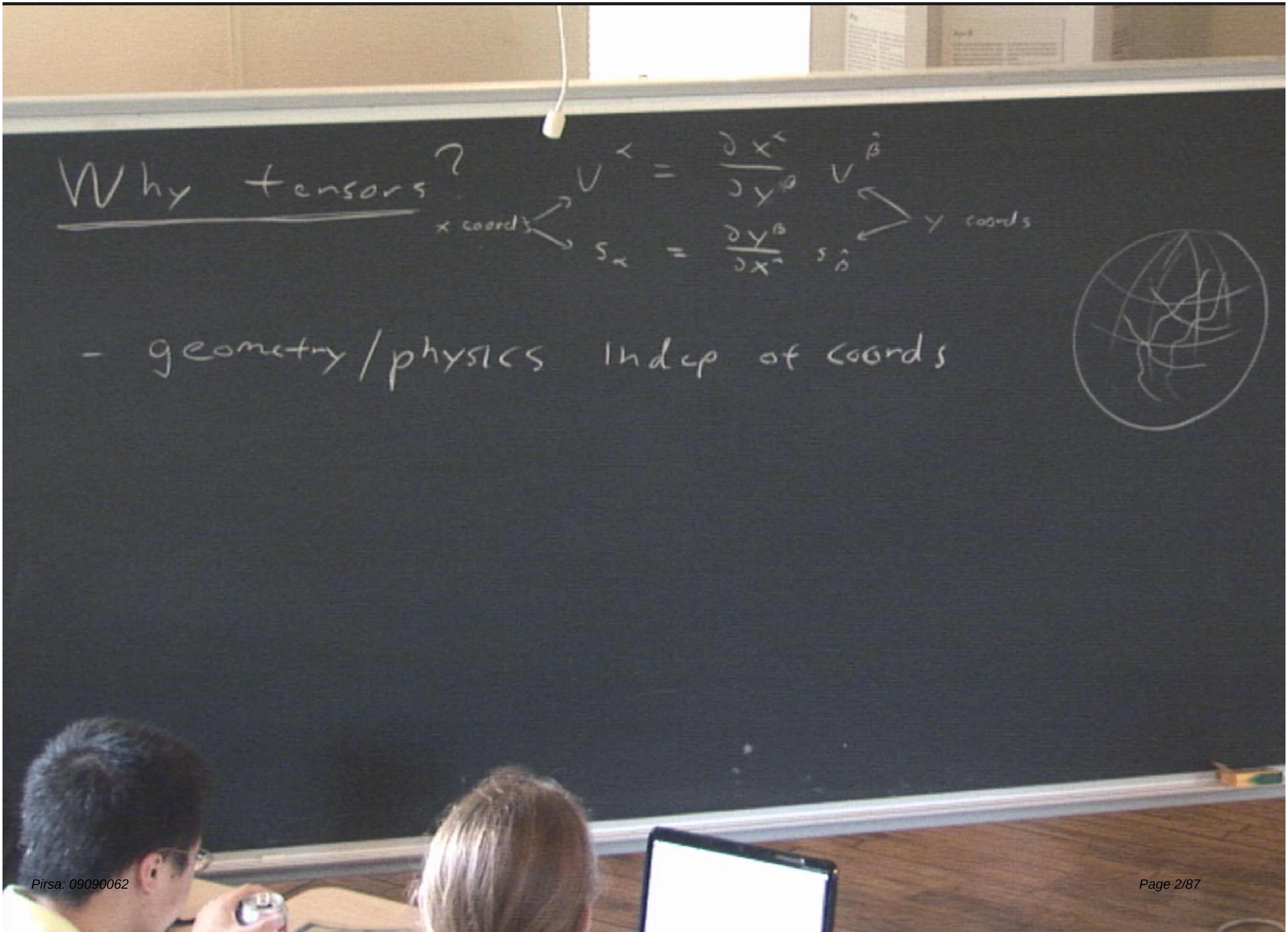
Title: Relativity - Core (PHYS 604) - Lecture 7

Date: Sep 11, 2009 09:00 AM

URL: http://pirsa.org/09090062

Abstract:


Why tensors?
$V^{\alpha} = \frac{\partial x^{\alpha}}{\partial y^{\hat\beta}} V^{\hat\beta}$
x coords
y coords
$S_{\alpha} = \frac{\partial y^{\hat\beta}}{\partial x^{\alpha}} S_{\hat\beta}$
- geometry/physics indep of coords

Why tensors?

$$V^{\alpha} = \frac{\partial x^{\alpha}}{\partial y^{\beta}} V^{\hat\beta}$$

x coord's

$$S_{\alpha} = \frac{\partial y^{\beta}}{\partial x^{\alpha}} S_{\hat\beta}$$

- geometry/physics indep of coord

$\longrightarrow$ coords important to calcula

Why tensors?

x coords

$$V^\alpha = \frac{\partial x^\alpha}{\partial y^\beta} V^{\hat\beta}$$

$$S_\alpha = \frac{\partial y^\beta}{\partial x^\alpha} S_{\hat\beta}$$

- geometry/physics indep of coord

$\longrightarrow$ coords important to calcula

Eg given a world line - $x^a(\tau) \longrightarrow$

might also calculate $\frac{d^2x^\alpha}{d\tau^2}$

ors ?

x coords $\qquad V^\alpha = \frac{\partial x^\alpha}{\partial y^{\hat\beta}} V^{\hat\beta}$ $\qquad$ y coords

$s_\alpha = \frac{\partial y^\beta}{\partial x^\alpha} s_{\hat\beta}$

/ physics indep of coords

important to calculate

a world line - $x^\alpha(\tau) \longrightarrow u^\alpha = \frac{dx^\alpha}{d\tau}$

ht also calculate $\frac{d^2x^\alpha}{d\tau^2} =$

rs ?

$$V^\alpha = \frac{\partial x^\alpha}{\partial y^\beta} V^{\hat\beta}$$

x coords y coords

$$S_\alpha = \frac{\partial y^\beta}{\partial x^\alpha} S_{\hat\beta}$$

physics indep of coords

important to calculate

world line - $x^\alpha(\tau) \longrightarrow u^\alpha = \frac{dx^\alpha}{d\tau}$

ht also calculate $\frac{d^2x^\alpha}{d\tau^2} = V^\alpha$

rs ?

x coords, y coords

$$V^{\alpha} = \frac{\partial x^{\alpha}}{\partial y^{\beta}} V^{\hat\beta} \quad (*)$$

$$s_{\alpha} = \frac{\partial y^{\beta}}{\partial x^{\alpha}} s_{\hat\beta}$$

physics indep of coords

important to calculate

world line - $x^{\alpha}(\tau) \longrightarrow u^{\alpha} = \frac{dx^{\alpha}}{d\tau}$

ht also calculate $\frac{d^2 x^{\alpha}}{d\tau^2} = V^{\alpha} \leftarrow$ this V^{α} doesn't satisfy (*)

tensors?

x coords $\quad$ y coords

$$V^\alpha = \frac{\partial x^\alpha}{\partial y^{\hat\beta}} V^{\hat\beta} \quad (*)$$

$$S_\alpha = \frac{\partial y^{\hat\beta}}{\partial x^\alpha} S_{\hat\beta}$$

ometry / physics indep of coords

coords important to calculate

given a world line - $x^\alpha(\tau) \longrightarrow u^\alpha = \frac{dx^\alpha}{d\tau}$

might also calculate $\frac{d^2 x^\alpha}{d\tau^2} = V^\alpha \leftarrow$ this

satis

calculate $\underset{\sim}{u} \cdot \underset{\sim}{V}$

tensors?

x coords

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x coords $\quad V^\alpha = \frac{\partial x^\alpha}{\partial y^\beta} V^{\hat\beta}$ $\quad$ (*) $\quad$ y coords

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coords important to calculate

given a world line - $x^\alpha(\tau) \longrightarrow u^\alpha = \frac{dx^\alpha}{d\tau}$

might also calculate $\frac{d^2x^\alpha}{d\tau^2} = V^\alpha \leftarrow$ this

satis

calculate $\quad \underset{\sim}{u}' \cdot \underset{\sim}{V} = f(\tau) \leftarrow$ x coords

