Title: Relativity - Core (PHYS 604) - Lecture 5

Date: Sep 09, 2009 10:30 AM

URL: http://pirsa.org/09090060

Abstract:

curved spacetime: $ds^2 = g_{\alpha\beta}(x)\, dx^\alpha \, dx^\beta$

1. $g_{\alpha\beta}$ symmetric + has inverse $g^{\alpha\beta} g_{\beta\gamma} = \delta^\alpha{}_\gamma$

2. ds^2 invariant under coord. transformations

$$\hat{g}_{\gamma\delta} = g_{\alpha\beta} \frac{\partial x^\alpha}{\partial y^\gamma} \frac{\partial x^\beta}{\partial y^\delta}$$

3. curved space $\equiv$ can't find coord's such that

$$\hat{g}_{\alpha\beta} = \eta_{\alpha\beta} \quad \underline{\text{everywhere}}$$

4. can always find coords <u>at a point</u>

$$\hat{g}_{\alpha\beta}\Big|_{x_0} = \eta_{\alpha\beta} \qquad \frac{\partial}{\partial y^\gamma} \hat{g}_{\alpha\beta}\Big|_{x_0} = 0$$

Curved spacetime: $ds^2 = g_{\alpha\beta}(x)\, dx^\alpha dx^\beta$

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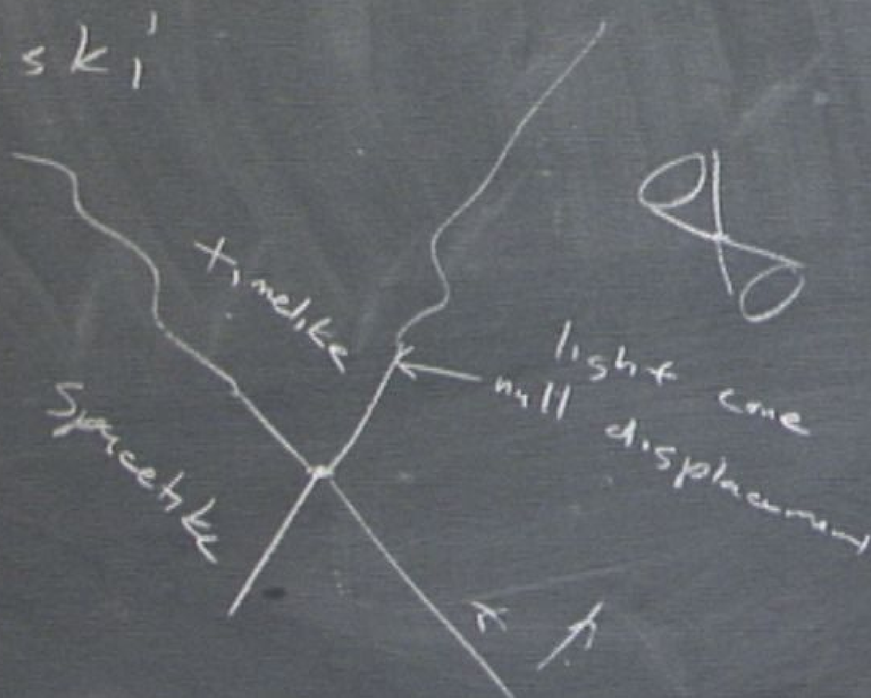
physical spacetime - metric has
1 minus & 3 plus eigenvalues

at given point structure like
Minkowski



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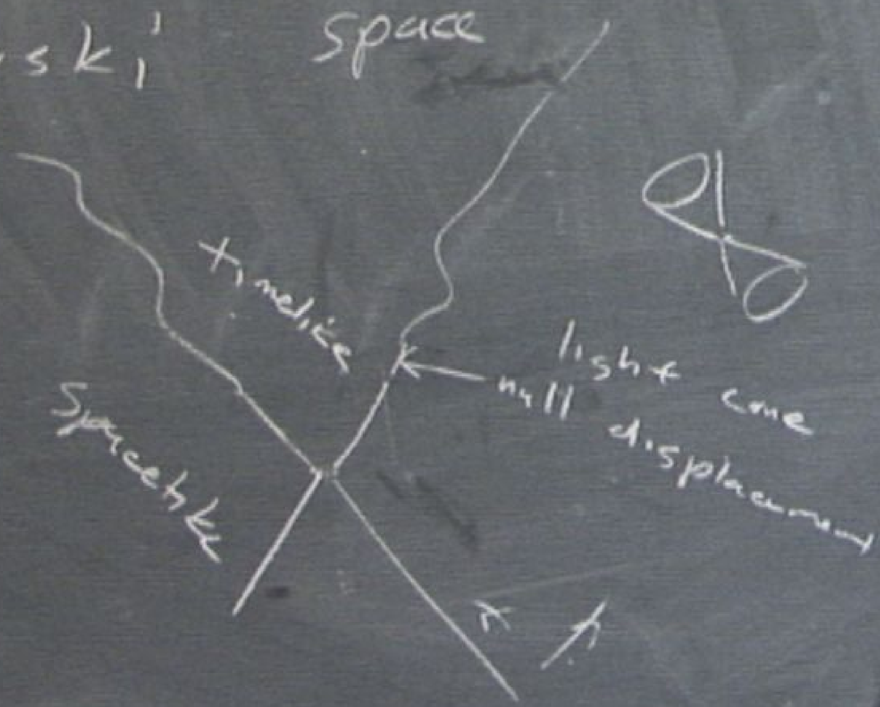
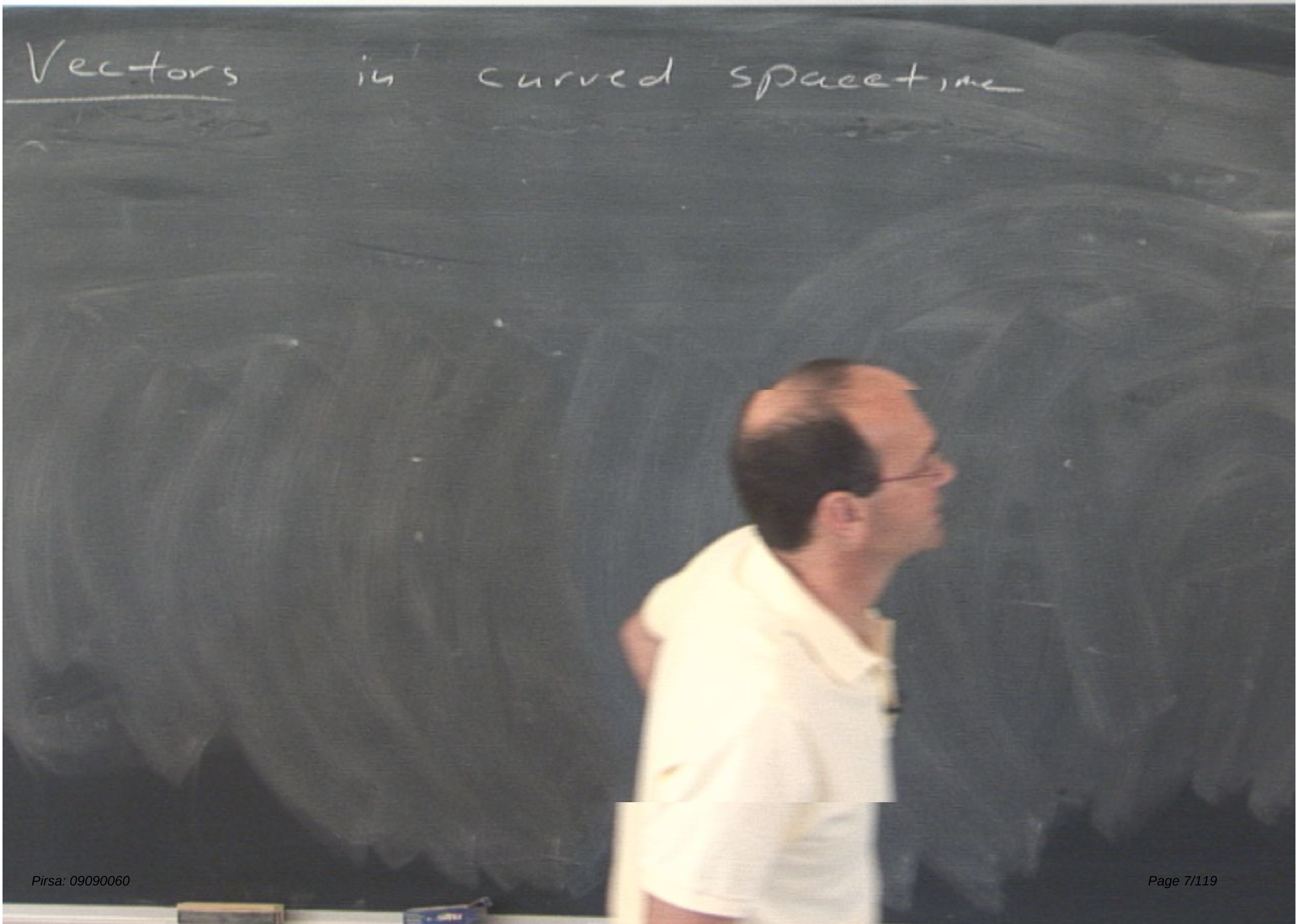


