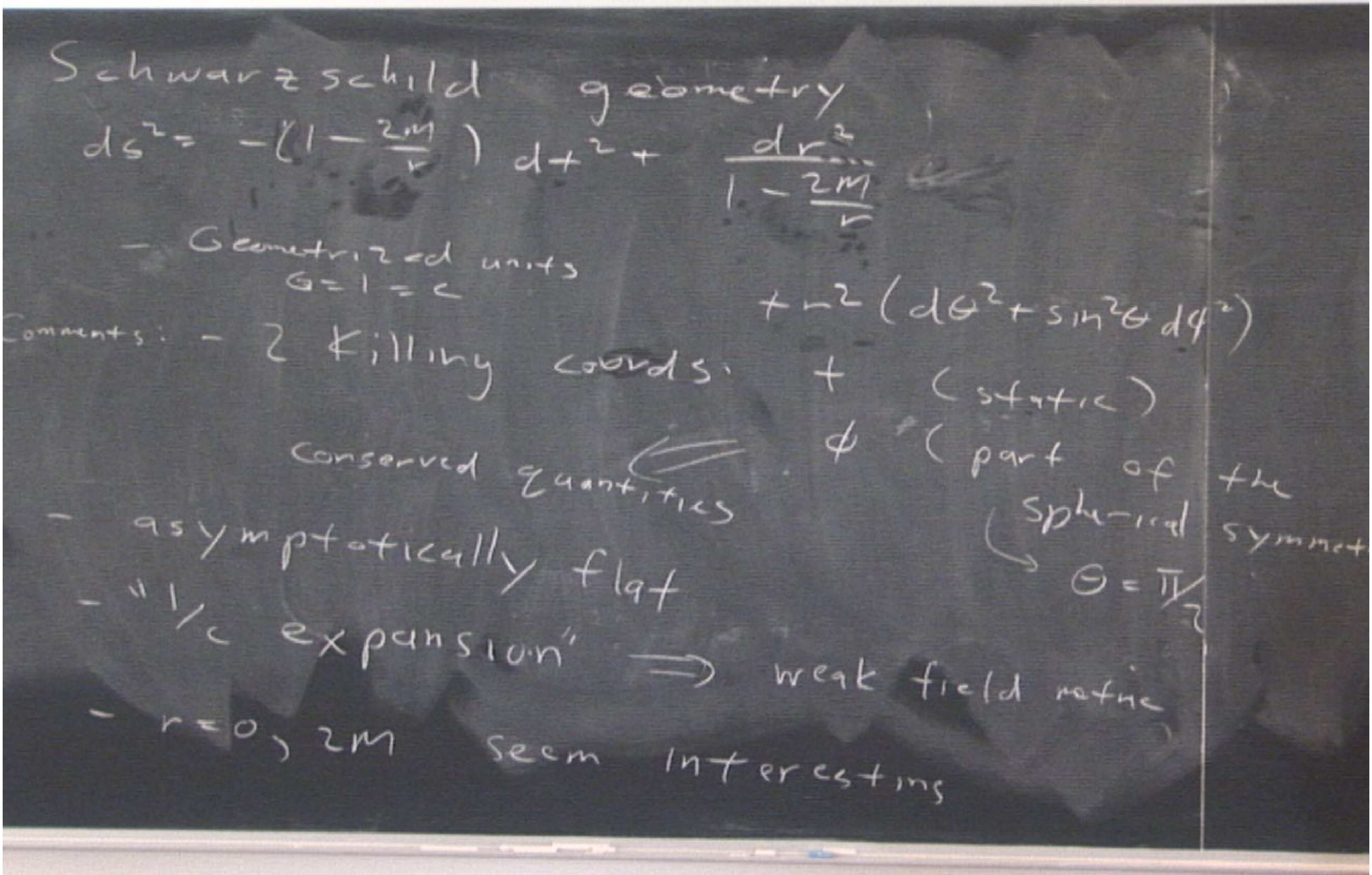

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Gravitational Red shift

- how does frequency measured at r_1 compare to that at r_2

$$\omega_1 / \omega_2 = ?$$

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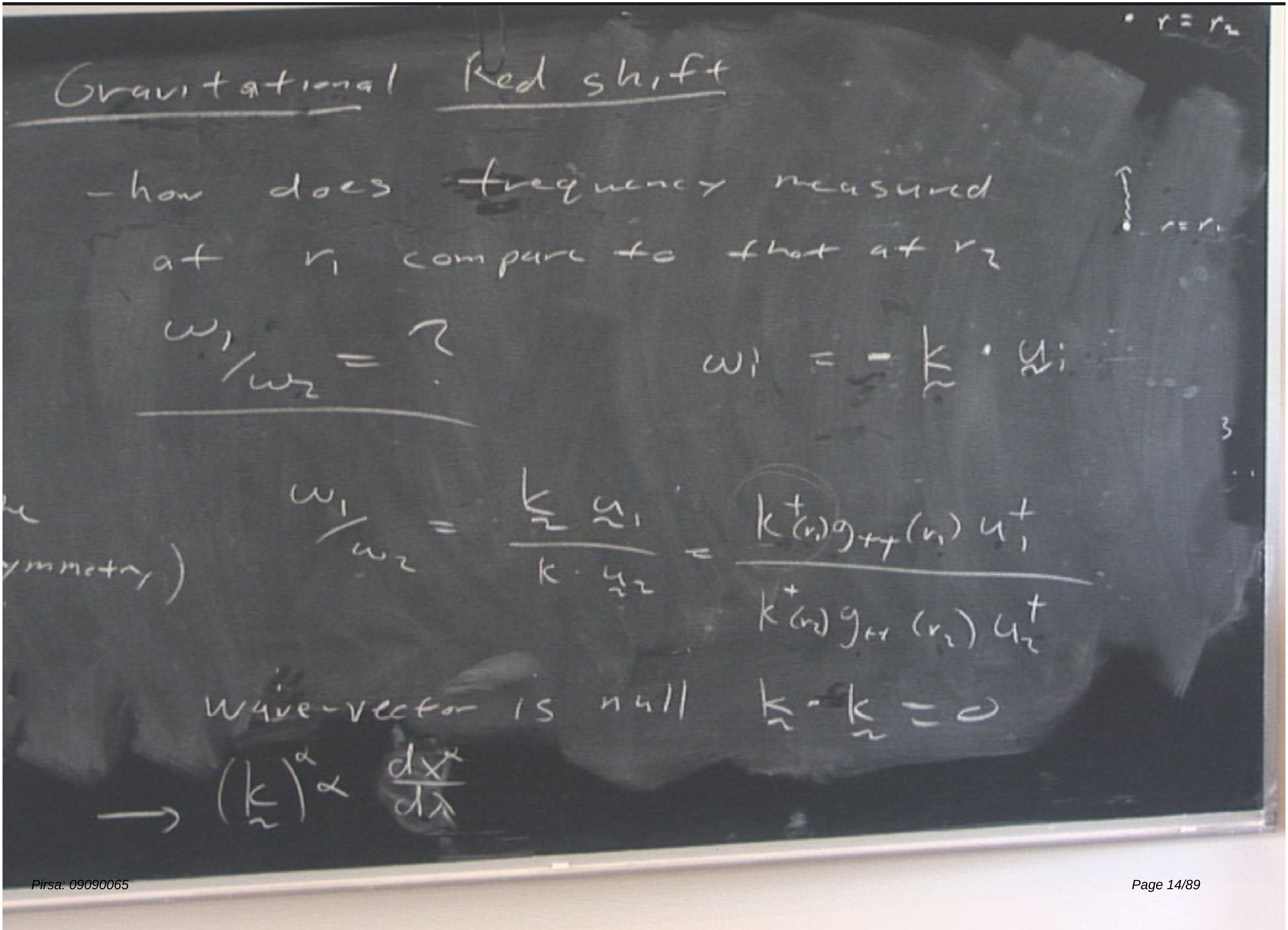
wave-vector is null $\underset{\sim}{k} \cdot \underset{\sim}{k} = 0$



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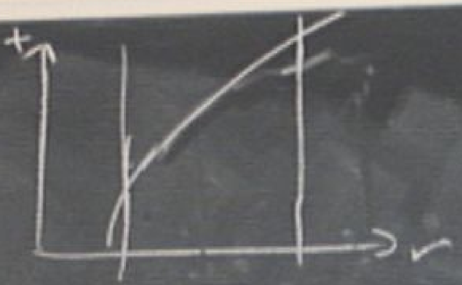

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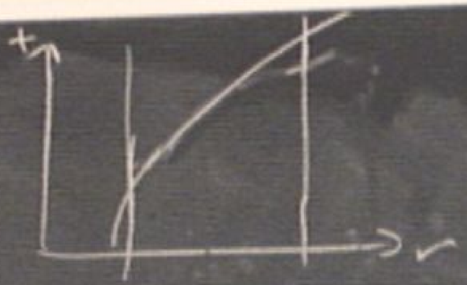

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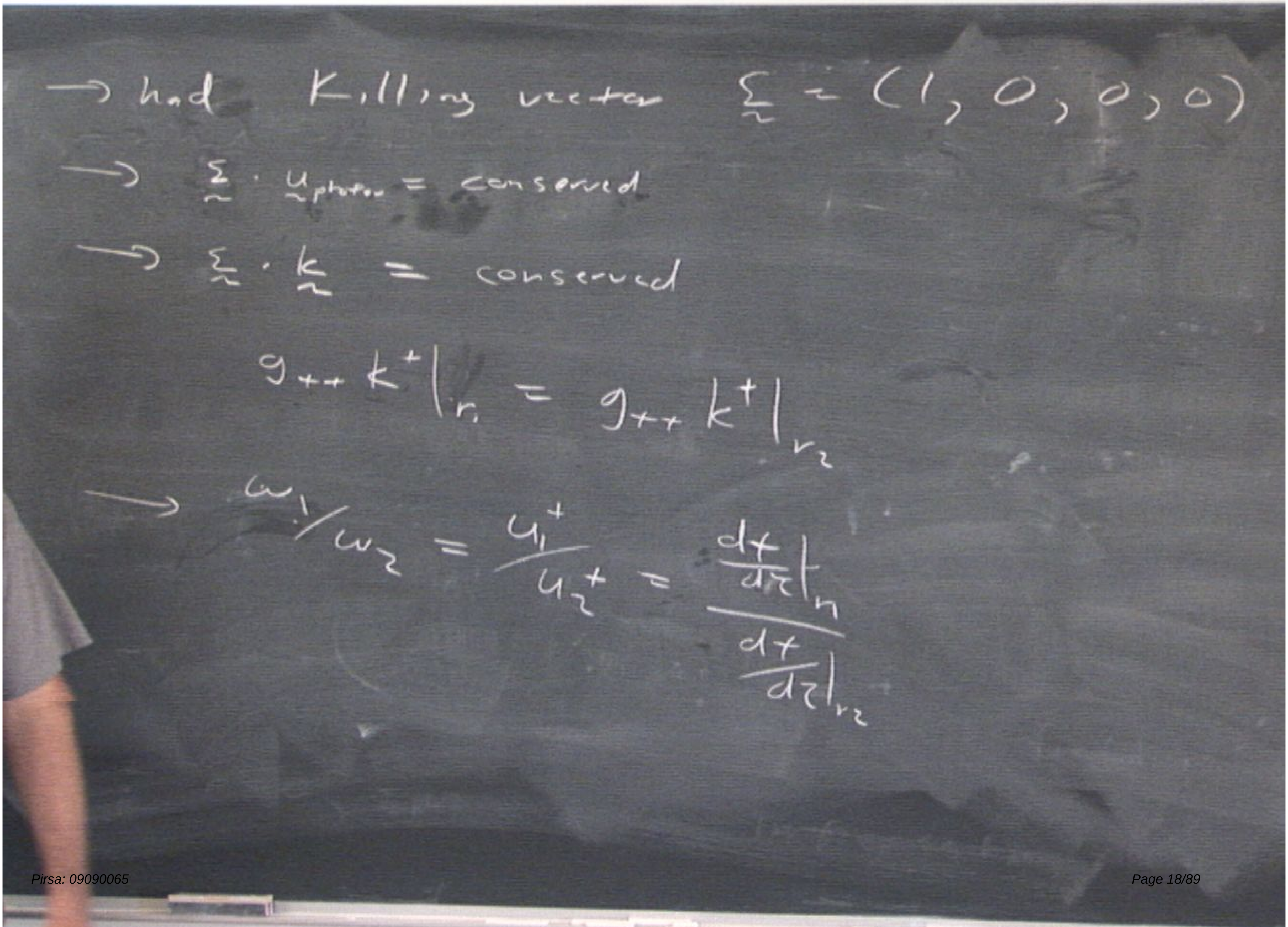
$\longrightarrow (\underset{\sim}{k})^\alpha \propto \frac{dx^\alpha}{d\lambda} \longrightarrow$ scale λ so $k^\alpha = \frac{dx^\alpha}{d\lambda}$

→ had Killing vector $\underset{\sim}{\xi} = (1, 0, 0, 0)$

→ $\underset{\sim}{k} \cdot \underset{\sim}{u}_{\text{photon}} =$ conserved



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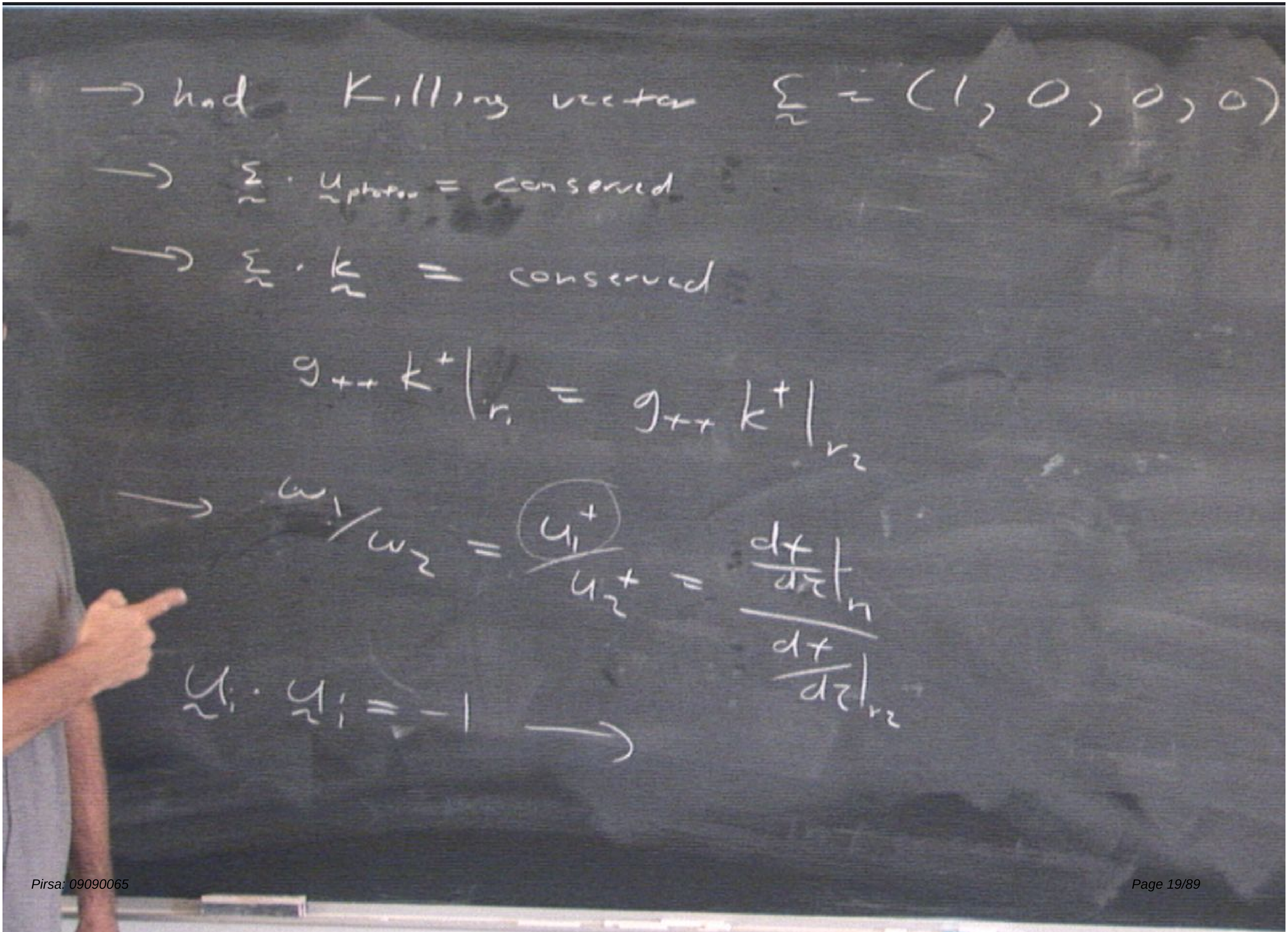
$$g_{tt} k^t \big|_{r_1} = g_{tt} k^t \big|_{r_2}$$

→ $\omega_1 / \omega_2 = \frac{u_1^t}{u_2^t} = \frac{dt/d\tau \big|_{r_1}}{dt/d\tau \big|_{r_2}}$



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$\underset{\sim}{u}_1 \cdot \underset{\sim}{u}_1 = -1$ →

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$u_i \cdot u_i = -1 \longrightarrow \longrightarrow u_1^t g_{tt} u_1^t = -1$

$u_1^a = (\alpha_1, 0, 0, 0) \nearrow \quad -(1 - \frac{2m}{r})\alpha^2 = -1 \nearrow \quad \alpha_1 = \frac{1}{\sqrt{1-2m/r_1}}$

...lling vector $\underline{\xi} = (1, 0, 0, 0)$

Gravi...

