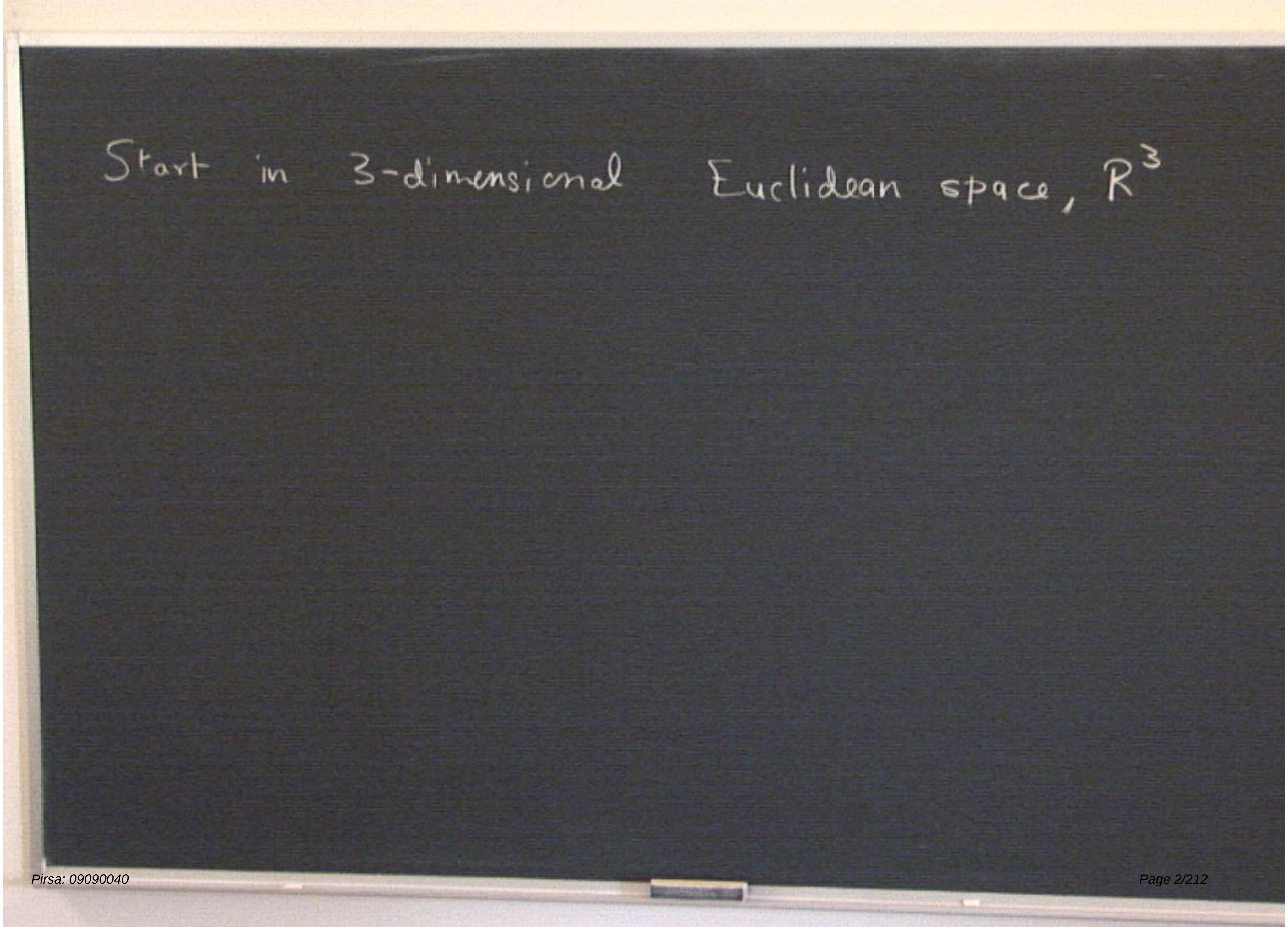
Title: Quantum Theory - Core (PHYS 605) - Lecture 1

Date: Sep 07, 2009 09:00 AM

URL: http://pirsa.org/09090040

Abstract:



Start in 3-dimensional Euclidean space, R^3

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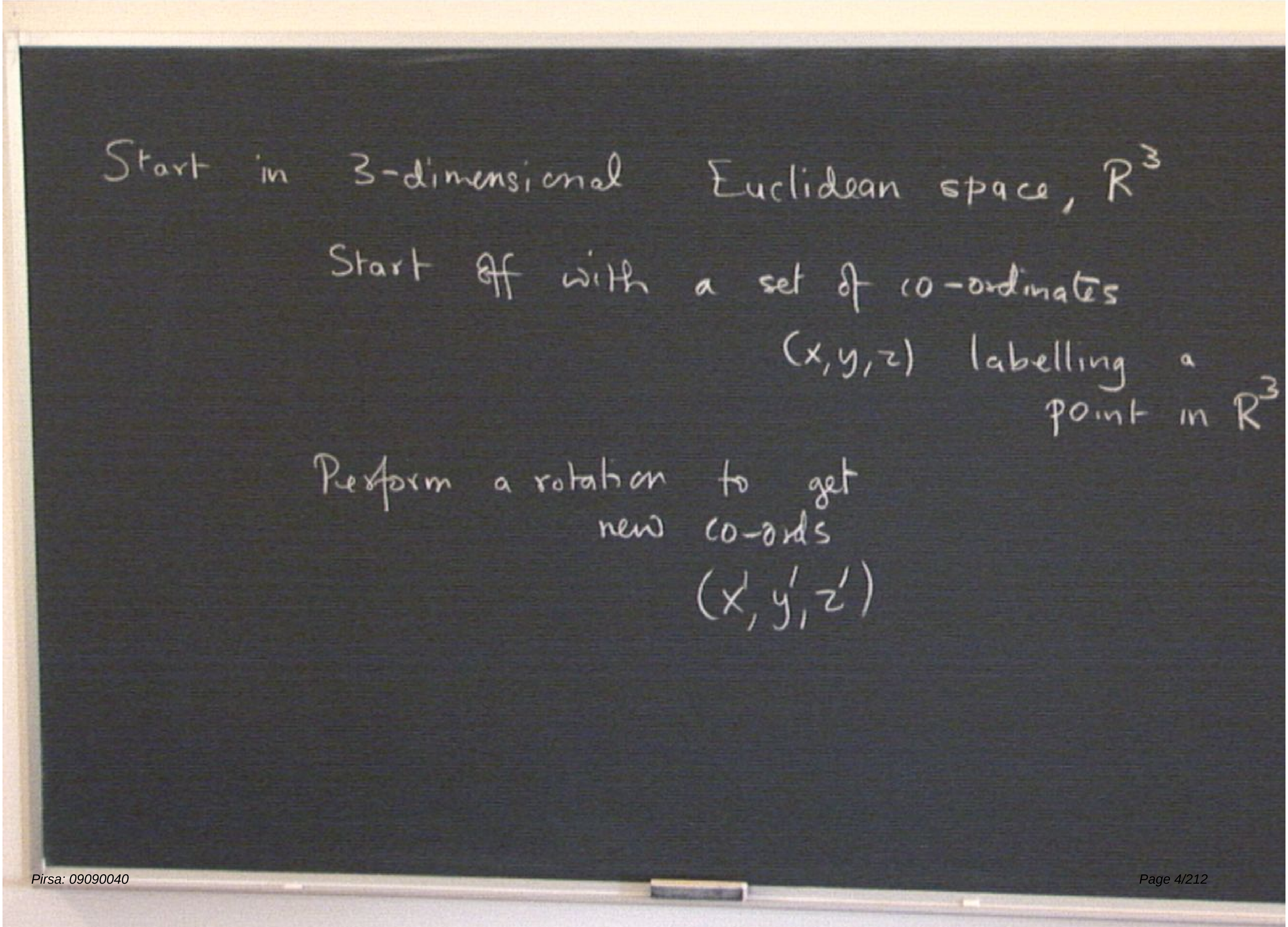
Start off with a set of co-ordinates (x, y, z) labelling a point in R^3



Start in 3-dimensional Euclidean space, R^3

Start off with a set of co-ordinates

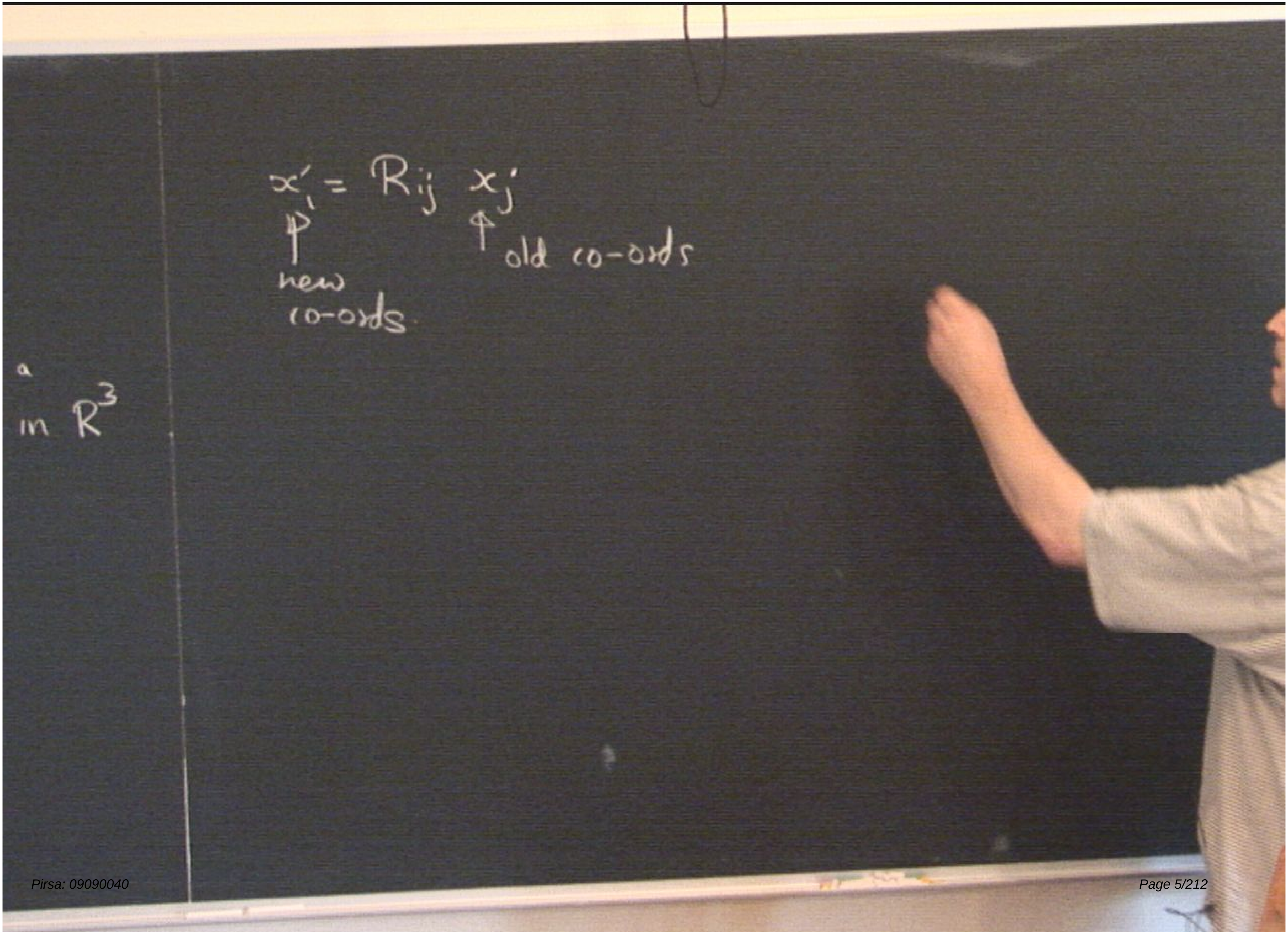
(x,y,z) labelling a

point in R^3

Perform a rotation to get

new co-ords

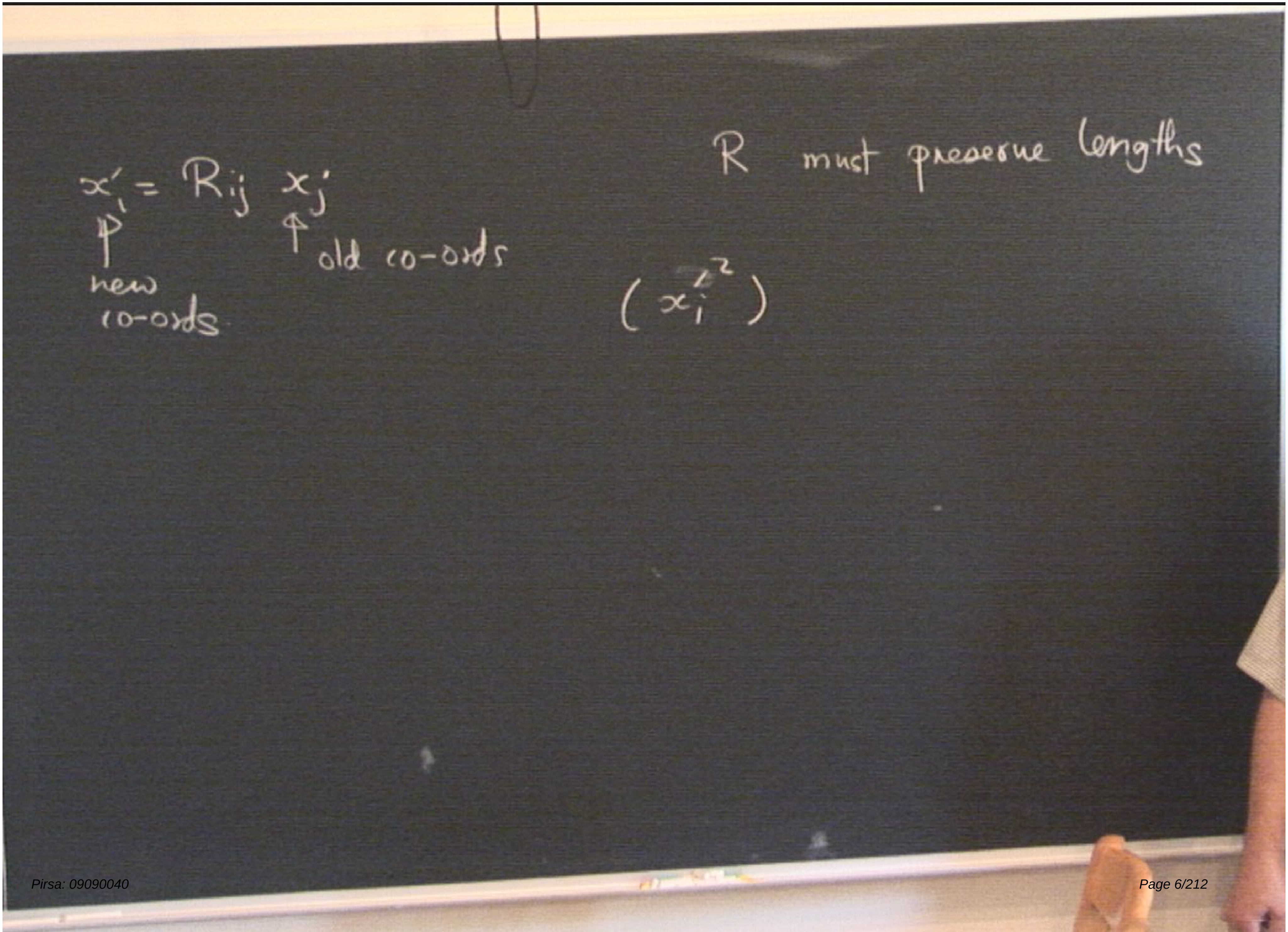
(x',y',z')



$$\underset{\substack{\uparrow\\ \text{new}\\ \text{co-ords}}}{x'_i} = R_{ij}\ \underset{\uparrow\ \text{old co-ords}}{x_j}$$

a

in R^3



$$\underset{\substack{\uparrow \\ \text{new} \\ \text{co-ords}}}{x_i'} = R_{ij}\, \underset{\uparrow\,\text{old co-ords}}{x_j}$$

R must preserve lengths

$$(x_i'^2)$$

$$\underset{\uparrow}{x'_i} = R_{ij} \; \underset{\uparrow}{x_j}$$

old co-ords

new co-ords.

R must preserve lengths

$$(x'^2_i) = x'_i x'_i$$

$$= R_{ij} x_j R_{ik} x_k$$

$$\underset{\substack{\uparrow \\ \text{new} \\ \text{co-ords}}}{x'_i} = R_{ij} \underset{\uparrow \text{ old co-ords}}{x_j}$$

R must preserve lengths

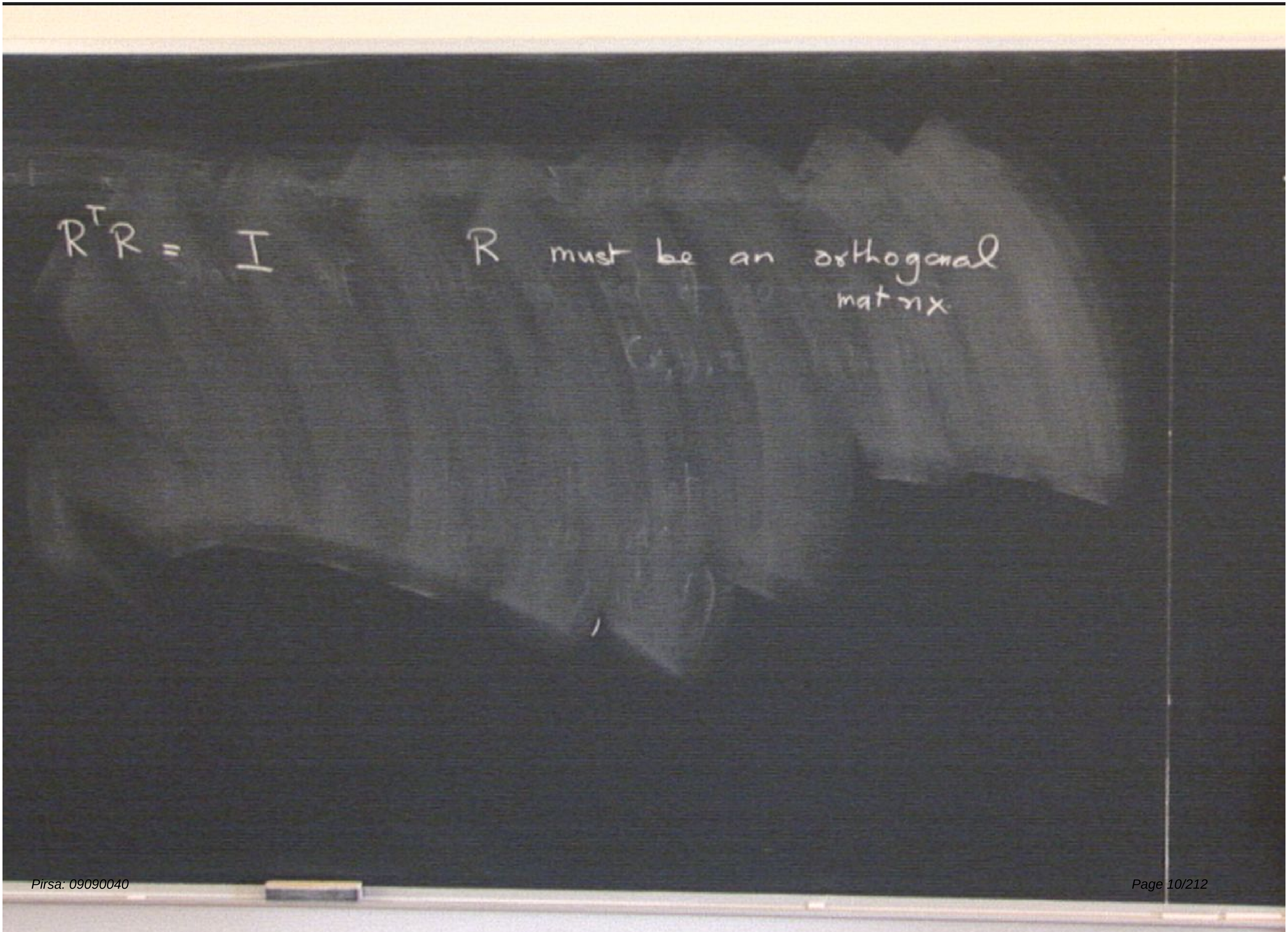
$$(x_i'^2) = x_i' x_i'$$

$$= R_{ij} x_j R_{ik} x_k$$

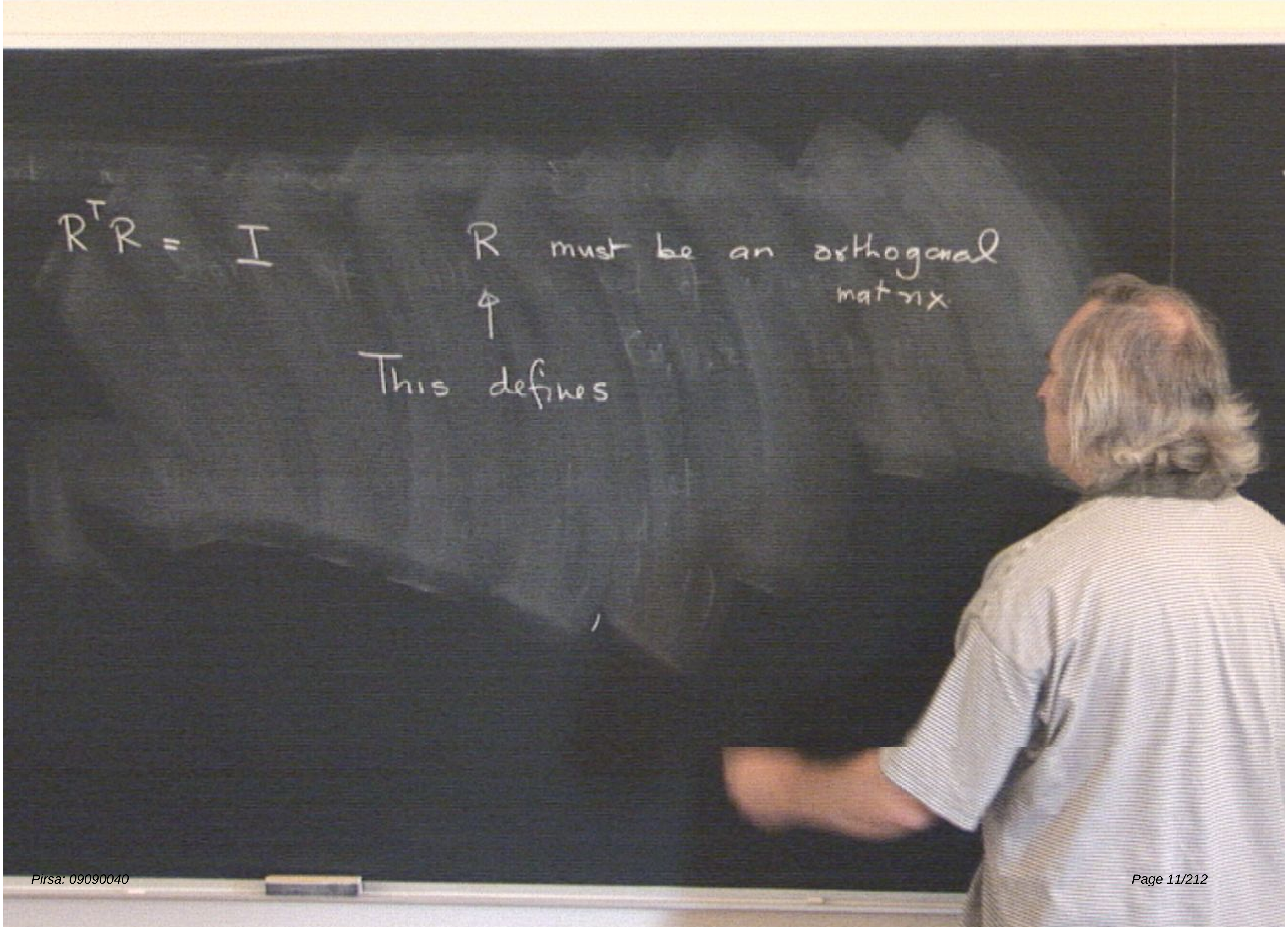
$$= x_j x_j$$

$$R_{ij} R_{ik} = \delta_{jk}$$

$$R^T R = \underline{I}$$



$R^T R = I$ R must be an orthogonal matrix.

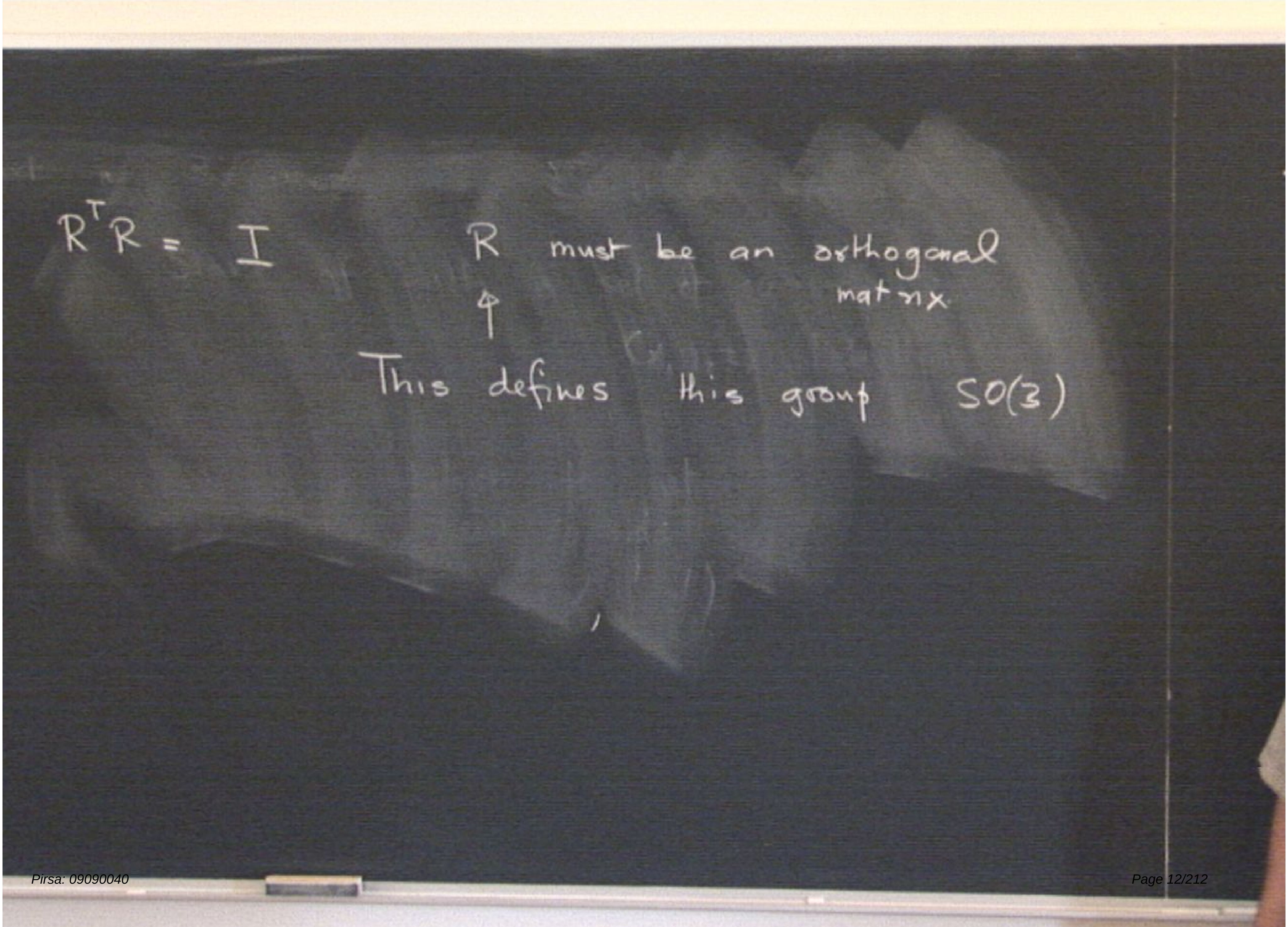


$R^T R = I$

R must be an orthogonal matrix

↑

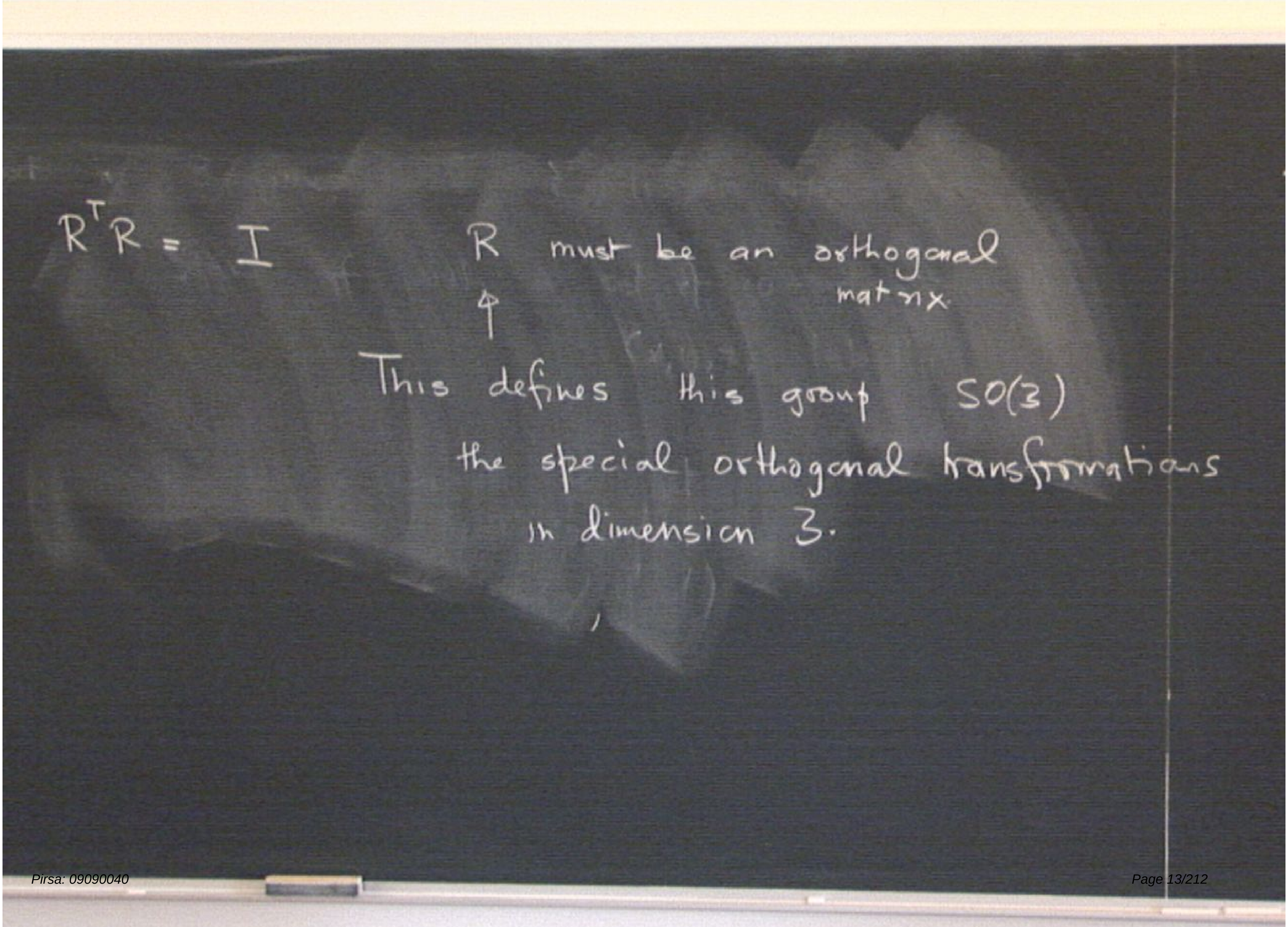
This defines



$R^T R = I$ R must be an orthogonal matrix

↑

This defines this group SO(3)



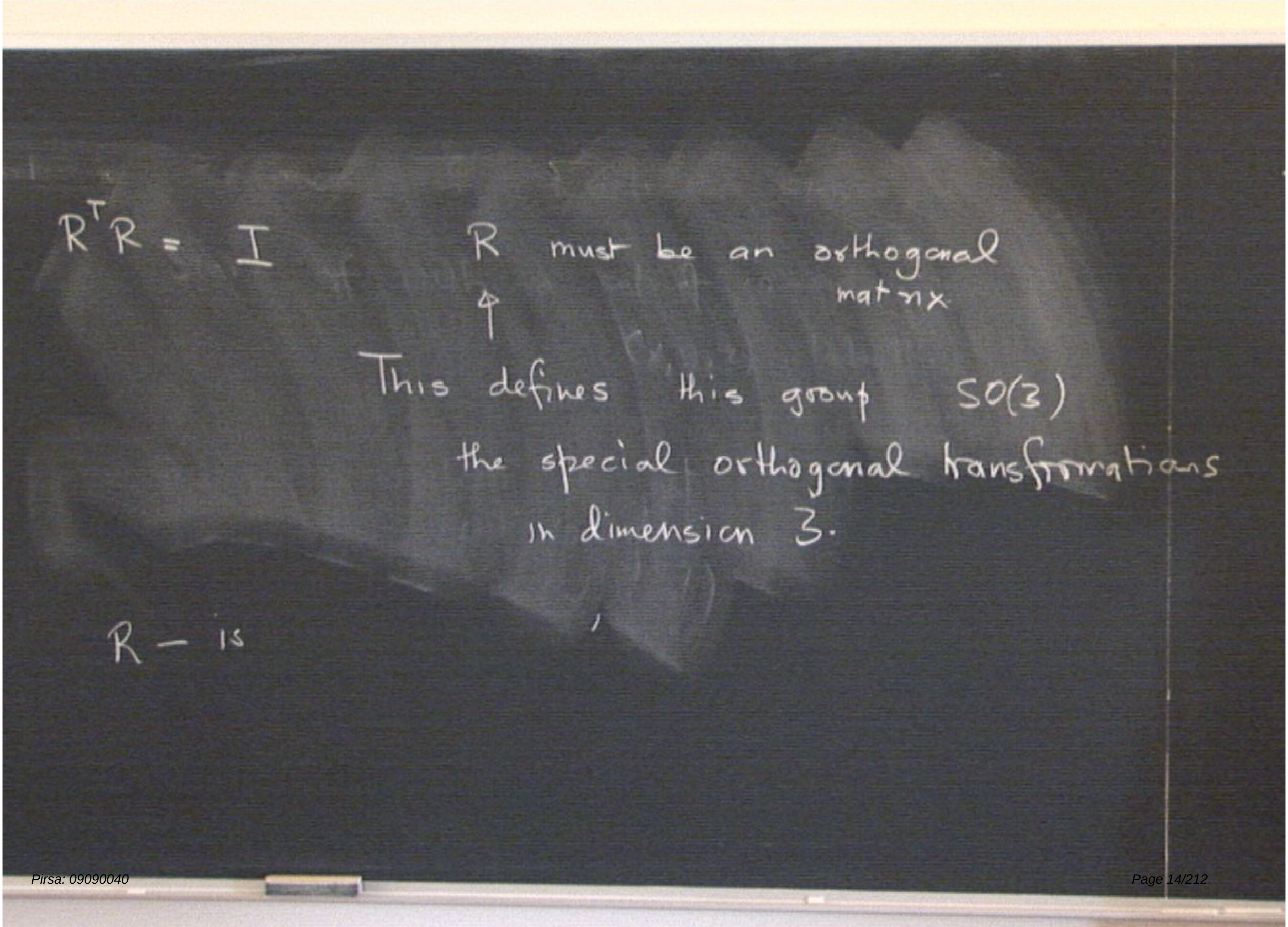
$R^T R = I$ R must be an orthogonal matrix

$\uparrow$

This defines this group SO(3)

the special orthogonal transformations

in dimension 3.



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This defines this group SO(3)

the special orthogonal transformations

in dimension 3.

R — is

$R^T R = I$ R must be an orthogonal matrix.

↑

This defines this group SO(3)

the special orthogonal transformations in dimension 3.

R – is the space of 3×3 orthogonal matrices

$$R^T R = I$$

$$(\det R)^2 = 1$$

$$\det R = \pm 1$$

$$\underset{\uparrow}{S}O(3)$$

tions

s

$$R^T R = I$$

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$$\underset{\substack{\uparrow \\ \det R = 1}}{S}O(3)$$

tions

s

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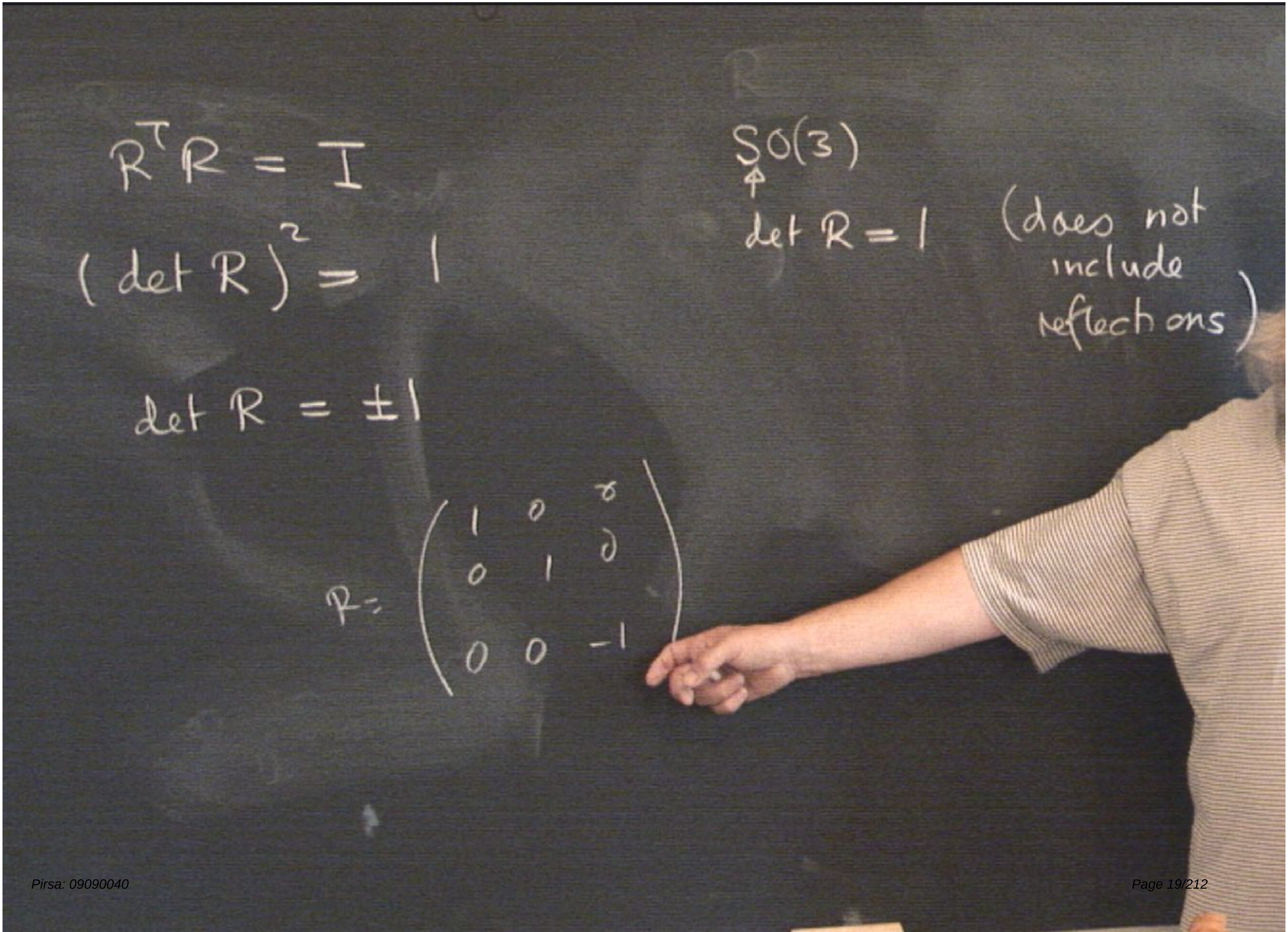
$$SO(3)$$

$$\uparrow$$

$\det R = 1$ (does not include reflections)



$$R^T R = I$$

$$(\det R)^2 = 1$$

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$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

SO(3)
↑
$\det R = 1$ (does not include reflections)

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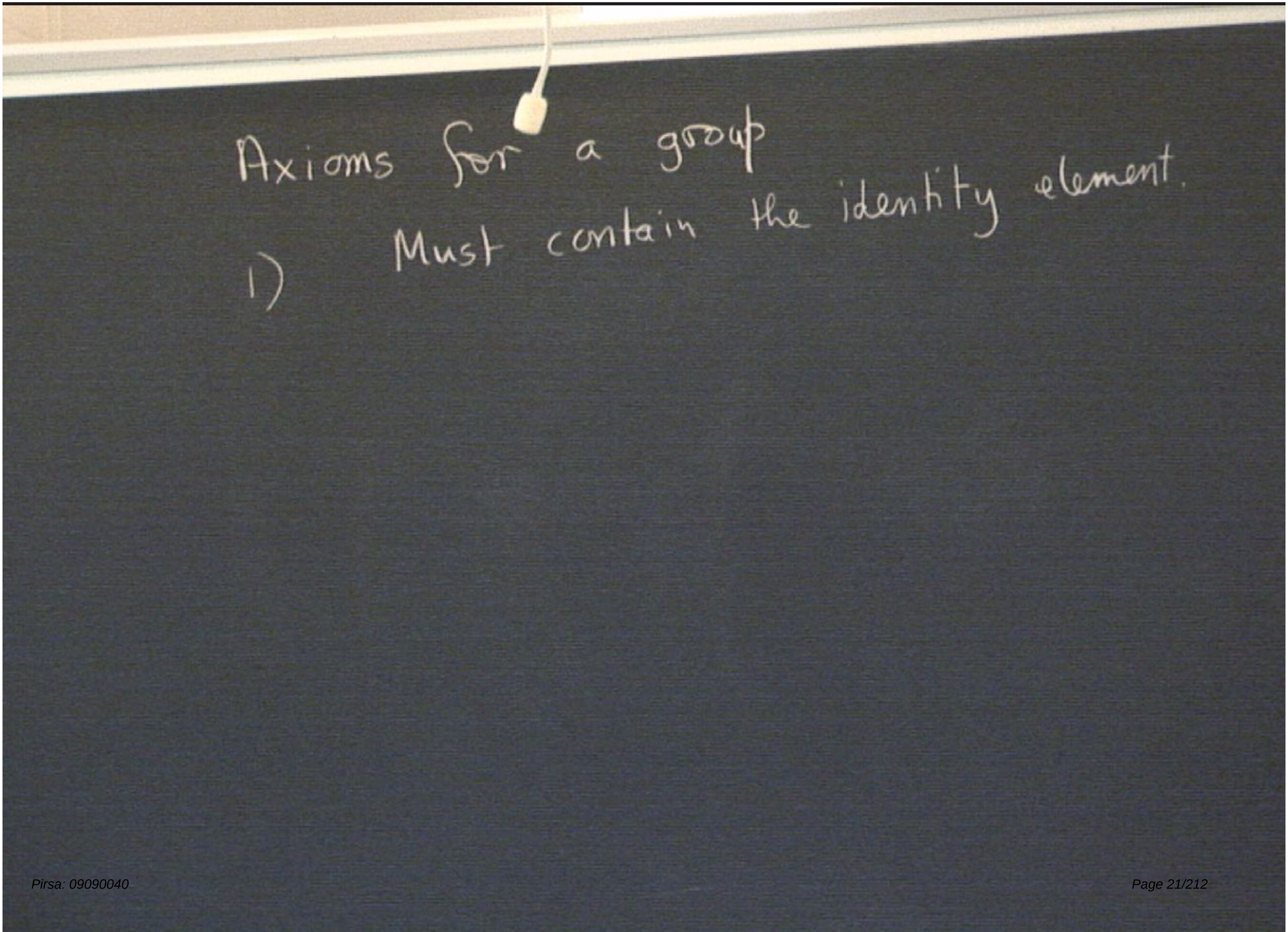

Axioms for a group

1) Must contain the identity element.

Axioms for a group

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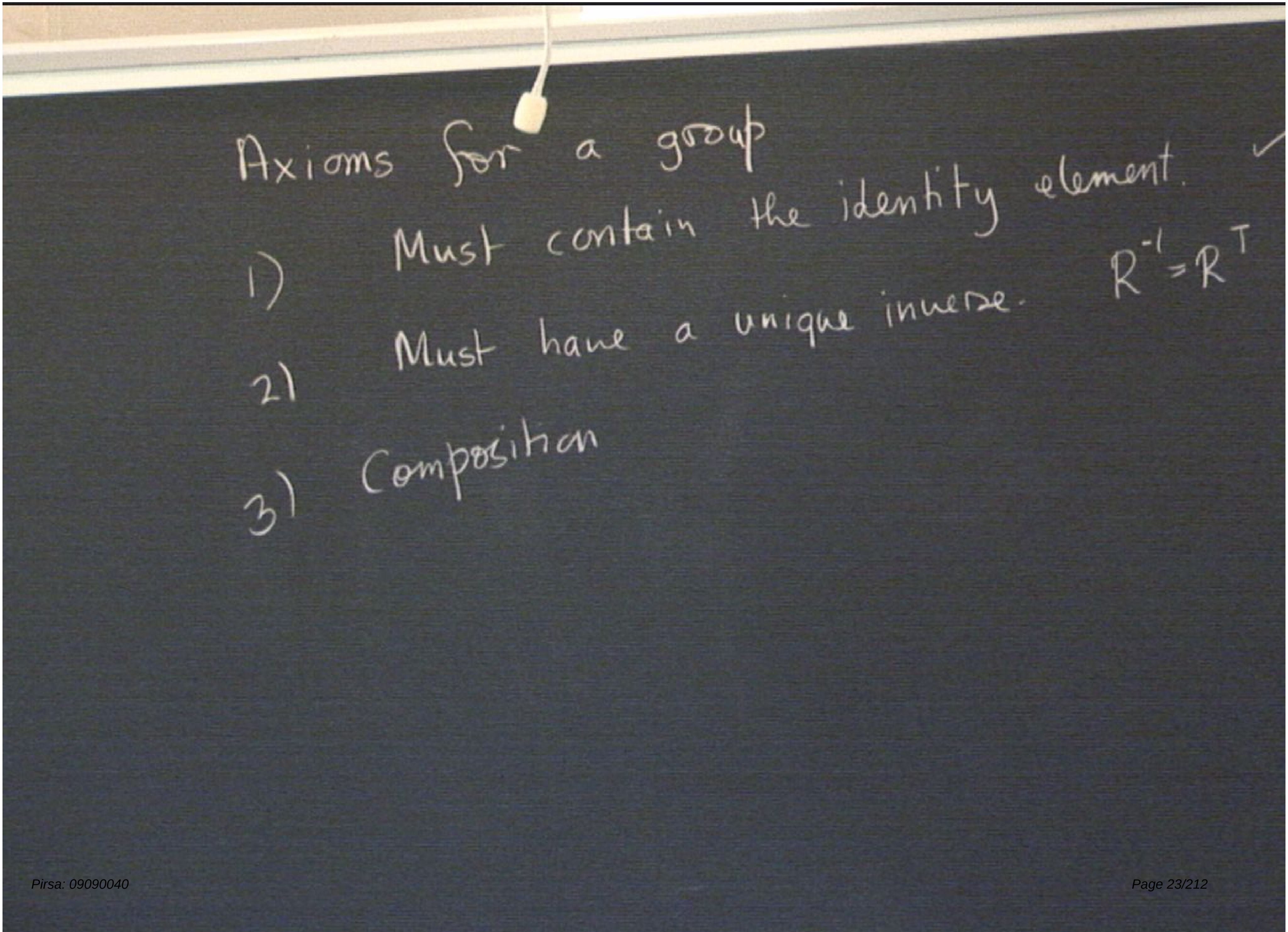
2) Must have a unique inverse.

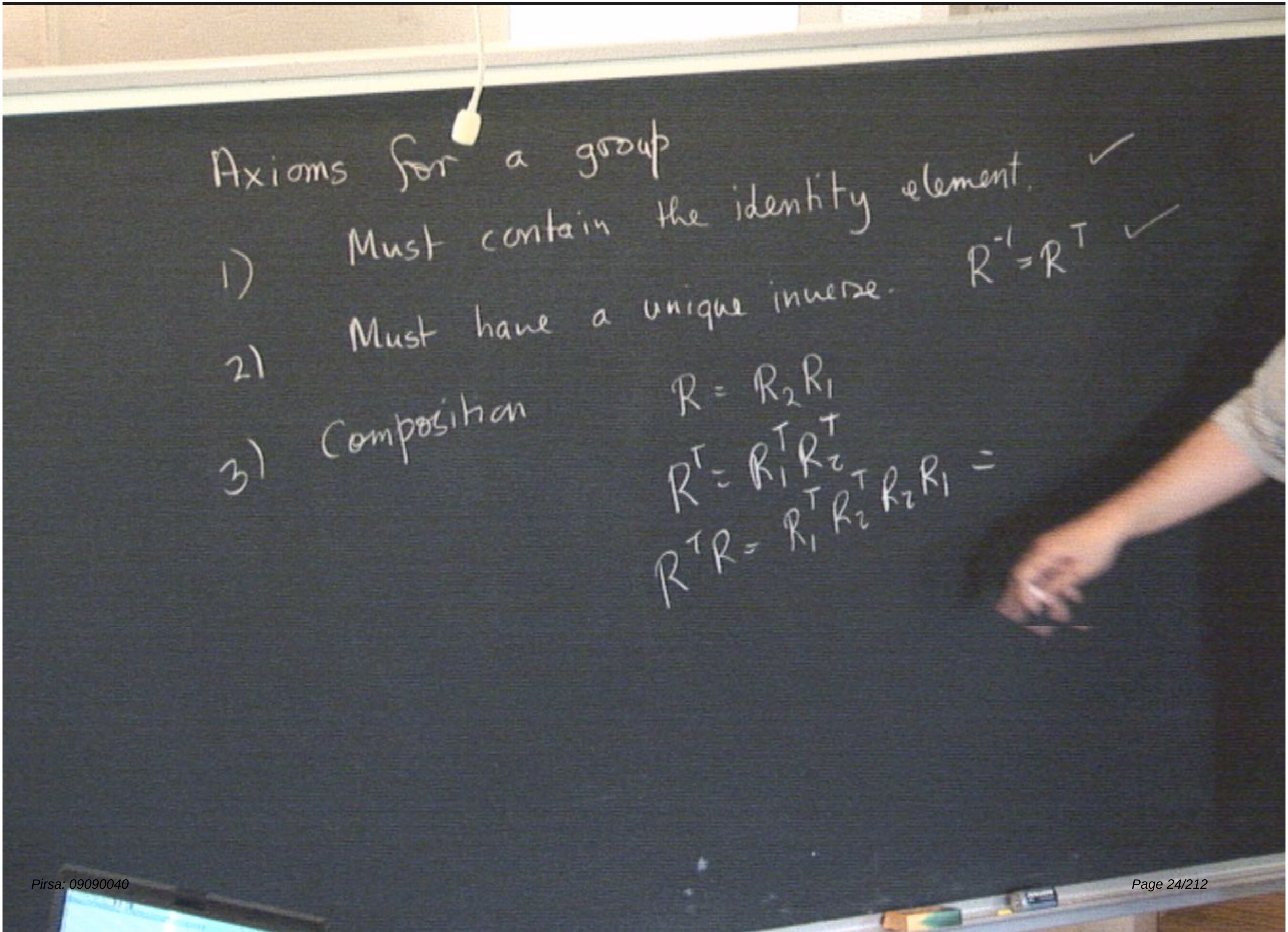


Axioms for a group

1) Must contain the identity element. ✓

2) Must have a unique inverse. $R^{-1} = R^T$

3) Composition



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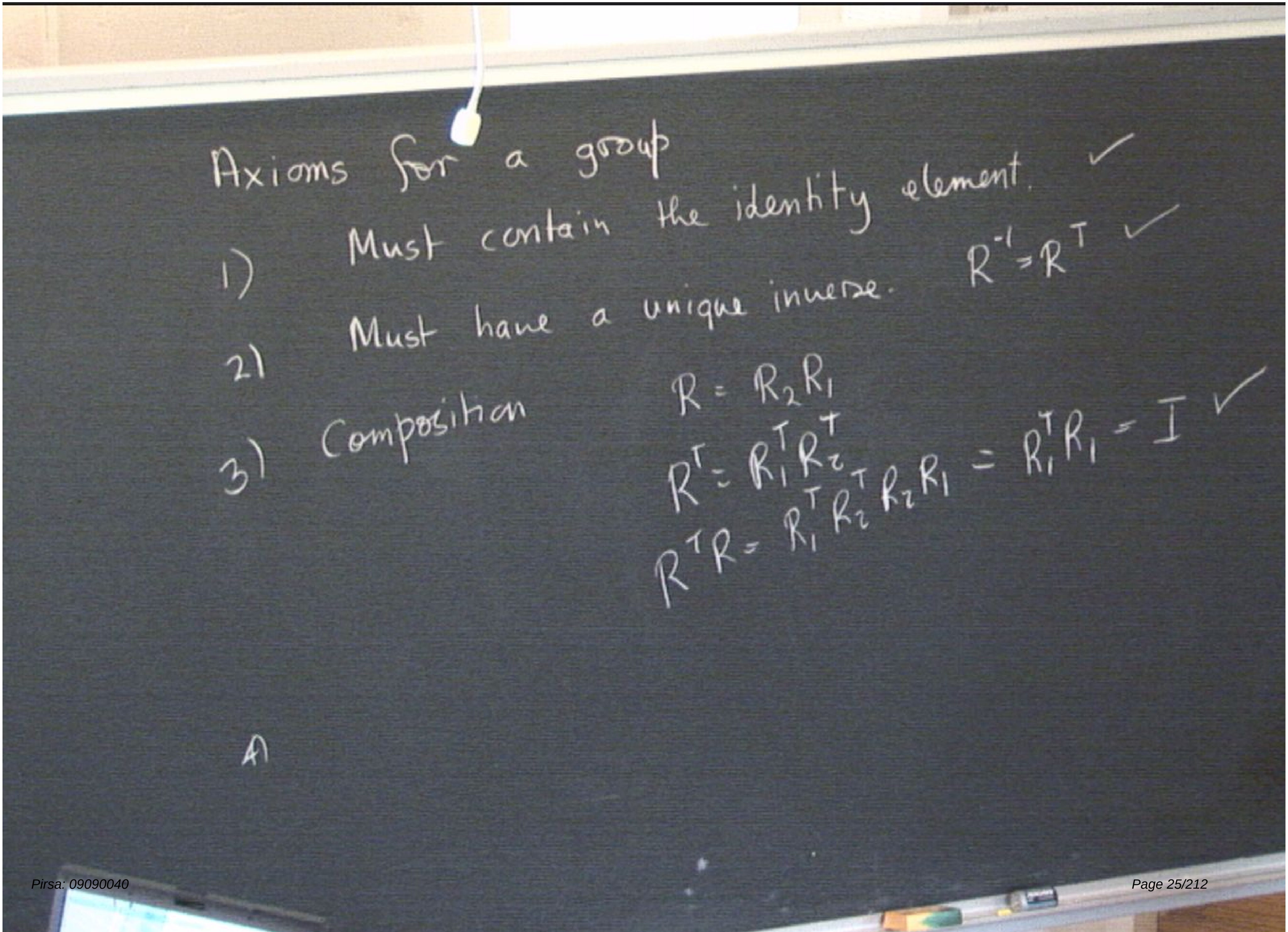
$$R^T = R_1^T R_2^T$$

$$R^T R = R_1^T R_2^T R_2 R_1 =$$



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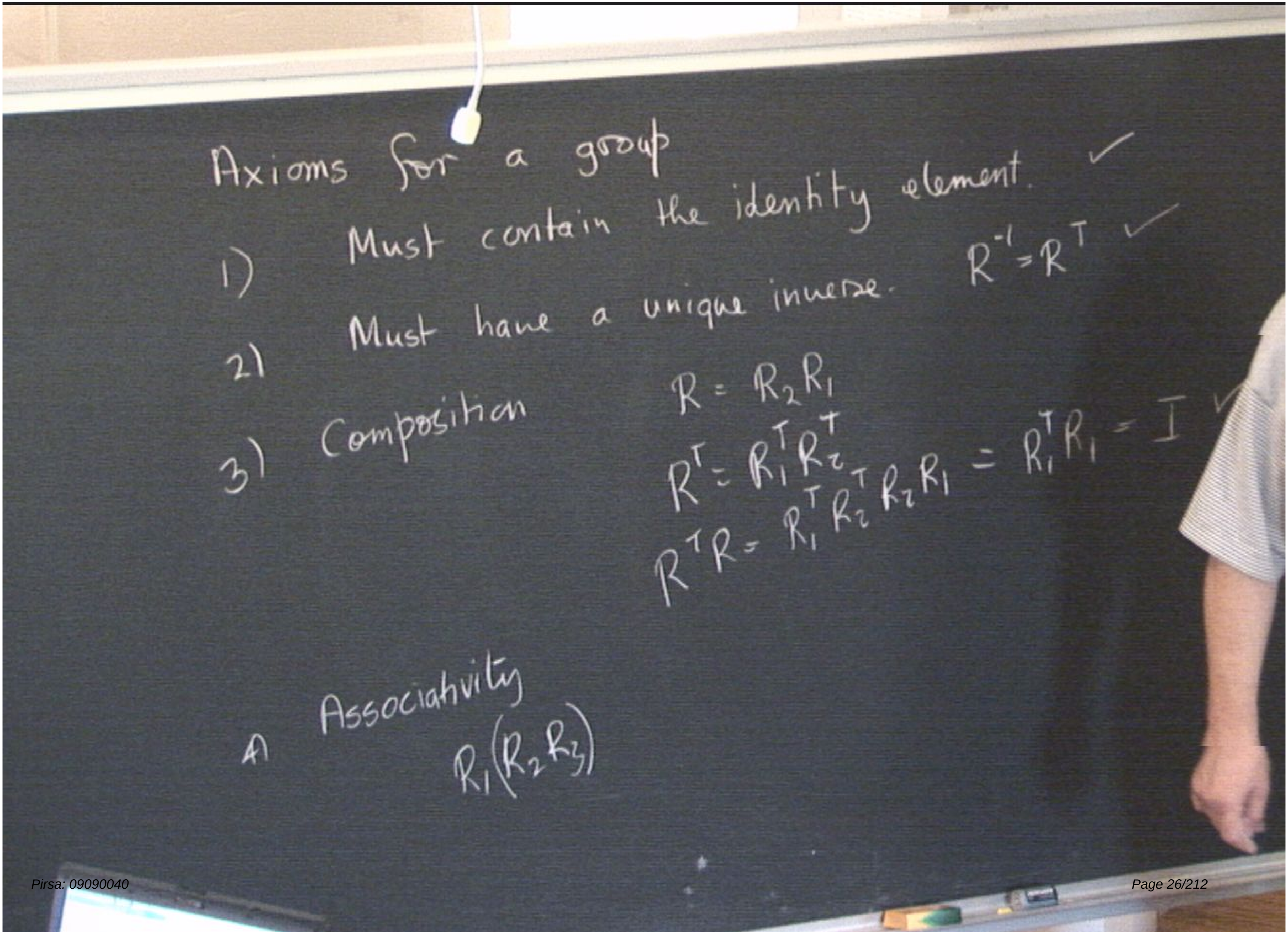
4)