$\underset{\sim}{u}' \cdot \underset{\sim}{V} = g(\tau) \leftarrow$ y coords

$f(\tau) \neq g(\tau)$

metric

$$g_{\alpha\beta} = \begin{array}{c|cccc} \alpha \backslash \beta & t & r & \theta & \phi \\ \hline t & & & & \\ r & & & & \\ \theta & & & & \\ \phi & & & & \end{array} \quad \begin{bmatrix} -c^2\left(1-\frac{2GM}{c^2 r}\right) & & & 0 \\ & \frac{1}{1-\frac{2GM}{c^2 r}} & & \\ & 0 & r^2 & \\ & & & r^2 \sin^2\theta \end{bmatrix}$$

Hence V^α is not useful in characterizing worldline

metric

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Hence V^α is not useful in characterizing worldline

$$a^\alpha = \frac{d^2 x^\alpha}{d\tau^2} + \Gamma^\alpha_{\mu\nu} \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} \leftarrow \text{this is a 4 vector}$$

$$\underset{\sim}{u} \cdot \underset{\sim}{a} = 0$$

metric

$$g_{\alpha\beta} = \begin{array}{c|c} {}_{\alpha}\backslash^{\beta} & \begin{matrix} t & r & \theta & \phi \end{matrix} \\ \begin{matrix} t \\ r \\ \theta \\ \phi \end{matrix} & \begin{bmatrix} -c^2\left(1-\frac{2GM}{c^2 r}\right) & & & 0 \\ & \frac{1}{1-\frac{2GM}{c^2 r}} & & \\ & 0 & r^2 & \\ & & & r^2 \sin^2\theta \end{bmatrix} \end{array}$$

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$$a^\alpha = \frac{d^2 x^\alpha}{d\tau^2} + \Gamma^\alpha{}_{\mu\nu} \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}$$ ← this is a 4 vector

So everyone agrees $\underset{\sim}{u} \cdot \underset{\sim}{a} = 0$

Schwarzschild Geometry

$$ds^2 = -\left(1 - 2\frac{GM}{c^2 r}\right)(c\,dt)^2 + \frac{dr^2}{1 - \frac{2GM}{c^2 r}} + r^2 d\theta^2 + r^2 \sin^2\theta$$

simplicity of invariant/scalar objects makes transformations easy

eg $S_\alpha dx^\alpha$

Schwarzschild Geometry

$$ds^2 = -\left(1 - 2\frac{GM}{c^2 r}\right)(c\,dt)^2 + \frac{dr^2}{1 - \frac{2GM}{c^2 r}} + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$$

simplicity of invariant/scalar objects makes transformations easy

eg $S_\alpha dx^\alpha$ rather than $**$

Schwarzschild Geometry

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- solution of "Einstein's eq's" describing geometry outside of a spherical star

metric

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Comments

1

metric

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g_{\alpha\beta} = \begin{array}{c|c}
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$$

Comments

(1) metric is static - independent of t

metric

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\end{array}\right]
$$

Comments

① metric is static - independent of t

Killing vector $\xi^\alpha = \delta^\alpha_t$, or $\underset{\sim}{\xi} = (1,0,0,0)$

Schwarzschild Geometry

$\longrightarrow \; \underset{\sim}{\xi} \cdot \underset{\sim}{u} = \text{const}$ for geodesics

2

Schwarzschild Geometry

$\longrightarrow \quad \underset{\sim}{\xi} \cdot \underset{\sim}{u} = \text{const}$ for geodesics

(2) metric independent of ϕ

$\longrightarrow \quad \underset{\sim}{\eta} = (0, 0, 0, 1)$ or $\eta^{\alpha} = \delta^{\alpha}_{\phi}$

