Title: PSI - Relativity (PHYS 604) - 6

Date: Sep 10, 2009 10:30 AM

URL: http://pirsa.org/09090061

Abstract:

Geodesics + Particle Motion

$$I = m \int d\lambda \; g_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda}$$

Geodesics + Particle Motion

$$I = m \int d\lambda \; g_{\alpha\beta}(x) \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda}$$

$$- \frac{\partial}{\partial\lambda} \frac{\delta L}{\delta \frac{\partial x^\mu}{\partial \lambda}} + \frac{\delta L}{\delta x^\mu} = 0$$

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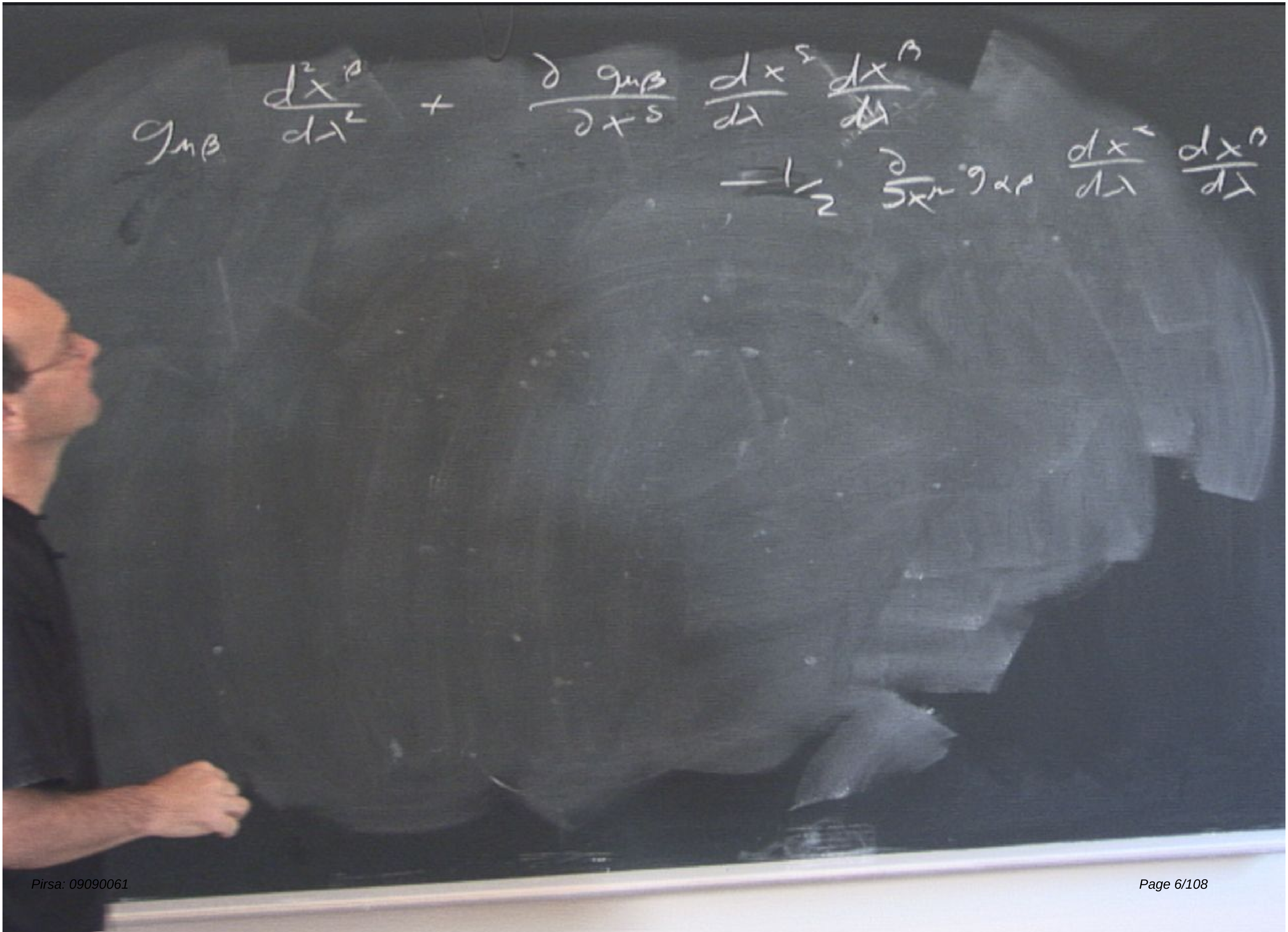
$$\frac{\partial}{\partial\lambda} g_{\mu\beta} = \frac{\partial}{\partial x^\delta} g_{\mu\beta} \frac{dx^\delta}{d\lambda}$$



$g_{\mu\beta} \frac{d^2x^\beta}{d\lambda^2} + \frac{\partial g_{\mu\beta}}{\partial x^\delta} \frac{dx^\delta}{d\lambda} \frac{dx^\beta}{d\lambda}$
$- 1/2 \frac{\partial}{\partial x^\mu} g_{\alpha\rho} \frac{dx^\alpha}{d\lambda} \frac{dx^\rho}{d\lambda}$

$$0 = g_{\mu\beta} \frac{d^2 x^\beta}{d\lambda^2} + \frac{\partial g_{\mu\beta}}{\partial x^\delta} \frac{dx^\delta}{d\lambda} \frac{dx^\beta}{d\lambda} - \frac{1}{2} \frac{\partial}{\partial x^\mu} g_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda}$$

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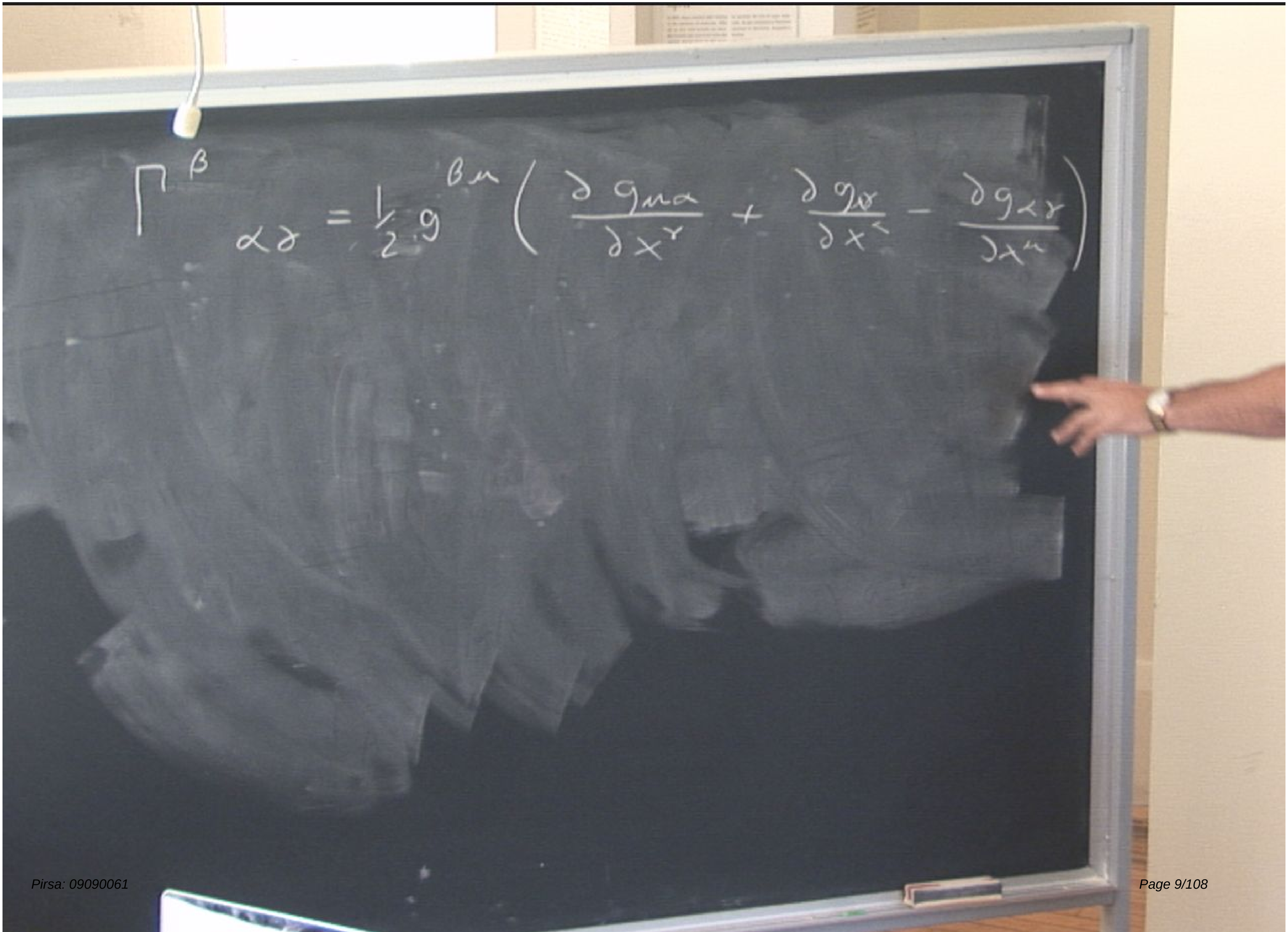
$$0 = \frac{d^2 x^\beta}{d\lambda} + \Gamma^\beta{}_{\alpha\gamma} \frac{dx^\alpha}{d\lambda} \frac{dx^\gamma}{d\lambda}$$


$\Gamma^{\beta}_{\alpha\gamma} = \frac{1}{2} g^{\beta\mu} \left(\frac{\partial g_{\mu\alpha}}{\partial x^{\gamma}} + \frac{\partial g_{\mu\gamma}}{\partial x^{\alpha}} - \frac{\partial g_{\alpha\gamma}}{\partial x^{\mu}} \right)$

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$$= \Gamma^{\beta}{}_{\gamma\alpha}$$

$$0 = g_{\mu\beta}\frac{d^2x}{d\lambda^2} + \frac{\partial g_{\mu\beta}}{\partial x^s}\frac{dx}{d\lambda}\frac{dx}{d\lambda}$$

$$-\frac{1}{2}\frac{\partial}{\partial x^\mu} g_{\alpha\rho}\frac{dx}{d\lambda}$$

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Christoffel symbol

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Christoffel symb

geodesic equation

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Christoffel symb

geodesic equation

– coordinate invariant express

$$g_{\mu\beta}\frac{d^2 x^\beta}{d\lambda^2} + \frac{\partial g_{\mu\beta}}{\partial x^\delta}\frac{dx^\delta}{d\lambda}\frac{dx^\beta}{d\lambda}$$

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$$= \frac{d^2 x^\beta}{d\lambda} + \underline{\Gamma^\beta{}_{\alpha\gamma}}\frac{dx^\alpha}{d\lambda}\frac{dx^\gamma}{d\lambda}$$

Christoffel symbol

geodesic equation

- coordinate invariant expression
- reduces to expected eq $\frac{d^2 x^\alpha}{d\lambda^2} = 0$

$$g_{\mu\beta}\frac{d^2x^\mu}{d\lambda^2} + \frac{\partial g_{\mu\beta}}{\partial x^\delta}\frac{dx^\delta}{d\lambda}\frac{dx^\mu}{d\lambda}$$

$$-\frac{1}{2}\frac{\partial}{\partial x^\mu} g_{\alpha\rho}\frac{dx^\alpha}{d\lambda}\frac{dx^\rho}{d\lambda}$$

$$= \frac{d^2x^\beta}{d\lambda} + \underline{\Gamma^\beta{}_{\alpha\gamma}}\frac{dx^\alpha}{d\lambda}\frac{dx^\gamma}{d\lambda}$$

Christoffel symbol

geodesic equation

- coordinate invariant expression
- reduces to expected S.R. eq $\frac{d^2x^\alpha}{d\lambda^2} = 0$

→ also reduces to this simple result when $\frac{\partial}{\partial x^\mu} g_{\alpha\beta} = 0$

- λ independence of L gives extra conserved quantity
-
-

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- $$\frac{dx^\alpha}{d\lambda} \frac{\delta L}{\delta \frac{dx^\alpha}{d\lambda}} - L = \underline{\text{const}}$$

-

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$$\frac{dx^\alpha}{d\lambda} \frac{\delta L}{\delta \frac{dx^\alpha}{d\lambda}} - L = \underline{\text{const}}$$

$\longrightarrow$

$$g_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} = \text{const}$$

-

- λ indopendence of L gives extra conserved quantity

- $$\frac{dx^\alpha}{d\lambda}\frac{\delta L}{\delta \frac{dx^\alpha}{d\lambda}} - L = \underline{\text{const}}$$

$\longrightarrow$

$$g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = \text{const}$$

make chace that's guided ~~bx~~ physic

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make chace that's guided by physics

$\rightarrow$ worldlines of observers/particles $\rightarrow$ timelike

$\rightarrow$ norm^2 of tangent < 0

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$\longrightarrow$

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make choice that's guided by physics

$\rightarrow$ worldlines of observers/particles $\rightarrow$ timelike

$\longrightarrow$ norm^2 of tangent < 0

$$g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = -k^2$$

$$g_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} = \text{const}$$

hace that's guided by physics

s of observers/particles $\longrightarrow$ timelike

rm^2 of tangent < 0

$$g_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} = -c^2 \leftarrow$$

choice amounts to

scaling $\lambda = \tau$

mathematically makes sense to choose positive or zero const. as well

$$= \frac{d^2 x^\beta}{d\lambda} + \underline{\Gamma^\beta{}_{\alpha\gamma}} \frac{dx^\alpha}{d\lambda}\frac{dx^\gamma}{d\lambda}$$

Christoffel symbol

geodesic equation

- coordinate invariant expression
- reduces to expected S.R. eq $\frac{d^2 x^\alpha}{d\lambda^2}$

→ mathematically makes sense fo

choose positive or zero con

as well

for general motion:

$$a^{\alpha} = \frac{d^2 x^{\beta}}{d\lambda} + \Gamma^{\beta}{}_{\alpha\gamma} \frac{dx^{\alpha}}{d\lambda} \frac{dx^{\gamma}}{d\lambda}$$

cs

nts to

→ mathematically makes sense fo[r]
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four-acceleration

→ mathematically makes sense fo[r]

choose positive or zero con[...]