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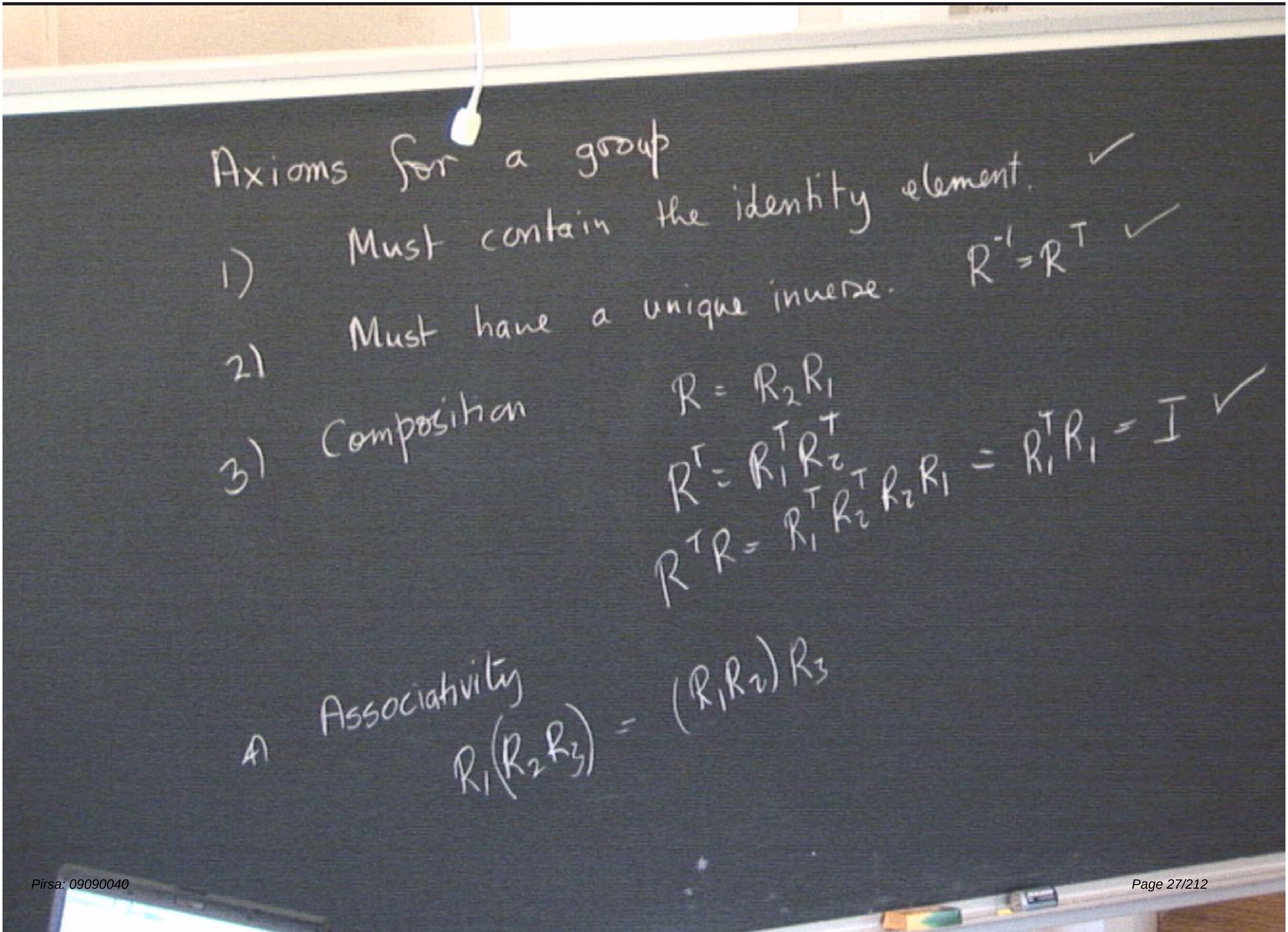
4 Associativity

$R_1(R_2 R_3)$



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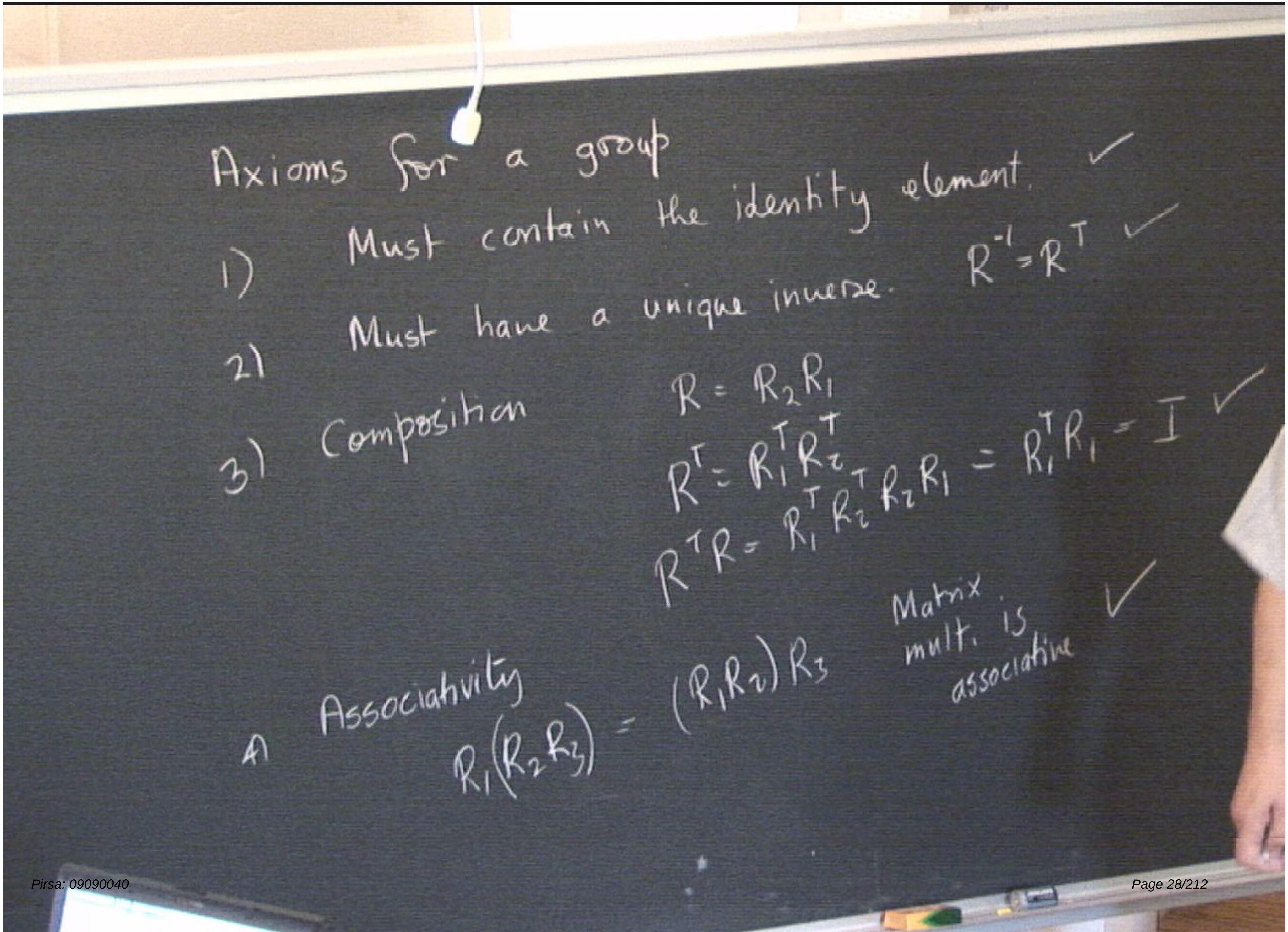
$$R^T R = R_1^T R_2^T R_2 R_1 = R_1^T R_1 = I$$ ✓

4) Associativity

$$R_1(R_2 R_3) = (R_1 R_2) R_3$$



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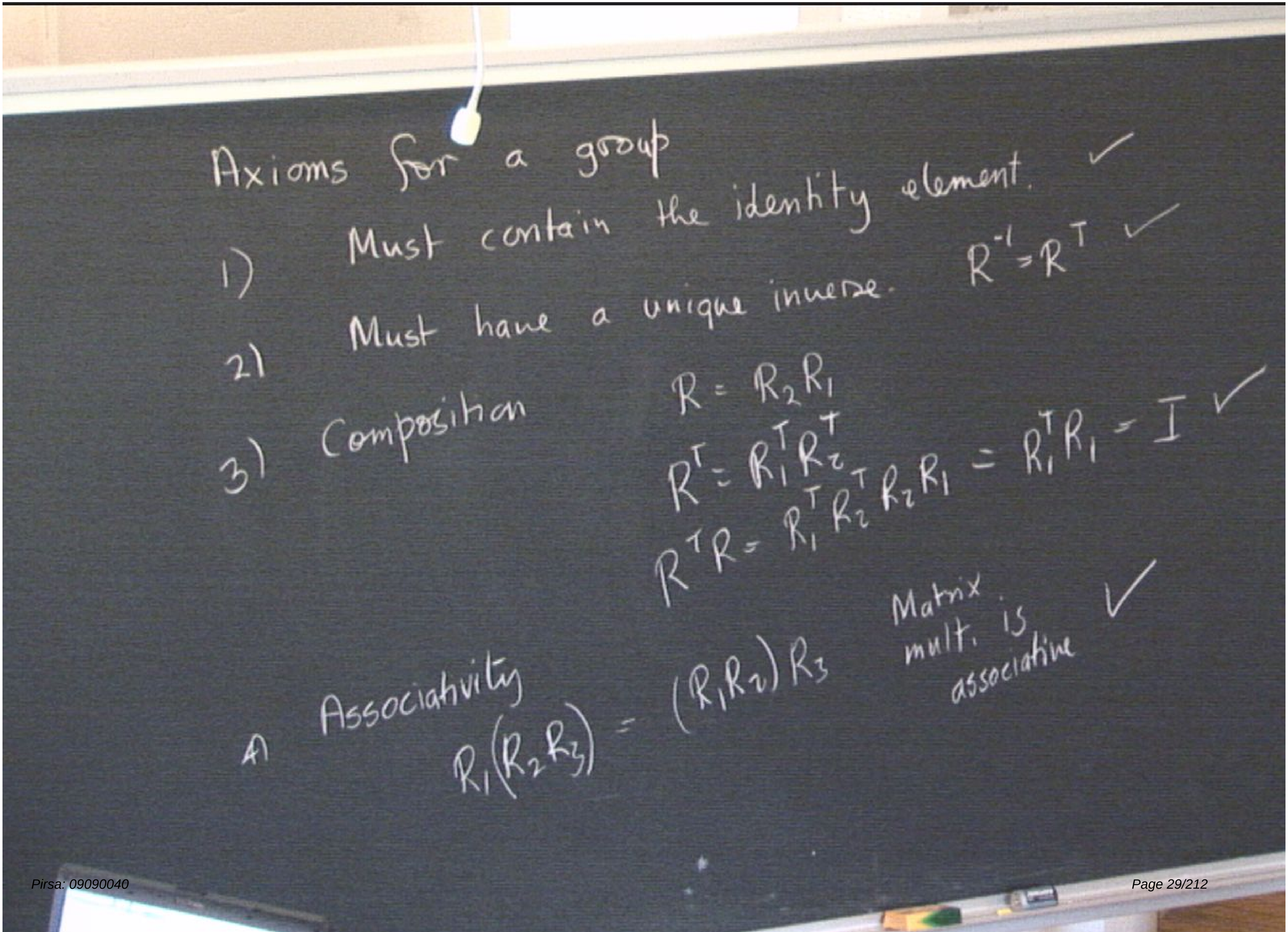
$R^T R = R_1^T R_2^T R_2 R_1 = R_1^T R_1 = I$ ✓

4 Associativity

$R_1(R_2 R_3) = (R_1 R_2) R_3$ Matrix mult. is associative ✓



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$R^T R = R_1^T R_2^T R_2 R_1 = R_1^T R_1 = I$ ✓

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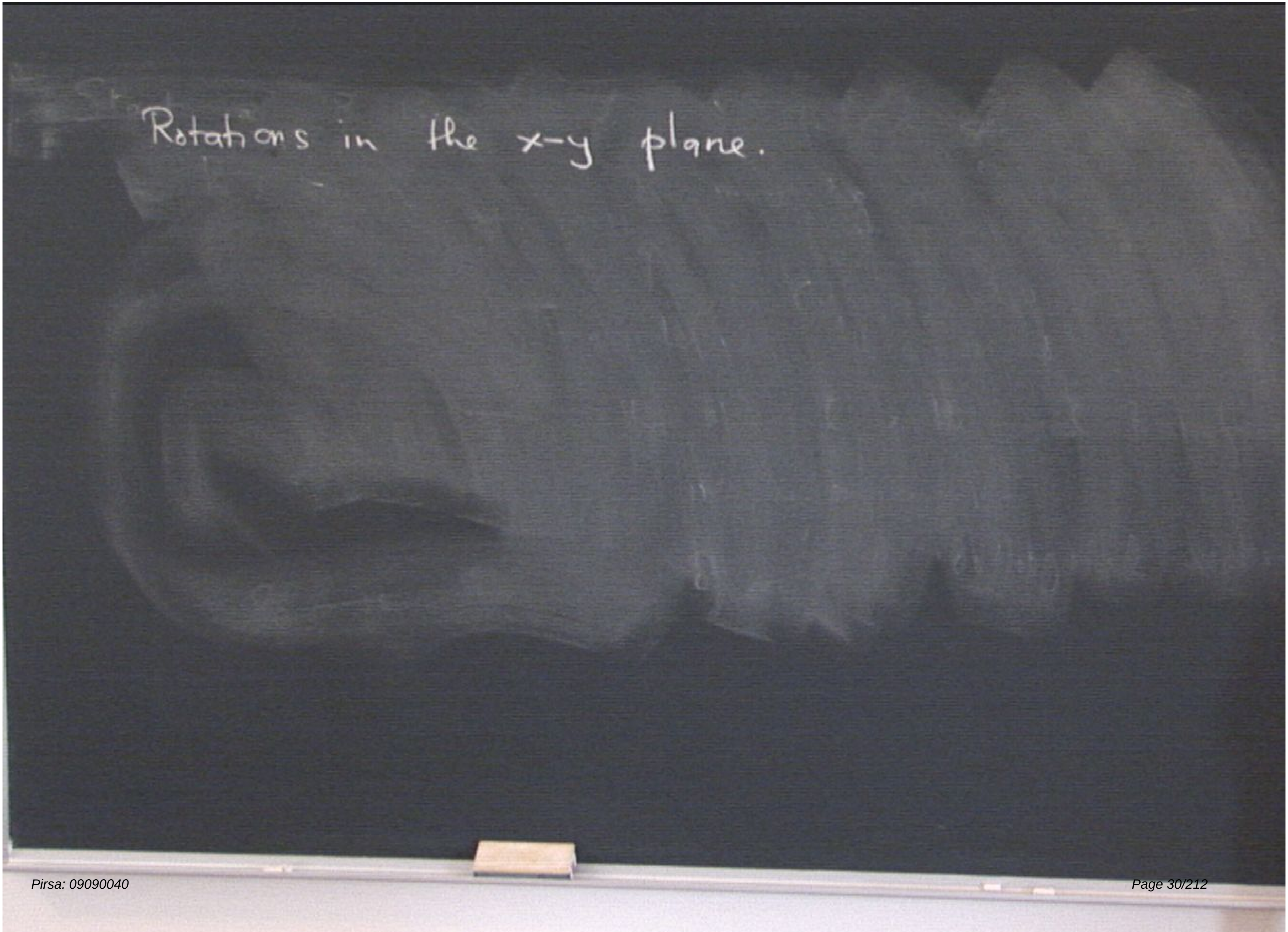
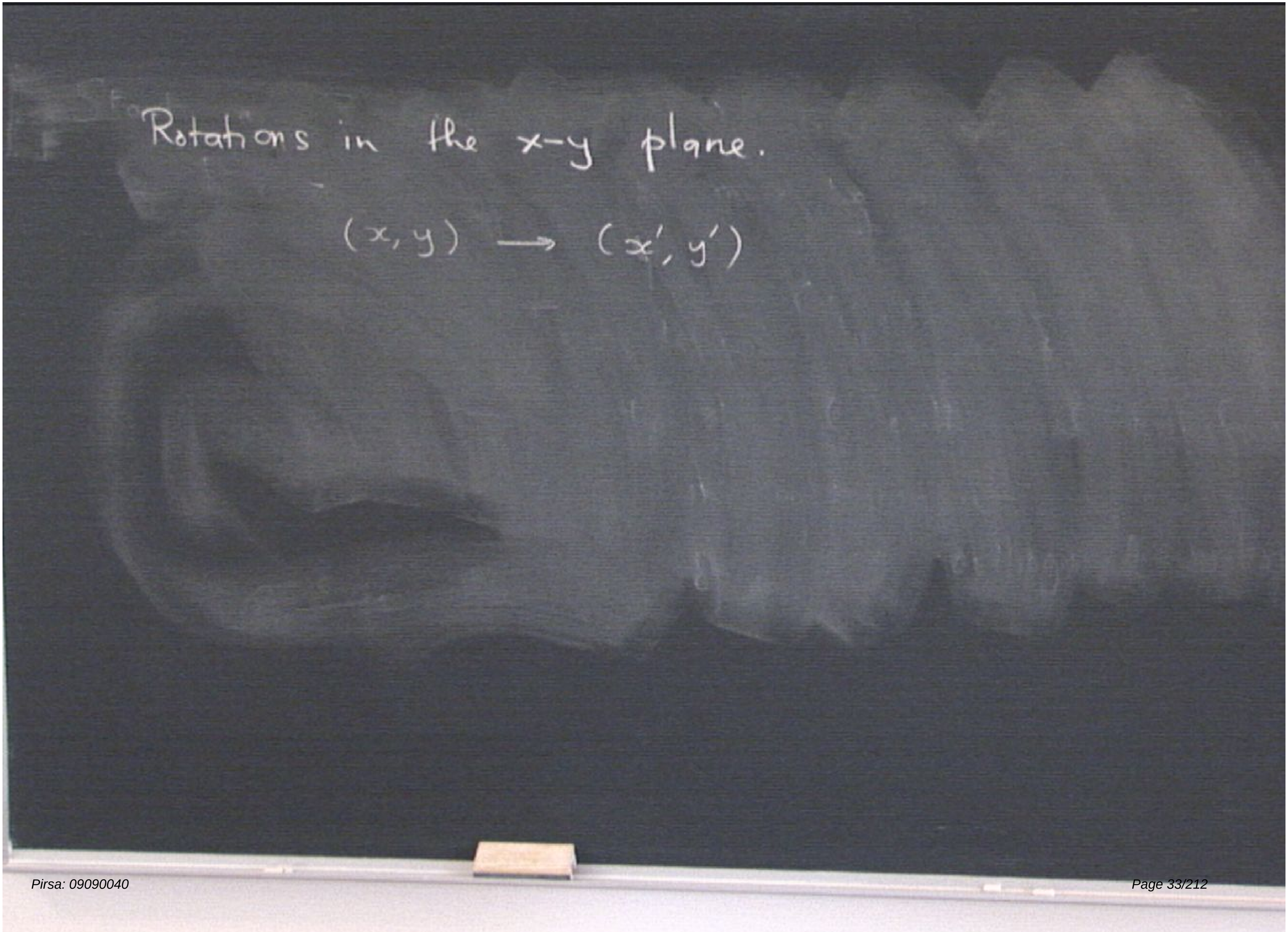


Rotations in the x-y plane.

Rotations in the x-y plane.

Rotations in the x-y plane.

$$(x, y) \longrightarrow (x', y')$$



Rotations in the x-y plane.

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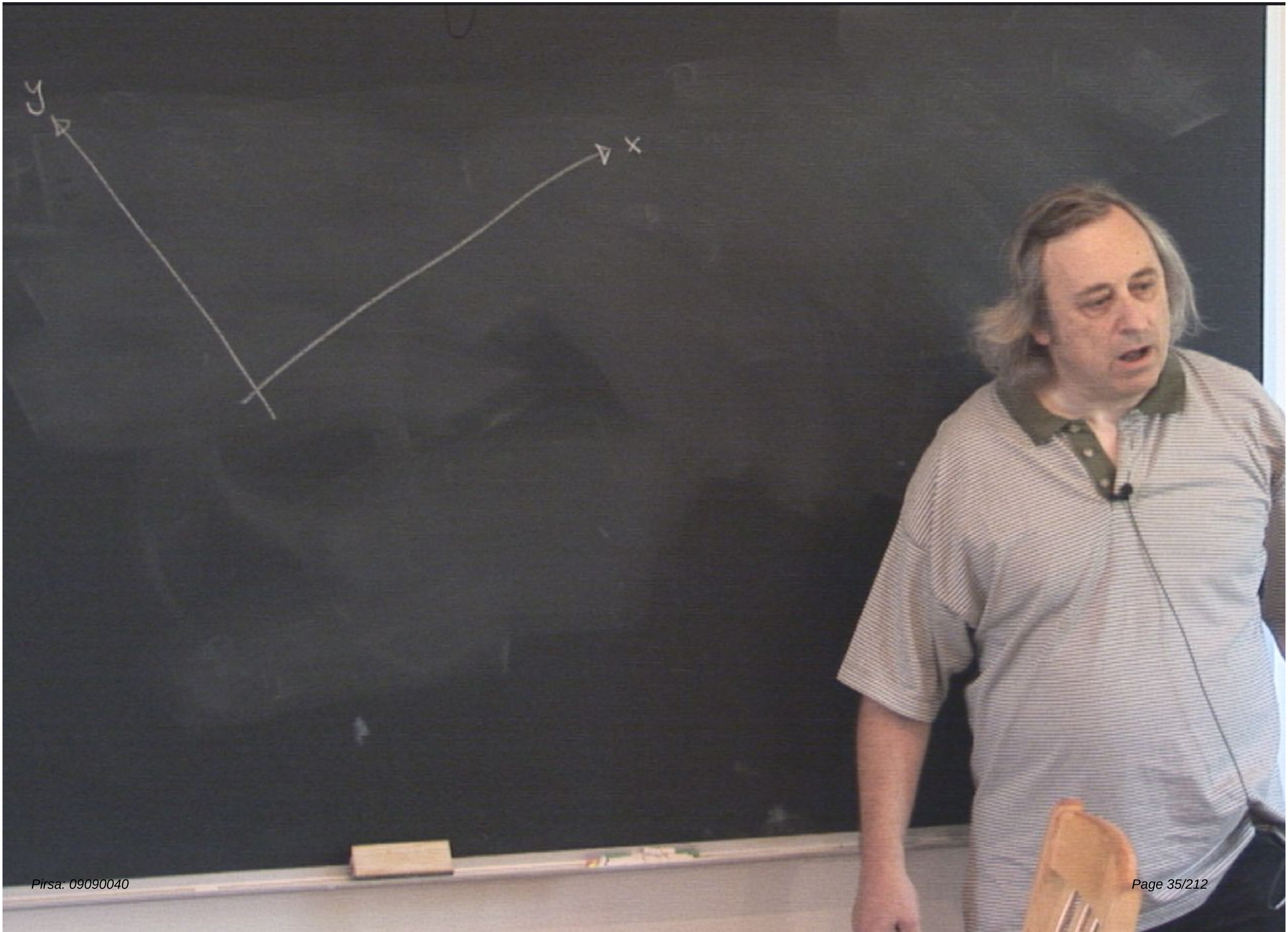
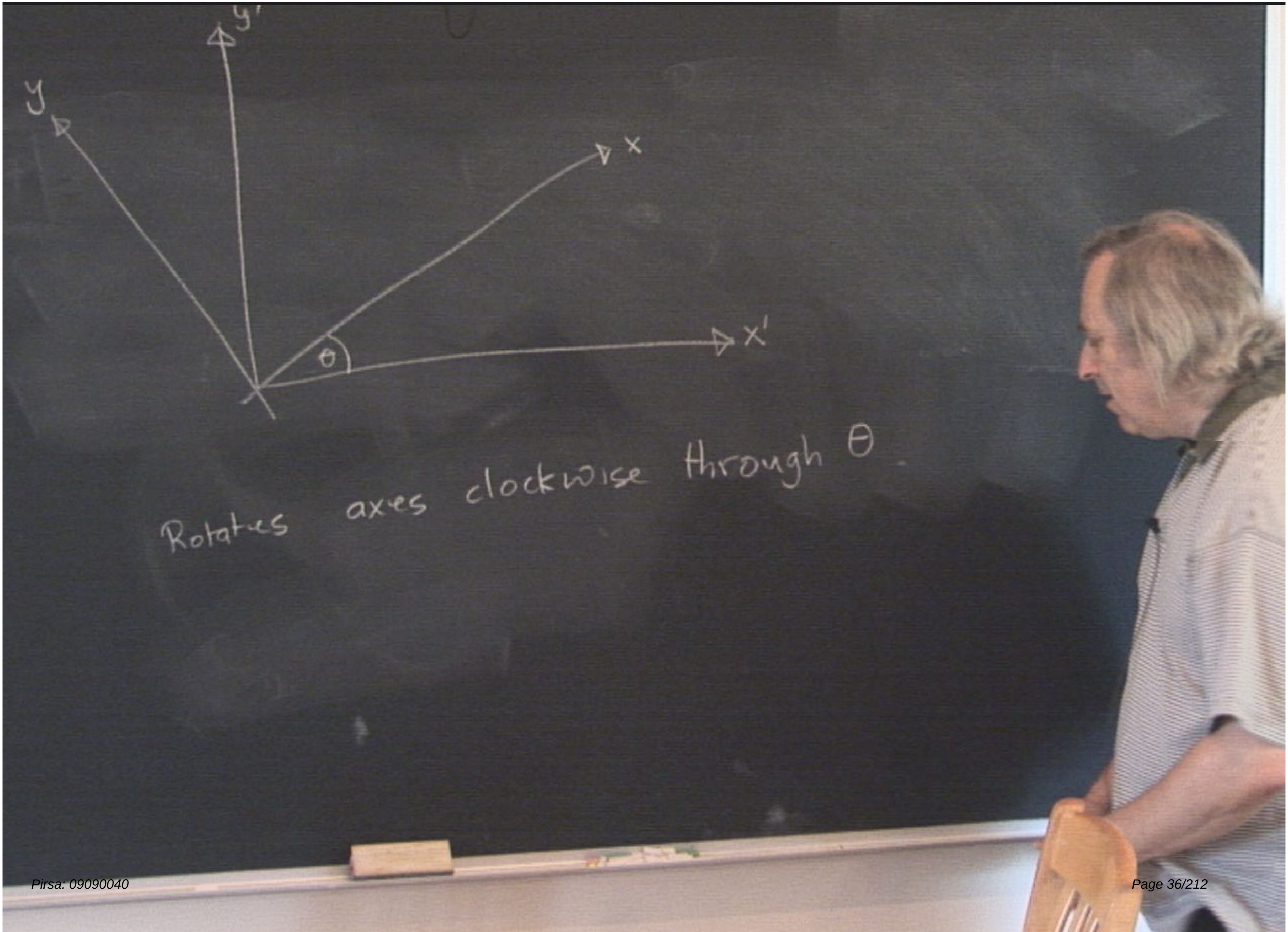
Rotations in the x-y plane.

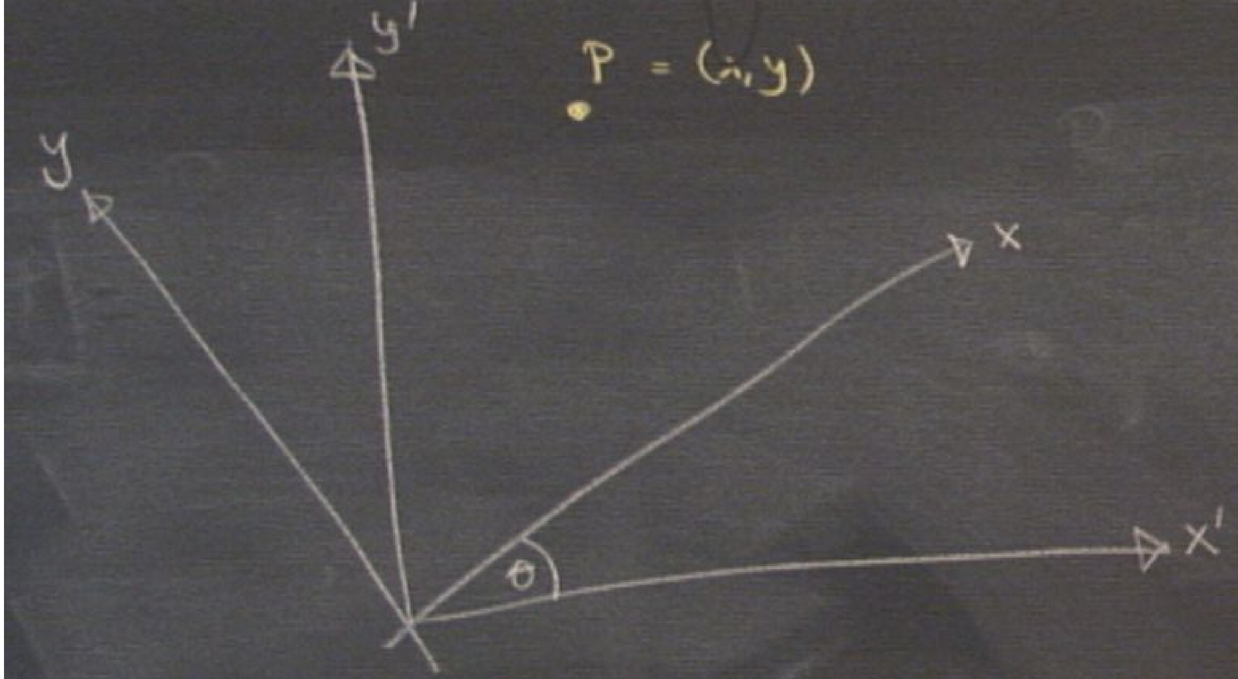
$(x, y) \longrightarrow (x', y')$ z-fixed.


y
x



Rotates axes clockwise through θ

$P = (x, y)$



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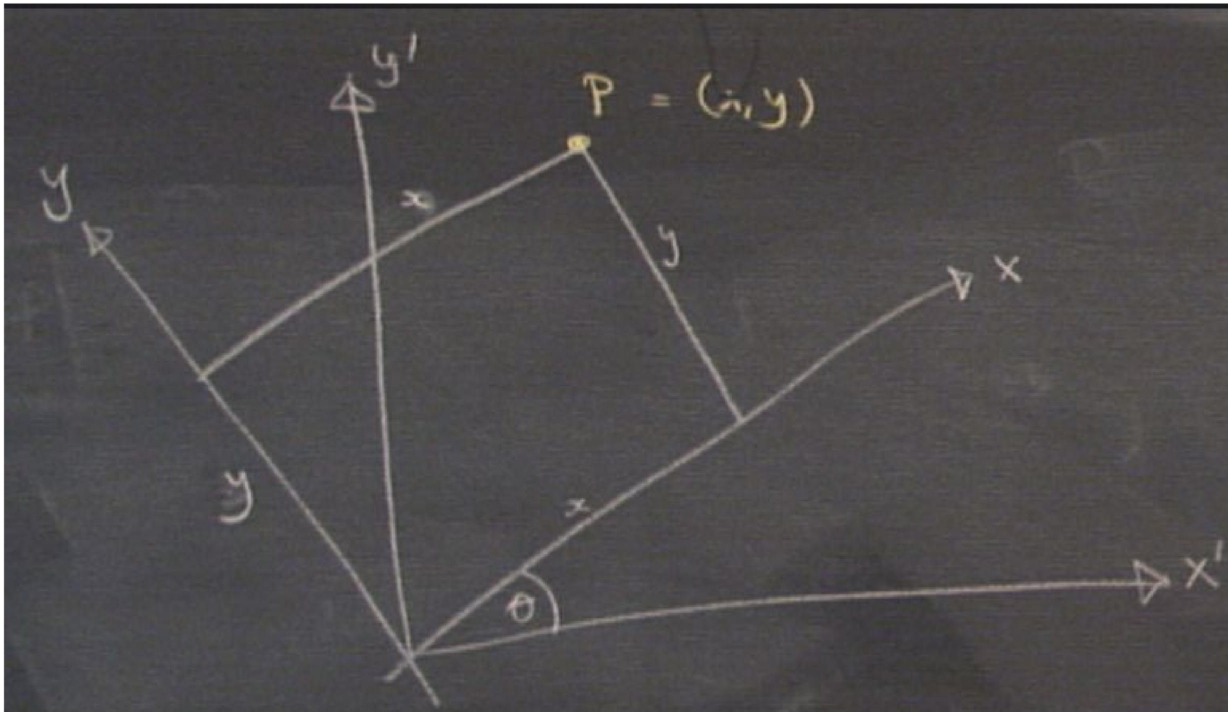

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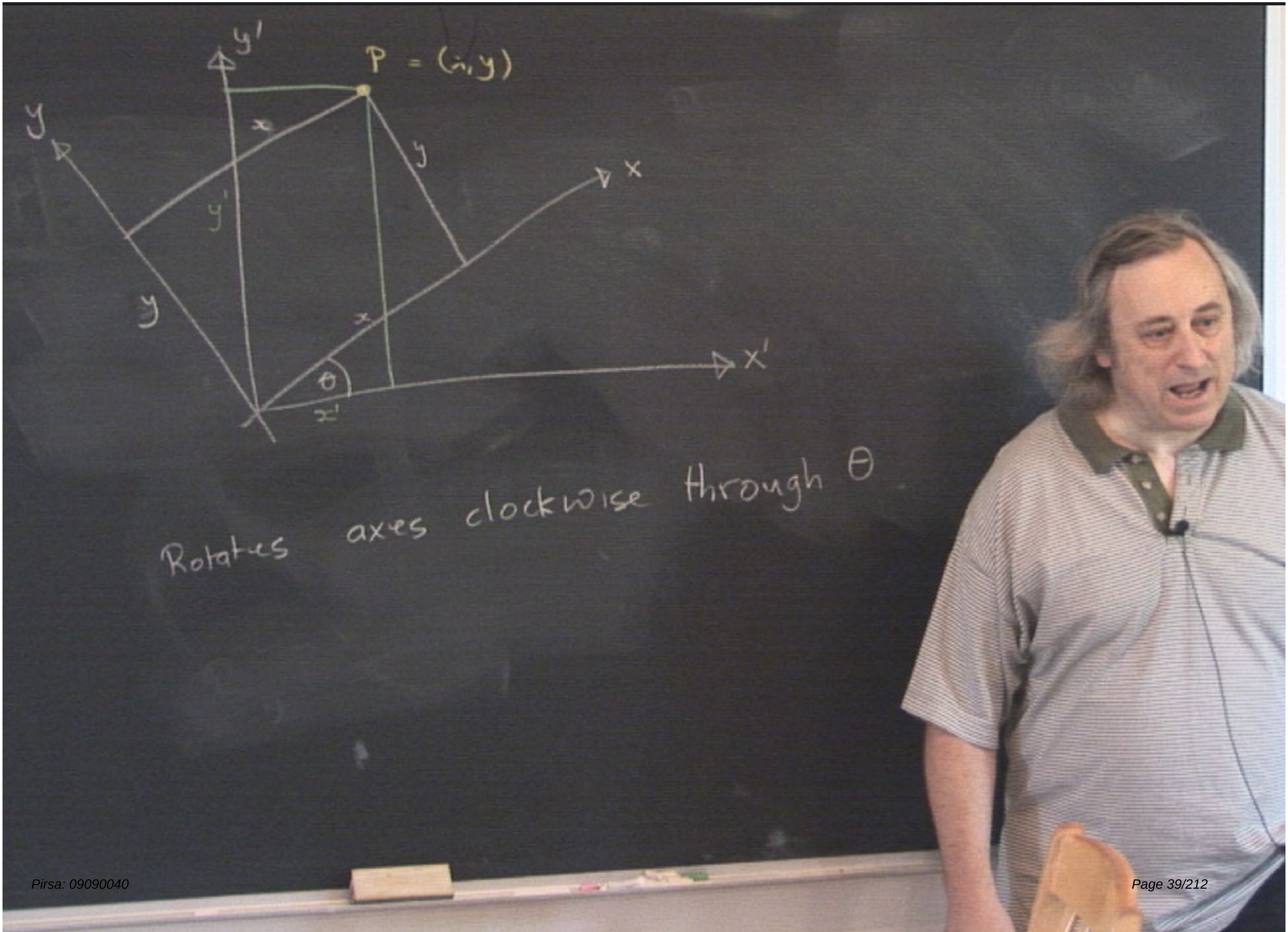

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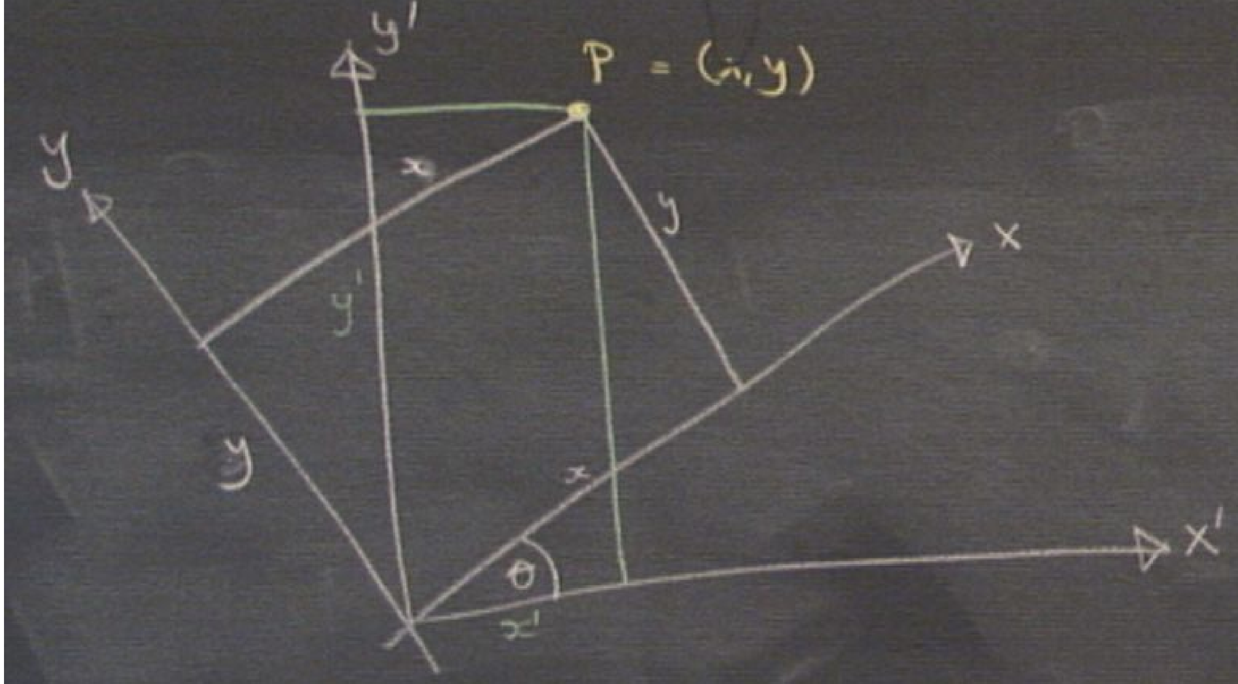
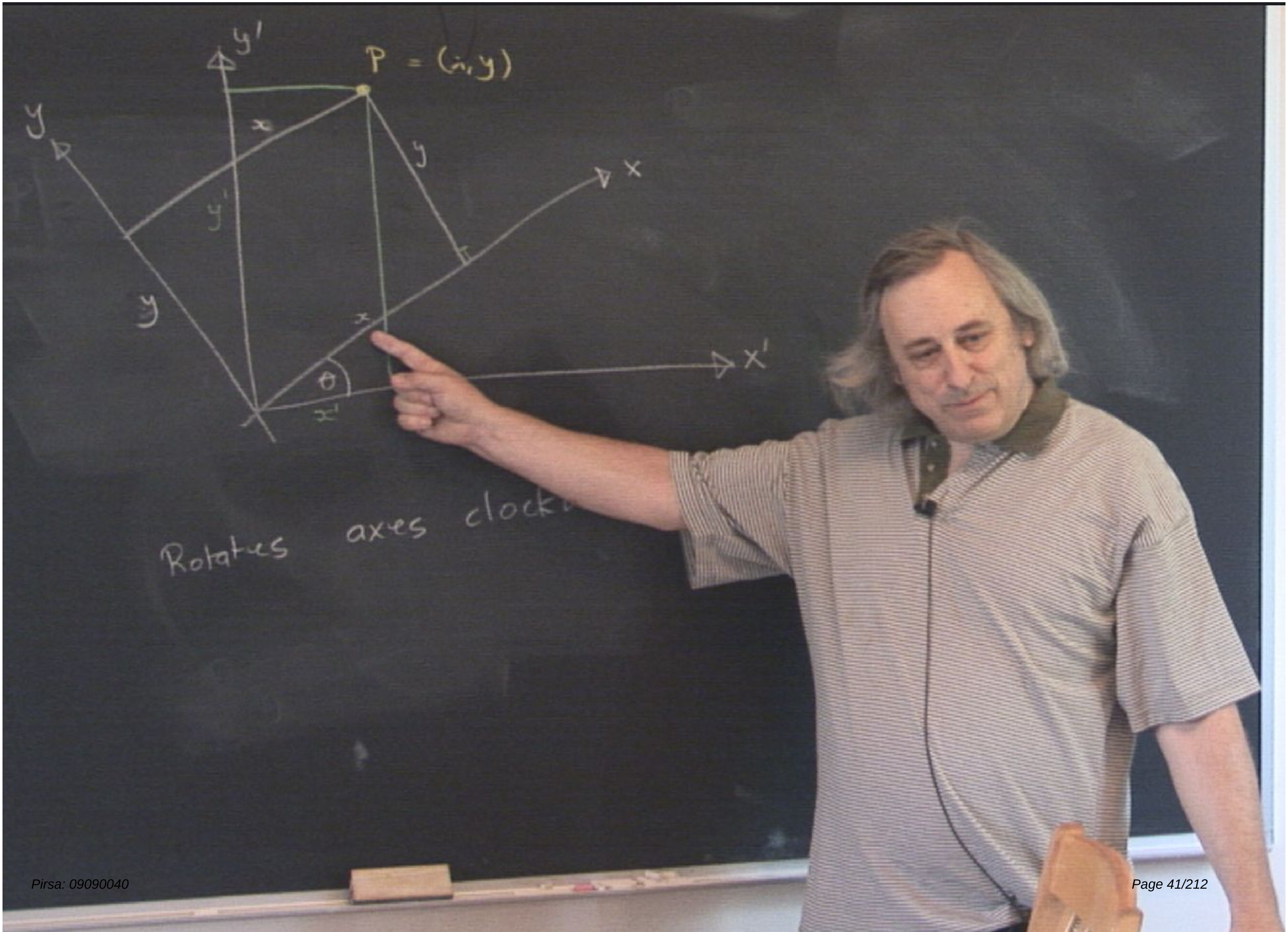
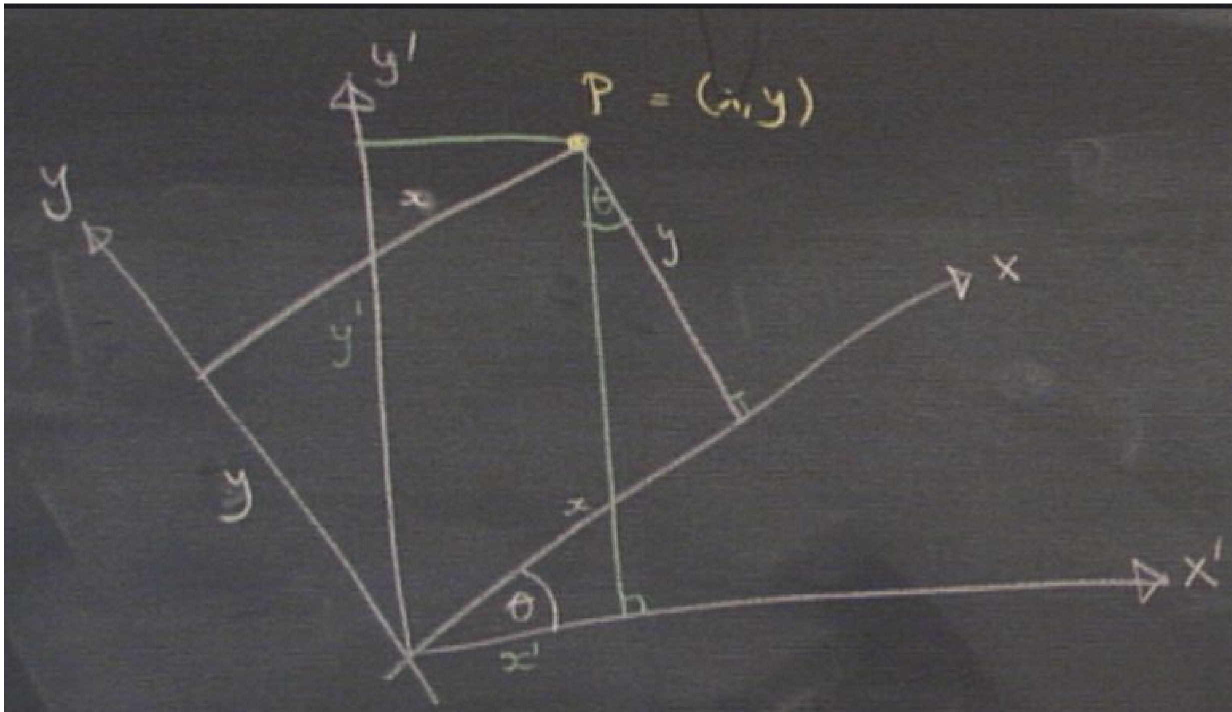
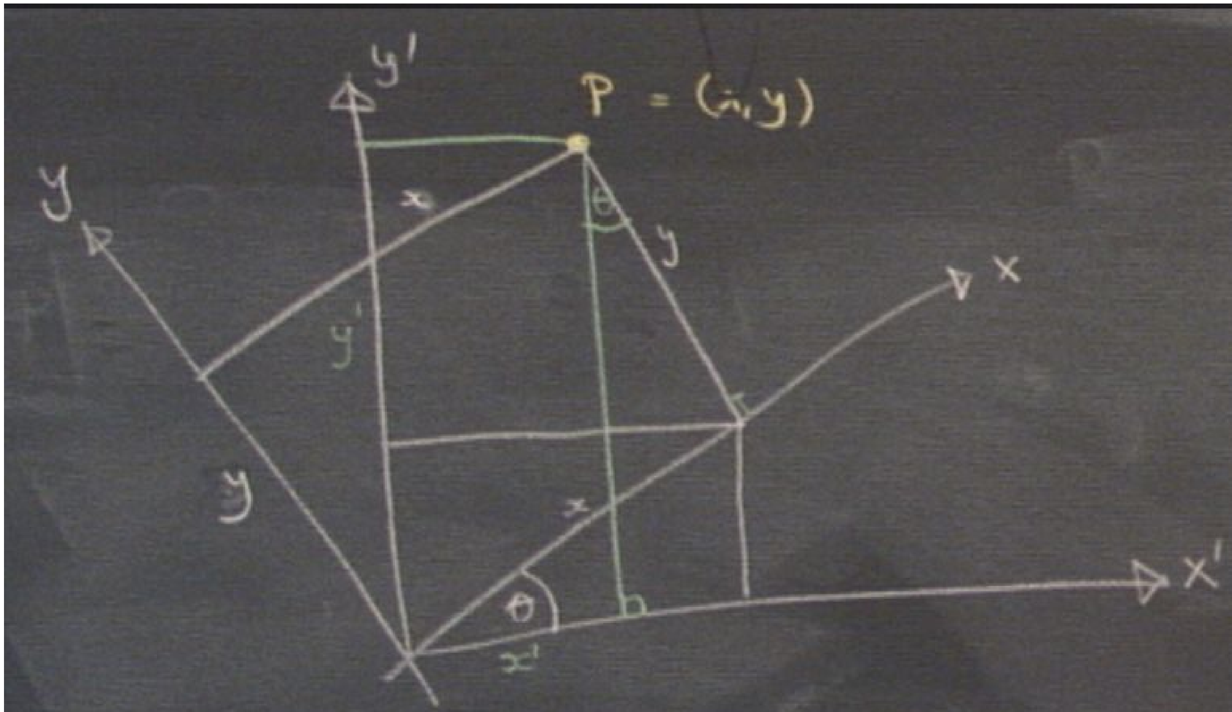


Rotates axes clockwise through θ.

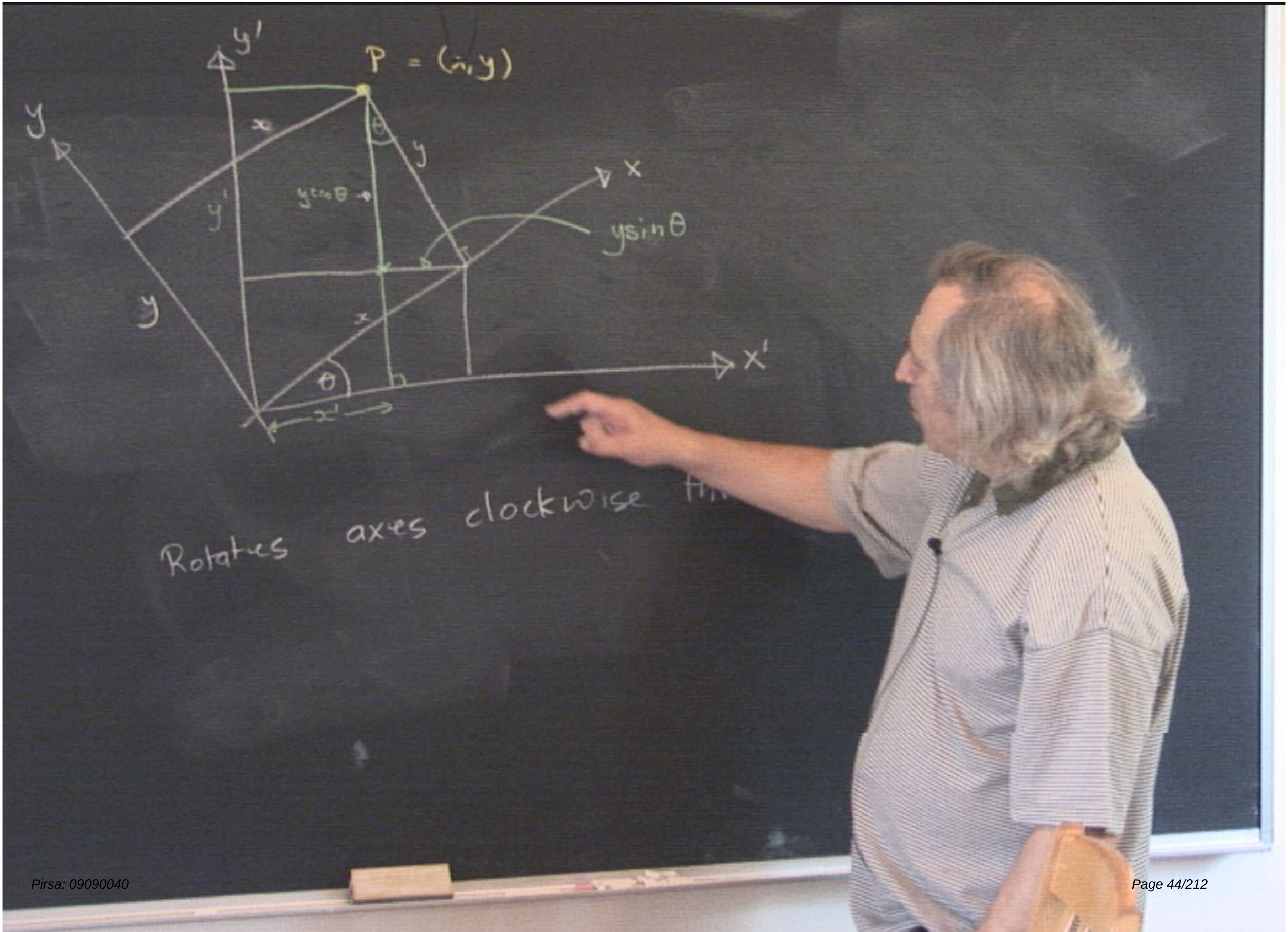

y'
P = (x, y)
y
x
y
x
y'
y
x
θ
x'
x'
Rotates axes clock

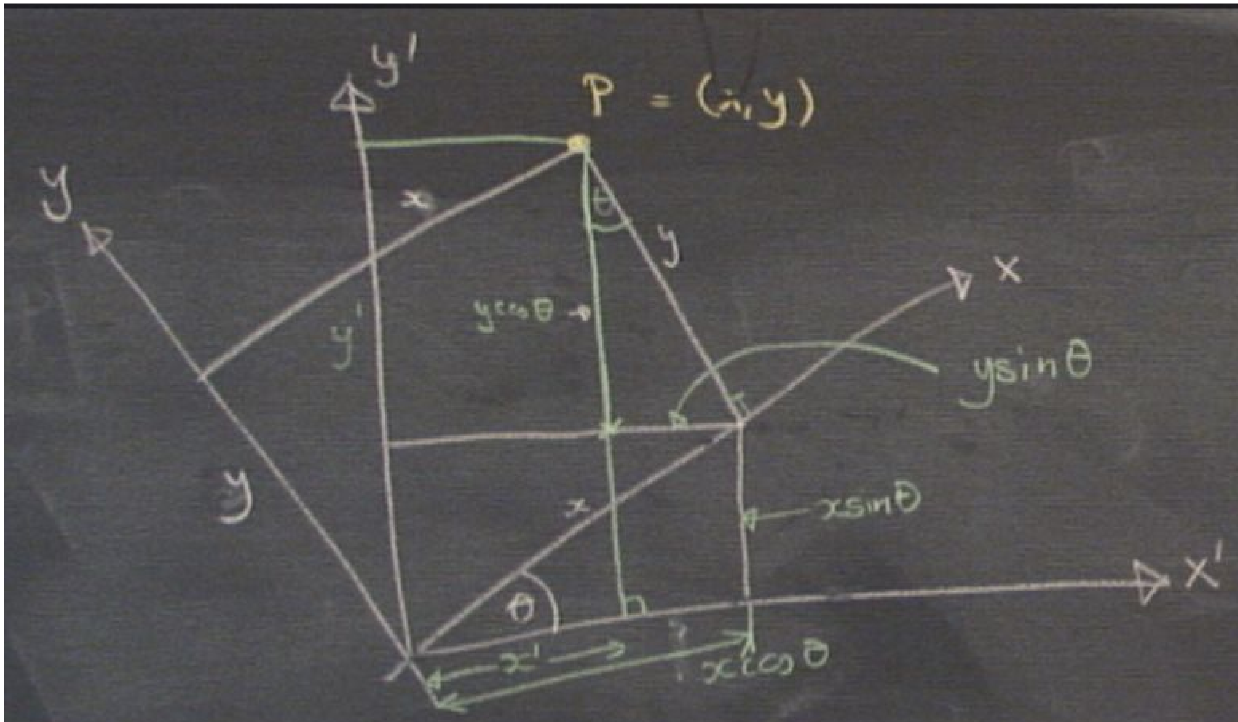


Rotates axes clockwise through θ.

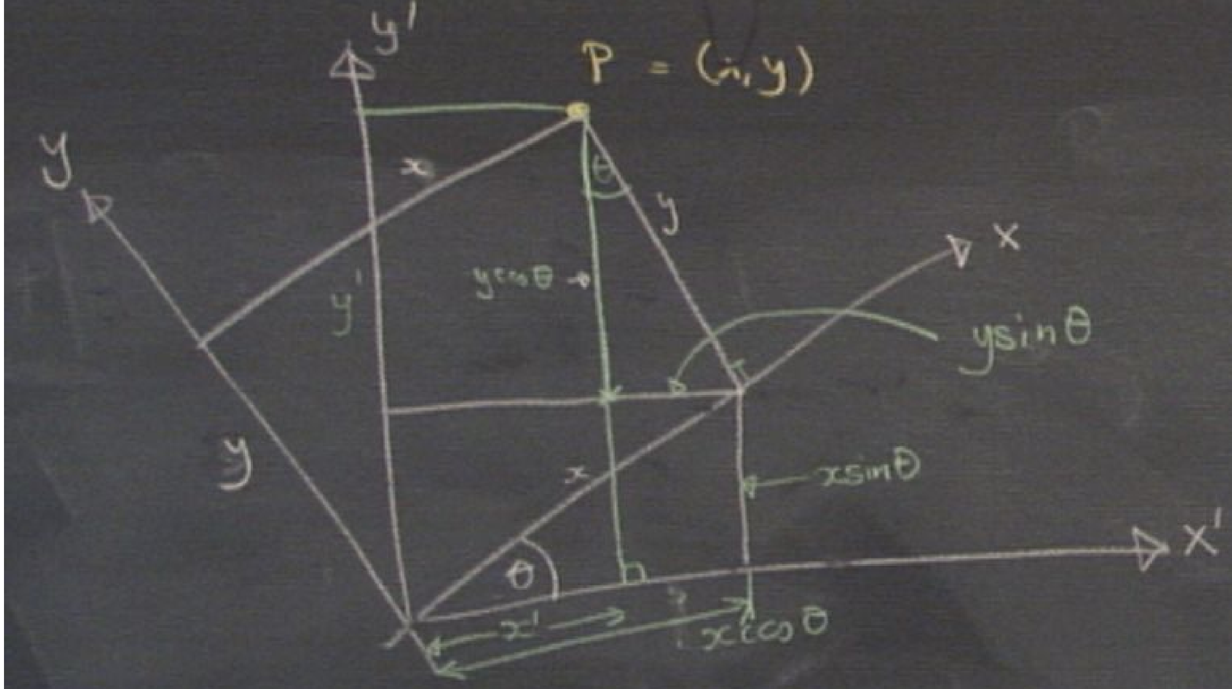


Rotates axes clockwise through θ.


y'
P = (x, y)
y
x
θ
y
X
y cos θ
y'
y sin θ
y
x
θ
x'
X'
Rotates axes clockwise th



Rotates axes clockwise through θ.