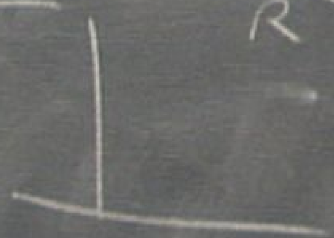

Vectors in curved spacetime

Vectors in curved spacetime

model in two dim's

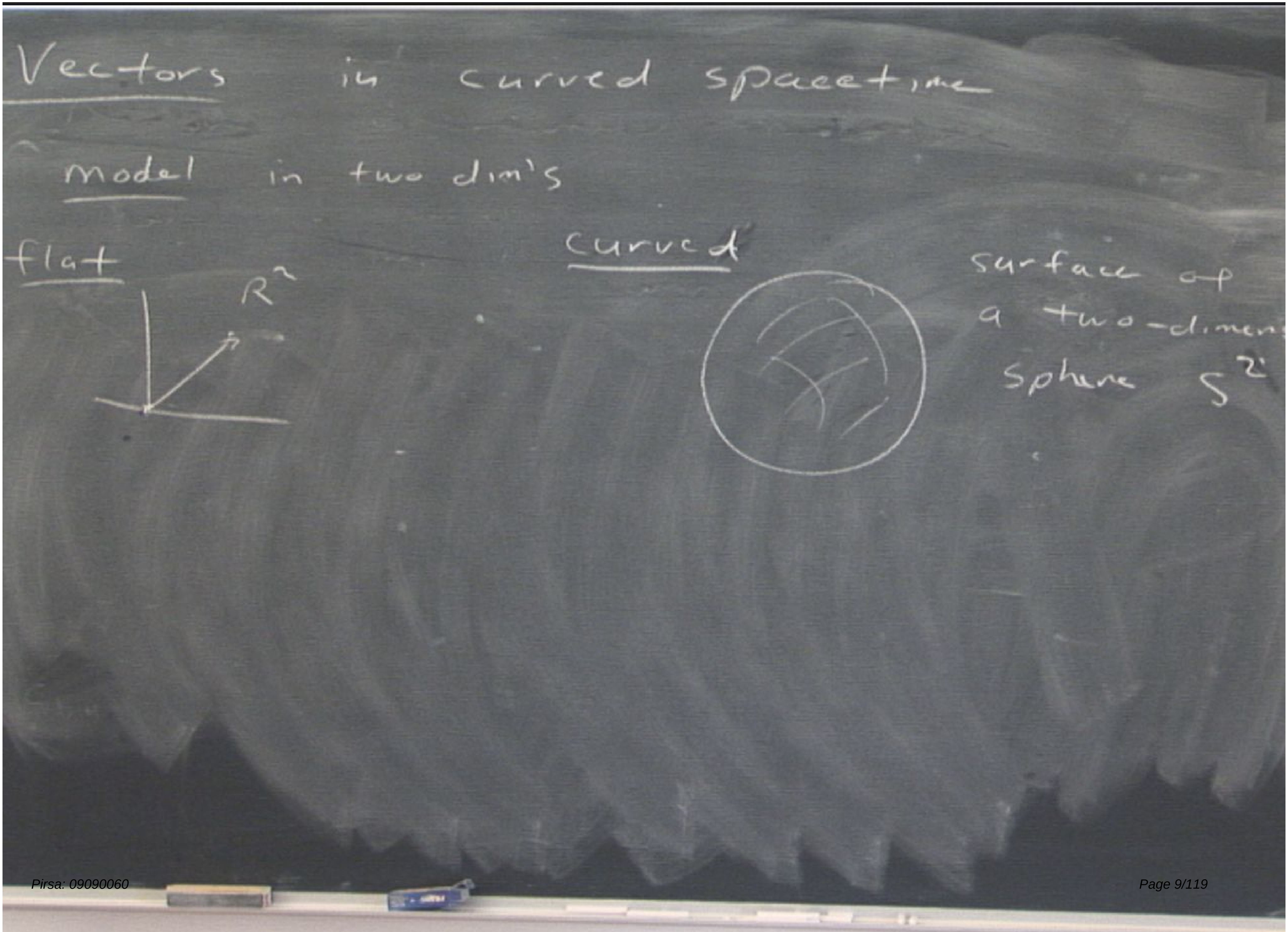
flat

R^2



curved

surface of a two-dimen
sphere S^2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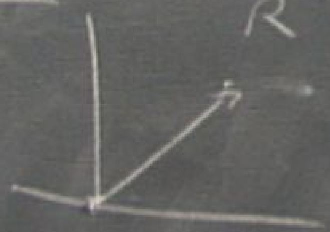