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$$a^{\beta} = \frac{d^2 x^{\beta}}{d\lambda} + \Gamma^{\beta}{}_{\alpha\gamma} \frac{dx^{\alpha}}{d\lambda} \frac{dx^{\gamma}}{d\lambda}$$

four-acceleration

→ just as S.R., $\underset{\sim}{u} \cdot \underset{\sim}{a} = 0$

Christoffel symbol

$$\Gamma^{\beta}{}_{\alpha\gamma} = \frac{1}{2} g^{\beta\mu} \left(\frac{\partial g_{\mu\alpha}}{\partial x^{\gamma}} + \frac{\partial g_{\mu\gamma}}{\partial x^{\alpha}} - \right.$$

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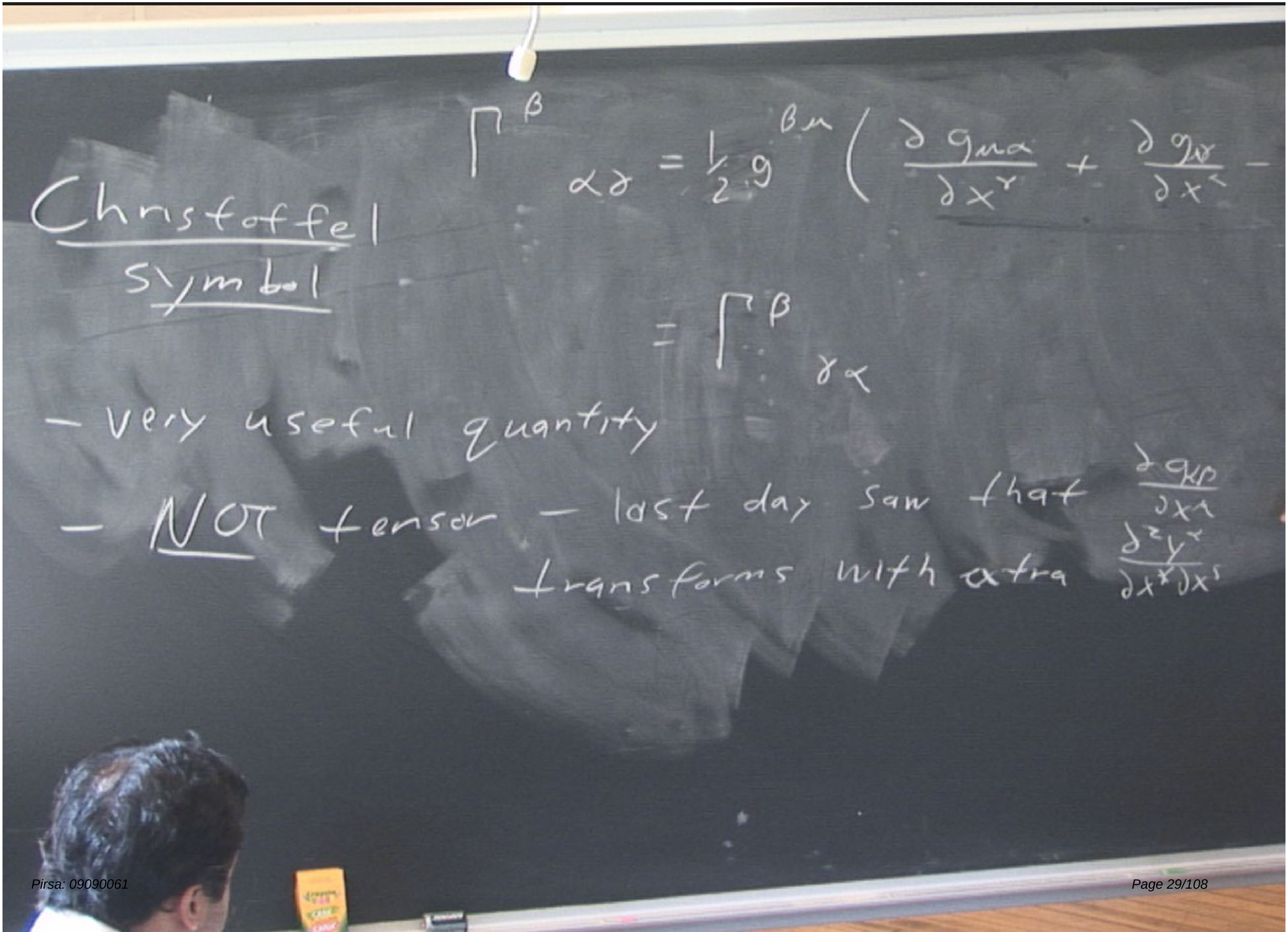
Christoffel symbol

$$= \Gamma^{\beta}{}_{\gamma\alpha}$$

- very useful quantity
- NOT tensor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toffel

mbol

$$= \Gamma^{\beta}{}_{\gamma\alpha}$$

useful quantity

OT tensor — last day saw that $\frac{\partial g_{\kappa\beta}}{\partial x^{\alpha}}$

transforms with extra $\frac{\partial^2 y^{\kappa}}{\partial x^{\gamma} \partial x^{\delta}}$ stuff r - acceleration

ast. as S.R.

)

$$\frac{\partial}{\partial \tau}\left(u^{\alpha} g_{\alpha\beta} u^{\beta}\right) = 0$$

$$\Gamma^{\beta}{}_{\alpha\gamma} = \frac{1}{2} g^{\beta\mu}\left(\frac{\partial g_{\mu\alpha}}{\partial x^{\gamma}} + \frac{\partial g_{\mu\gamma}}{\partial x^{\alpha}} - \frac{\partial g_{\alpha\gamma}}{\partial x^{\mu}}\right)$$

fel

l

$$= \Gamma^{\beta}{}_{\gamma\alpha}$$

seful quantity

tensor — last day saw that $\frac{\partial g_{\kappa\rho}}{\partial x^{\lambda}}$

transforms with extra $\frac{\partial^2 y^{\alpha}}{\partial x^{\gamma}\partial x^{s}}$ stuff

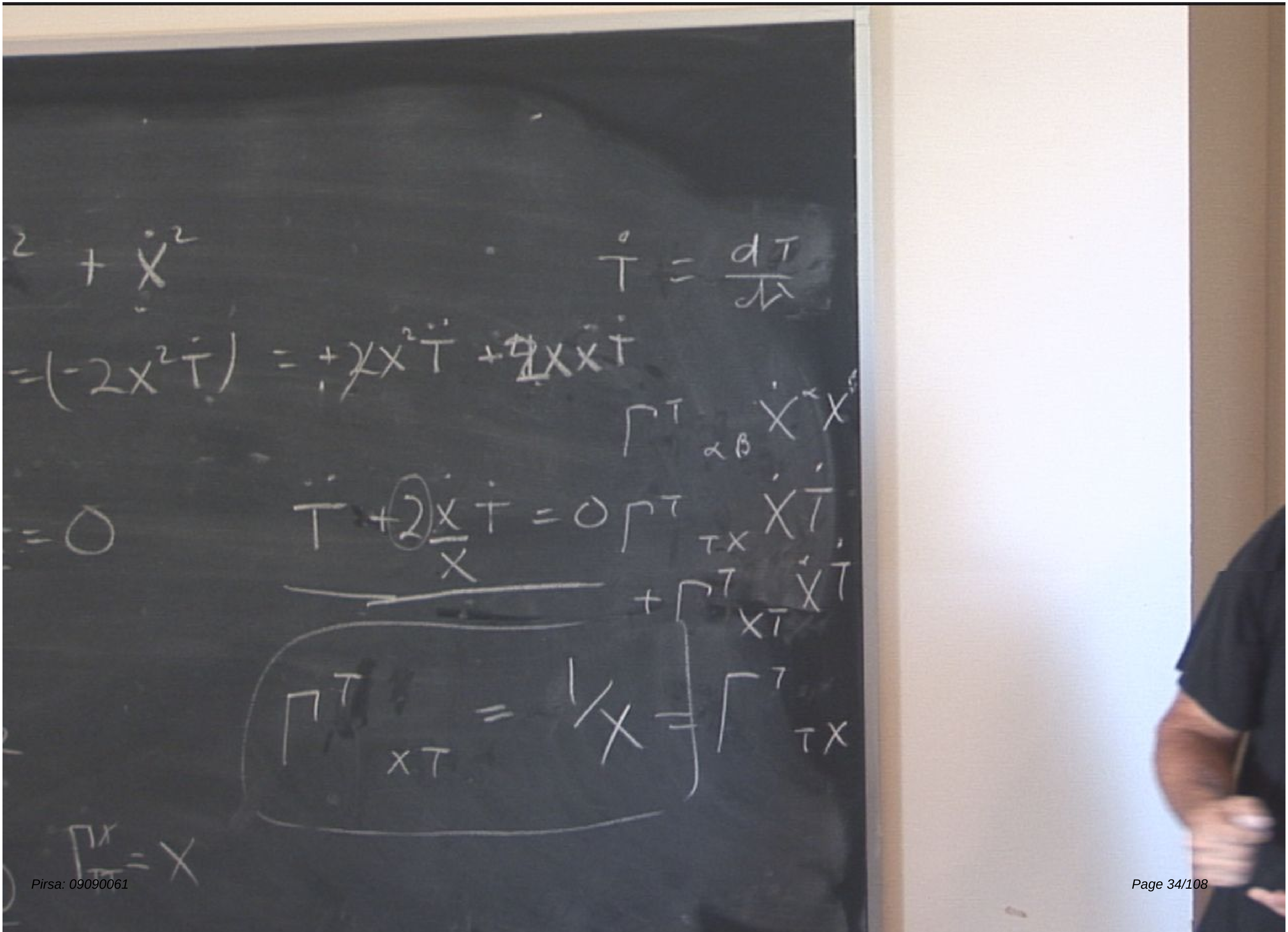
- entire geodesic eq defines a tensor

- don't use this eq. → instead use

action → vary to produce

geodesic eq → read off Γ

xample : $ds^2 = -X^2 dT^2 + dX^2$



+ Ẋ²
Ṫ = dT/dλ
=(-2x²Ṫ) = +2xx²Ṫ + 2xẋṪ
Γ^T_αβ Ẋ^α Ẋ^β
= 0
T̈ + 2 ẋ/x Ṫ = 0
Γ^T_TX Ẋ Ṫ
+ Γ^T_XT Ẋ Ṫ
Γ^T_XT = 1/X = Γ^T_TX
Γ^X_TT = X

$$g_{\mu\nu} = \begin{pmatrix} -X^2 & 0 \\ 0 & 1 \end{pmatrix}$$

$$L = g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = -X^2\dot{T}^2 + \dot{X}^2 \qquad \dot{T} = \frac{dT}{d\lambda}$$

$$\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{T}}\right) = (-2X^2\dot{T}) = +2X^2\ddot{T} + 4X\dot{X}\dot{T}$$

$$\frac{\partial L}{\partial \dot{x}^\alpha} = g_{\mu\nu}\frac{\partial \dot{x}^\mu}{\partial \dot{x}} \qquad \Gamma^T{}_{\alpha\beta}\dot{X}^\alpha\dot{X}^\beta$$

$$\frac{\partial L}{\partial T} = 0 \qquad \ddot{T} + 2\frac{\dot{X}}{X}\dot{T} = 0 \qquad \Gamma^T{}_{TX}\dot{X}\dot{T} + \Gamma^T{}_{XT}\dot{X}\dot{T}$$

$$\frac{\partial L}{\partial \dot{X}} = 2\dot{X} =$$

$$\boxed{\Gamma^T{}_{XT} = 1/X = \Gamma^T{}_{TX}}$$

$$2\ddot{X} = -2X\dot{T}^2$$

$$\ddot{X} + X\dot{T}^2 = 0 \qquad \Gamma^X_{TT} = X$$

fo

Null geodesics

- for a null trajectory

$$g_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} = 0$$

→ consider $\lambda = a\sigma^3 + b\sigma^2 + c$

Null geodesics

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→ but we can also use geodesic

$$\frac{d^2x^{\beta}}{d\lambda^2} + \Gamma^{\beta}_{\alpha\gamma}\,\frac{dx^{\alpha}}{d\lambda}\,\frac{dx^{\beta}}{d\lambda} = 0$$

Null geodesics

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$$\frac{d^2x^\beta}{d\lambda^2} + \Gamma^\beta{}_{\alpha\gamma} \frac{dx^\alpha}{d\lambda} \frac{dx^\gamma}{d\lambda} = 0$$