... = conserved

- how

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$$k^t|_{r_1} = g_{tt} k^t|_{r_2}$$

$$\nu_2 = \frac{\boxed{u_1^t}}{u_2^t} = \frac{\frac{dt}{d\tau}\Big|_{r_1}}{\frac{dt}{d\tau}\Big|_{r_2}} = \frac{\sqrt{1 - \frac{2M}{r_2}}}{\sqrt{1 - 2M/r_1}}$$

$$= -1 \longrightarrow \longrightarrow$$

$$u_1^t g_{tt} u_1^t = -1$$

$$\alpha(1, 0, 0, 0)$$

$$-\left(1 - \frac{2M}{r_1}\right)\alpha^2 = -1 \longrightarrow \alpha_1 = \frac{1}{\sqrt{1 - 2M/r_1}} = u_1^t \longrightarrow ($$

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conserved

conserved

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$\longrightarrow \quad \longrightarrow$

$0, 0, 0)$

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$$-\left(1-\frac{2M}{r_1}\right)\alpha^2 = -1 \nearrow \quad \alpha_1 = \frac{1}{\sqrt{1-2M/r_1}} = u_1^t \longrightarrow (\underset{\sim}{k})^\alpha \alpha$$

Gravitati

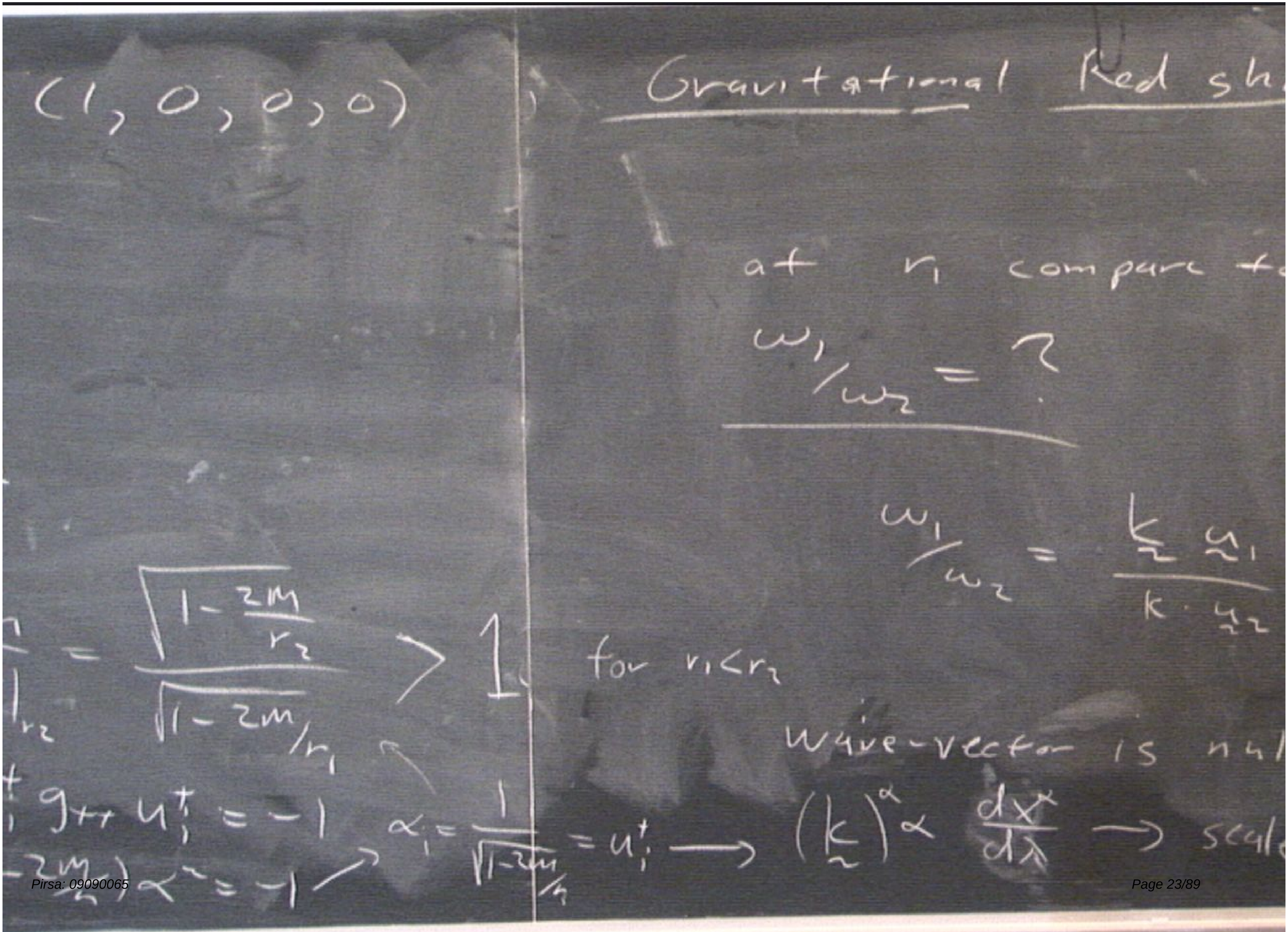
- how d

at r

ω_1 / ω_2

ω

wave-



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Gravitational Red sh

at r_1 compare t

$$\omega_1 / \omega_2 = ?$$

$$\omega_1 / \omega_2 = \frac{\underset{\sim}{k} \cdot \underset{\sim}{u}_1}{k \cdot \underset{\sim}{u}_2}$$

$$= \frac{\sqrt{1 - \frac{2M}{r_2}}}{\sqrt{1 - 2M/r_1}} > 1 \quad \text{for } r_1 < r_2$$

r_2

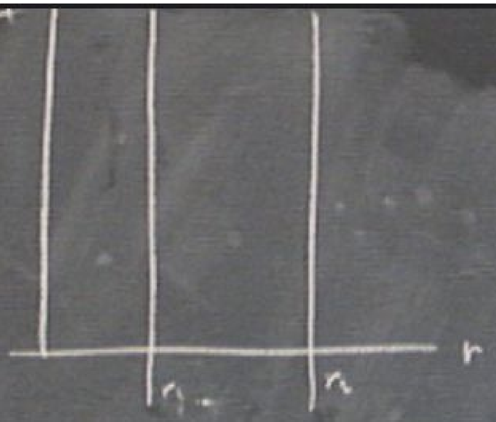
wave-vector is nul

$$g_{tt}\, u^t_i = -1$$

$$\alpha_i = \frac{1}{\sqrt{1 - 2M/r_i}} = u^t_i \longrightarrow (\underset{\sim}{k})^\alpha \frac{dx^\alpha}{d\lambda} \longrightarrow \text{scal}$$

$$-\frac{2M}{r})\alpha^2 = -1 \nearrow$$

Gravitational Red shift



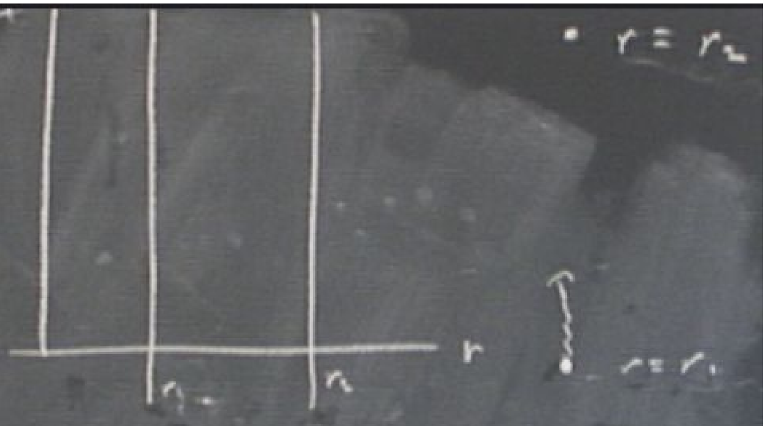
$\longrightarrow$ receiver appropriate weak field formula when $M/r_i \ll 1$

$\longrightarrow$

for $r_1 < r_2$

$\frac{}{M/r_1} = u^t_i \longrightarrow$

Gravitational Red shift



→ recover appropriate weak field formula when $M/r_i \ll 1$

→ but gives exact results when $M/r_i \sim 1$

→ note: $r_1 \to 2M$

for $r_1 < r_2$

$= u^t_i \longrightarrow$

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or $\omega_2/\omega_1 \to 0$ ie infinite redshift

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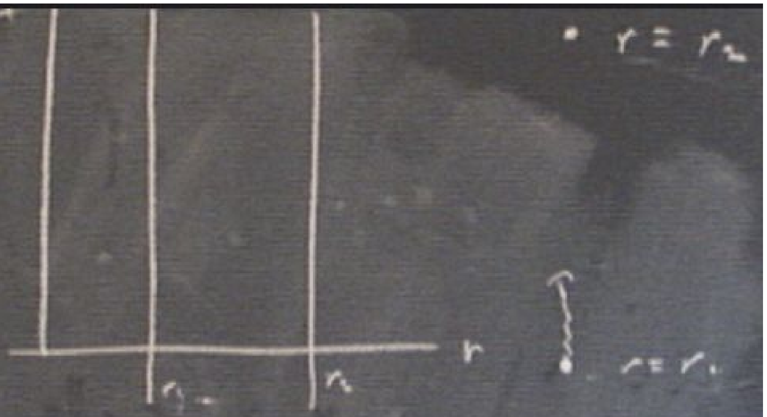

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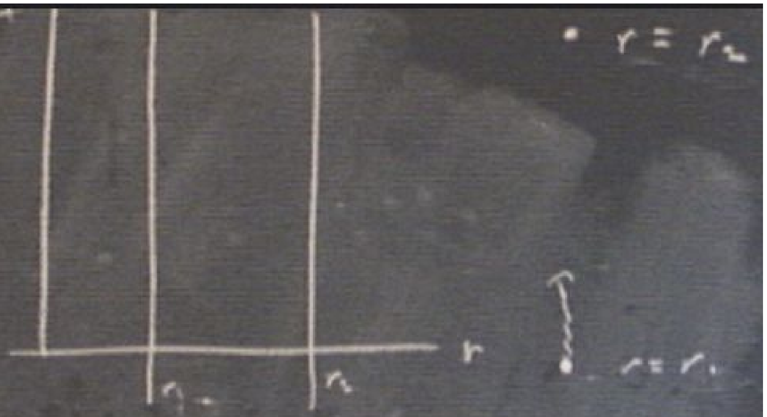

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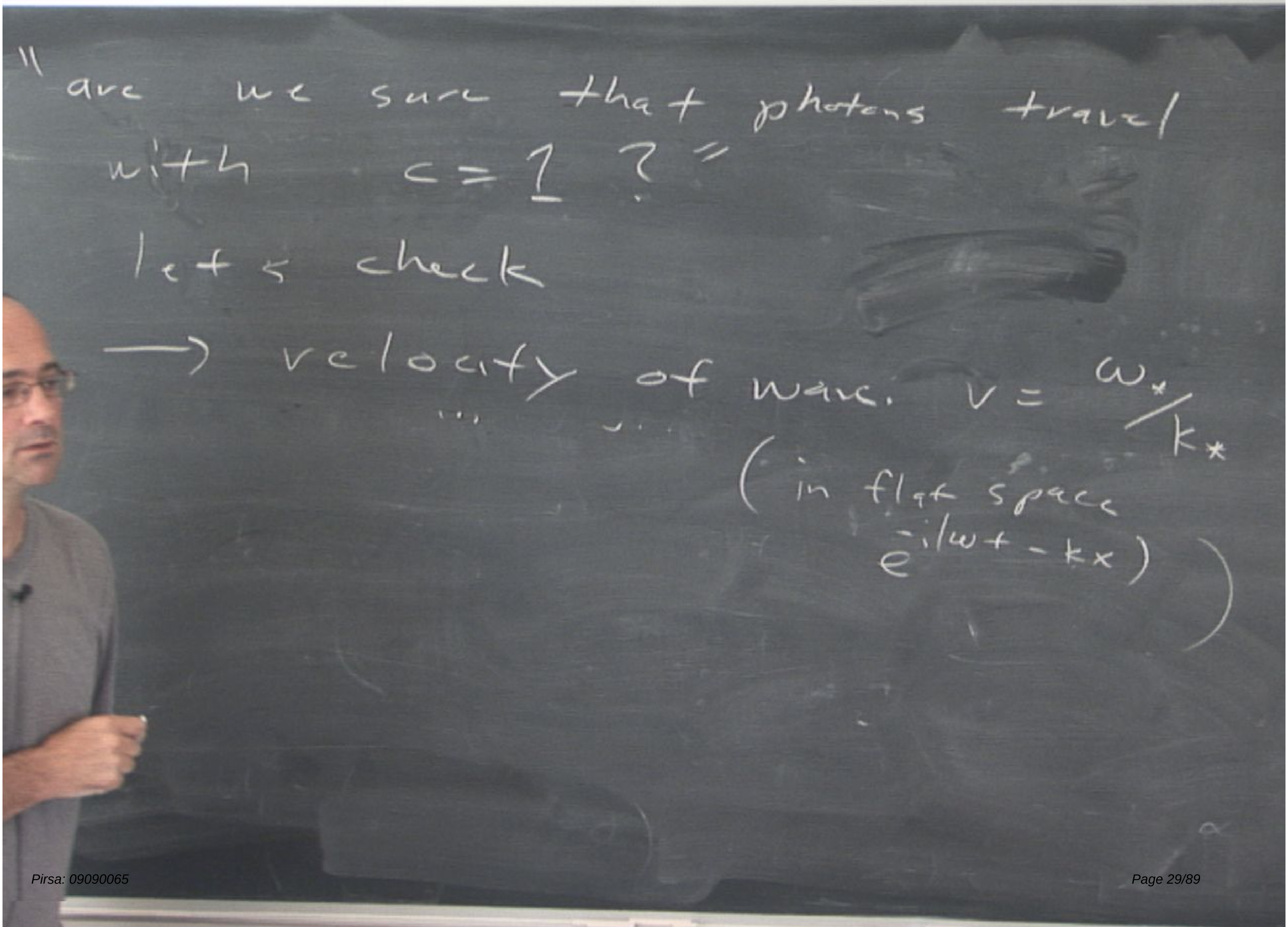
or $\omega_2/\omega_1 \longrightarrow 0$ ie infinite redshift

⟶ for $r_1 < 2M \longrightarrow$ formula doesn't make sense – return later

$= u^t_i \longrightarrow$

M/r_i

" are we sure that photons travel with $c = 1$? "

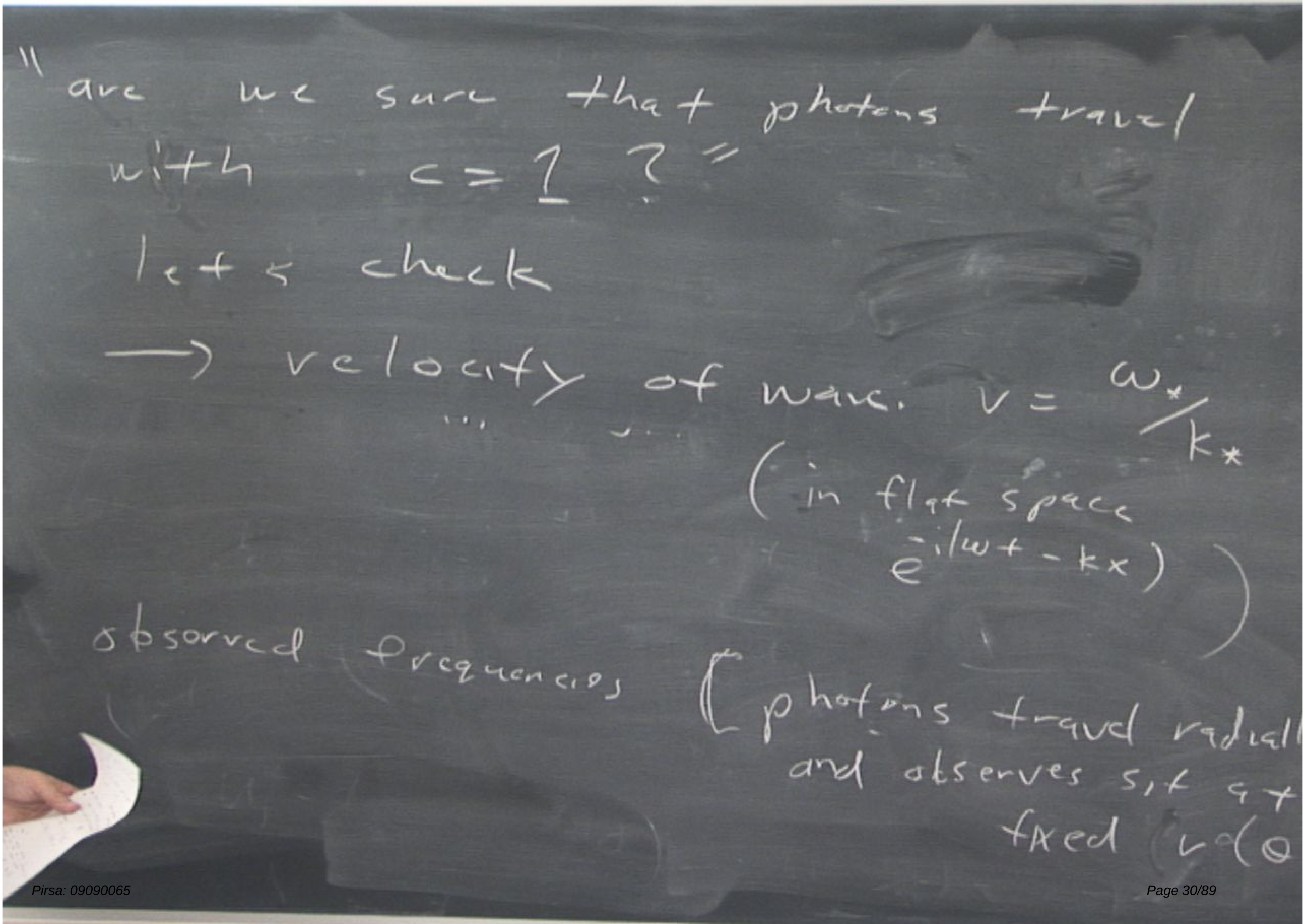


" are we sure that photons travel with $c = \underline{1}$? "

let's check

$\longrightarrow$ velocity of wave. $v = \omega_* / k_*$

(in flat space $e^{-i(\omega t - kx)}$)



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observed frequencies (photons travel radiall and observes s,t at fixed r,(θ

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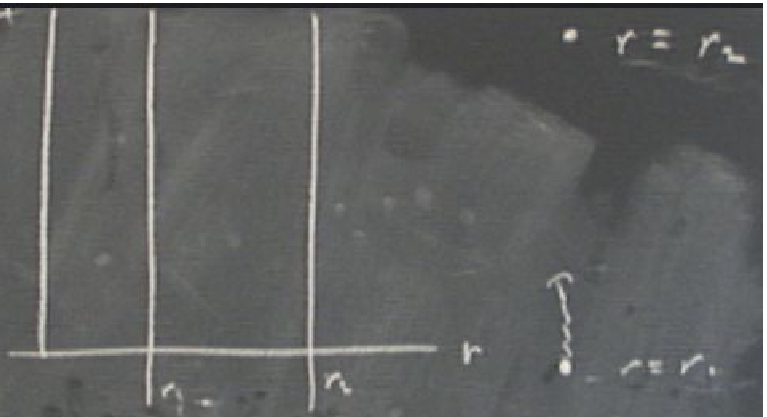

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$\longrightarrow$ for $r_1 < 2M \longrightarrow$ formula doesn't make sense – return later

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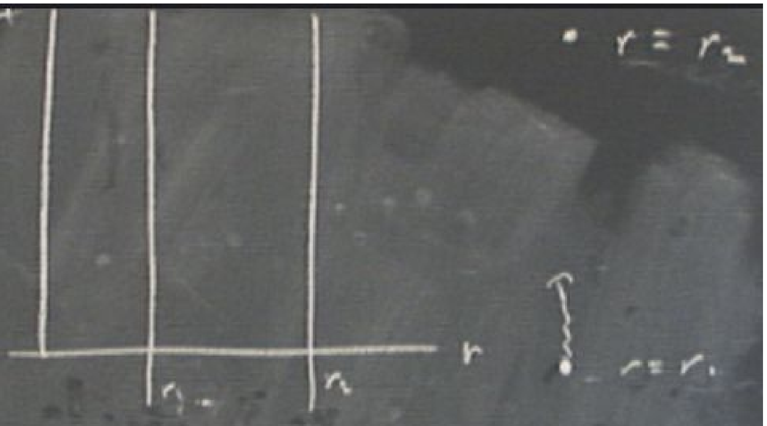

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$d\phi)_0]$

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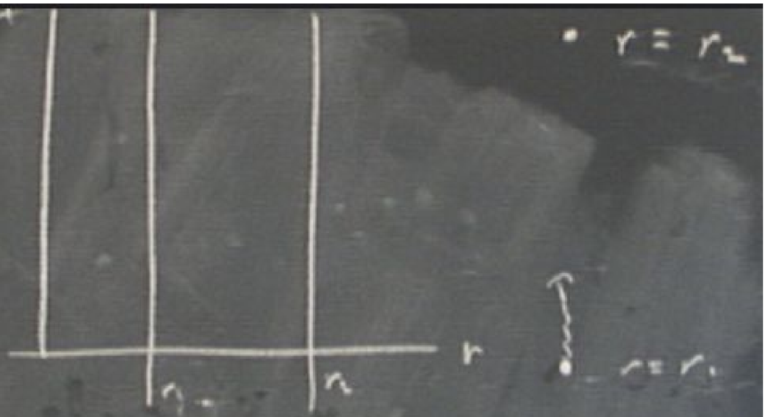

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[illegible]

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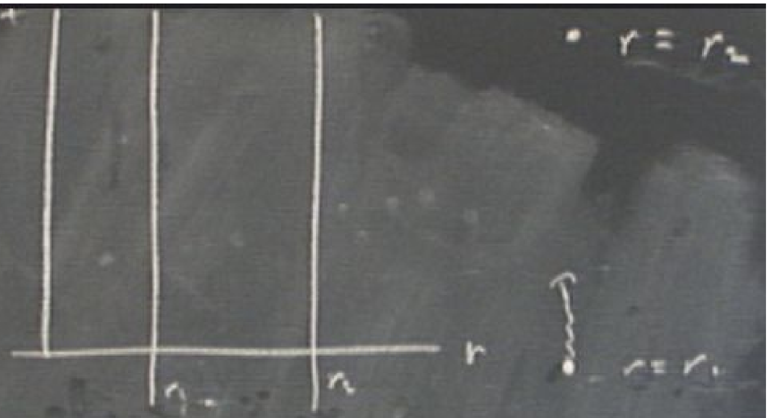
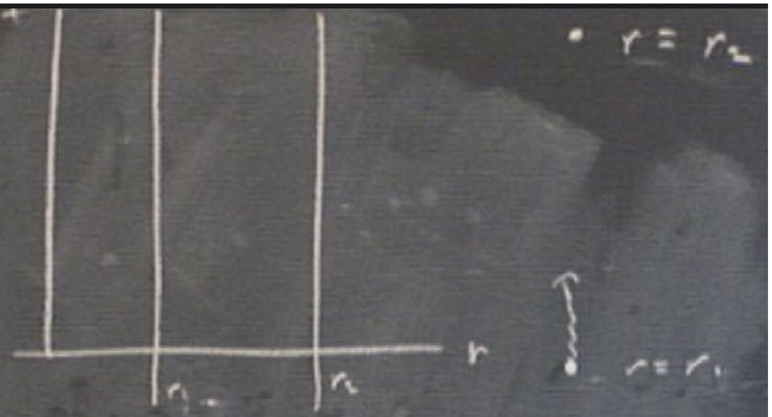


$$\omega_* = k^t \sqrt{1 - 2M/r}$$

Gravitational Red shift



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what is k_*?