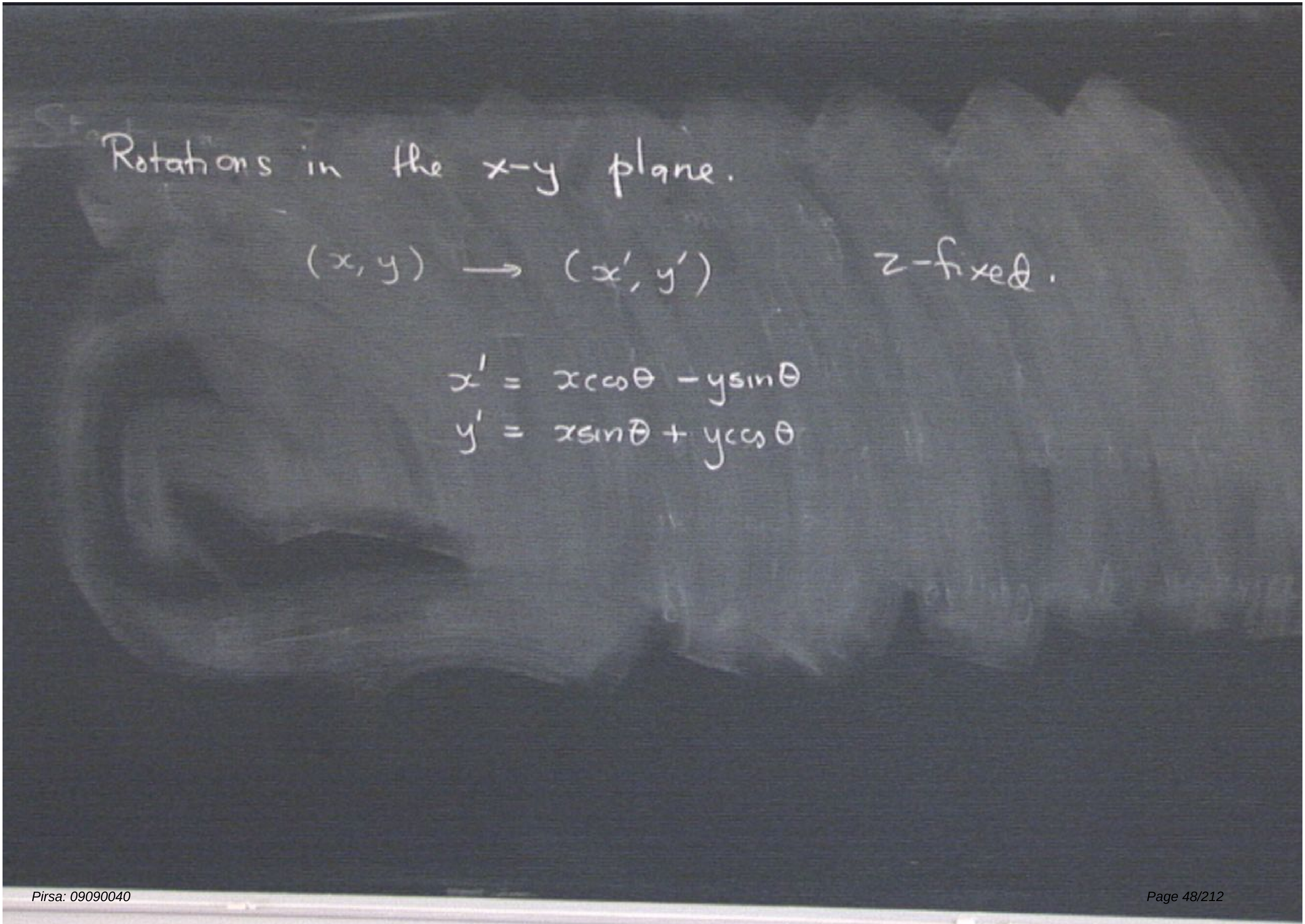


Rotates axes clockwise through θ.

Rotations in the x-y plane.

$(x, y) \longrightarrow (x', y')$ z-fixed.

$$x' = x\cos\theta - y\sin\theta$$
$$y' =$$



Rotations in the x-y plane.

$(x, y) \longrightarrow (x', y')$ z-fixed.

$$x' = x\cos\theta - y\sin\theta$$
$$y' = x\sin\theta + y\cos\theta$$

Rotations in the x-y plane.

$$(x, y) \longrightarrow (x', y') \qquad z\text{-fixed.}$$

$$x' = x\cos\theta - y\sin\theta$$
$$y' = x\sin\theta + y\cos\theta$$

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Rotations in the x-y plane.

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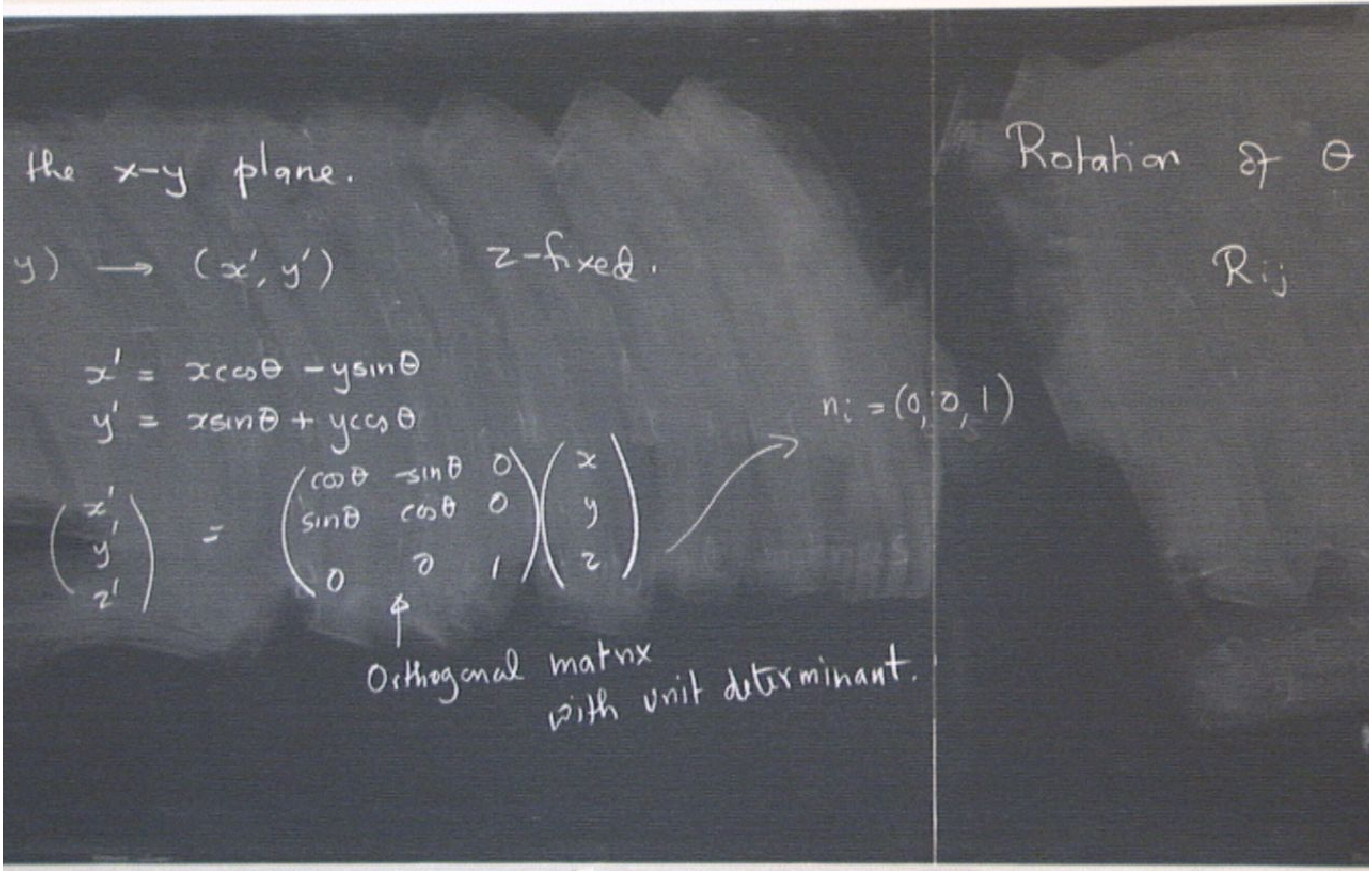
$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \underset{\substack{\uparrow \\ \text{Orthogonal matrix} \\ \text{with unit determinant.}}}{\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$



the x-y plane.

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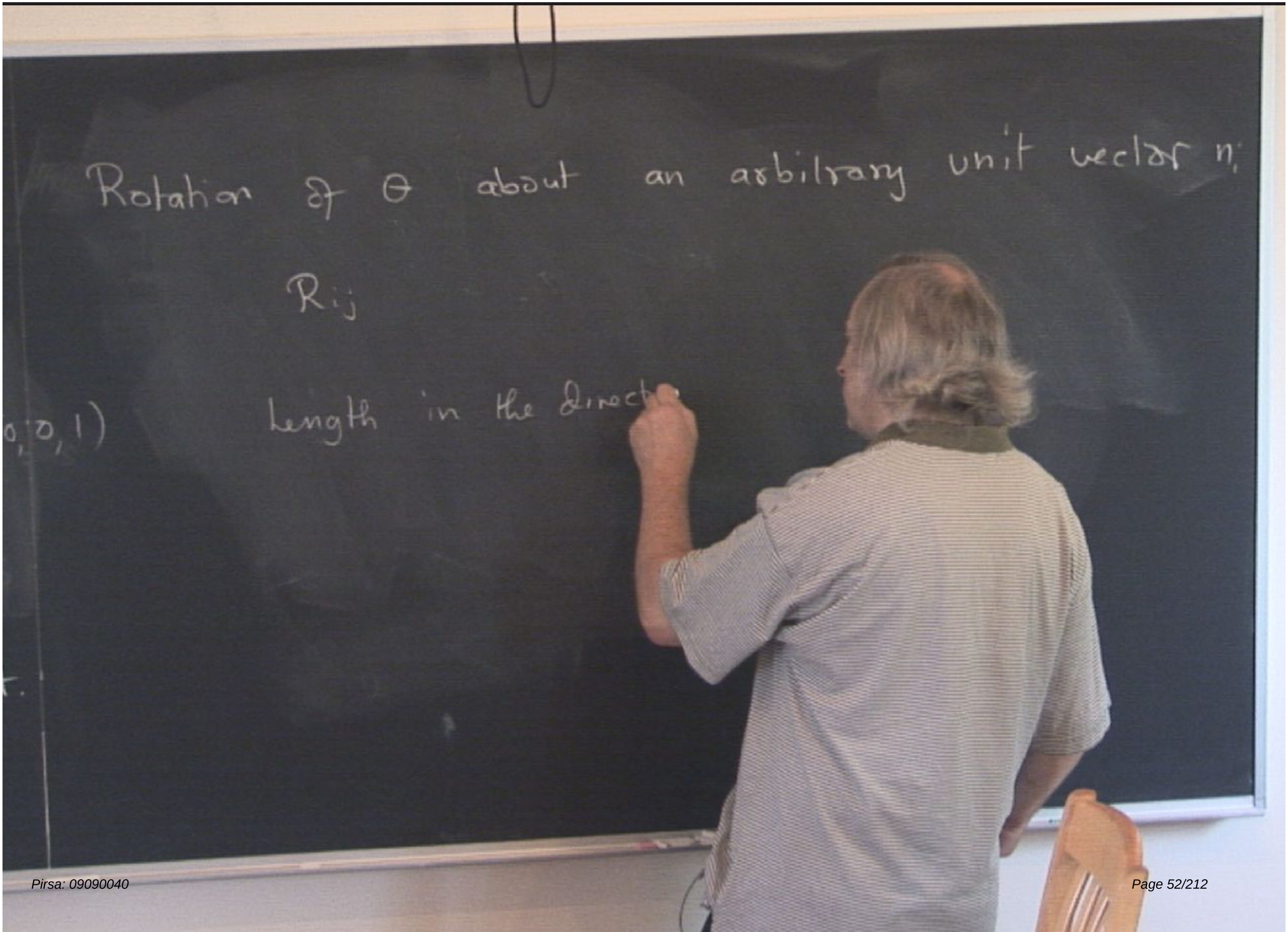
$n_i = (0, 0, 1)$

Orthogonal matrix with unit determinant.

Rotation of θ

R_{ij}



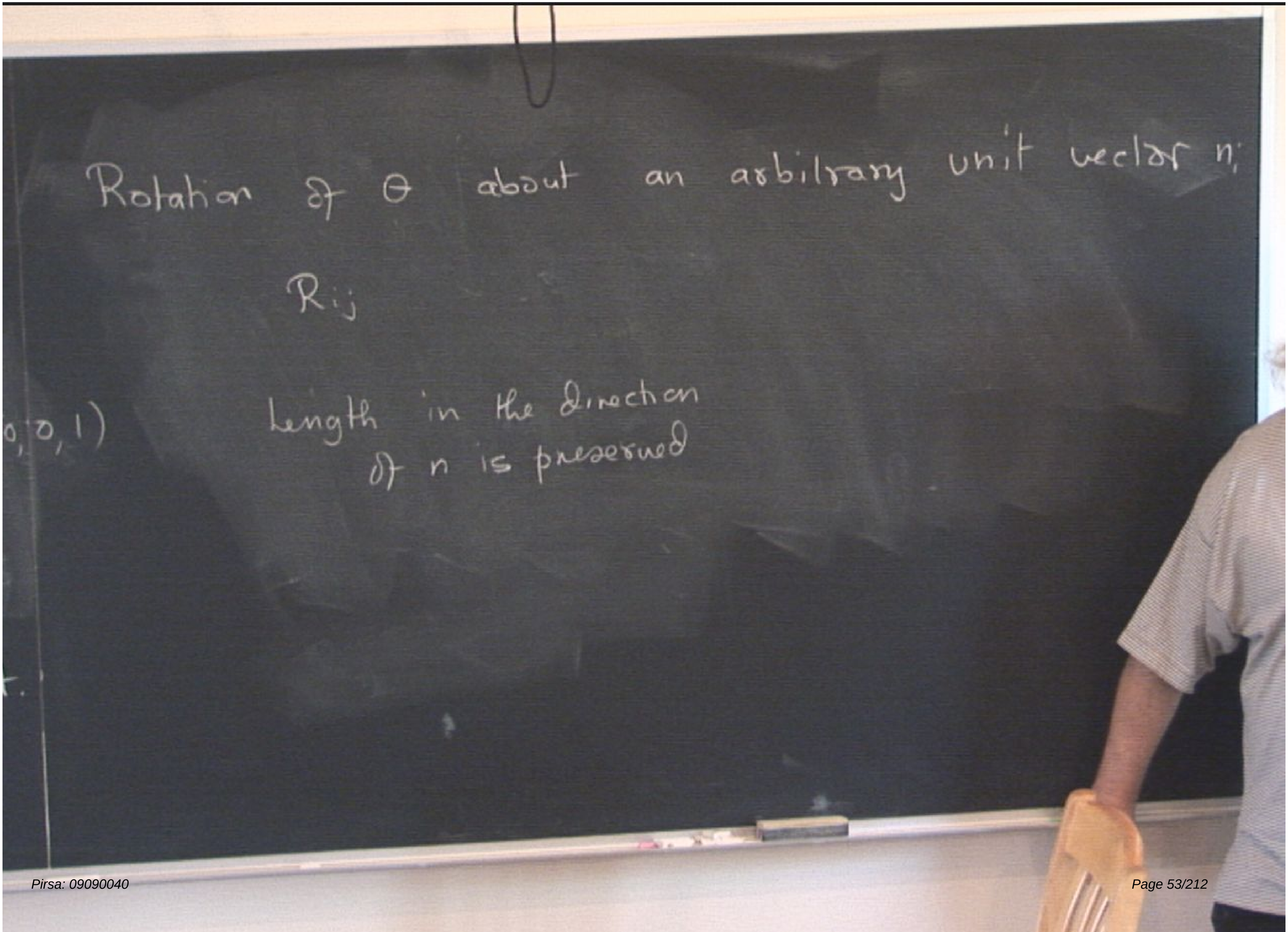
Rotation of θ about an arbitrary unit vector n_i

R_{ij}

0, 0, 1)

Length in the direct

r.



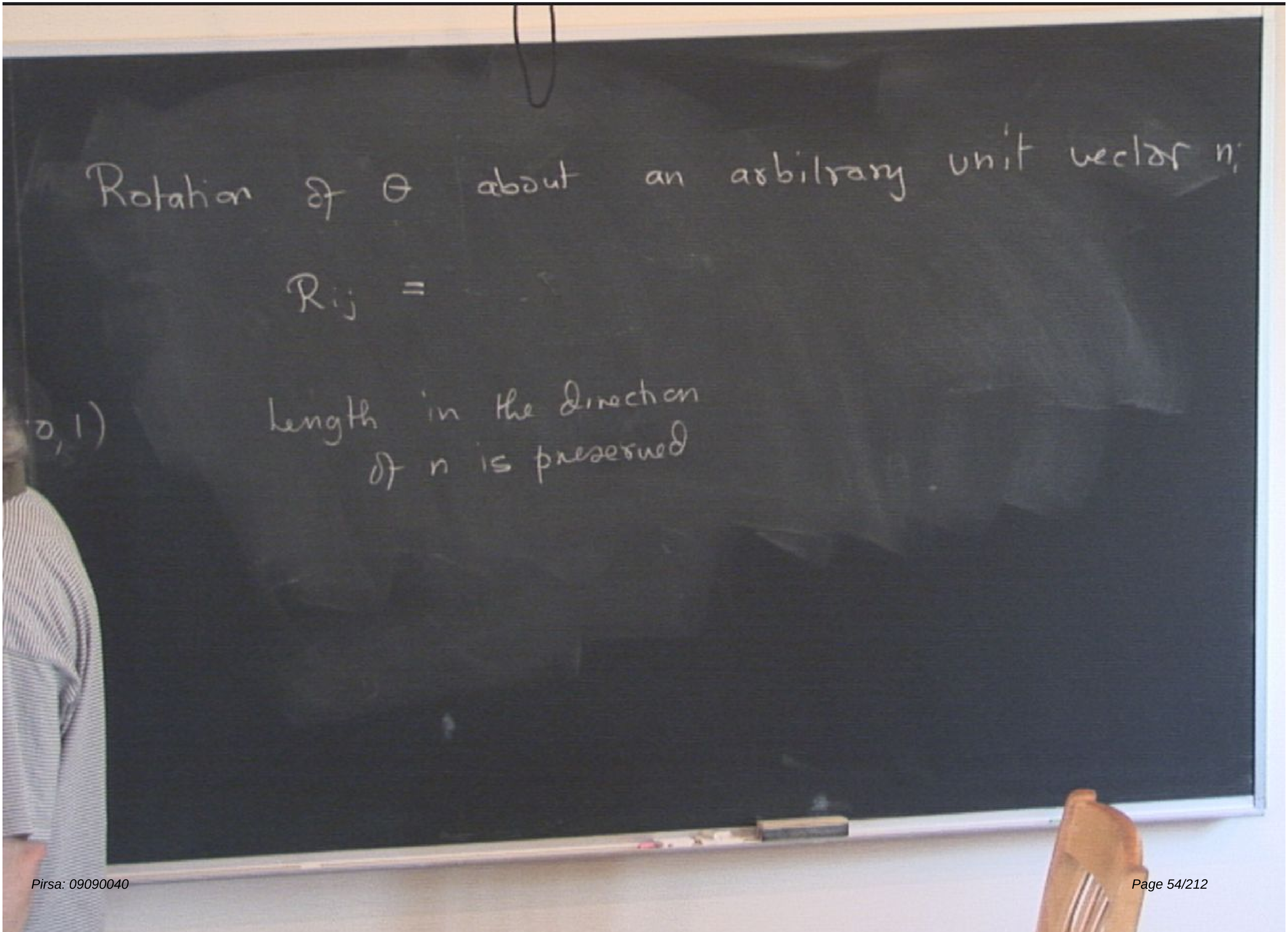
Rotation of θ about an arbitrary unit vector n_i

R_{ij}

$0, 0, 1)$

Length in the direction of n is preserved

t.

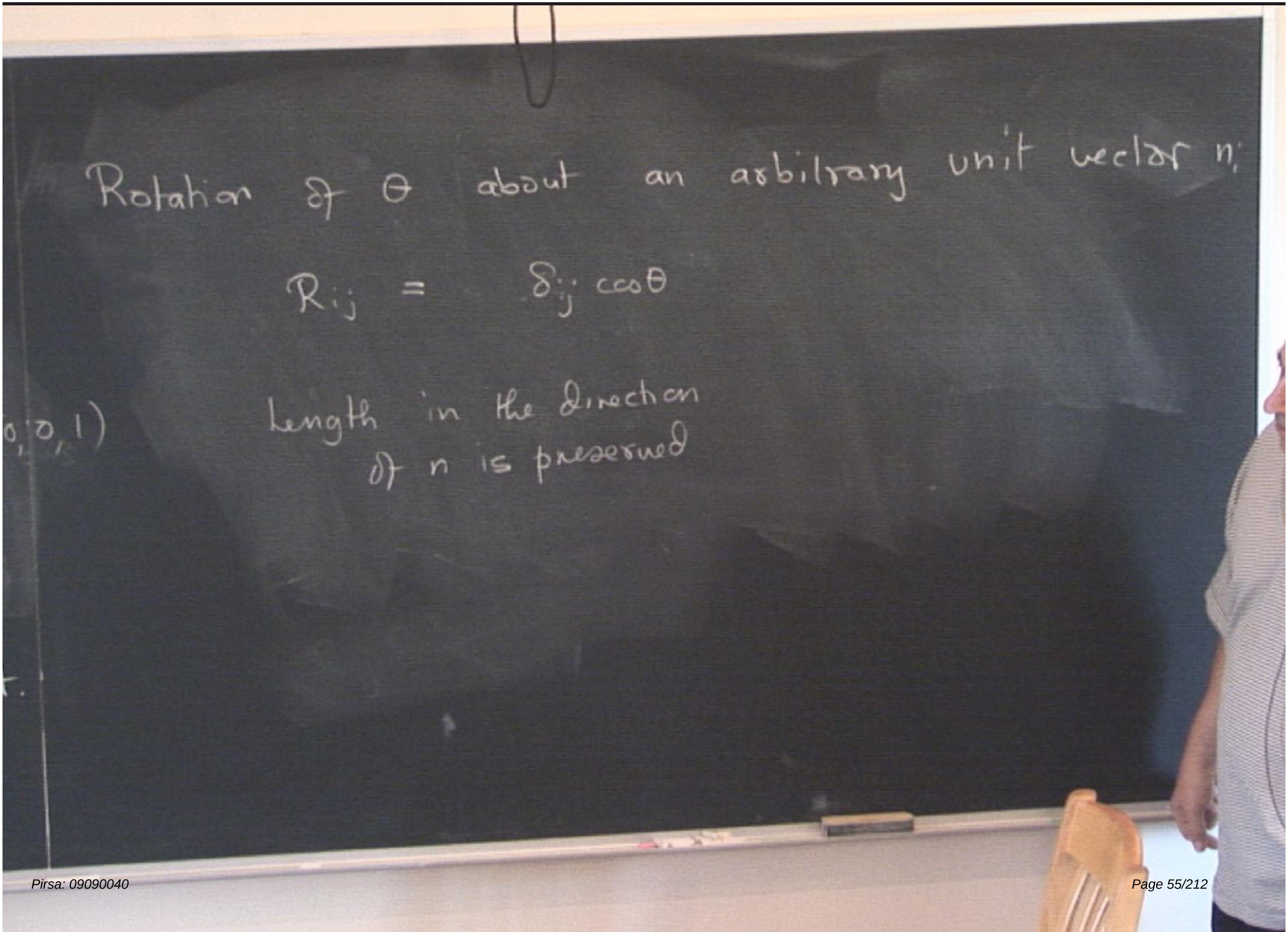


Rotation of θ about an arbitrary unit vector n_i

$R_{ij} =$

o, 1)

Length in the direction
of n is preserved



Rotation of θ about an arbitrary unit vector n_i:

$$R_{ij} = \delta_{ij} \cos\theta$$

Length in the direction of n is preserved

0, 0, 1)

t.

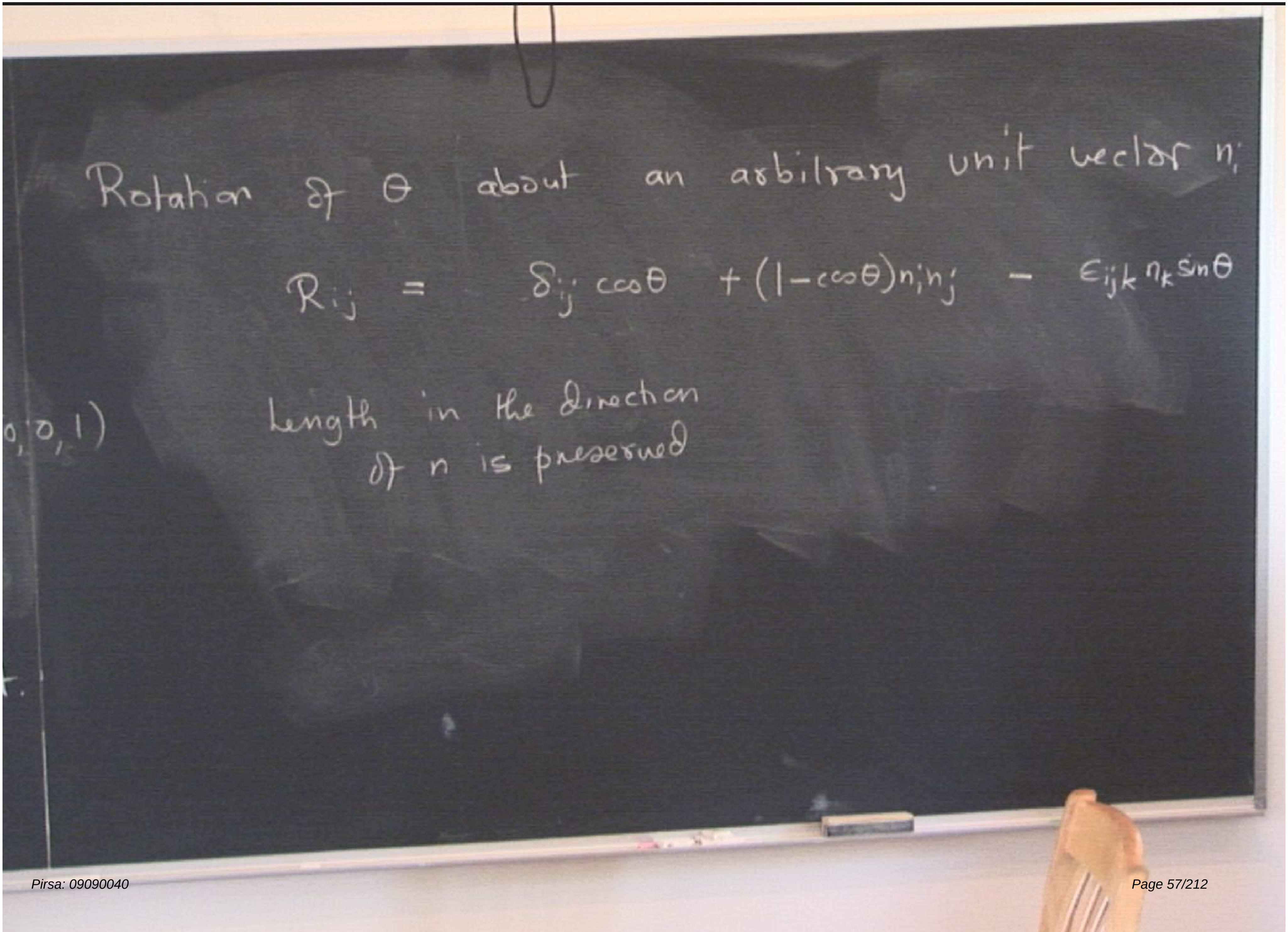
Rotation of θ about an arbitrary unit vector n_i

$$R_{ij} = \delta_{ij}\cos\theta + (1-\cos\theta)n_i n_j$$

0, 0, 1)

Length in the direction of n is preserved

r.



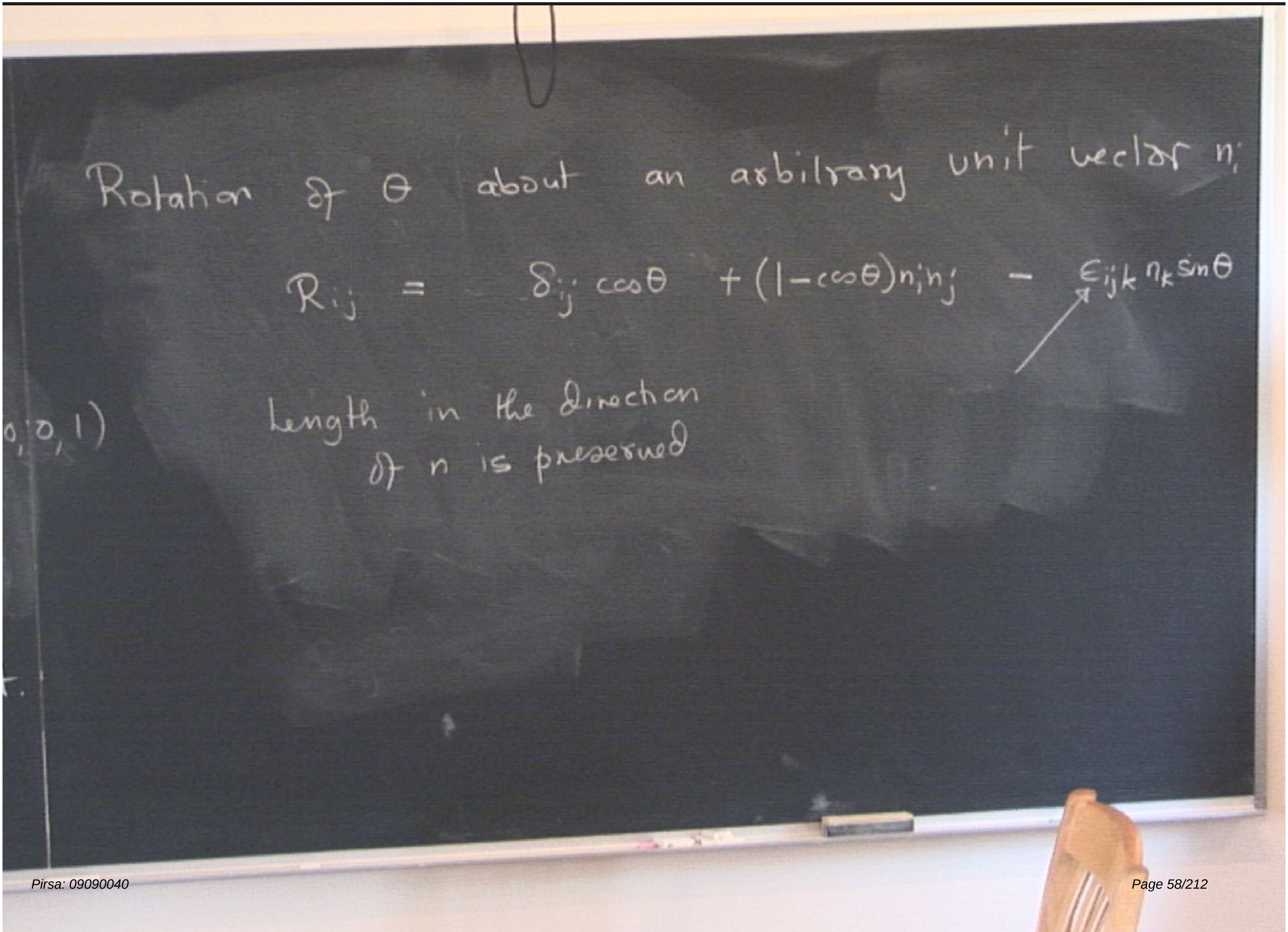
Rotation of θ about an arbitrary unit vector n_i

$$R_{ij} = \delta_{ij} \cos\theta + (1-\cos\theta) n_i n_j - \epsilon_{ijk} n_k \sin\theta$$

0, 0, 1)

Length in the direction of n is preserved

r.



Rotation of θ about an arbitrary unit vector n_i

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0, 0, 1)

t.

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0, 0, 1)

Length in the direction
of n is preserved

Alternating symbol

t.

Rotation of θ about an arbitrary unit vector n_i

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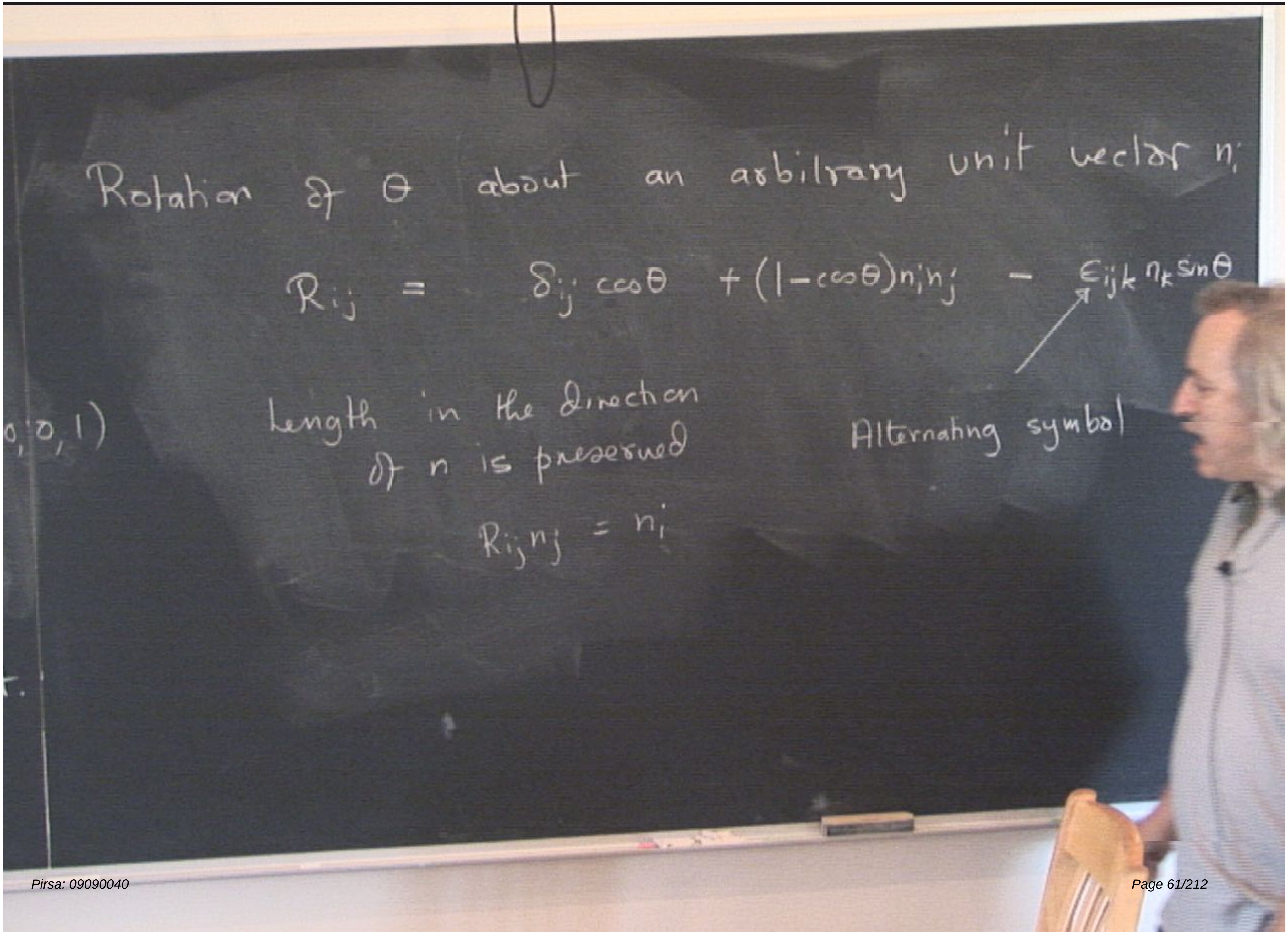
Alternating symbol

0, 0, 1)

Length in the direction of n is preserved

$$R_{ij} n_j = n_i$$

t.



Rotation of θ about an arbitrary unit vector n_i

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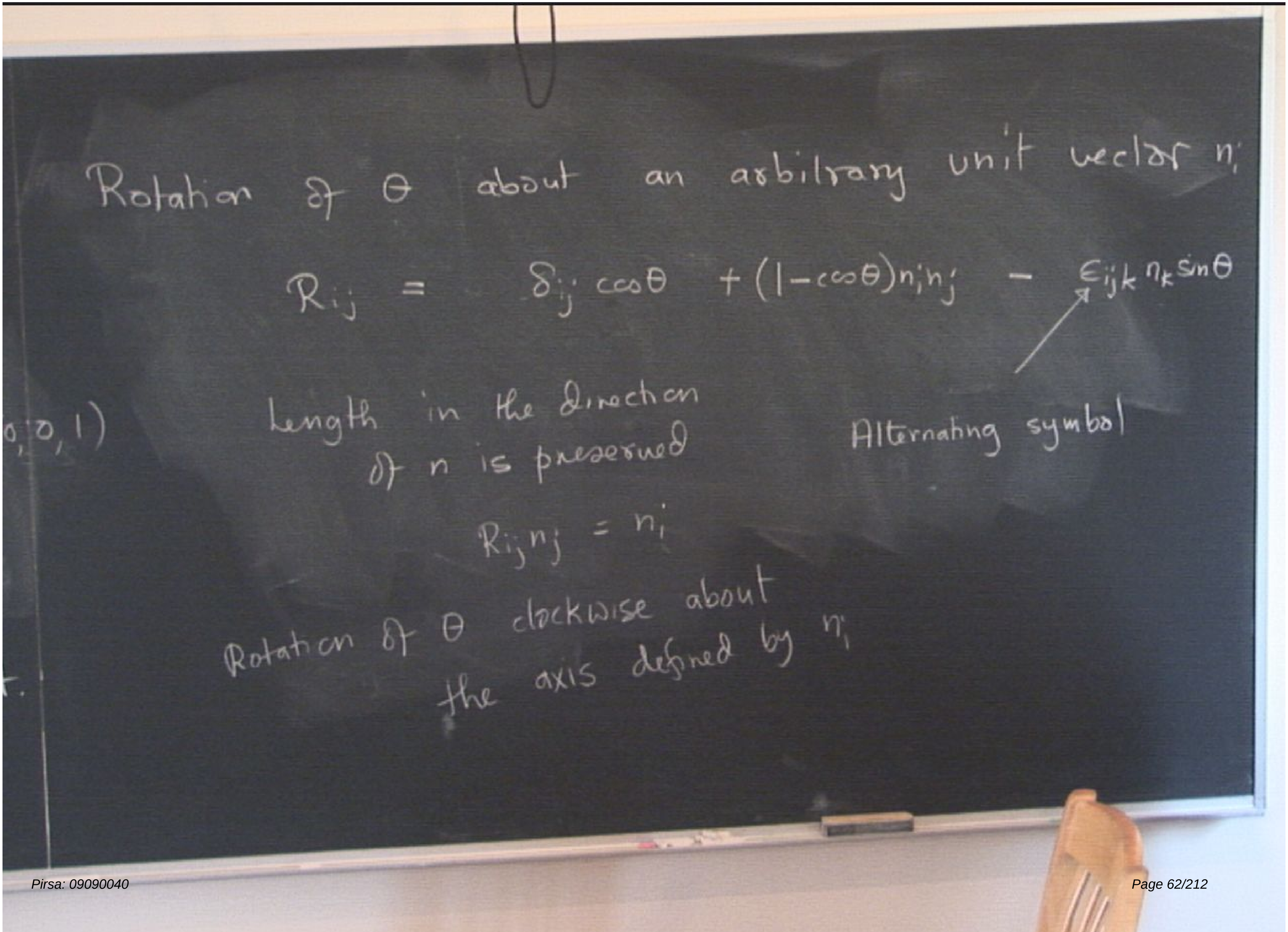
Alternating symbol

$0, 0, 1)$

Length in the direction of n is preserved

$$R_{ij} n_j = n_i$$

t.



Rotation of θ about an arbitrary unit vector n_i

$$R_{ij} = \delta_{ij} \cos\theta + (1-\cos\theta) n_i n_j - \epsilon_{ijk} n_k \sin\theta$$

0, 0, 1)

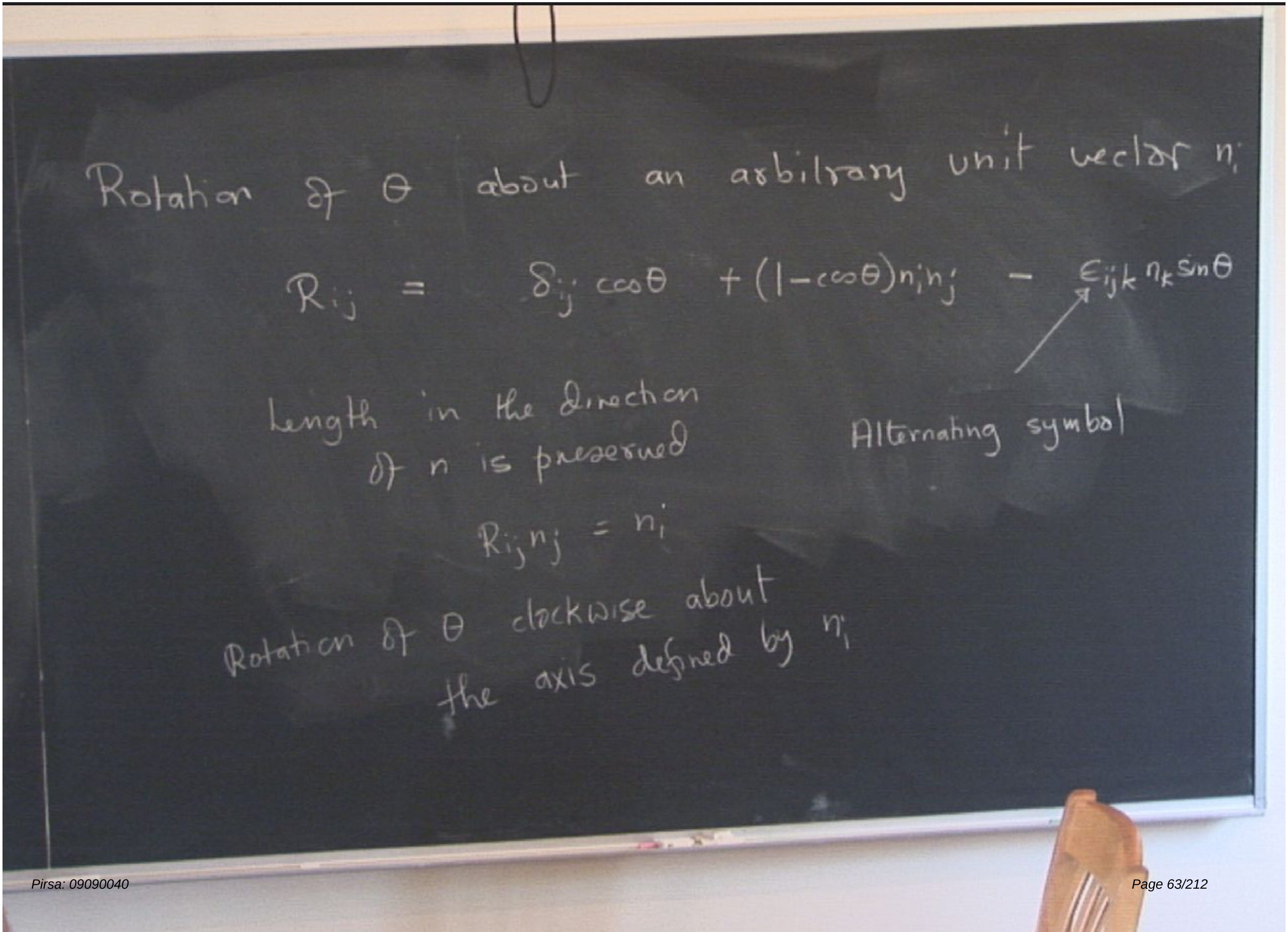
Length in the direction of n is preserved

Alternating symbol

$$R_{ij} n_j = n_i$$

Rotation of θ clockwise about the axis defined by n_i

r.



Rotation of θ about an arbitrary unit vector n_i

$$R_{ij} = \delta_{ij} \cos\theta + (1-\cos\theta) n_i n_j - \epsilon_{ijk} n_k \sin\theta$$

Alternating symbol

Length in the direction of n is preserved

$$R_{ij} n_j = n_i$$

Rotation of θ clockwise about the axis defined by n_i

Lie groups, have continuous parameters defining them,

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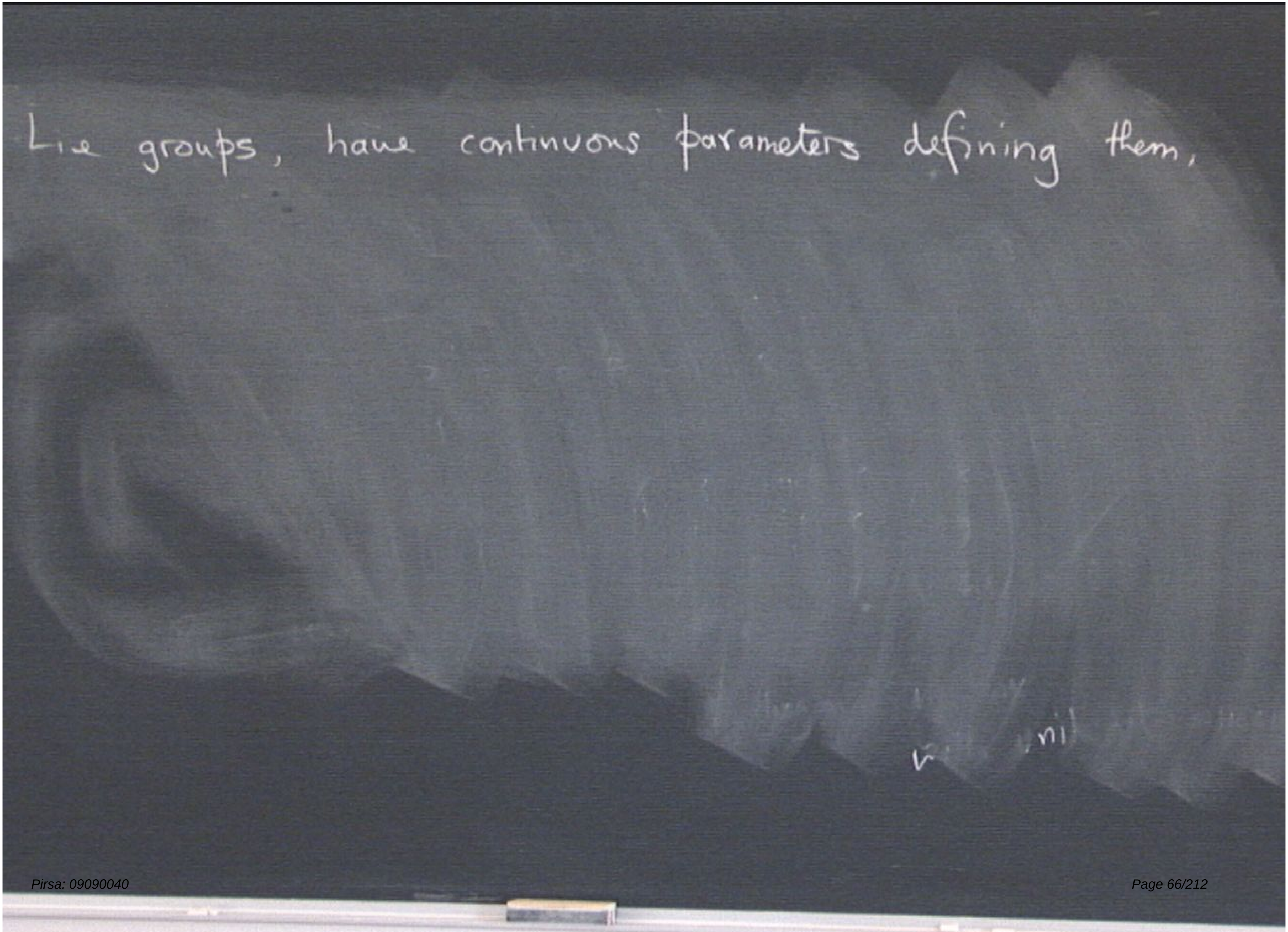

Lie groups, have continuous parameters defining them,

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$R =$

Lie groups, have continuous parameters defining them,

$$R = \exp(-i \lambda_i p_i)$$

parameters of the group

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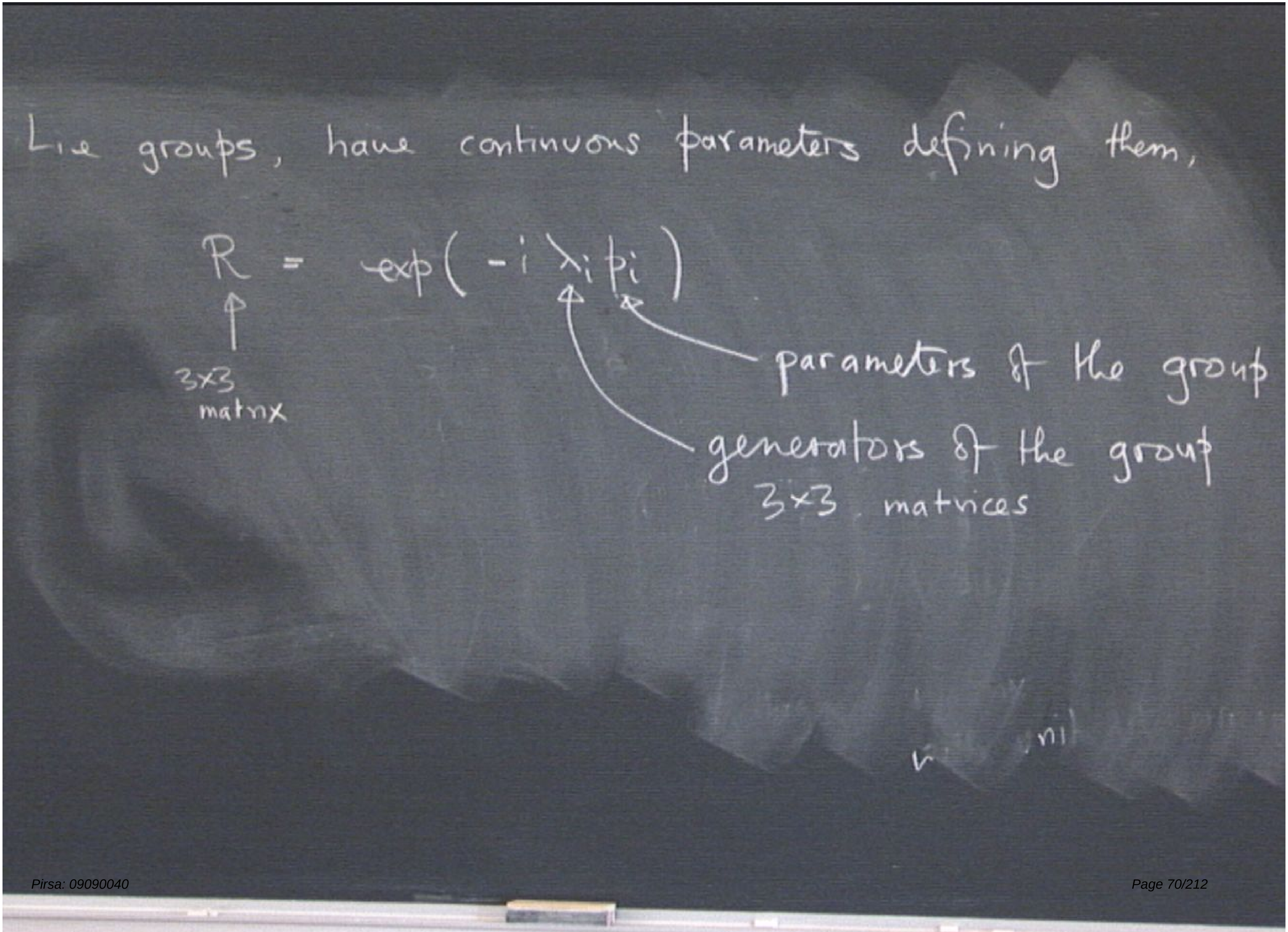
parameters of the group

generators of the group



Lie groups, have continuous parameters defining them,

$$\underset{\substack{\uparrow \\ 3\times3 \\ \text{matrix}}}{R} = \exp(-i\,\lambda_i\, p_i)$$

parameters of the group

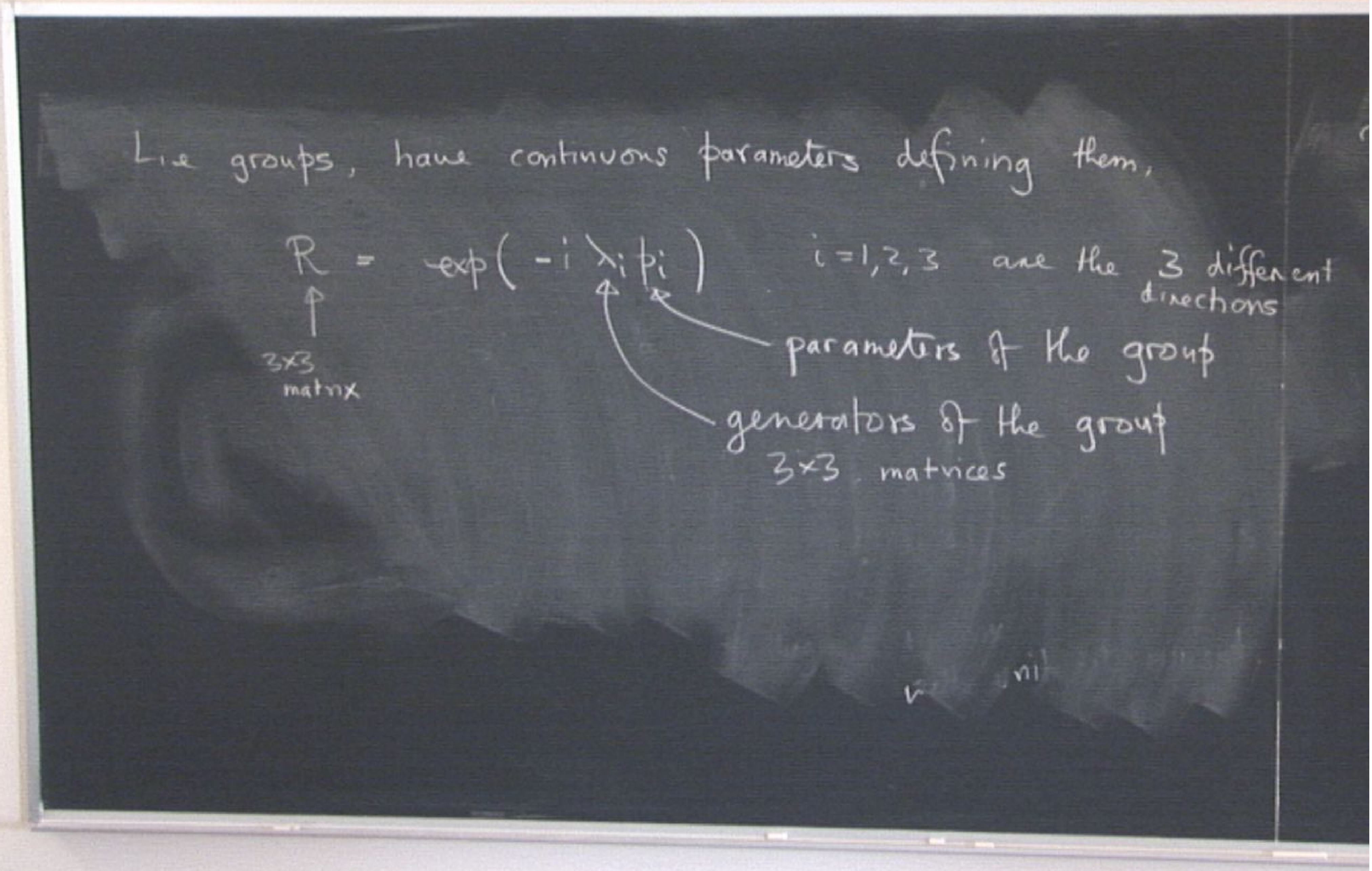
generators of the group
3×3 matrices

Lie groups, have continuous parameters defining them,

$$\underset{\substack{\uparrow \\ 3\times3 \\ \text{matrix}}}{R} = \exp(-i\,\lambda_i\,p_i) \qquad i=1,2,3$$