→ compatible with constraint → picks one particular convenient set of para

1) eq. in coord invariant

2) reduces to $\frac{d^2x^\alpha}{d\lambda^2} = 0$ in

SR or ~~for~~ freely falling frame

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Note photon

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<u>Note</u> photon has k^α

→ $k^\alpha \parallel \frac{dx^\alpha}{d\lambda}$

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Note photon has k^α $\parallel$ k^1, $\frac{dx^1}{d\lambda}$ points direction

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Note photon has k^α k^i, $\frac{dx^i}{d\lambda}$ points direction in space

$$\rightarrow \quad k^\alpha \parallel \frac{dx^\alpha}{d\lambda}$$

$\frac{dx^0}{d\lambda} > 0$

1) eq. in coord invariant

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Note photon has k^α $\parallel$ k^i, $\frac{dx^i}{d\lambda}$ points direction in space

$\rightarrow$ $k^\alpha \parallel \frac{dx^\alpha}{d\lambda}$

$\frac{dx^0}{d\lambda} > 0$ towards future

$k^0 > 0 \leftarrow$ positive energy

Symmetries + Conserved quantities

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– analog applies for particle motion

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→ if $g_{\alpha\beta}(x)$ is independent of one of the coordinates, then there is an extra conserved quant

consider $L = m\, g_{\alpha\beta}(x) \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda}$

→ imagine $g_{\alpha\beta}$ is independent of x^A

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→ eom from δx^A →

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→ imagine $g_{\alpha\beta}$ is independent of x^A

→ eom from $\delta x^A \to \frac{d}{d\lambda}\left(g_{A\beta}\frac{dx^\beta}{d\lambda}\right) = 0$

$$g_{A\beta}\frac{dx^\beta}{d\lambda} = \underline{\text{const}}$$

→ can still shift $\lambda = a\hat{\lambda} + b$

__Note__ photon has k^α

$k^i, \frac{dx^i}{d\lambda}$ points direction in space

→ $k^\alpha \parallel \frac{dx^\alpha}{d\lambda}$

$\frac{dx^0}{d\lambda} > 0$ towards future

$k^0 > 0$ ← positive energy

$$g_{A\beta} \frac{dx^\beta}{d\lambda} = \underline{\text{const}}$$

construct a vector k^α

$\rightarrow$ can still shift $\lambda = a\hat{\lambda} + b$

__Note__ photon has k^α $\left(k^i, \frac{dx^i}{d\lambda} \right.$ points direction in space

$\rightarrow k^\alpha \parallel \frac{dx^\alpha}{d\lambda}$

towards future

$\frac{dx^0}{d\lambda} > 0$

$k^0 > 0 \leftarrow$ positive energy

$$g_{A\beta} \frac{dx^\beta}{d\lambda} = \underline{\text{const}}$$

construct a vector K^α which picks out special direction $K^\alpha = \delta^\alpha_A$

$\rightarrow$ can still shift $\lambda = a\hat{\lambda} + b$

<u>Note</u> photon has k^α k^i, $\frac{dx^i}{d\lambda}$ points direction in space

$\longrightarrow k^\alpha \parallel \frac{dx^\alpha}{d\lambda}$

towards future

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special direction $K^\alpha = \delta^\alpha_A$

$$K^\alpha g_{\alpha\beta}\frac{dx^\beta}{d\lambda} = \text{const}$$

$$\underset{\sim}{K}\cdot\underset{\sim}{u} = \underline{\text{const}}$$

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construct a vector K^α which picks out special direction $K^\alpha = \delta^\alpha_A$

$$K^\alpha g_{\alpha\beta}\frac{dx^\beta}{d\lambda} = \text{const}$$

$$\underset{\sim}{K}\cdot\underset{\sim}{u} = \underline{\text{const}} \leftarrow \underline{\text{looks coord}}$$

$\underline{\text{Invariant}}$

$\rightarrow$ had special case.

$$K^\alpha \partial_\alpha g_{\gamma\delta} = 0$$

$$g_{A\beta}\frac{dx^\beta}{d\lambda} = \underline{\text{const}}$$

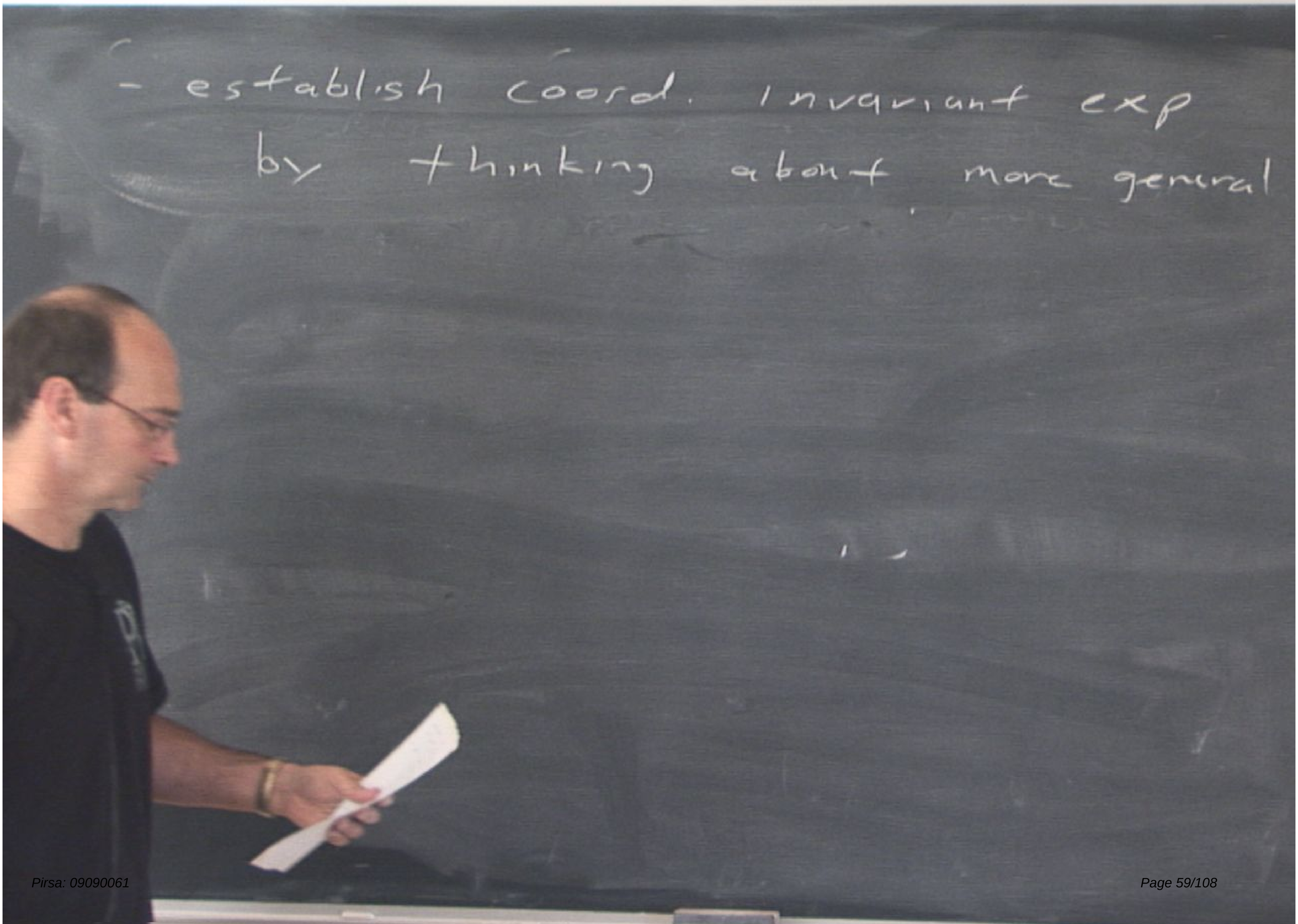
construct a vector K^α which picks out special direction $K^\alpha = \delta^\alpha_A$

$$K^\alpha g_{\alpha\beta}\frac{dx^\beta}{d\lambda} = \text{const}$$

$$\underset{\sim}{K}\cdot\underset{\sim}{u} = \underline{\text{const}} \leftarrow \underline{\text{looks coord}}\ \underline{\text{Invariant}}$$

→ had special case:

$$K^\alpha \partial_\alpha g_{\gamma\delta} = 0 \quad \partial_\beta K^\alpha = 0$$


- establish coord. invariant exp
by thinking about more general

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  by thinking about more general case

→ any path $x^\alpha(\lambda)$ (not necessarily geodesics)

→ K^α associated with symmetry in geometry

$$\hat{x}^\alpha = x^\alpha(\lambda) + \varepsilon K^\alpha$$

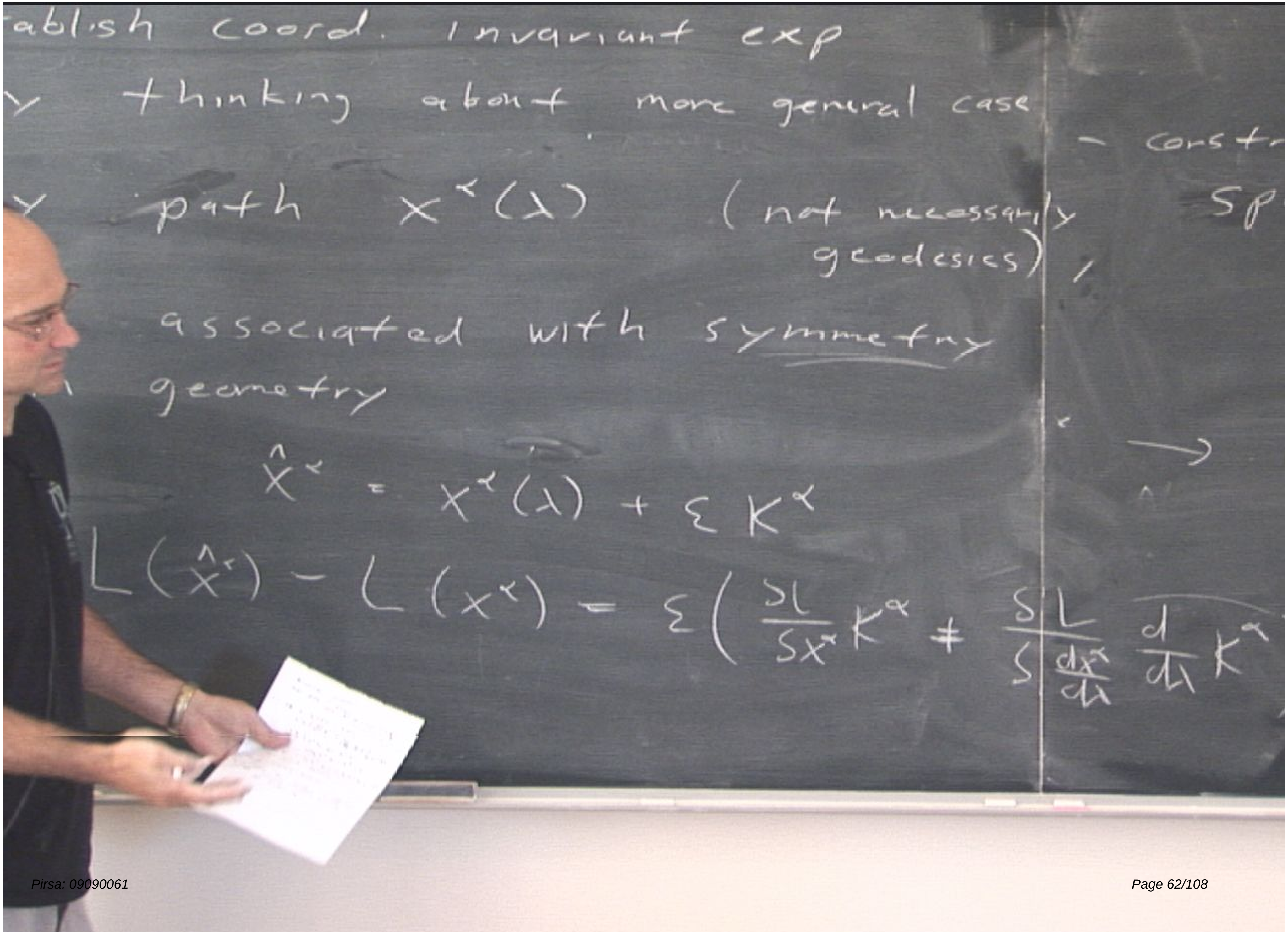
- establish coord. invariant exp
  by thinking about more general cas[e]

→ any path $x^\alpha(\lambda)$ (not necessa[rily] geodesics)

→ K^α associated with symmetry in geometry

$$\hat{x}^\alpha = x^\alpha(\lambda) + \varepsilon K^\alpha$$

$$\Rightarrow \delta L = L(\hat{x}^\alpha) - L(x^\alpha) =$$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Sp
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$\hat{X}^\alpha = x^\alpha(\lambda) + \varepsilon K^\alpha$
$L(\hat{X}^\alpha) - L(x^\alpha) = \varepsilon\left(\frac{\delta L}{\delta x^\alpha} K^\alpha + \frac{\delta L}{\delta \frac{dx^\alpha}{d\lambda}} \frac{d}{d\lambda} K^\alpha \right.$