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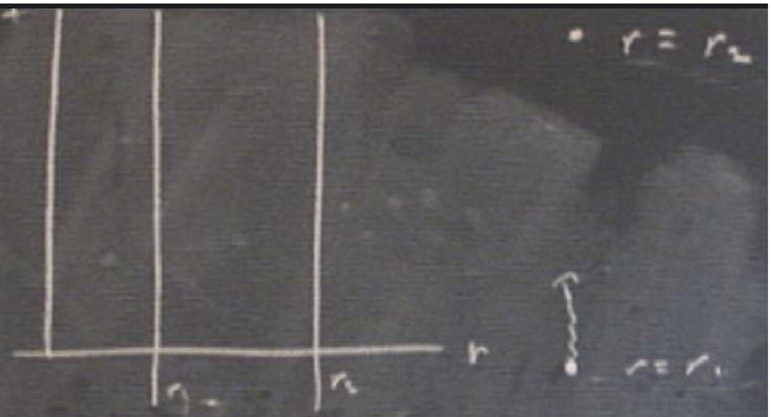

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$$\omega_* = k^t\sqrt{1-2M/r}$$

what is k_*?

$$k_* = \underset{\sim}{k}\cdot\underset{\sim}{E}'$$

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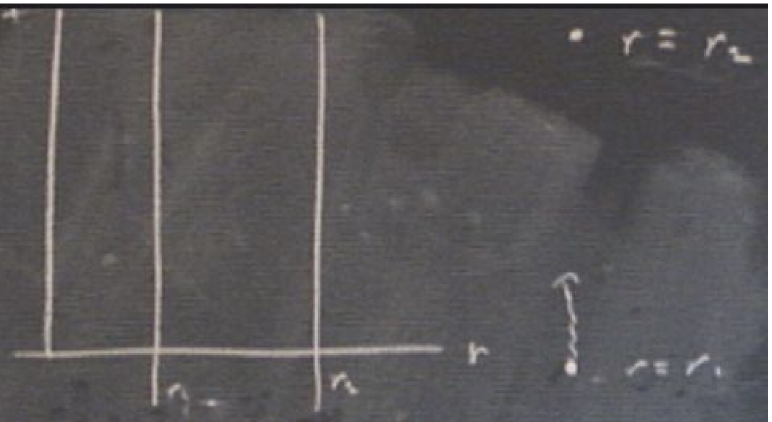
$$= - \underset{\sim}{k} \cdot \underset{\sim}{E}^0$$

Gravitational Red shift



$$\underset{\sim}{u}\cdot\underset{\sim}{u} = -1$$

$$\longrightarrow \quad g_{tt}\, {u^{t}}^{2} = -1$$

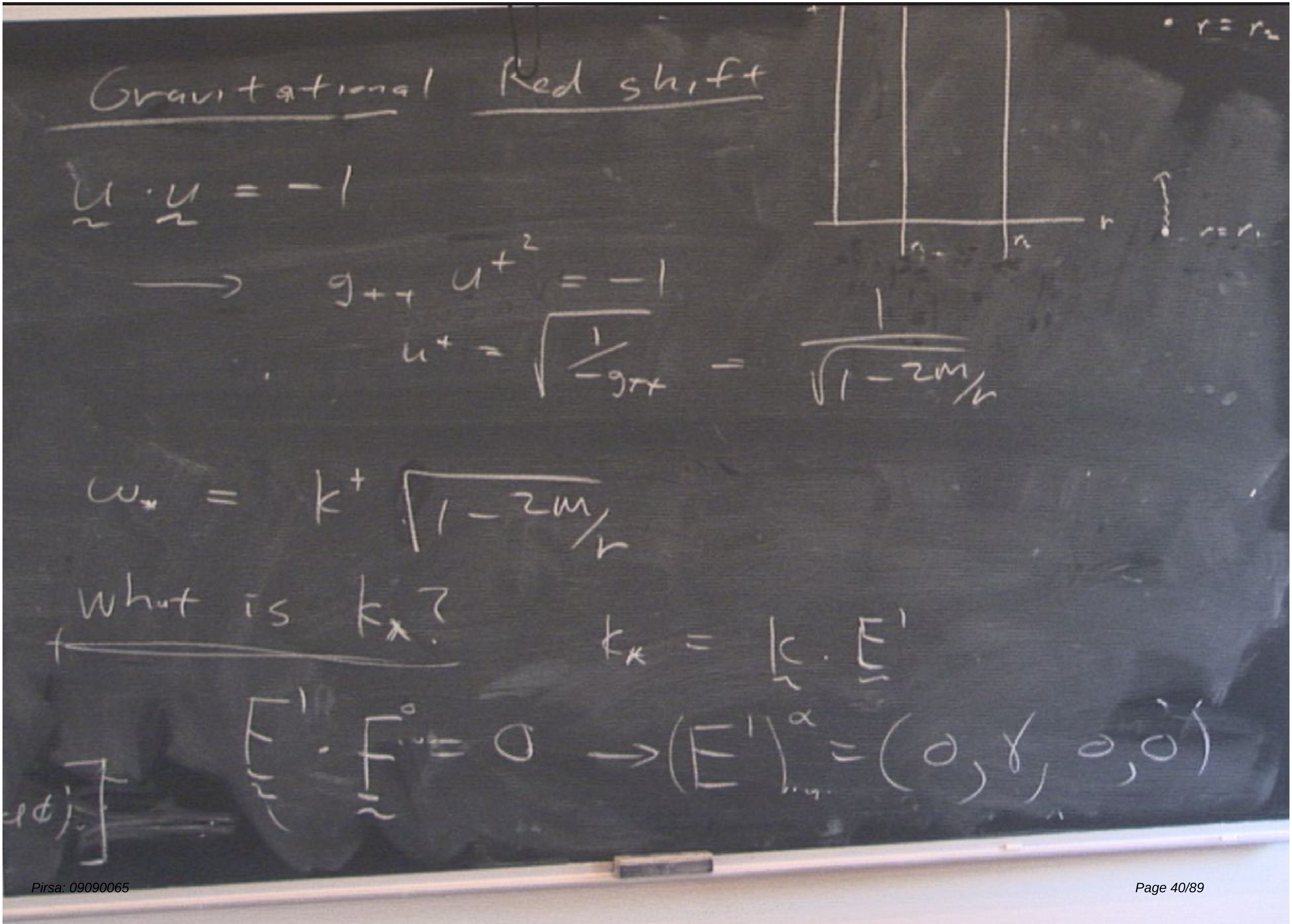
$$u^{t} = \sqrt{\frac{1}{-g_{tt}}} = \frac{1}{\sqrt{1-2M/r}}$$

$$\omega_{*} = k^{t}\sqrt{1-2M/r}$$

what is k_{*}?

$$k_{*} = \underset{\sim}{k}\cdot\underset{\sim}{E}^{1}$$

$$\underset{\sim}{E}^{1}\cdot\underset{\sim}{E}^{0} = 0$$



Gravitational Red shift

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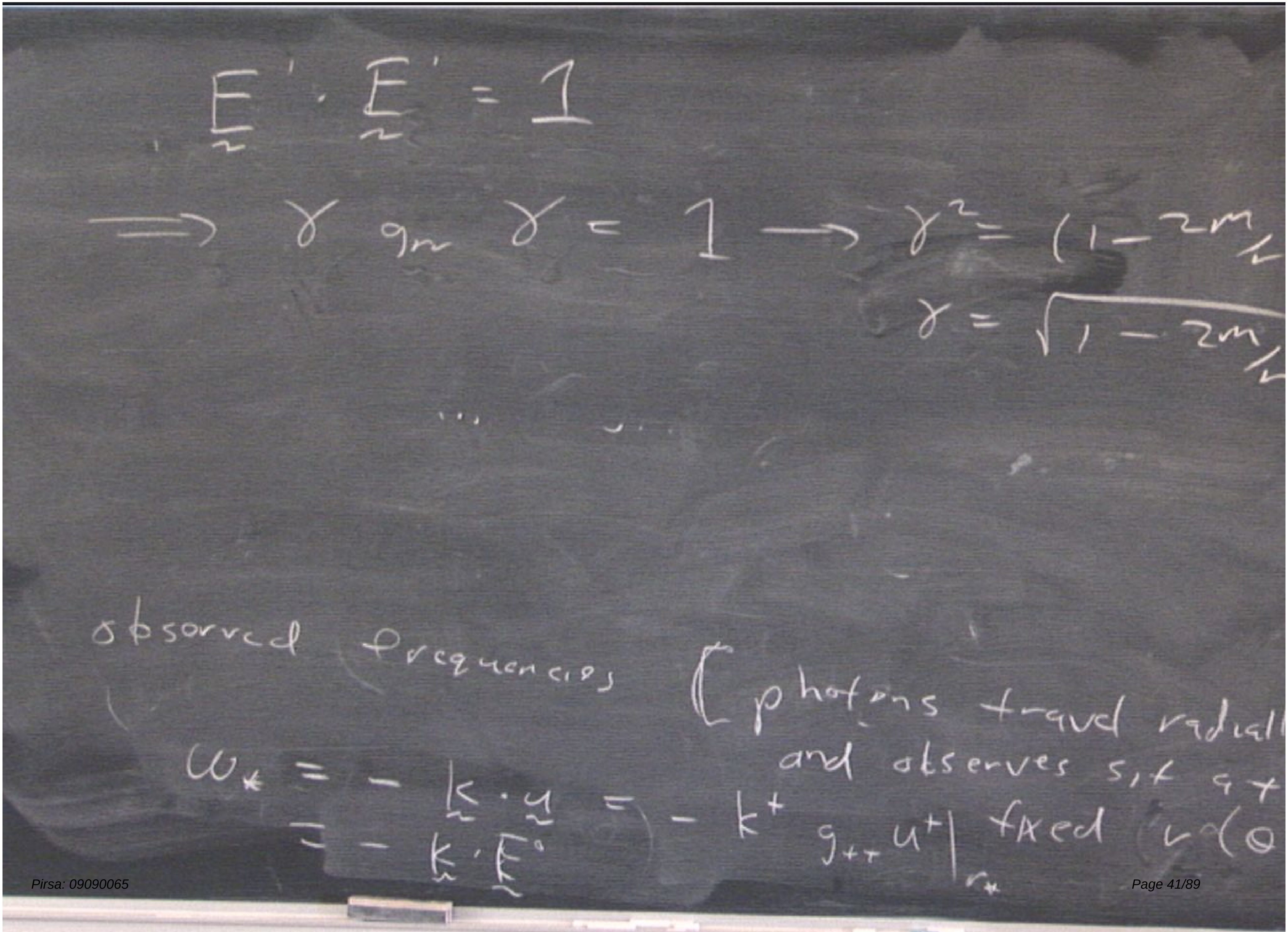
$$\omega_* = k^t \sqrt{1-2M/r}$$

What is k_*?

$$k_* = \underset{\sim}{k} \cdot \underset{\sim}{E}'$$

$$\underset{\sim}{E}^1 \cdot \underset{\sim}{E}^0 = 0 \longrightarrow (E^1)^\alpha = (0, \gamma, 0, 0)$$



$$\underset{\sim}{E}' \cdot \underset{\sim}{E}' = 1$$

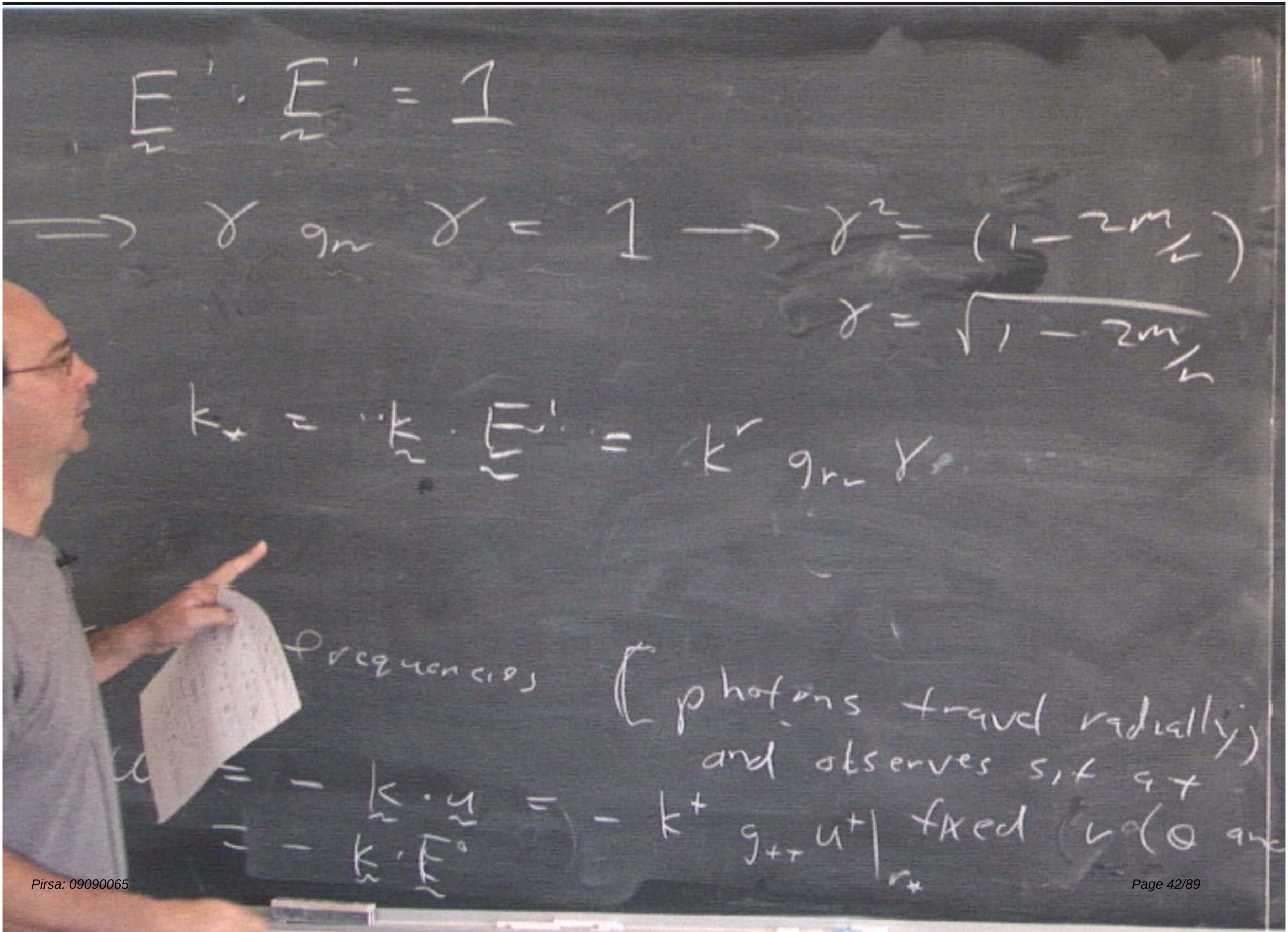
$$\Longrightarrow \gamma \; g_{\mu\nu} \; \gamma = 1 \longrightarrow \gamma^2 = (1 - 2m/r$$

$$\gamma = \sqrt{1 - 2m/r}$$

observed frequencies [photons travel radiall
and observes s.t θ +

$$\omega_* = -\underset{\sim}{k} \cdot \underset{\sim}{u} = -k^t \; g_{tt} \, u^t \Big|_{r_*} \text{ fixed } (r, (\theta$$

$$= -\underset{\sim}{k} \cdot \underset{\sim}{E}^{0}$$



$$\underset{\sim}{E}^1 \cdot \underset{\sim}{E}^1 = 1$$

$$\Longrightarrow \gamma \, g_{rr} \, \gamma = 1 \longrightarrow \gamma^2 = \left(1 - {}^{2m}/_{r}\right)$$

$$\gamma = \sqrt{1 - {}^{2m}/_{r}}$$

$$k_* = \underset{\sim}{k} \cdot \underset{\sim}{E}^1 = k^r \, g_{rr} \, \gamma$$

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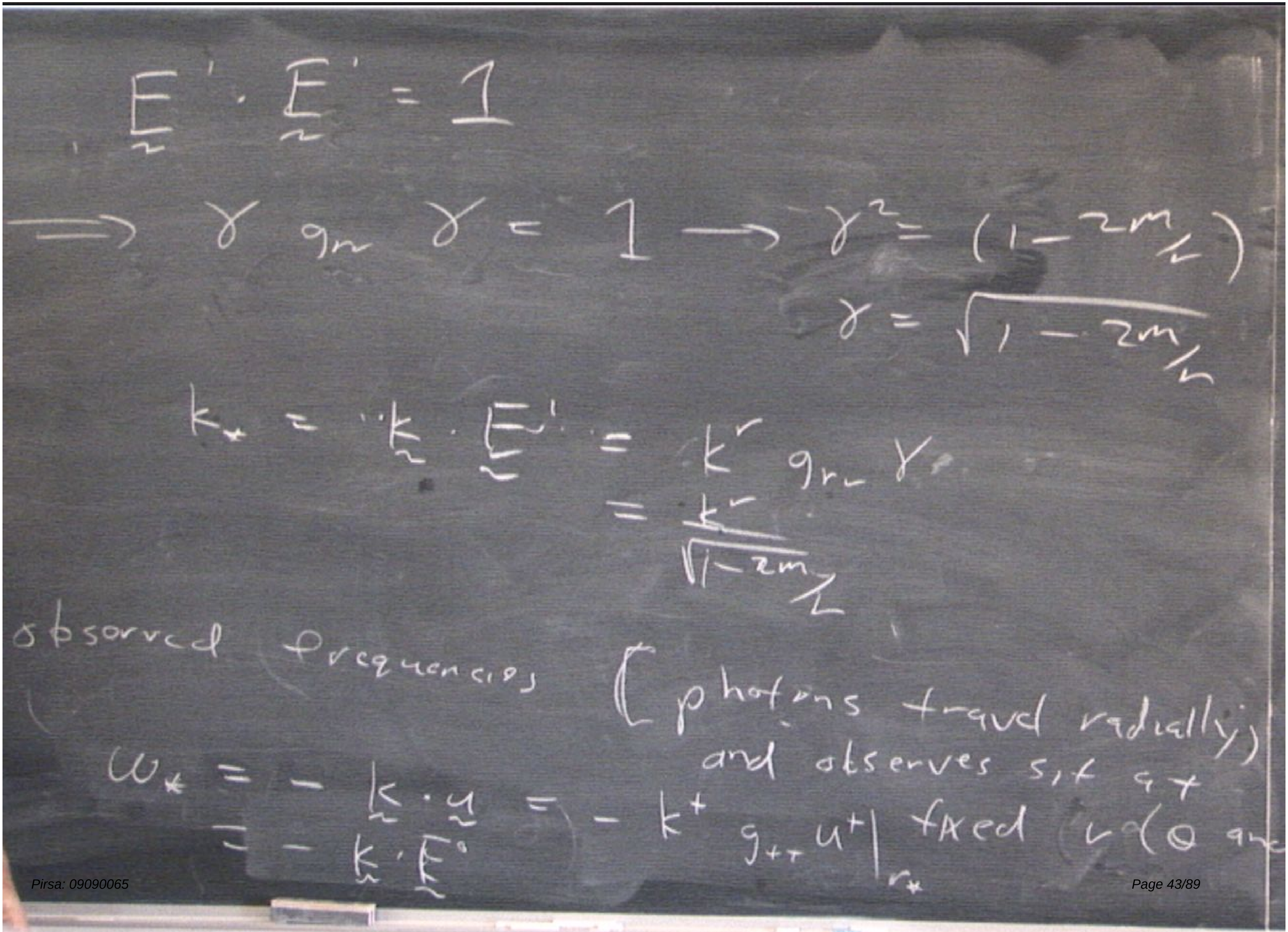