p_i ← parameters of the group

λ_i ← generators of the group
3×3 matrices



Lie groups, have continuous parameters defining them,

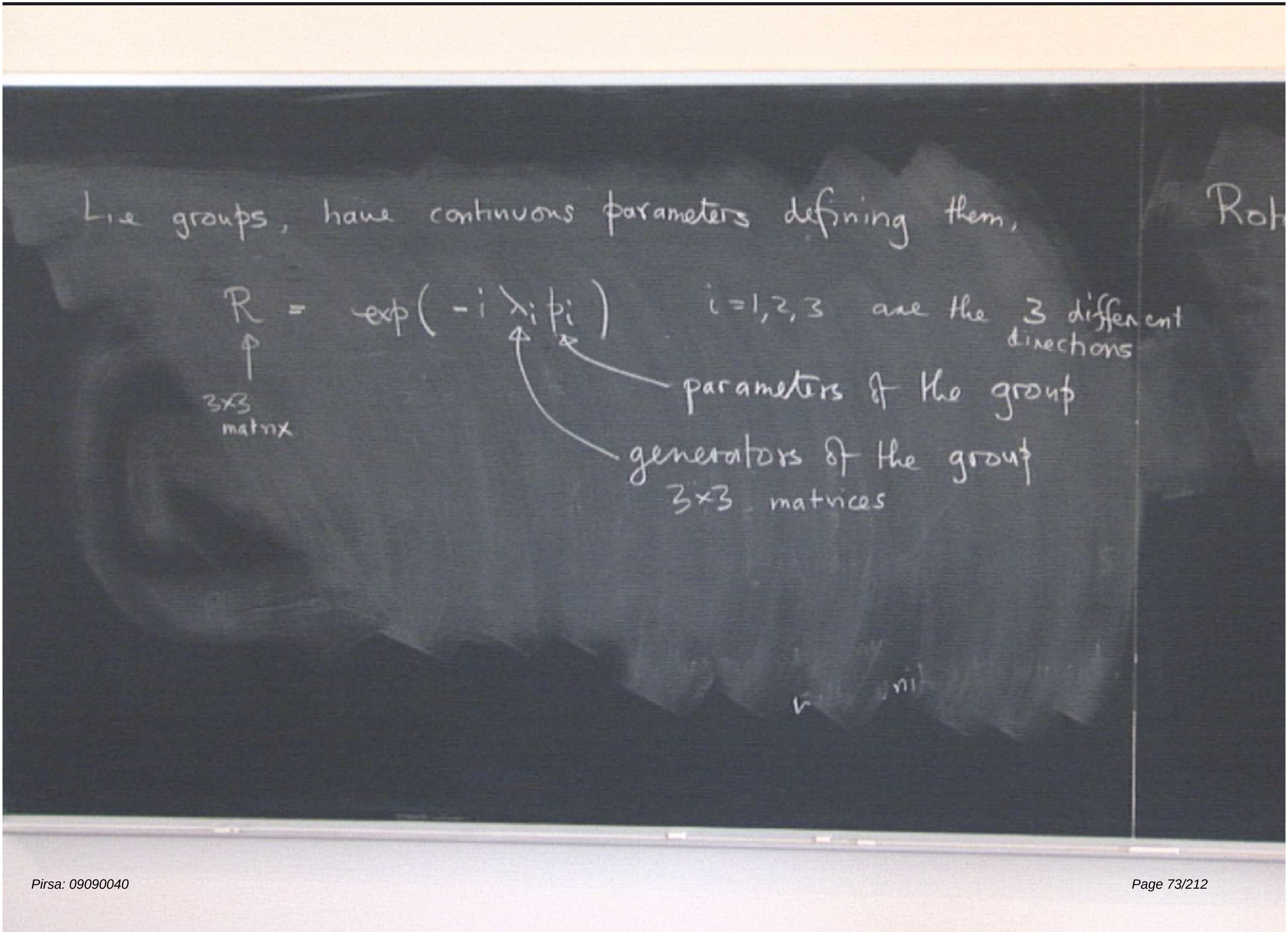
$$R = \exp(-i \lambda_i p_i)$$

R: 3×3 matrix

$i = 1,2,3$ are the 3 different directions

p_i: parameters of the group

λ_i: generators of the group, 3×3 matrices



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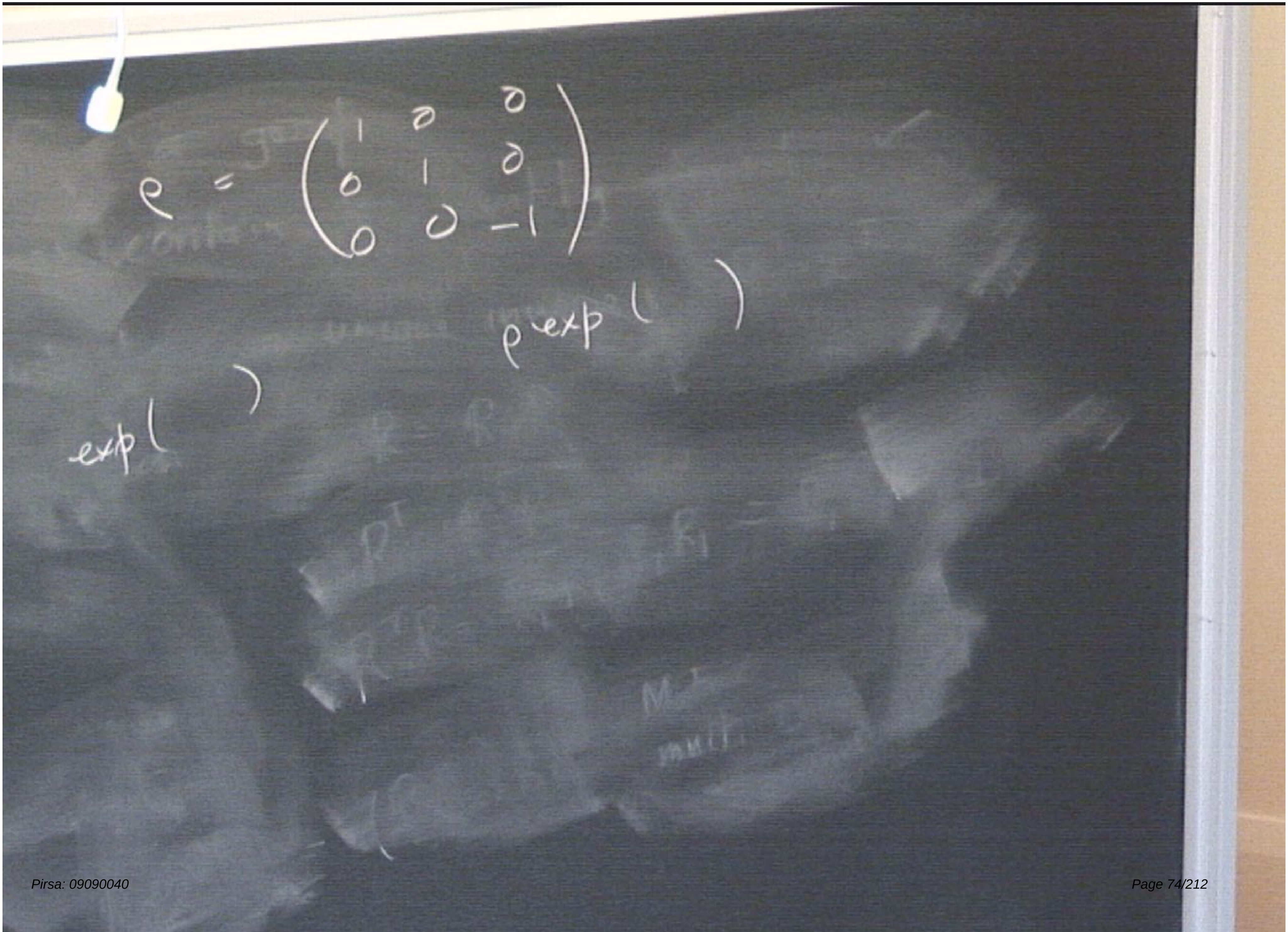
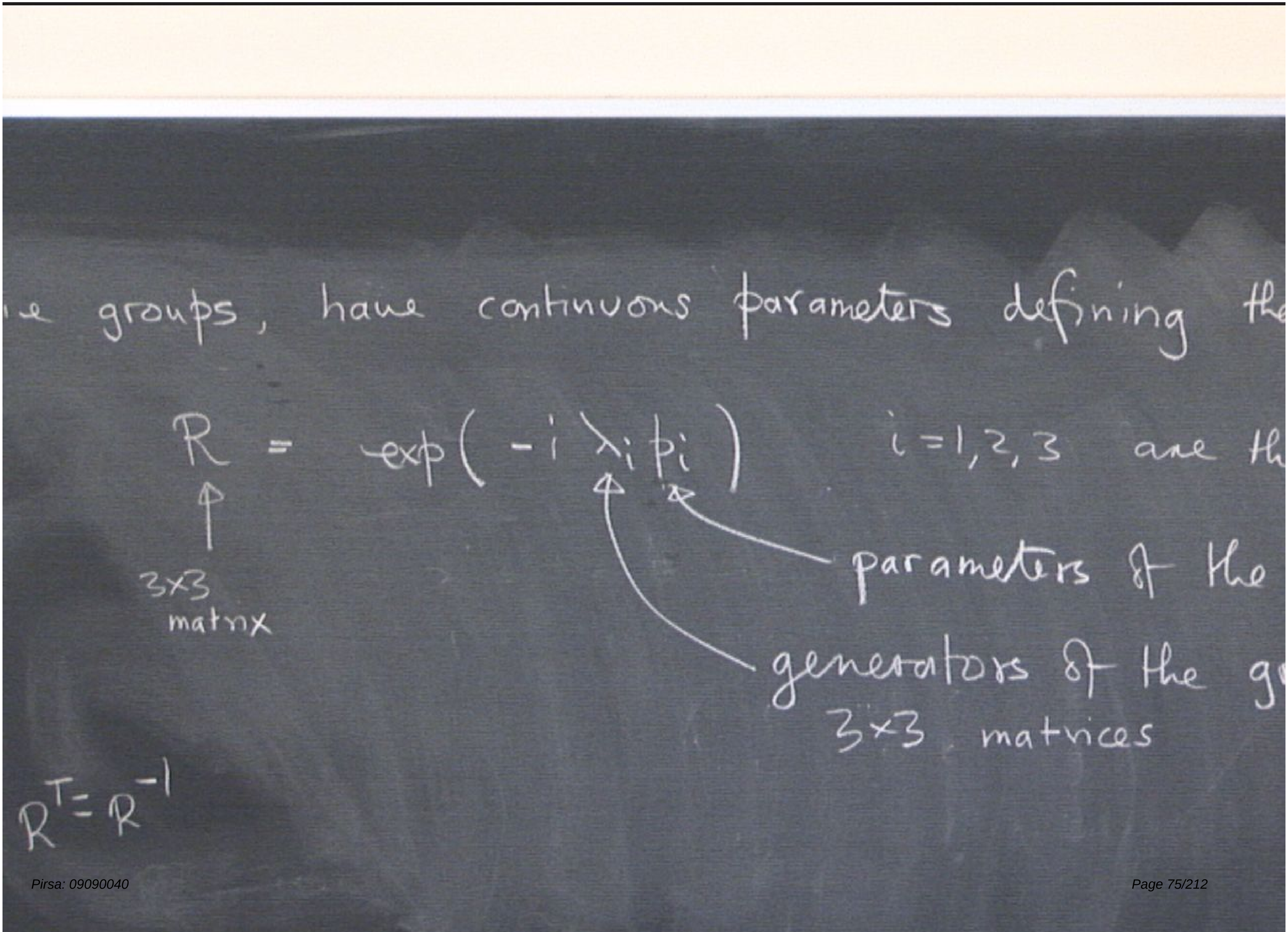


$$\rho = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\rho \exp(\quad)$$

$$\exp(\quad)$$



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$$R = \exp(-i \lambda_i p_i) \qquad i = 1, 2, 3 \quad \text{are th}$$

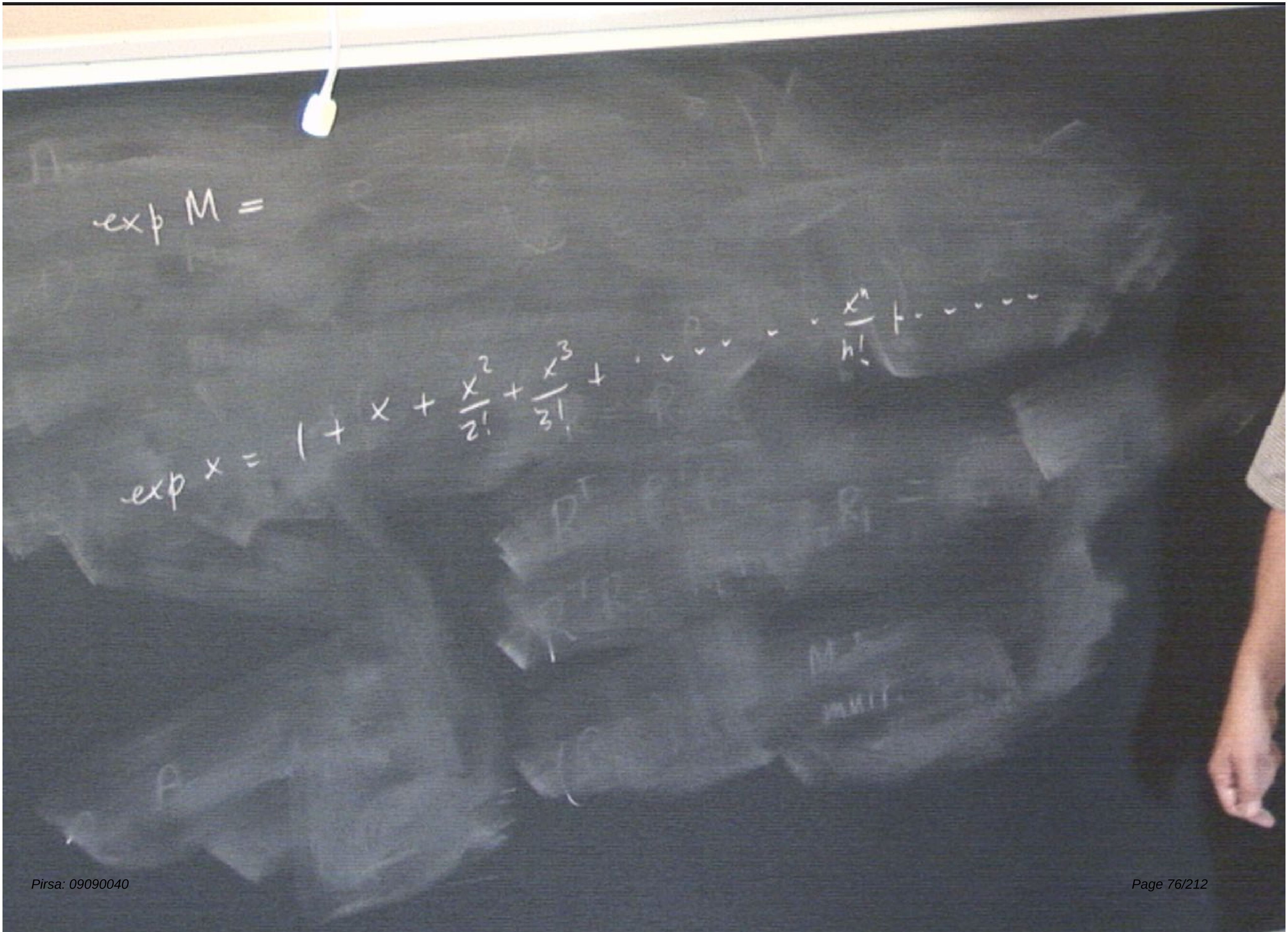
3x3 matrix

parameters of the

generators of the g

3×3 matrices

$R^T = R^{-1}$

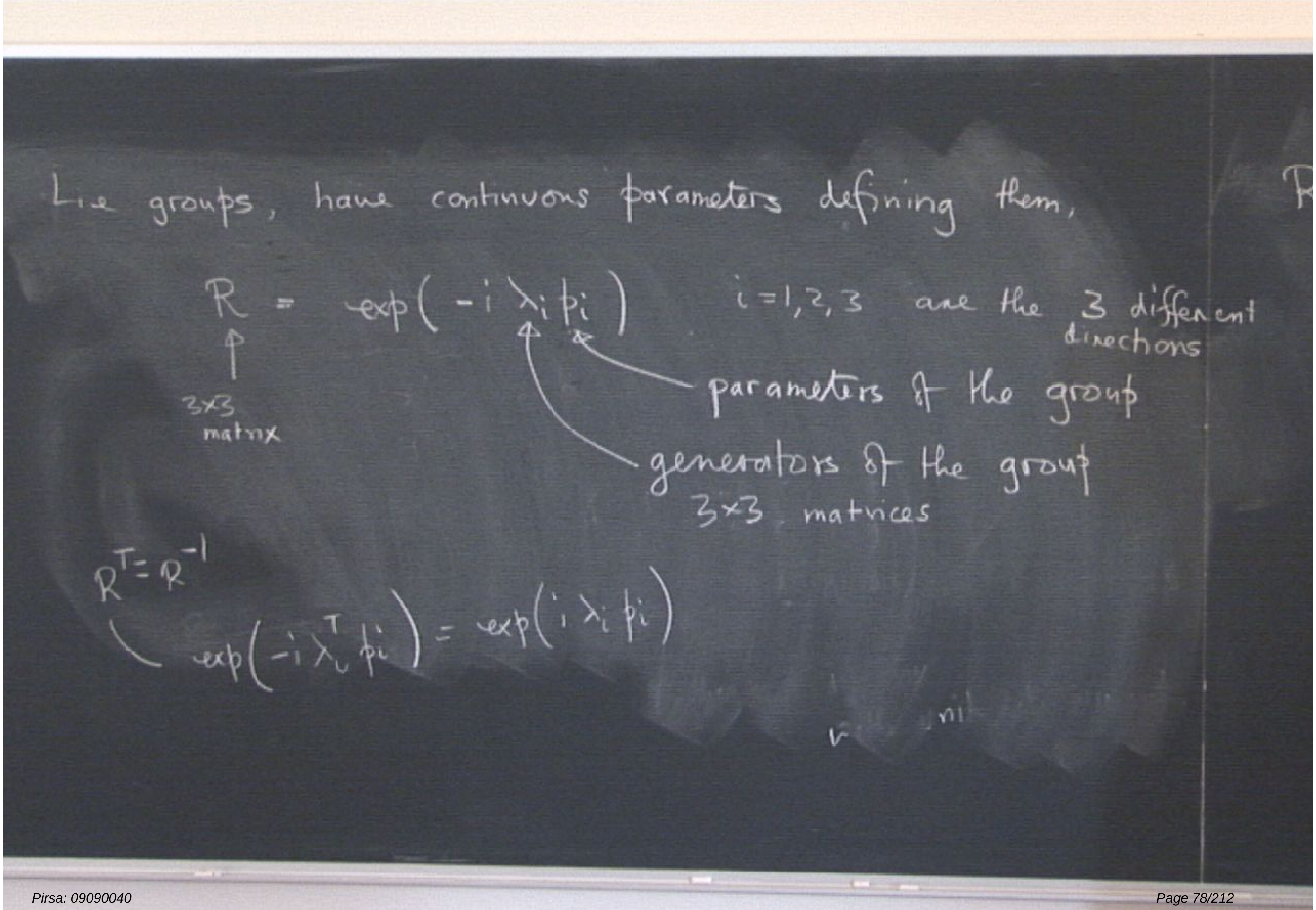


$\exp M =$

$\exp x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\cdots \frac{x^n}{n!} + \cdots\cdots$

$$\exp M = 1 + M + \frac{1}{2!}M^2 + \frac{1}{3!}M^3 + \cdots + \frac{1}{r!}M^r + \cdots$$

$$\exp x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \frac{x^n}{n!} + \cdots$$



Lie groups, have continuous parameters defining them,

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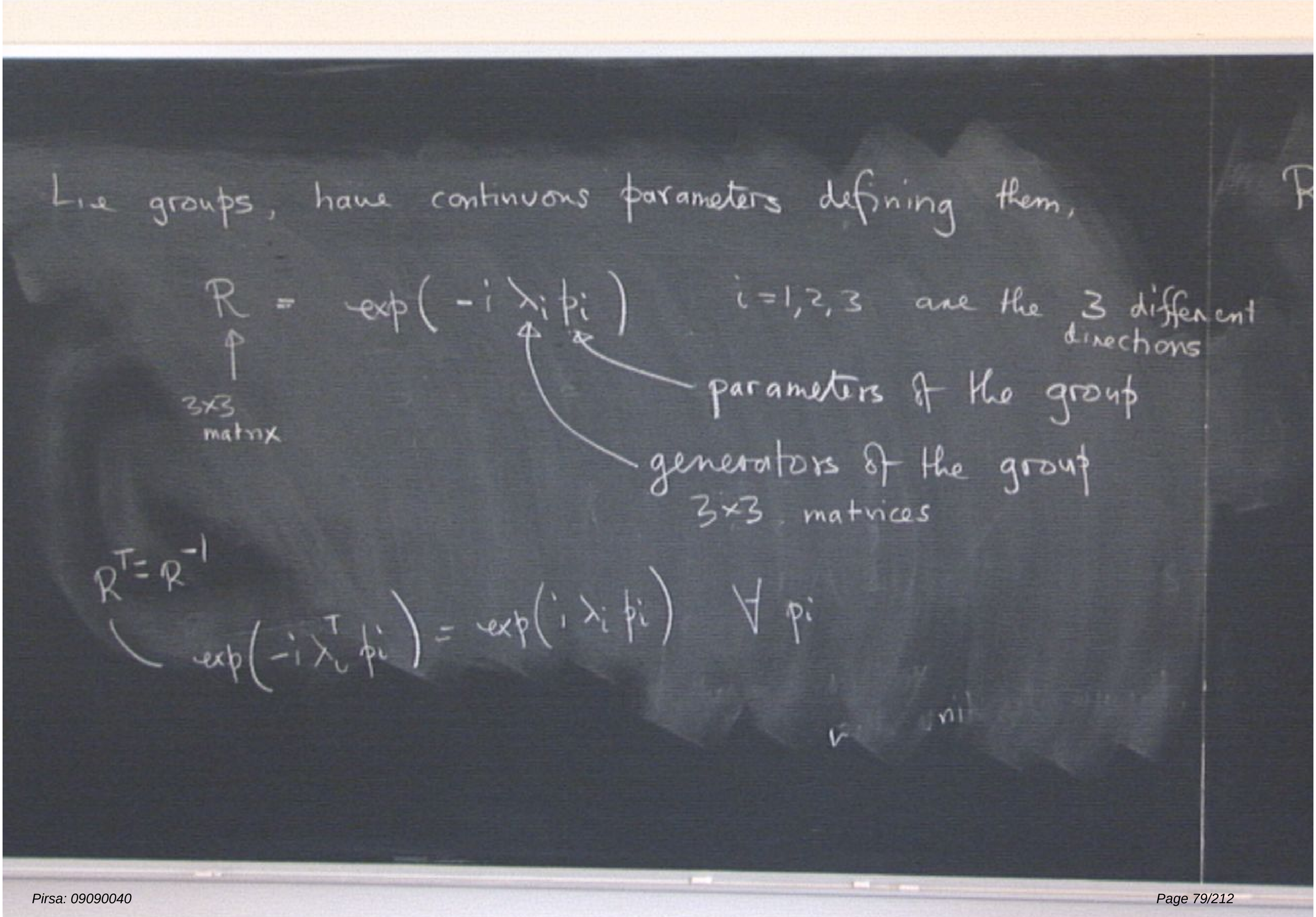
p_i: parameters of the group

λ_i: generators of the group, 3×3 matrices

$$R^T = R^{-1}$$

$$\exp(-i\,\lambda_i^T\, p_i) = \exp(i\,\lambda_i\, p_i)$$



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$R \uparrow$ 3×3 matrix

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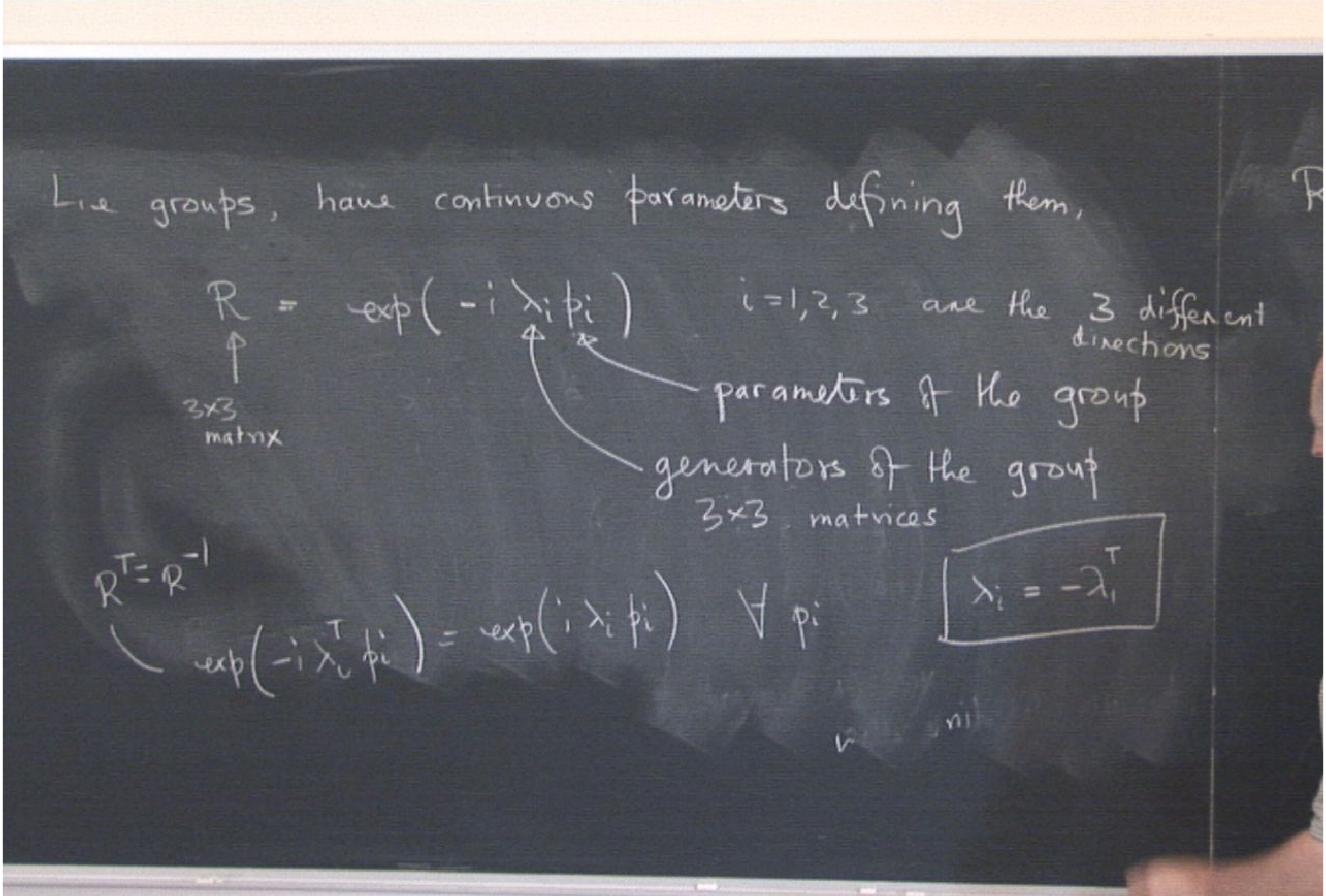

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3×3 matrix

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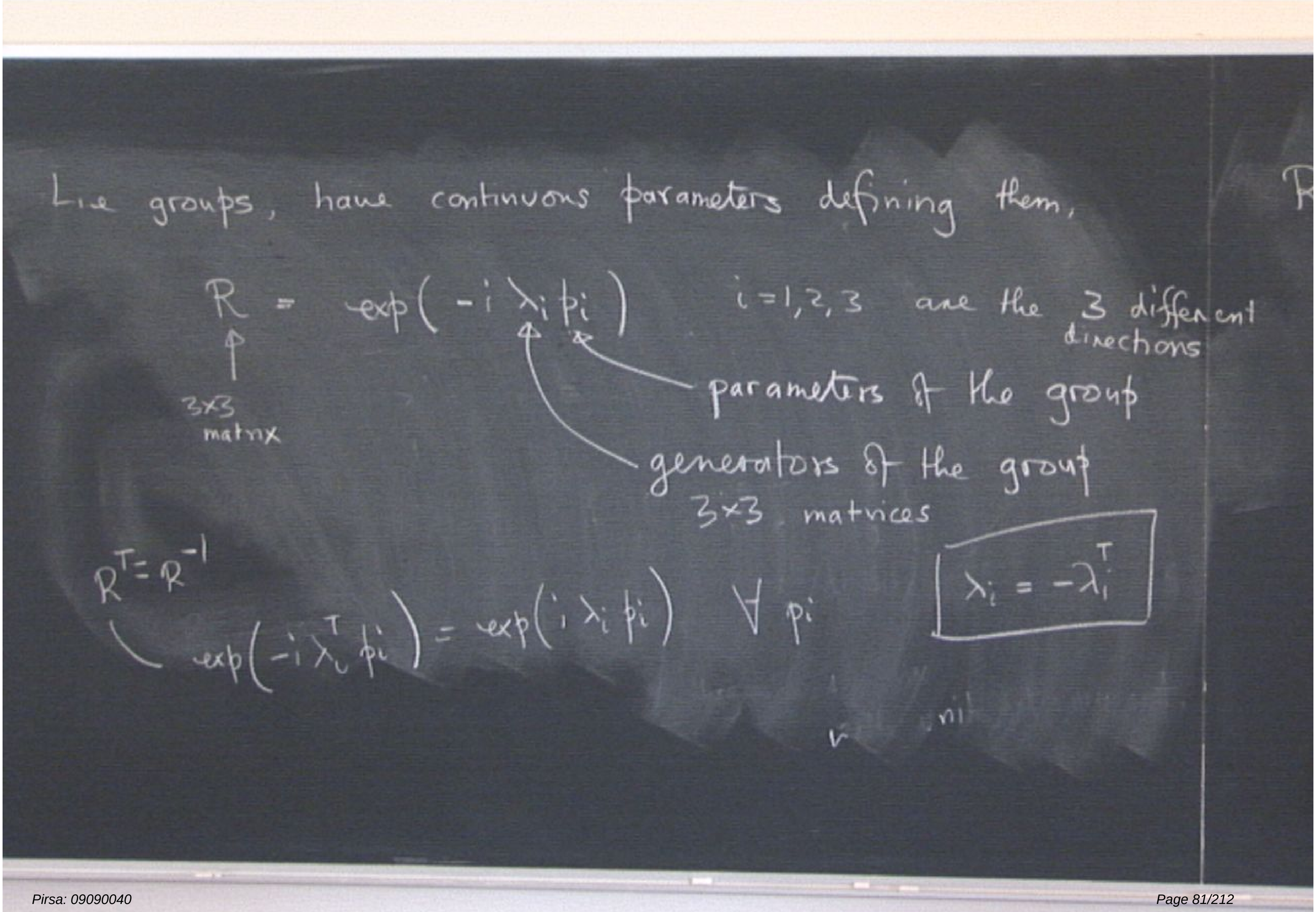
$$\exp(-i\lambda_i^T p_i) = \exp(i\lambda_i p_i) \quad \forall\, p_i$$

$$\boxed{\lambda_i = -\lambda_i^T}$$



Lie groups, have continuous parameters defining them,

$$R = \exp(-i \lambda_i \phi_i)$$

$i = 1,2,3$ are the 3 different directions

$\uparrow$ 3×3 matrix

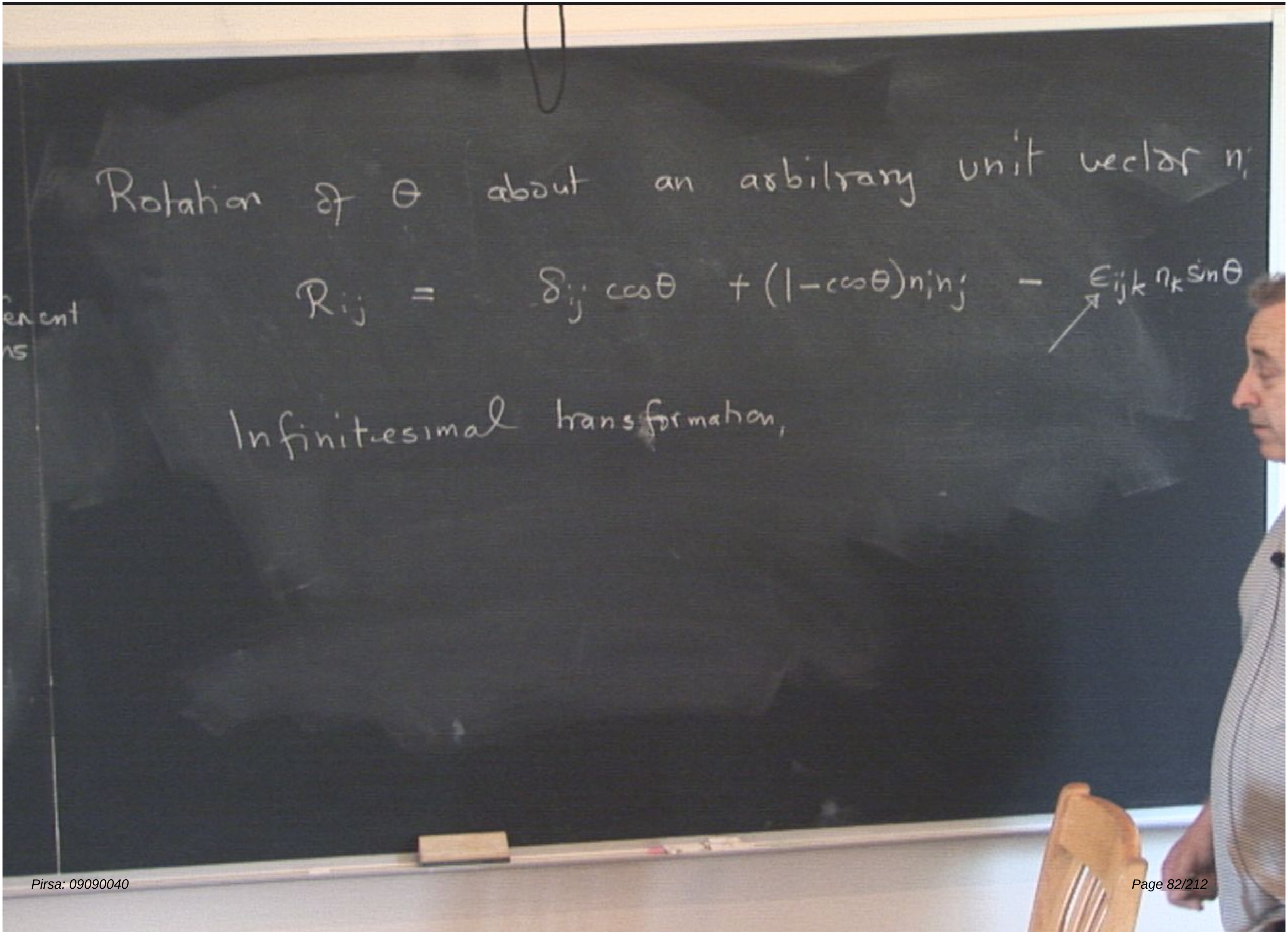
parameters of the group

generators of the group 3×3 matrices

$$R^T = R^{-1}$$

$$\exp(-i \lambda_i^T \phi_i) = \exp(i \lambda_i \phi_i) \quad \forall \phi_i$$

$$\boxed{\lambda_i = -\lambda_i^T}$$

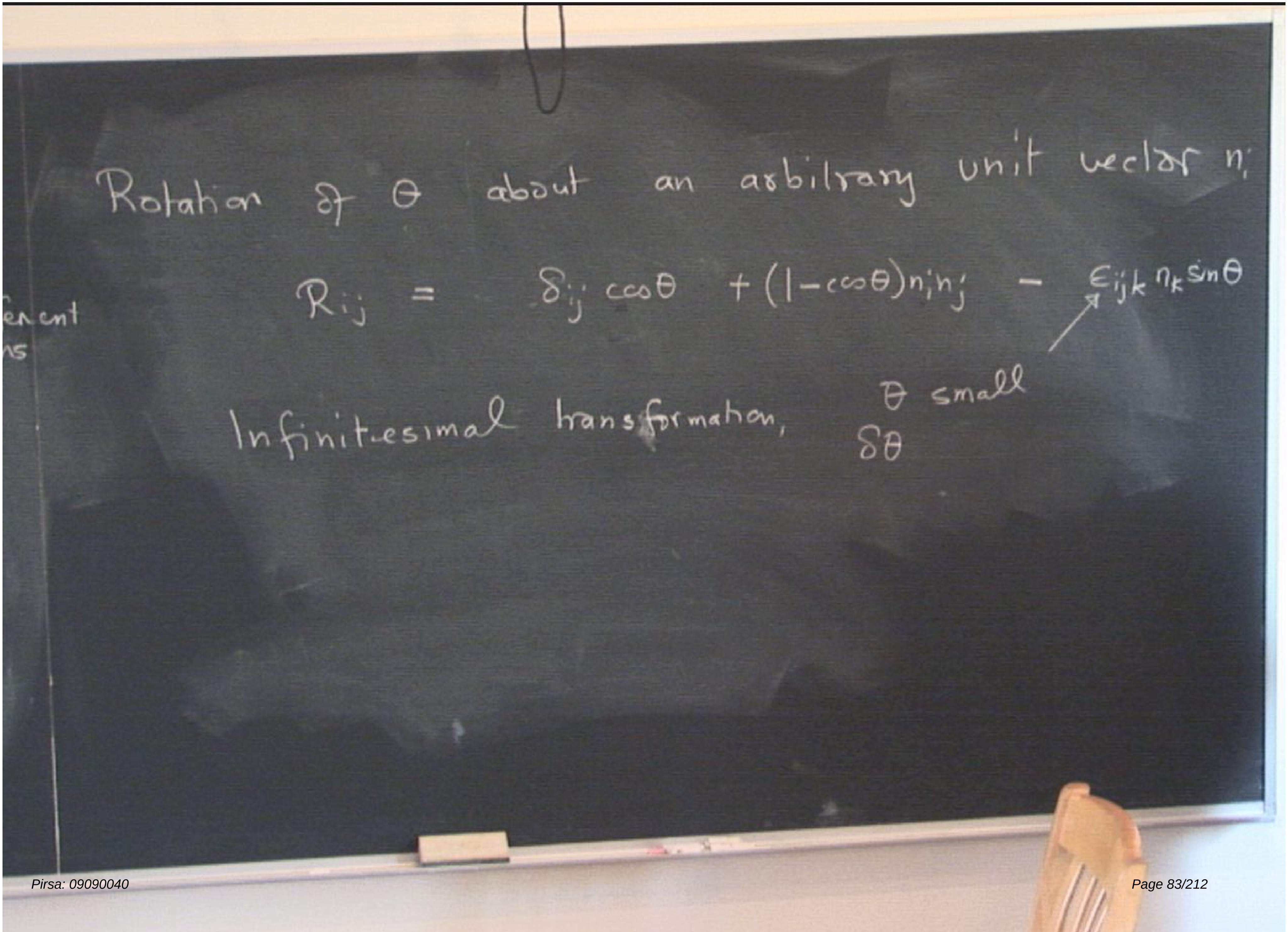


ferent
ns

Rotation of θ about an arbitrary unit vector n_i

$$R_{ij} = \delta_{ij}\cos\theta + (1-\cos\theta)n_i n_j - \epsilon_{ijk} n_k \sin\theta$$

Infinitesimal transformation,



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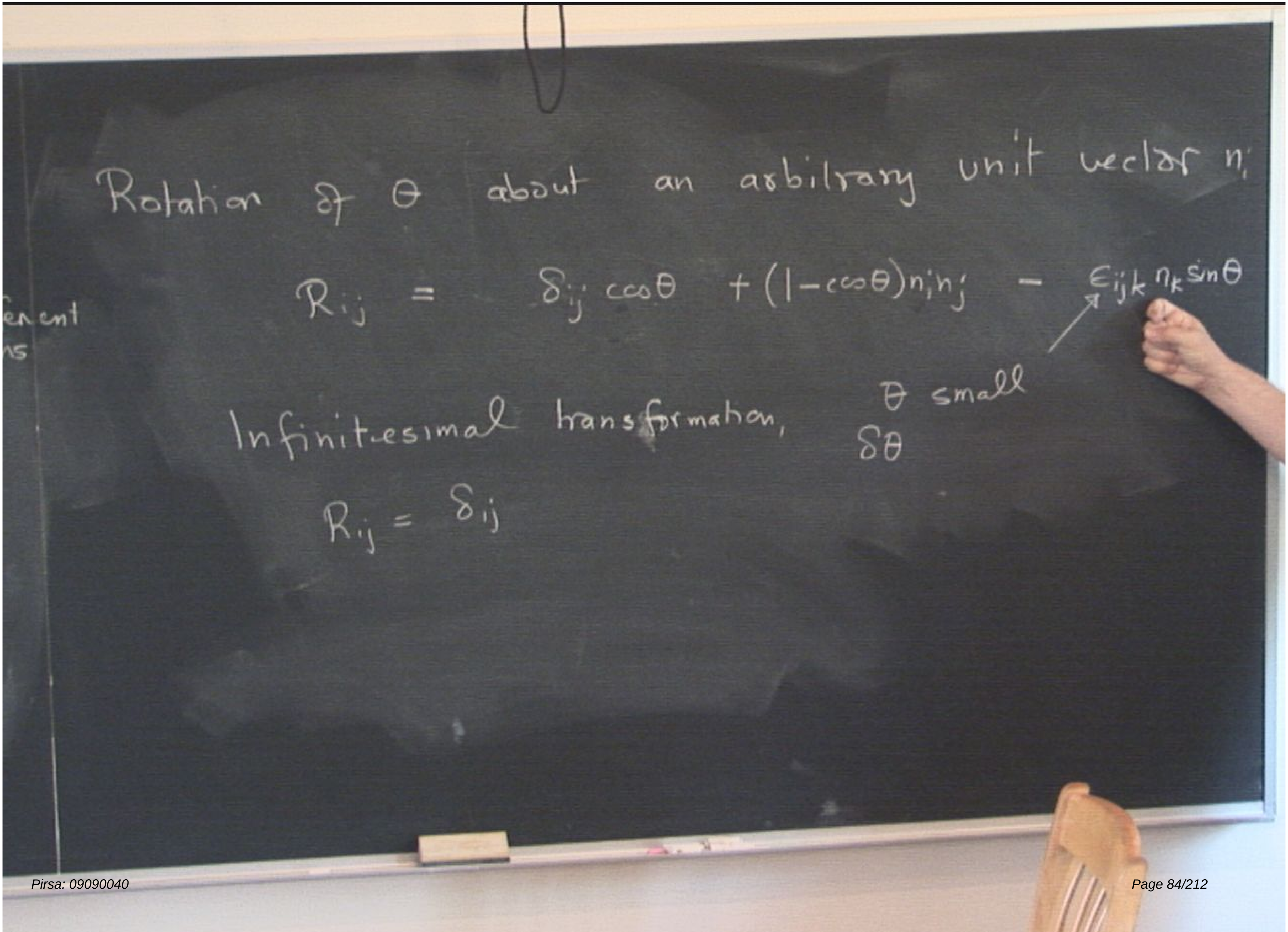
Rotation of θ about an arbitrary unit vector n_i

$$R_{ij} = \delta_{ij}\cos\theta + (1-\cos\theta)n_i n_j - \epsilon_{ijk} n_k \sin\theta$$

Infinitesimal transformation, θ small

$\delta\theta$



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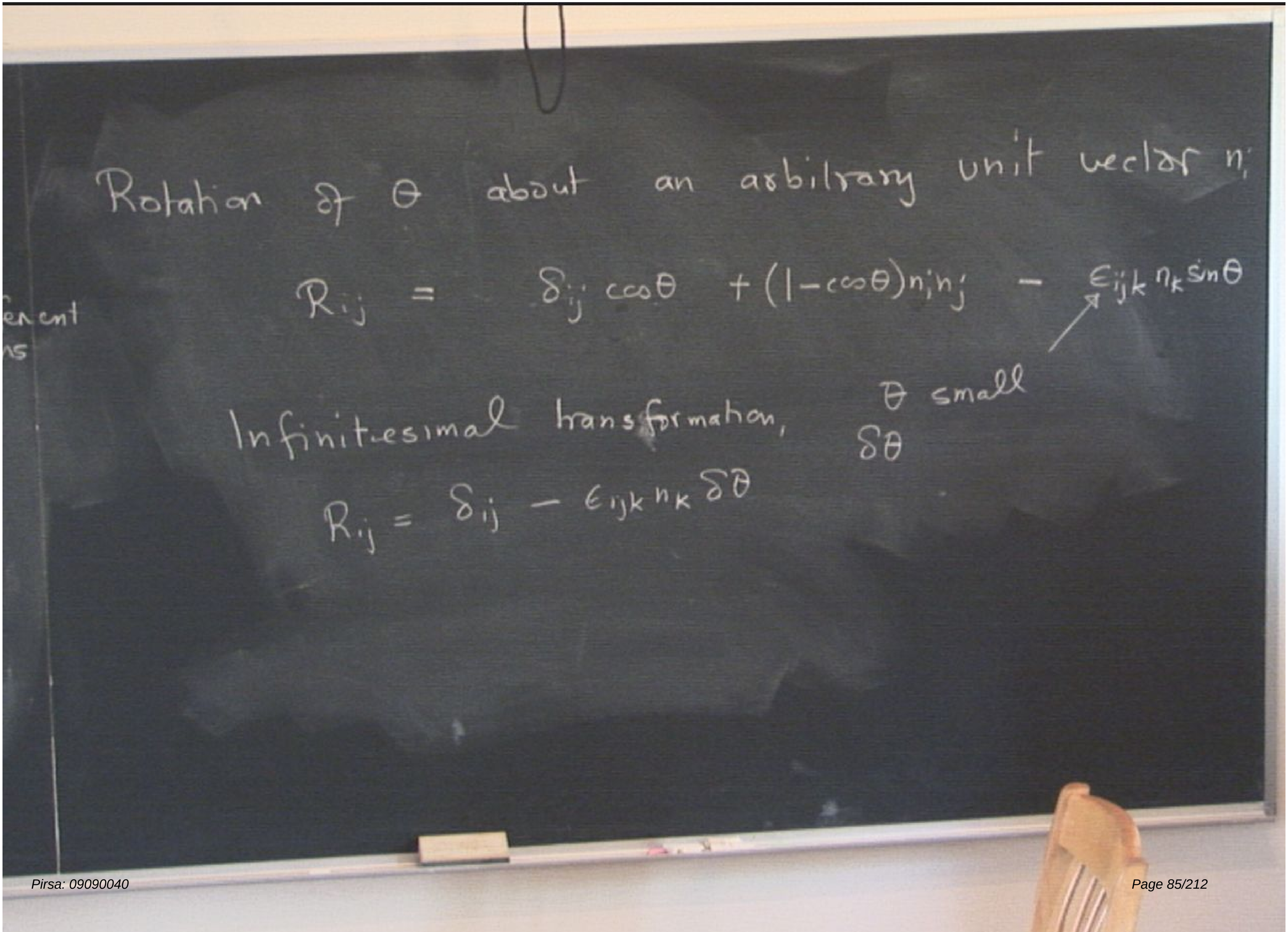
θ small

Infinitesimal transformation, $\delta\theta$

$$R_{ij} = \delta_{ij}$$



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$$R_{ij} = \delta_{ij}\cos\theta + (1-\cos\theta)n_i n_j - \epsilon_{ijk} n_k \sin\theta$$

erent

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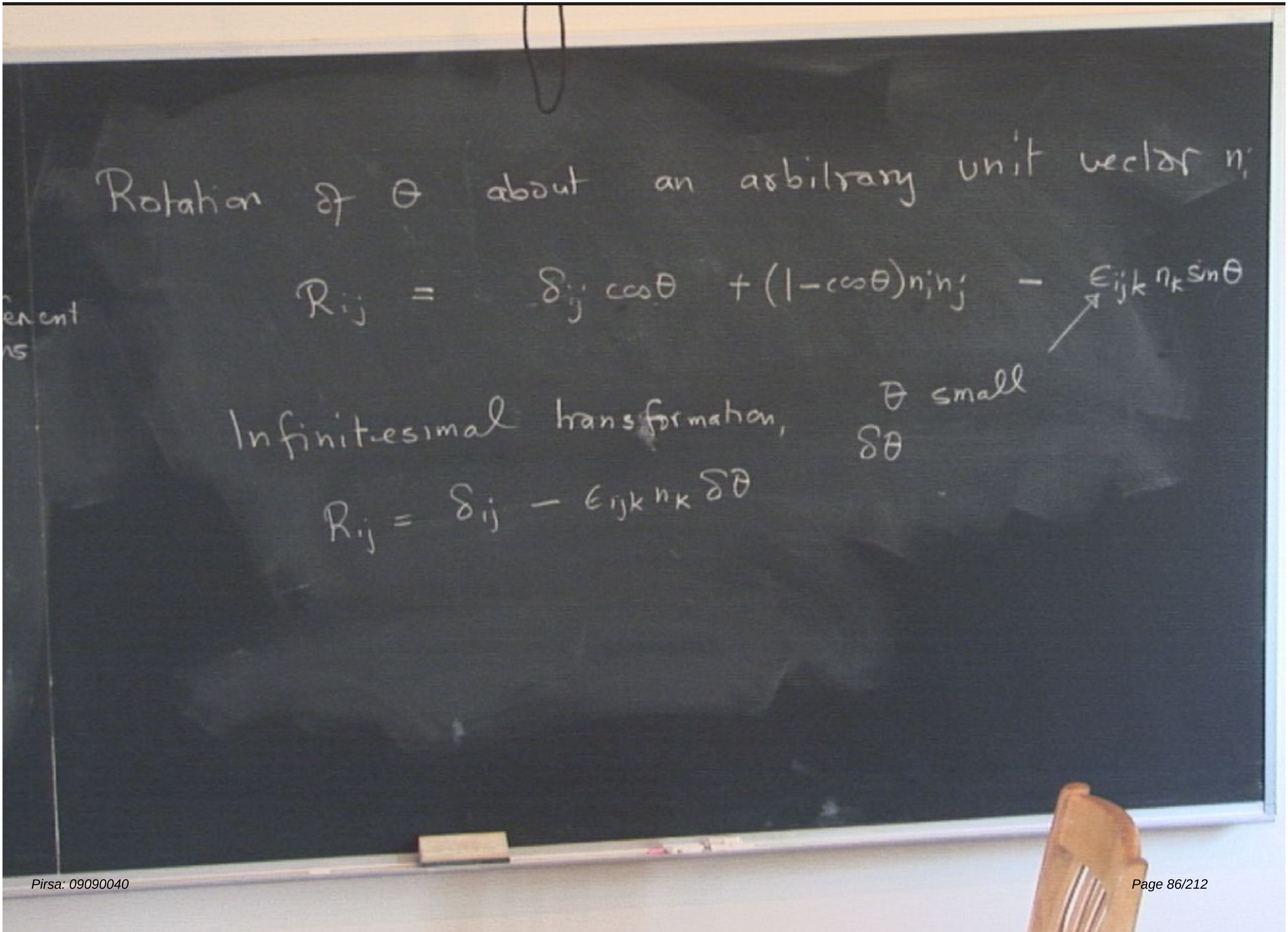
Infinitesimal transformation, θ small

$\delta\theta$

$$R_{ij} = \delta_{ij} - \epsilon_{ijk} n_k \delta\theta$$



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Infinitesimal transformation, $\delta\theta$

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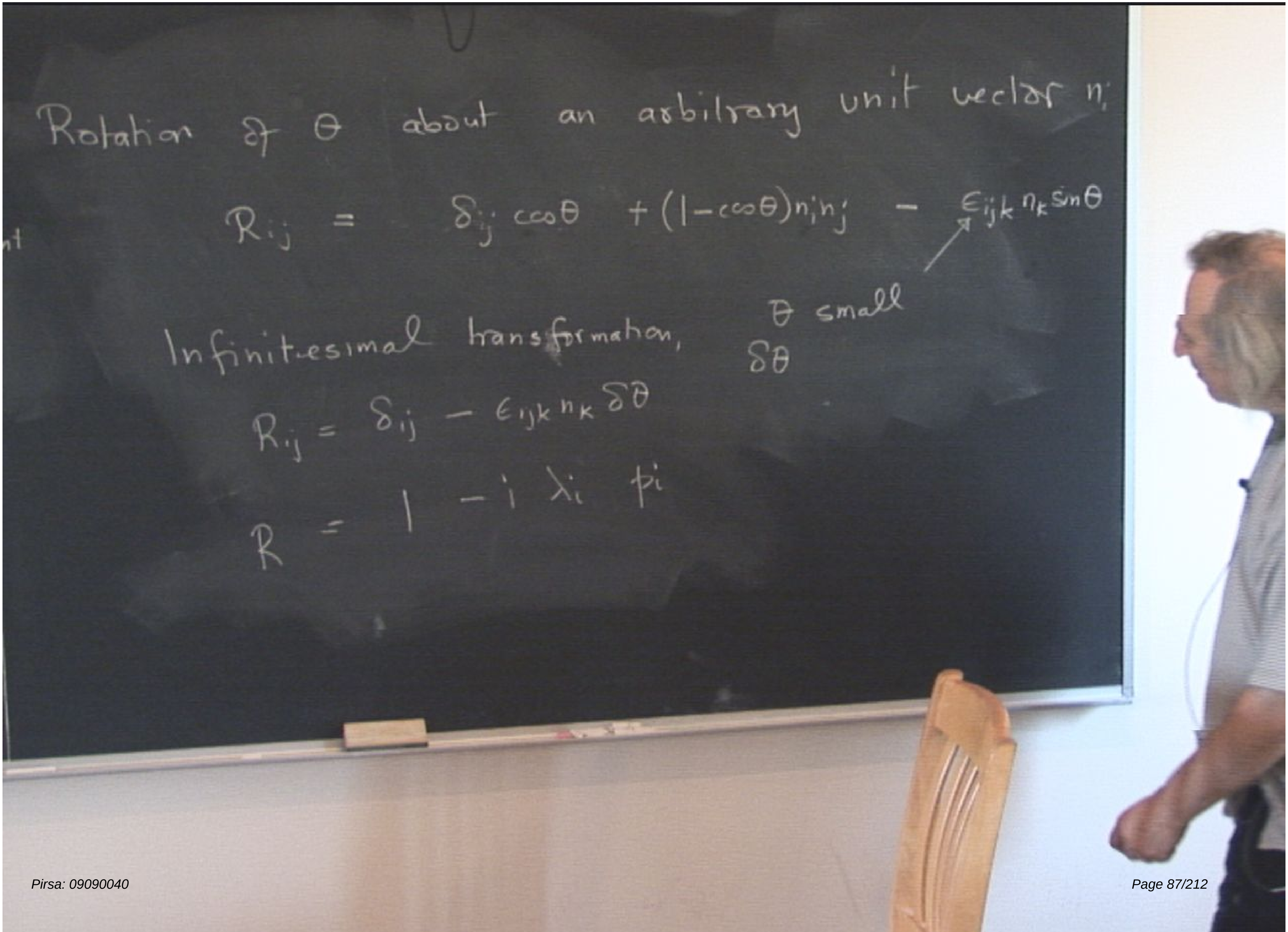


Rotation of θ about an arbitrary unit vector n_i

nt

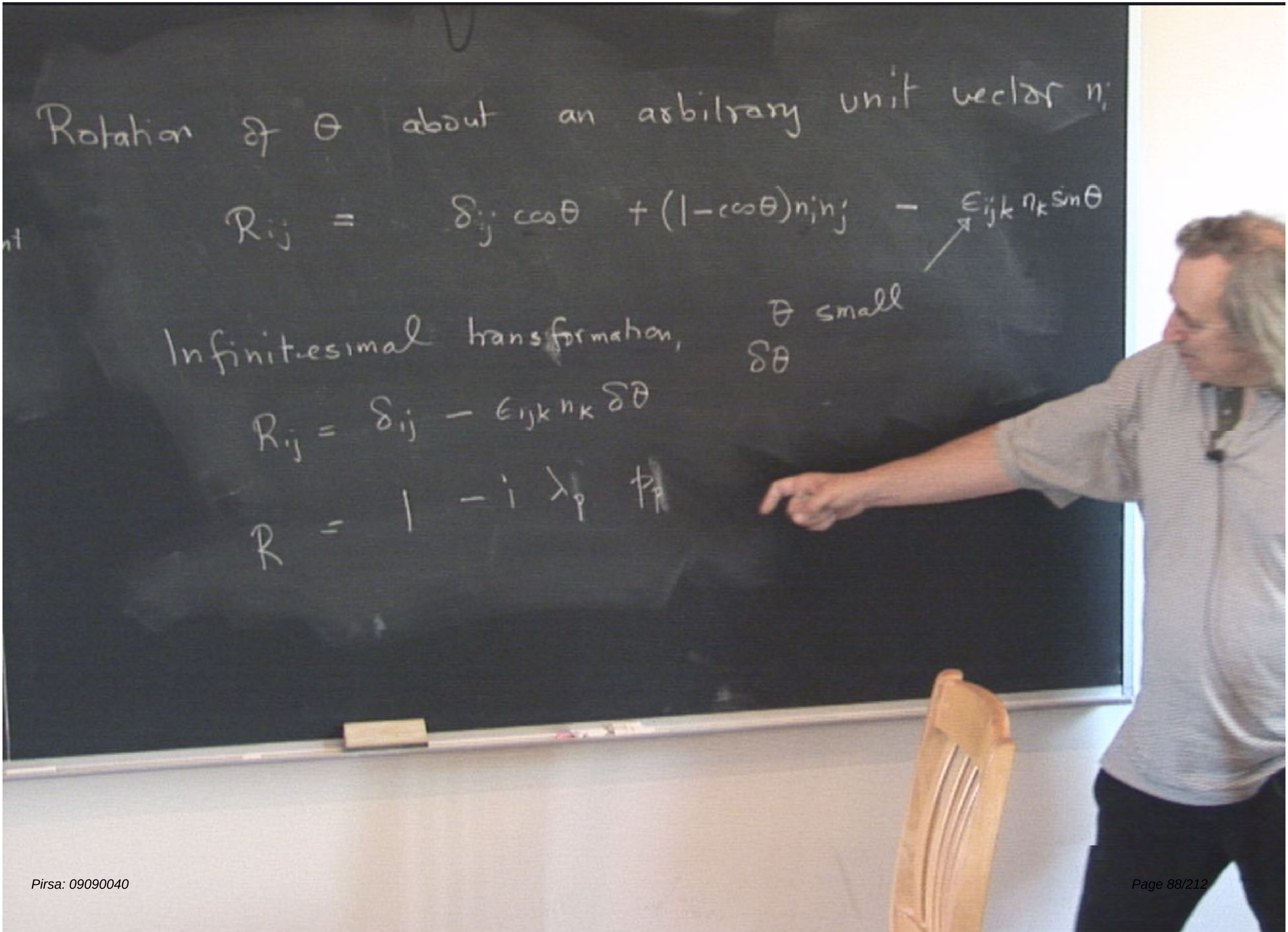
$$R_{ij} = \delta_{ij}\cos\theta + (1-\cos\theta)n_i n_j - \epsilon_{ijk} n_k \sin\theta$$