-ing about more general case

— construct a ve

h $x^\alpha(\lambda)$ (not necessarily geodesics)

special dir

K^α

ociated with symmetry

etry

$\underset{\sim}{K}$

$\longrightarrow$ had spec

$$\hat{x}^\alpha = x^\alpha(\lambda) + \varepsilon K^\alpha$$

$$\frac{d\hat{x}^\beta}{d\lambda}\partial_\beta K^\alpha \qquad K^\alpha \partial_\alpha g_{\gamma\delta}$$

$$- L(x^\alpha) = \varepsilon\left(\frac{\delta L}{\delta x^\alpha}K^\alpha + \frac{\delta L}{\delta \frac{dx^\alpha}{d\lambda}}\frac{d}{d\lambda}K^\alpha\right)$$

$$= \Sigma \Big(K^\alpha \partial_\alpha g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu$$

$$\frac{dx^\beta}{d\lambda} \partial_\beta K^\alpha$$

$$\frac{d}{d\lambda} K^\alpha \Big)$$

$$= \Sigma \Big(K^{\alpha} \partial_{\alpha} g_{\mu\nu} \dot{x}^{\mu} \dot{x}^{\nu}$$

$$+ \dot{x}^{\beta} \partial_{\beta} K^{\alpha} g_{\alpha\gamma} \dot{X}^{\gamma} \times 2$$

$$\frac{dx^{\beta}}{d\lambda} \partial_{\beta} K^{\alpha}$$

$$\frac{d}{d\lambda} K^{\alpha}$$

$$0 = \epsilon \Big(K^\alpha \partial_\alpha g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu + \dot{x}^\beta \partial_\beta K^\alpha g_{\alpha\gamma} \dot{x}^\gamma \times 2 \Big)$$

If true for any path:

$$K^\alpha \partial_\alpha g_{\mu\nu} + \partial_\mu K^\alpha g_{\alpha\nu} + \partial_\nu K^\alpha g_{\alpha\mu} = 0$$

$$\frac{dx^\beta}{d\lambda} \partial_\beta K^\alpha$$

$$\|$$

$$\frac{d}{d\lambda} K^\alpha$$

$$0 = \varepsilon \left(K^\alpha \partial_\alpha g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu + \dot{x}^\beta \partial_\beta K^\alpha g_{\alpha\gamma} \dot{x}^\gamma \times 2 \right)$$

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$\Longrightarrow$ Killing equation

$$\frac{d}{d\lambda} K^\alpha$$

$$0 = \Sigma \left(K^\alpha \partial_\alpha g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu + \dot{x}^\beta \partial_\beta K^\alpha g_{\alpha\gamma} \dot{x}^\gamma \times 2 \right)$$

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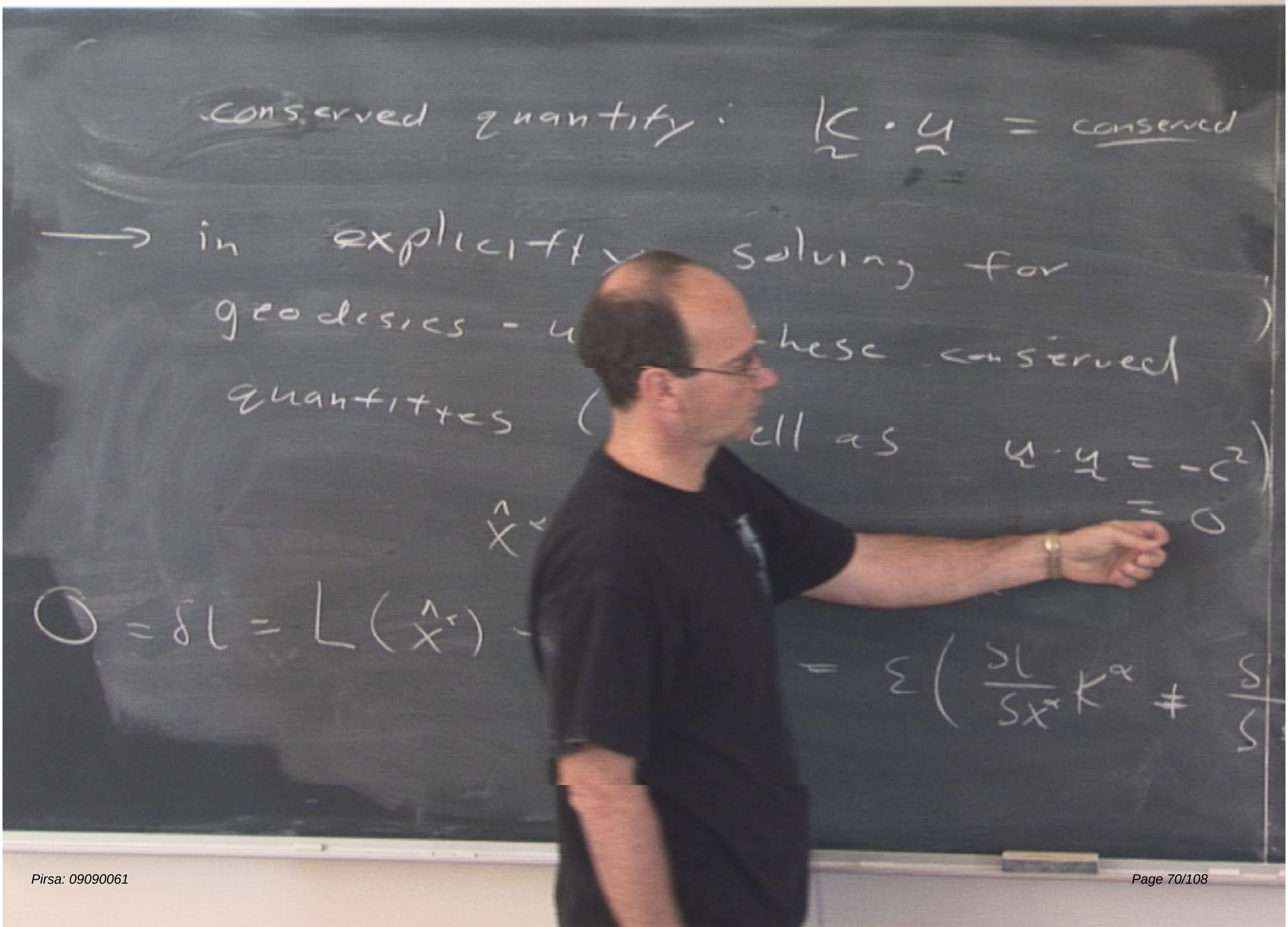
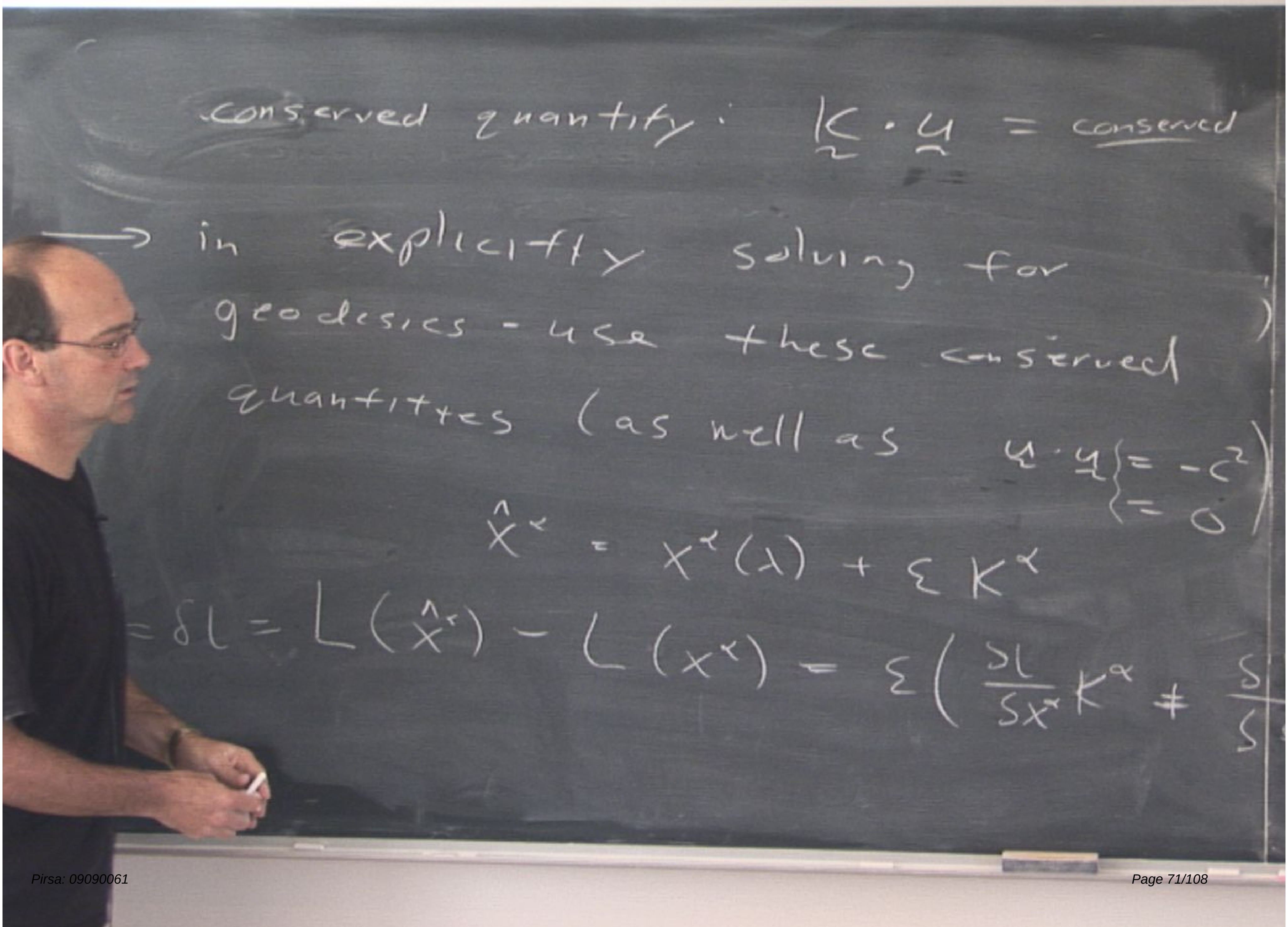
$\Longrightarrow$ Killing equation

$\longrightarrow$ coordinate invariant expression that picks symmetries conserved quantity: $\underset{\sim}{K} \cdot \underset{\sim}{u} = \underline{\text{conserved}}$

$$\hat{x}^{\alpha} = x^{\alpha}(\lambda) + \varepsilon K^{\alpha}$$

$$0 = \delta L = L(\hat{x}^{\alpha}) - L(x^{\alpha}) = \varepsilon\left(\frac{\partial L}{\delta x^{\alpha}} K^{\alpha} + \frac{\delta}{\delta}\right.$$


conserved quantity: K · u = conserved
in explicit
solving for
geodesics - u
these conserved
quantities (
ell as u · u = -c²
= 0
0 = δL = L(x̂
= Σ(∂L/∂x K


conserved quantity: $\underset{\sim}{K} \cdot \underset{\sim}{u}$ = conserved
→ in explicitly solving for
geodesics - use these conserved
quantities (as well as $\underset{\sim}{u} \cdot \underset{\sim}{u} \begin{cases} = -c^2 \\ = 0 \end{cases}$)
$\hat{x}^\alpha = x^\alpha(\lambda) + \varepsilon K^\alpha$
$= \delta L = L(\hat{x}^\alpha) - L(x^\alpha) = \varepsilon\left(\frac{\partial L}{\partial x^\alpha} K^\alpha + \frac{\partial}{\partial}\right.$