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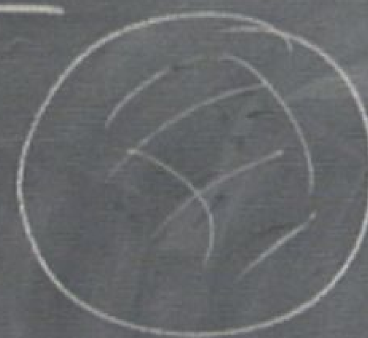

surface of a two-dimens[...] sphere S^2

- in flat space, can think of vectors as a directed line segment joining two points δx^α

Vectors in curved spacetime

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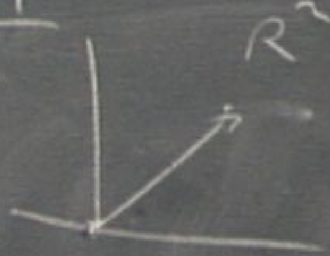
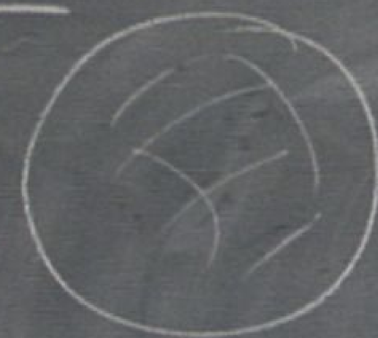


curved



surface of a two-dimens[ional] sphere S^2

- in flat space, can think of vectors as a directed line segment joining two points - δx^α

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Vectors in curved spacetime

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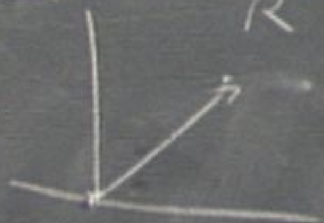

curved

surface of a two-dimen[...] sphere S^2

- in flat space, can think of vectors as a directed line segment joining two points - δx^α
- not useful model in curved space, except for infinitesimal displacements

– use instead the idea of a velocity

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→ one treats with same rule

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eg addition + multiplication by constant

$$m\vec{v}_1 + m\vec{v}_2 = m\vec{v}_3$$

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$$m_1\vec{v}_1 + m_2\vec{v}_2 = m_3\vec{v}_3$$

→ defined locally - tangent to some trajectory at a given point

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$x^\alpha(\lambda=0) = x_0^\alpha$

$$V^\alpha = \frac{dx^\alpha}{d\lambda}\Big|_{x^0}$$

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–collection of all such curves builds up the collection of all possible vectors at x_0 → "tangent space"

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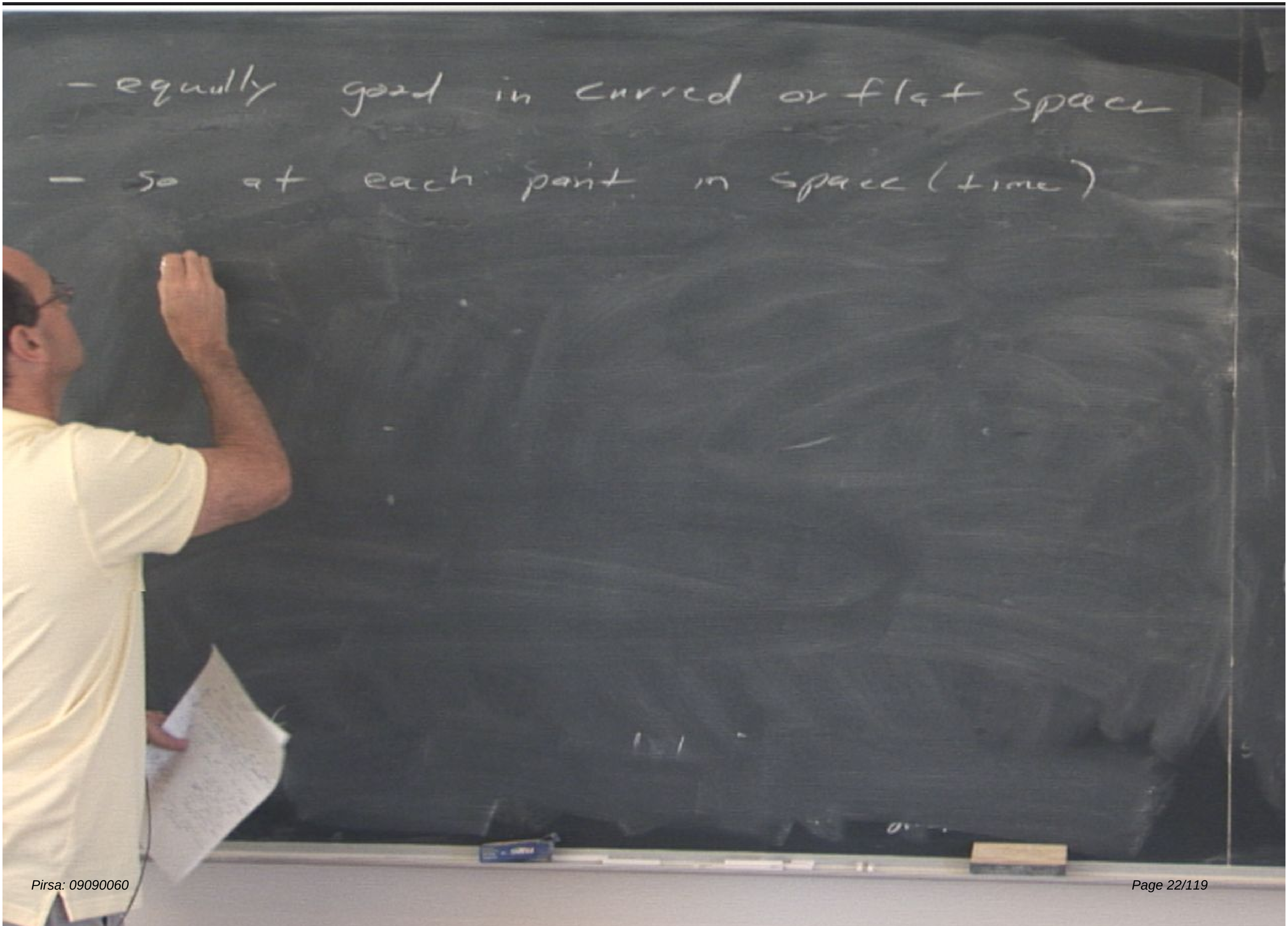
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  space when flat but not
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Canonical model . 4-velocity

in Mink space

$\underset{\sim}{u} =$

Canonical model: 4-velocity

in Mink space

$$\underset{\sim}{u} = (u^\alpha) = (u^0, u^1, u^2, u^3)$$

Canonical model . 4-velocity

in Mink space

$$\underset{\sim}{u} = (u^\alpha) = (u^0, u^1, u^2, u^3)$$

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$$= u^\alpha \underset{\sim}{e}_\alpha$$