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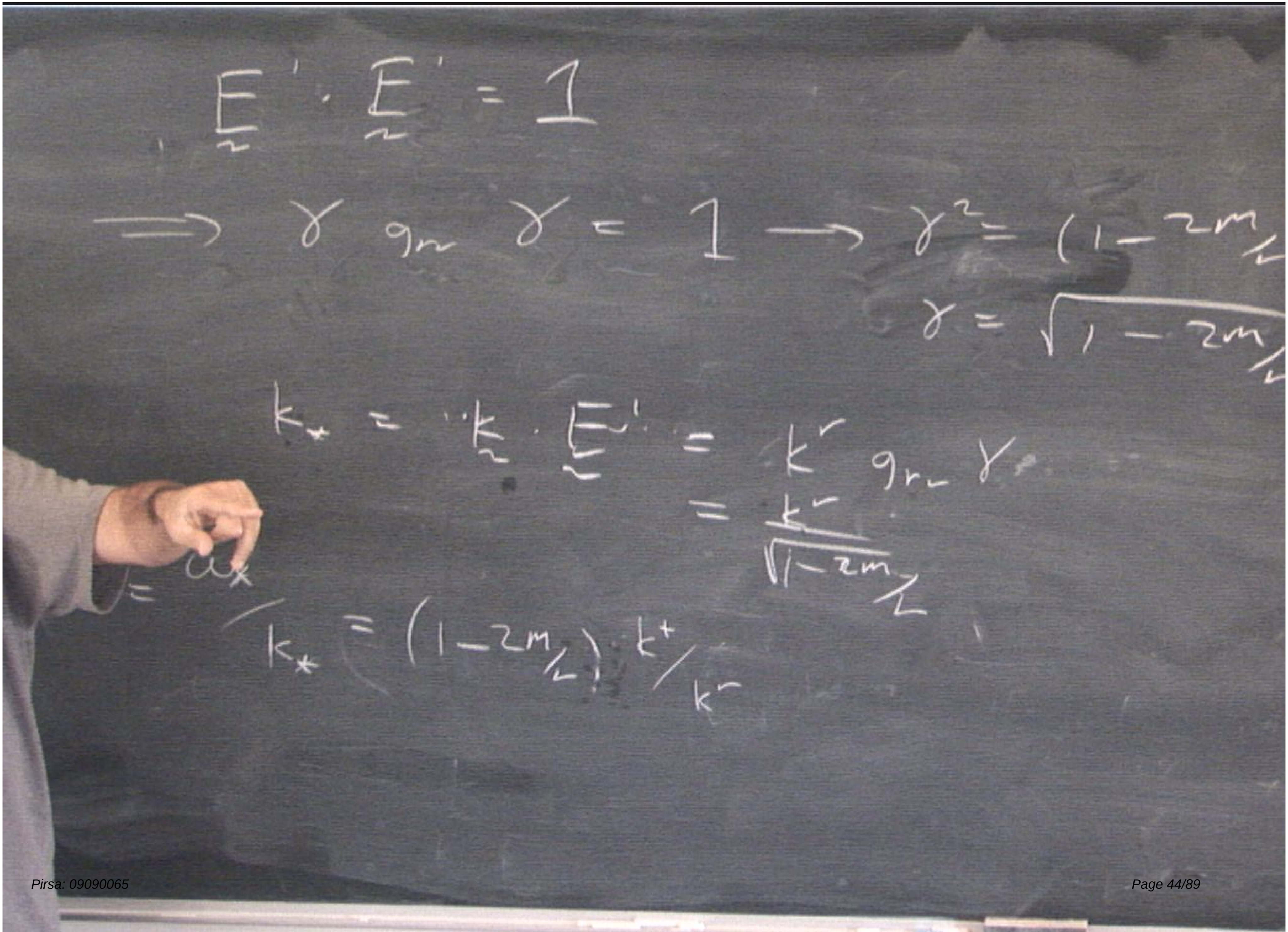

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$$k_* = \underset{\sim}{k} \cdot \underset{\sim}{E}' = k^r g_{rr} \gamma$$

$$= \frac{k^r}{\sqrt{1 - 2m/r}}$$

$$= \omega_* / k_* = (1 - 2m/r) \, k^t / k^r$$

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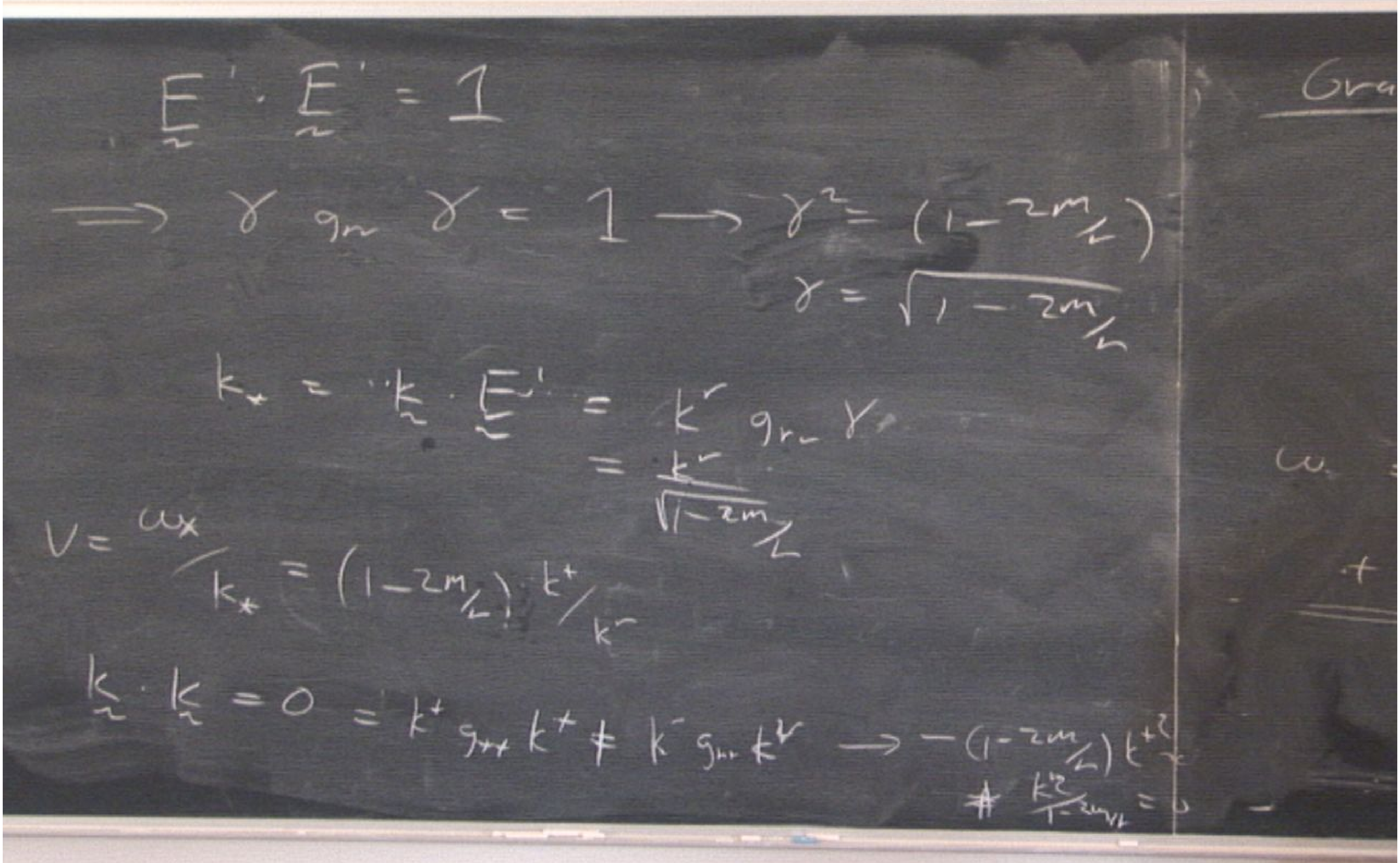
$$V = {}^{\omega_x}/_{k_*} = \left(1 - {}^{2m}/_{r}\right) {}^{k^t}/_{k^r}$$

$$\underset{\sim}{k} \cdot \underset{\sim}{k} = 0 = k^t g_{tt} k^t + k^r g_{rr} k^r$$



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Gra

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$$k_* = \underset{\sim}{k} \cdot \underset{\sim}{E}' = k^r \, g_{rr} \, \gamma$$

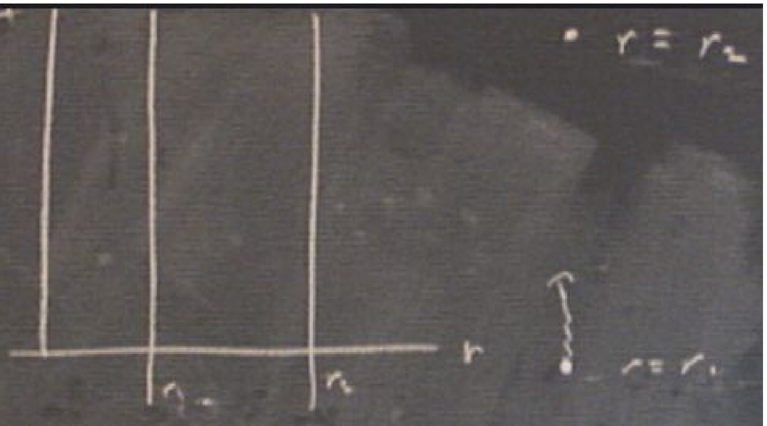
$$= \frac{k^r}{\sqrt{1 - {}^{2m}/_{r}}}$$

$$V = {}^{\omega_*}/_{k_*} = \left(1 - {}^{2m}/_{r}\right) {}^{k^+}/_{k^r}$$

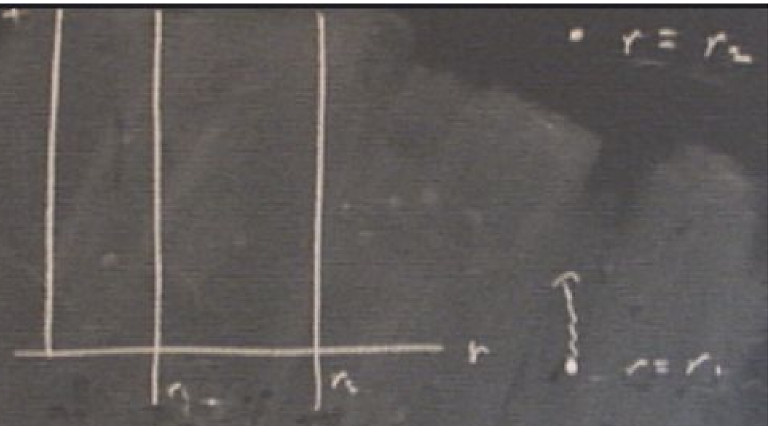
$$\underset{\sim}{k} \cdot \underset{\sim}{k} = 0 = k^+ g_{++} k^+ + k^r g_{rr} k^r \longrightarrow -\left(1 - {}^{2m}/_{r}\right) k^{+2} + \frac{k^{r2}}{1 - {}^{2m}/_{r}} = 0$$

Gravitational Red shift

$$k^r = \left(1 - \frac{2m}{r}\right) k^t$$



Gravitational Red shift

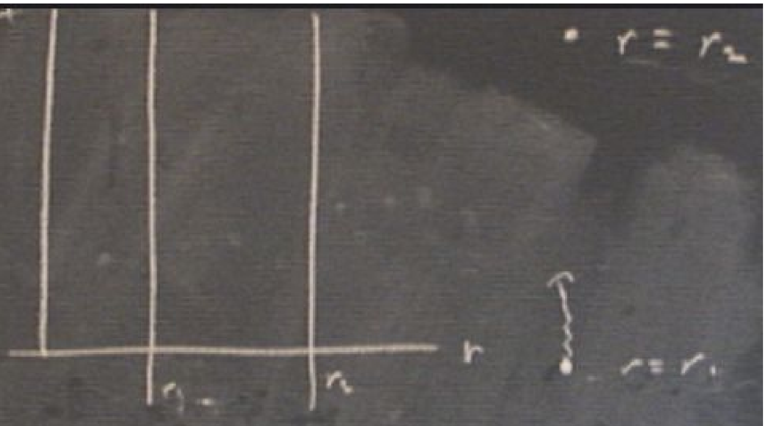


$$k^r = \left(1 - 2m/r\right) k^t$$

$$V = \omega_* / k_x = 1$$

$$\underset{\sim}{k} = \omega\, \underset{\sim}{E}^0 + k^1 \underset{\sim}{E}^1 + k^2 \underset{\sim}{E}^2 + k^3 \underset{\sim}{E}^3$$

Gravitational Red shift



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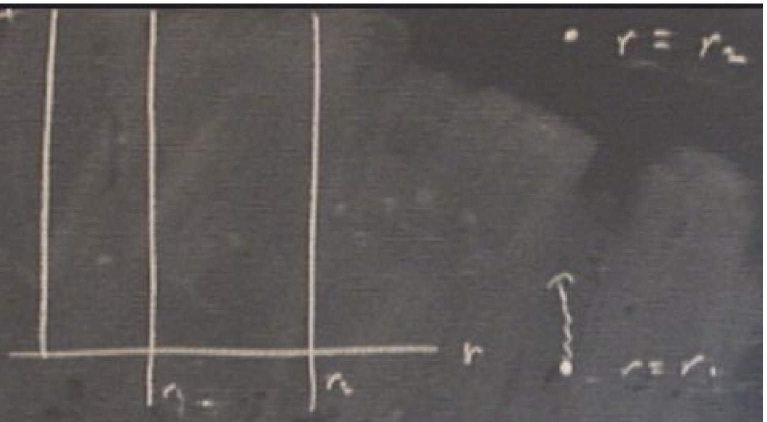

$$V = {}^{\omega_*}/_{k_*} = 1$$

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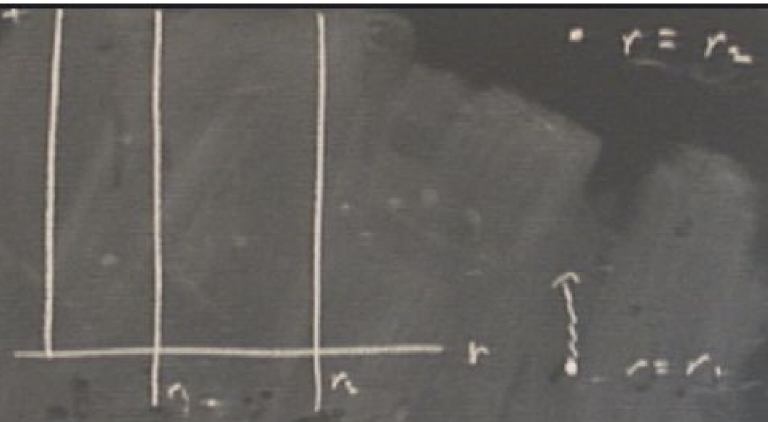
$$\underset{\sim}{k} \cdot \underset{\sim}{k} = 0 \qquad \underset{\sim}{E}^a \cdot \underset{\sim}{E}^b = \eta_{ab}$$

Gravitational Red shift



$$k^r = (1 - {}^{2m}/_{r})\, k^t$$

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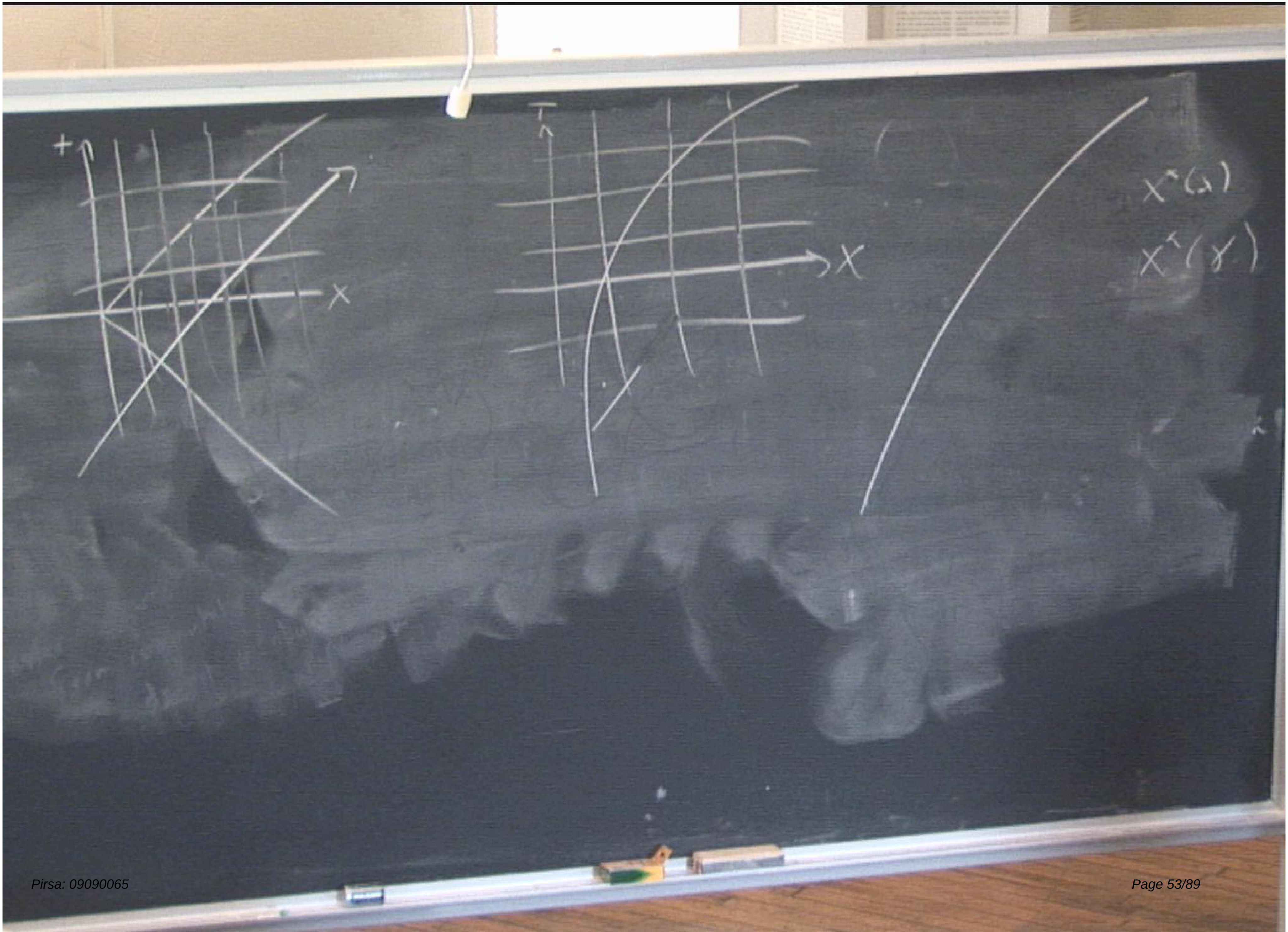
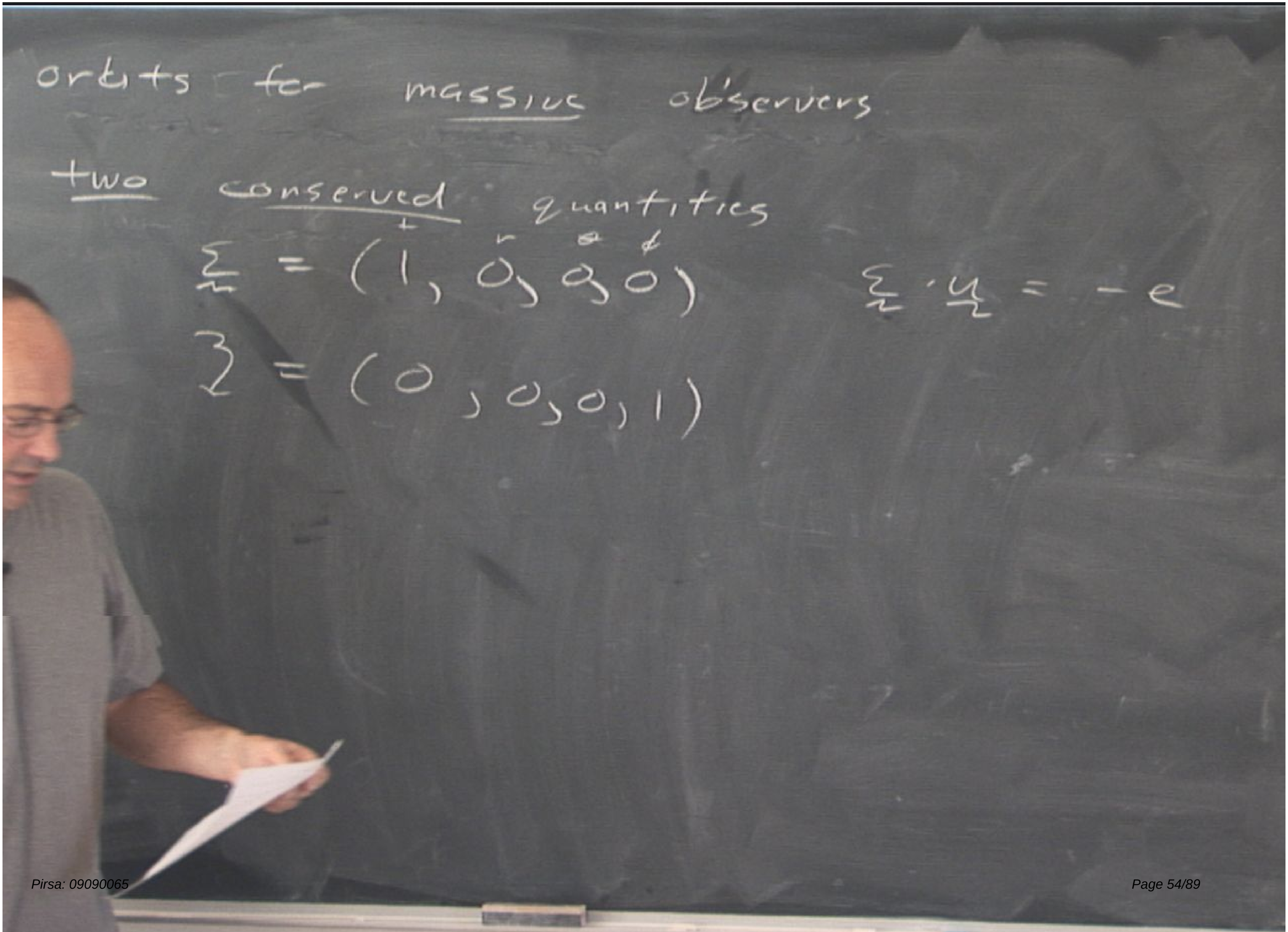
$$\underset{\sim}{k} \cdot \underset{\sim}{k} = 0 \qquad \underset{\sim}{E}^a \cdot \underset{\sim}{E}^b = \eta_{ab}$$