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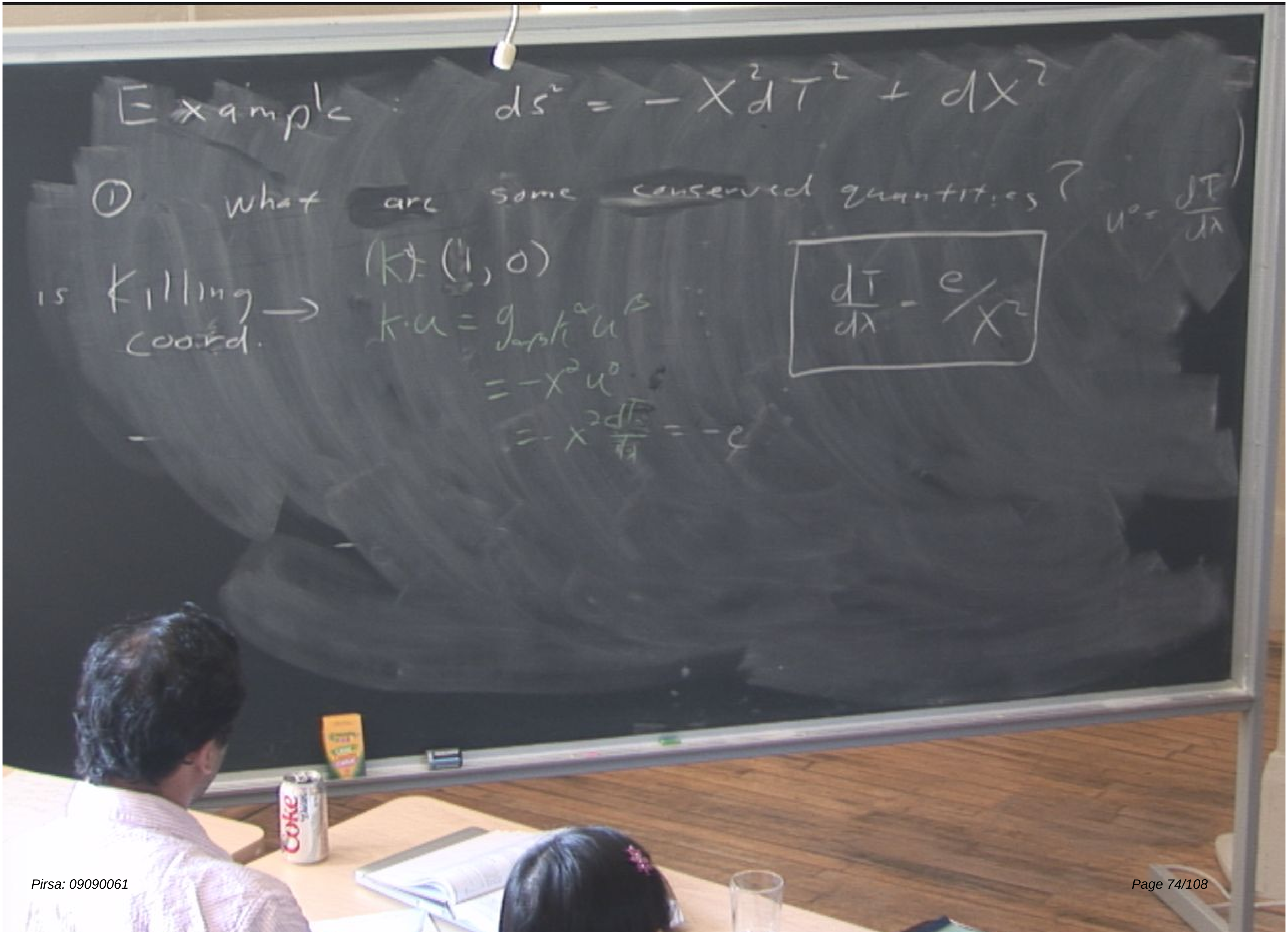
→ in explicitly solving for geodesics - use these conserved quantities (as well as $\underset{\sim}{u} \cdot \underset{\sim}{u} \begin{cases} = -c^2 \\ = 0 \end{cases}$)

→ NOT geodesic eq

→ simplifies analysis → 1st order eq's rather than 2nd order eq's

xample : $ds^2 = -X^2 dT^2 + dX^2$

what are some conserved quantities?


Example: $ds^2 = -X^2 dT^2 + dX^2$
① what are some conserved quantities?
$u^0 = \frac{dT}{d\lambda}$
$(k) = (1, 0)$
is Killing → coord.
$k \cdot u = g_{\alpha\beta} k^\alpha u^\beta$
$= -X^2 u^0$
$= -X^2 \frac{dT}{d\lambda} = -e$
$\boxed{\frac{dT}{d\lambda} = e/X^2}$

Example: $ds^2 = -X^2 dT^2 + dX^2$

① What are some conserved quantities

Killing coord. $\longrightarrow$ $(K^\alpha) = (1, 0)$ $\quad K^\alpha = \delta^\alpha_T$

$$K \cdot u = g_{\alpha\beta} K^\alpha u^\beta$$
$$= -X^2 u^0$$
$$= -X^2 \frac{dT}{d\lambda} = -e$$

constant

$$\boxed{\frac{dT}{d\lambda} = e/X^2}$$

$$u \cdot u = \begin{cases} -1 \\ 0 \end{cases}$$

Example: $ds^2 = -X^2 dT^2 + dX$

① What are some conserved quantities

Killing coord. $\longrightarrow$

$(K) = (1, 0)$ $\quad K^\alpha = \delta^\alpha_T$

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$$\underset{\sim}{u} \cdot \underset{\sim}{u} = \begin{cases} -1 & \text{timelike} \\ 0 & \text{null} \end{cases}$$

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…ing →
…rd.

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$$\boxed{\frac{dT}{d\lambda} = e/X^2}$$

$u^0 =$

$u \cdot u = \begin{cases} -1 & \text{timelike} \\ 0 & \text{null} \end{cases}$

$-X\dot{T}^2 + \dot{X}^2 = \begin{cases} -1 \\ 0 \end{cases}$

geodesic: $$\frac{d^2X}{d\lambda^2} + X\underbrace{\left(\frac{dT}{d\lambda}\right)^2}_{=\left(e^2/X^4\right)} = 0$$

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$\longrightarrow$ 2nd order equation for X

instead we turn to constraint:

Null $$-X^2\dot{T}^2 + \dot{X}^2 = 0$$

$$-e^2/X^2 + \dot{X}^2 = 0$$

geodesic: $\frac{d^2X}{d\lambda^2} + X\underbrace{\left(\frac{dT}{d\lambda}\right)^2}_{= \left(e^2/X^4\right)} = 0$

$\longrightarrow$ 2nd order equation for X

instead we turn to constraint:

Null

$$-X^2\dot{T}^2 + \dot{X}^2 = 0$$

$$-e^2/X^2 + \dot{X}^2 = 0 \longrightarrow \dot{X} = e/X$$

geodesic: $$\frac{d^2X}{d\lambda^2} + X \underbrace{\left(\frac{dT}{d\lambda}\right)^2}_{=\,(e^2/X^4)} = 0$$

$\longrightarrow$ 2nd order equation for X

instead we turn to constraint:

Null $$-X^2\dot{T}^2 + \dot{X}^2 = 0$$

$$-\frac{e^2}{X^2} + \dot{X}^2 = 0 \longrightarrow \frac{dX}{d\lambda} = \pm e/X$$

geodesic: $$\frac{d^2X}{d\lambda^2} + X\underbrace{\left(\frac{dT}{d\lambda}\right)^2}_{= \left(e^2/X^4\right)} = 0$$ second

$\longrightarrow$ 2nd order equation for X

instead we turn to constraint:

Null $$-X^2\dot{T}^2 + \dot{X}^2 = 0$$

$$-e^2/X^2 + \dot{X}^2 = 0 \longrightarrow \frac{dX}{d\lambda} = \pm e/X$$

$$X^2 = \pm 2e\lambda$$

$$+ X \underbrace{\left(\frac{dT}{d\lambda}\right)^2}_{\left(e^2/X^4\right)} = 0 \quad \text{served}$$

order equation for X

turn to constraint:

$$+ \dot{X}^2 = 0$$

$$+ \dot{X}^2 = 0 \longrightarrow \frac{dX}{d\lambda} = \pm e/X$$

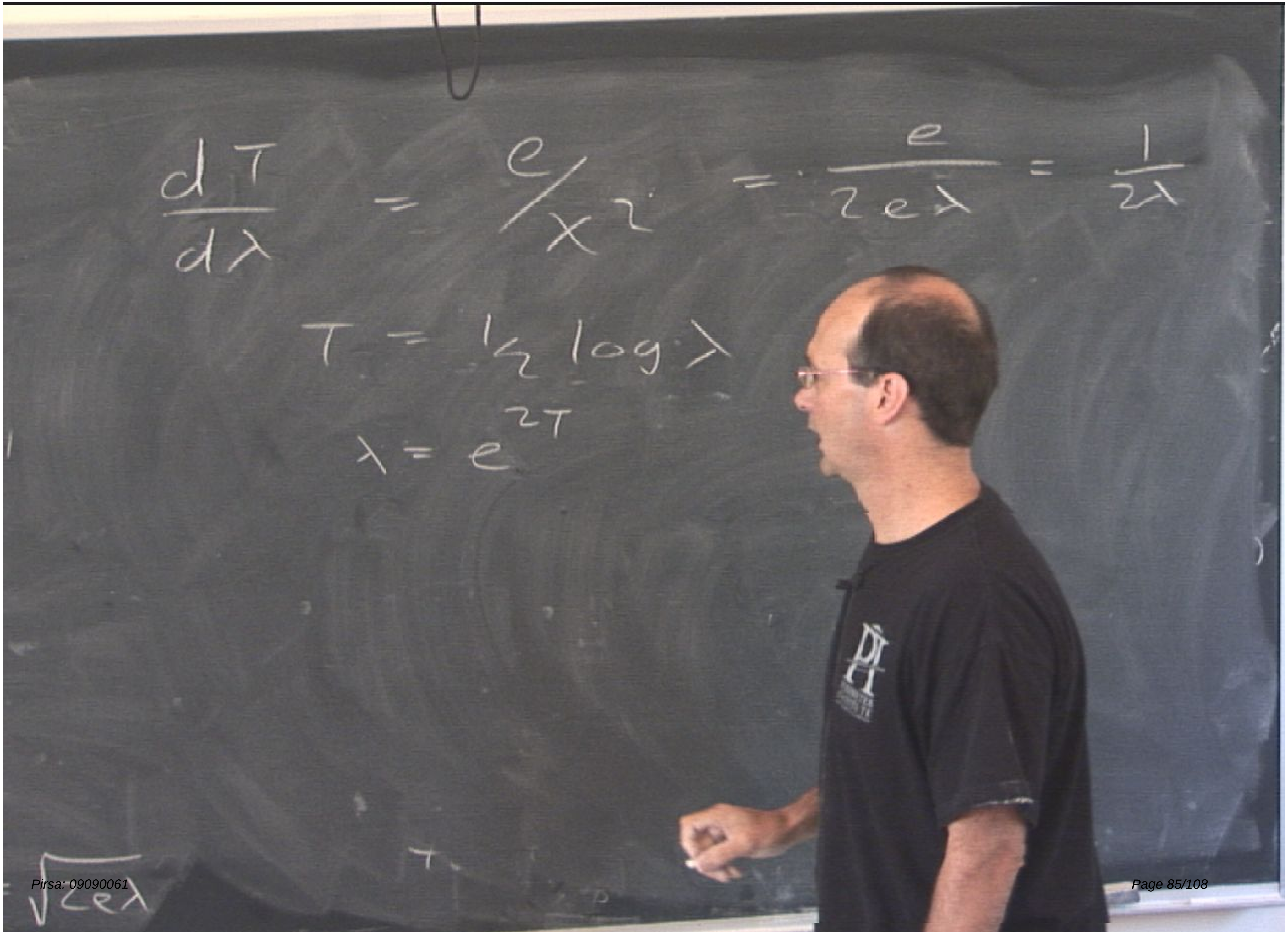
$$X^2 = \underset{>0}{\pm 2e\lambda} \longrightarrow X = \sqrt{2e\lambda}$$

If true

$K^\alpha \partial_\alpha$

$$\frac{d\tau}{d\lambda} = e/x^2 = \frac{e}{2e\lambda} = \frac{1}{2\lambda}$$

$$\sqrt{2e\lambda}$$


$\frac{dT}{d\lambda} = e/x^2 = \frac{e}{2e\lambda} = \frac{1}{2\lambda}$
$T = 1/2 \log \lambda$
$\lambda = e^{2T}$
$\sqrt{2e\lambda}$