Schwarzschild Geometry

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for fixed (t, r), geometry of surfaces described by

$$d\sigma^2$$

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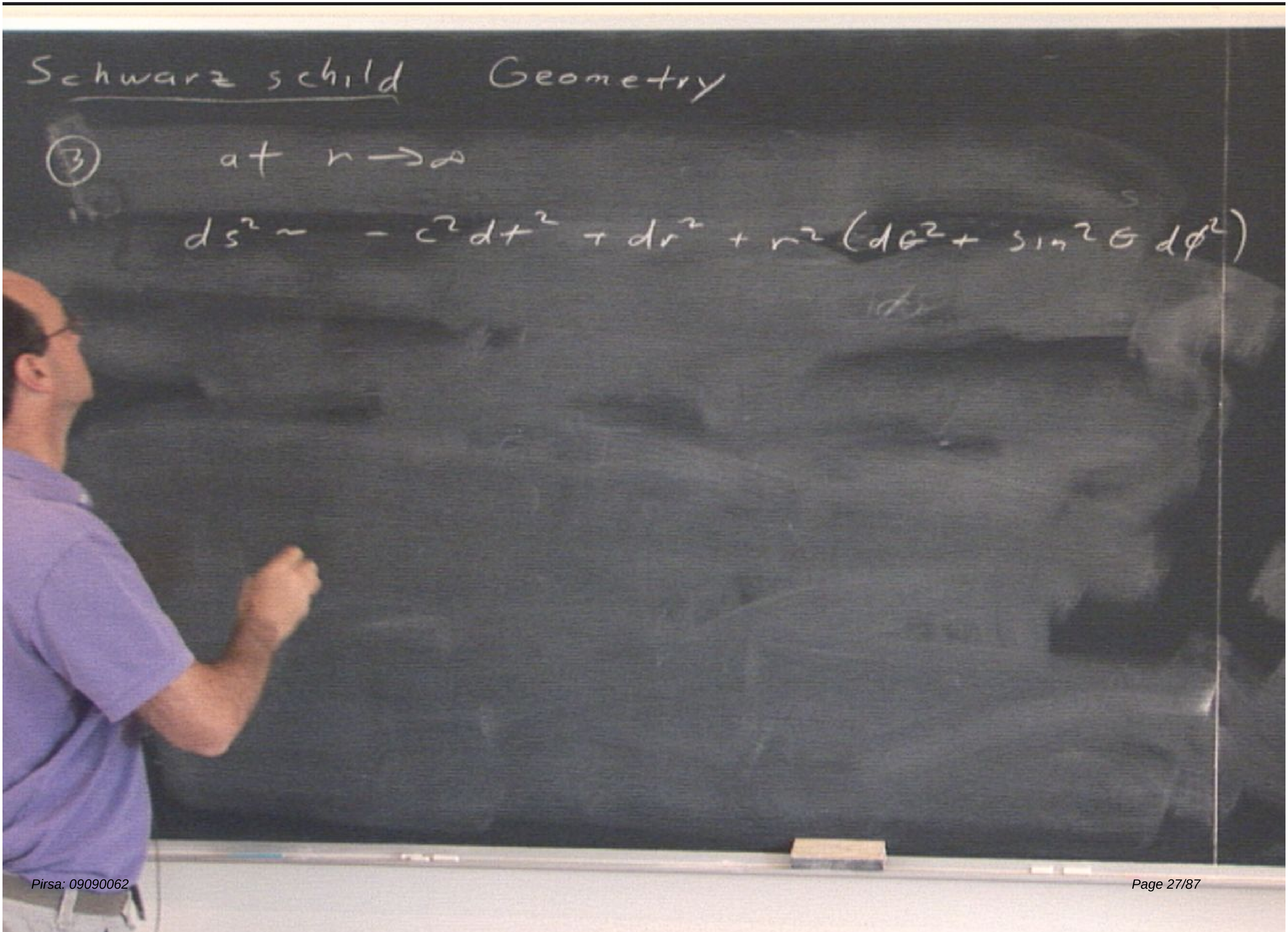
for fixed (t, r), geometry of surfaces described by

$$d\sigma^2 = r^2 (d\theta^2 + \sin^2\theta \, d\phi^2)$$

$\longrightarrow$ same as spherical "polar coord's

$\longrightarrow$ geometry is spherically symmetric

→ so actually would have symmetries under
basis of three independent Killing vectors
($\underset{\sim}{\eta}$ is one of these)


Schwarzschild Geometry
3 at $r \to \infty$
$ds^2 \sim -c^2dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$