$$-\omega^2 + |\vec{k}|^2 = 0 \longrightarrow v = {}^{\omega}/_{|\vec{k}|} = 1$$


t
x
T
X
$X^\mu(\sigma)$
$X^\mu(\gamma)$



orbits for <u>massive</u> observers

<u>two</u> <u>conserved</u> quantities

$$\underset{\sim}{\xi} = (\overset{t}{1}, \overset{r}{0}, \overset{\theta}{0}, \overset{\phi}{0}) \qquad \underset{\sim}{\xi} \cdot \underset{\sim}{u} = -e$$

$$\eta = (0, 0, 0, 1)$$

orbits for massive observers

two conserved quantities

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θ eom + spherical symmetry $\Rightarrow$ $\theta = \pi/2$

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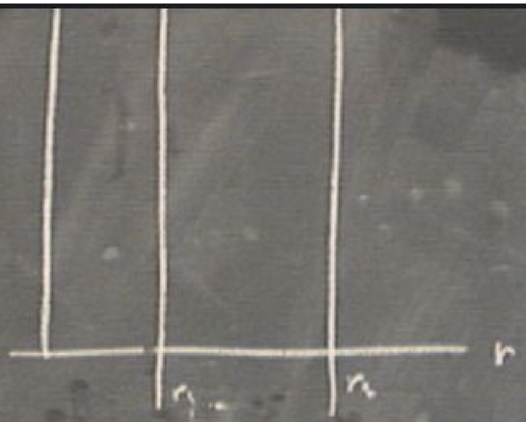
$\underline{\theta \text{ eom}}$ + <u>spherical symmetry</u> $\Rightarrow \theta = \pi/2$

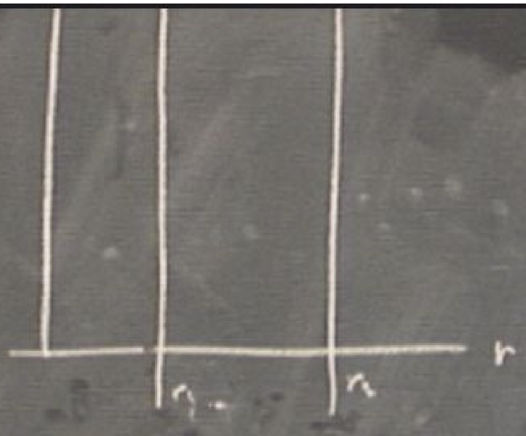
<u>extra</u> constraint → $\underset{\sim}{u} \cdot \underset{\sim}{u} = -1$

Gravitational Red shift

$$\frac{dt}{d\tau}$$



Gravitational Red shift



$$\frac{dt}{d\tau} = \frac{e}{1-\frac{2m}{r}}$$

$$\frac{d\phi}{d\tau} = l/r^2$$

considering $r \longrightarrow \infty$

$$\frac{dt}{d\tau} = \frac{e}{1-\frac{2m}{r}} \quad \ldots \qquad e = \text{energy}/\text{mass}$$

$$\frac{d\phi}{d\tau} = \ell/r^2$$

considering $r \Longrightarrow \rho$

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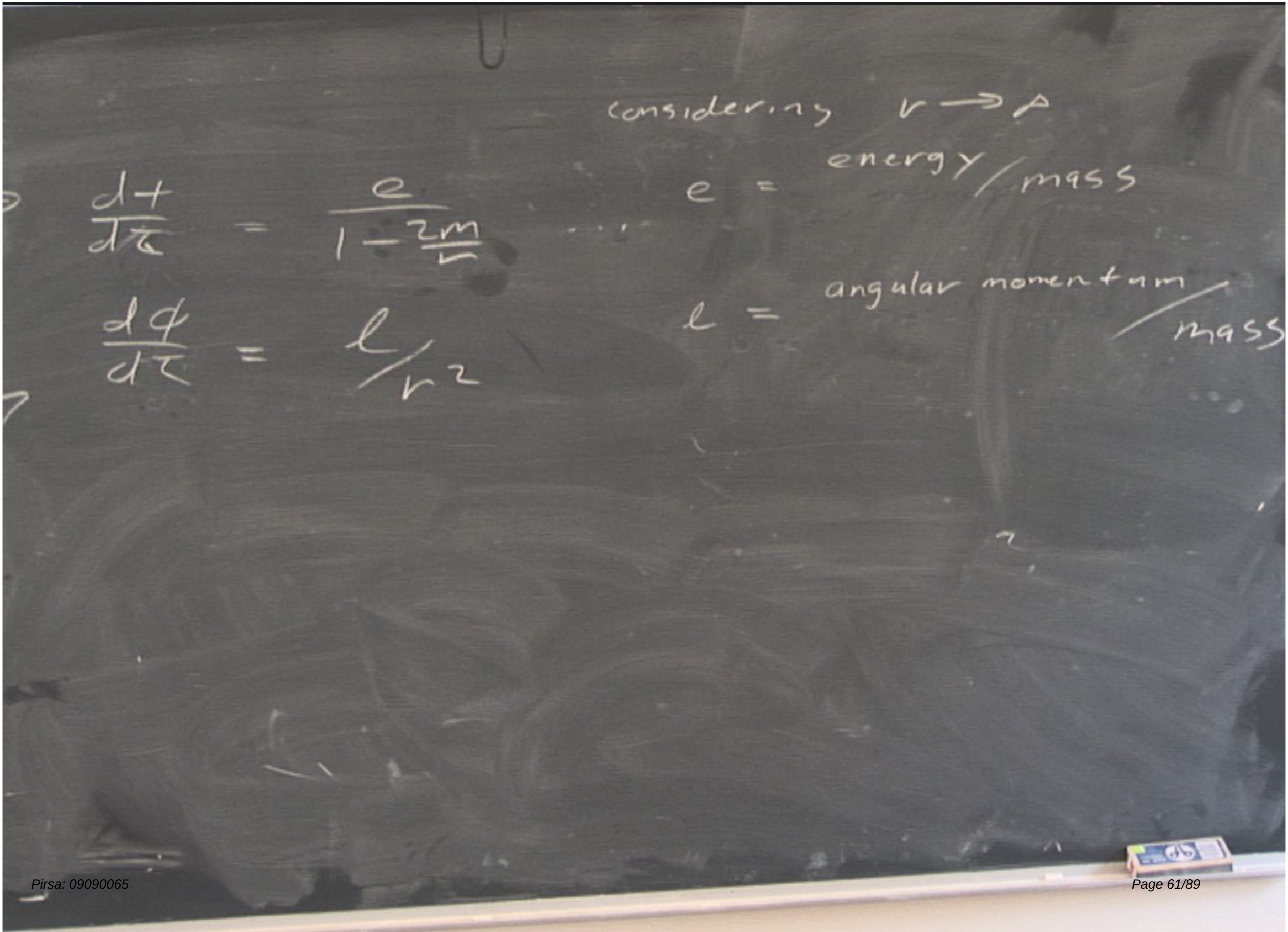
$e =$ energy/mass

$$\frac{d\phi}{d\tau} = \ell / r^2$$

$\ell =$ angular momentu



considering $r \longrightarrow \rho$

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③ along ① + ② combine to give

an equation that looks like the Hamiltonian constraint for a classical particle in a potential

$$= \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2$$

considering $r \rightarrow \rho$

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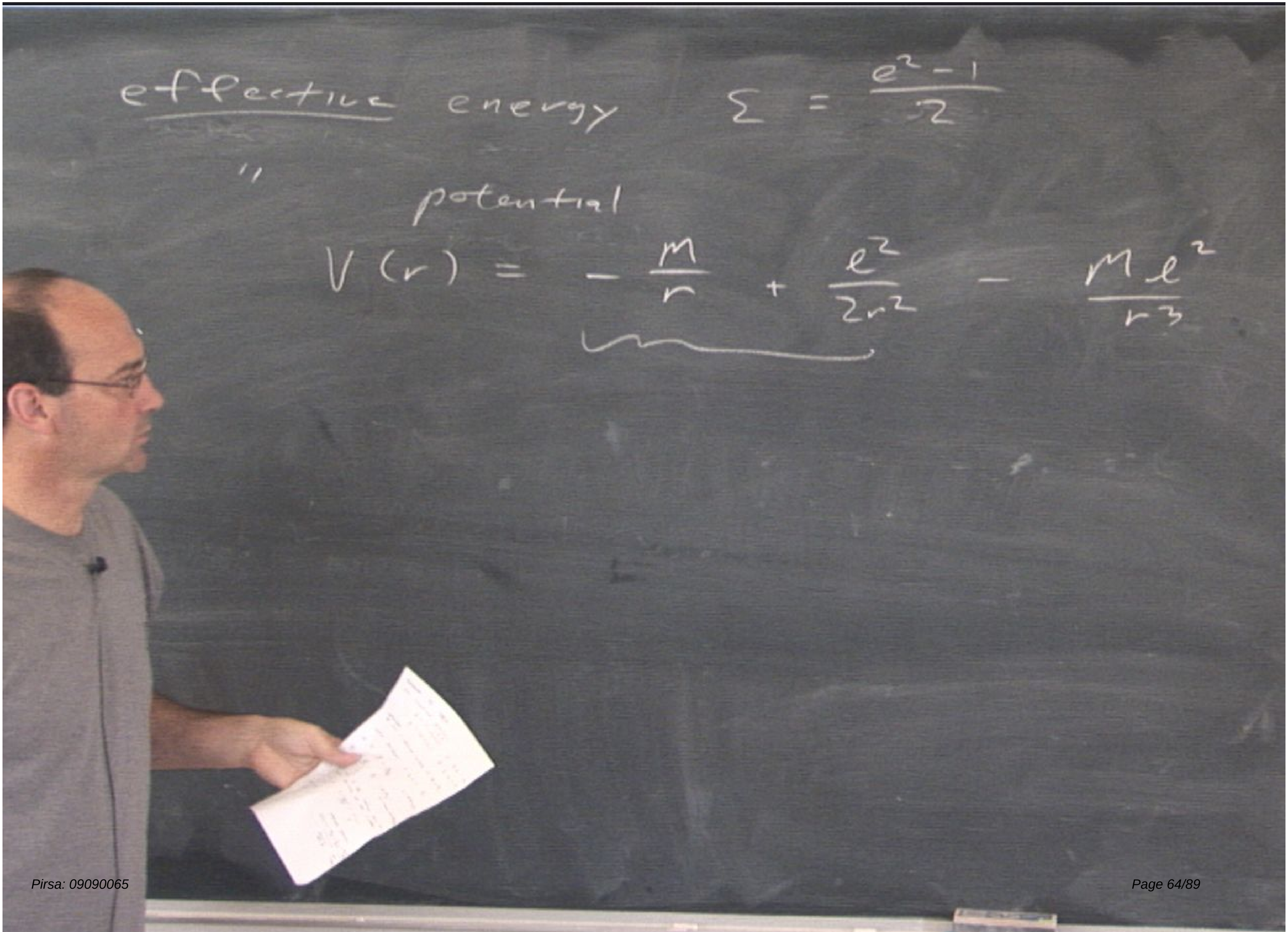
$$\mathcal{E} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V(r)$$



effective energy $\Sigma = \frac{e^2-1}{2}$

" potential

$$V(r) = \underbrace{-\frac{M}{r} + \frac{\ell^2}{2r^2}} - \frac{M\ell^2}{r^3}$$

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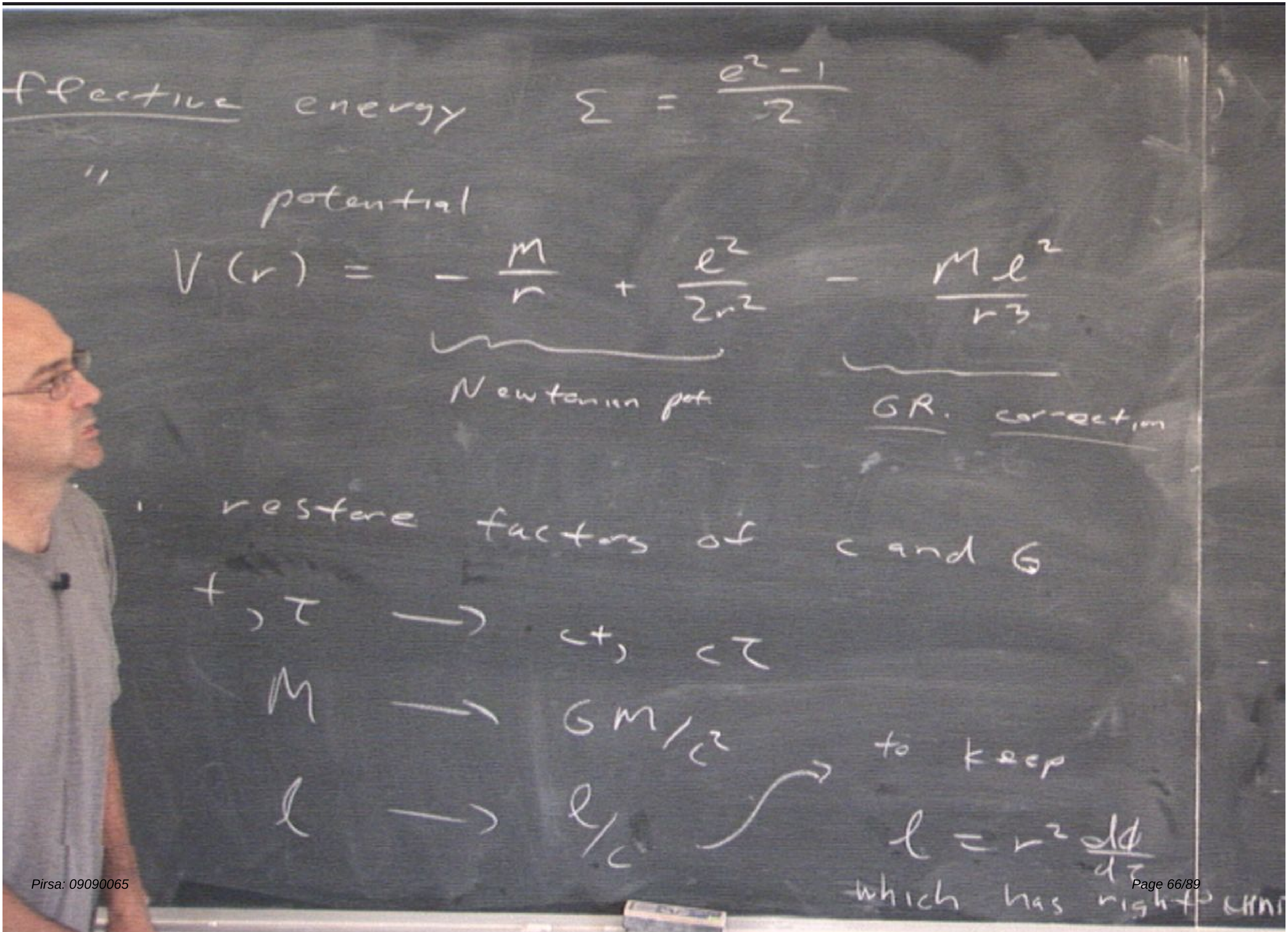
$$V(r) = \underbrace{-\frac{M}{r} + \frac{\ell^2}{2r^2}}_{\text{Newtonian pot.}} - \underbrace{\frac{M\ell^2}{r^3}}_{\text{G.R. correction}}$$

check : restore factors of c and G

$$t, \tau \longrightarrow ct, c\tau$$

$$M \longrightarrow GM/c^2$$

$$\ell \longrightarrow$$



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$$M \longrightarrow GM/c^2$$

$$\ell \longrightarrow \ell/c$$

to keep $\ell = r^2 \frac{d\phi}{d\tau}$

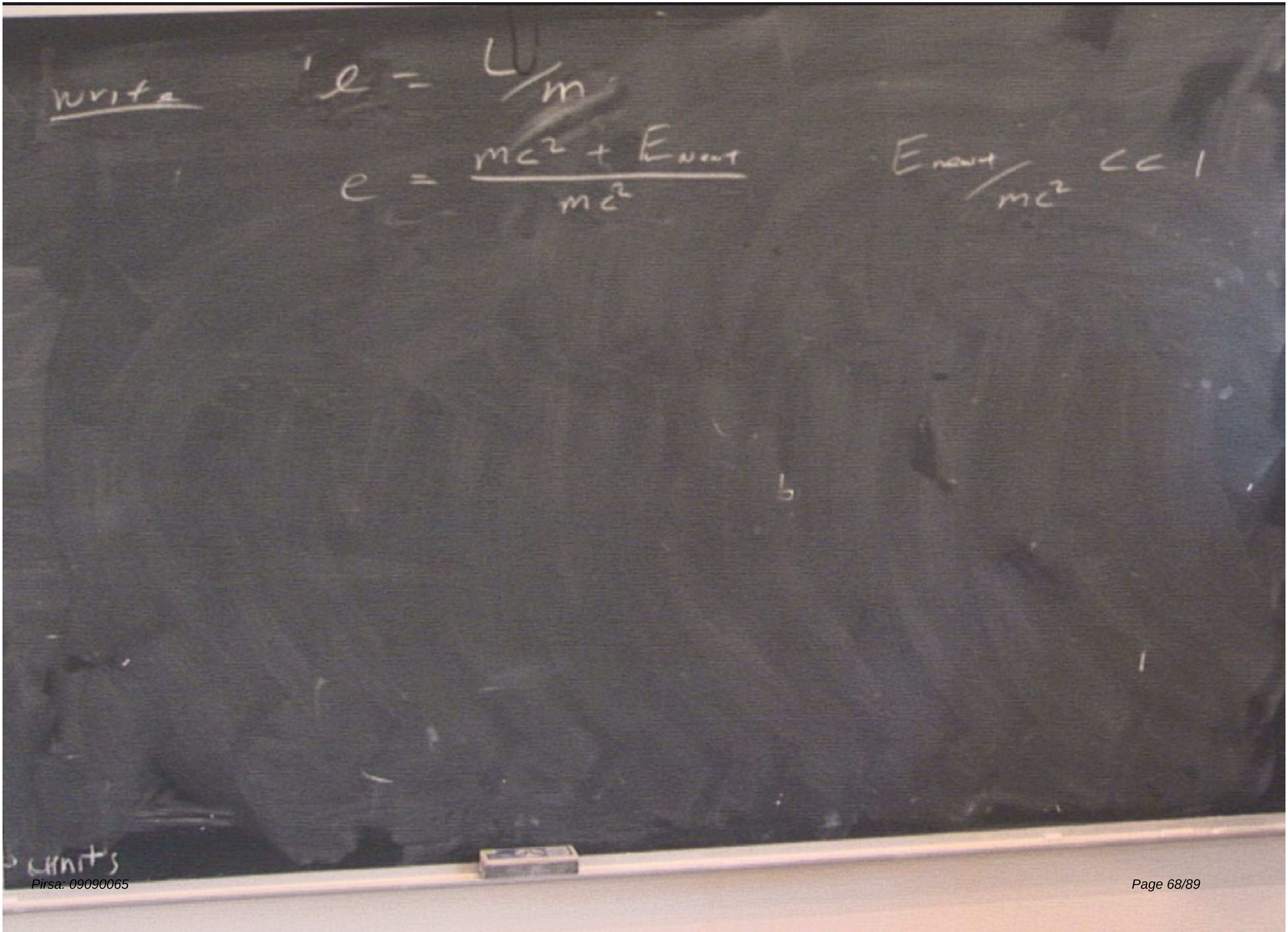
which has right un

τ

to keep

$$\ell = r^2 \frac{d\phi}{d\tau}$$

which has right units



write

$$\ell = L/m$$

$$e = \frac{mc^2 + E_{Newt}}{mc^2} \qquad E_{newt}/mc^2 << 1$$

units

write

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$$e = \frac{mc^2 + E_{Newt}}{mc^2}$$

will assume

$$E_{newt}/mc^2 \ll 1$$

plug + grind away.