$$\frac{d\tau}{d\lambda} = e/x^2 = \frac{e}{2e\lambda} = \frac{1}{2\lambda}$$

$$\tau = \tfrac{1}{2}\log \lambda$$

$$\lambda = e^{2\tau}$$

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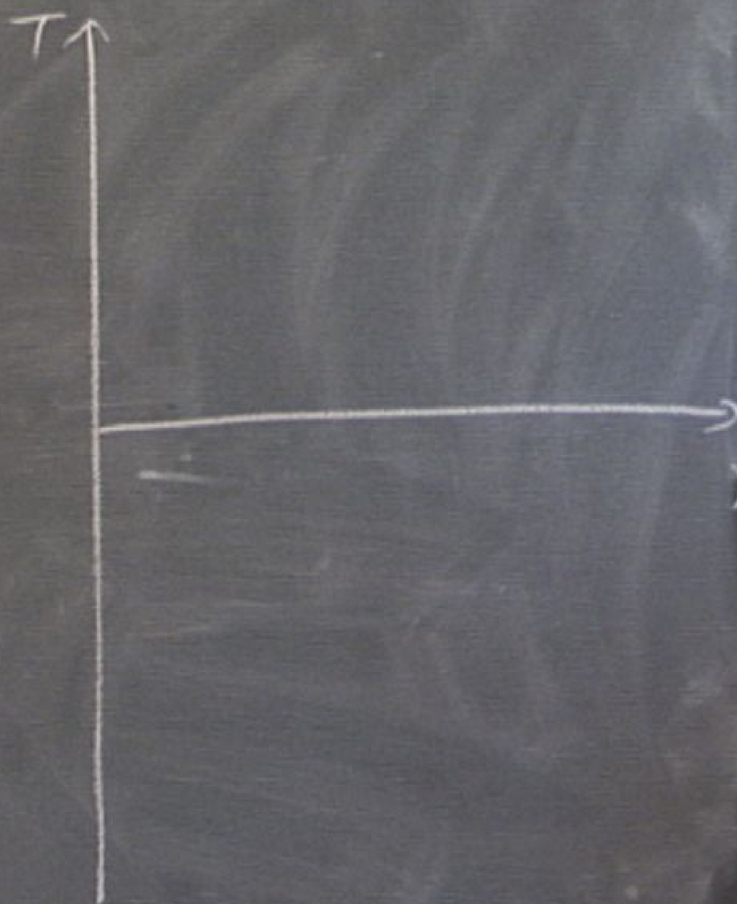
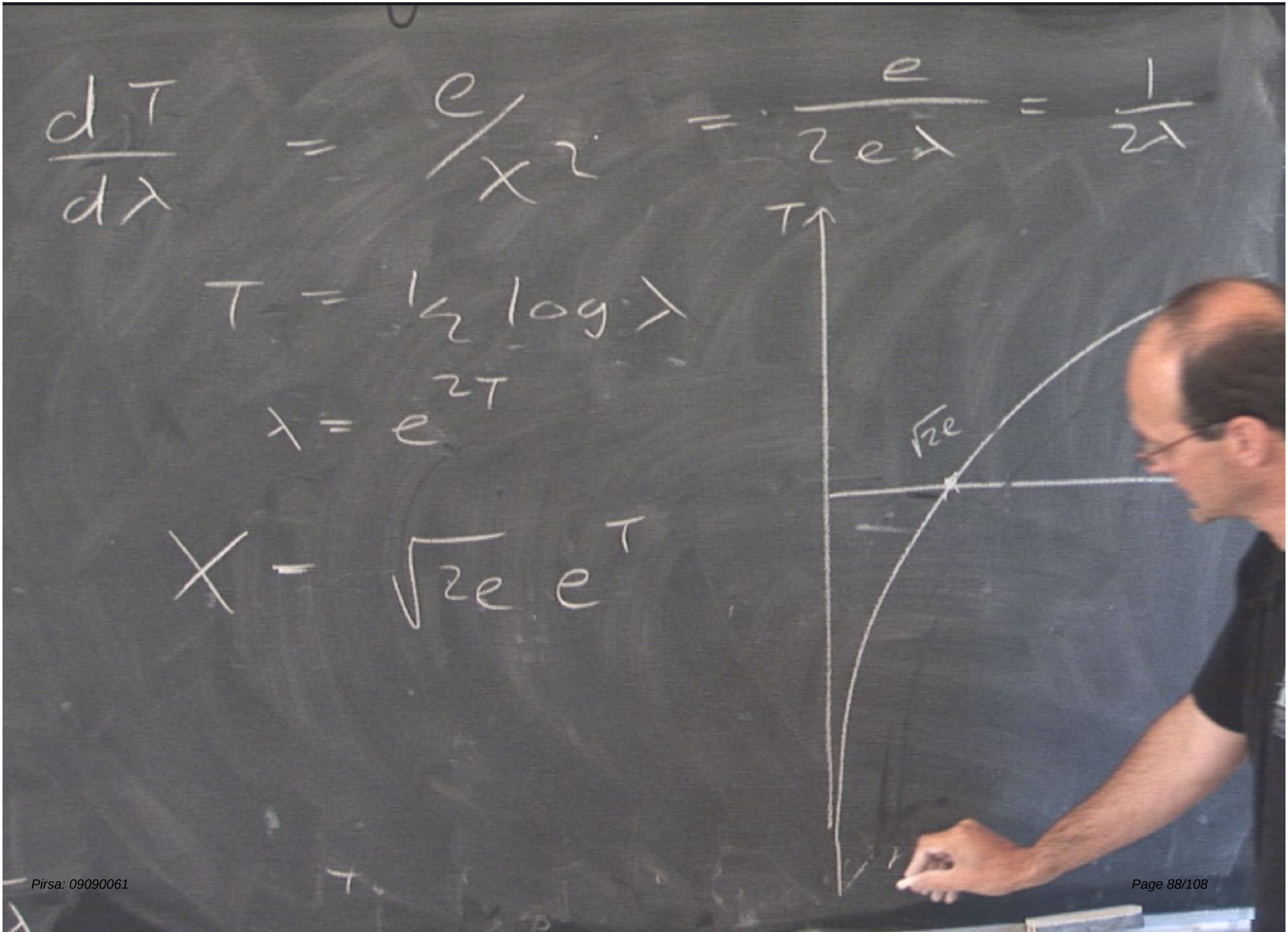


$= \sqrt{2e\lambda}$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τ
$\sqrt{2e}$

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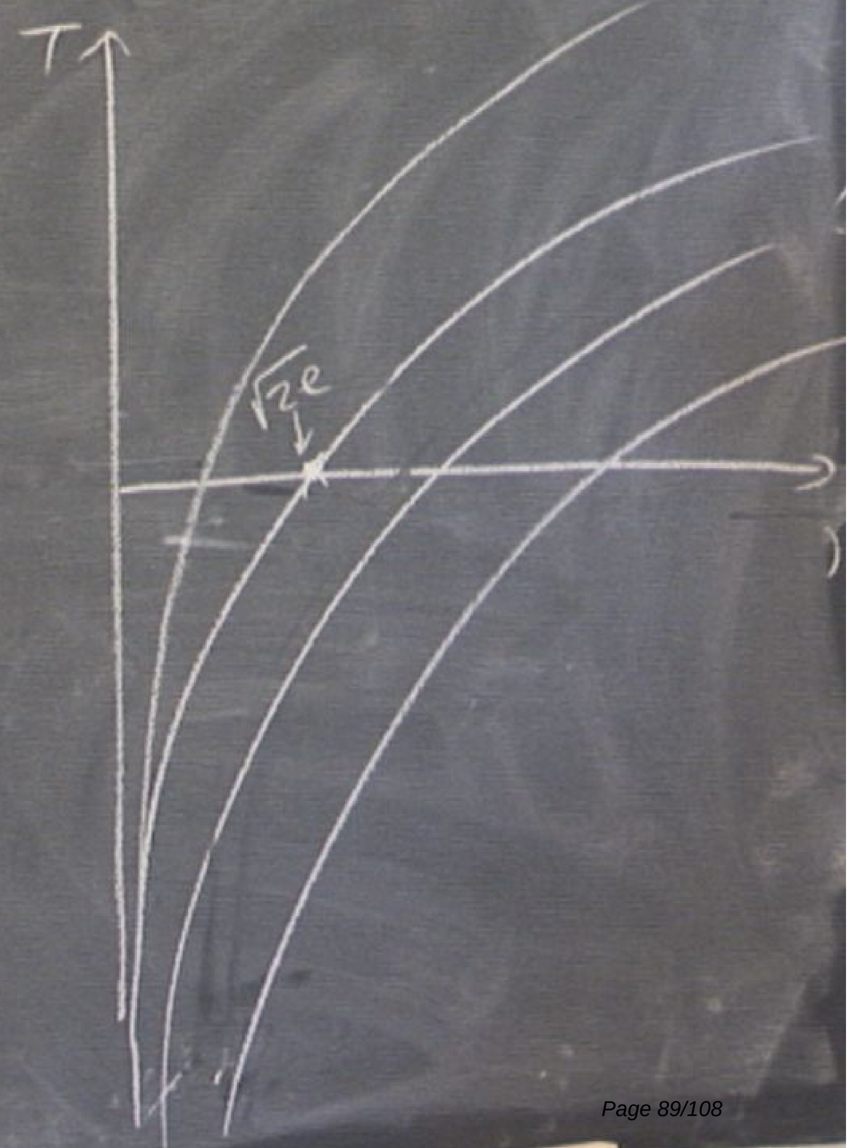

$$\tau = \tfrac{1}{2}\log\lambda$$

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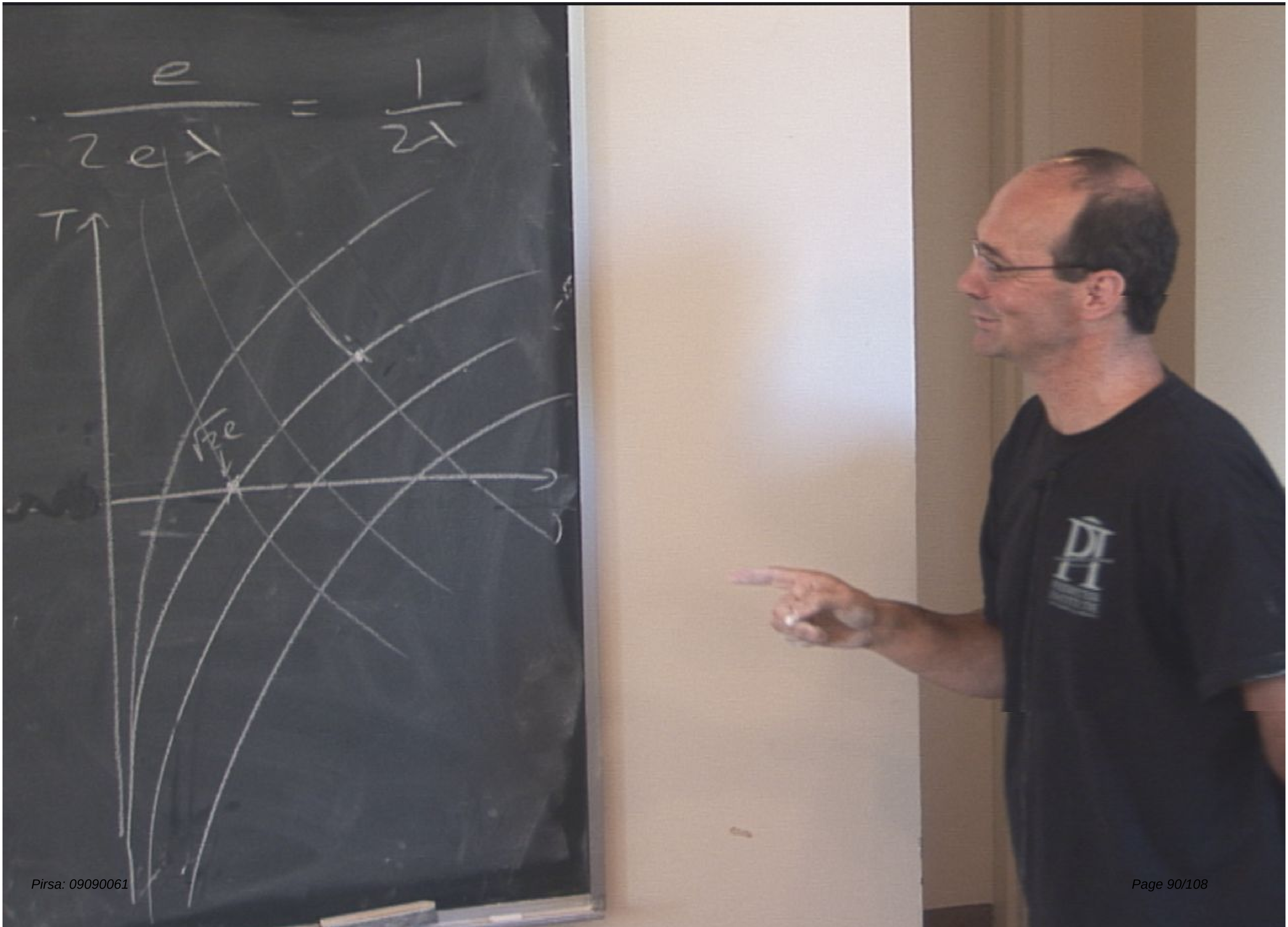

e / 2eλ = 1 / 2λ
T
√2e

$$\frac{dT}{d\lambda} = \quad \frac{\quad}{X^2} = \quad \frac{\quad}{2e\lambda}$$

$$T = \tfrac{1}{2}\log\lambda$$

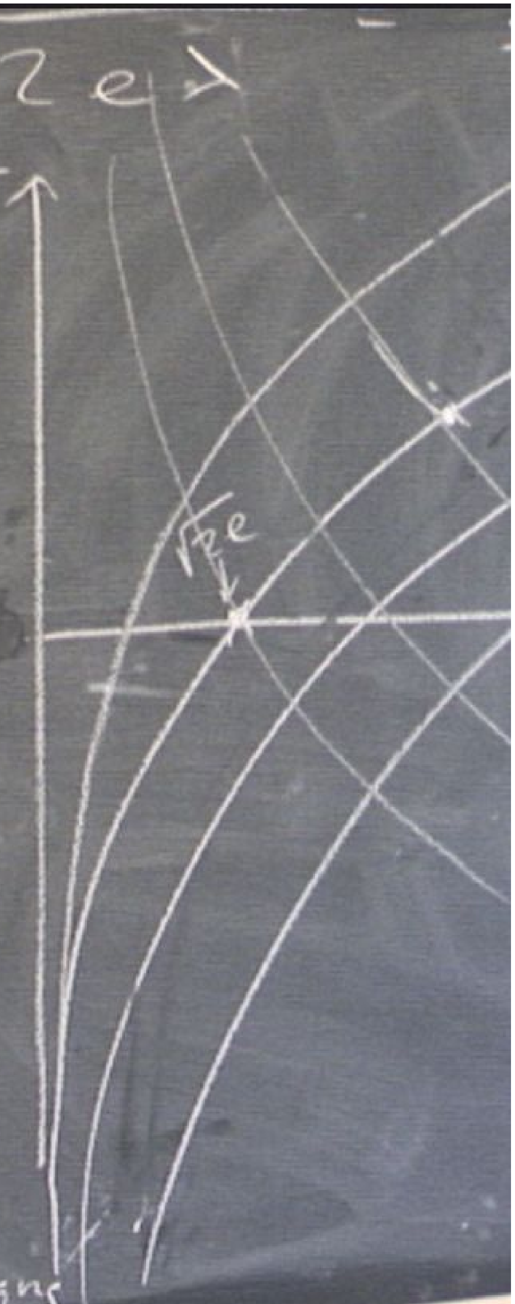
$$\lambda = e^{2T}$$

utward null rays $X = \sqrt{2e}\, e^{T}$

inward null rays: $X = \sqrt{2e}\, e^{-T}$

$= \sqrt{2e\lambda}$

$\longrightarrow$ time reversal or be careful with signs



Example : $ds^2 = -X^2 dT^2 + dX^2$

$\longrightarrow$ this is " just flat space

$$\left.\begin{aligned} t &= X \sinh T \\ x &= X \cosh T \end{aligned}\right\} \longrightarrow ds^2 = -dt^2 + dx^2$$