Schwarzschild Geometry

③ at $r \to \infty$

$$ds^2 \sim -c^2 dt^2 + dr^2 + r^2 (d\theta^2 + \sin^2\theta \, d\phi^2)$$

$\longrightarrow$ Minkowski space (which is flat)

$\longrightarrow$ asymptotically flat

④

Schwarzschild Geometry

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4. expanding in $1/c^2$ for large c (NR limit)

$$ds^2 \simeq -\left(c^2 + 2\left(-\frac{GM}{r}\right)\right) dt^2 +$$

Schwarzschild Geometry

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$\longrightarrow$ recover the N.R geometry that we discussed earlier for $\Phi/c^2 \ll 1$ and $v^2/c^2 \ll 1$

Schwarzschild Geometry

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→ recover the N.R geometry that we discussed earlier for $\Phi/c^2 \ll 1$ and $v^2/c^2 \ll 1$ for a spherical mass M

Schwarzschild Geometry

(3) at $r \to \infty$

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$\longrightarrow$ recover the N.R geometry that we discussed earlier for $\Phi/c^2 \ll 1$ and $v^2/c^2 \ll 1$ for a spherical mass $\underline{M} \longleftarrow M$ really is mass

$\theta\, d\phi^2)$

$(1-t)$

$)$

$\theta^2 + \sin^2\theta\, d\phi)^2$

we

$v^2/c^2 \ll 1$

is mass

metric

$$g_{\alpha\beta} = \begin{array}{c|cccc} \alpha \backslash \beta & t & r & \theta & \phi \\ \hline t & -c^2\left(1-\frac{2GM}{c^2 r}\right) & & 0 & \\ r & & \frac{1}{1-\frac{2GM}{c^2 r}} & & \\ \theta & & 0 & r^2 & \\ \phi & & & & r^2 \sin \end{array}$$

④ Something interesting happens at r

metric

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④ Something interesting happens at $r=0$ and $r = 2GM/c^2$

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$$

④ something interesting happens at $r=0$
and $r = 2GM/c^2$ ⟵ return to this later

$ds^2 \simeq -c^2 dt^2 + dr^2$

$\longrightarrow$ Minkowski space

$\longrightarrow$ asymptotically

expanding in $1/c^2$ for larg

④

$\boxed{\frac{GM}{c^2 r}}$

$$ds^2 \simeq -\left(c^2 + 2\left(-\frac{GM}{r}\right)\right) dt^2$$

$$-\frac{GM}{r} = \Phi$$

$\longrightarrow$ recover the N.R

discussed earlier f

for a spherical mass

at $r \to \infty$

$$ds^2 \sim -c^2 dt^2 + dr^2 + r^2 (d\theta^2 + \sin^2\theta \, d\phi^2)$$

→ Minkowski space (which is flat)

→ asymptotically flat

expanding in $1/c^2$ for large c (NR limit)

$$ds^2 \simeq -\Big(c^2 + 2\underbrace{\Big(-\frac{GM}{r}\Big)}_{=\Phi}\Big) dt^2 + \underbrace{dr^2 + r^2 (d\theta^2 + \sin^2\theta \, d\phi)^2}_{(dx^i)^2} \qquad 4$$

dt^2

$$\simeq dt^2 \left(-(c^2 + 2\Phi) + \frac{dr^2}{dt^2} + r^2\Big(\frac{d\theta^2}{dt^2} + \ldots\Big) \right)$$

g

③ at $r \to \infty$

$ds^2 \sim -c^2dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi$

$\longrightarrow$ Minkowski space (which is flat)

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④ expanding in $1/c^2$ for large c (NR limit)

Φ/c^2

$$ds^2 \approx -\left(c^2 + 2\left(-\frac{GM}{r}\right)\right)dt^2 + \underbrace{dr^2 + r^2(d\theta^2 + \sin^2\theta}_{(dx^i)^2}$$

dt^2 $\qquad \Phi$

$$\approx c^2dt^2\left(-\left(1 + 2\frac{\Phi}{c^2}\right) + \frac{1}{c^2}\frac{dr^2}{dt^2} + \frac{r^2}{c^2}\left(\frac{d\theta}{dt}\right)^2 +\right.$$