$$E_{Newt} + \frac{E_{newt}^2}{2mc^2} = \frac{m}{2}\left(\frac{dr}{d\tau}\right)^2 + \frac{L^2}{2mr^2} - \frac{GMm}{r}$$

$$- \frac{GML^2}{c^2 r^3}$$

units

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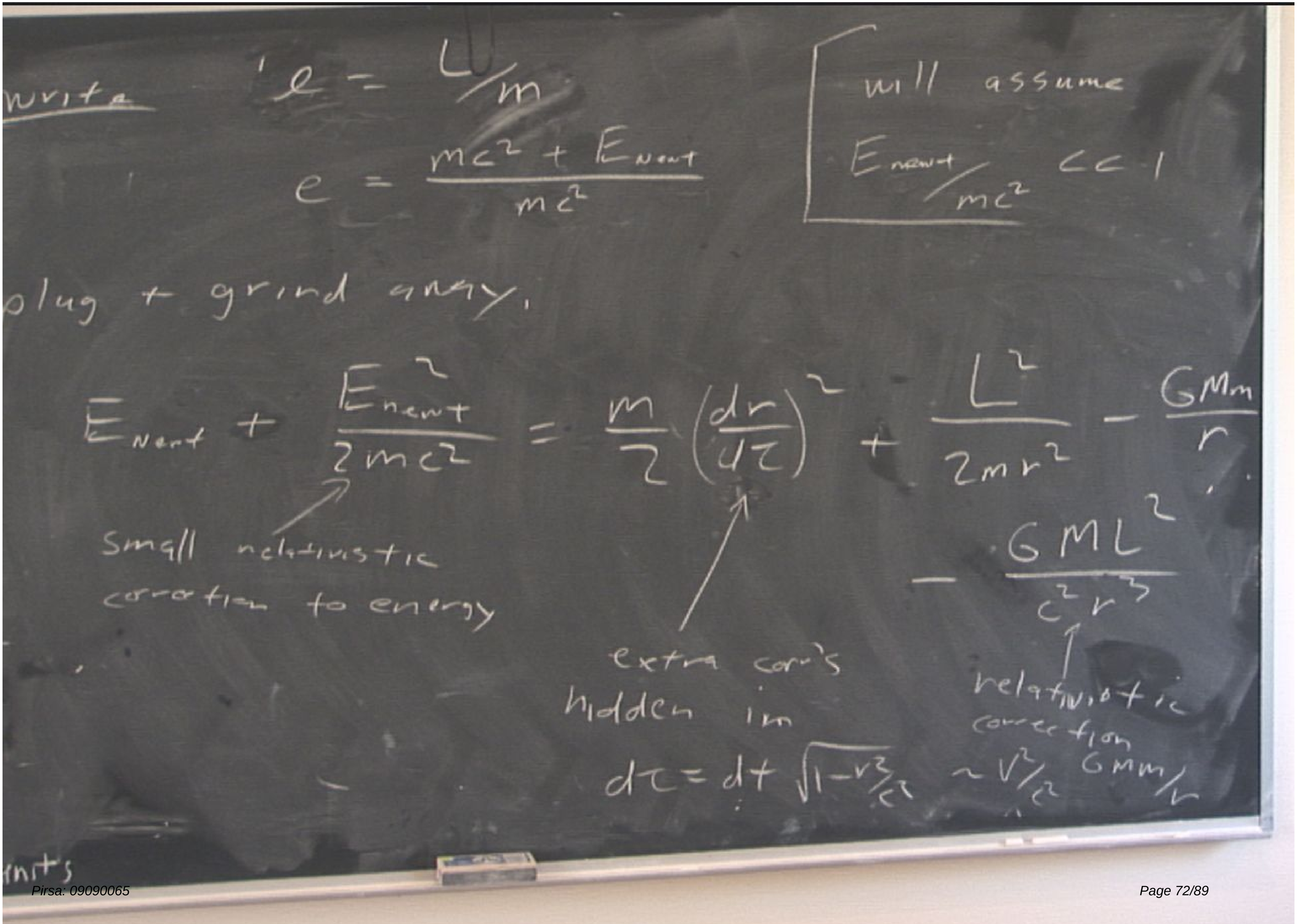
small relativistic correction to energy

relativistic correction $\sim V^2/c^2 \; GMm/r$



write $\ell = L/m$

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will assume

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small relativistic correction to energy

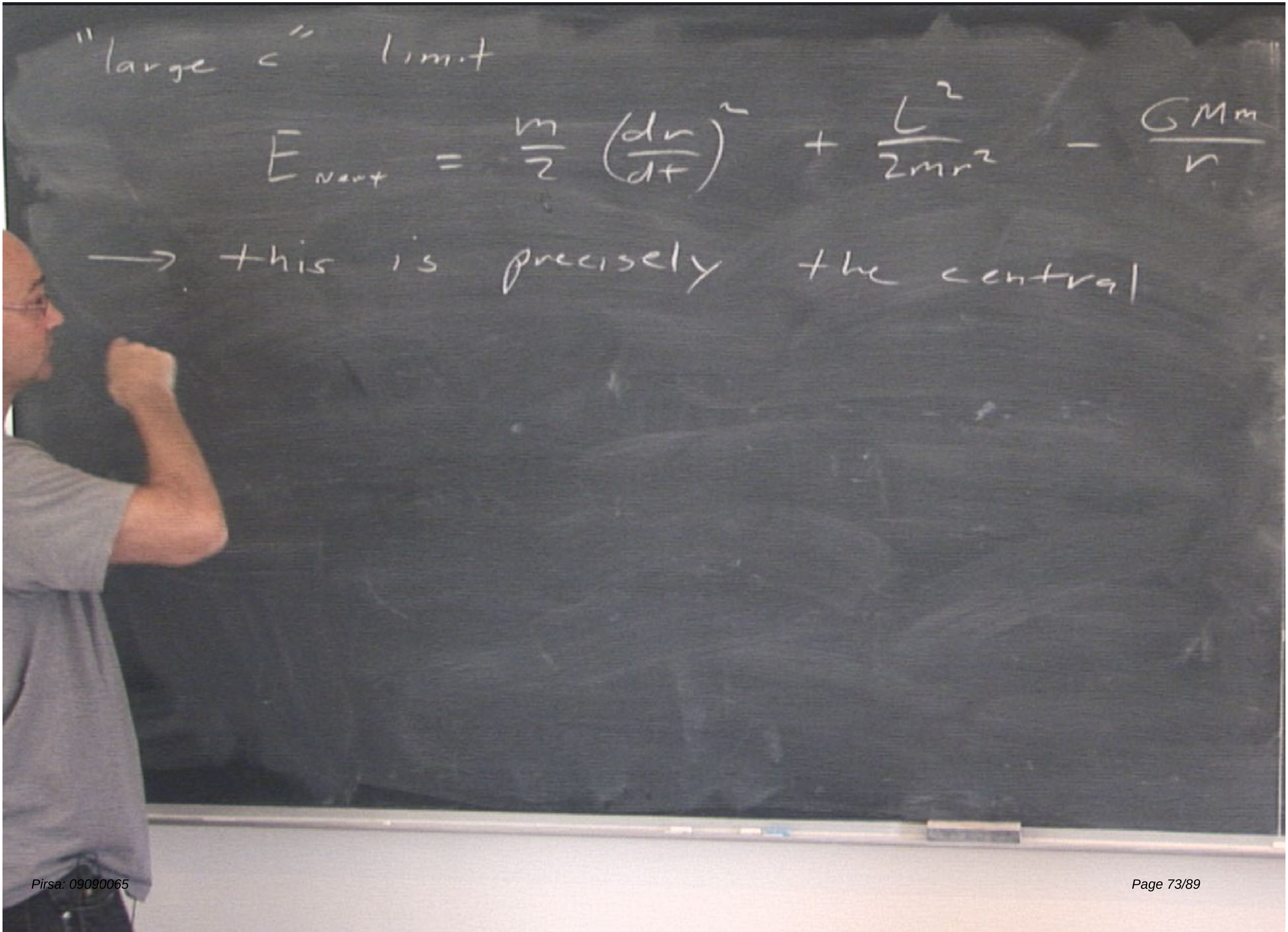
extra corr's hidden in

$d\tau = dt\sqrt{1-v^2/c^2}$

relativistic correction

$\sim V^2/c^2 \; GMm/r$

nits



"large c" limit

$$E_{Newt} = \frac{m}{2}\left(\frac{dr}{dt}\right)^2 + \frac{L^2}{2mr^2} - \frac{GMm}{r}$$

→ this is precisely the central

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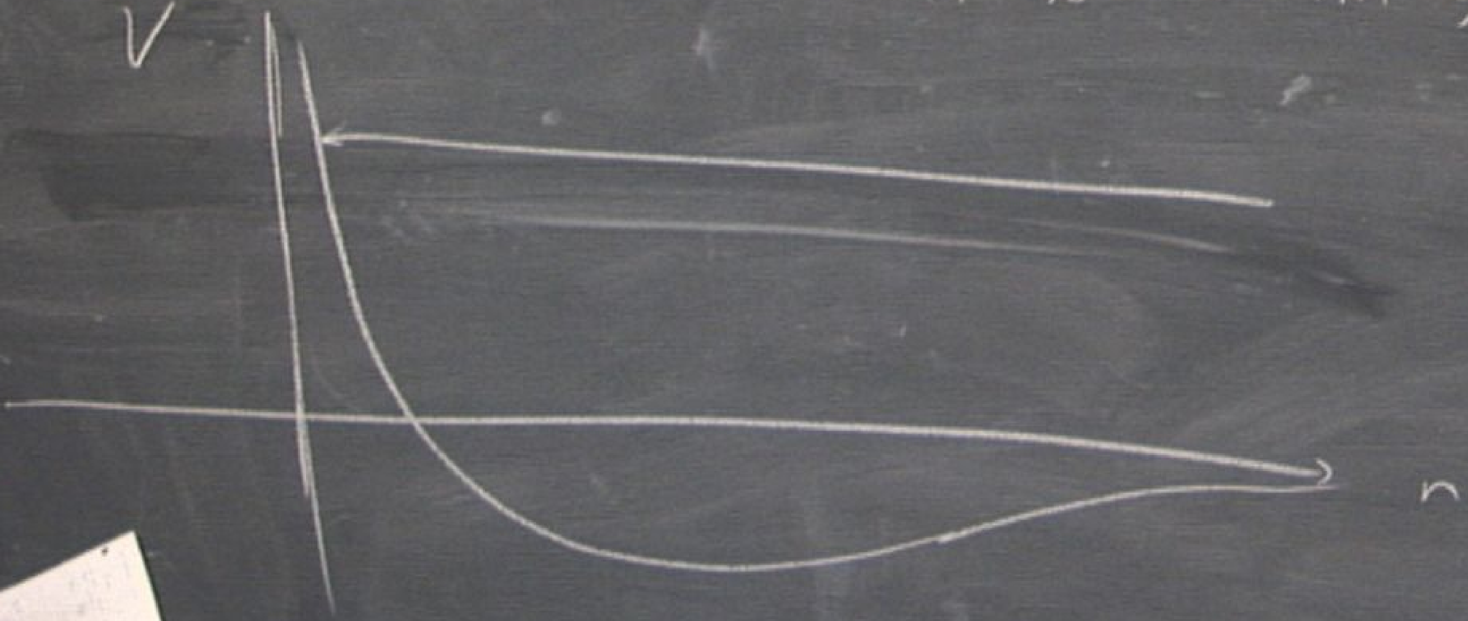

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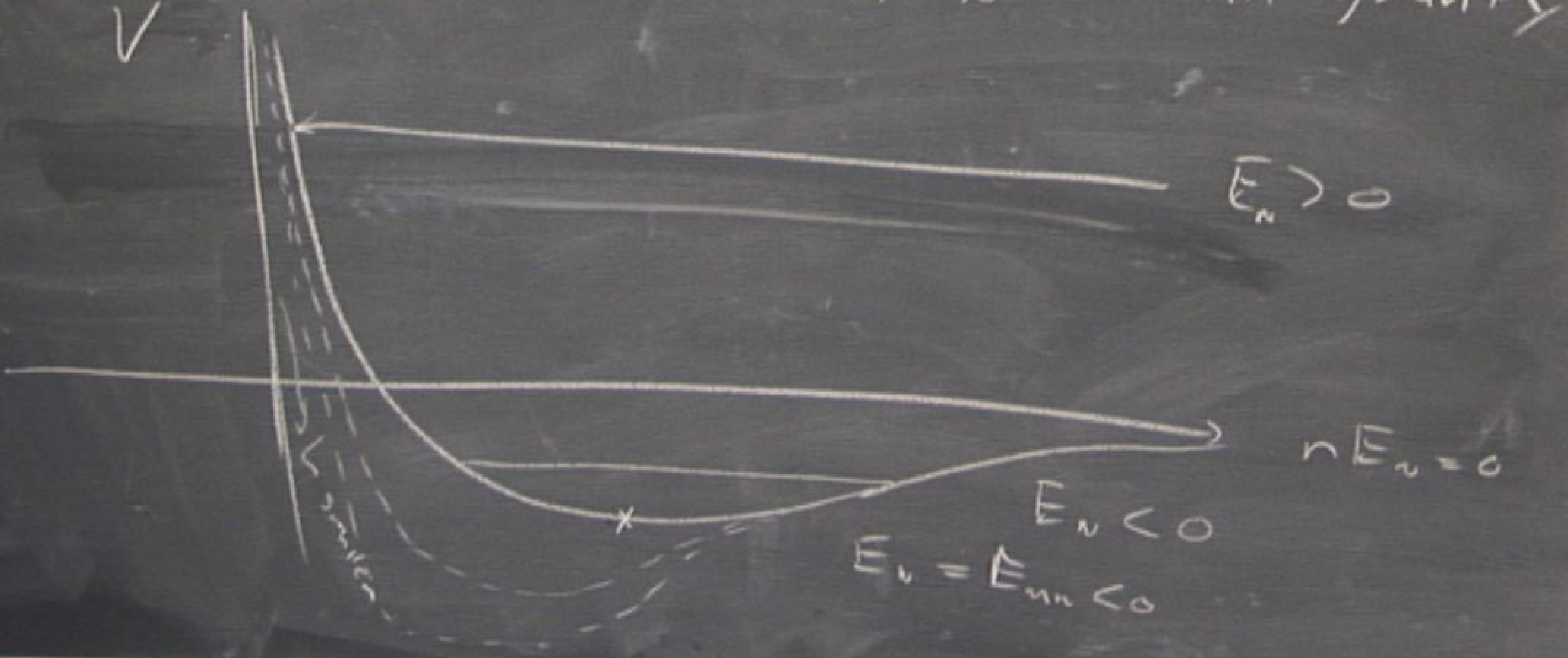
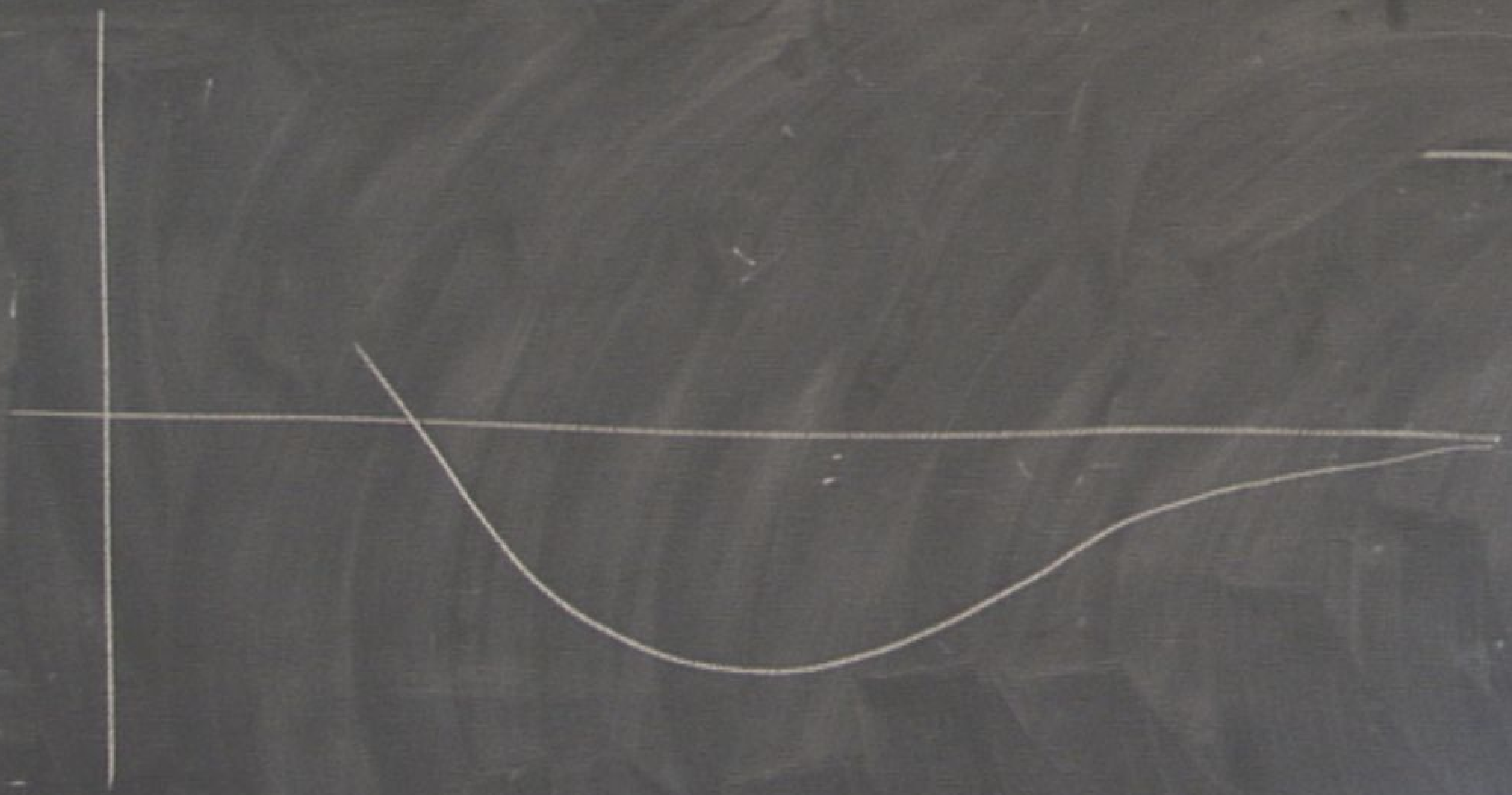