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$$t/x = \tanh T$$

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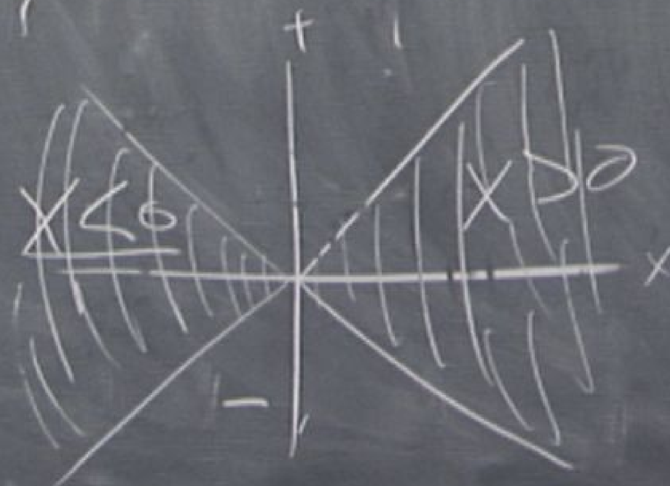
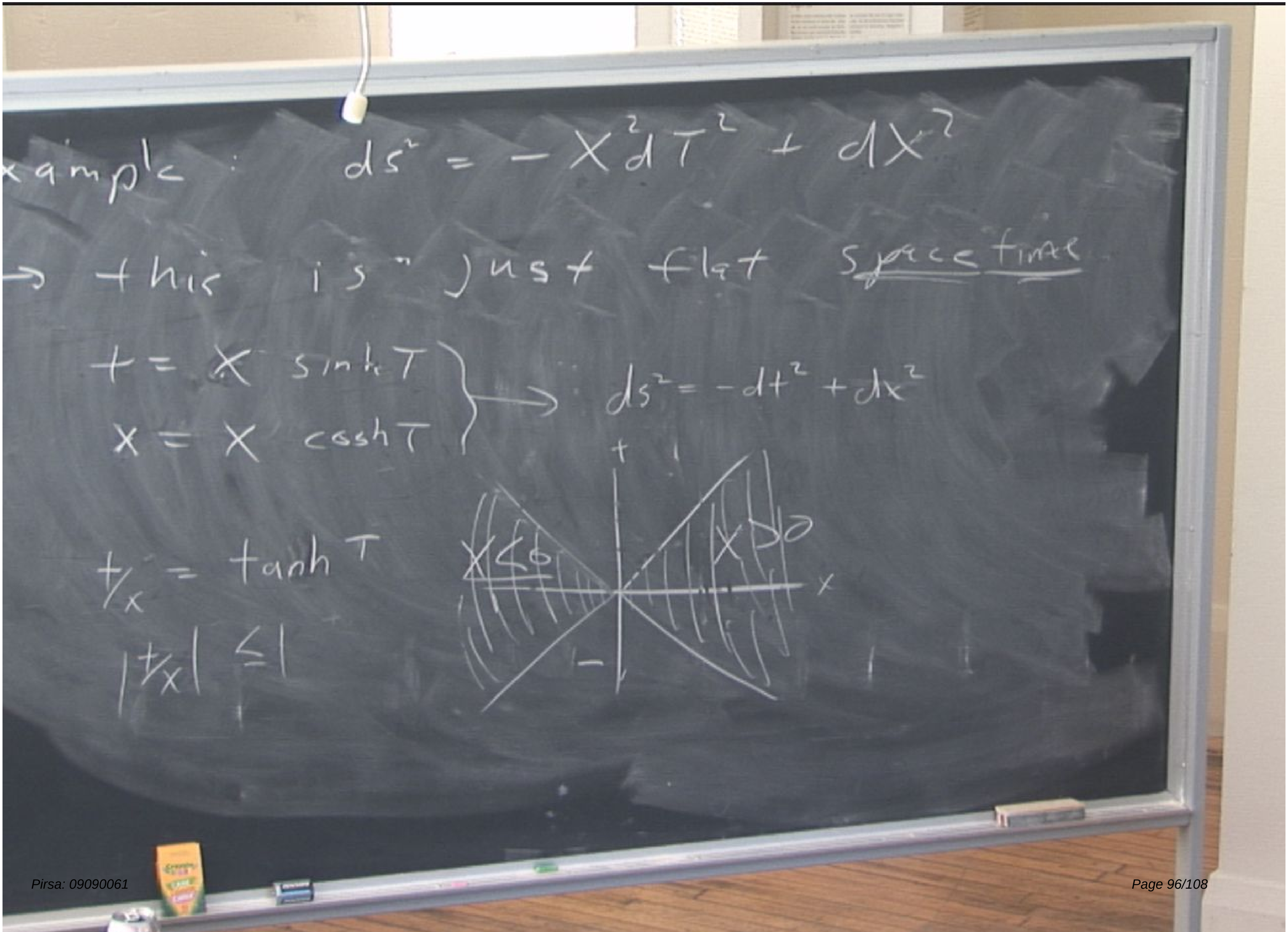



xample : $ds^2 = -X^2 dT^2 + dX^2$
→ this is just flat spacetime
$t = X \sinh T$
$x = X \cosh T$
$ds^2 = -dt^2 + dx^2$
$t/x = \tanh T$
$|t/x| \leq 1$
$X<0$
$X>0$
t
x

$u(x, t)$ null geodesics trivial

$$x = x_0 \pm t$$

in (x, t) null geodesics trivial

$$x = x_0 \pm t$$

$$X \cosh T = x_0 \pm X \sinh T$$

$$X(\cosh T \pm \sinh T) = x_0$$

$$+ \longrightarrow X e^{T} = x_0$$

$$- \longrightarrow X e^{-T} = x_0$$

time like

$$-x^2 \dot{t}^2 + \dot{x}^2 = -1$$

$$-\frac{e^2}{x^2} + \dot{x}^2 = -1$$

time like

$$-x^2\,\dot{t} + \dot{x}^2 = -1$$

$$-\frac{e^2}{x^2} + \dot{x}^2 = -1$$

$$\boxed{\dot{x}^2 - \frac{e^2}{x^2} = -1}$$

time like

$$- x^2 \dot{t} + \dot{x}^2 = -1$$

$$- \frac{e^2}{x^2} + \dot{x}^2 = -1$$

$$\boxed{\dot{x}^2 - \frac{e^2}{x^2} = -1}$$

$\longrightarrow$ looks like a Hamiltonian con

for a classical particle m

in 1-dim (

time like

$$-x^2 \quad + \dot{x}^2 = -1$$

$$-\frac{e^2}{x^2} + \dot{x}^2 = -1$$

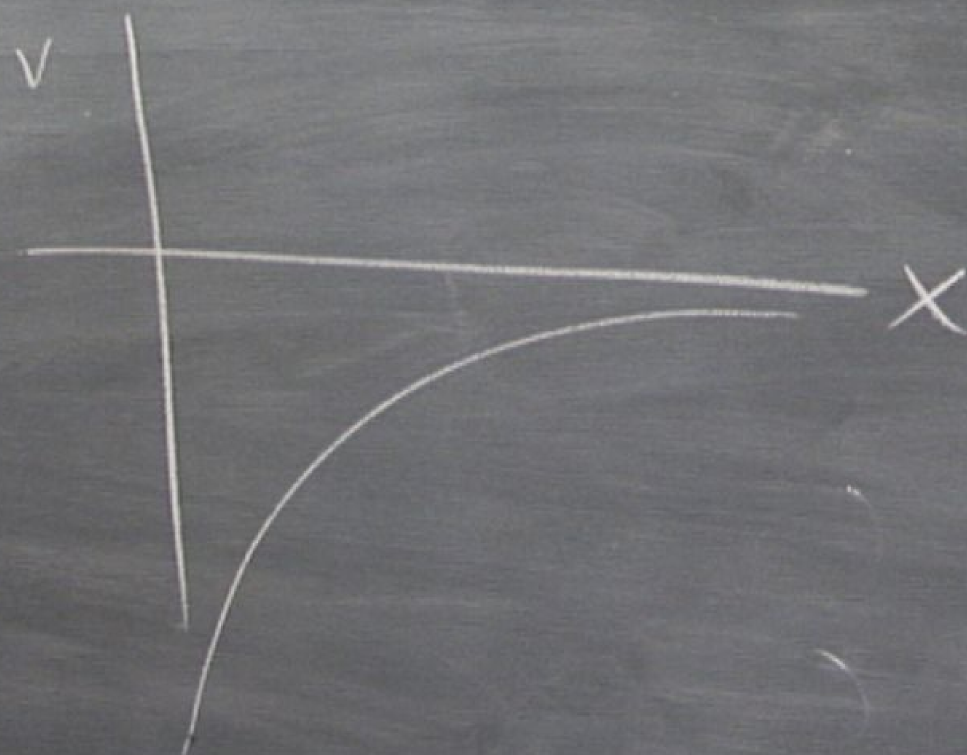
$$\boxed{\dot{x}^2 - \frac{e^2}{x^2} = -1}$$

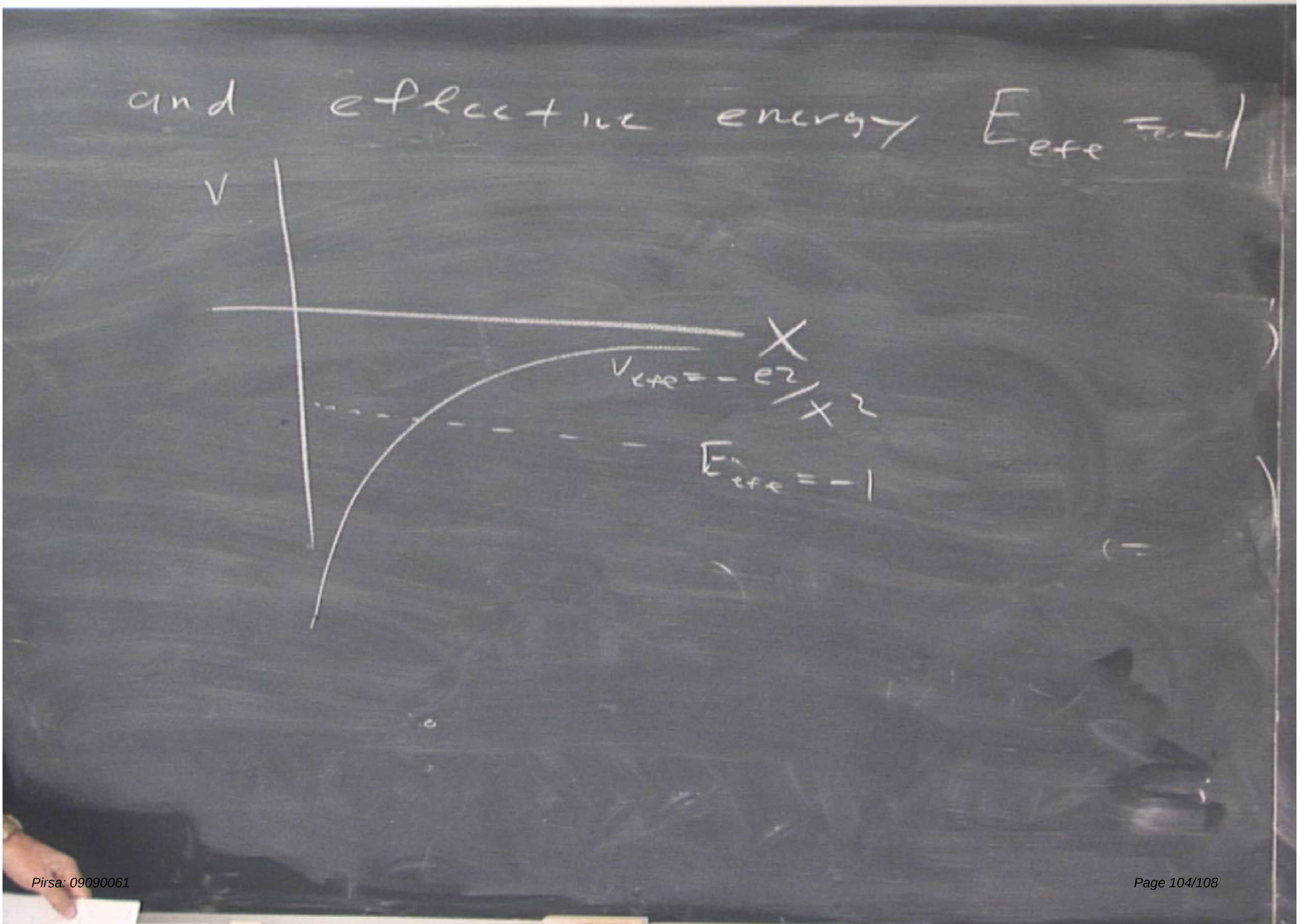
$\longrightarrow$ looks like a Hamiltonian con