③ at $r \to \infty$

$$ds^2 \sim -c^2 dt^2 + dr^2 + r^2\left(d\theta^2 + \sin^2\theta \, d\phi^2\right)$$

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④ expanding in $1/c^2$ for large c (NR limit)

$\Phi/c^2 \ll 1$

$v^2/c^2 \ll 1$

$$ds^2 \simeq -\Big(c^2 + 2\underbrace{\left(-\frac{GM}{r}\right)}_{=\Phi}\Big)\, dt^2 + \underbrace{dr^2 + r^2\left(d\theta^2 + \sin^2\theta \dots\right)}_{(dx^i)^2}$$

dt^2

$$\simeq c^2 dt^2 \left(-\left(1 + 2\frac{\Phi}{c^2}\right) + \frac{1}{c^2}\left(\frac{dr}{dt}\right)^2 + \frac{1}{c^2} r^2 \left(\frac{d\theta}{dt}\right)^2 + \dots \right.$$
$$\left. + \cancel{2\frac{\Phi}{c^2}\frac{1}{c^2}\left(\frac{dr}{dt}\right)^2} \right)$$

metric

Schwarzschild coord's

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④ Something interesting happens at $r=0$
and $r = 2GM/c^2$ ← return to this lat

Geometrized Units

- recall in SR, c

Geometrized Units

- recall in SR, c was a conversion constant between standard time and length units

$\rightarrow$ measure time in length units $\rightarrow c=1$

$$\frac{1}{1-\frac{2GM}{c^2 r}}$$

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→ similarly in GR find M

Geometrized Units

- recall in SR, c was a conversion constant between standard time and length units

$\rightarrow$ measure time in length units $\rightarrow c = 1$

$\rightarrow$ similarly in GR find M accompanied by G/c^2 $\rightarrow$ conversion constant to change mass units (kg) to length units (metres)

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- recall in SR, c was a conversion constant between standard time and length units

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- recall in SR, c was a conversion constant between standard time and length units

→ measure time in length units → $c = 1$

→ similarly in GR find M accompanied by G/c^2 → conversion constant to change mass units (kg) to length units (metres)

→ if mass measured in length units → $G/c^2 = 1$

Schwarzschild Geometry

$\Longrightarrow \quad G = 1 \,, \quad c = 1$

"geometrized" units

$\longrightarrow$ asymptotically flat

(4) expanding in $1/c^2$ for large c (NR limit)

$$ds^2 = -\Big(c^2 + 2\underbrace{\left(-\frac{GM}{r}\right)}_{=\,\Phi}\Big)\, dt^2 + \underbrace{dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)}_{(dx^i)^2}$$

$\Phi/c^2 \ll 1$

$v^2/c^2 \ll 1$

$$\simeq c^2 dt^2 \left(-\left(1 + 2\frac{\Phi}{c^2}\right) + \frac{1}{c^2}\left(\frac{dr}{dt}\right)^2 + \frac{1}{c^2} r^2 \left(\left(\frac{d\theta}{dt}\right)^2 + \ldots\right) \right.$$
$$\left. + \cancel{2\frac{\Phi}{c^2}\frac{1}{c^2}\left(\frac{dr}{dt}\right)^2} \right)$$

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\end{array}
$$

Schwarzschild coord's

$\theta\, d\phi)^2$

(4) Something interesting happens at $r=0$ and $r = 2GM/c^2$ ← return to this later

$\ldots))$

metric

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Schwarzschild coord's
→ in geom. units