Canonical model. 4-velocity

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implicitly have a basis of unit vectors poin[ting]
along coord axes in <u>Mink</u>

canonical model: 4-velocity

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$$\underset{\sim}{e}_\alpha \cdot \underset{\sim}{e}_\beta = \eta_{\alpha\beta} \longleftarrow$$

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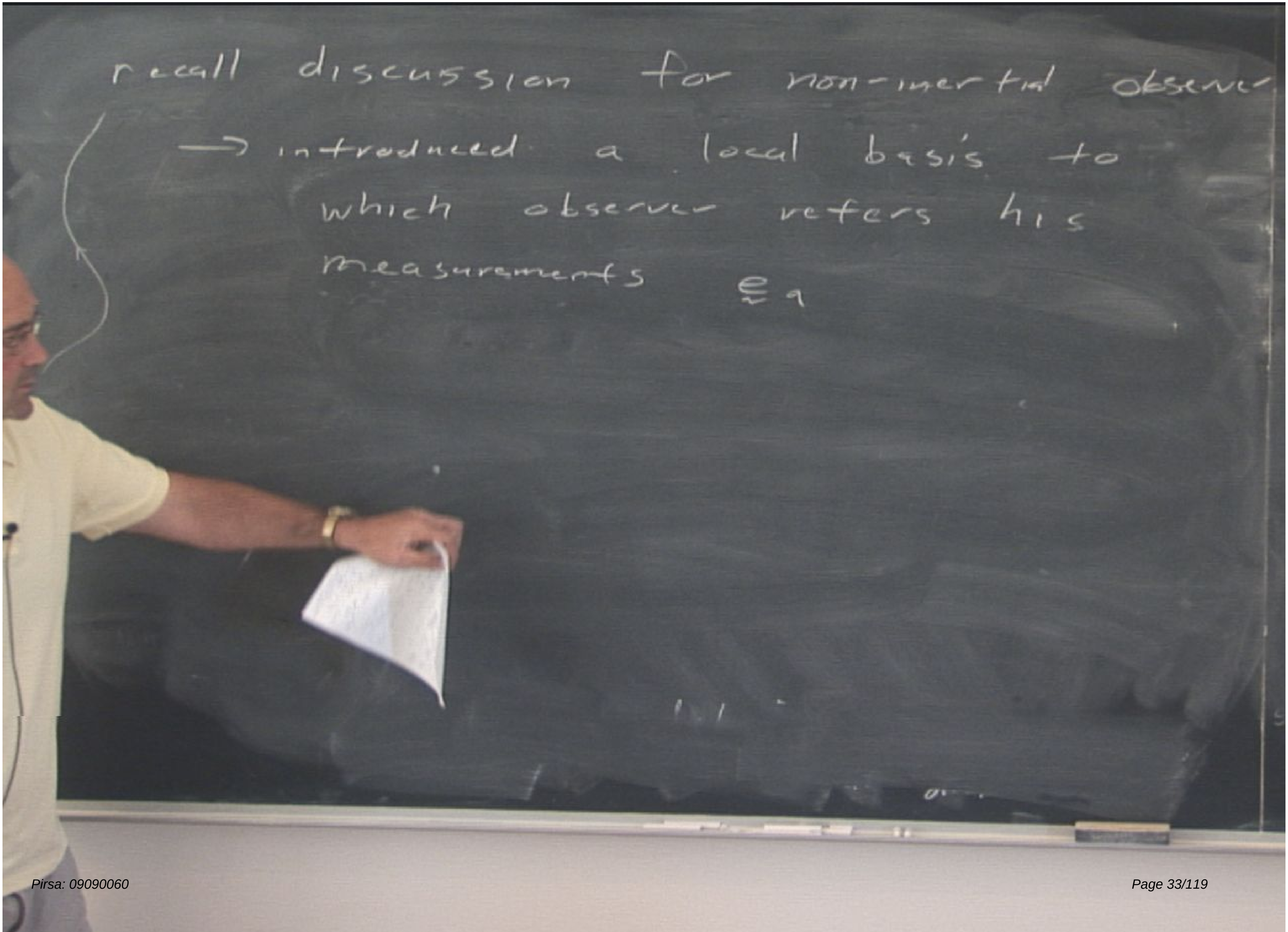
$\underset{\sim}{e}_\alpha \cdot \underset{\sim}{e}_\beta = \eta_{\alpha\beta} \longleftarrow$ coordinate basis

or

orthonormal basis


recall discussion for non-inertial observer
→ introduced a local basis to
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as in SR, $\quad |u|^2 = g_{\alpha\beta} u^\alpha u^\beta$

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spacelike vector $|u|^2 > 0$

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null like vectors $|u|^2 = 0 \leftarrow$ tangent t[o] photon traje[ctory]

d...

$$\underset{\sim}{e}_\alpha \cdot \underset{\sim}{e}_\beta = g_{\alpha\beta}(x) \Leftarrow \text{coord basis } \underset{\sim}{e}_\alpha$$

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eg just as in S.R: for a passing particle with 4-momentum $\underset{\sim}{p}$

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Simple example

infinitesimal displacements $\underset{\sim}{dx}$

Simple example

infinitesimal displacements $\underset{\sim}{dx} = dx^{\alpha}\, \underset{\sim}{e}_{\alpha}$

$ds^2 =$

simple example

infinitesimal displacements $\underset{\sim}{dx} = dx^\alpha \underset{\sim}{e}_\alpha$

$$ds^2 = \underset{\sim}{dx} \cdot \underset{\sim}{dx} = dx^\alpha \underbrace{\underset{\sim}{e}_\alpha \cdot \underset{\sim}{e}_\beta}_{g_{\alpha\beta}} dx^\beta$$

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dx

simple example

standard coord basis

infinitesimal displacements $\underset{\sim}{dx} = dx^\alpha \underset{\sim}{e}_\alpha$

$= (dx^a)\, \underset{\sim}{e}_a$

orthonormal basis

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simple example

infinitesimal displacements $\underset{\sim}{dx} = dx^\alpha \underset{\sim}{e}_\alpha$ (standard coord basis)

$= (dx^a)\, \underset{\sim}{e}_a$ (orthonormal basis)