compare: $V = -\frac{M}{r} + \frac{\ell^2}{2r^2} - \frac{M\ell^2}{r^3}$



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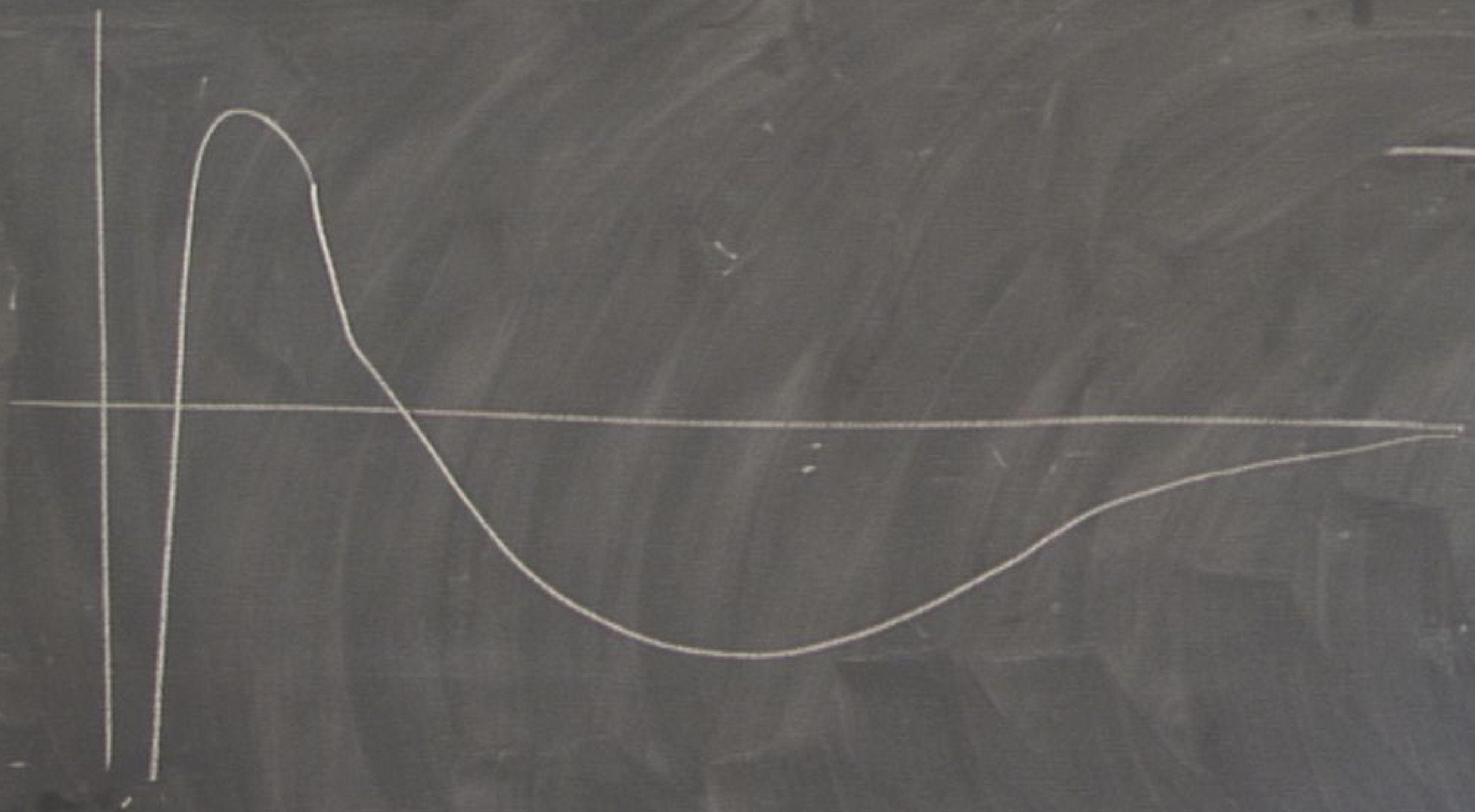

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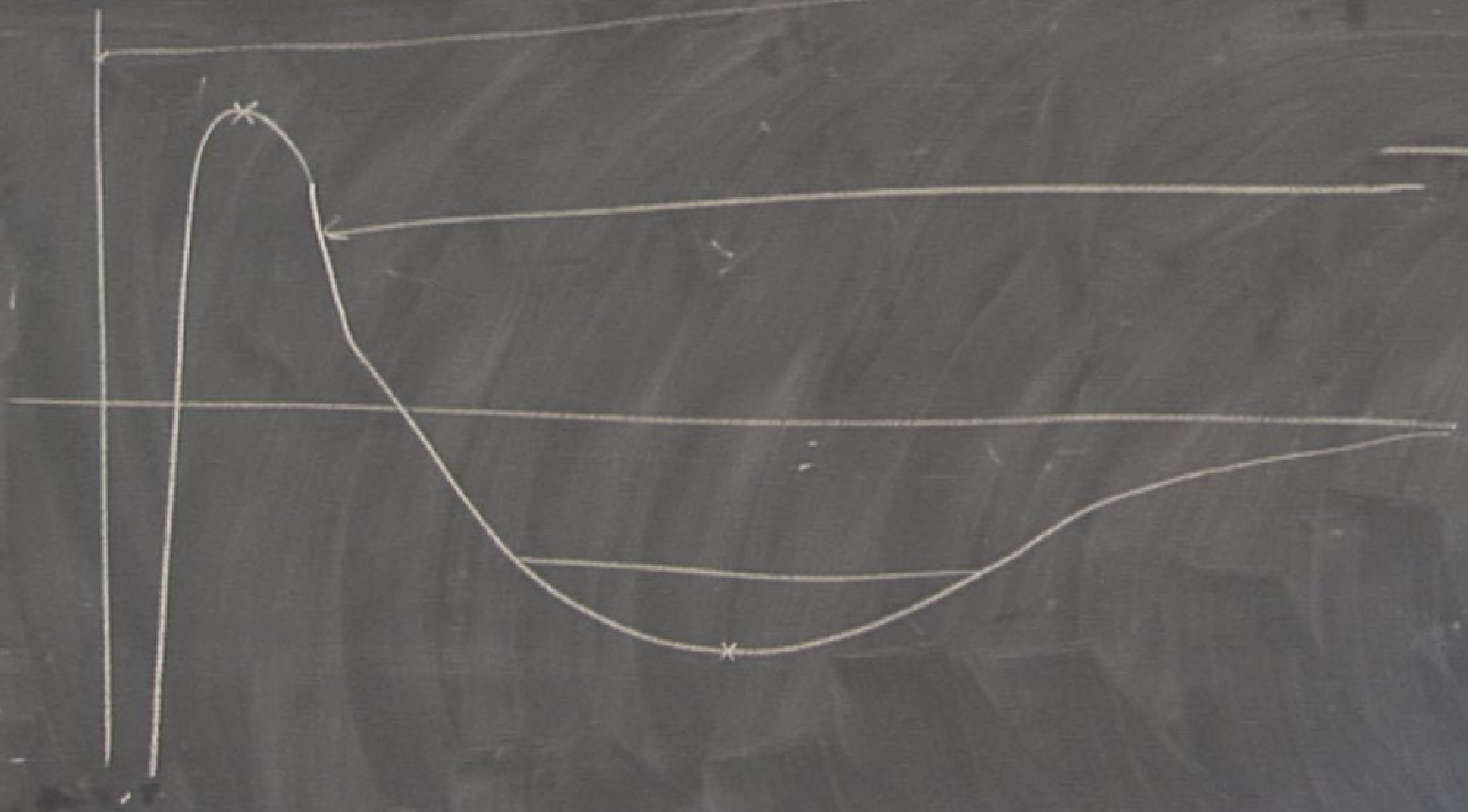

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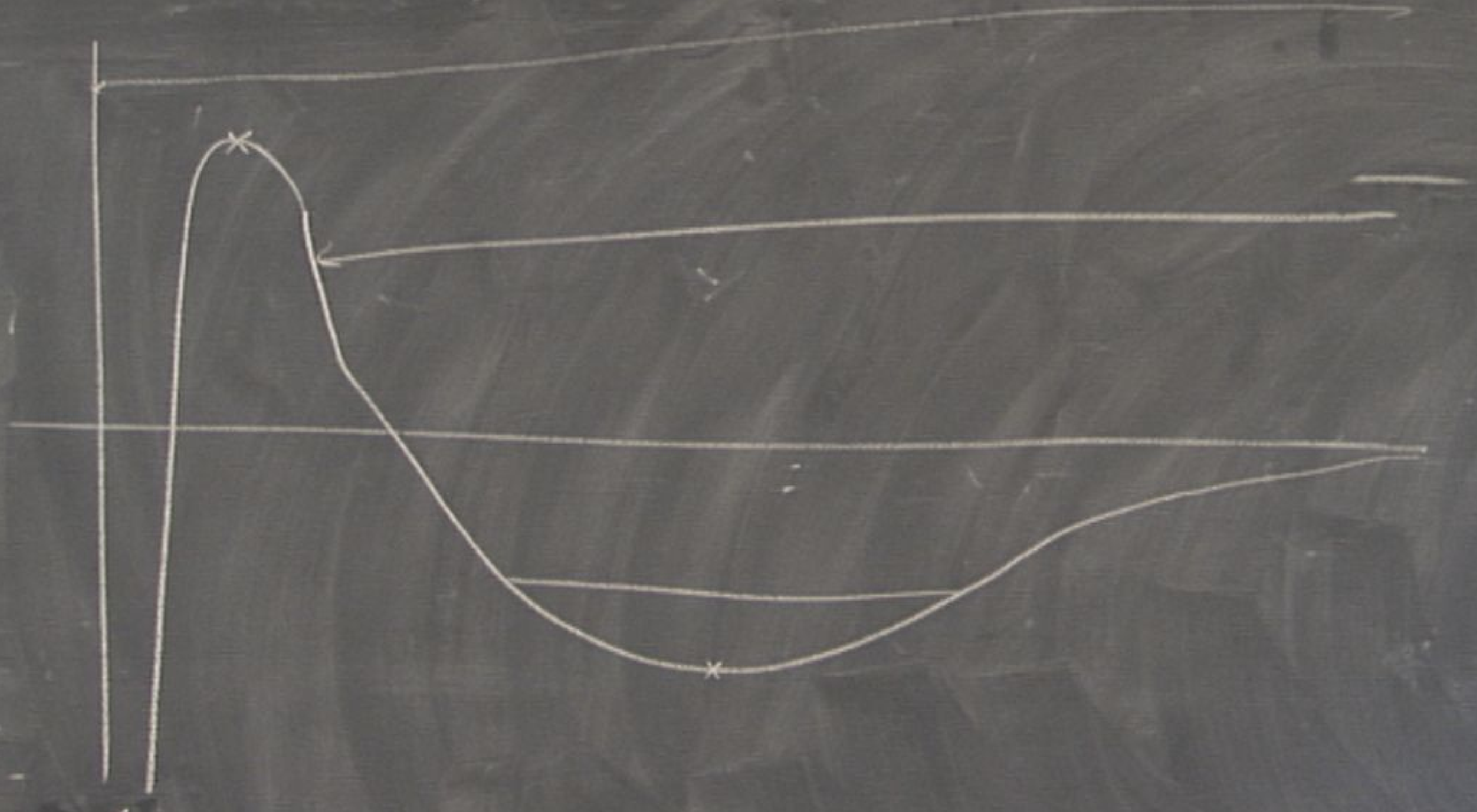
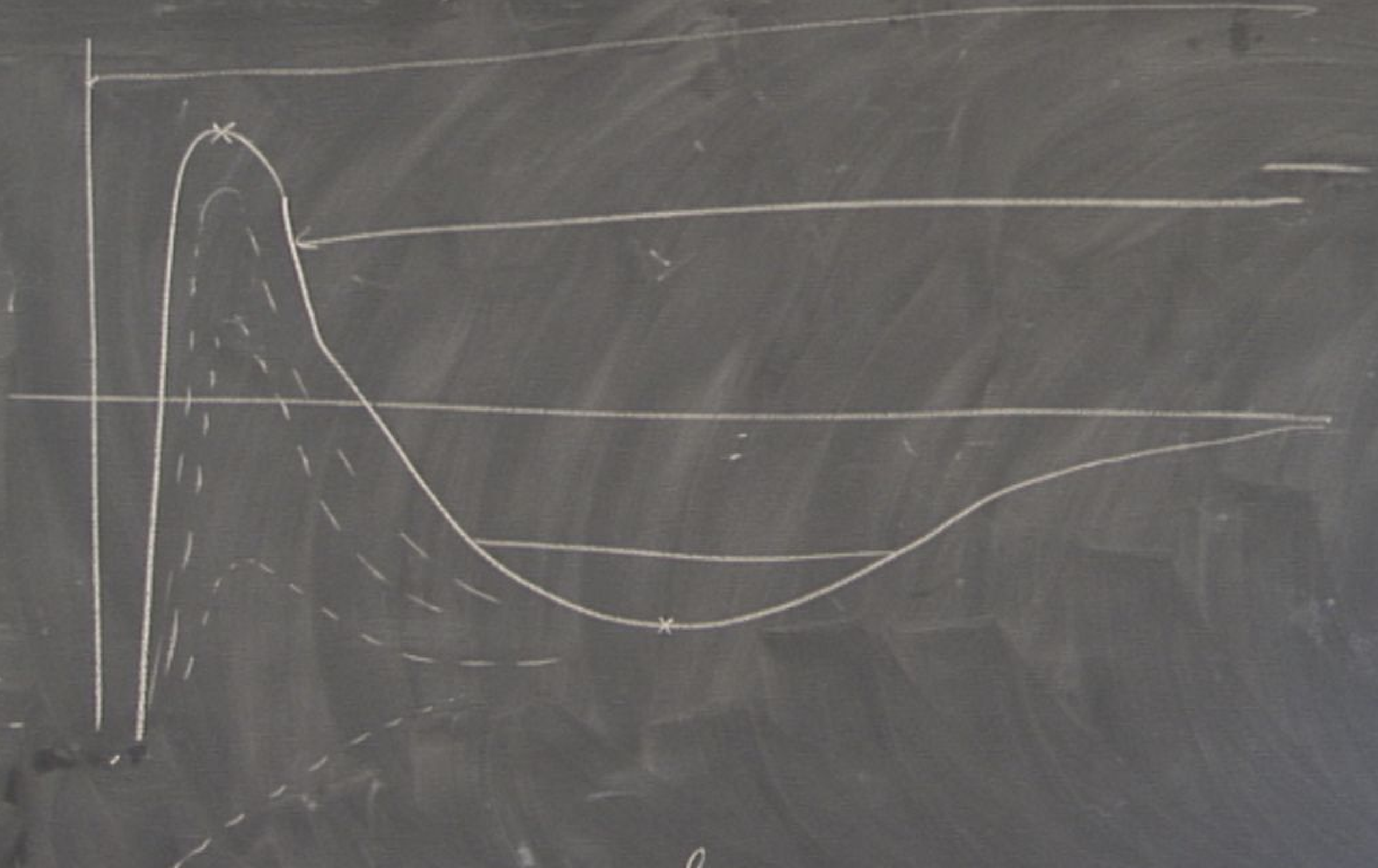
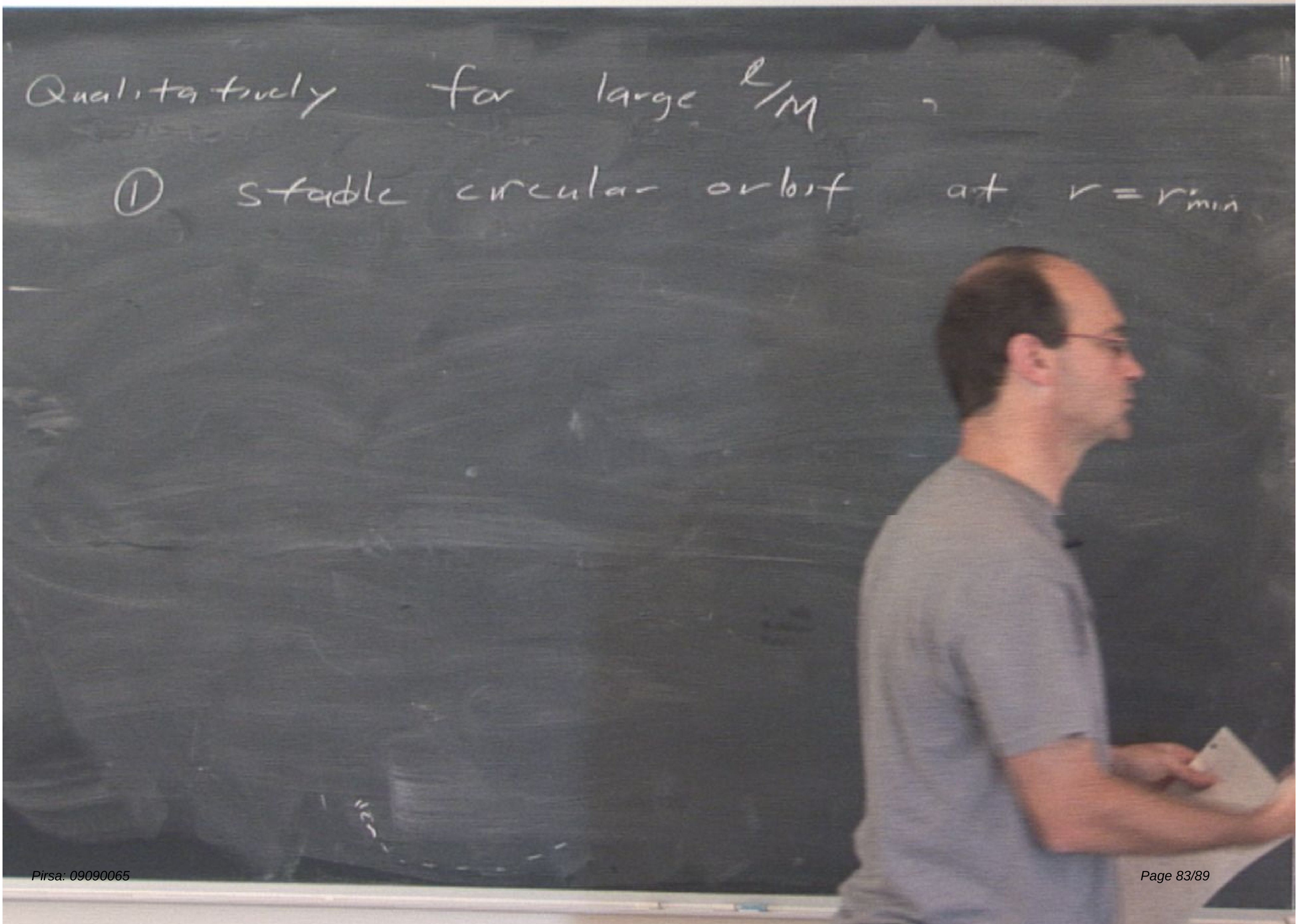