θ small

Infinitesimal transformation, $\delta\theta$

$$R_{ij} = \delta_{ij} - \epsilon_{ijk} n_k \delta\theta$$

$$R = 1 - i\lambda_i \ p_i$$



Rotation of θ about an arbitrary unit vector n_i

nt

$$R_{ij} = \delta_{ij}\cos\theta + (1-\cos\theta)n_i n_j - \epsilon_{ijk} n_k \sin\theta$$

θ small

Infinitesimal transformation, $\delta\theta$

$$R_{ij} = \delta_{ij} - \epsilon_{ijk} n_k \delta\theta$$

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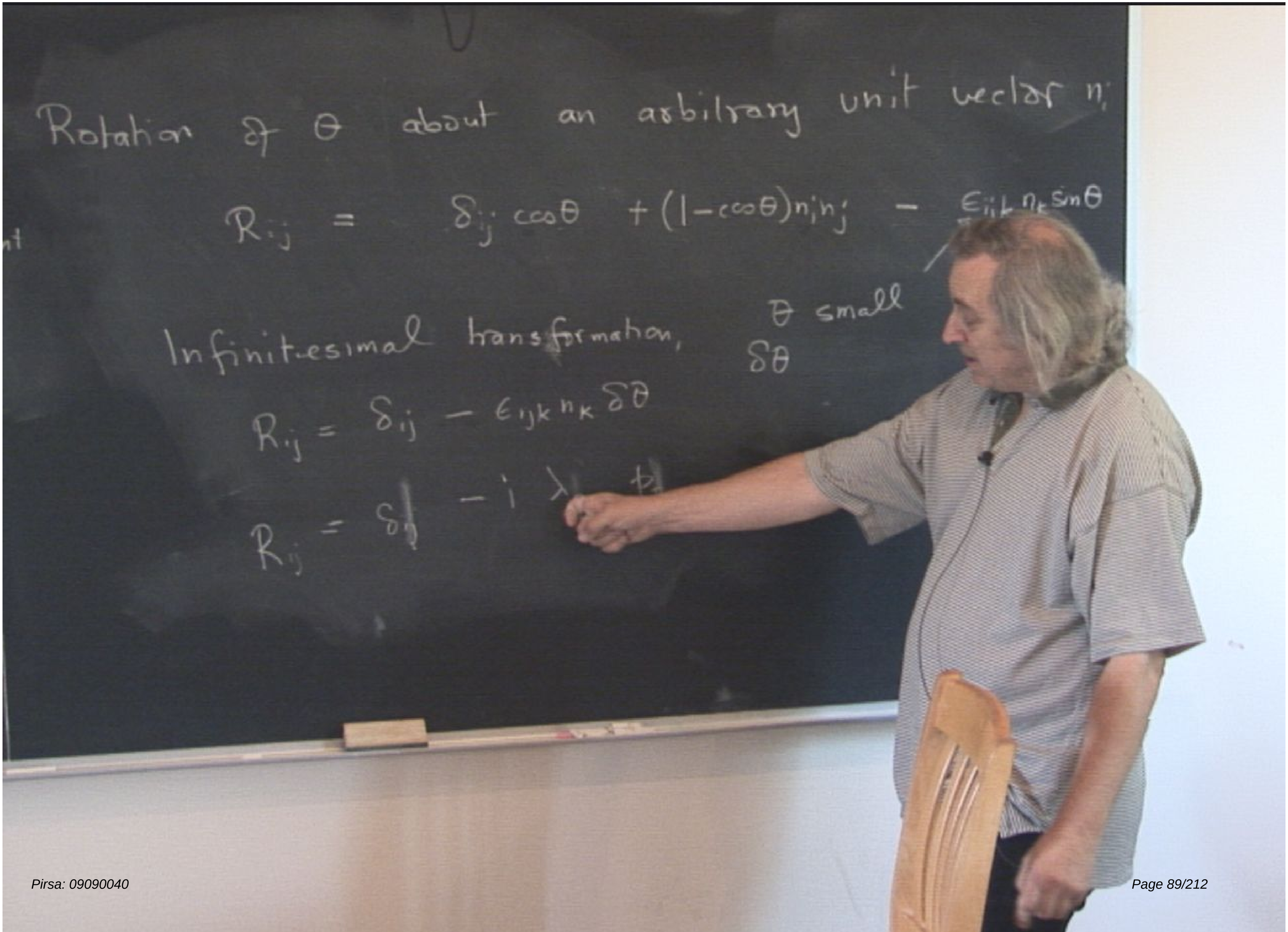


Rotation of θ about an arbitrary unit vector n_i

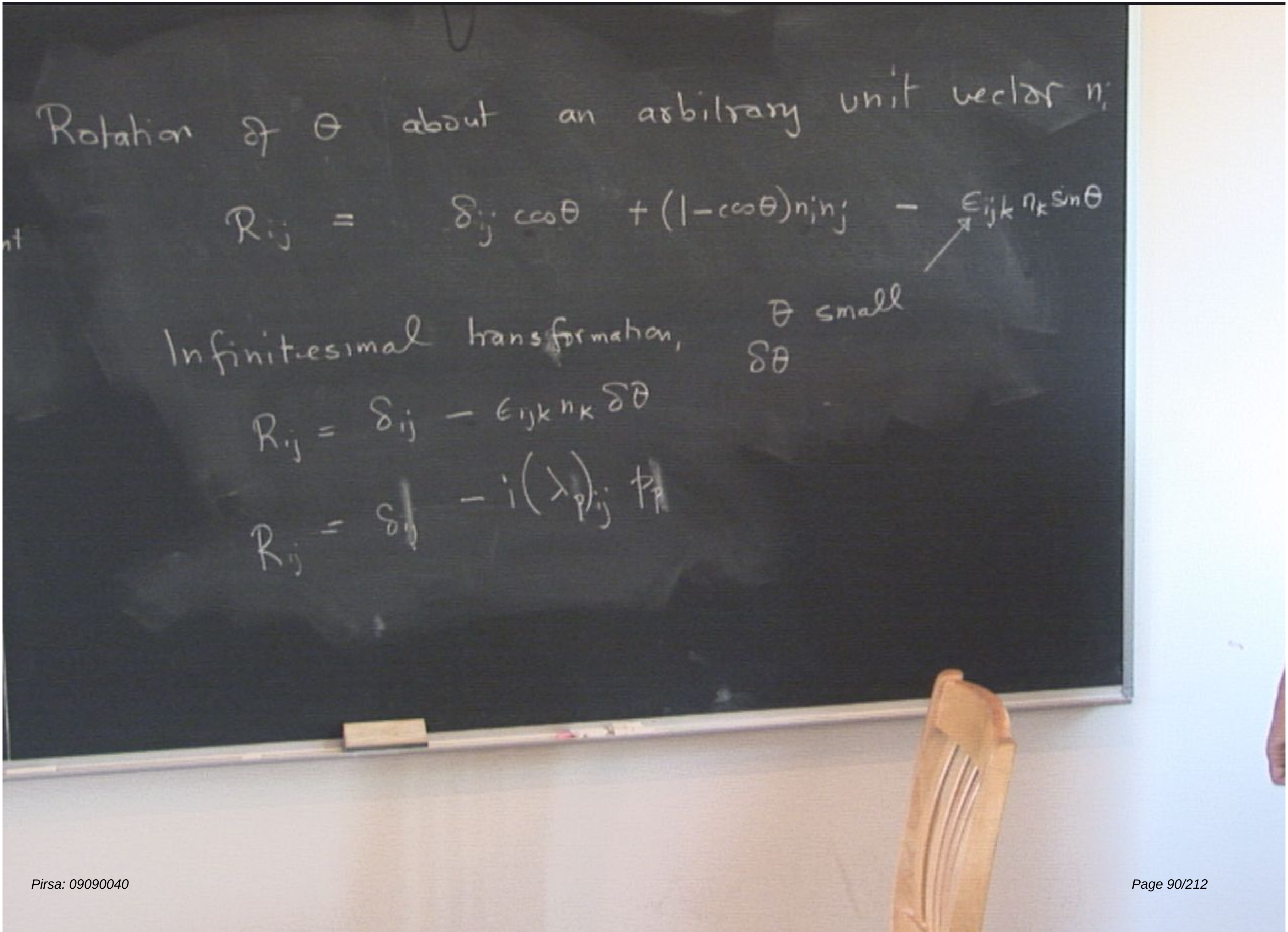
$$R_{ij} = \delta_{ij}\cos\theta + (1-\cos\theta)n_i n_j - \epsilon_{ijk}n_k\sin\theta$$

Infinitesimal transformation, θ small

$\delta\theta$

$$R_{ij} = \delta_{ij} - \epsilon_{ijk}n_k\delta\theta$$

$$R_{ij} = \delta_{ij} - i\lambda$$



Rotation of θ about an arbitrary unit vector n_i

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θ small

Infinitesimal transformation, $\delta\theta$

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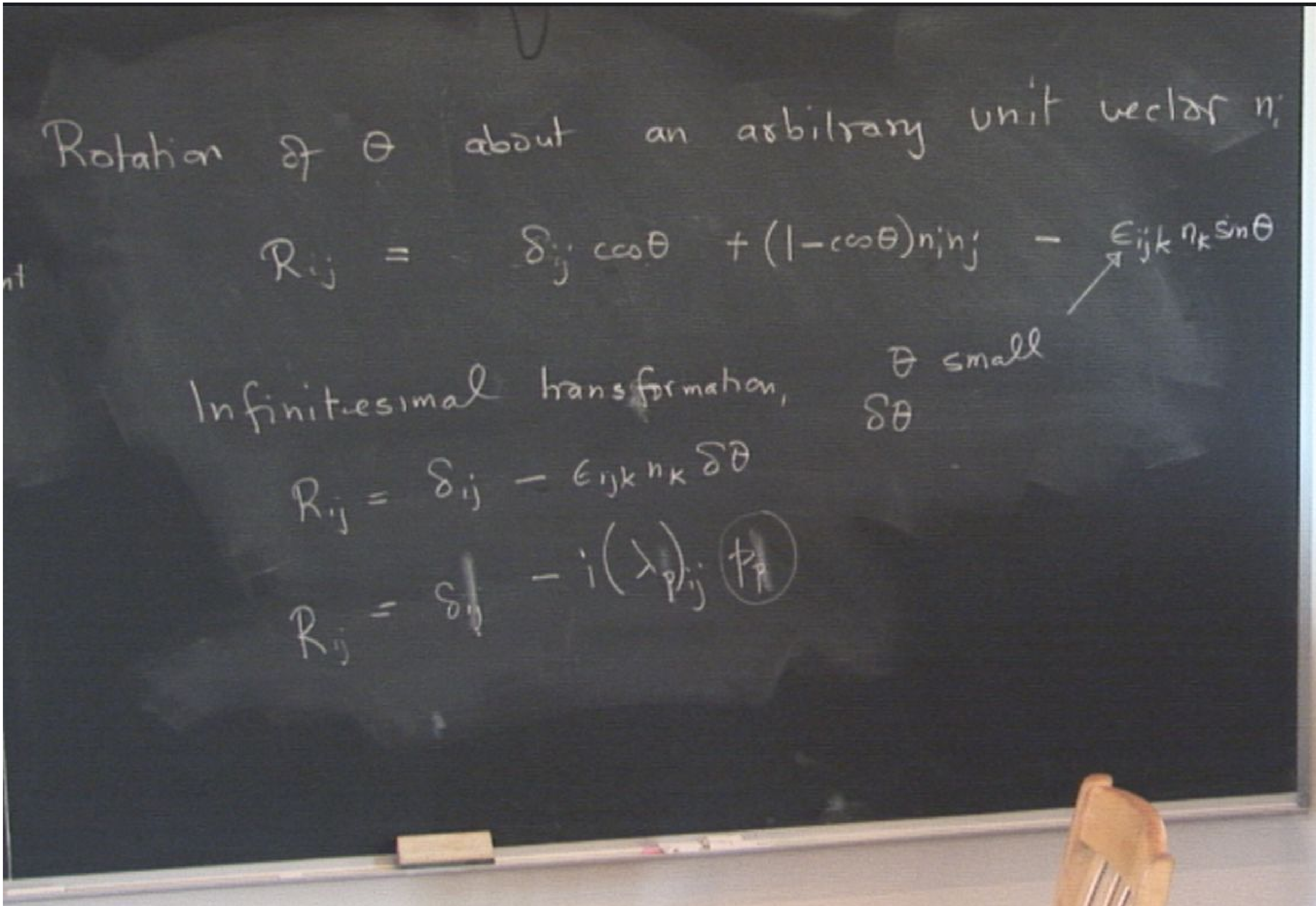
$$R_{ij} = \delta_{ij} - i(\lambda_p)_{ij}\, p_p$$



Rotation of θ about an arbitrary unit vector n_i

nt

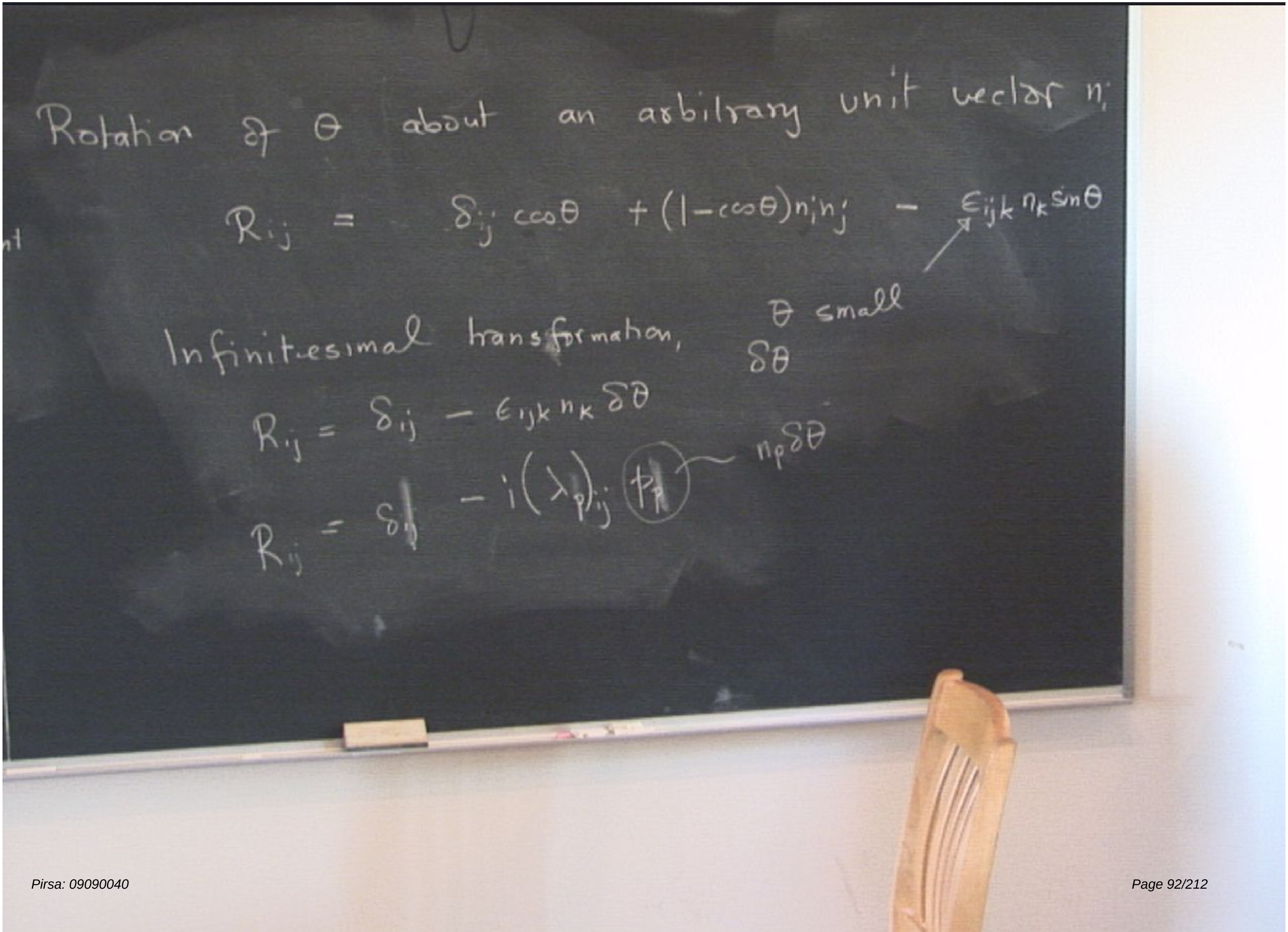
$$R_{ij} = \delta_{ij} \cos\theta + (1-\cos\theta) n_i n_j - \epsilon_{ijk} n_k \sin\theta$$

θ small

Infinitesimal transformation, $\delta\theta$

$$R_{ij} = \delta_{ij} - \epsilon_{ijk} n_k \delta\theta$$

$$R_{ij} = \delta_{ij} - i(\lambda_p)_{ij} \theta_p$$



Rotation of θ about an arbitrary unit vector n_i

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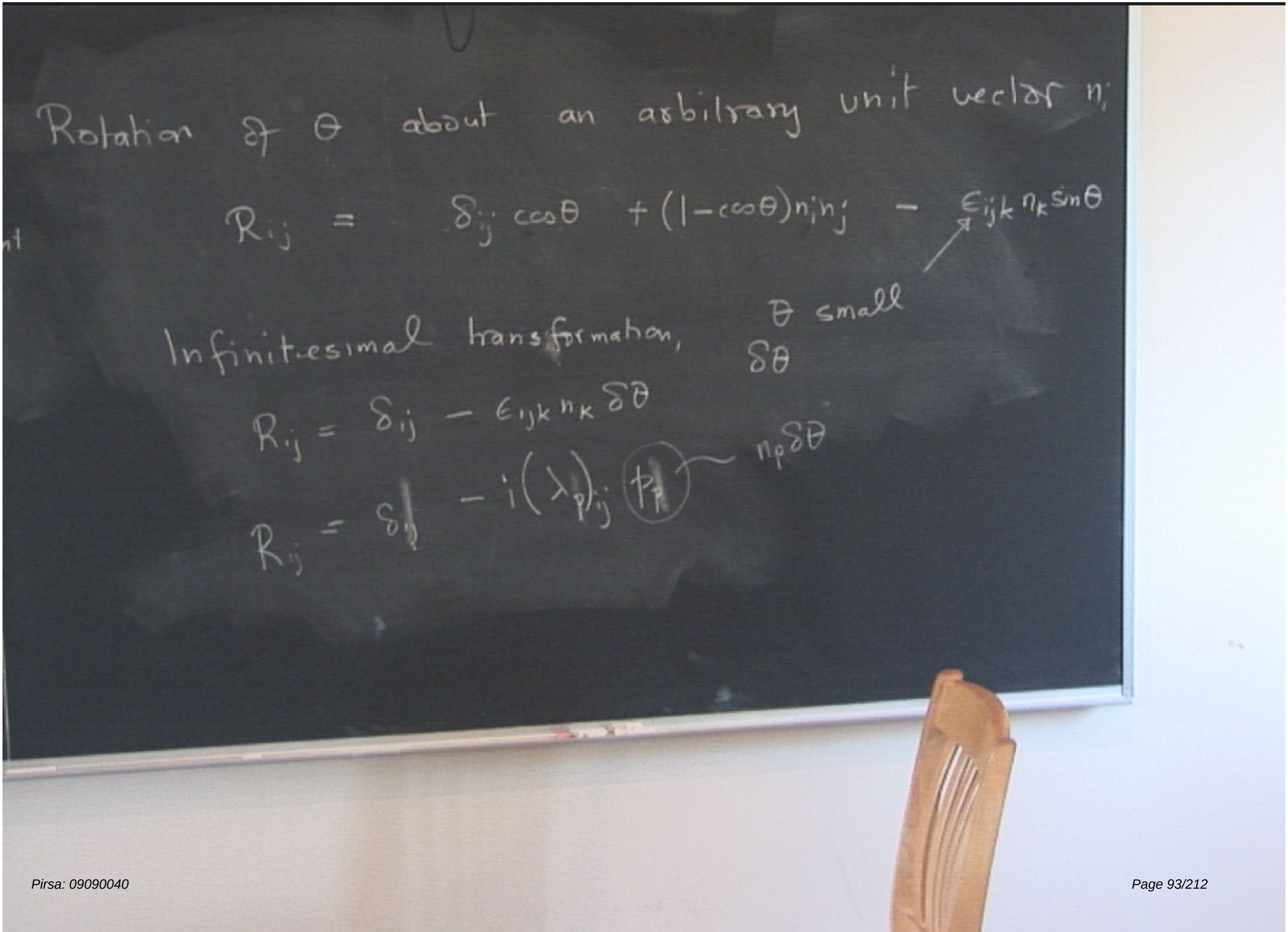
$$R_{ij} = \delta_{ij}\cos\theta + (1-\cos\theta)n_i n_j - \epsilon_{ijk}n_k \sin\theta$$

θ small

Infinitesimal transformation, $\delta\theta$

$$R_{ij} = \delta_{ij} - \epsilon_{ijk}n_k\,\delta\theta$$

$$R_{ij} = \delta_{ij} - i(\lambda_p)_{ij}\, p_p \sim n_p\delta\theta$$



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Infinitesimal transformation, θ small

$\delta\theta$

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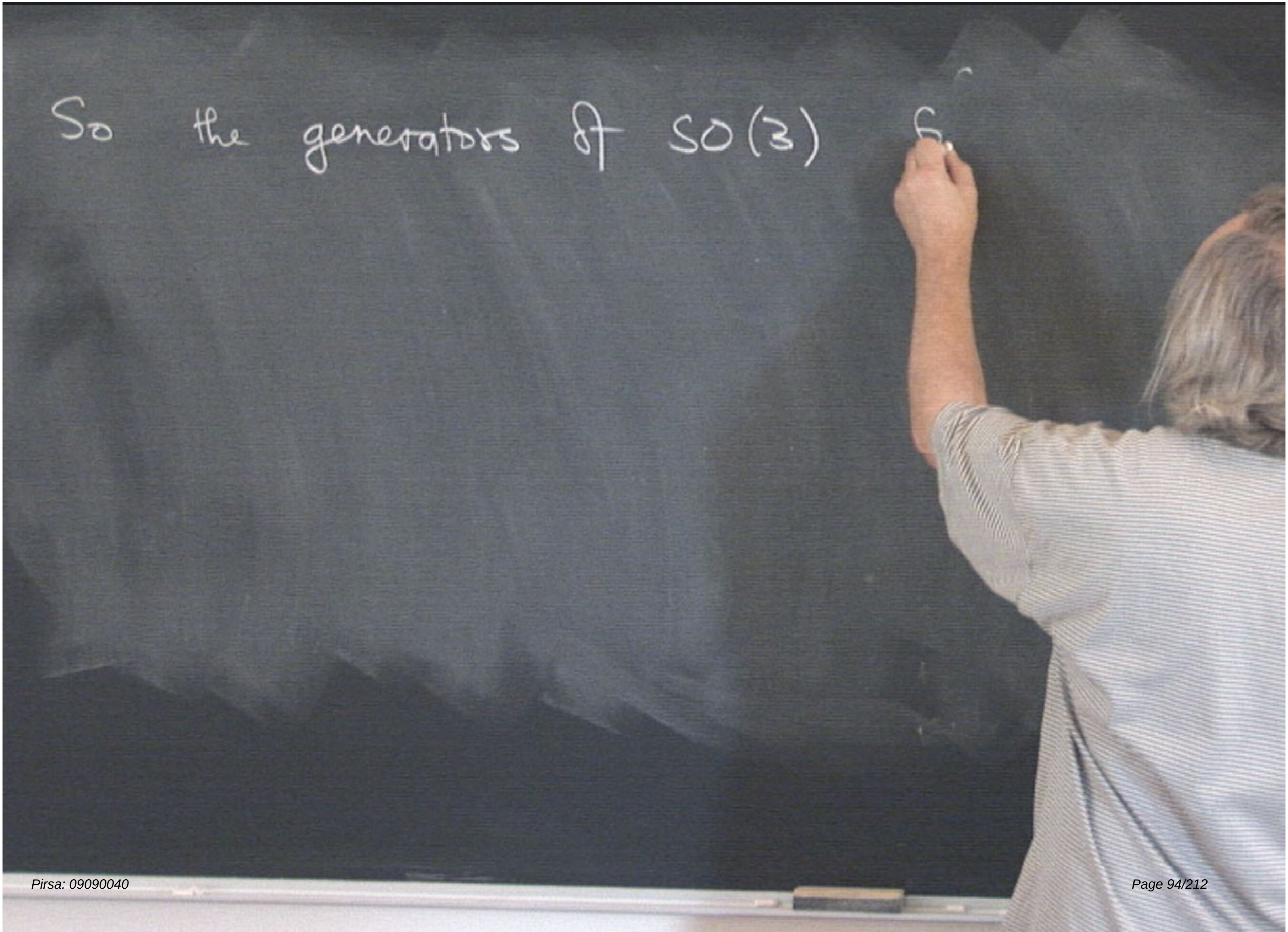
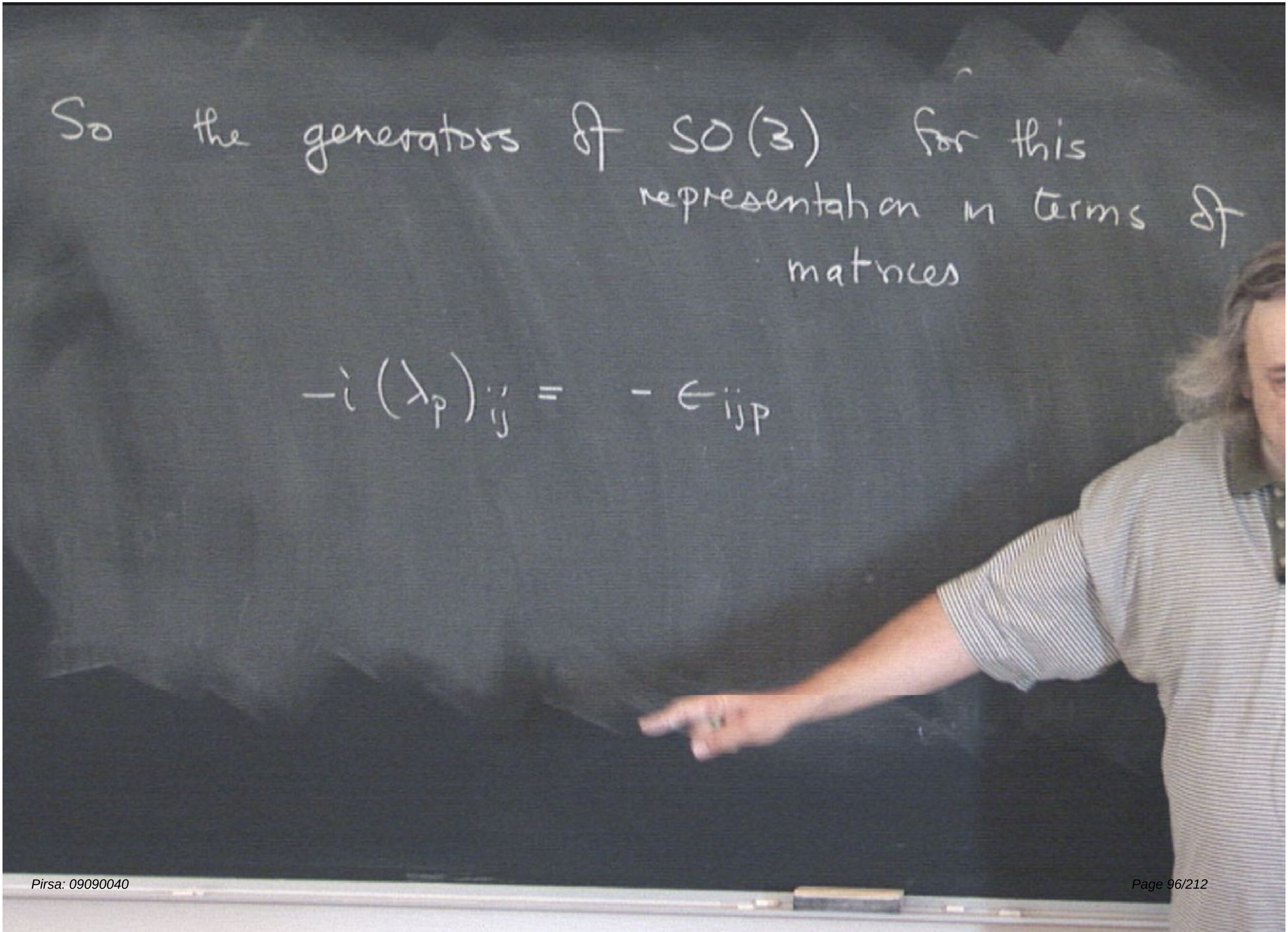


So the generators of SO(3)

So the generators of SO(3) for this representation in terms of matrices



So the generators of SO(3) for this representation in terms of matrices

$$-i(\lambda_p)_{ij} = -\epsilon_{ijp}$$

So the generators of SO(3) for this representation in terms of matrices

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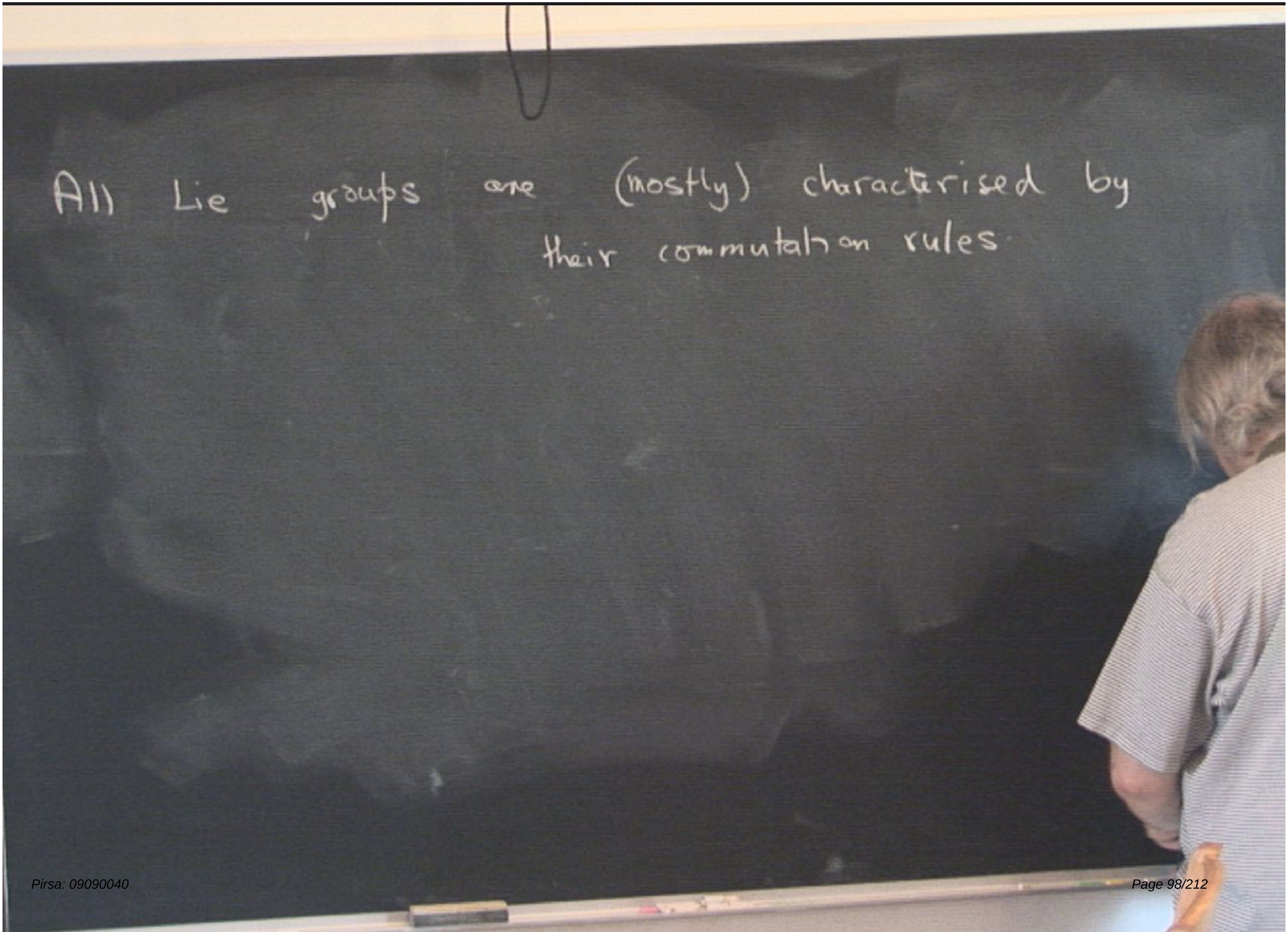
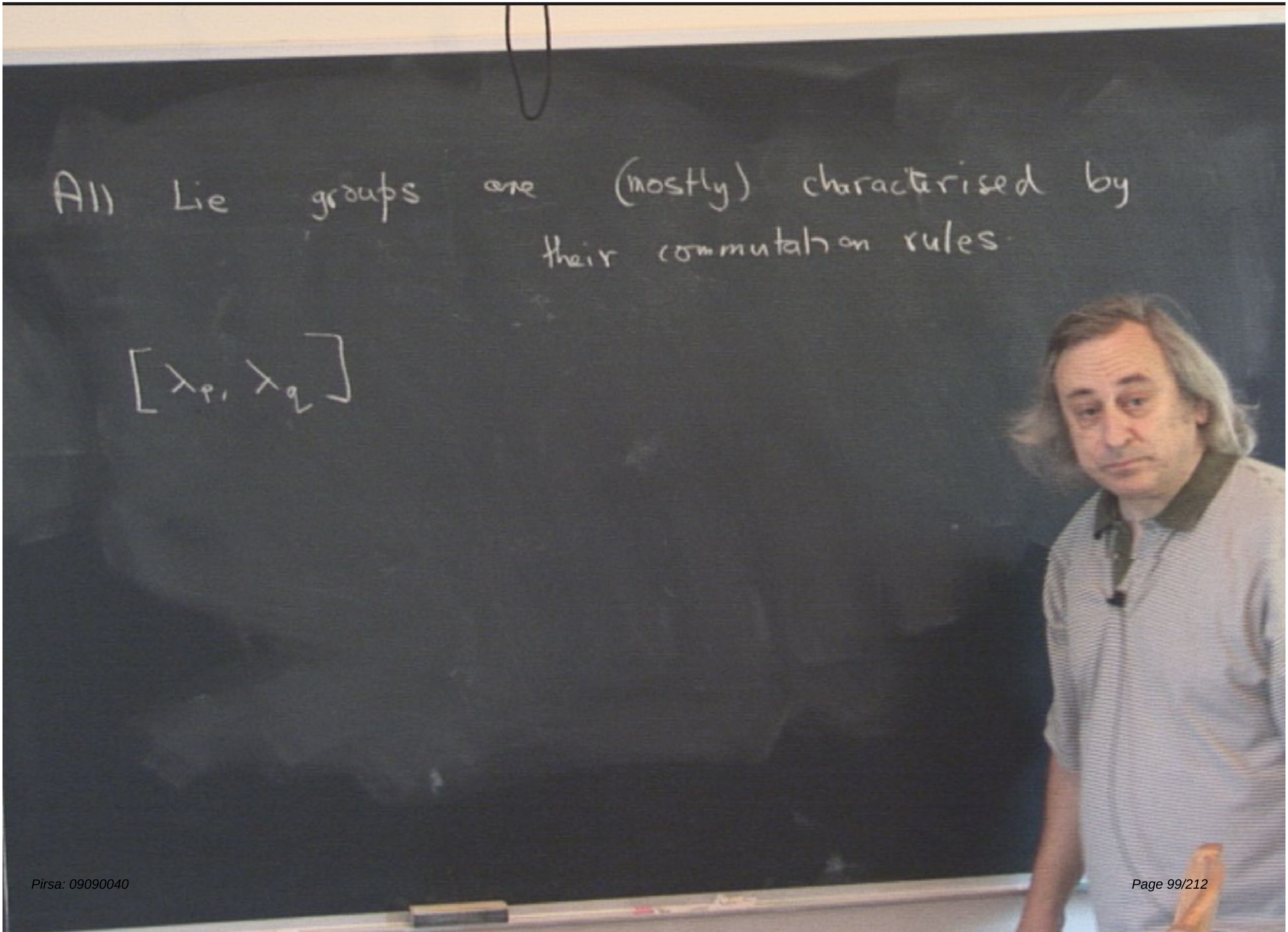
$$\boxed{(\lambda_p)_{ij} = -i\epsilon_{ijp}}$$

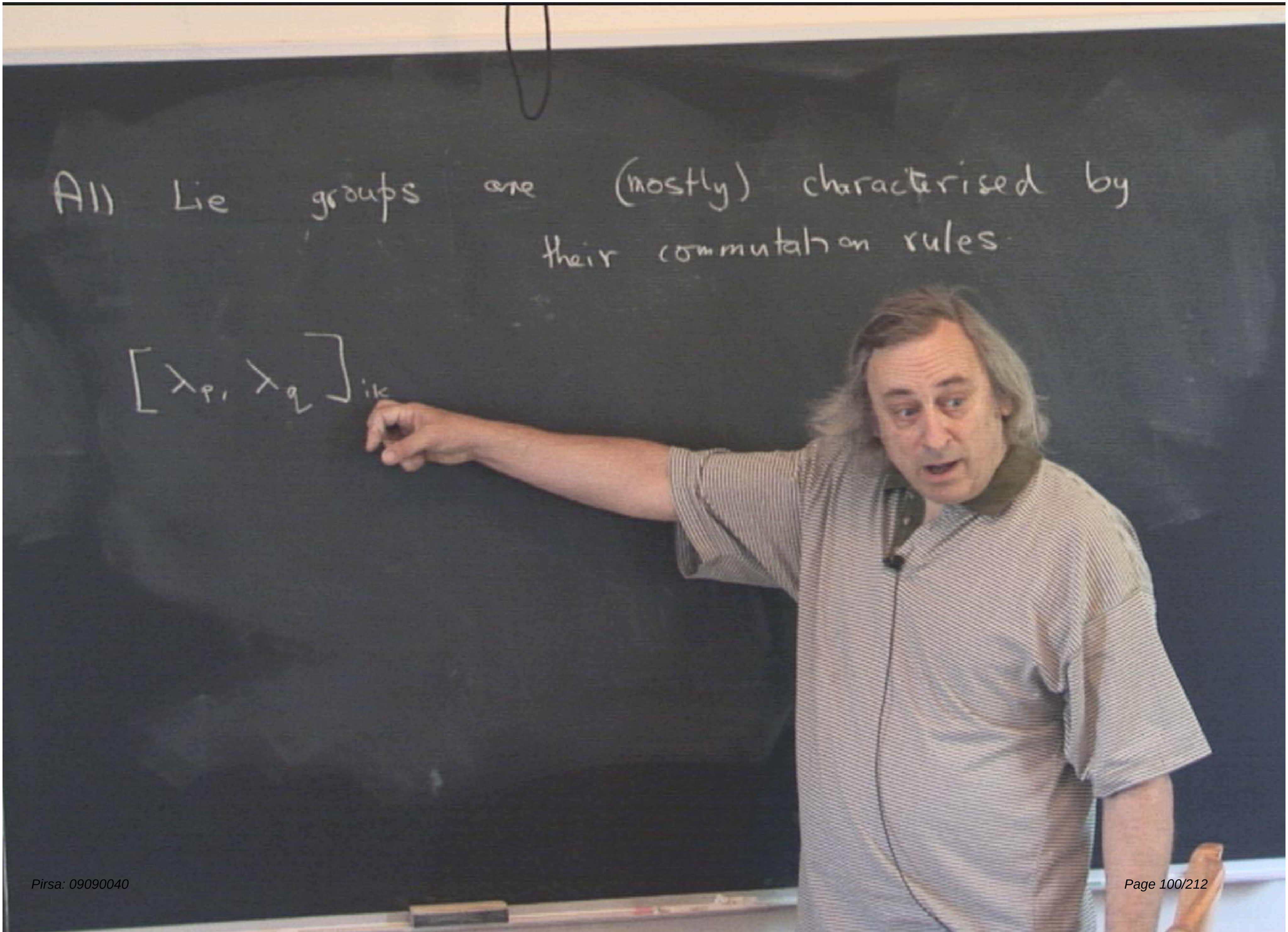


All Lie groups are (mostly) characterised by their commutation rules.



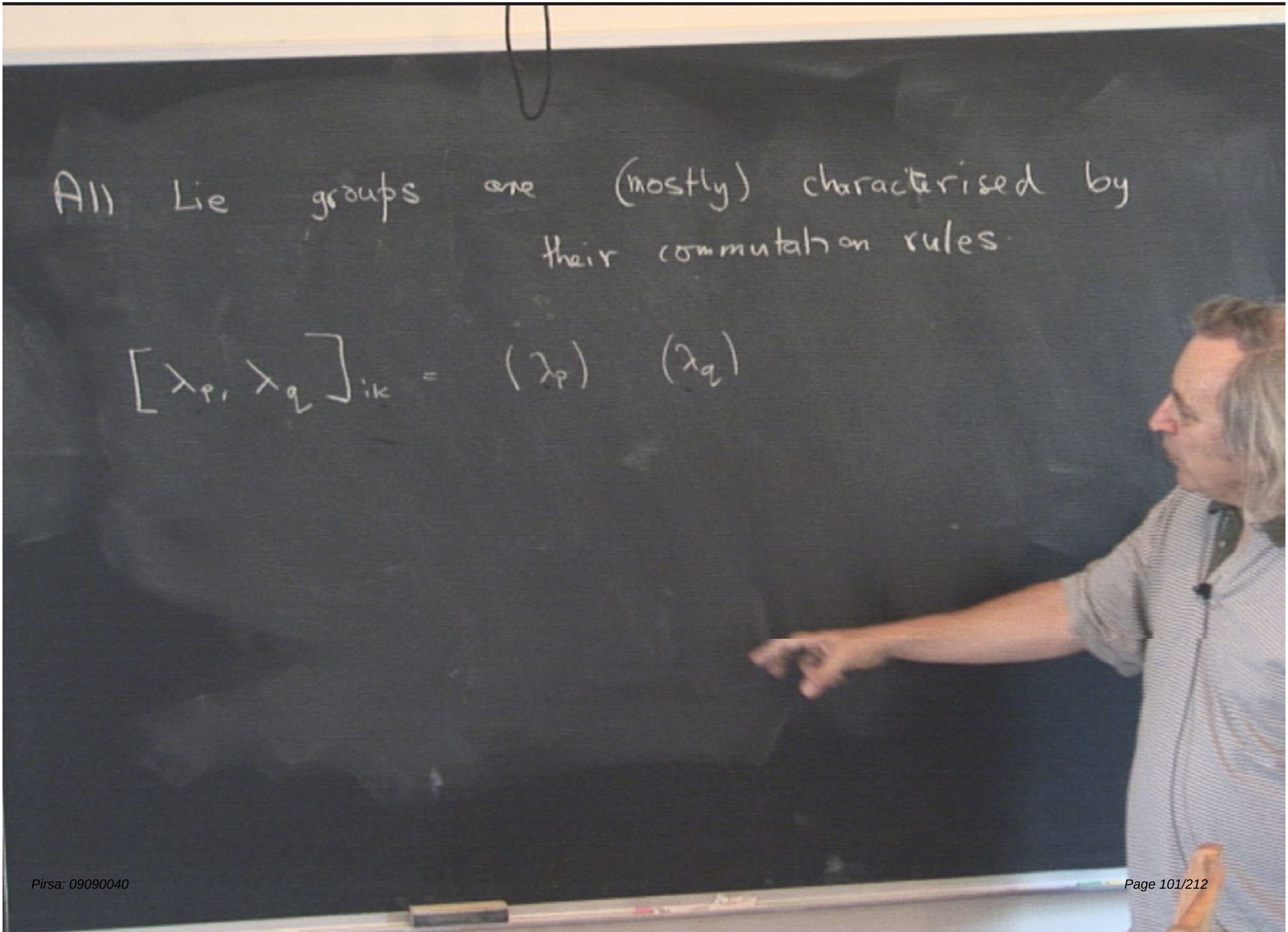
All Lie groups are (mostly) characterised by their commutation rules.

$$[\lambda_p, \lambda_q]$$



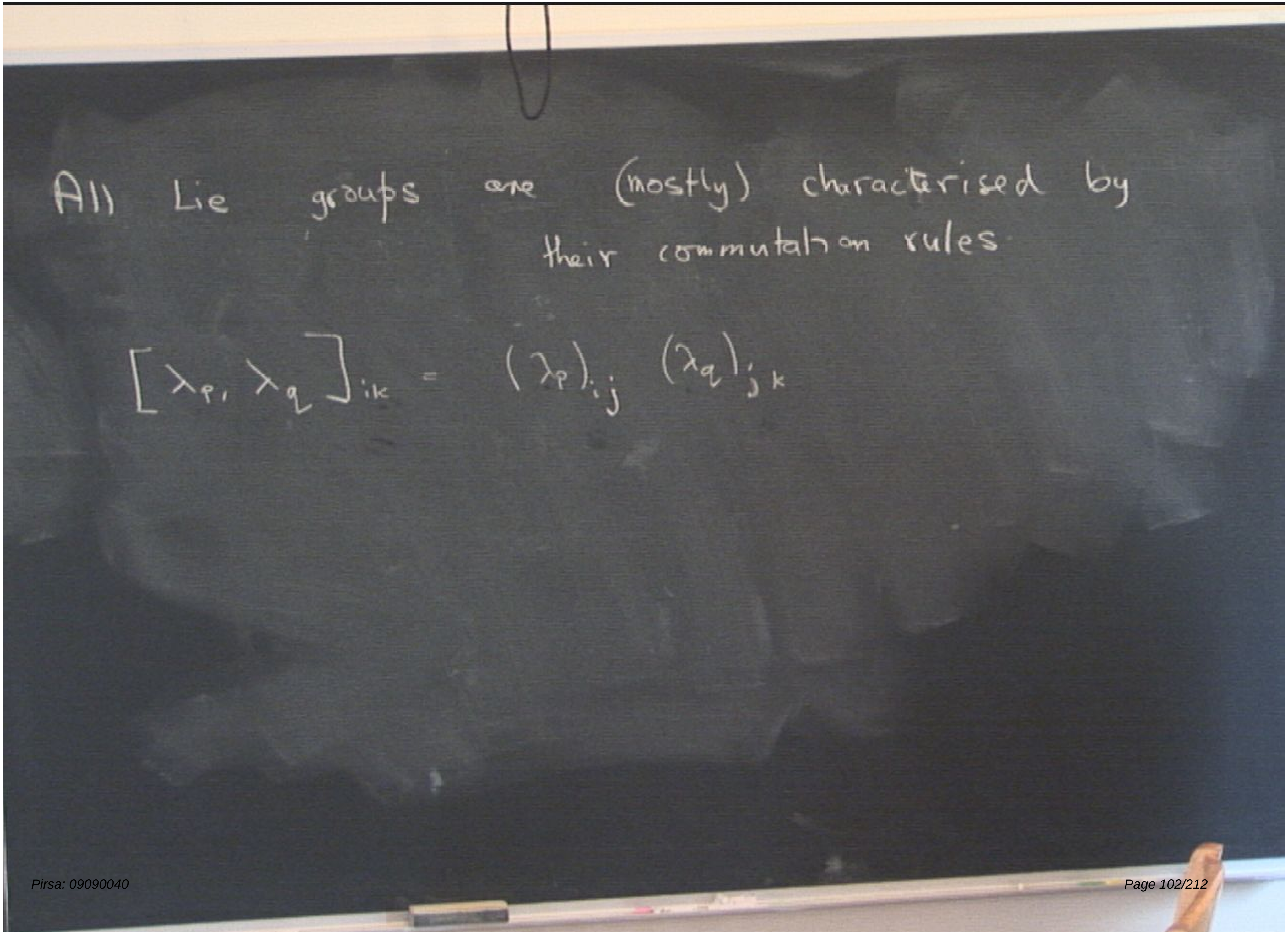
All Lie groups are (mostly) characterised by their commutation rules.

$$[\lambda_p, \lambda_q]_{ik}$$



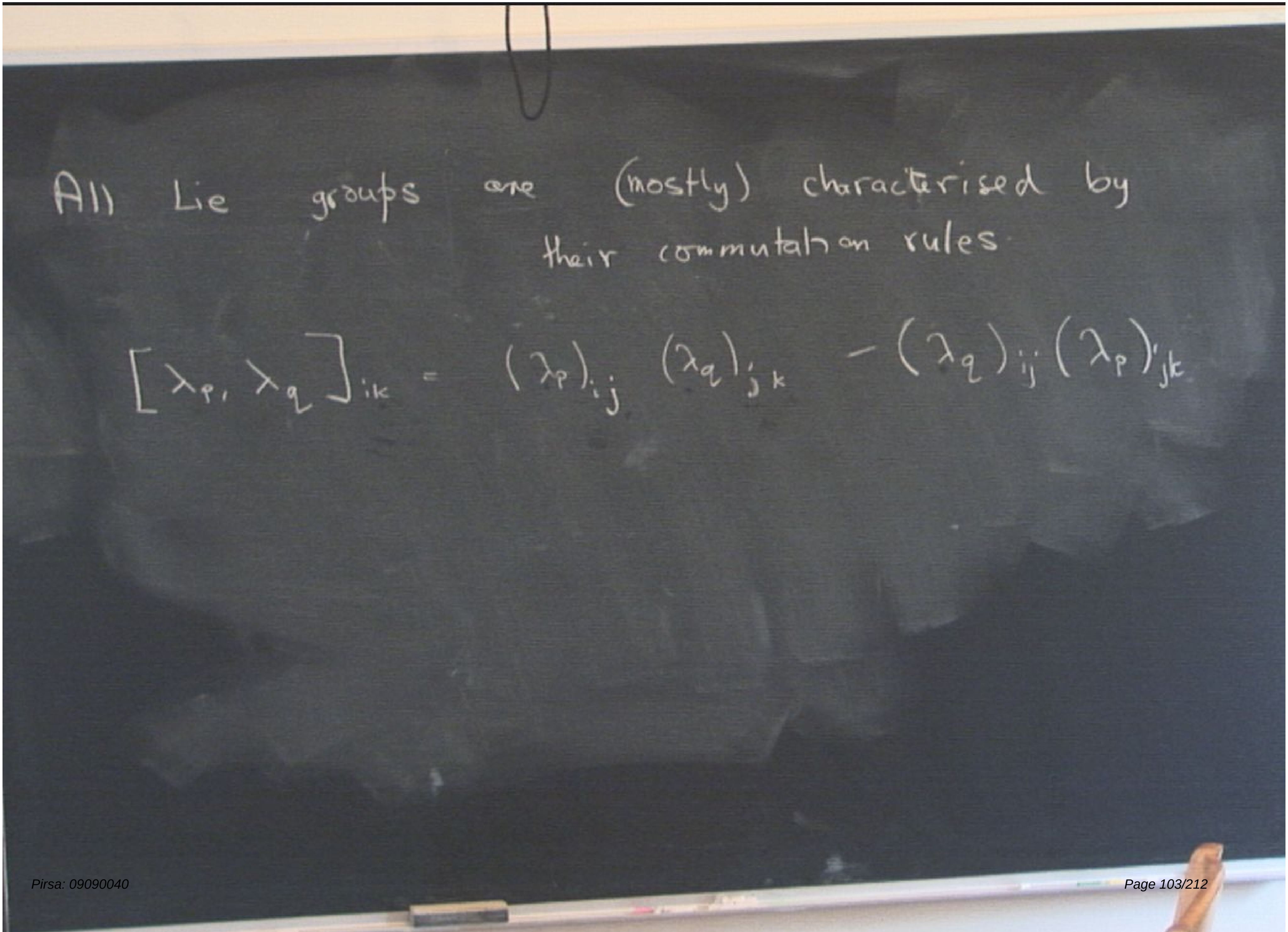
All Lie groups are (mostly) characterised by their commutation rules.

$$[\lambda_p, \lambda_q]_{ik} = (\lambda_p) \quad (\lambda_q)$$



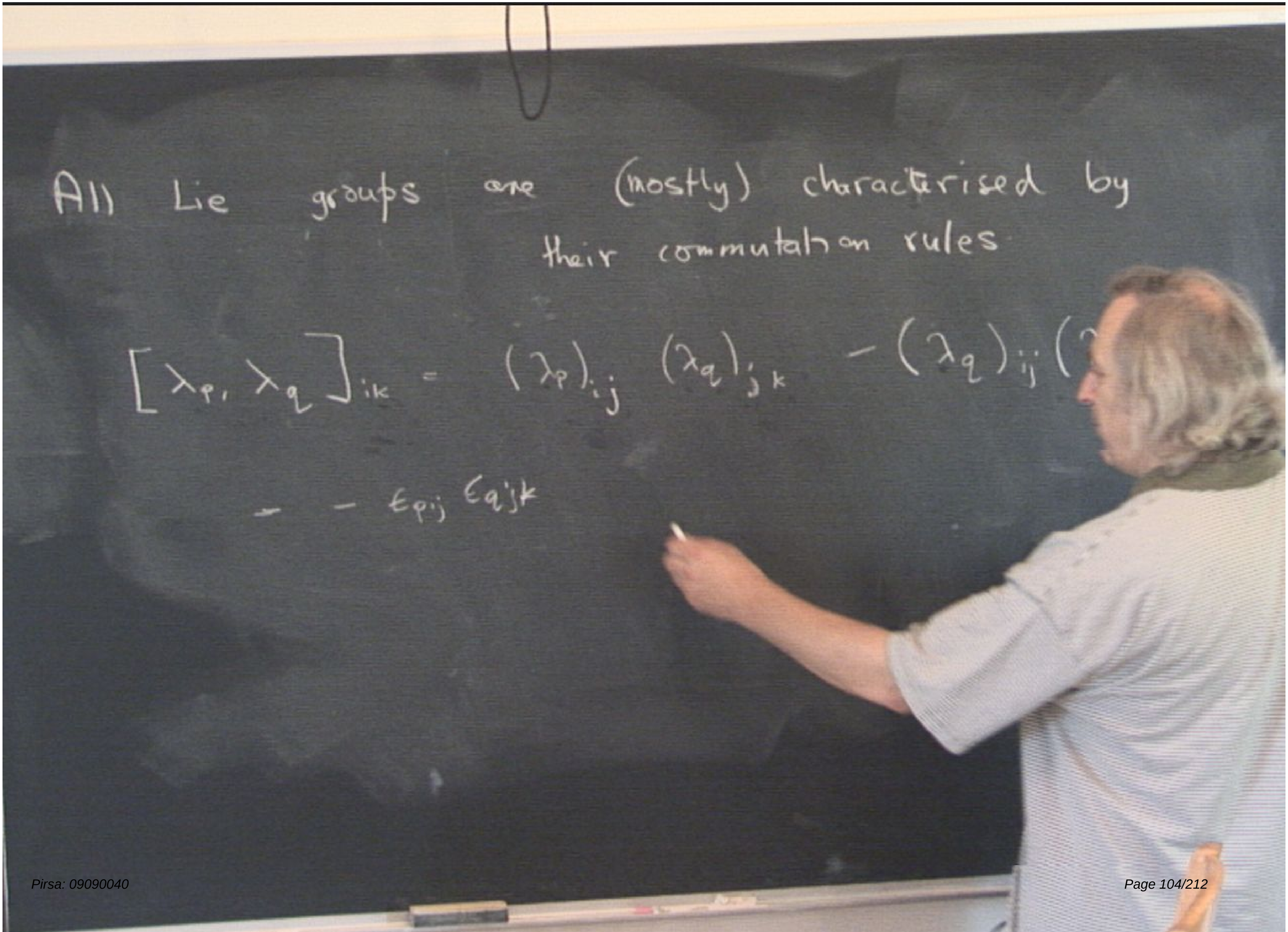
All Lie groups are (mostly) characterised by their commutation rules

$$[\lambda_p, \lambda_q]_{ik} = (\lambda_p)_{ij} (\lambda_q)_{jk}$$



All Lie groups are (mostly) characterised by their commutation rules.

$$[\lambda_p, \lambda_q]_{ik} = (\lambda_p)_{ij} (\lambda_q)_{jk} - (\lambda_q)_{ij} (\lambda_p)_{jk}$$