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$$= dx^a \underbrace{\underset{\sim}{e}_a \cdot \underset{\sim}{e}_b}_{\eta_{ab}} dx^b$$

$$= \eta_{ab}\, dx^a dx^b$$

polar coords

$$ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d$$

$$g_{\alpha\beta} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & r^2 & \\ & & & r^2\sin^2\theta \end{pmatrix}$$

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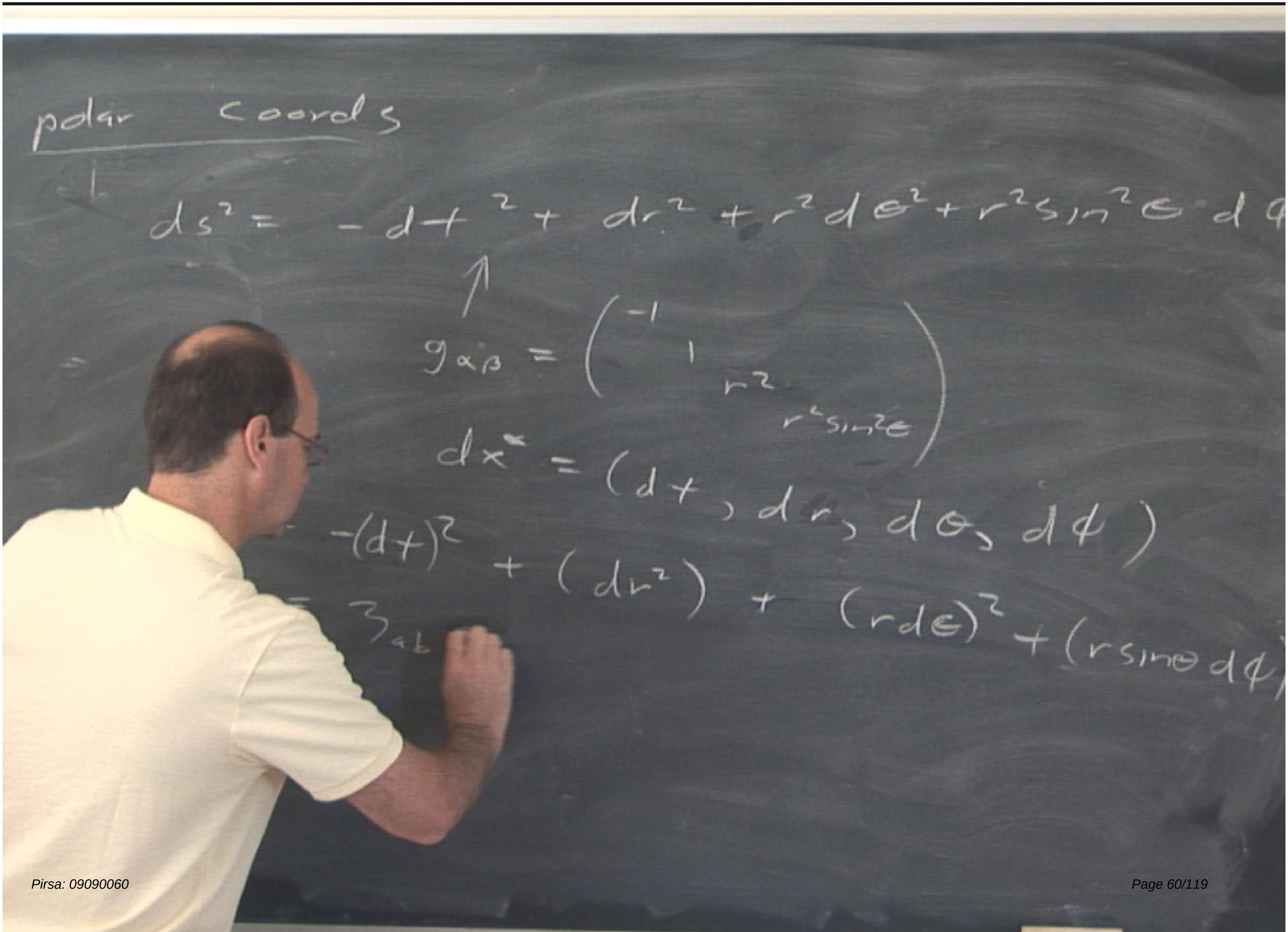
$$dx^\alpha = (dt, dr, d\theta, d\phi)$$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$dx^\alpha = (dt, dr, d\theta, d\phi)$
$= -(dt)^2 + (dr^2) + (r d\theta)^2 + (r \sin\theta \, d\phi)$
$= \eta_{ab}$

dlar coords

$$ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$$

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$$dx^\alpha = (dt, dr, d\theta, d\phi)$$

$$= -(dt)^2 + (dr^2) + (r\,d\theta)^2 + (r\sin\theta\, d\phi)^2$$

$$= \eta_{ab}\, dx^a dx^b$$

$$\eta_{ab} = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix} \qquad dx^a = (dt, dr, r\,d\theta,$$

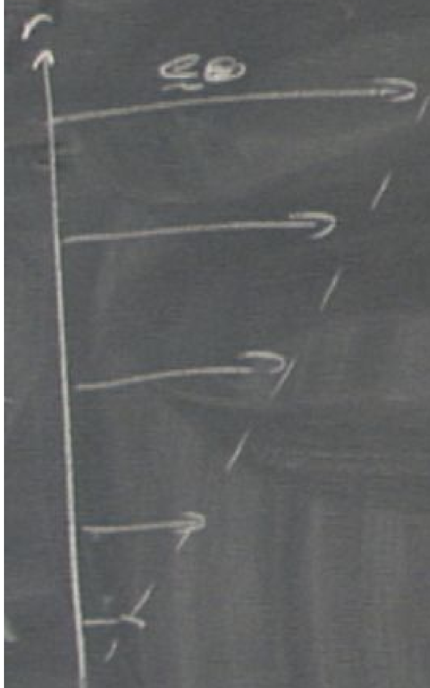
e_θ . $e_\theta \cdot e_\theta = g_{\theta\theta} = r^2$

$\sin\theta d\theta)$

$\underset{\sim}{e}_\theta$.