Increase ratio of ℓ/M

compare: $V = -\frac{M}{r} + \frac{\ell^2}{2r^2} - \frac{M\ell^2}{r^3}$

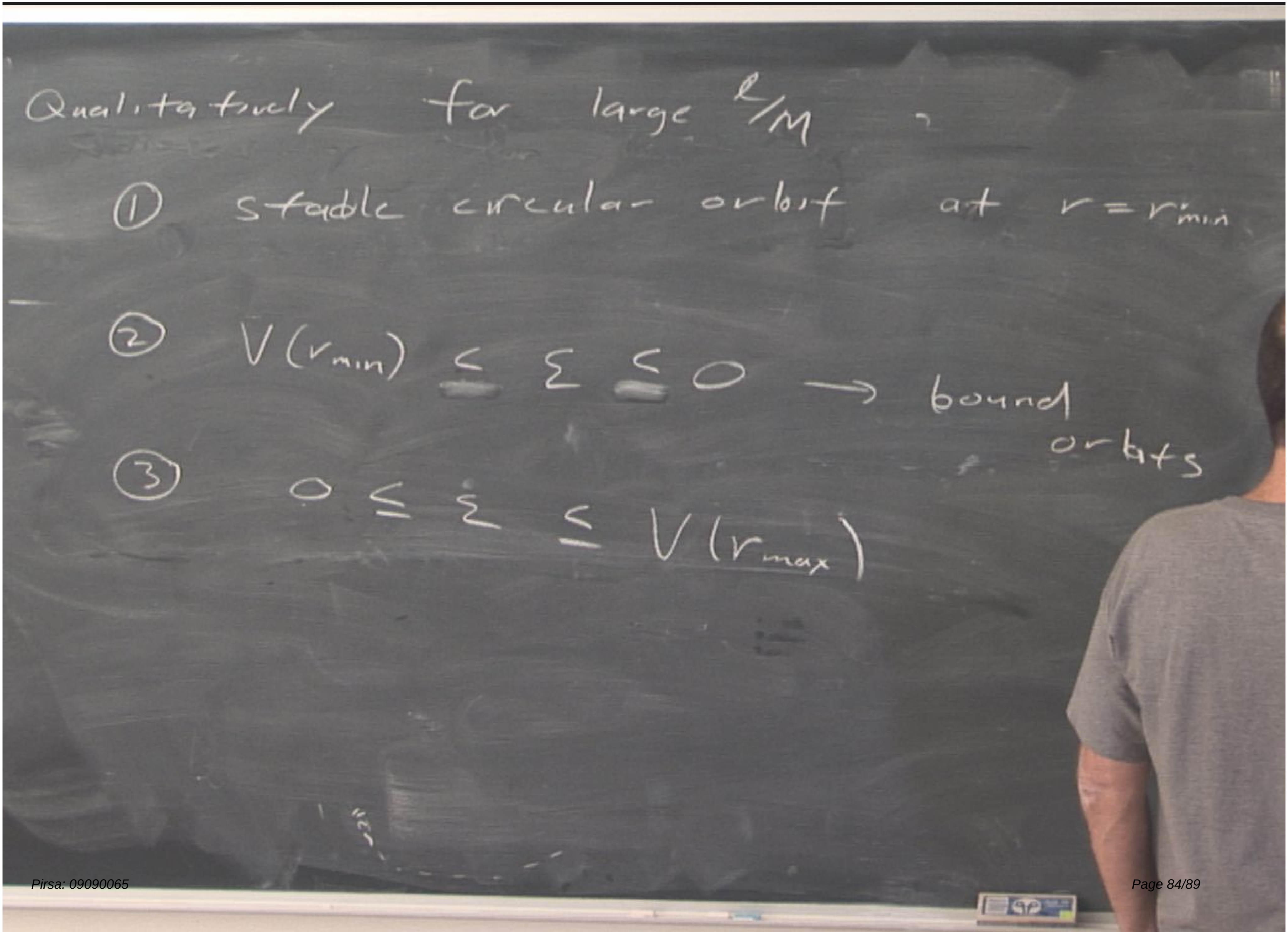


Increase ratio of ℓ/M



Qualitatively for large ℓ/M ,

① stable circular orbit at $r = r_{min}$

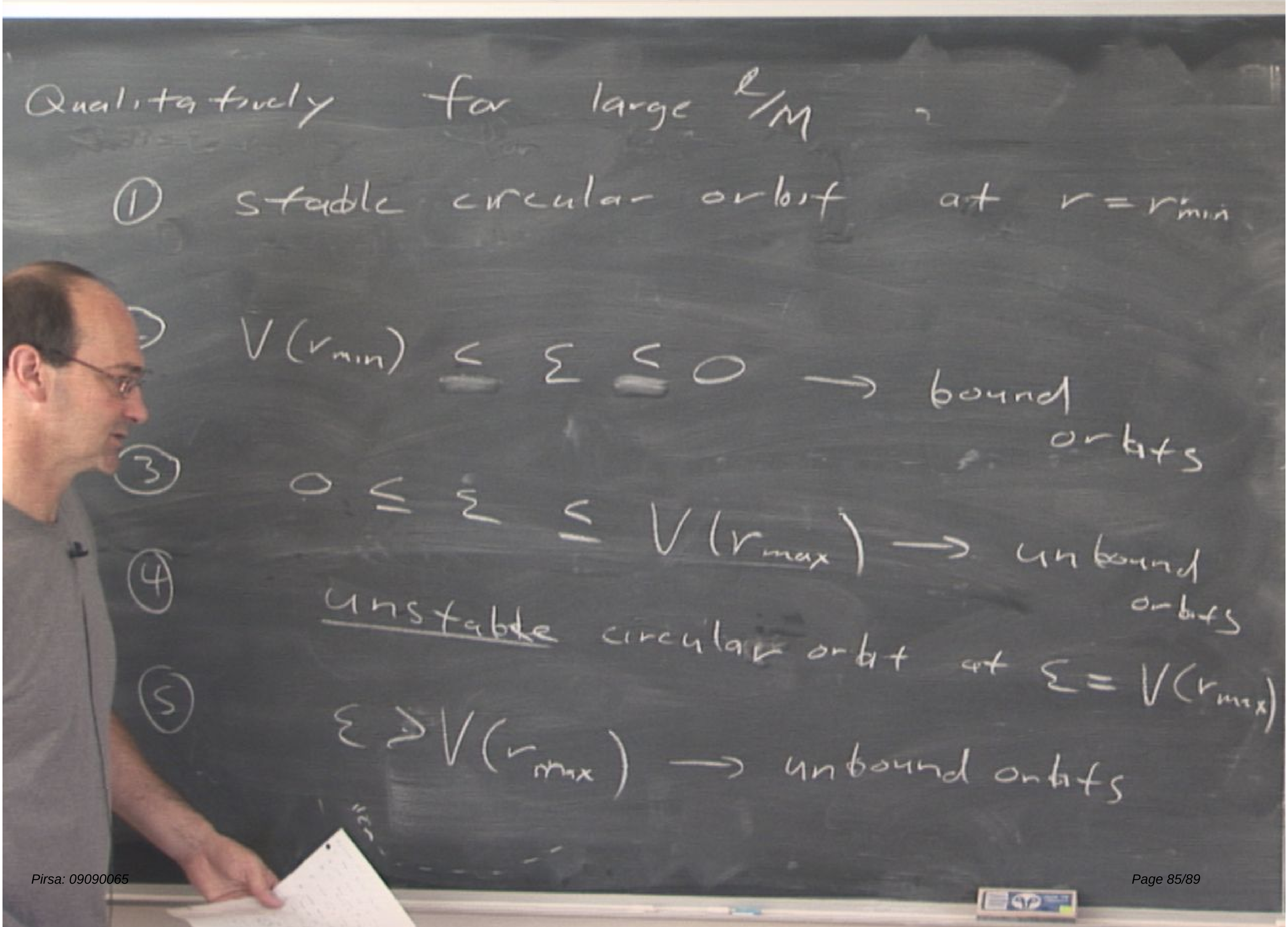


Qualitatively for large ℓ/M ,

① stable circular orbit at $r = r_{min}$

② $V(r_{min}) \leq \mathcal{E} \leq 0 \longrightarrow$ bound orbits

③ $0 \leq \mathcal{E} \leq V(r_{max})$



Qualitatively for large ℓ/M,

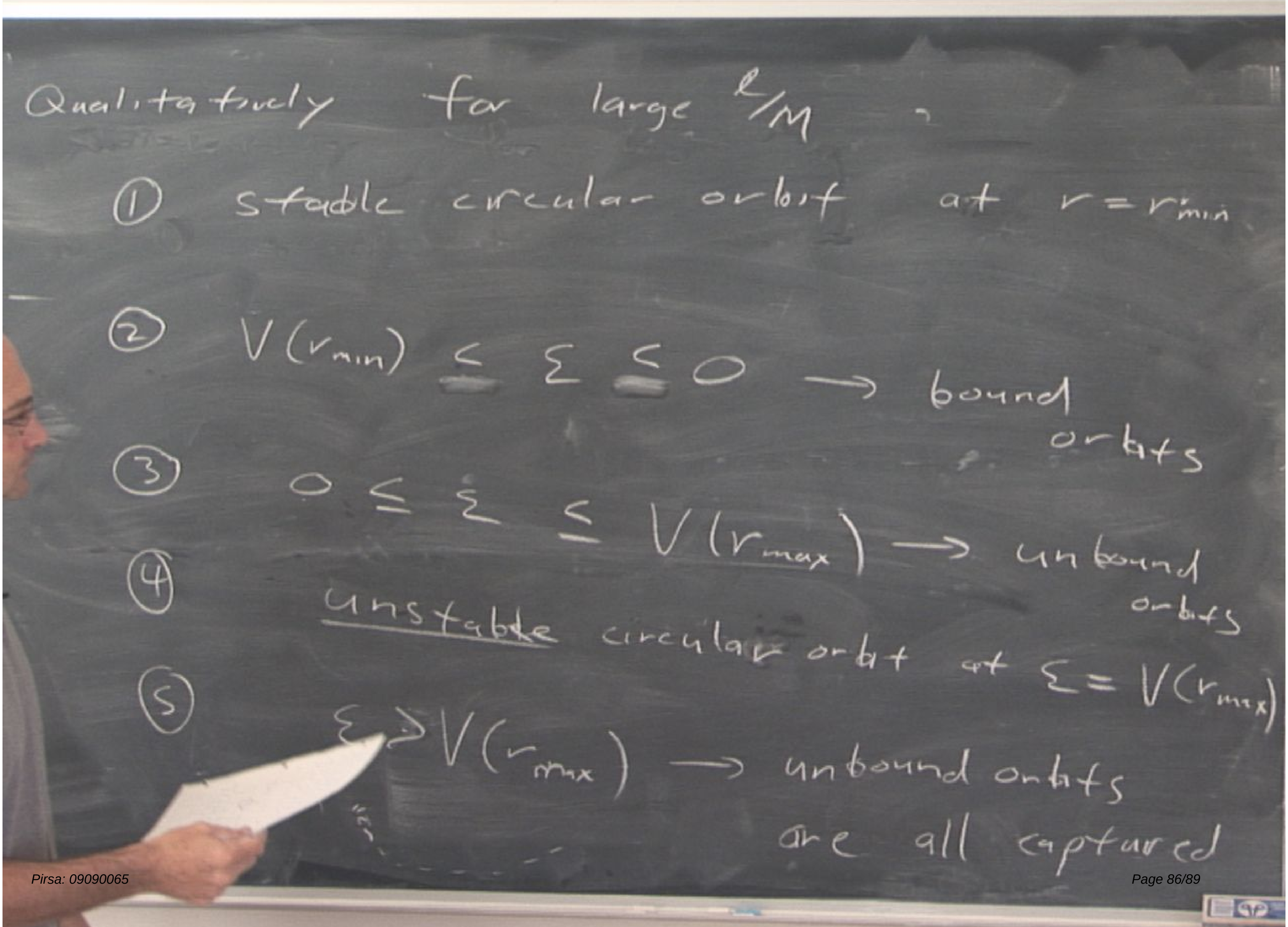
① stable circular orbit at $r = r_{min}$

② $V(r_{min}) \le \mathcal{E} \le 0 \longrightarrow$ bound orbits

③ $0 \le \mathcal{E} \le V(r_{max}) \longrightarrow$ unbound orbits

④ unstable circular orbit at $\mathcal{E} = V(r_{max})$

⑤ $\mathcal{E} > V(r_{max}) \longrightarrow$ unbound orbits



Qualitatively for large ℓ/M :

① stable circular orbit at $r = r_{min}$

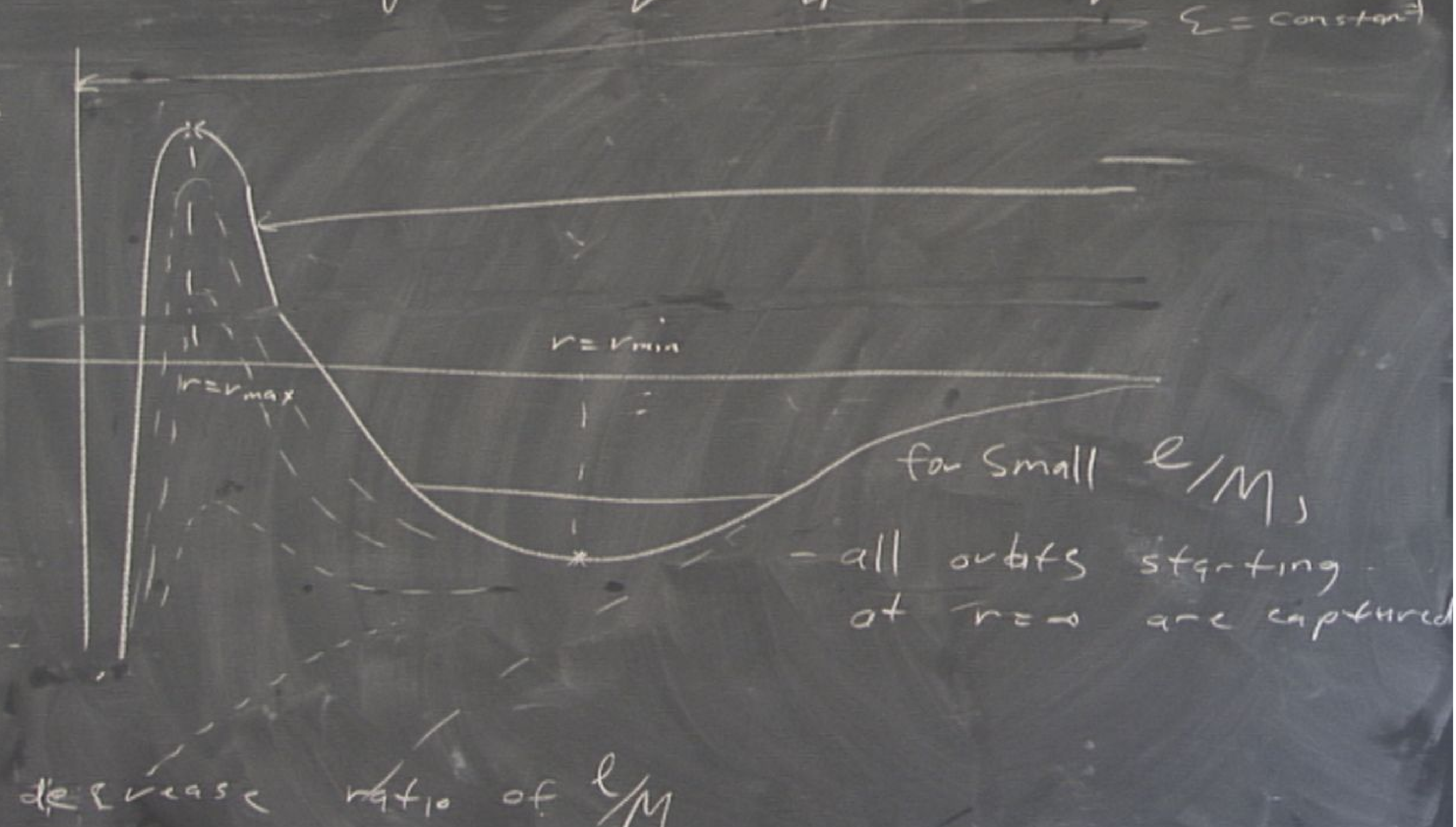
② $V(r_{min}) \le \mathcal{E} \le 0 \longrightarrow$ bound orbits

③ $0 \le \mathcal{E} \le V(r_{max}) \longrightarrow$ unbound orbits

④ <u>unstable</u> circular orbit at $\mathcal{E} = V(r_{max})$

⑤ $\mathcal{E} > V(r_{max}) \longrightarrow$ unbound orbits are all captured

compare: $V = -\frac{M}{r} + \frac{\ell^2}{2r^2} - \frac{M\ell^2}{r^3}$

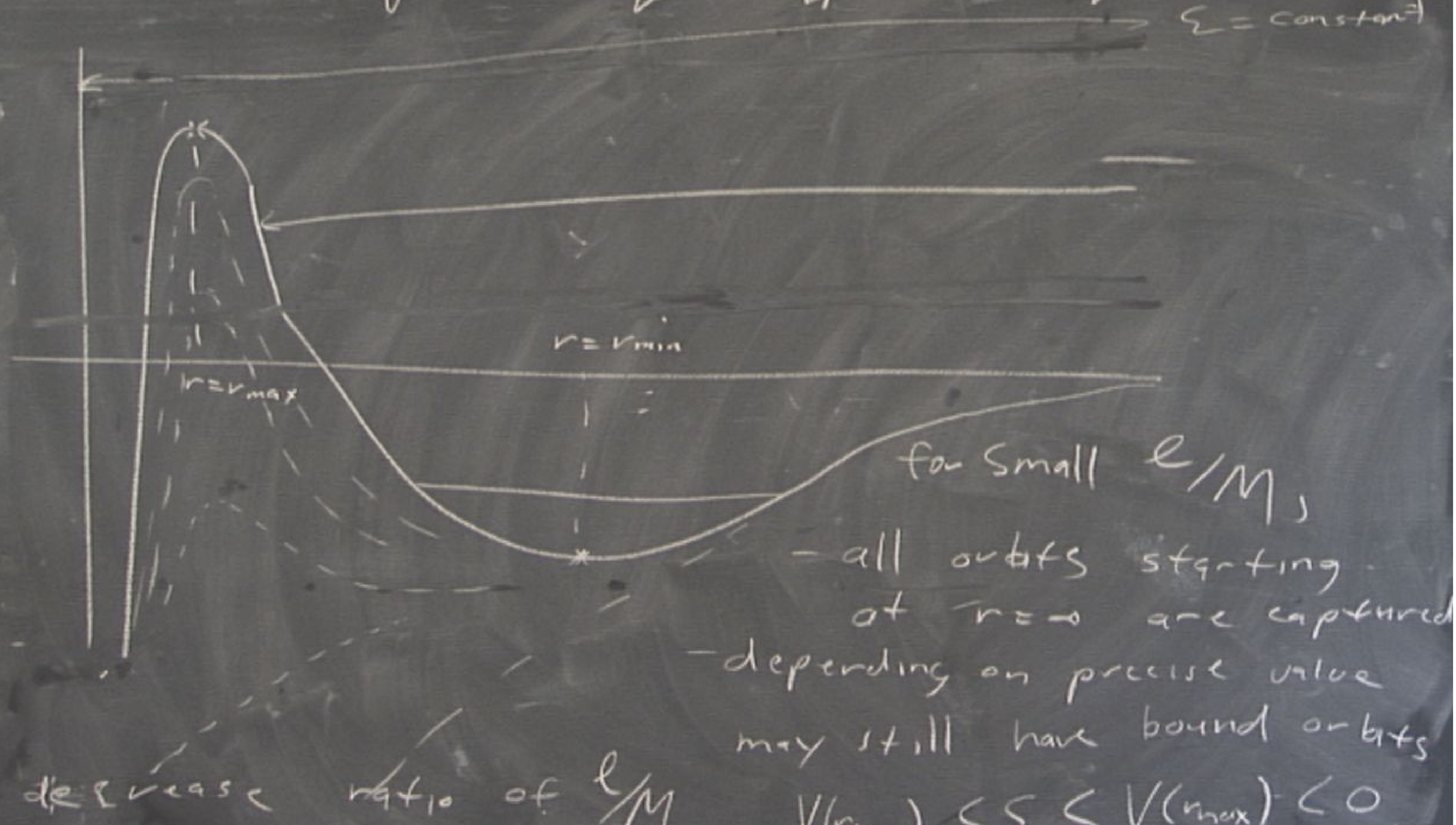


for small ℓ/M,

- all orbits starting at $r=\infty$ are captured

decrease ratio of ℓ/M

compare: $V = -\frac{M}{r} + \frac{\ell^2}{2r^2} - \frac{M\ell^2}{r^3}$



for small ℓ/M,

- all orbits starting at $r=\infty$ are captured
- depending on precise value may still have bound orbits

$V(r_{min}) < \xi < V(r_{max}) < 0$

for sufficiently small ℓ/M

V has no local extrema

→ all orbits are captured

③ $0 \le \mathcal{E} \le V(r_{max}) \longrightarrow$ unbound orbits

④ unstable circular orbit at $\mathcal{E} = V(r_{max})$

⑤ $\mathcal{E} > V(r_{max}) \longrightarrow$ unbound orbits are all captured