All Lie groups are (mostly) characterised by their commutation rules

$$[\lambda_p, \lambda_q]_{ik} = (\lambda_p)_{ij} (\lambda_q)_{jk} - (\lambda_q)_{ij} (\lambda$$

$$- - \epsilon_{pij} \epsilon_{qjk}$$

So the generators of SO(3) for this representation in terms of matrices

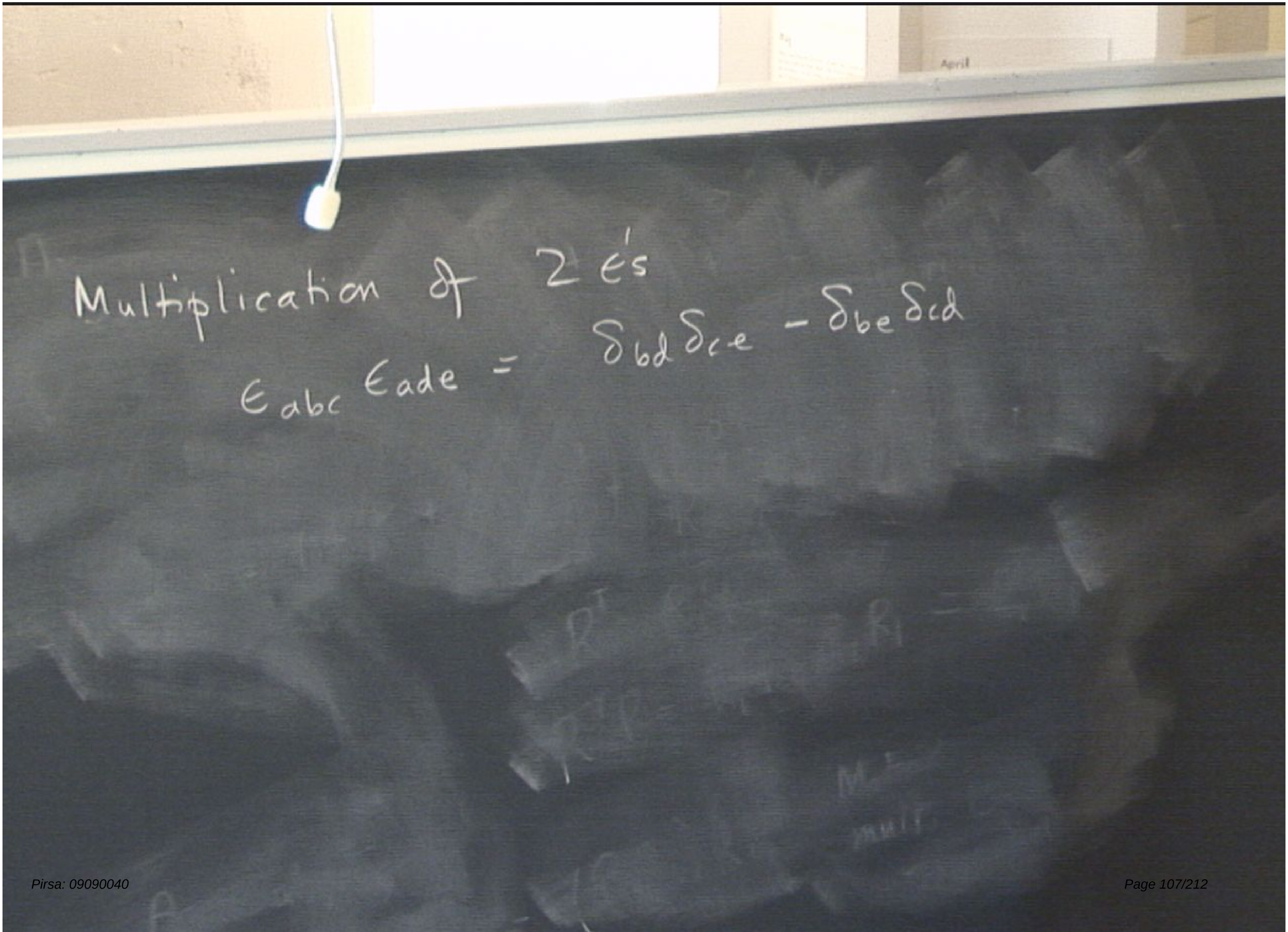
$$-i\,(\lambda_p)_{ij} = -\epsilon_{ijp}$$

$$\boxed{(\lambda_p)_{ij} = -i\,\epsilon_{ijp}}$$

All Lie groups are (mostly) characterised by their commutation rules

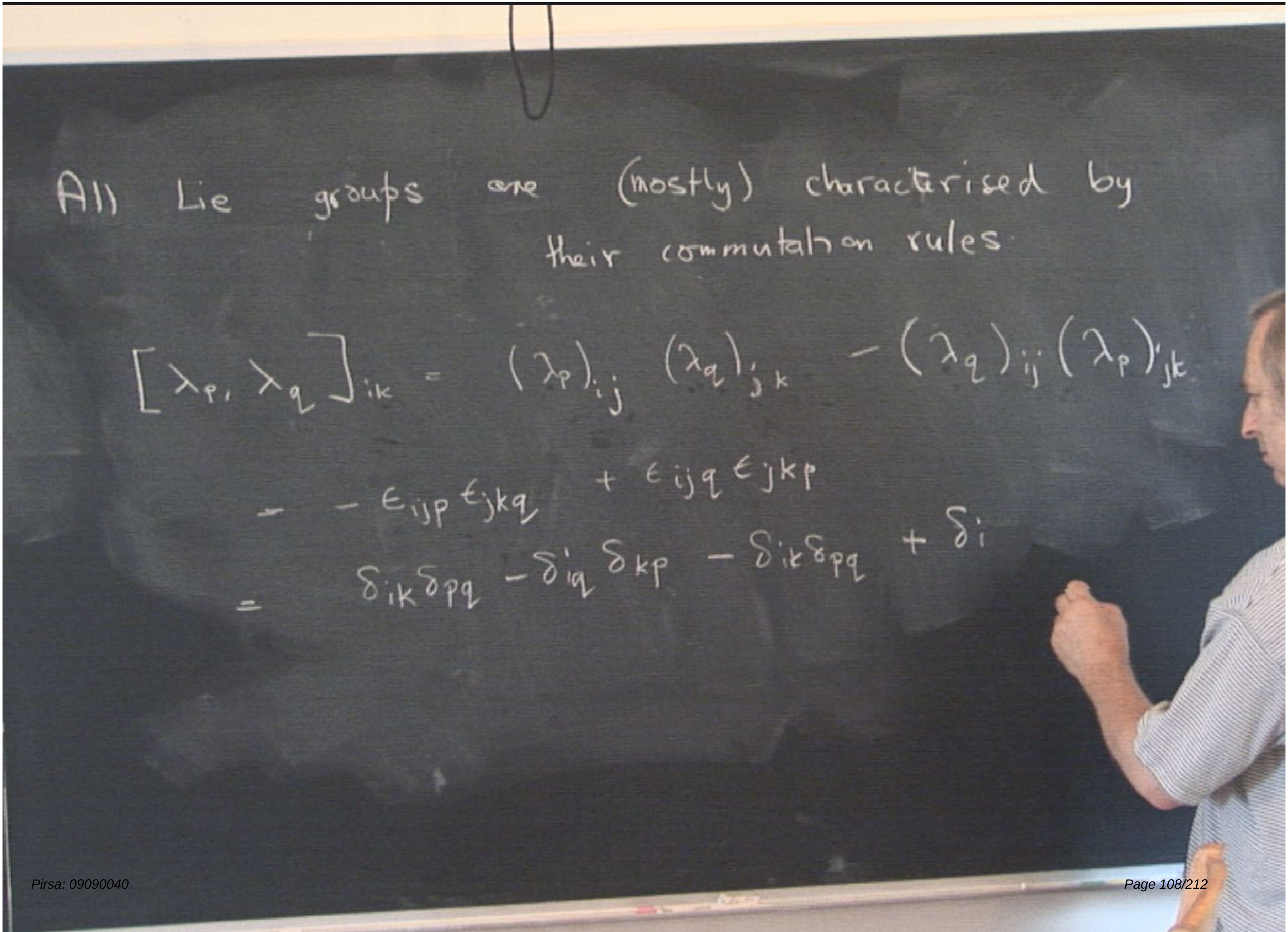
$$[\lambda_p, \lambda_q]_{ik} = (\lambda_p)_{ij} (\lambda_q)_{jk} - (\lambda_q)_{ij} (\lambda_p)_{jk}$$

$$= - \epsilon_{ijp} \epsilon_{jkq} + \epsilon_{ijq} \epsilon_{jkp}$$



Multiplication of 2 ϵ's

$$\epsilon_{abc}\,\epsilon_{ade} = \delta_{bd}\,\delta_{ce} - \delta_{be}\,\delta_{cd}$$



All Lie groups are (mostly) characterised by their commutation rules.

$$[\lambda_p, \lambda_q]_{ik} = (\lambda_p)_{ij} (\lambda_q)_{jk} - (\lambda_q)_{ij} (\lambda_p)_{jk}$$

$$= -\epsilon_{ijp}\epsilon_{jkq} + \epsilon_{ijq}\epsilon_{jkp}$$

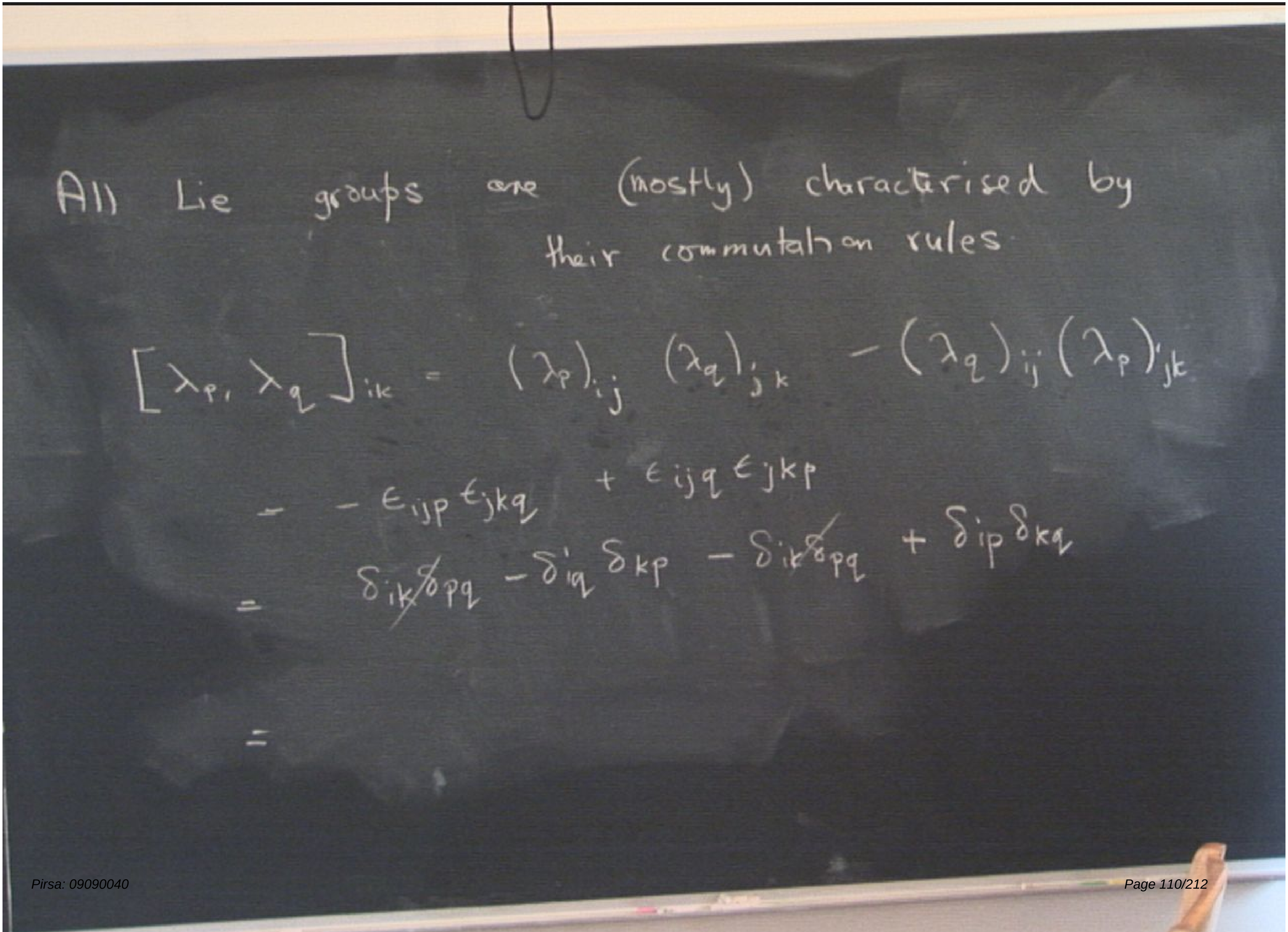
$$= \delta_{ik}\delta_{pq} - \delta_{iq}\delta_{kp} - \delta_{ik}\delta_{pq} + \delta_{i}$$

All Lie groups are (mostly) characterised by their commutation rules.

$$[\lambda_p, \lambda_q]_{ik} = (\lambda_p)_{ij} (\lambda_q)_{jk} - (\lambda_q)_{ij} (\lambda_p)_{jk}$$

$$= - \epsilon_{ijp} \epsilon_{jkq} + \epsilon_{ijq} \epsilon_{jkp}$$

$$= \delta_{ik} \delta_{pq} - \delta_{iq} \delta_{kp} - \delta_{ik} \delta_{pq} + \delta_{ip} \delta_{kq}$$



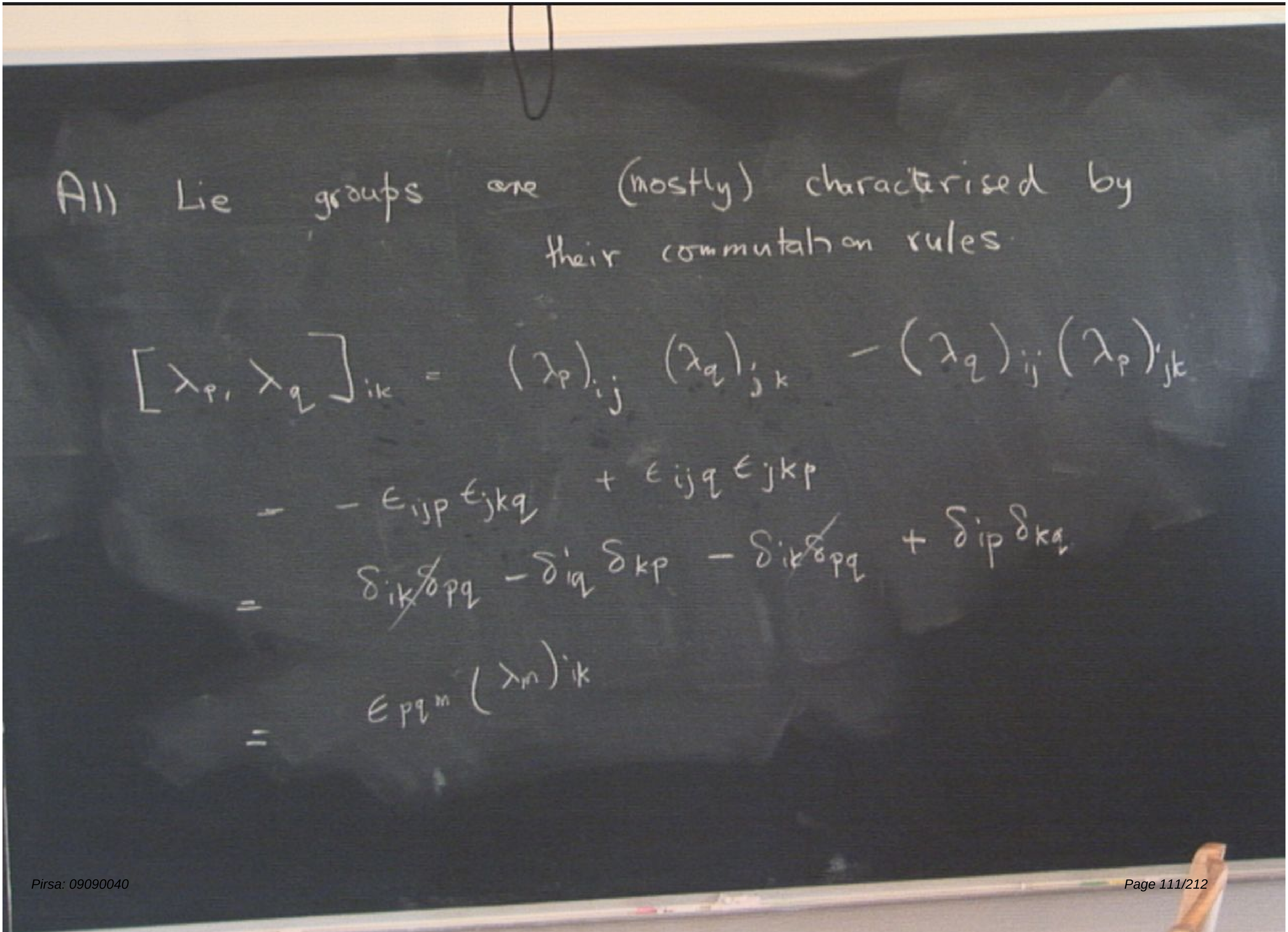
All Lie groups are (mostly) characterised by their commutation rules.

$$[\lambda_p, \lambda_q]_{ik} = (\lambda_p)_{ij} (\lambda_q)_{jk} - (\lambda_q)_{ij} (\lambda_p)_{jk}$$

$$= - \epsilon_{ijp} \epsilon_{jkq} + \epsilon_{ijq} \epsilon_{jkp}$$

$$= \cancel{\delta_{ik} \delta_{pq}} - \delta_{iq} \delta_{kp} - \cancel{\delta_{ik} \delta_{pq}} + \delta_{ip} \delta_{kq}$$

$$=$$



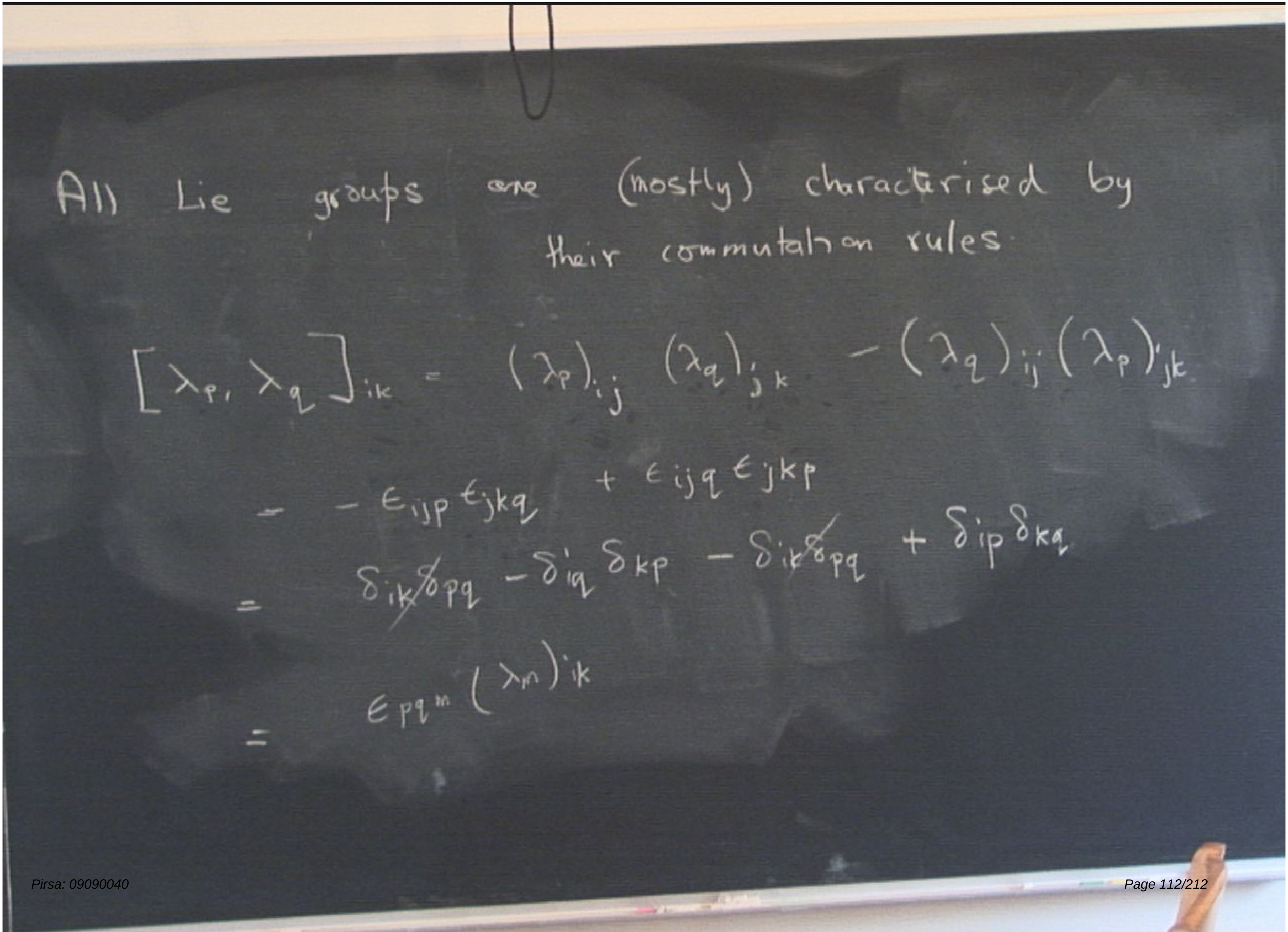
All Lie groups are (mostly) characterised by their commutation rules.

$$[\lambda_p, \lambda_q]_{ik} = (\lambda_p)_{ij} (\lambda_q)_{jk} - (\lambda_q)_{ij} (\lambda_p)_{jk}$$

$$= -\epsilon_{ijp}\epsilon_{jkq} + \epsilon_{ijq}\epsilon_{jkp}$$

$$= \delta_{i\cancel{k}}\cancel{\delta}_{pq} - \delta_{iq}\delta_{kp} - \delta_{i\cancel{k}}\cancel{\delta}_{pq} + \delta_{ip}\delta_{kq}$$

$$= \epsilon_{pq}{}^{m} (\lambda_m)_{ik}$$



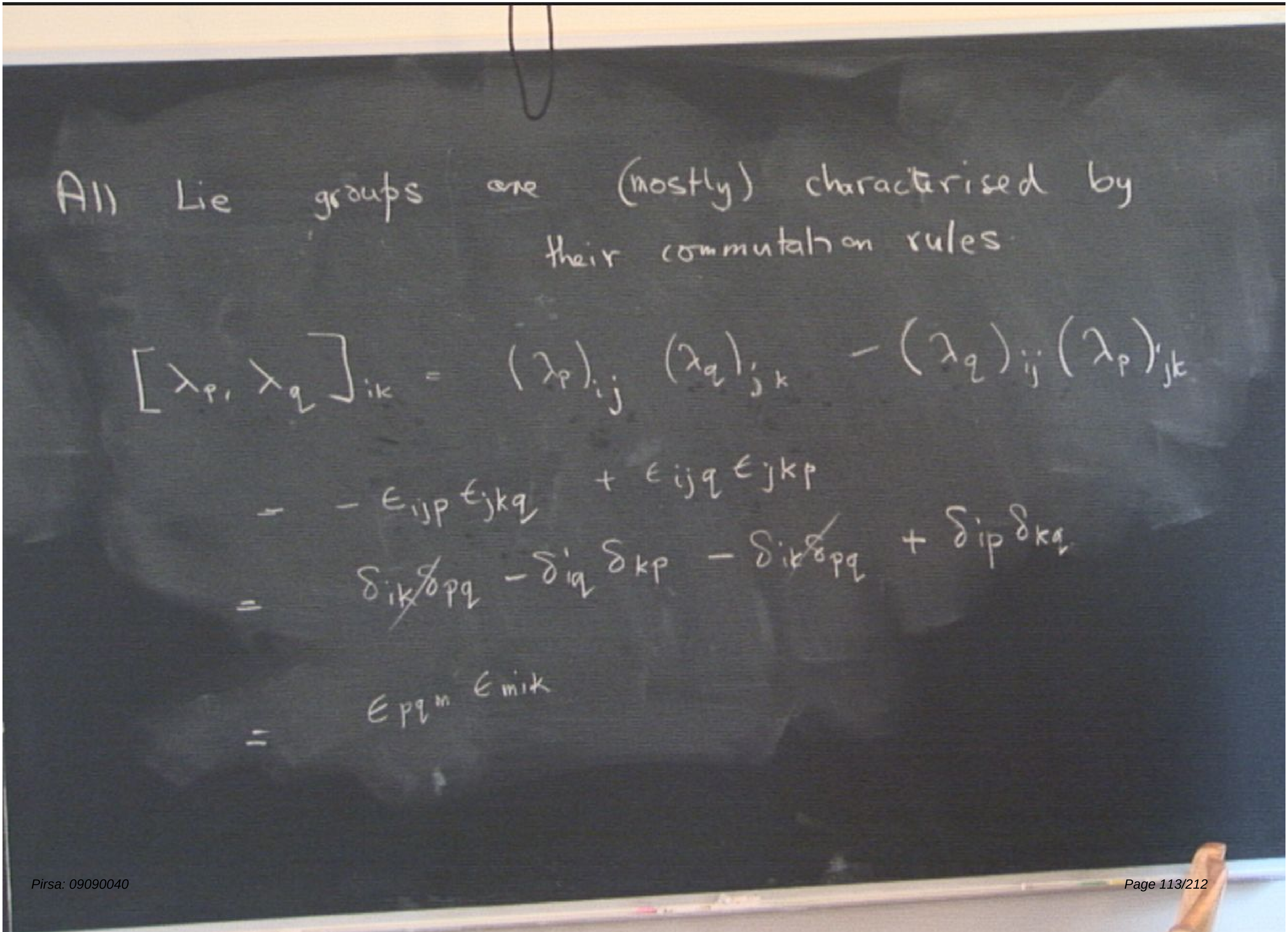
All Lie groups are (mostly) characterised by their commutation rules.

$$[\lambda_p, \lambda_q]_{ik} = (\lambda_p)_{ij} (\lambda_q)_{jk} - (\lambda_q)_{ij} (\lambda_p)_{jk}$$

$$= -\epsilon_{ijp}\epsilon_{jkq} + \epsilon_{ijq}\epsilon_{jkp}$$

$$= \cancel{\delta_{ik}\delta_{pq}} - \delta_{iq}\delta_{kp} - \cancel{\delta_{ik}\delta_{pq}} + \delta_{ip}\delta_{kq}$$

$$= \epsilon_{pqm} (\lambda_m)_{ik}$$



All Lie groups are (mostly) characterised by their commutation rules.

$$[\lambda_p, \lambda_q]_{ik} = (\lambda_p)_{ij}(\lambda_q)_{jk} - (\lambda_q)_{ij}(\lambda_p)_{jk}$$

$$= -\epsilon_{ijp}\epsilon_{jkq} + \epsilon_{ijq}\epsilon_{jkp}$$

$$= \cancel{\delta_{ik}\delta_{pq}} - \delta_{iq}\delta_{kp} - \cancel{\delta_{ik}\delta_{pq}} + \delta_{ip}\delta_{kq}$$

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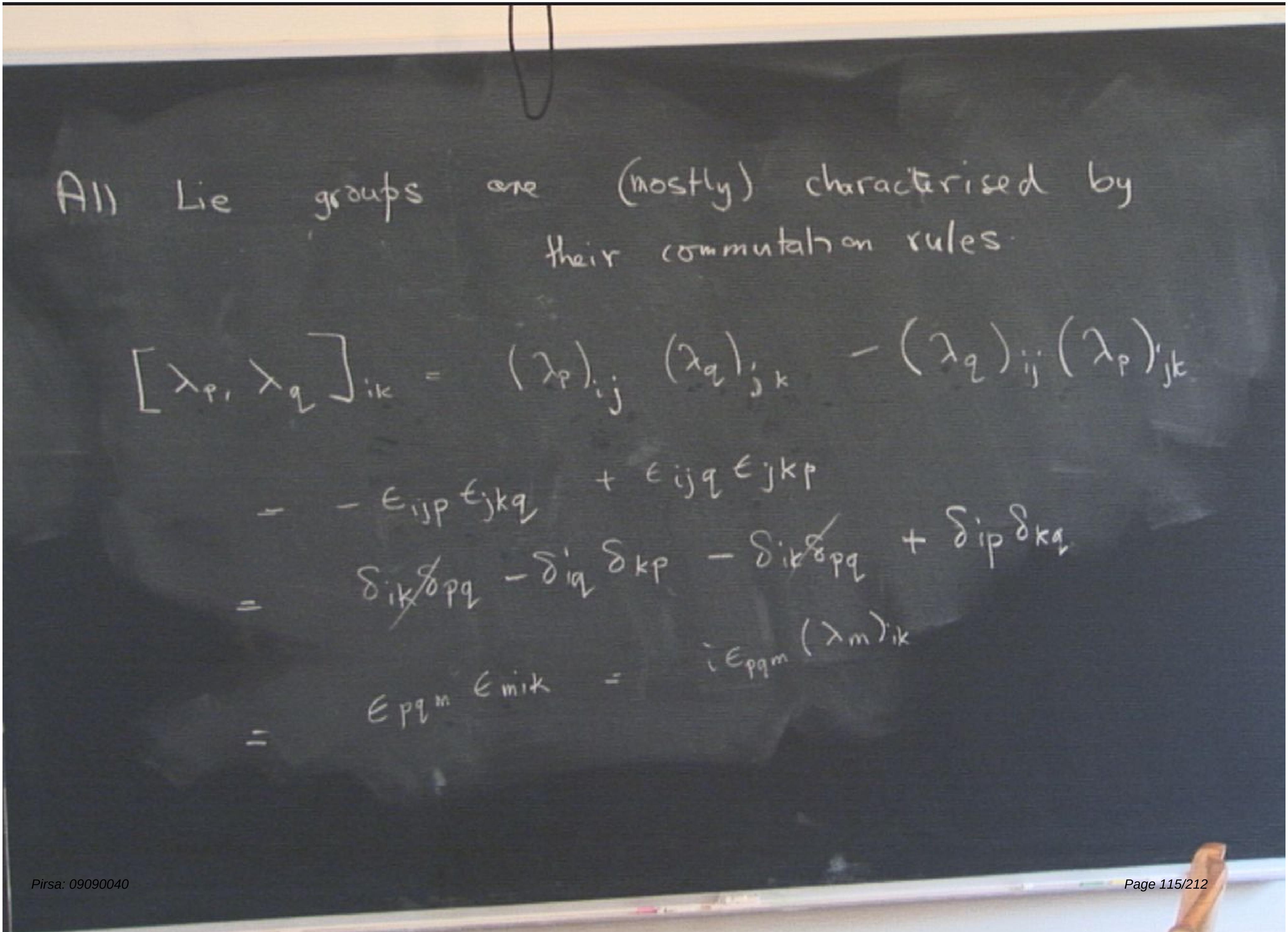

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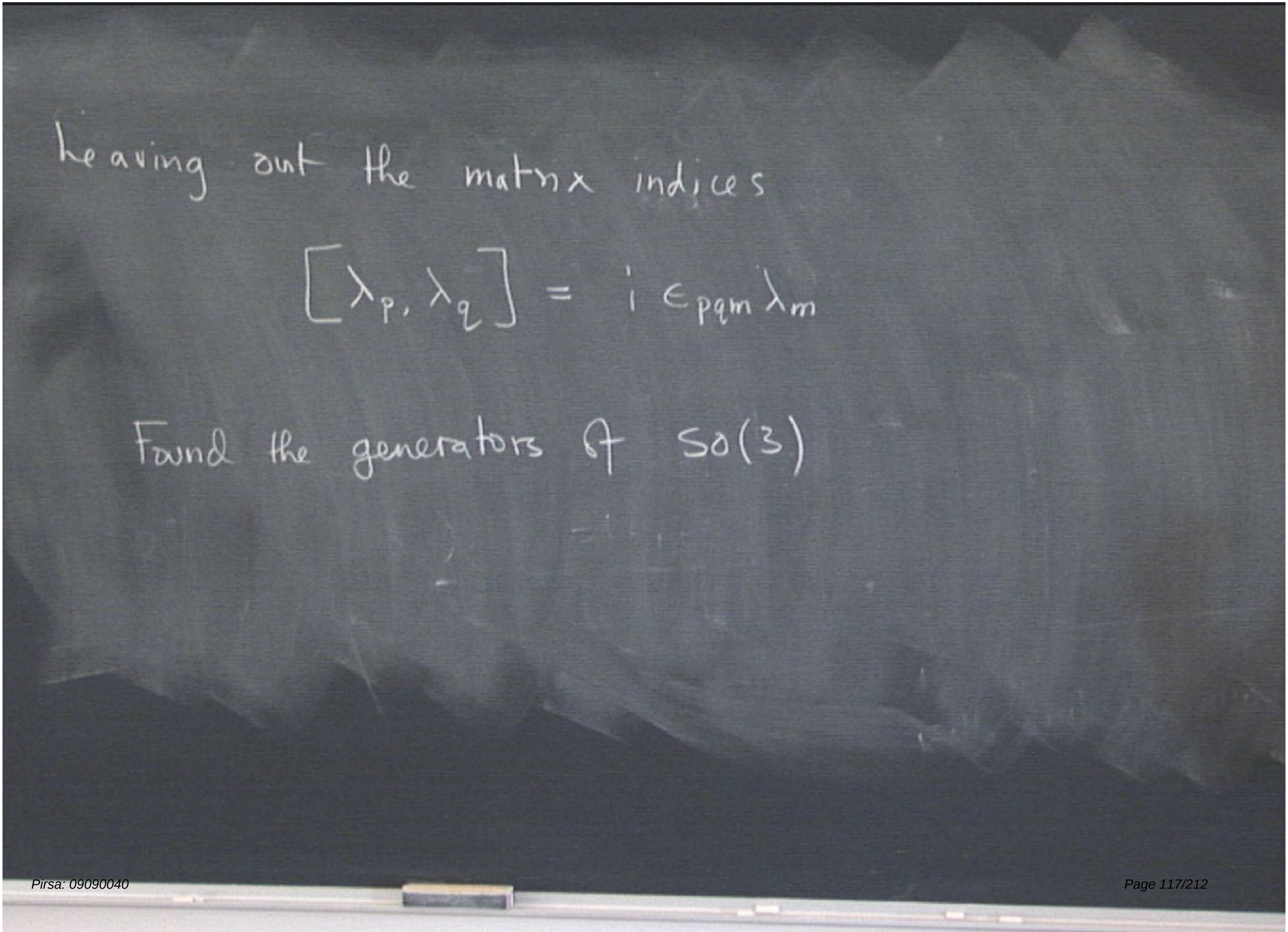
$$=$$

Leaving out the matrix indices

$$[\lambda_p, \lambda_q] = i\,\epsilon_{pqm}\,\lambda_m$$



Leaving out the matrix indices

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Found the generators of so(3)

Leaving out the matrix indices

$$[\lambda_p, \lambda_q] = i \epsilon_{pqm} \lambda_m$$

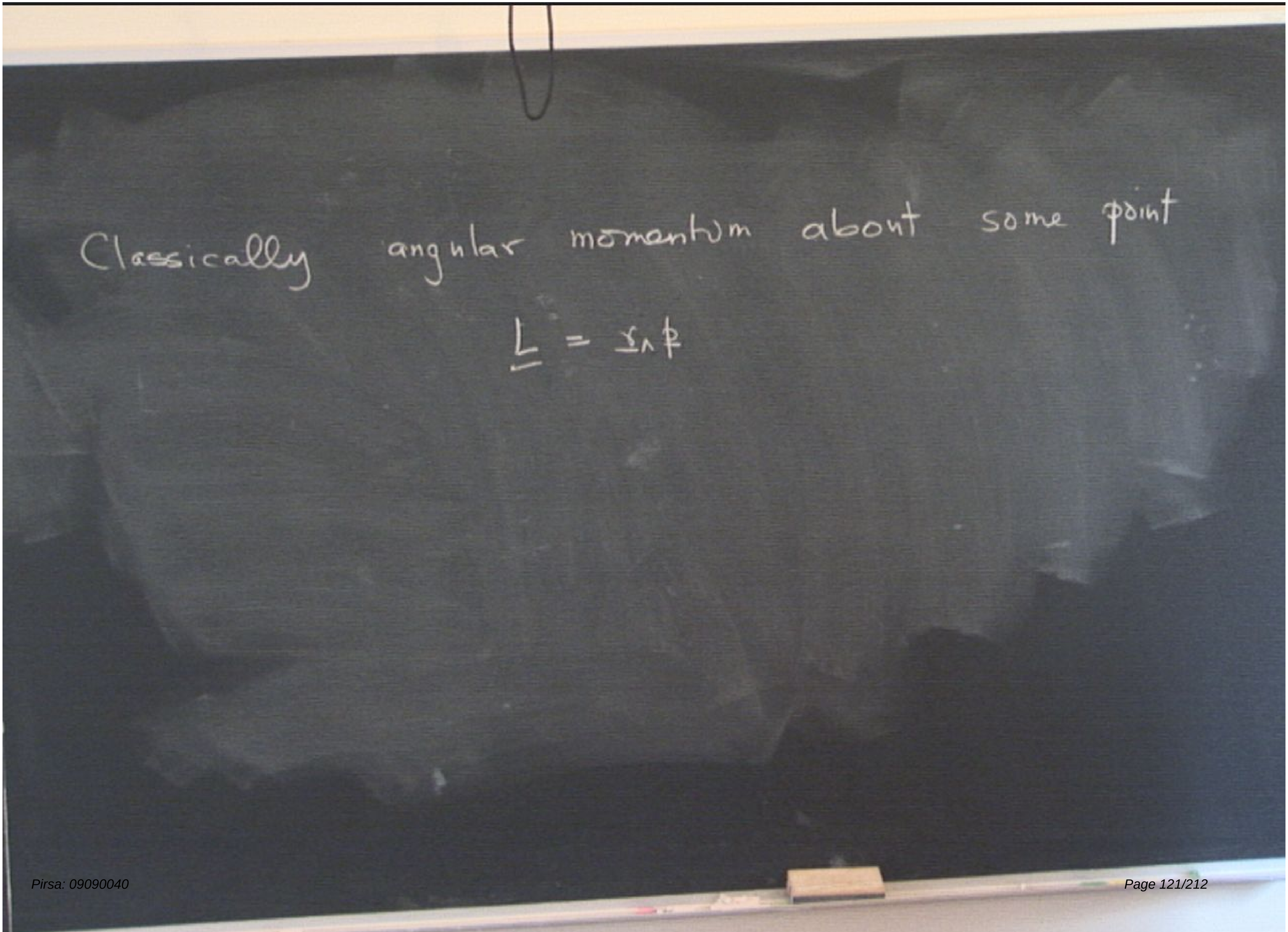
Found the generators of so(3) in the vector representation.

Leaving out the matrix indices

$$[\lambda_p, \lambda_q] = i \epsilon_{pqm} \lambda_m$$

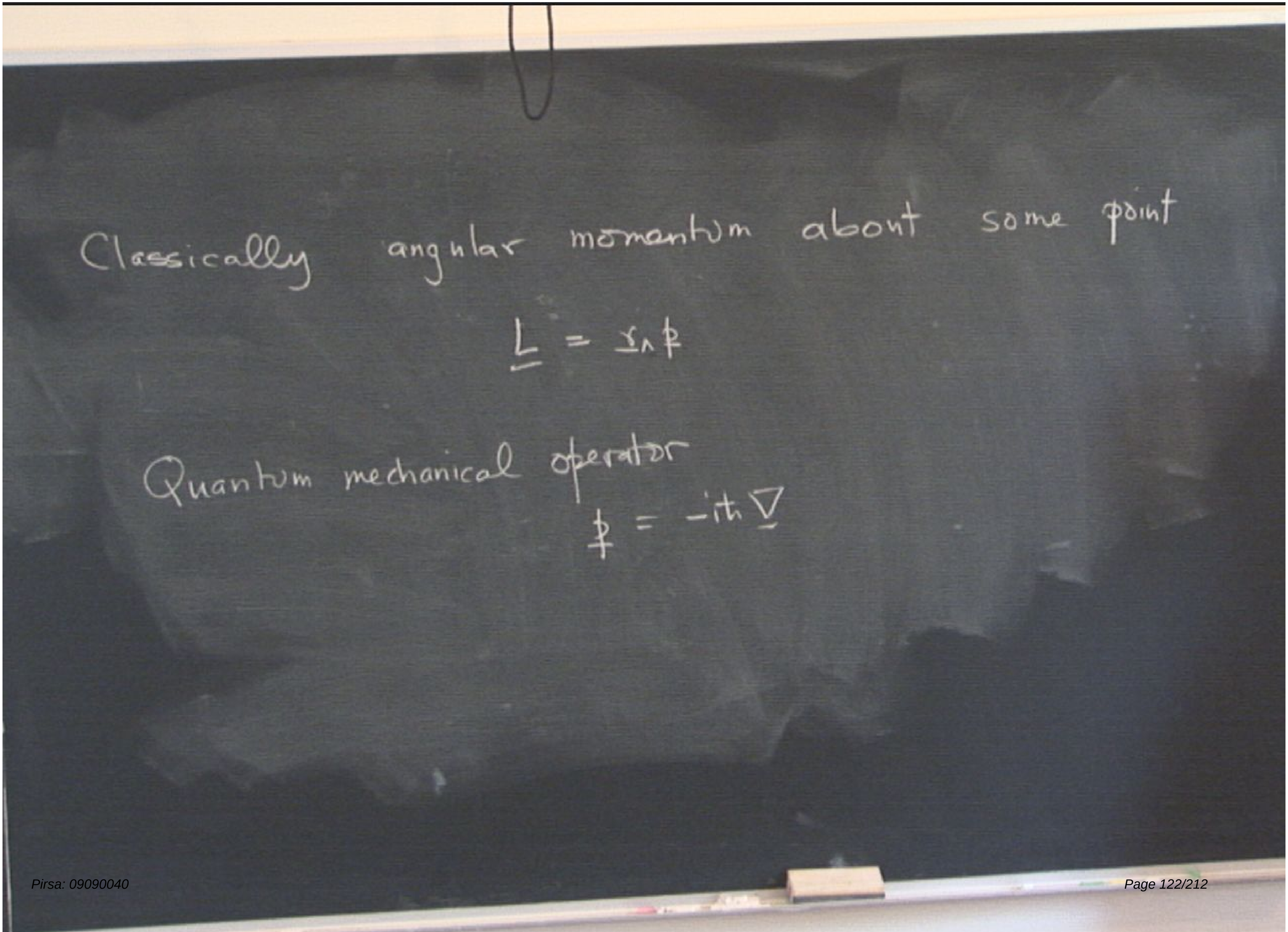
Found the generators of $SO(3)$ in the vector representation.

Classically angular momentum about some point



Classically angular momentum about some point

$$\underline{L} = \underline{r} \wedge \underline{p}$$

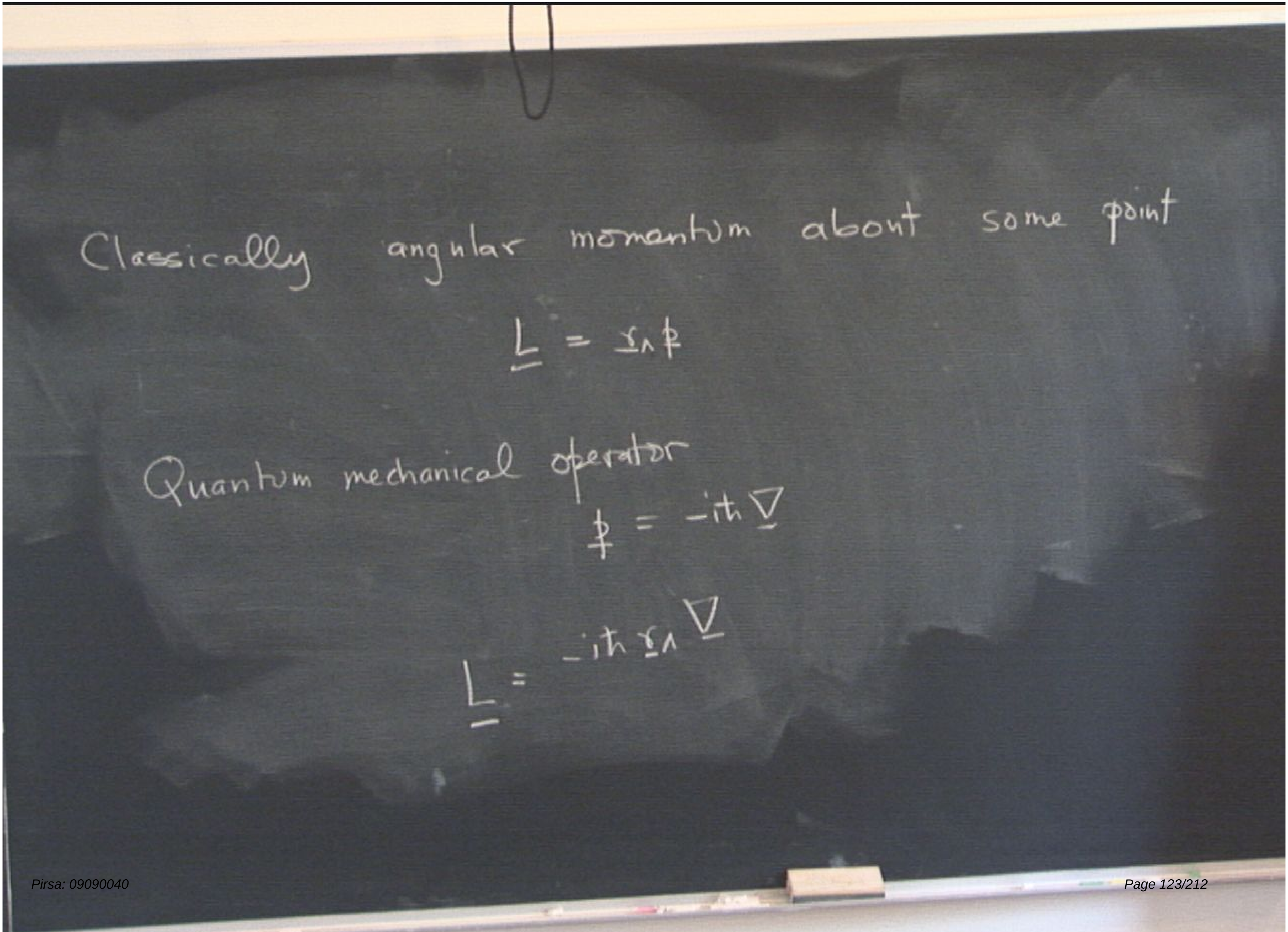


Classically angular momentum about some point

$$\underline{L} = \underline{r} \wedge \underline{p}$$

Quantum mechanical operator

$$\underline{p} = -i\hbar \underline{\nabla}$$



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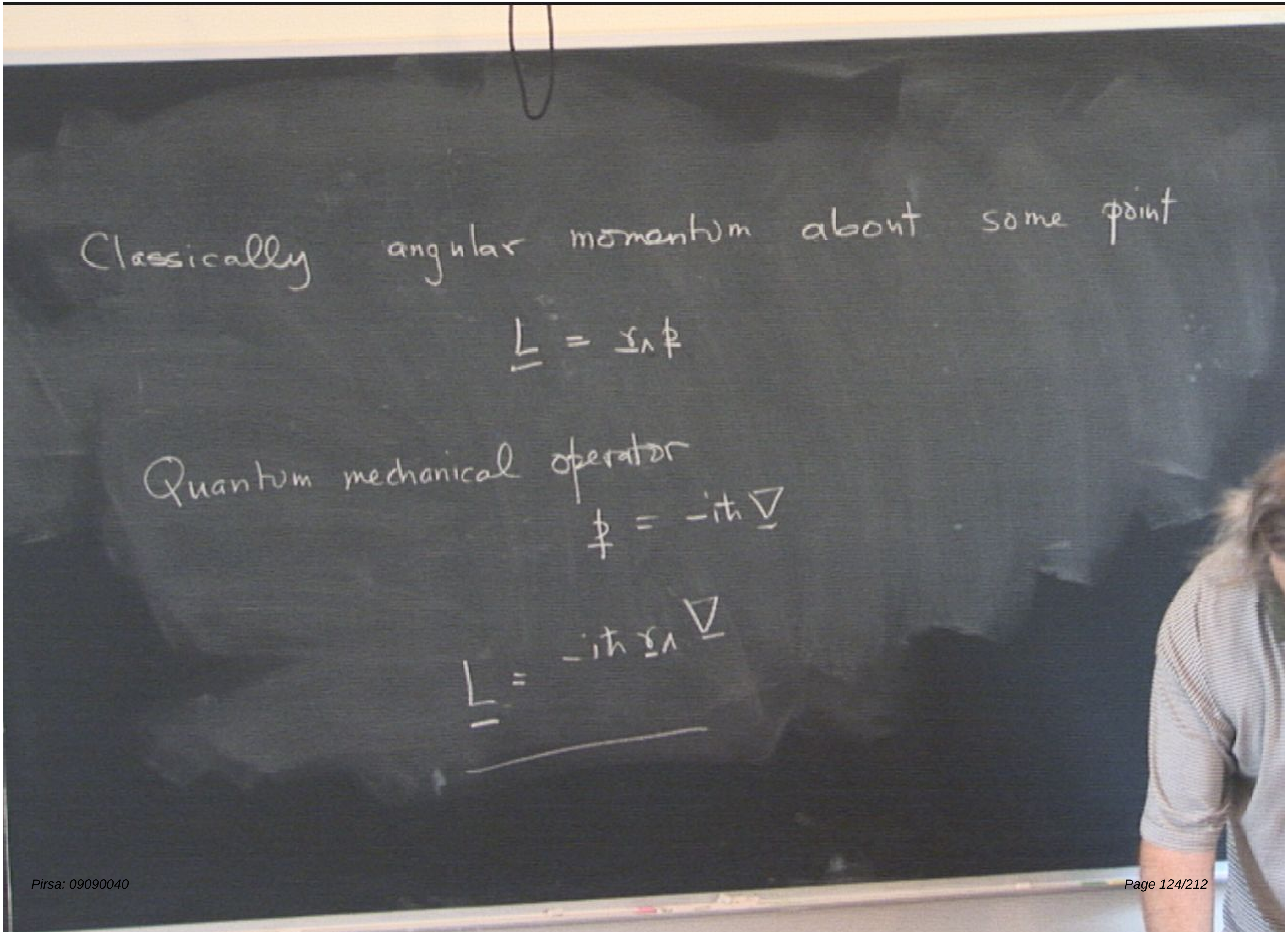

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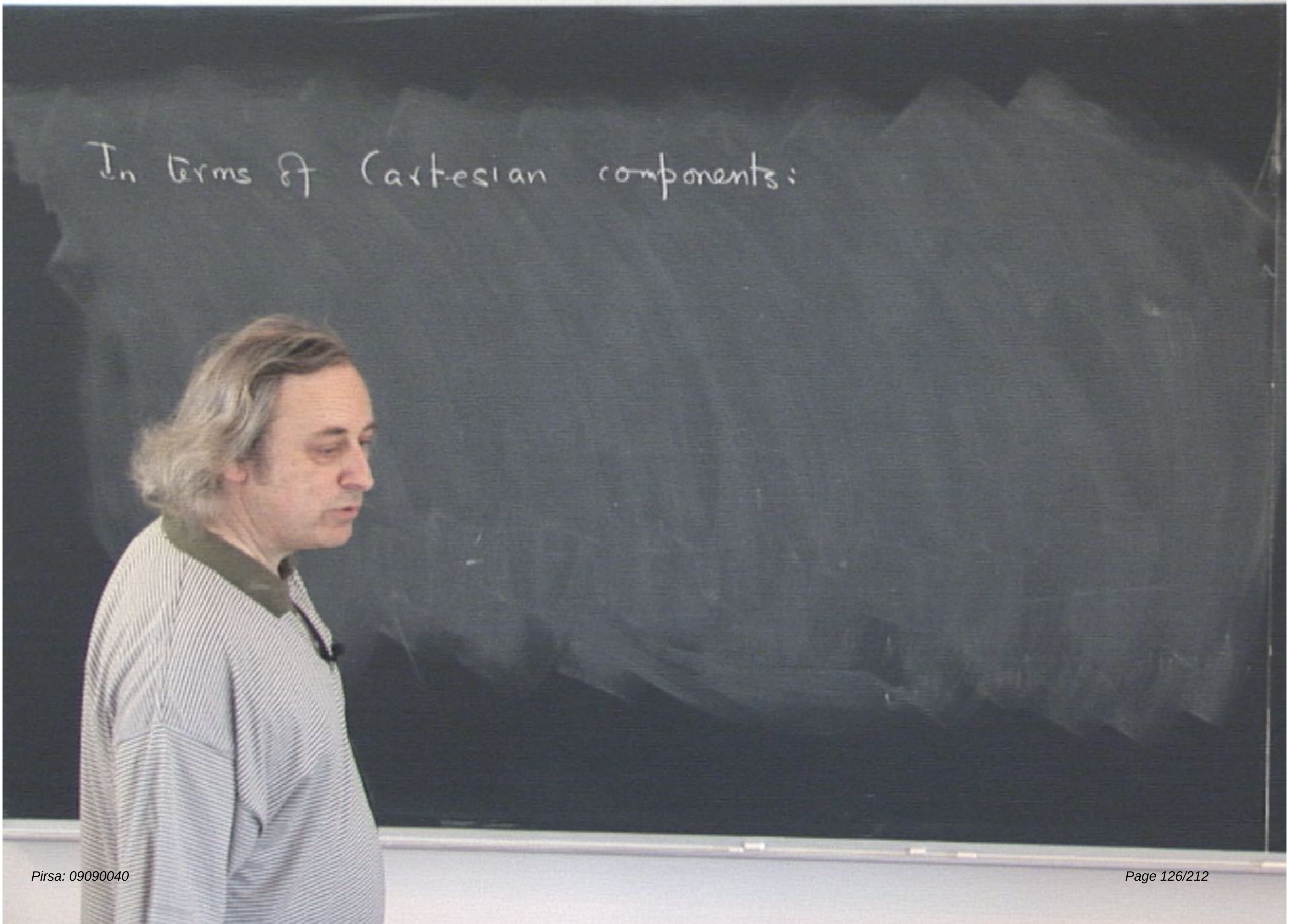

In terms of Cartesian components:

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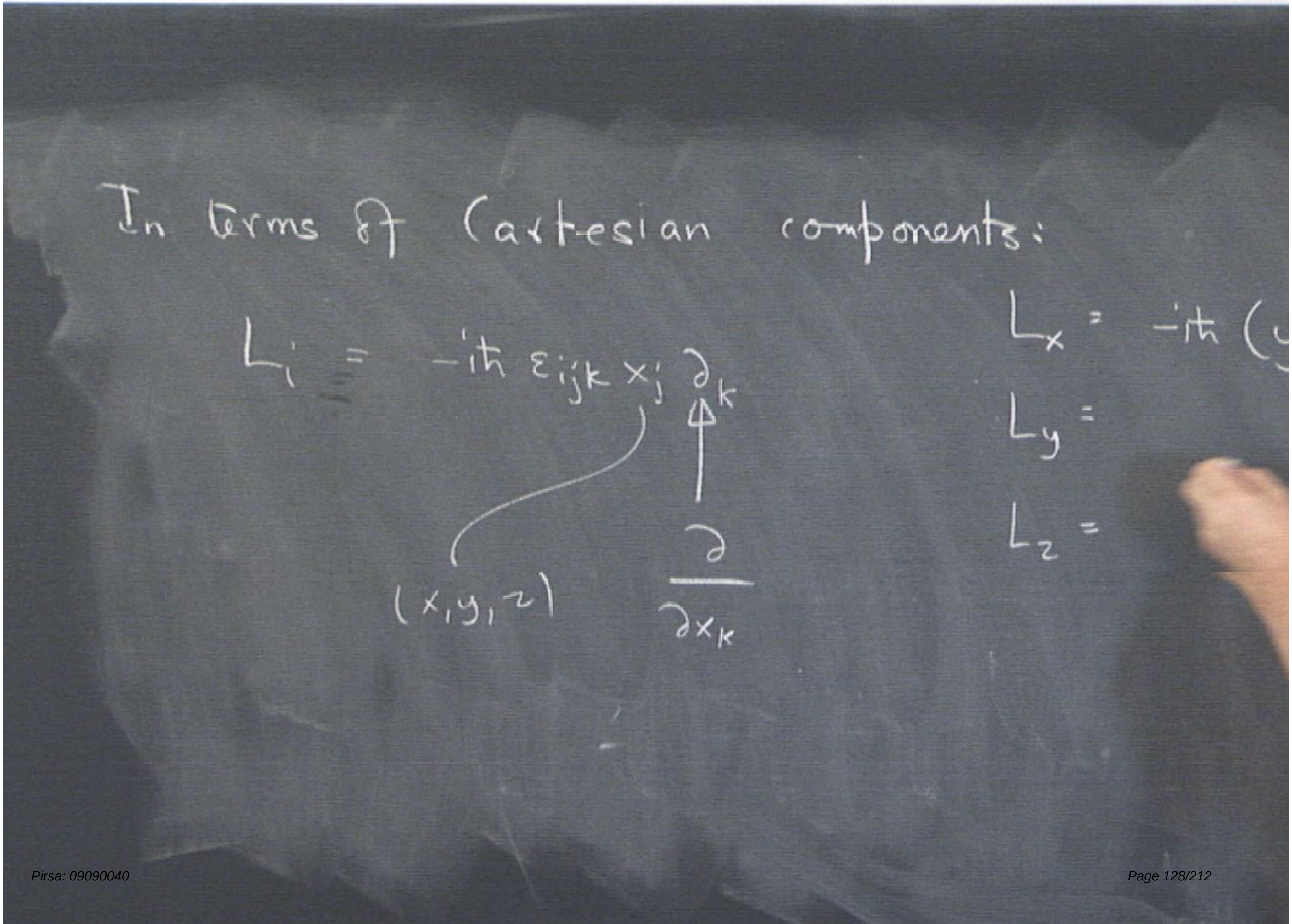
$$L_i = -i\hbar\, \varepsilon_{ijk}\, x_j\, \underset{\underset{\displaystyle\frac{\partial}{\partial x_k}}{\Big\uparrow}}{\partial_k}$$



In terms of Cartesian components:

$$L_i = -i\hbar\, \varepsilon_{ijk}\, \underset{(x,y,z)}{x_j}\, \underset{\frac{\partial}{\partial x_k}}{\partial_k}$$

$$L_x = -i\hbar\,(y$$

$$L_y =$$

$$L_z =$$

In terms of Cartesian components:

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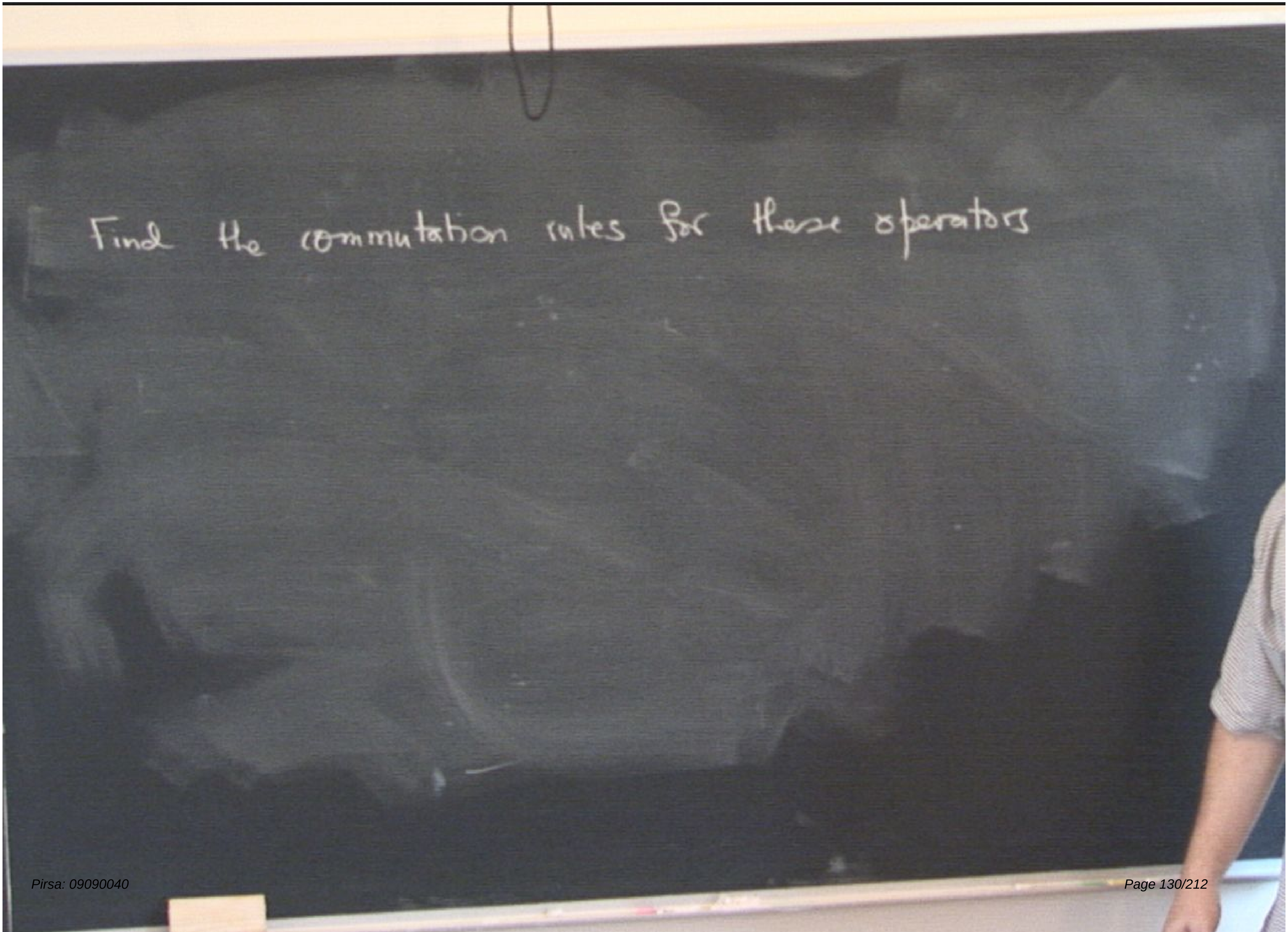
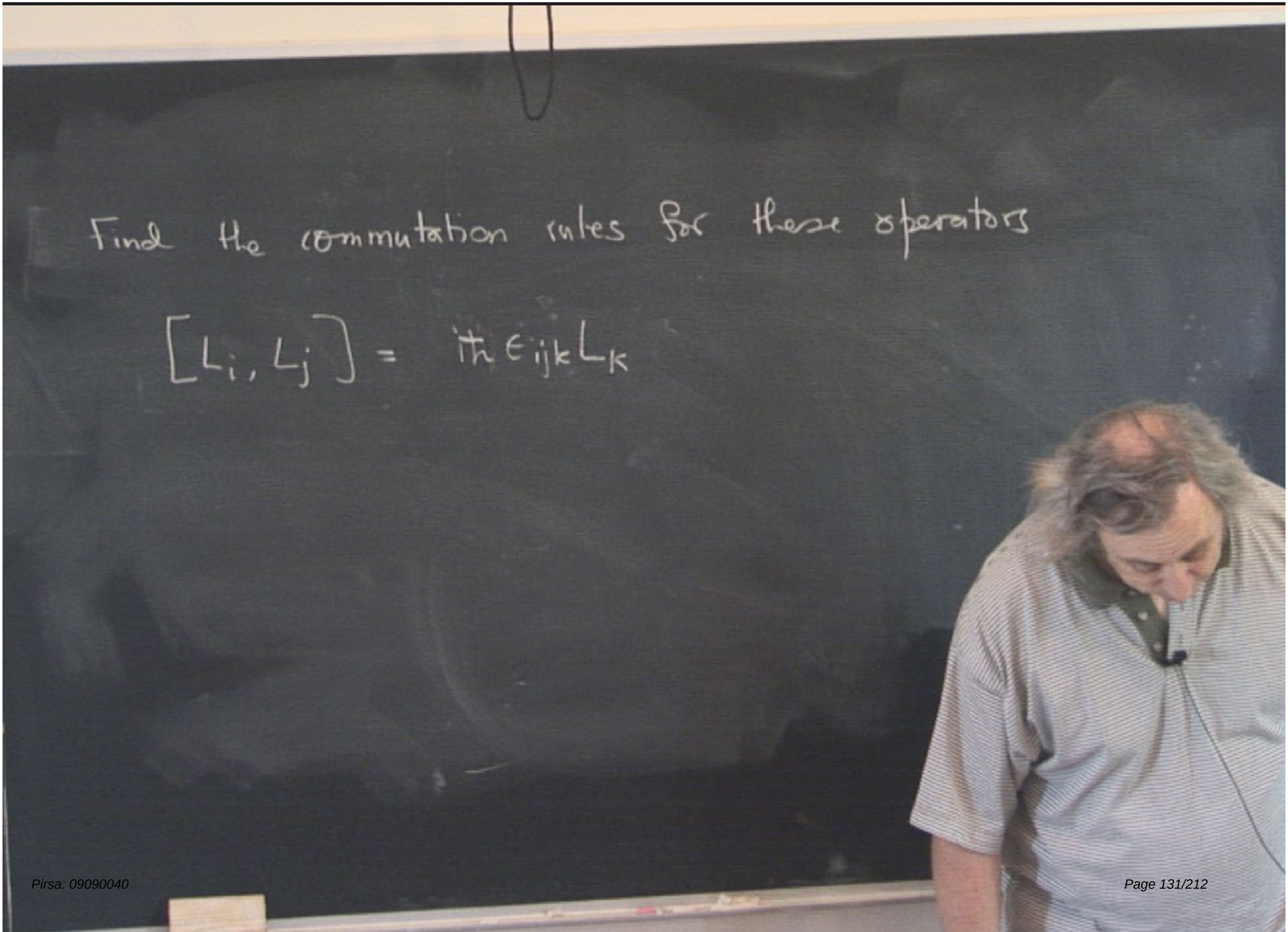
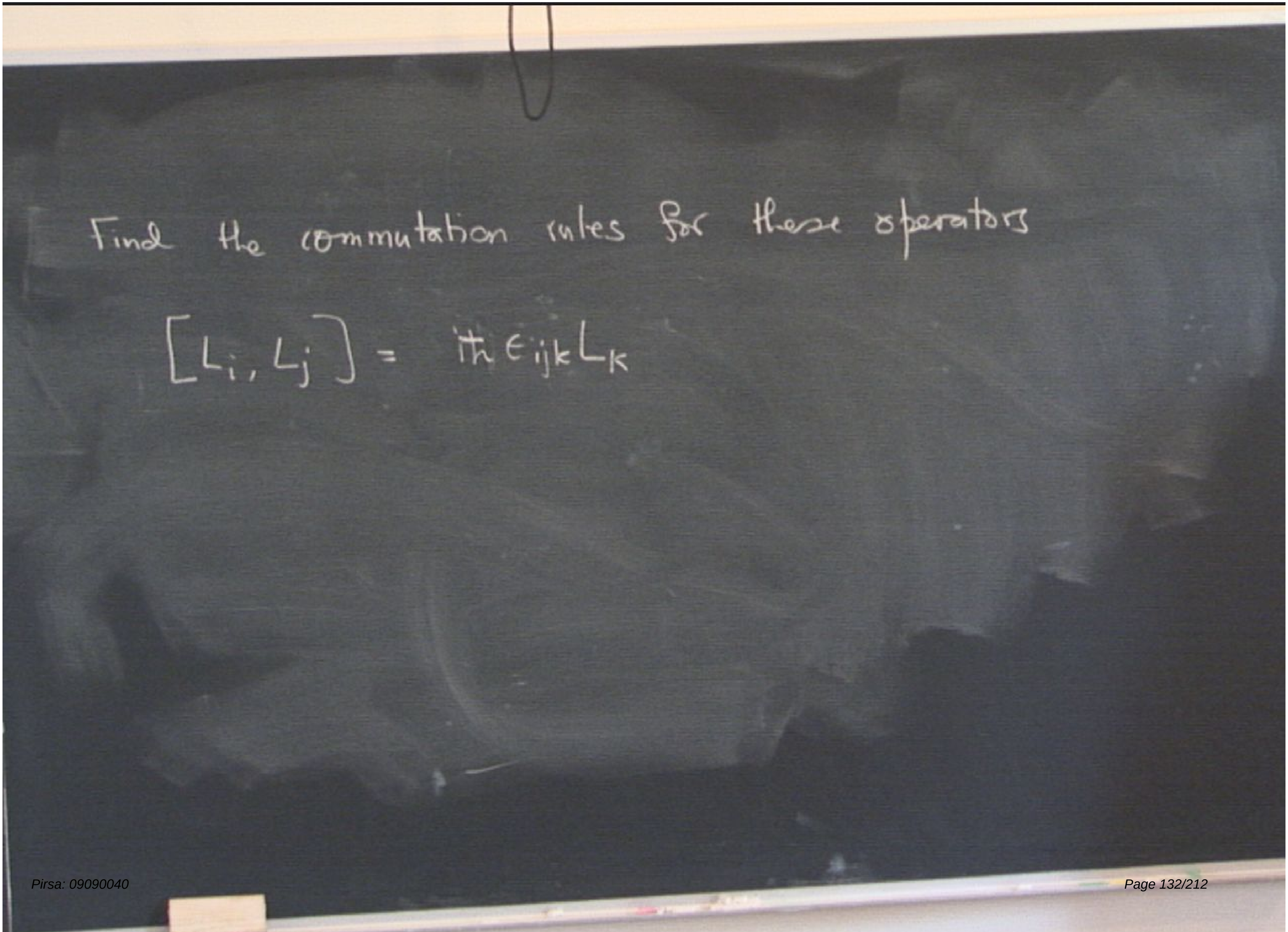


Find the commutation rules for these operators



Find the commutation rules for these operators

$$[L_i, L_j] = i\hbar\,\epsilon_{ijk} L_k$$



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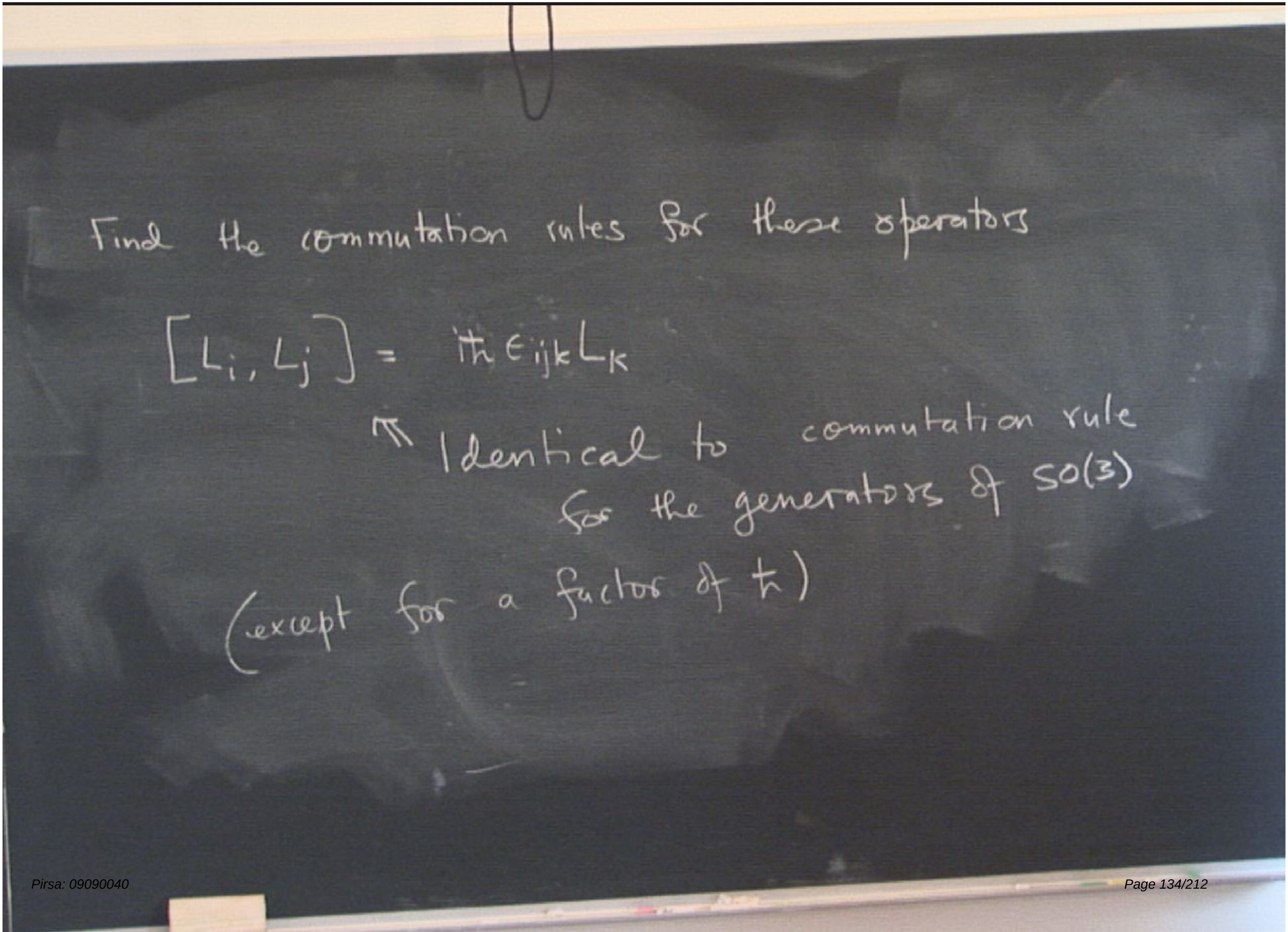
⇖ Identical to commutation rule for the generators of SO(3)



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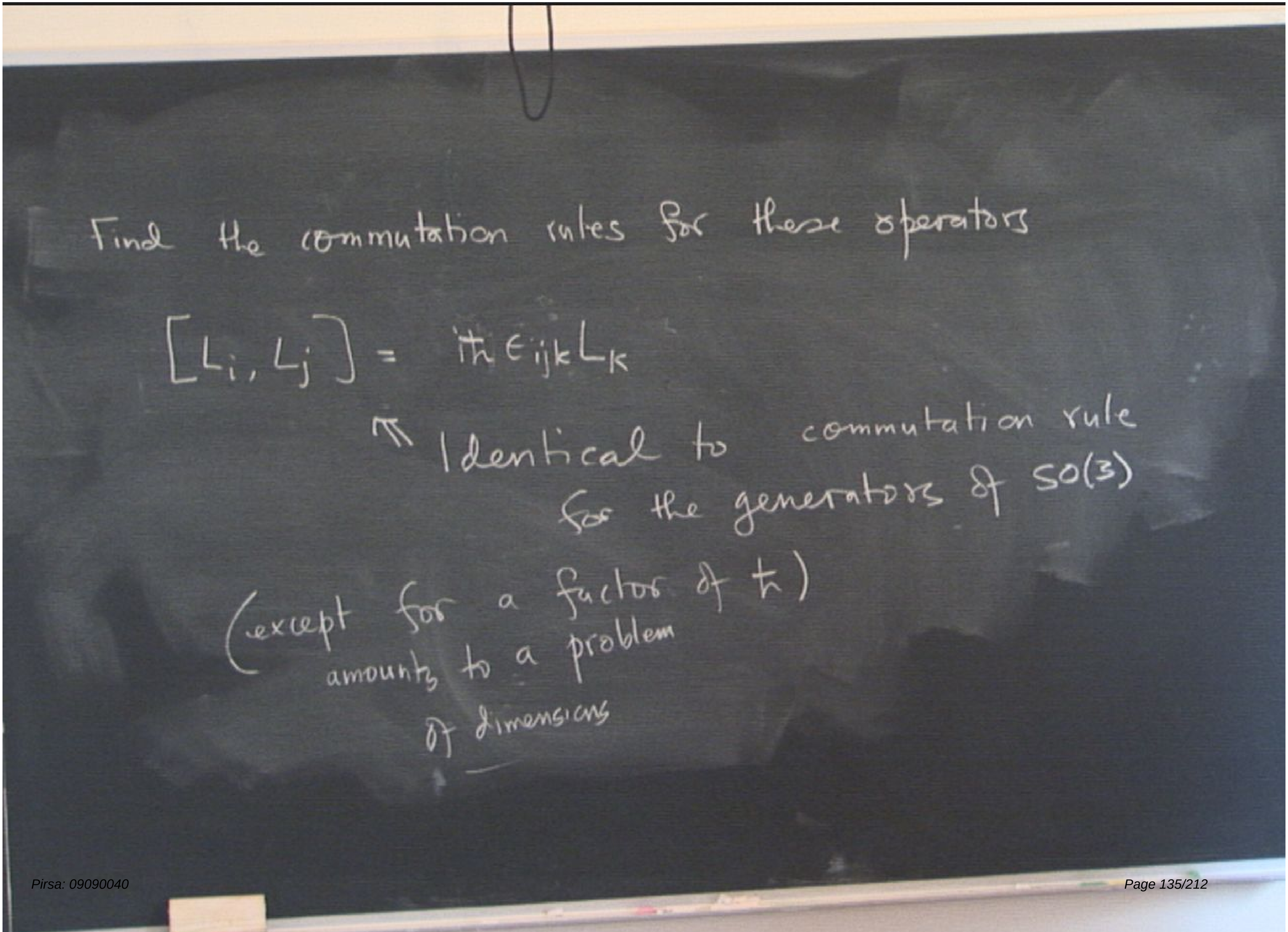
(except for a factor of $\hbar$)



Find the commutation rules for these operators

$$[L_i, L_j] = i\hbar \epsilon_{ijk} L_k$$

⇈ Identical to commutation rule for the generators of $SO(3)$

(except for a factor of $\hbar$)
amounts to a problem of dimensions

In terms of Cartesian components:

$$L_i = -i\hbar\, \varepsilon_{ijk}\, \underbrace{x_j}_{(x,y,z)}\, \underbrace{\partial_k}_{\frac{\partial}{\partial x_k}}$$

$$L_x = -i\hbar\,(y\partial_z - z\partial_y)$$

$$L_y = -i\hbar\,(z\partial_x - x\partial_z)$$

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Representation of so(3)
on the space of all

In terms of Cartesian components:

$$L_i = -i\hbar\, \varepsilon_{ijk}\, \underset{(x,y,z)}{\underbrace{x_j}}\ \underset{\overset{\uparrow}{\frac{\partial}{\partial x_k}}}{\partial_k}$$

$$L_x = -i\hbar\,(y\partial_z - z\partial_y)$$

$$L_y = -i\hbar\,(z\partial_x - x\partial_z)$$

$$L_z = -i\hbar\,(x\partial_y - y\partial_x)$$

Representation of so(3)
on the space of all

differentiable

In terms of Cartesian components:

$$L_i = -i\hbar\, \varepsilon_{ijk}\, \underbrace{x_j}_{(x,y,z)}\, \underbrace{\partial_k}_{\frac{\partial}{\partial x_k}}$$

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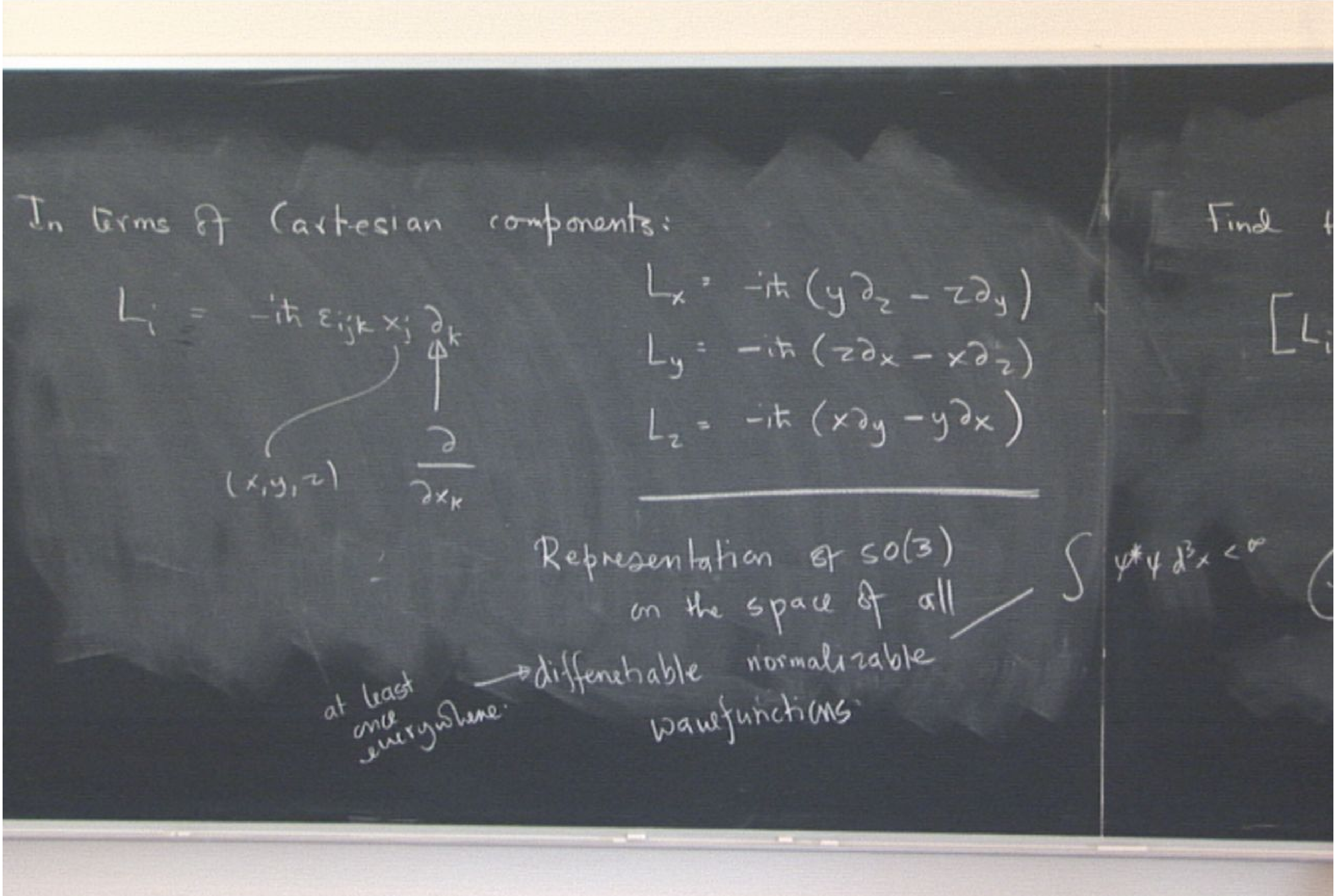
Representation of so(3) on the space of all differentiable normalizable wavefunctions



In terms of Cartesian components:

$$L_i = -i\hbar\, \varepsilon_{ijk}\, \underbrace{x_j}_{(x,y,z)}\, \underbrace{\partial_k}_{\frac{\partial}{\partial x_k}}$$

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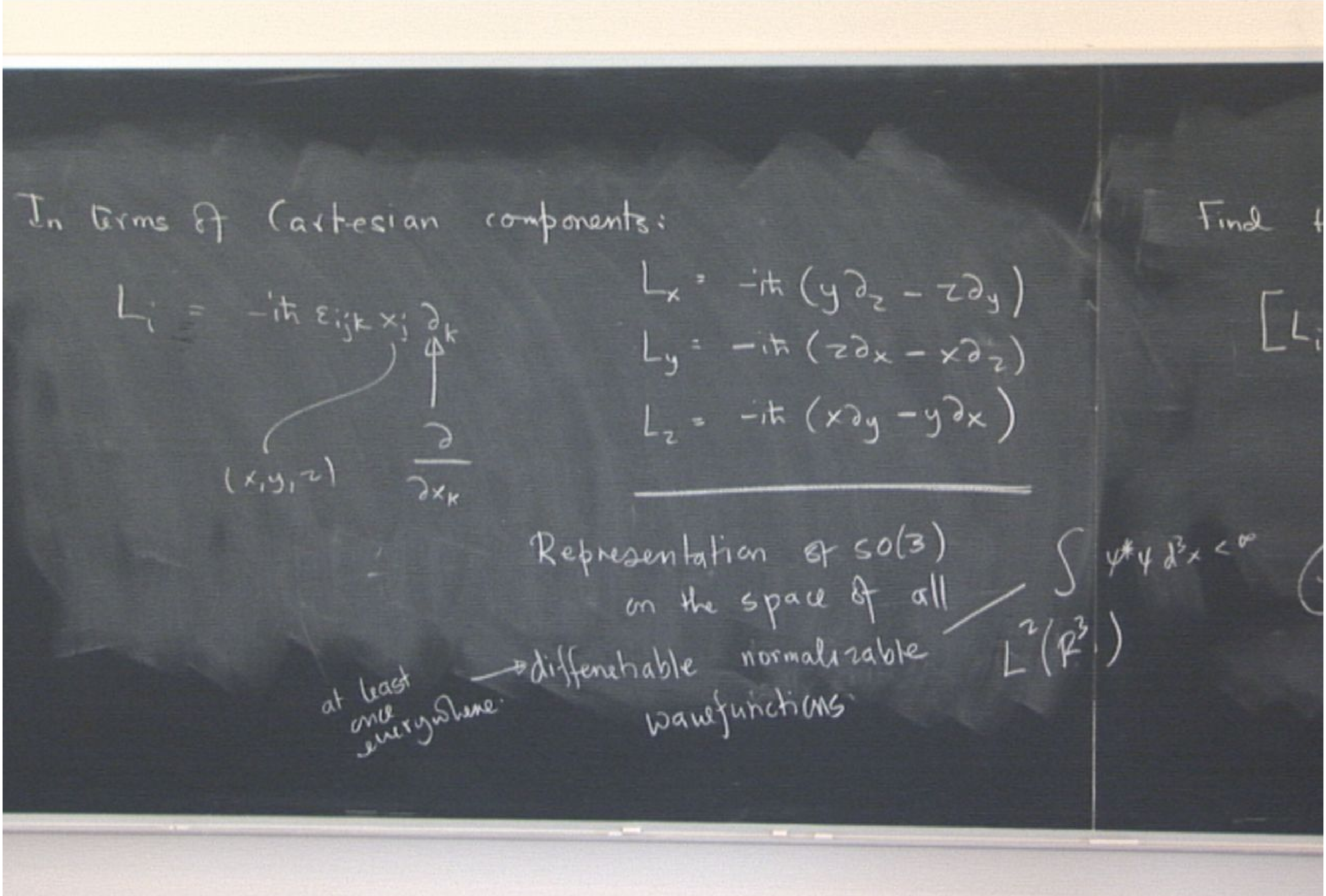
Representation of so(3)
on the space of all
differentiable normalizable
wavefunctions

at least once everywhere → differentiable

normalizable → $\int \psi^*\psi\, d^3x < \infty$

Find

$[L_i$



In terms of Cartesian components:

$$L_i = -i\hbar\, \varepsilon_{ijk}\, x_j\, \partial_k$$

x_j: (x, y, z)

∂_k: $\frac{\partial}{\partial x_k}$

$$L_x = -i\hbar\,(y\partial_z - z\partial_y)$$
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Representation of so(3)
on the space of all
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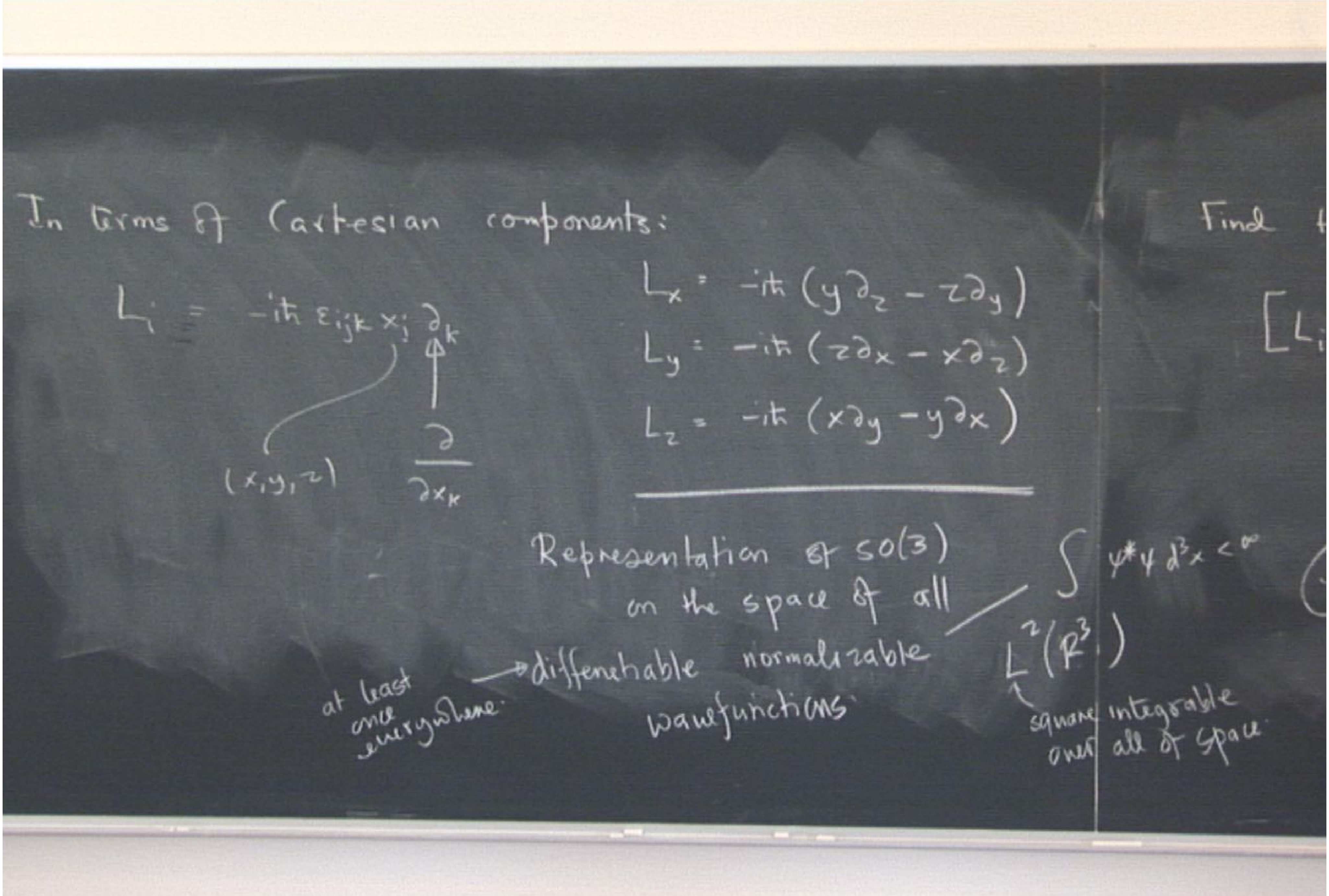
at least once everywhere → differentiable

$\int \psi^* \psi \, d^3x < \infty$

$L^2(\mathbb{R}^3)$

Find t

$[L_i$



In terms of Cartesian components:

$$L_i = -i\hbar\, \varepsilon_{ijk}\, x_j\, \partial_k$$

(x, y, z) $\qquad$ $\dfrac{\partial}{\partial x_k}$

$$L_x = -i\hbar\,(y\partial_z - z\partial_y)$$

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Representation of so(3) on the space of all differentiable normalizable wavefunctions

at least once everywhere

$\int \psi^* \psi \, d^3x < \infty$

$L^2(\mathbb{R}^3)$

square integrable over all of space

Find

$[L_i$

Can also think of these in spherical co-ordinates

$$x = r \sin\theta \cos\phi$$

$$y = r \sin\theta \sin\phi$$

$$z = r \cos\theta$$

Can also think of these in spherical co-ordinates

$$x = r\sin\theta\cos\phi$$

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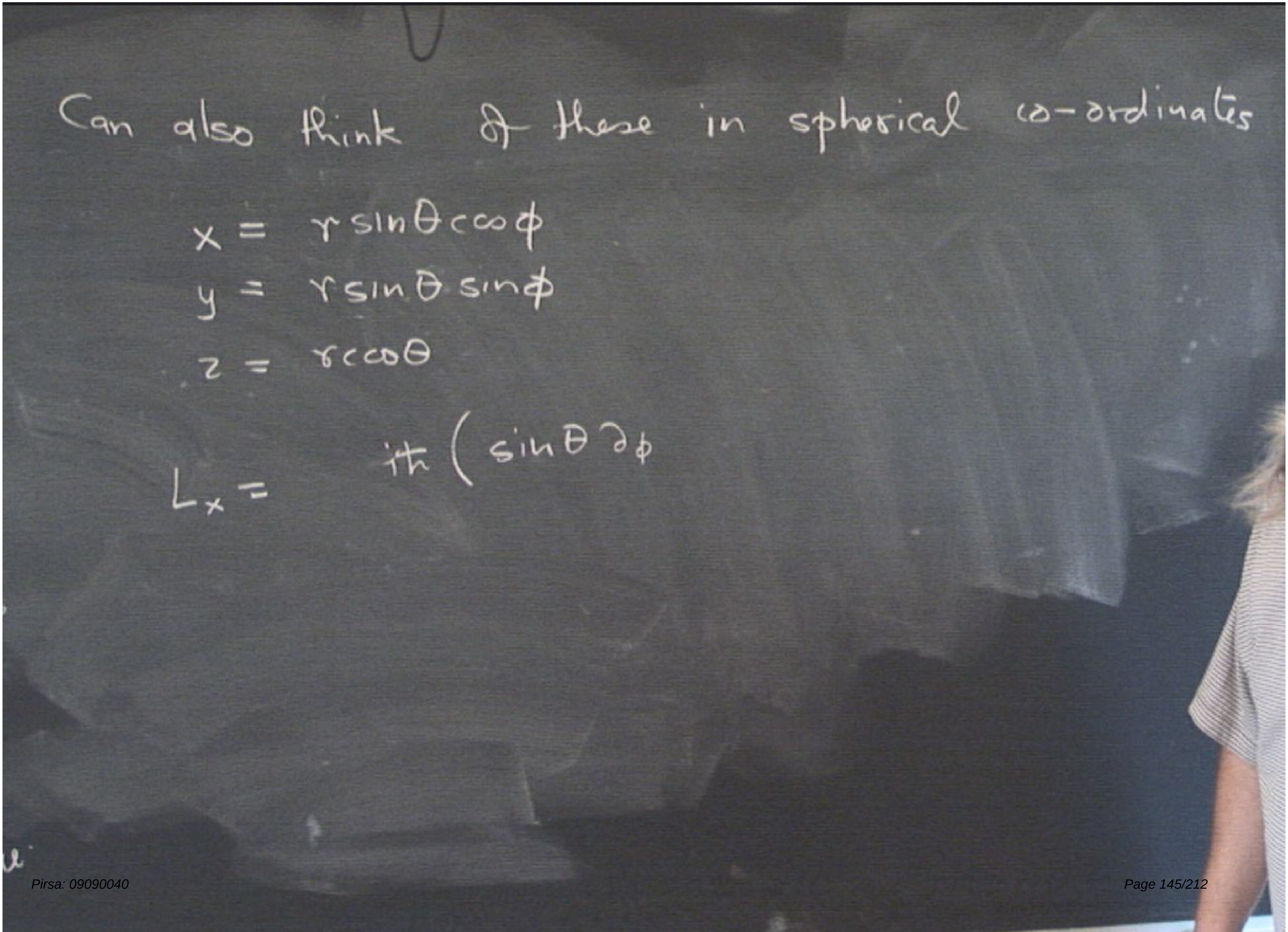
$$z = r\cos\theta$$

$$L_x =$$



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$$L_x = i\hbar\left(\sin\theta\,\partial_\phi\right.$$

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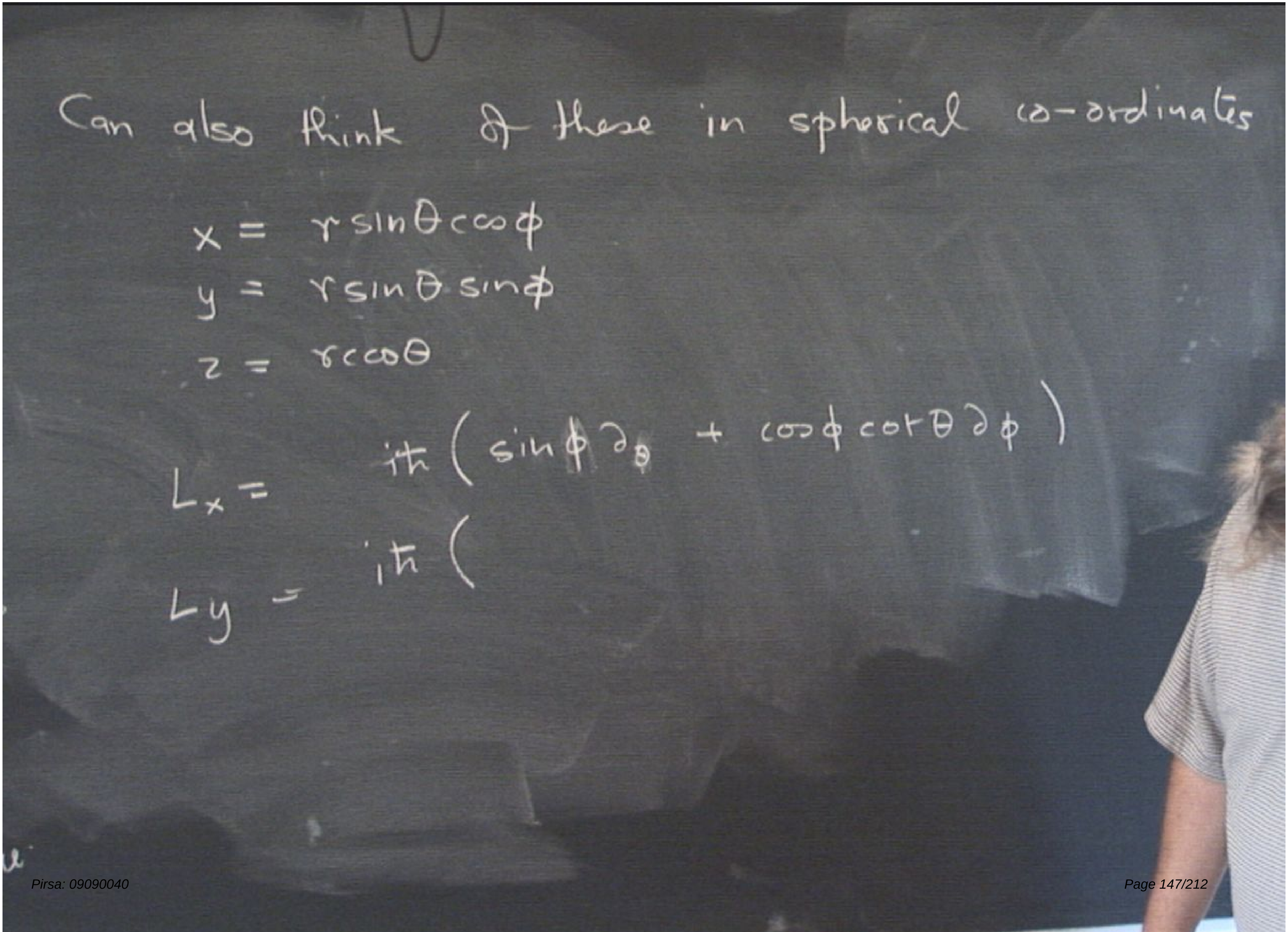
$$L_x = i\hbar \left(\sin\theta \, \partial_\phi + \cos\phi \right.$$



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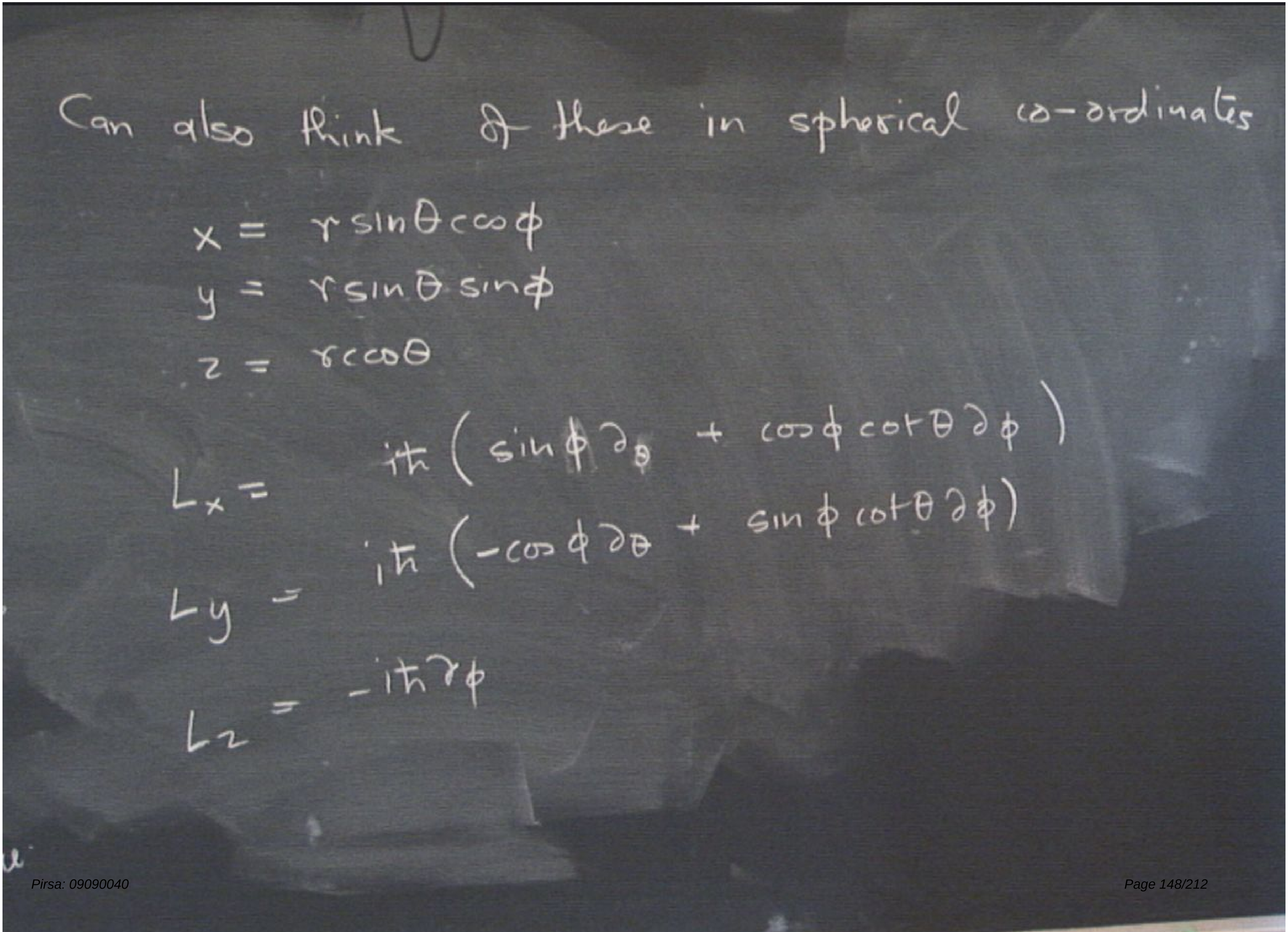
$$z = r \cos\theta$$

$$L_x = i\hbar \left(\sin\phi \, \partial_\theta + \cos\phi \cot\theta \, \partial_\phi \right)$$

$$L_y = i\hbar \left(\right.$$



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$$z = r\cos\theta$$

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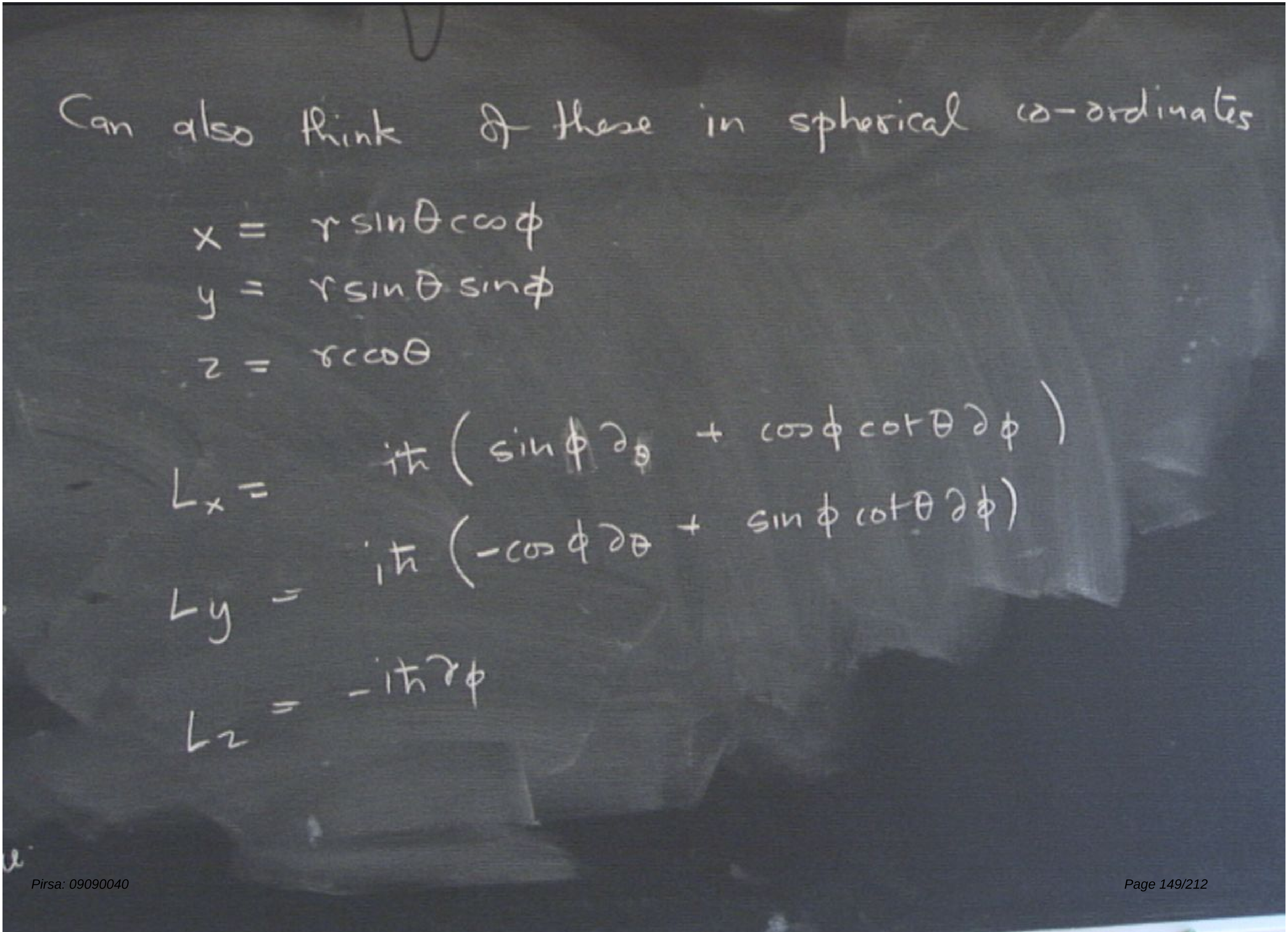
$$L_y = i\hbar\left(-\cos\phi\,\partial_\theta + \sin\phi\cot\theta\,\partial\phi\right)$$

$$L_z = -i\hbar\partial\phi$$



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$$L^2 = L_x^2 + L_y^2 + L_z^2$$

$$L^2 = L_x^2 + L_y^2 + L_z^2$$

✓

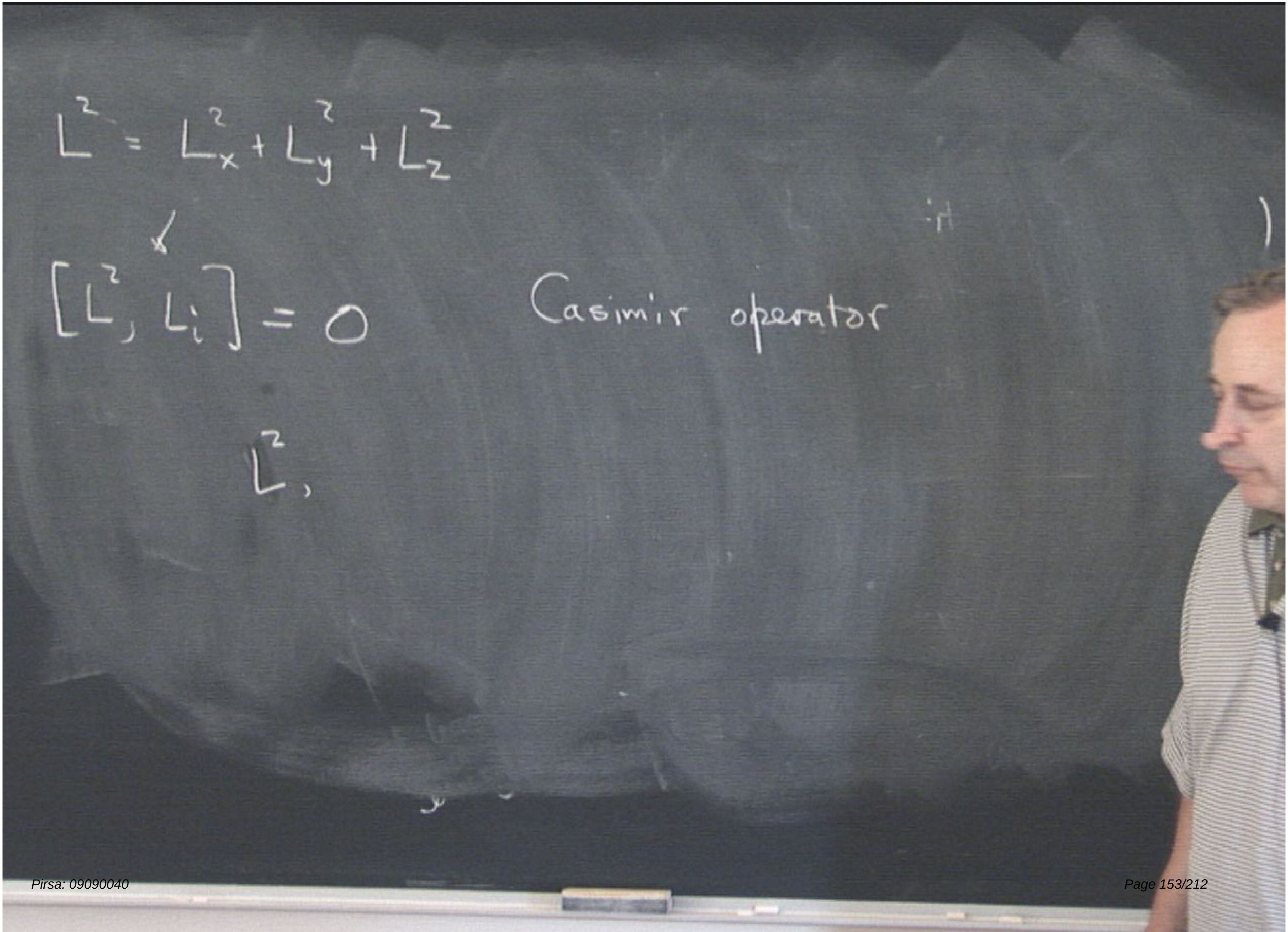
$$[L^2, L_i] = 0$$

$$L^2 = L_x^2 + L_y^2 + L_z^2$$

✓

$$[L^2, L_i] = 0$$

Casimir operator



$$L^2 = L_x^2 + L_y^2 + L_z^2$$

✓

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Casimir operator

$$L^2,$$

$$L^2 = L_x^2 + L_y^2 + L_z^2$$

✓

$$[L^2, L_i] = 0$$

Casimir operator

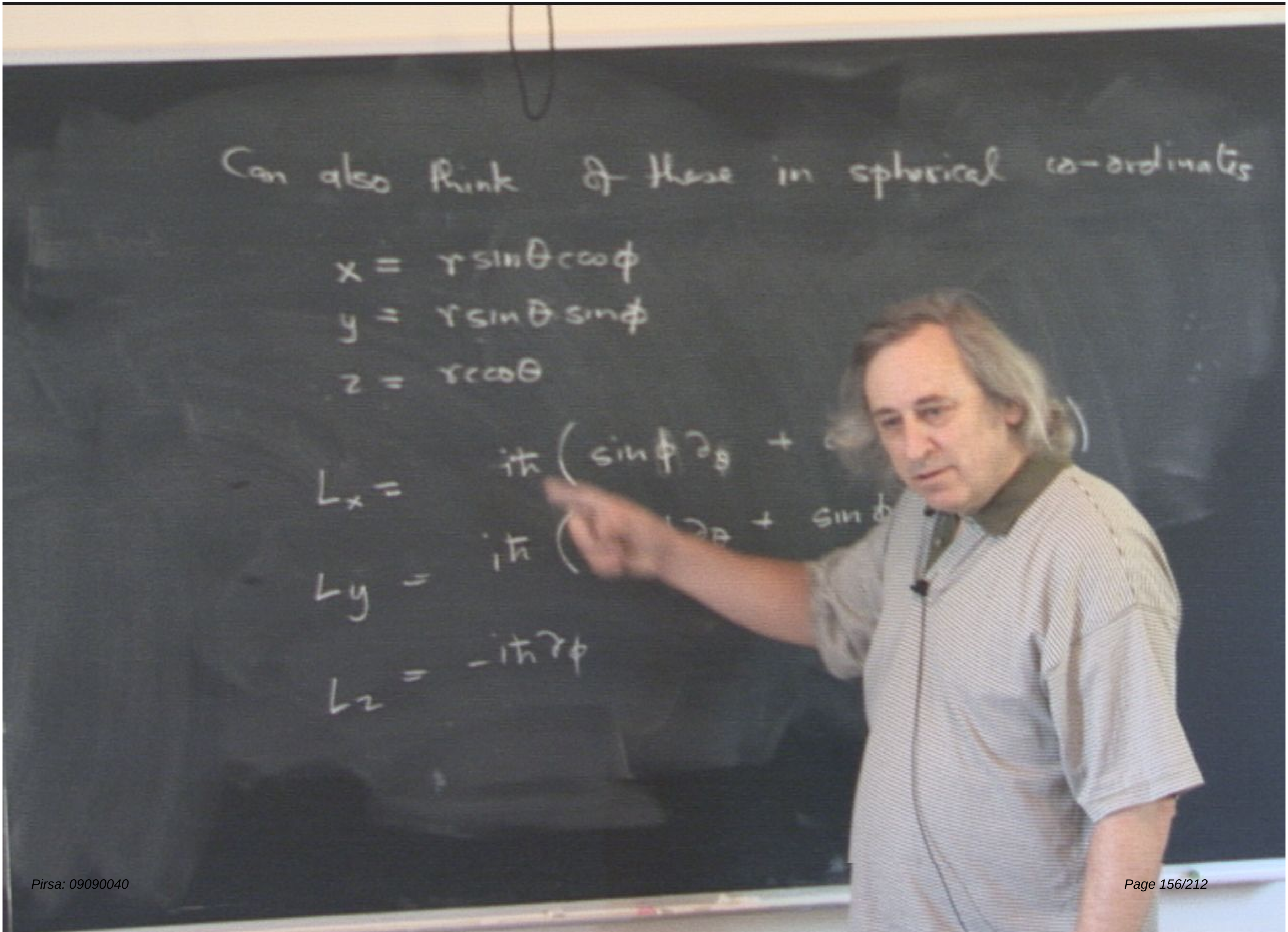
$$L^2, \; L_z$$

$$L^2 = L_x^2 + L_y^2 + L_z^2$$

✓

$$[L^2, L_i] = 0$$ Casimir operator

L^2, L_z — their eigenvalues describe all possible wavefunctions



Can also think of these in spherical co-ordinates

$$x = r\sin\theta\cos\phi$$

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$$z = r\cos\theta$$

$$L_x = i\hbar\left(\sin\phi\,\partial_\theta + \right.$$

$$L_y = i\hbar\left(\quad\partial_\theta + \sin\phi\right.$$

$$L_z = -i\hbar\partial_\phi$$

Can also think of these in spherical co-ordinates

$$x = r\sin\theta\cos\phi$$

$$y = r\sin\theta\sin\phi$$

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$$L_x = i\hbar\left(\sin\phi\,\partial_\theta + \cos\phi\cot\theta\,\partial\phi\right)$$