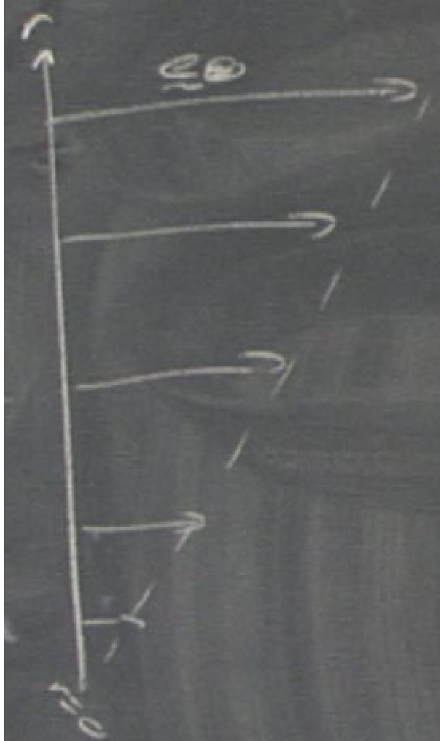
$\underset{\sim}{e}_\theta \cdot \underset{\sim}{e}_\theta = g_{\theta\theta} = r^2$

$|\underset{\sim}{e}_\theta| = r$



$\sin\theta d\theta)$

$\underset{\sim}{e}_\theta$. $\underset{\sim}{e}_\theta \cdot \underset{\sim}{e}_\theta = g_{\theta\theta} = r^2$

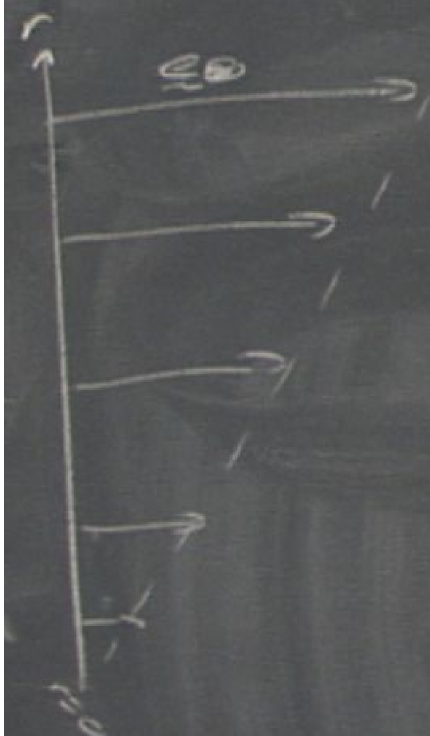


$|\underset{\sim}{e}_\theta| = r \leftarrow$ grows with r

$\sin\theta\, d\phi)$

$\underset{\sim}{e}_\phi$.

$\underset{\sim}{e}_\theta \cdot \underset{\sim}{e}_\theta = g_{\theta\theta} = r^2$



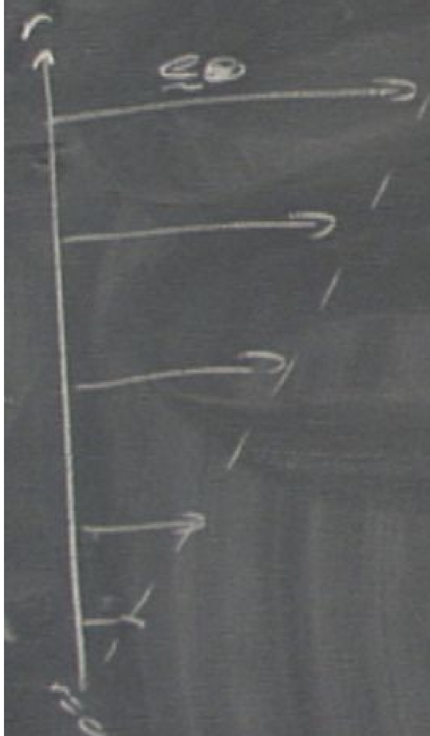
$|\underset{\sim}{e}_\theta| = r \leftarrow$ grows with r

in orthonormal basis

$\underset{\sim}{e}_{\hat\theta} \cdot \underset{\sim}{e}_{\hat\theta} = +1$

$\sin\theta d\phi)$

$e_\theta \cdot$ $\quad e_\theta \cdot e_\theta = g_{\theta\theta} = r^2$



$|e_\theta| = r \longleftarrow$ grows with r

in orthonormal basis

$e_{\hat\theta} \cdot e_{\hat\theta} = +1$

$|e_{\hat\theta}| = 1 \longrightarrow$ fixed

$\sin\theta d\theta)$

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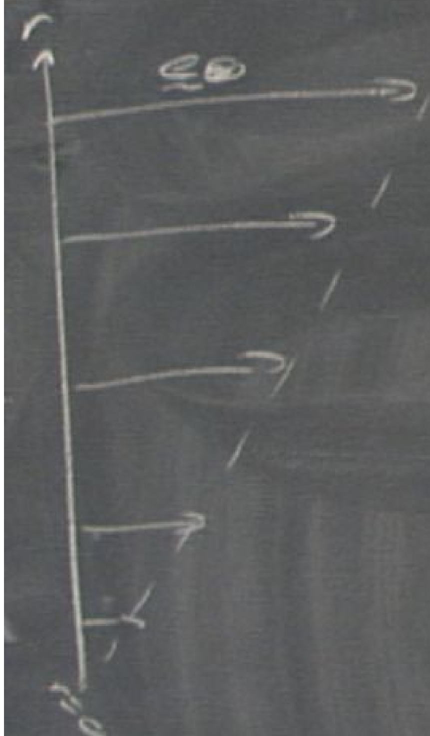
$\underset{\sim}{e}_\theta \cdot \underset{\sim}{e}_\theta = g_{\theta\theta} = r^2$



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$|e_{\hat\theta}| = 1 \xrightarrow[\text{fixed}]{}$

"put 'growth'" in

$(dx)^{\hat\theta} = r\,d\theta$

$\sin\theta d\phi)$

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- similar in curved spacetime

d ex)