$\theta \, d\phi)^2$

$\ldots))$

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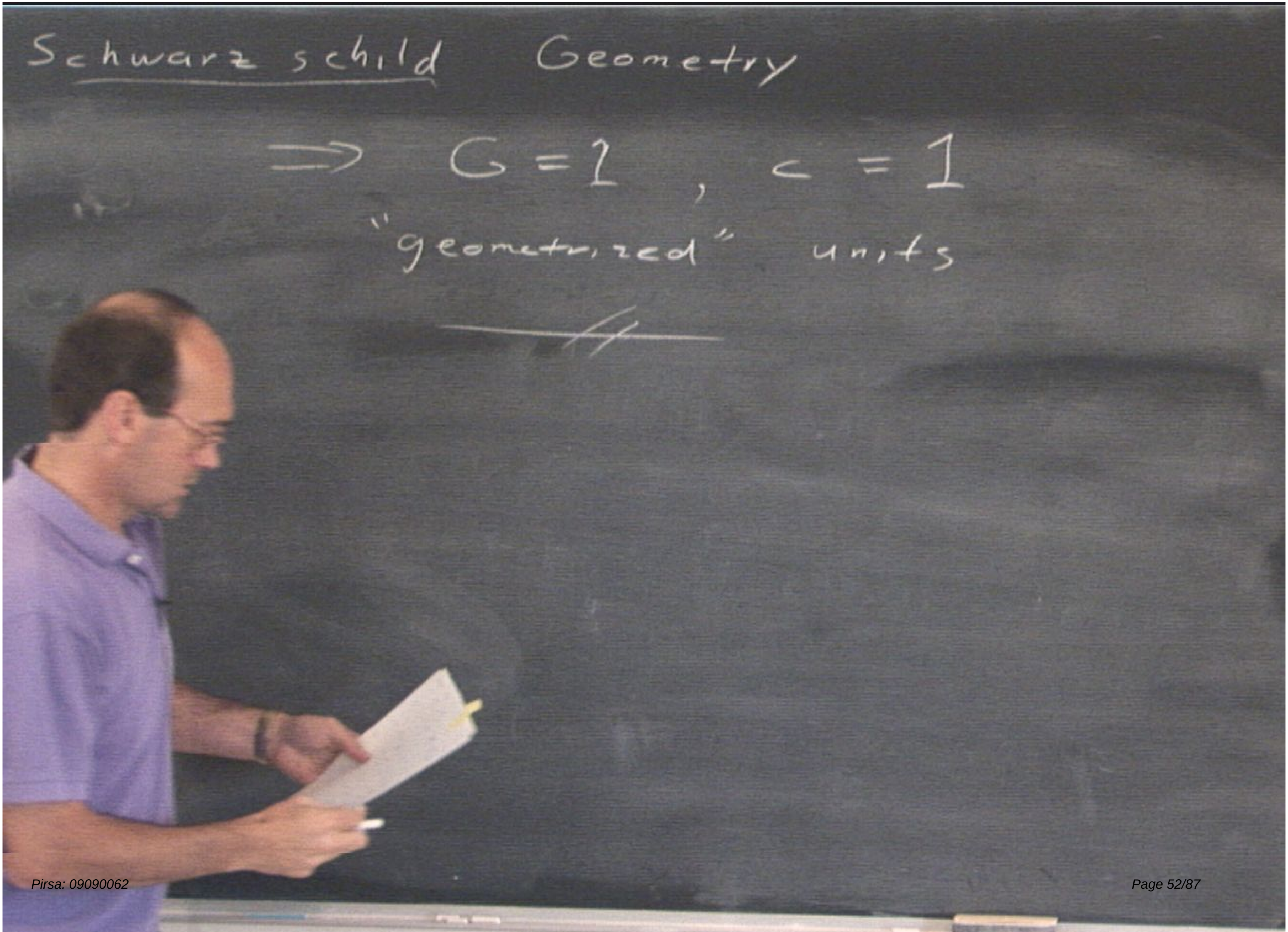
Schwarzschild Geometry
⇒ G = 1 , c = 1
"geometrized" units

Schwarzschild Geometry

$$\Rightarrow \quad G = 1 \ , \quad c = 1$$

"geometrized" units

particle motion

$$I = m \int d\lambda \; \underbrace{g_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda}}_{L}$$

hwarzschild Geometry

$$\Rightarrow \quad G = 1 \,, \quad c = 1$$

"geometrized" units

particle motion

$$I = m \int d\lambda \underbrace{g_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda}}_{L}$$

$$\tilde{L} = -\left(1-\frac{2m}{r}\right)\left(\frac{dt}{d\tau}\right)^2 + \frac{(dr/d\tau)^2}{1-2m/r} + r^2\left(\left(\frac{d\theta}{d\tau}\right)^2 + \sin^2\theta\left(\frac{d\phi}{d\tau}\right)^2\right)$$