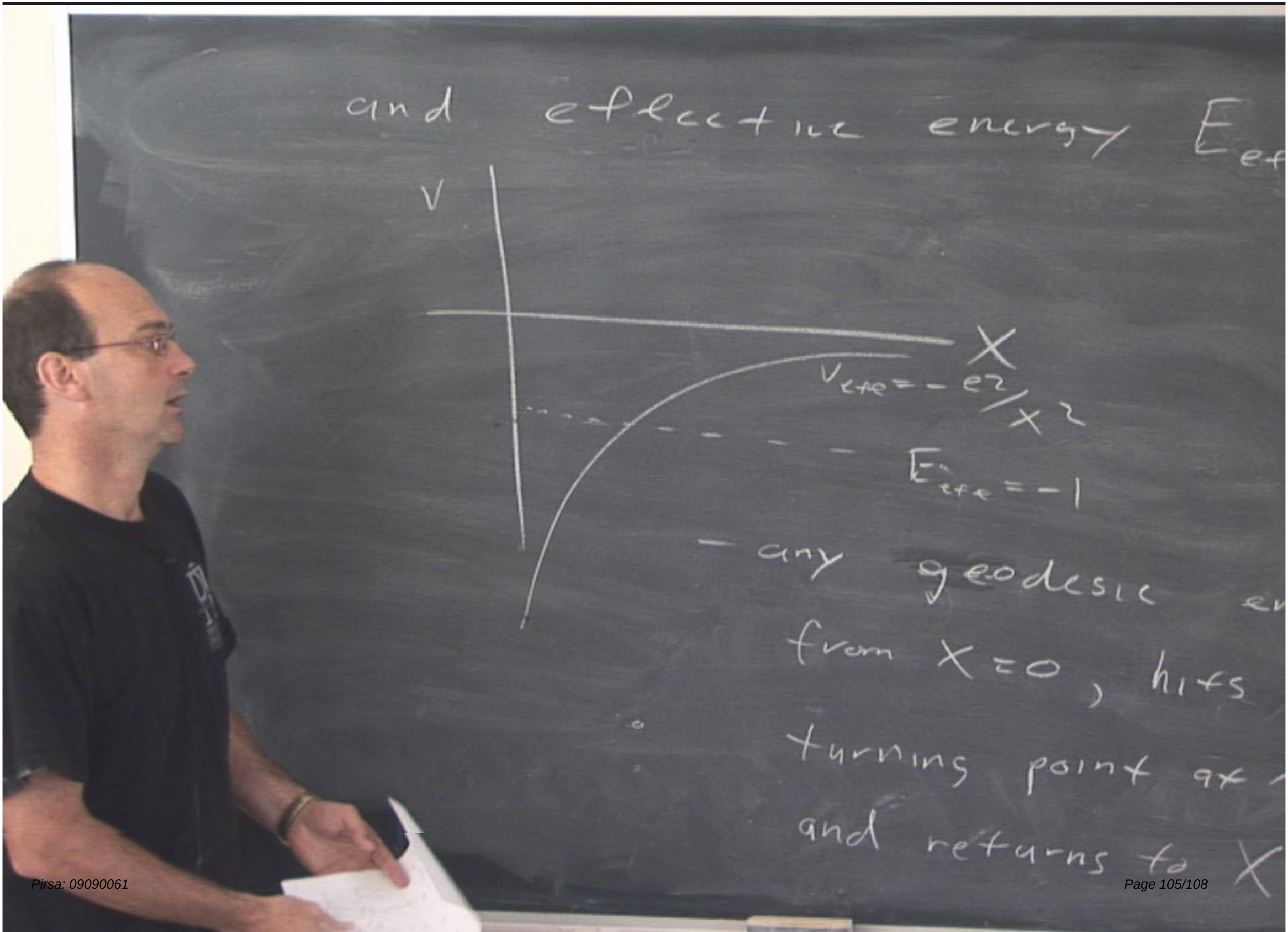
for a classical particle m

in 1-dim ($M_{eff} = 2$) with $V_{eff} =$

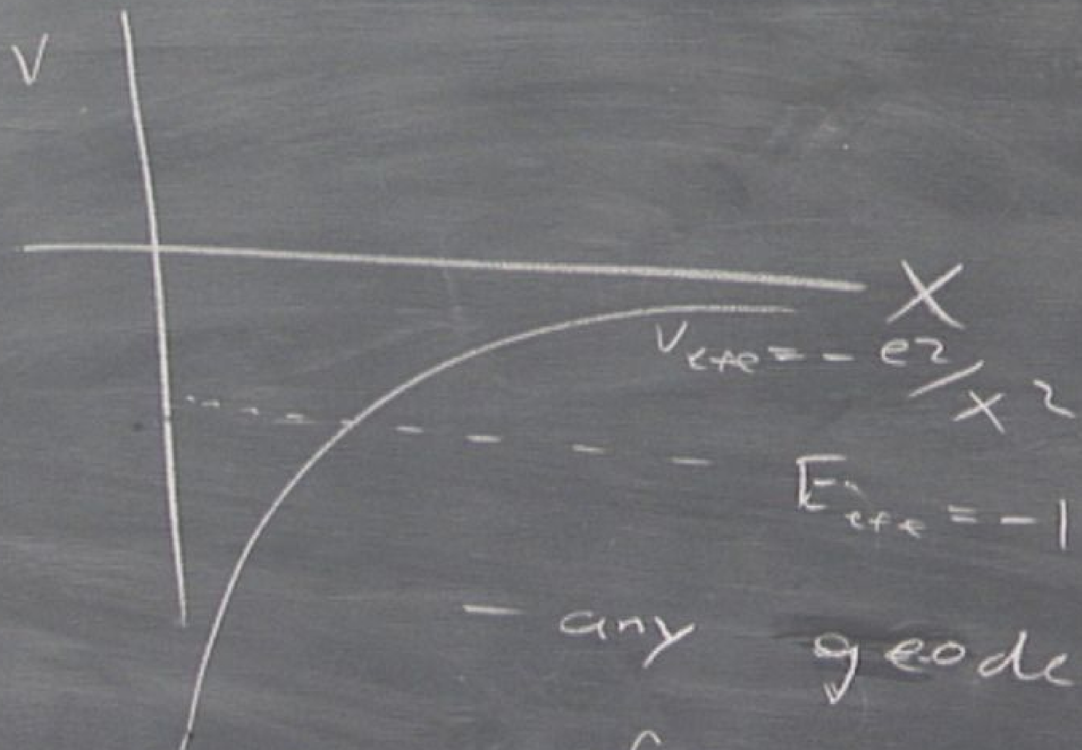
and effective energy $E_{eff} =$




and effective energy $E_{efe} = -1$
V
X
$V_{efe} = -e^2/x^2$
$E_{efe} = -1$


and effective energy E
V
X
Veff = - e2/x2
Eeff = -1
— any geodesic
from x=0, hits
turning point at
and returns to X

and effective energy $E_{eff} = -1$

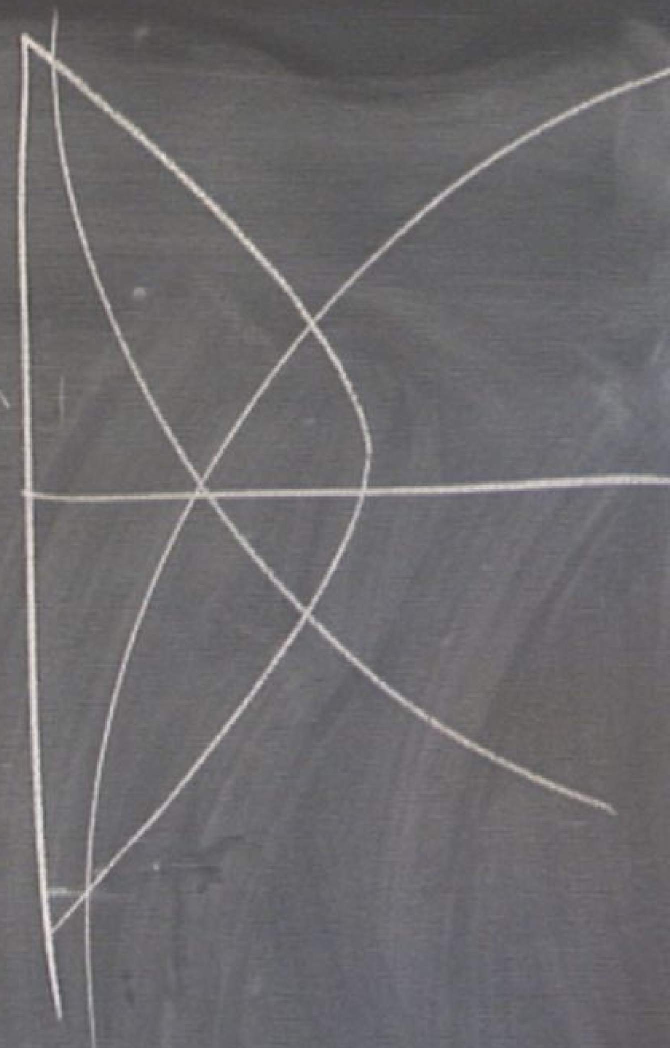


— any geodesic emerges from $X=0$, hits a turning point at $X=a$ and returns to $X=0$

$\dot{t}^2 + \dot{x}^2 = -1$

$\frac{e^2}{x^2} + \dot{x}^2 = -1$

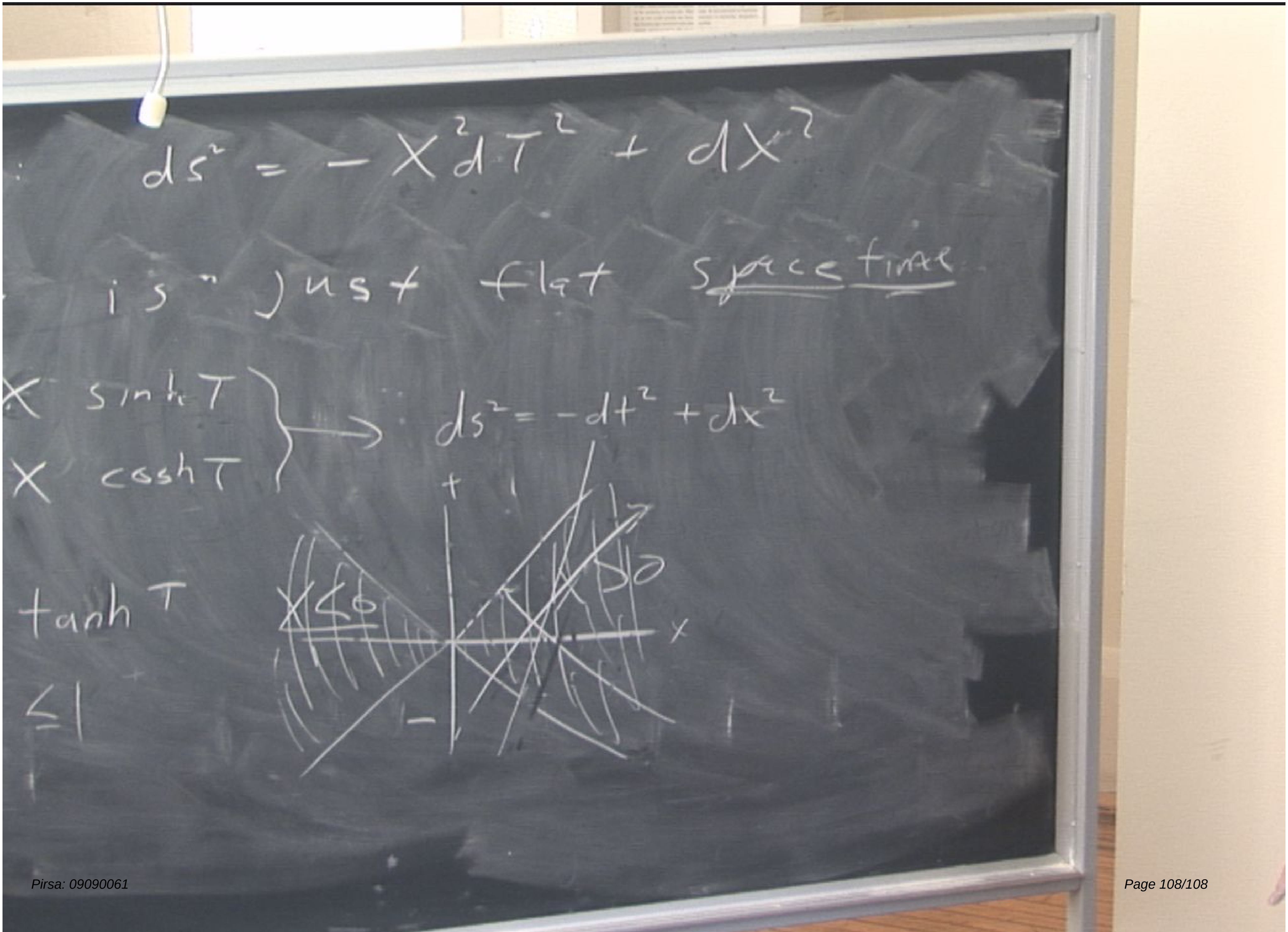
$\left(\dot{x}^2 - \frac{e^2}{x^2} \right) = -1$



like a Hamiltonian constraint

a classical particle moving

with $V = -\frac{e^2}{x^2}$


ds² = −X²dT² + dX²
is just flat spacetime
X sinh T
X cosh T
ds² = −dt² + dx²
t
x
tanh T
≤ 1