$S_\alpha u^\alpha = 0$

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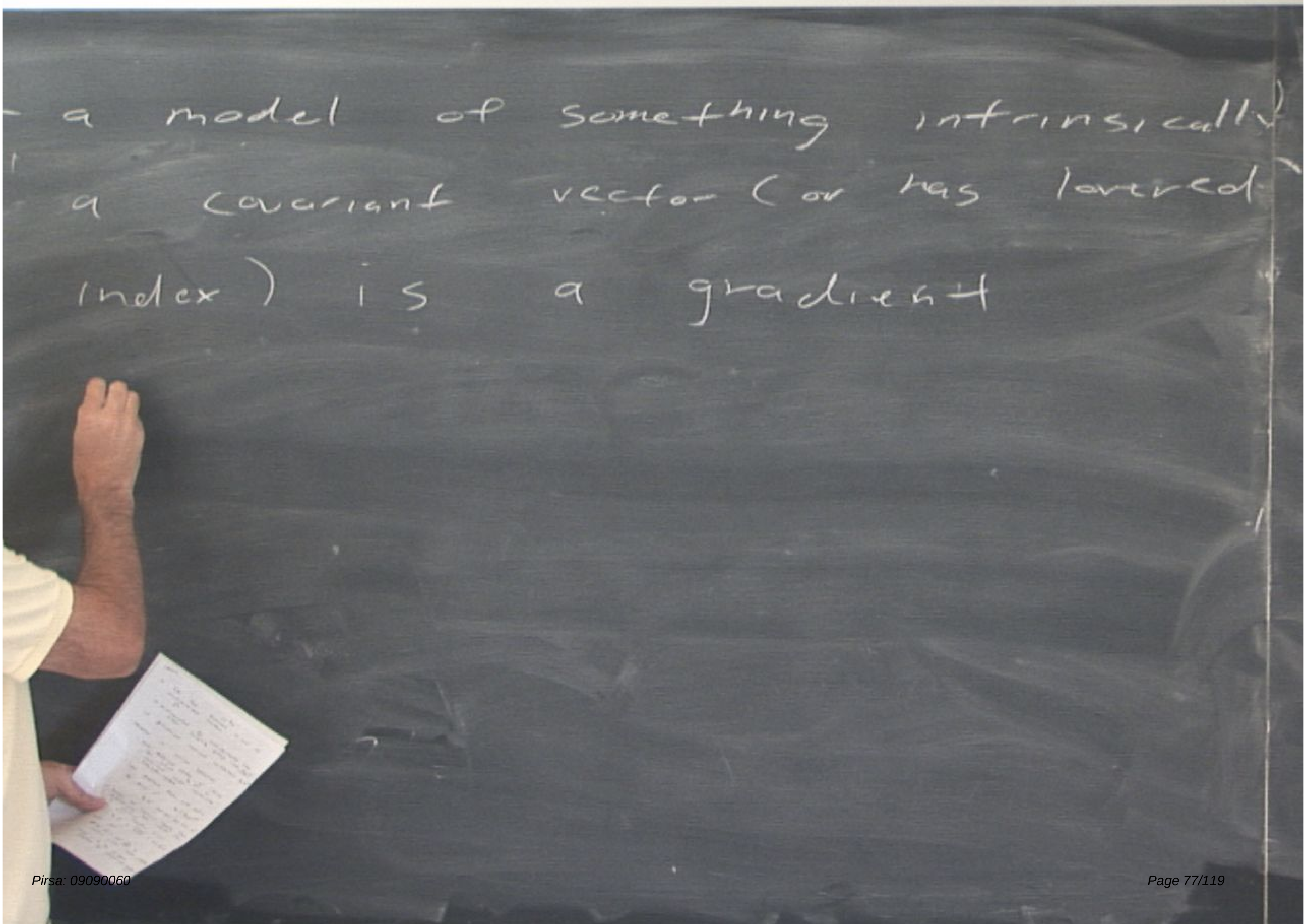
yield $S_\alpha u^\alpha$ is invariant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eg from a function $f(x^\tau)$

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<u>check</u> combine with infinitesimal displacement dx^α

$$df = dx^\alpha \partial_\alpha f$$

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- basis of one-forms : $\hat{\omega}^\alpha$

-rule

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n-rule

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n-rule

has lowered

n H

$$\underset{\sim}{u} = u^r \underset{\sim}{e}_r$$

(x^r)

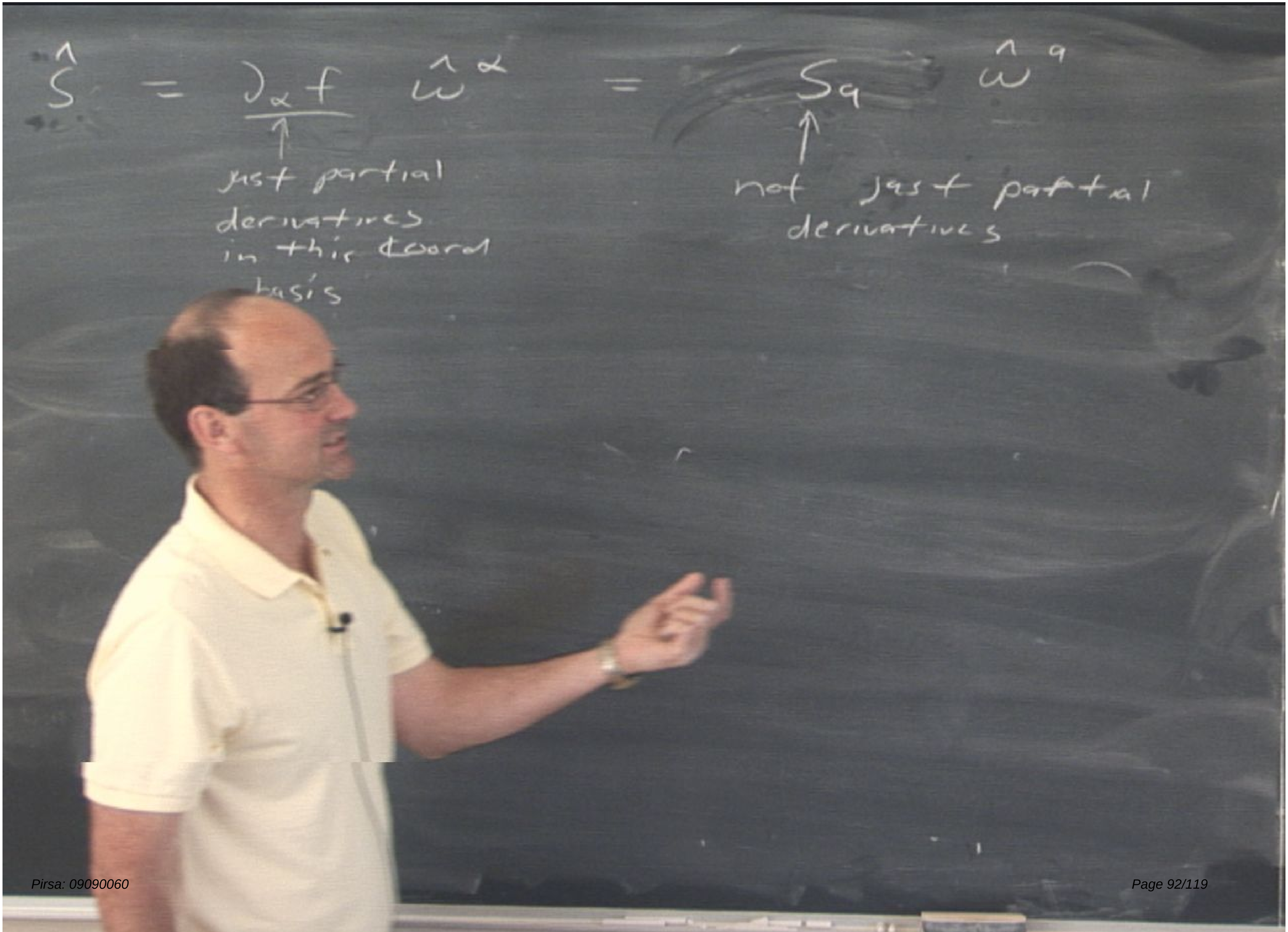
onstants) vectors

basis of one-forms

$$\hat{S} = S_\alpha \hat{\omega}^\alpha$$

$$\hat{S} = \underline{\partial_\alpha f} \ \hat{\omega}^\alpha = S_a \ \hat{\omega}^a$$

just partial
derivatives
in this coord
basis



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just partial derivatives in this coord basis
not just partial derivatives