Schwarzschild Geometry

$\Rightarrow \quad G = 1 \, , \quad c = 1$

"geometrized" units

+particle motion

$$I = m \int d\lambda \; \underbrace{g_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda}}_{L}$$

$$\tilde{L} = -\left(1 - \frac{2m}{r}\right)\left(\frac{dt}{d\lambda}\right)^2 + \frac{(dr/d\lambda)^2}{1 - 2m/r} + r^2\left(\left(\frac{d\theta}{d\lambda}\right)^2 + \sin^2\theta\left(\frac{d\phi}{d\lambda}\right)^2\right)$$

→ first look for conserved quantities

2 Killing vectors:

$$\underset{\sim}{\xi}^{\alpha} = \delta^{\alpha}_{t} \longrightarrow \underset{\sim}{\xi} \cdot \underset{\sim}{u} = \text{const}$$

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→ as a ~~const~~ constraint, impose

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as a constraint impose $g_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} = -1$

$\Rightarrow$ first look for conserved quantities

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$$-\left(1-\frac{2M}{r}\right)\frac{dt}{d\tau} =$$

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$$\frac{dt}{d\tau} = \frac{e}{1 - \frac{2m}{r}}$$

particle motion

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$$\frac{dt}{d\tau} = \frac{e}{1-\frac{2M}{r}} \xrightarrow[r\to\infty]{} e = \text{energy}/\text{mass}$$

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$$\frac{dt}{d\tau} = \frac{e}{1 - \frac{2M}{r}} \xrightarrow[r \to \infty]{} e = \text{relativistic energy/mass}$$

$$\xi^\alpha = \delta^\alpha_\phi \longrightarrow \underset{\sim}{\xi} \cdot \underset{\sim}{u} = \text{const}$$

$$g_{\phi\phi} \frac{d\phi}{d\tau} = \text{const}$$

$$r^2 \sin^2\theta \, \frac{d\phi}{d\tau} =$$

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$$\frac{d\phi}{d\tau} = \frac{\ell}{r^2 \sin^2\theta} \longleftarrow \text{angular momentum / mass}$$

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– need one more eq: look at geodesic eq for Θ

$$\delta\theta \rightarrow \quad \frac{d}{d\tau}\frac{\delta\tilde{L}}{\delta\dot{\theta}} = \frac{\delta\tilde{L}}{\delta\theta}$$

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$\longrightarrow$ but using spherical symmetry to orient coord's can always set any orbit to be in plane $\theta = \pi/2$

(orbits were planar)

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with some algebra,

$$\frac{e^2-1}{2}$$

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$$\frac{e^2-1}{2} = \frac{1}{2}\dot{r}^2 + \frac{1}{2}\left[\left(1-\frac{2m}{r}\right)\left(1+e^2/r^2\right) - 1\right]$$

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looks like 1-dim classical motion in potential

$$\mathcal{E} = \frac{1}{2}\dot{r}^2 + V(r)$$

effective energy $\mathcal{E} = \frac{e^2-1}{2}$

effective potential $V(r) =$

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in Newtonian grav

effective potential $V(r) = \overbrace{-\frac{M}{r} + \frac{\ell^2}{2r^2}}$

$$-\frac{M\ell^2}{r^3}$$

GR correction

$$\mathcal{E} = \frac{1}{2}\dot{r}^2 + V(r)$$

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Prob 7-7

show: $\left(\frac{d\phi}{dt}\right)^2 R^3 = \text{const}$ for some set of orbit

$$\left(\frac{d\phi}{dt}\right)^2 R^3 = \mathfrak{M}(M, r, e, \ell)$$

Prob 7-7 ...

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$$= \mathfrak{M}(M)$$

Prob ?-?...

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$$\left(\frac{d\phi}{dt}\right)^2 R^3 = \mathfrak{M}(M, R, e, \ell)$$

$$= \mathfrak{M}(M) \leftarrow \text{physics}$$

algebra