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relation to vectors, and tangent space

$$\langle \hat{\omega}^\alpha, \underset{\sim}{e}_\beta \rangle = \delta^\alpha{}_\beta$$

$$\langle \hat{\omega}^a, \underset{\sim}{e}_a \rangle = \delta^a{}_b$$

consistency gives then

$$\hat{\omega}^\alpha \cdot \hat{\omega}^\beta = g^{\alpha\beta}$$

$$\hat{\omega}^a \cdot \hat{\omega}^b = \eta^{ab}$$

$$\langle \hat{S}, \underset{\sim}{u} \rangle = S_\alpha \langle \hat{\omega}^\alpha, \underset{\sim}{e}_\beta \rangle u^\beta .$$

$$= S_\alpha \, \delta^\alpha{}_\beta \, u^\beta$$

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---

given $S^\alpha \longrightarrow S_\alpha = g_{\alpha\beta} S^\beta$

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transformation rule

$$\langle \hat{s}, \underset{\sim}{u} \rangle = s_\alpha \langle \hat{\omega}^\alpha, \underset{\sim}{e}_\beta \rangle u^\beta$$

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Coord.

transformation rule

vector $\underset{\sim}{u} = \left(\frac{dx^\alpha}{d\tau} \right)$

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vector $\underset{\sim}{u} = \left(\frac{dx^\alpha}{d\tau} \right)$

coord trans. $x^\alpha = x^\alpha(y)$

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$\rightarrow$ invertible $\longrightarrow$ $y^\beta = y^\beta(x)$
(at least
in some region)

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$$\rightarrow \quad u^\alpha = \frac{\partial x^\alpha}{\partial y^\beta}\, u^\beta$$

Con

gives then

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$$\frac{dx^\alpha}{d\tau} = \frac{\partial x^\alpha}{\partial y^\beta} \frac{dy^\beta}{d\tau}$$

$$\rightarrow \quad u^\alpha = \frac{\partial x^\alpha}{\partial y^\beta} u^{\hat\beta} \quad \leftarrow \text{holds for general } u^\alpha$$

in x coord

y coords

$$\hat{s} = \frac{\partial f}{\partial x^\alpha} \longrightarrow \frac{\partial f}{\partial x^\alpha} = \frac{\partial y^\beta}{\partial x^\alpha} \frac{\partial f}{\partial y^\beta}$$

$$s_\alpha = \frac{\partial y^\beta}{\partial x^\alpha} s_{\hat\beta}$$

x coords

y coords

matrix $\frac{\partial y^\beta}{\partial x^\alpha}$ is inverse of $\frac{\partial x^\gamma}{\partial y^\delta}$.

$$\Longrightarrow \frac{\partial y^\beta}{\partial x^\alpha} \frac{\partial x^\delta}{\partial y^\beta}$$

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in general meet objects with lots of indices

$$T^{\alpha_1 \cdots\cdots \alpha_n}{}_{\beta_1 \cdots\cdots \beta_k}$$

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$$T^{\alpha_1 \cdots\cdots \alpha_n}{}_{\beta_1 \cdots \beta_k} =$$

$$= \frac{\partial x^{\alpha_1}}{\partial y^{\gamma_1}} \cdots\cdots \frac{\partial x^{\alpha_n}}{\partial y^{\gamma_n}} \; T^{\gamma_1 \cdots \gamma_n}$$

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Tensor transform rule

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Tensor transform rule

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example lives in tensor product
of n copies of tangent space
and k " " cotangent space

$\longrightarrow$ $g_{\alpha\beta}$ is an example

$$\hat{S} = \frac{\partial f}{\partial x^\alpha} \longrightarrow \frac{\partial f}{\partial x^\alpha} = \frac{\partial x^\beta}{\partial x^\alpha}\frac{\partial f}{\partial y^\beta}$$

$$S_\alpha = \frac{\partial y^\beta}{\partial x^\alpha} S_{\hat{\beta}}$$

x coords

y coord

ample

$$t = T$$
$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
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$$S_\alpha = (0, 0, 0, x^2+y^2)$$
$$= (S_t, S_x, S_y, S_z)$$

$$S_\alpha = \frac{\partial y^\beta}{\partial x^\alpha} S_\beta$$

x coords

y coords

$$S_\alpha \, dx^\alpha = (x^2+y^2)\, dz$$

$$= (r^2 \sin^2\theta)(dr \cos\theta - r\sin\theta \, d\theta)$$

$$= r^2 \sin^2\theta \cos\theta \, dr - r^3 \sin^3\theta \, d\theta$$

$$\boxed{S_\alpha = \frac{\partial y^\beta}{\partial x^\alpha} S_{\hat{\beta}}}$$

x coords

y coords

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$$S_t = 0$$

$$S_r = r^2\sin^2\theta\cos\theta$$

$$S_\theta = $$

$$\boxed{S_\alpha = \frac{\partial y^\beta}{\partial x^\alpha} S_{\hat\beta}}$$

x coords　　y coords

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$$= r^2 \sin^2\theta \cos\theta\, dr - r^3 \sin^3\theta\, d\theta$$

$$\boxed{\begin{aligned} S_t &= 0 \\ S_r &= r^2 \sin^2\theta \cos\theta \\ S_\theta &= -r^3 \sin^3\theta \\ S_\phi &= 0 \end{aligned}}$$

$$S_\alpha = \frac{\partial y^\beta}{\partial x^\alpha} S_{\hat\beta}$$

x coords   y coords

$z = r\cos\theta$

$$(S_\alpha) = (\,0,\ 0,\ 0,\ x$$

$$= (S_t,\ S_x,\ S_y,$$

similarly for vectors

build an invariant o

$$u^\alpha \partial_\alpha f$$

$y =$

$z = r\cos\theta$

$(S_\alpha) = (0, 0, 0, x^2+y^2)$

$= (S_t, S_x, S_y, S_z)$

similarly for vectors u^α

build an invariant object

$u^\alpha \partial_\alpha f$

$$) = (0, 0, 0, x^2 + y^2)$$

$$= (S_t, S_x, S_y, S_z)$$

ilarly for vectors u^α

uild an invariant object

$u^\alpha \partial_\alpha f$ eg, $(u^\alpha) = (0, 0, z, 0)$