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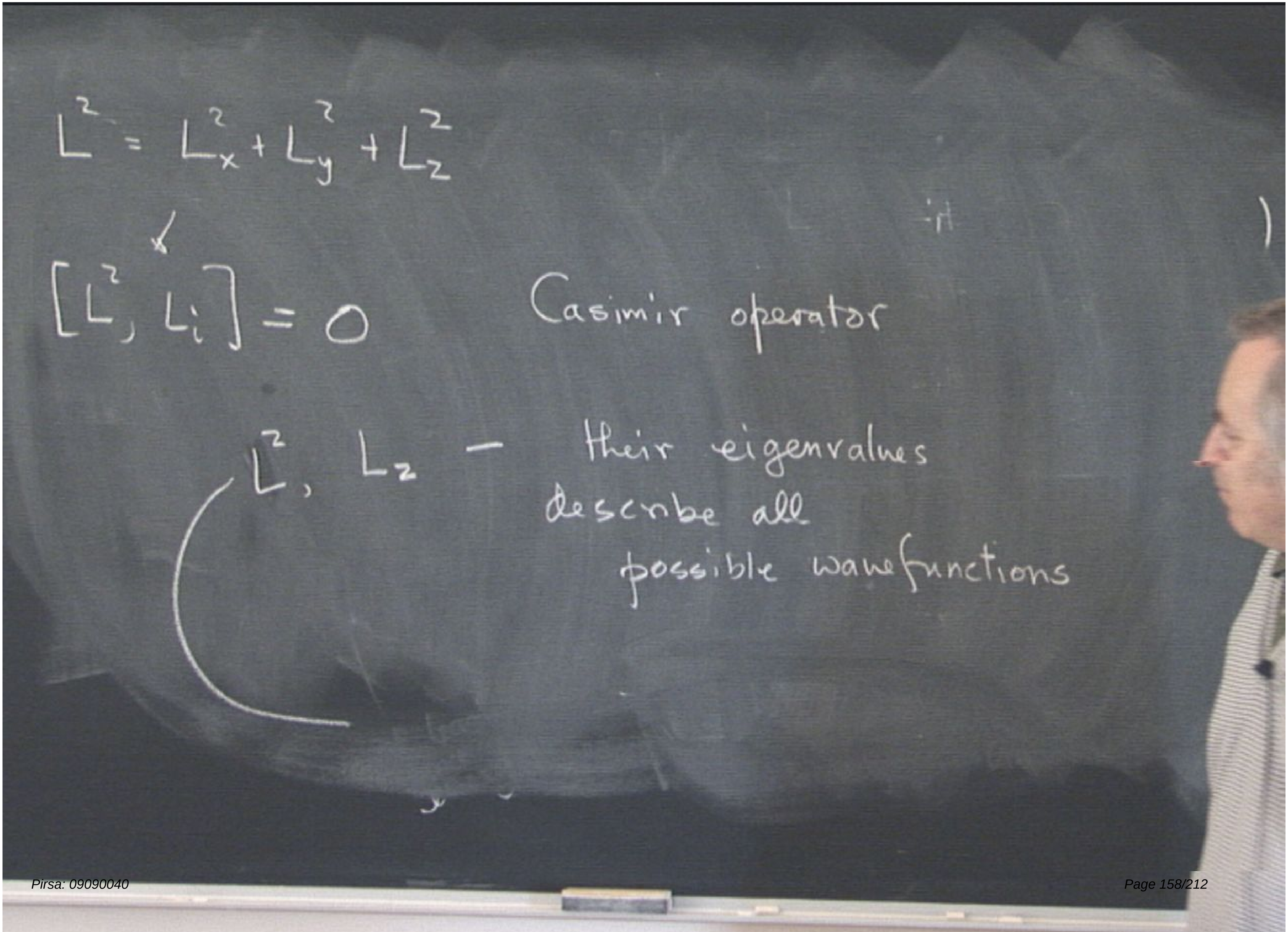
$$L_z = -i\hbar\,\partial\phi$$



$$L^2 = L_x^2 + L_y^2 + L_z^2$$

✓

$$[L^2, L_i] = 0$$

Casimir operator

L^2, L_z — their eigenvalues describe all possible wavefunctions

$$L^2 = L_x^2 + L_y^2 + L_z^2$$

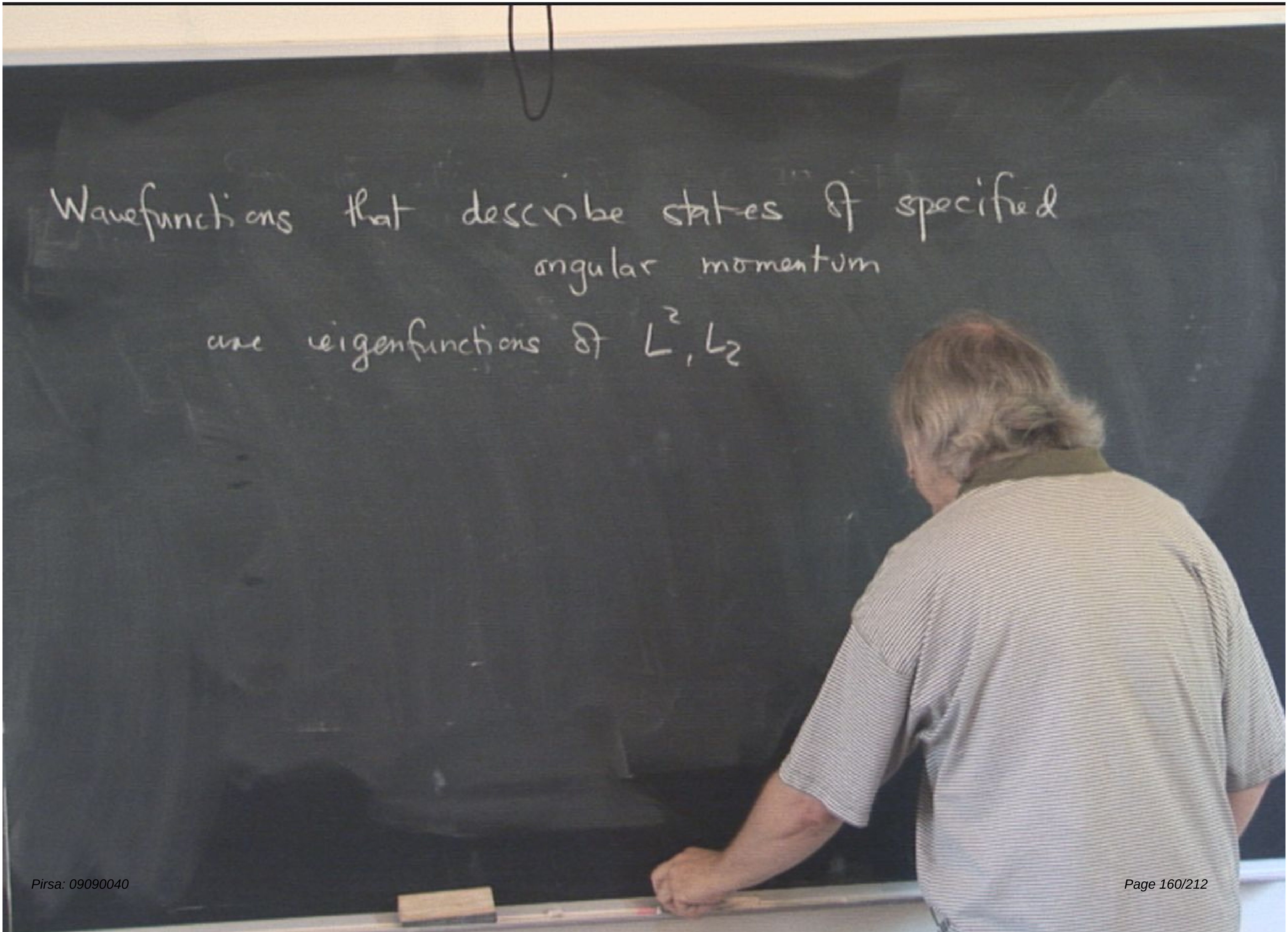
✓

$$[L^2, L_i] = 0$$

Casimir operator

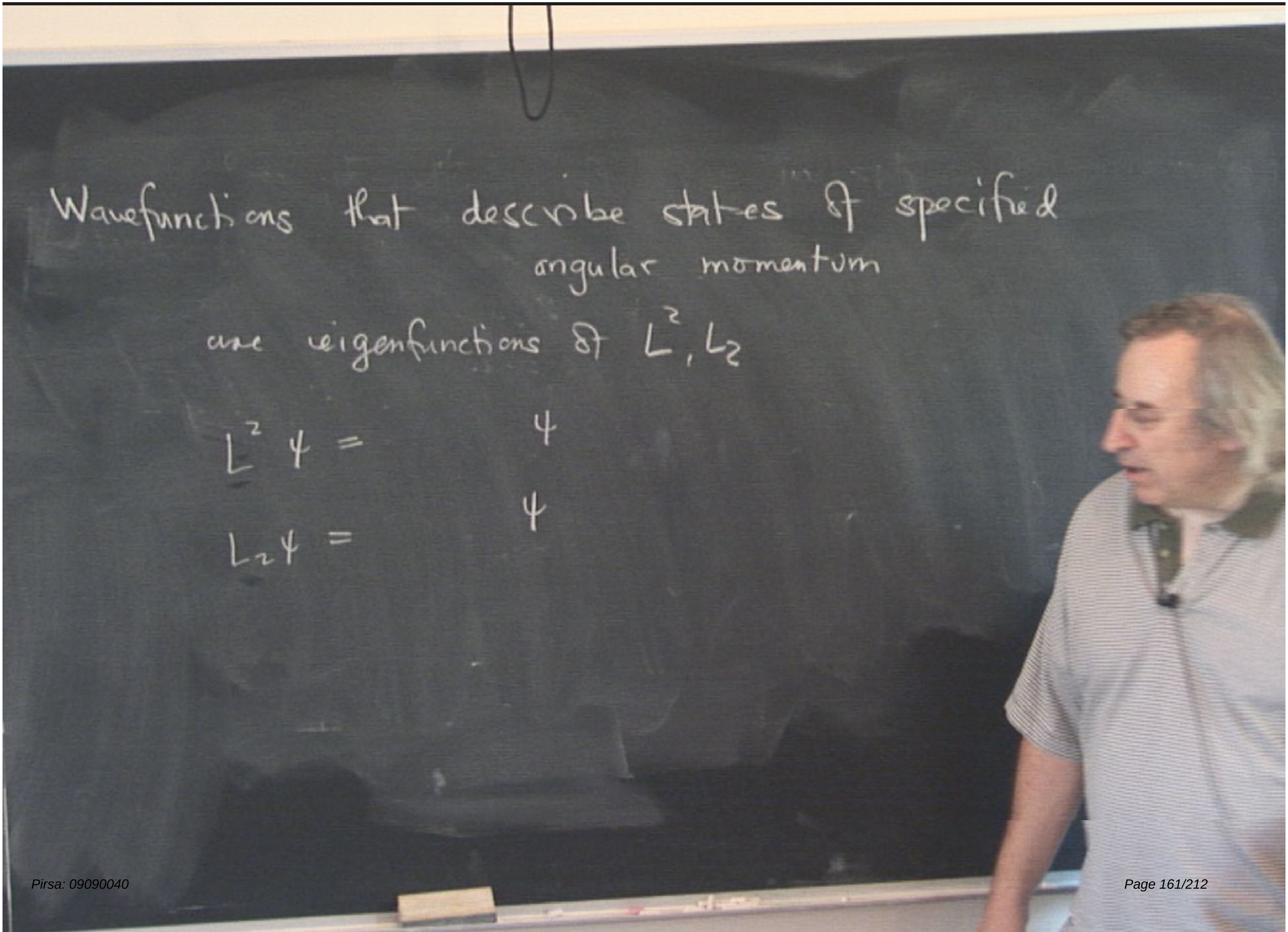
L^2, L_z — their eigenvalues describe all possible wavefunctions

$$-\hbar^2 \left[\frac{\partial^2}{\partial\theta^2} + \cot\theta \frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} \right]$$



Wavefunctions that describe states of specified angular momentum

are eigenfunctions of L^2, L_z

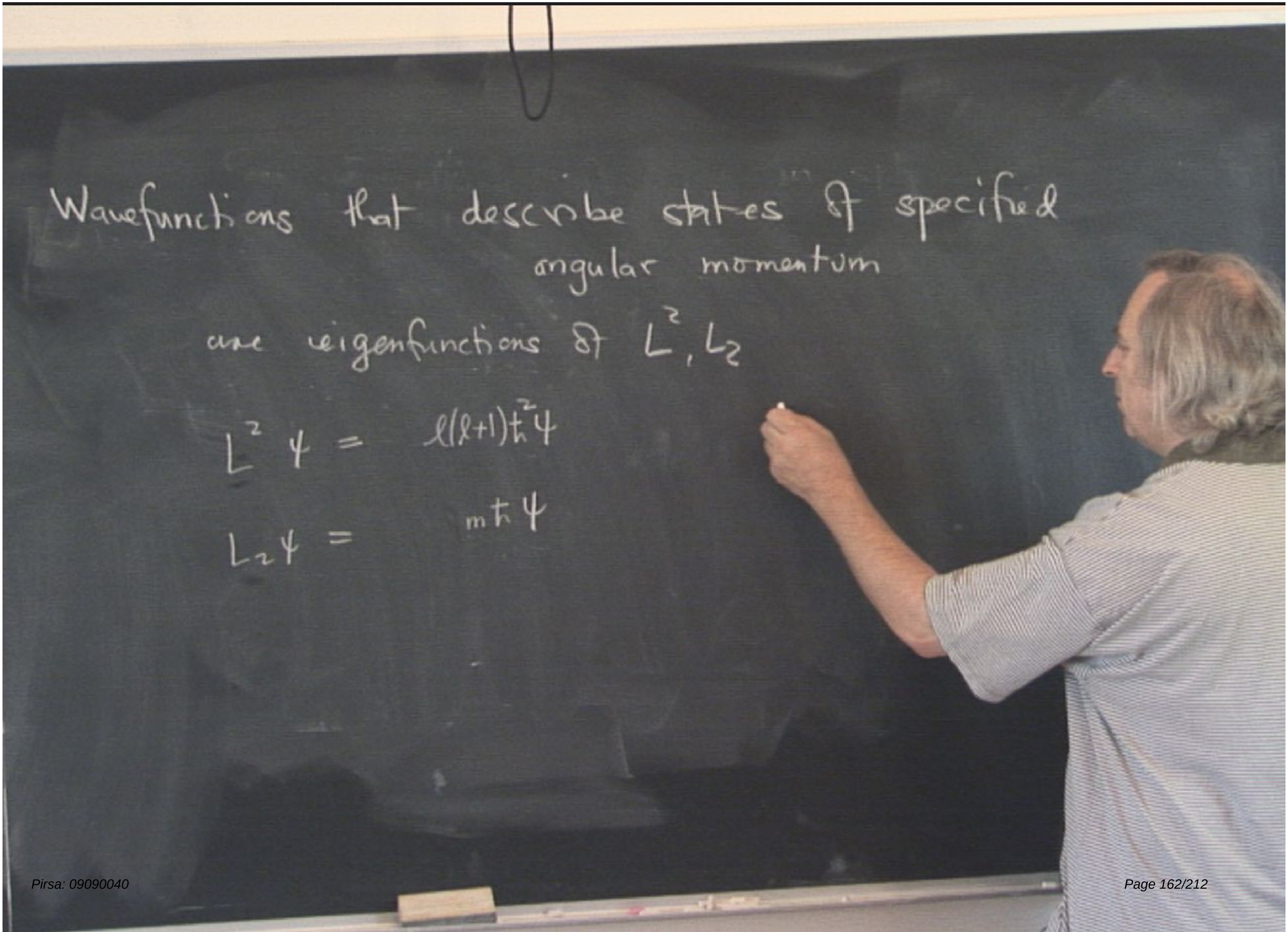


Wavefunctions that describe states of specified angular momentum

are eigenfunctions of L^2, L_z

$$L^2 \psi = \qquad \psi$$

$$L_z \psi = \qquad \psi$$

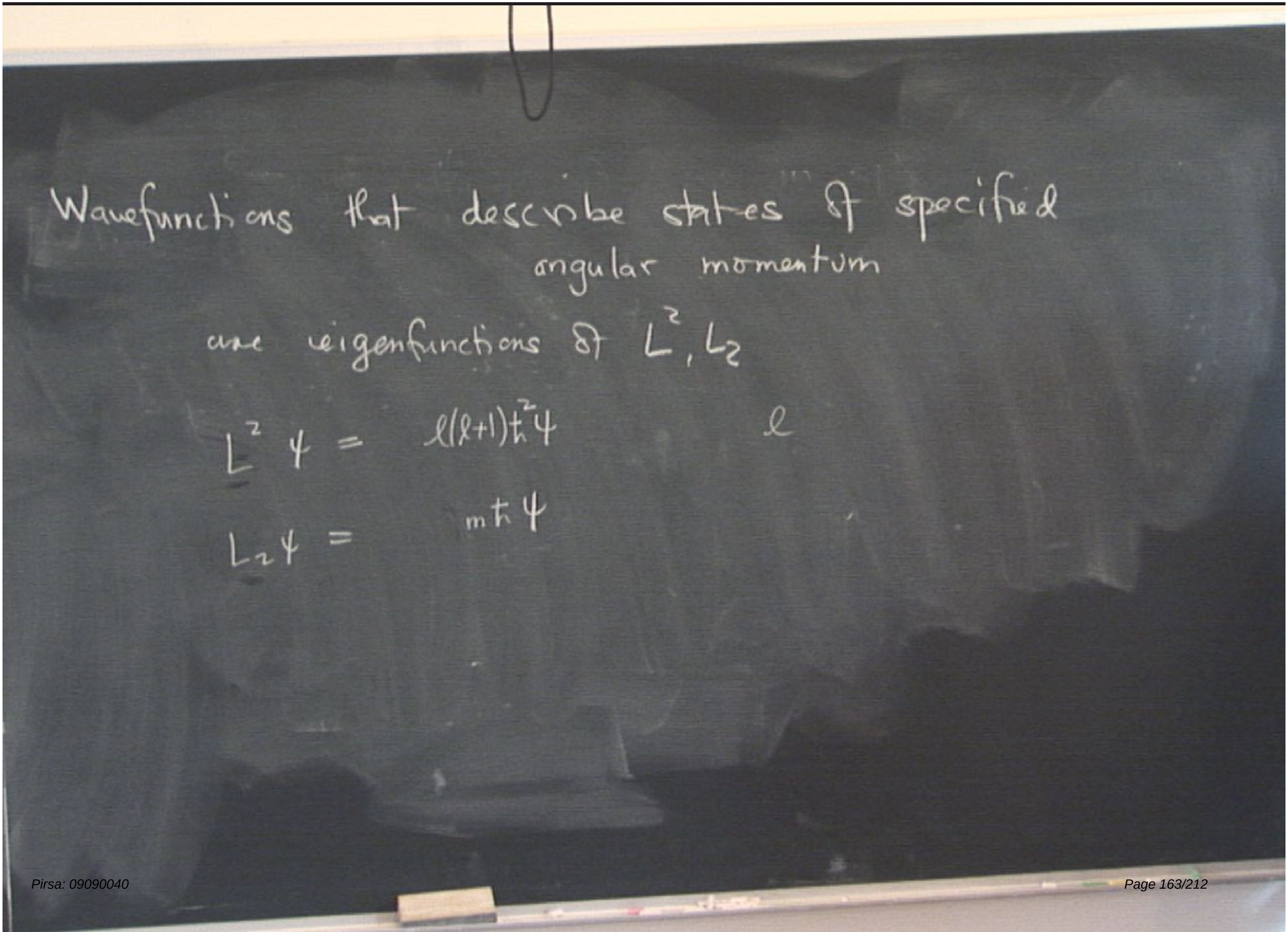


Wavefunctions that describe states of specified angular momentum

are eigenfunctions of L^2, L_z

$$L^2 \psi = \ell(\ell+1)\hbar^2 \psi$$

$$L_z \psi = m\hbar \psi$$



Wavefunctions that describe states of specified angular momentum

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$L^2 \psi = \ell(\ell+1)\hbar^2 \psi$ ℓ

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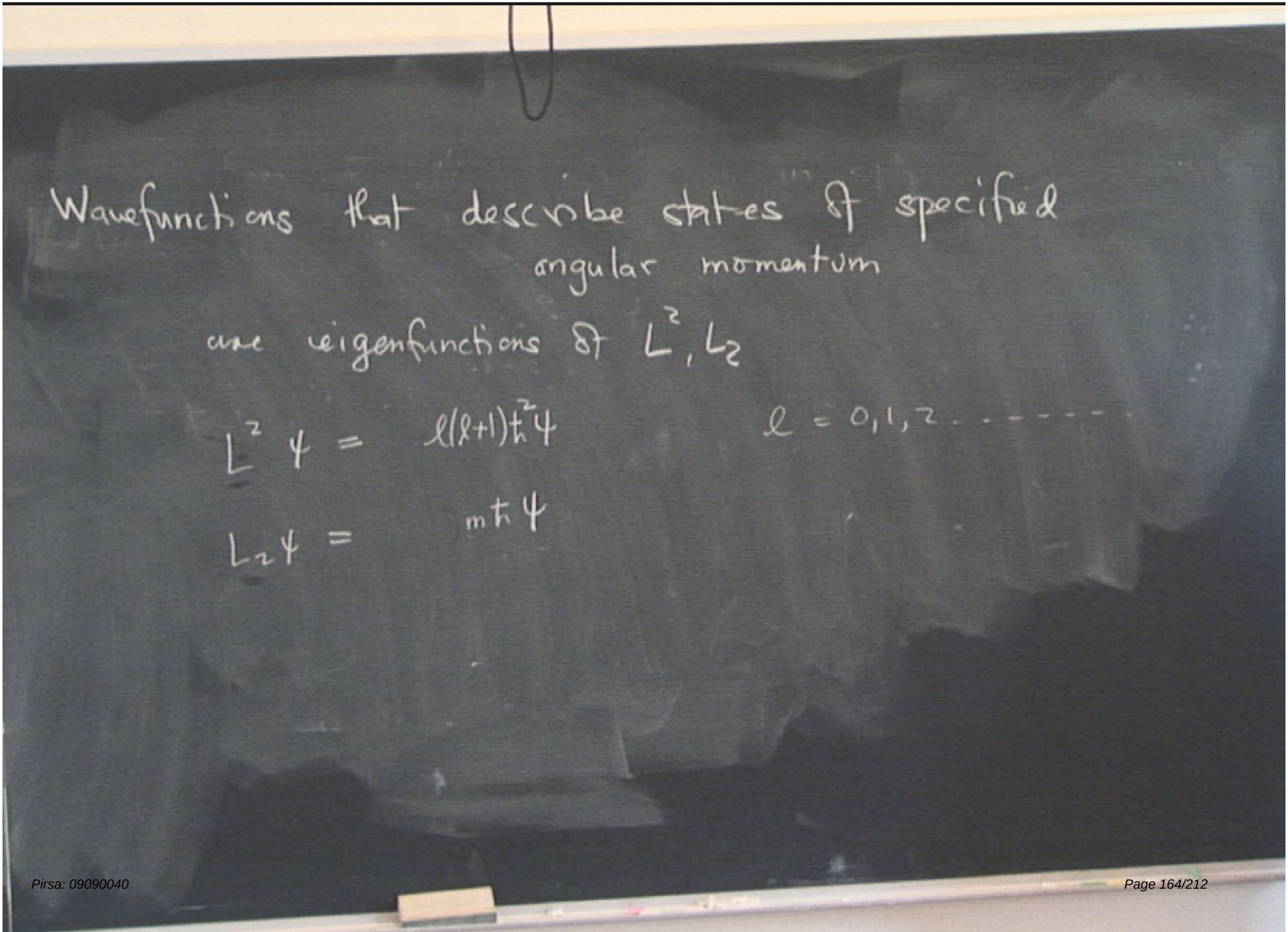
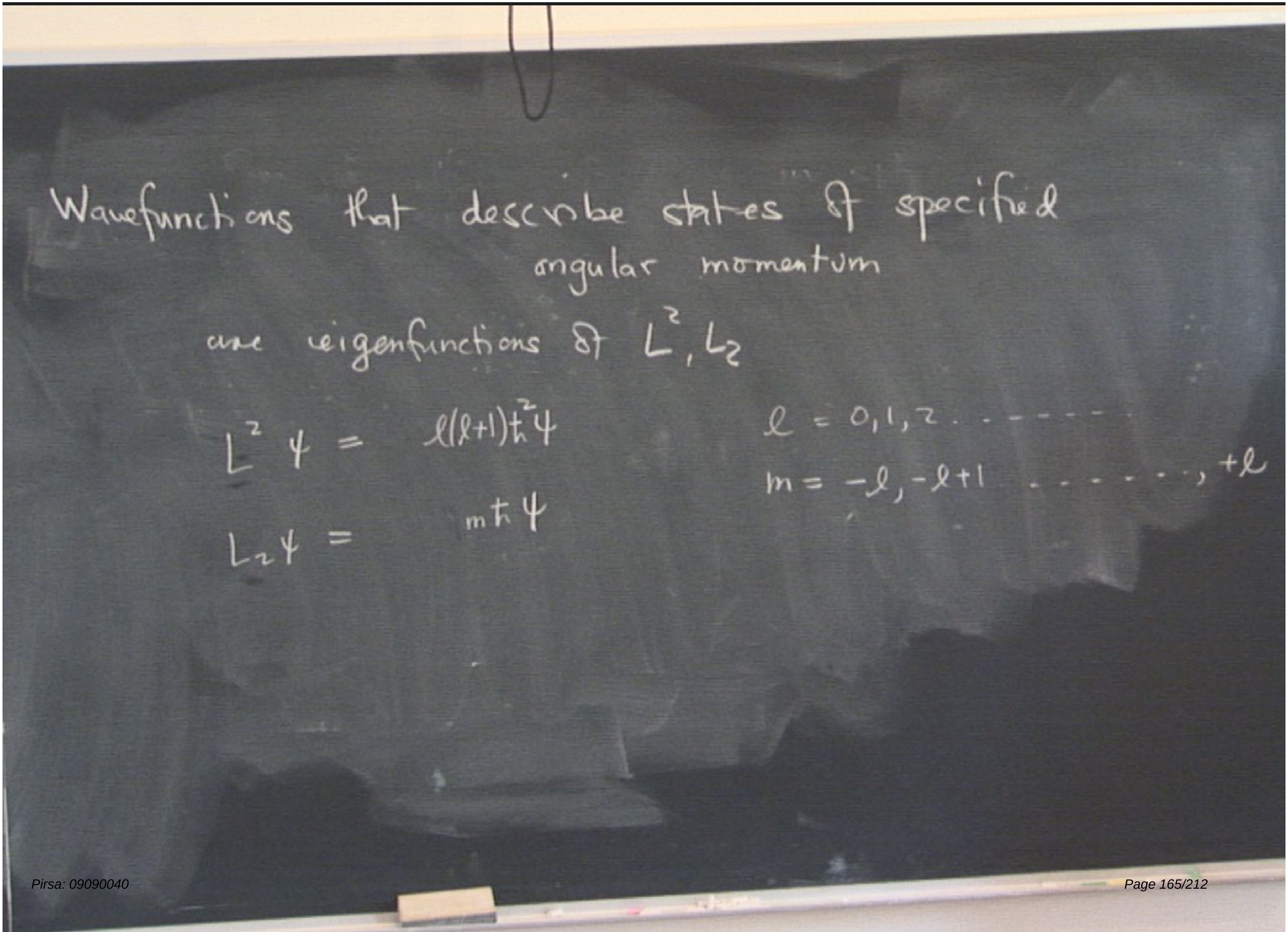


Wavefunctions that describe states of specified angular momentum

are eigenfunctions of L^2, L_z

$L^2 \psi = l(l+1)\hbar^2 \psi$ $\qquad l = 0,1,2 \ldots\ldots$

$L_z \psi = m\hbar \psi$



Wavefunctions that describe states of specified angular momentum

are eigenfunctions of L^2, L_z

$$L^2 \psi = \ell(\ell+1)\hbar^2 \psi \qquad \ell = 0, 1, 2 \ldots\ldots$$

$$L_z \psi = m\hbar \psi \qquad m = -\ell, -\ell+1 \ldots\ldots, +\ell$$

Wavefunctions that describe states of specified angular momentum

are eigenfunctions of L^2, L_z

$$L^2 \psi = \ell(\ell+1)\hbar^2 \psi$$

$$L_z \psi = m\hbar\psi$$

$$\ell = 0, 1, 2 \ldots\ldots$$

$$m = \underbrace{-\ell, -\ell+1 \ldots\ldots, +\ell}_{2\ell+1 \text{ possibilities}}$$

Wavefunctions that describe states of specified angular momentum

are eigenfunctions of L^2, L_z

$$L^2 \psi = \ell(\ell+1)\hbar^2\psi$$

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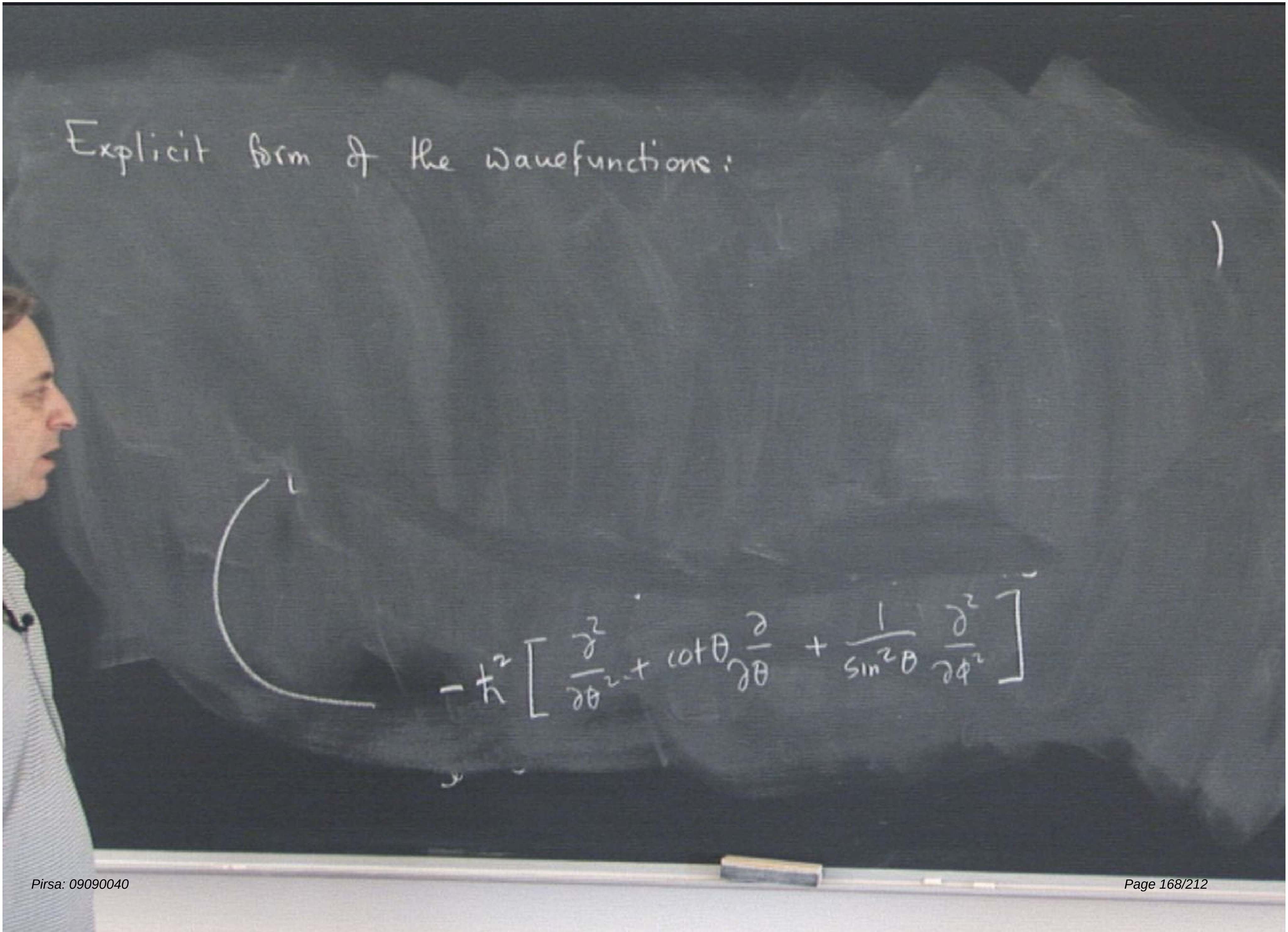
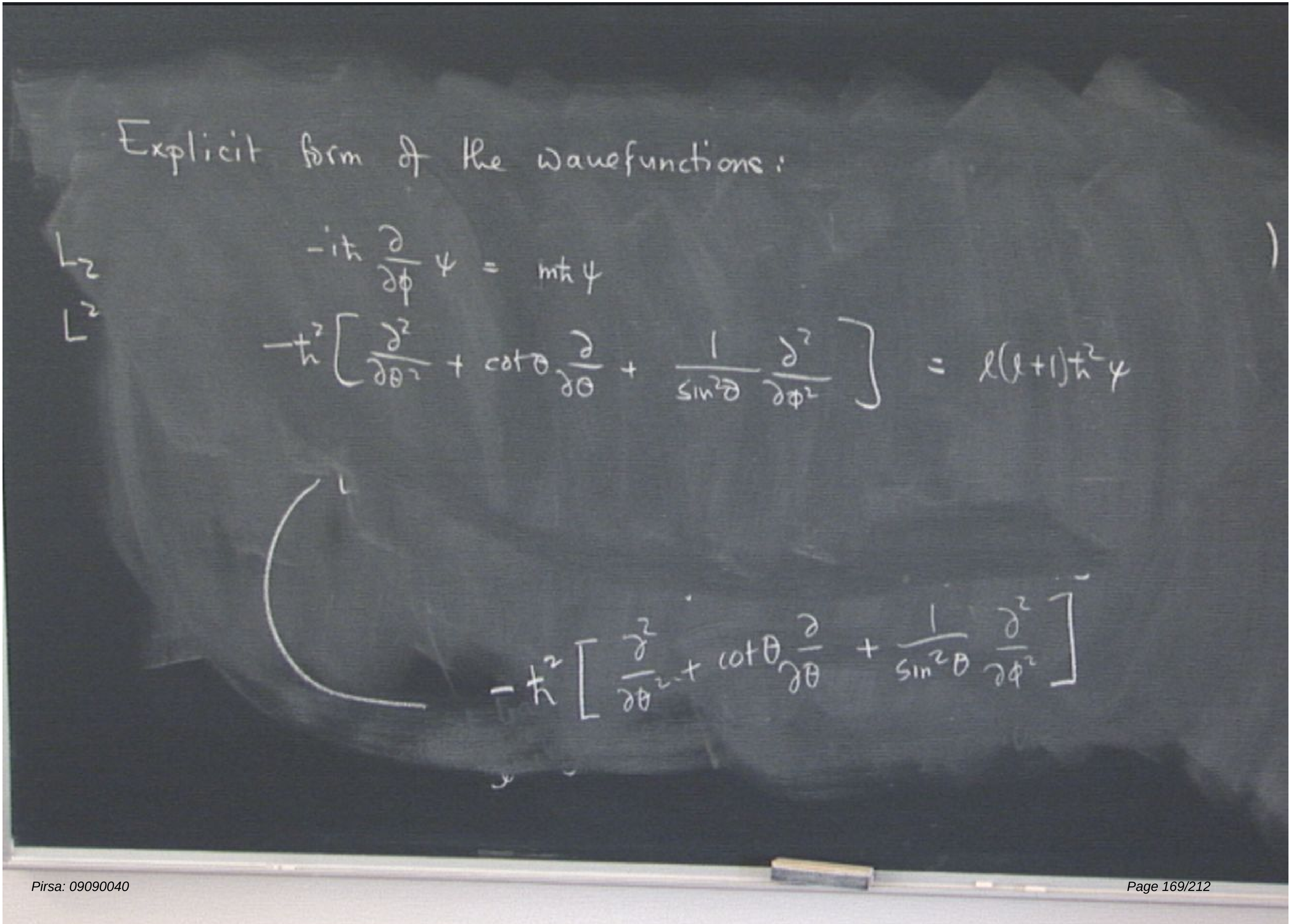


Explicit form of the wavefunctions:

$$-\hbar^2\left[\frac{\partial^2}{\partial\theta^2}+\cot\theta\frac{\partial}{\partial\theta}+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$$



Explicit form of the wavefunctions:

L_z $$-i\hbar \frac{\partial}{\partial\phi}\psi = m\hbar\psi$$

L^2 $$-\hbar^2\left[\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] = \ell(\ell+1)\hbar^2\psi$$

$$-\hbar^2\left[\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$$

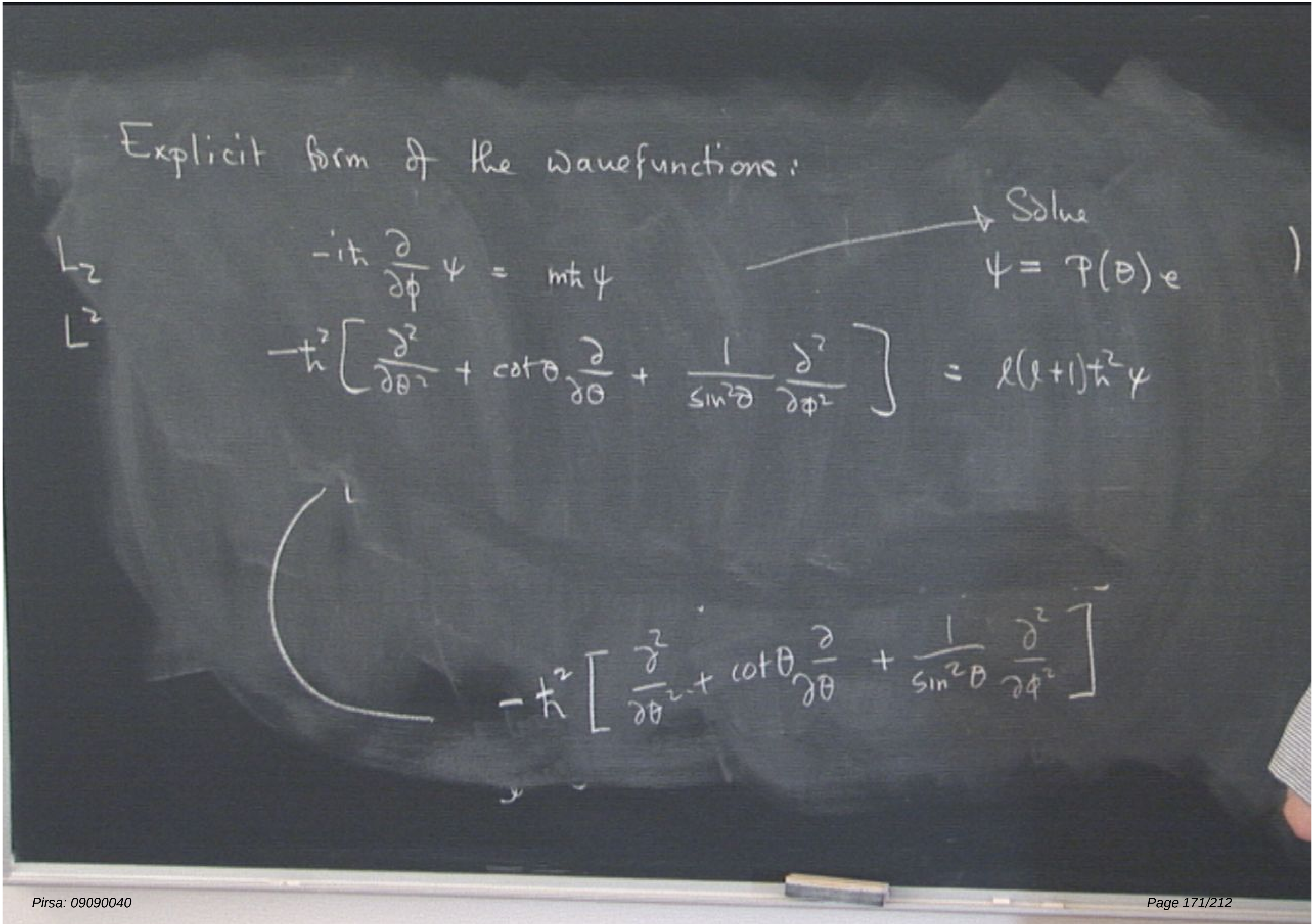
Explicit form of the wavefunctions:

L_z $\quad -i\hbar \frac{\partial}{\partial\phi}\psi = m\hbar\psi \longrightarrow$ Solve

$\psi =$

L^2 $\quad -\hbar^2\left[\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] = l(l+1)\hbar^2\psi$

$$-\hbar^2\left[\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$$



Explicit form of the wavefunctions:

L_z $$-i\hbar \frac{\partial}{\partial\phi}\psi = m\hbar\psi$$

Solve

$$\psi = P(\theta)\, e$$

)

L^2 $$-\hbar^2\left[\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] = \ell(\ell+1)\hbar^2\psi$$

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Explicit form of the wavefunctions:

L_z $$-i\hbar \frac{\partial}{\partial\phi}\psi = m\hbar\psi$$ → Solve $$\psi = P(\theta)\, e^{im\phi}$$

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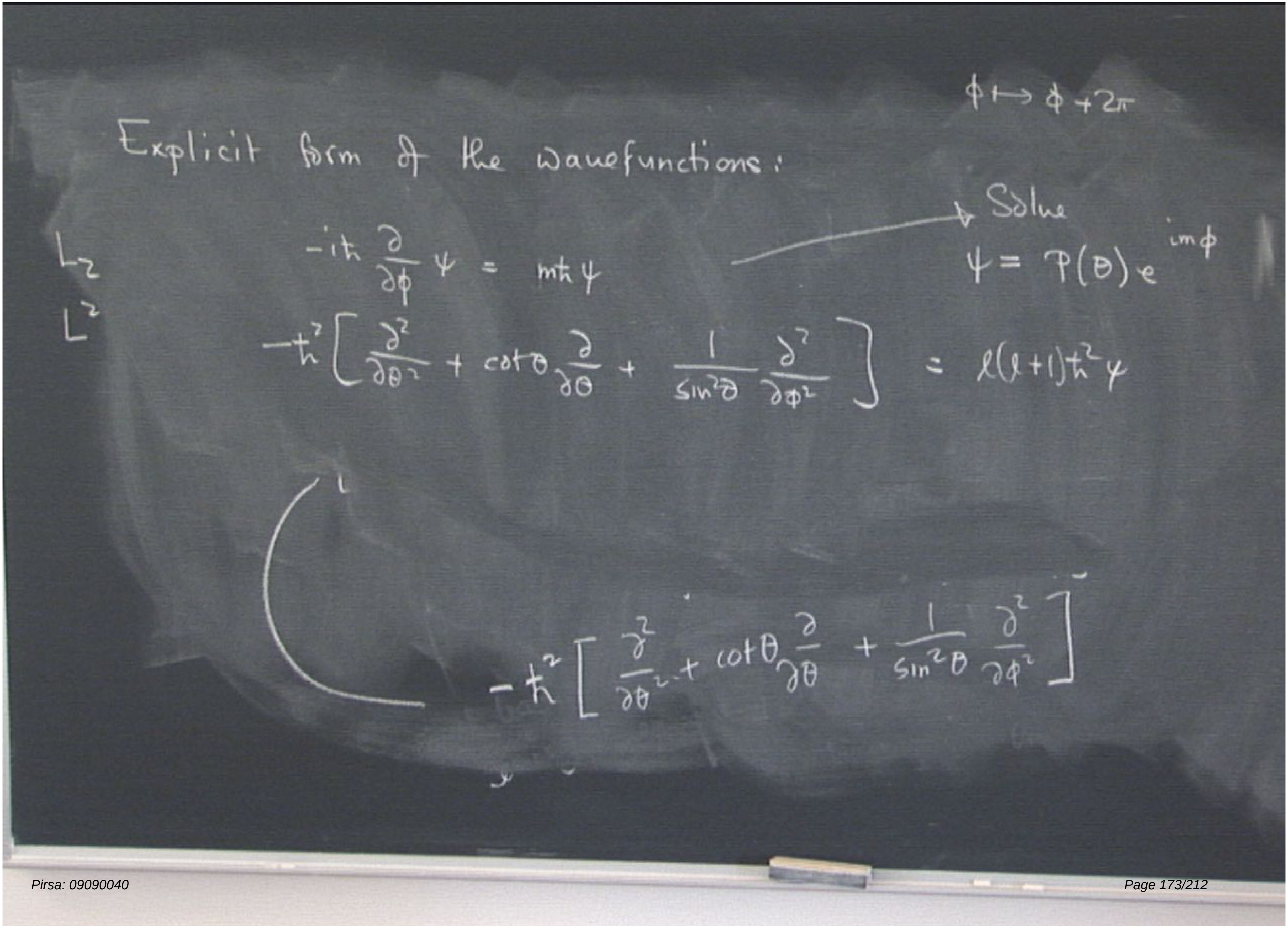

$\phi \mapsto \phi + 2\pi$

Explicit form of the wavefunctions:

Solve

L_z $$-i\hbar \frac{\partial}{\partial \phi}\psi = m\hbar\psi$$

$$\psi = P(\theta)\, e^{im\phi}$$

L^2 $$-\hbar^2\left[\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] = \ell(\ell+1)\hbar^2\psi$$

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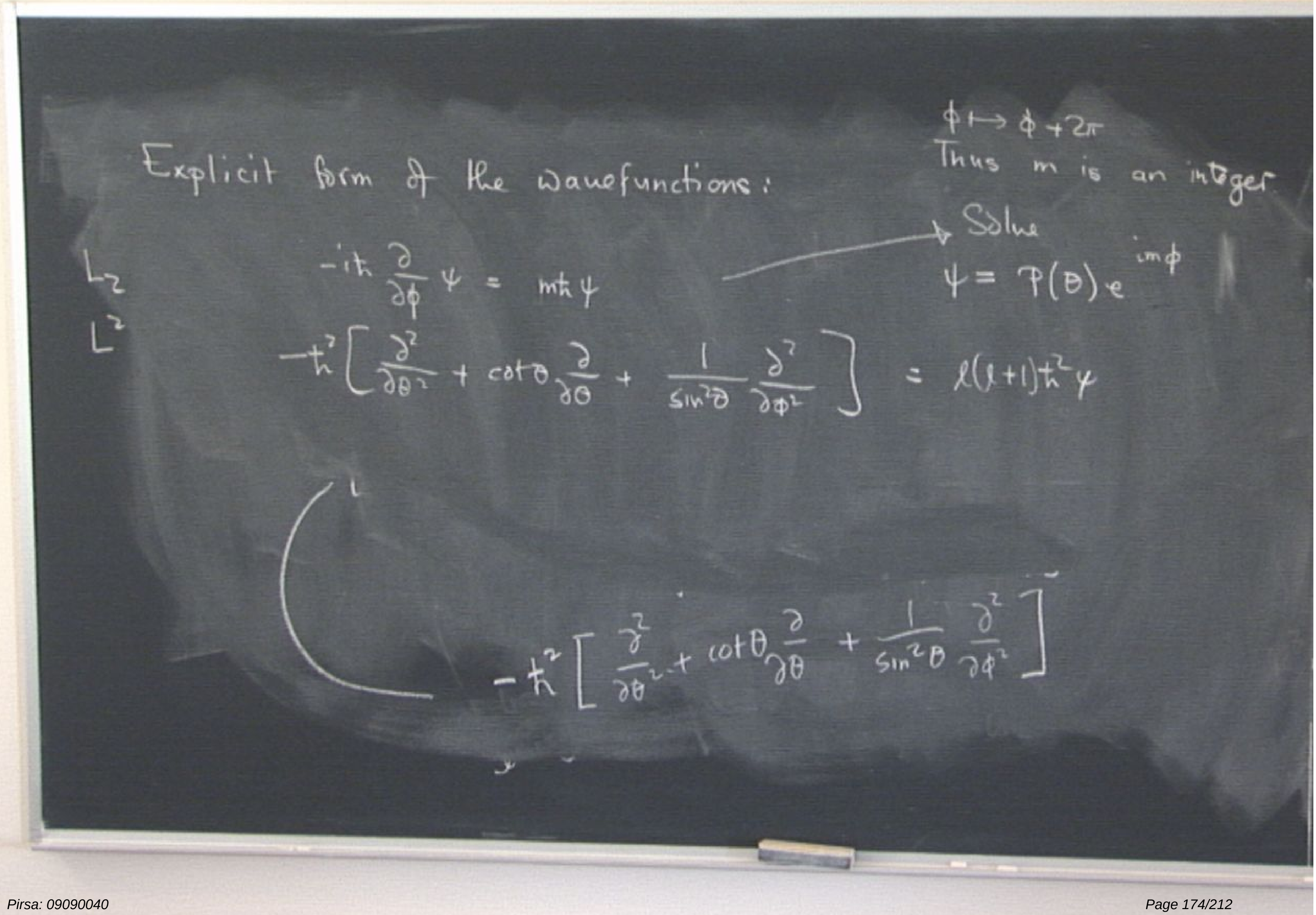

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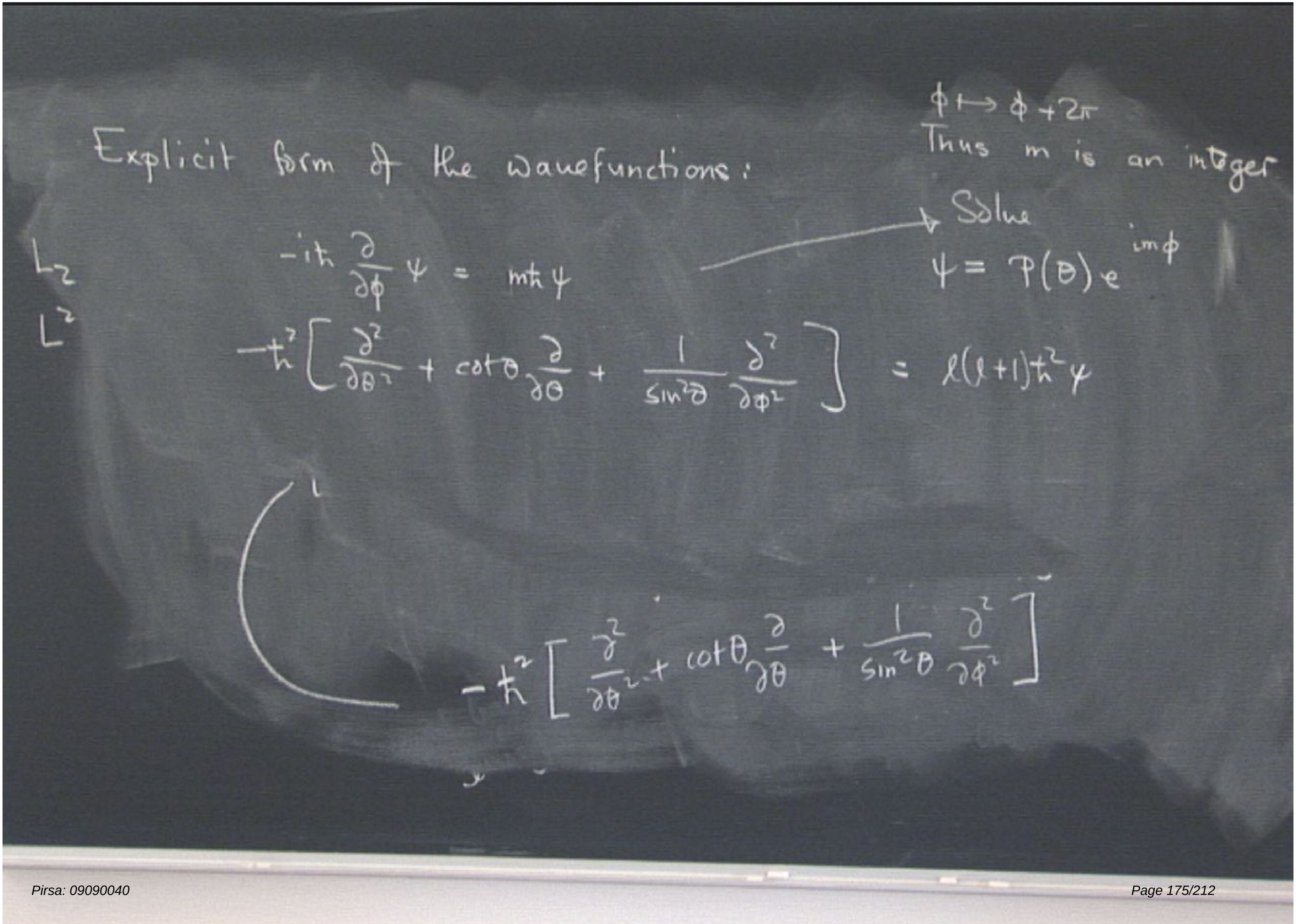
$\phi \mapsto \phi + 2\pi$

Thus m is an integer

Solve

$$\psi = P(\theta)e^{im\phi}$$

$$-\hbar^2\left[\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$$



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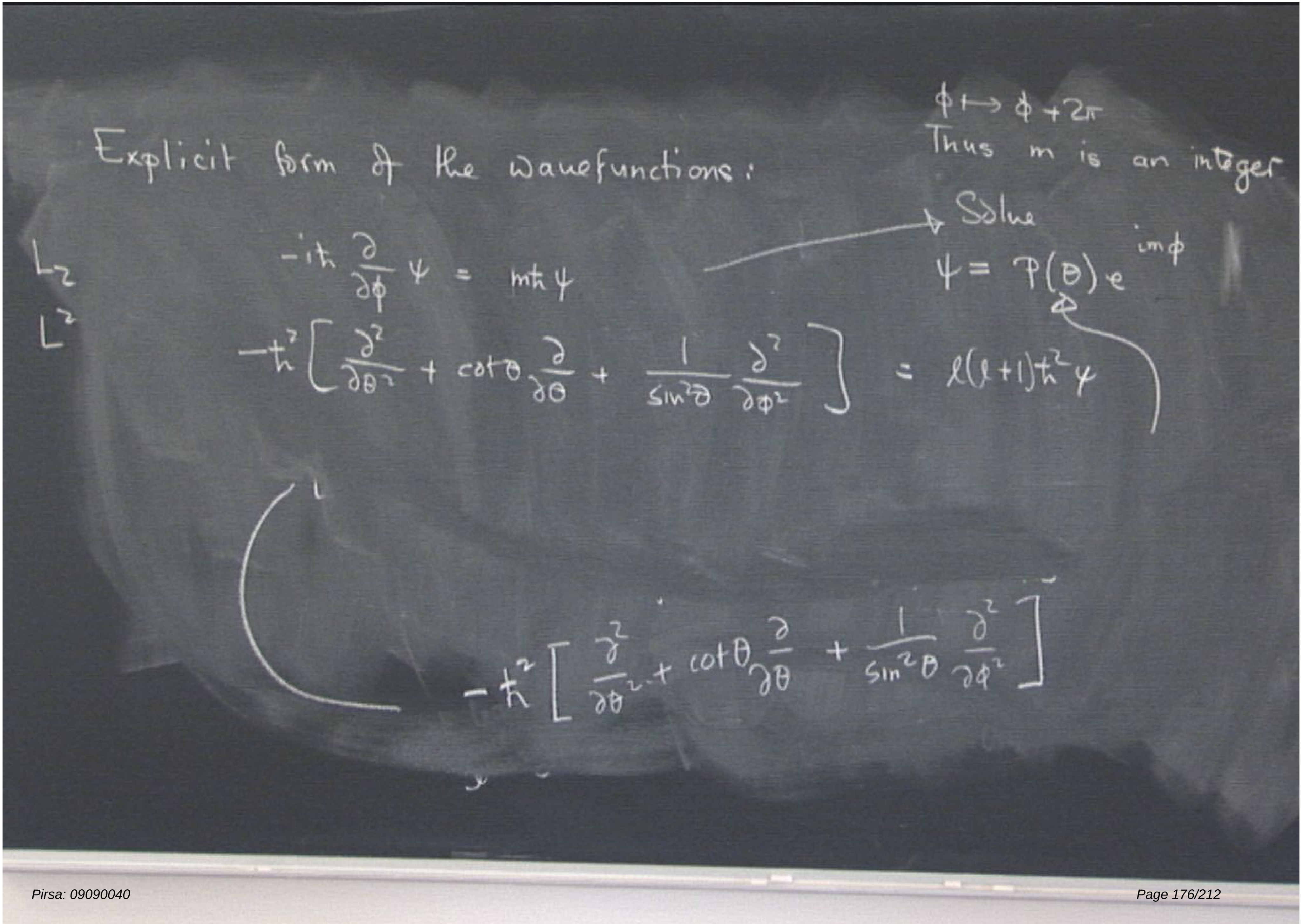

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Solve

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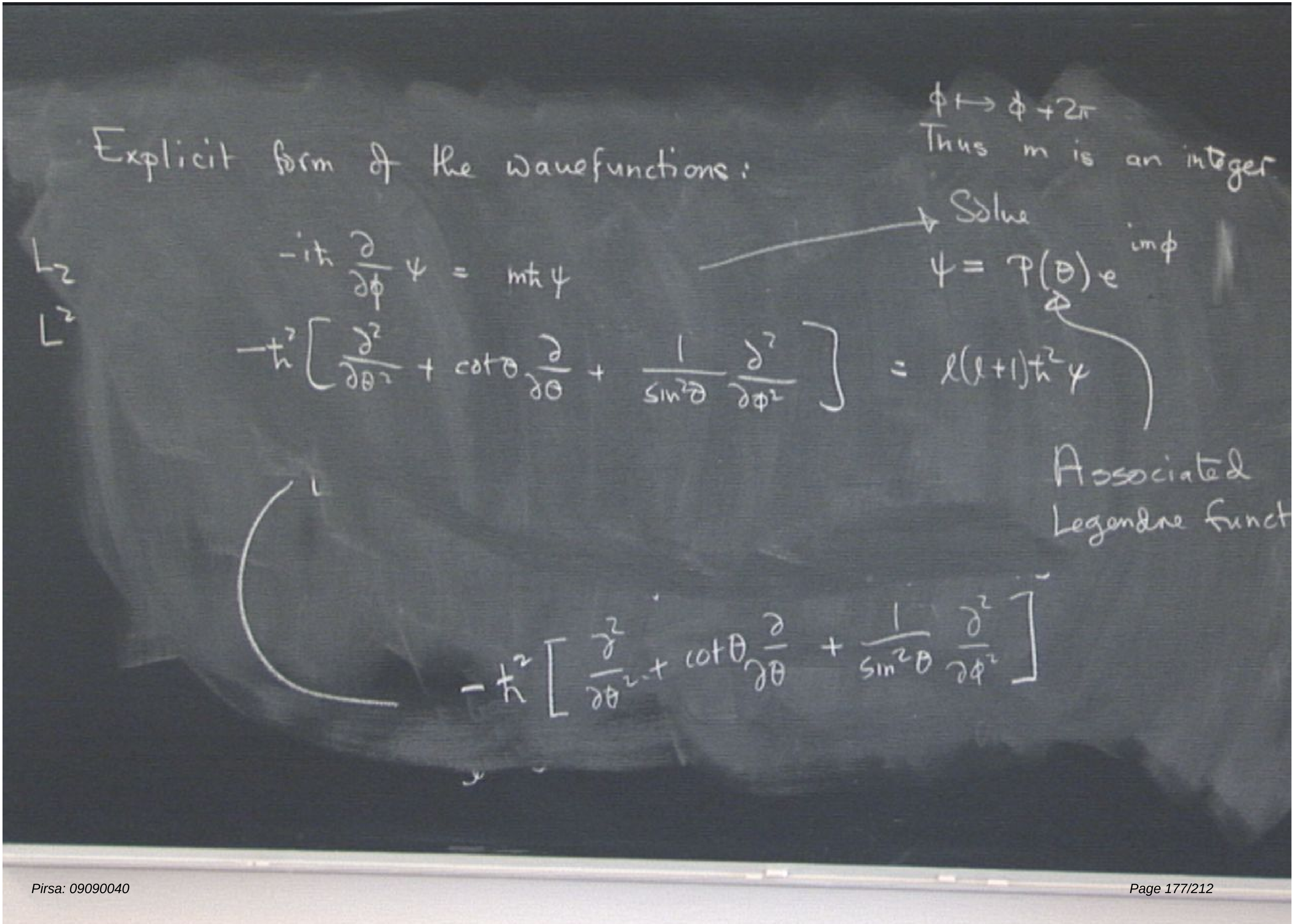

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Associated Legendre funct

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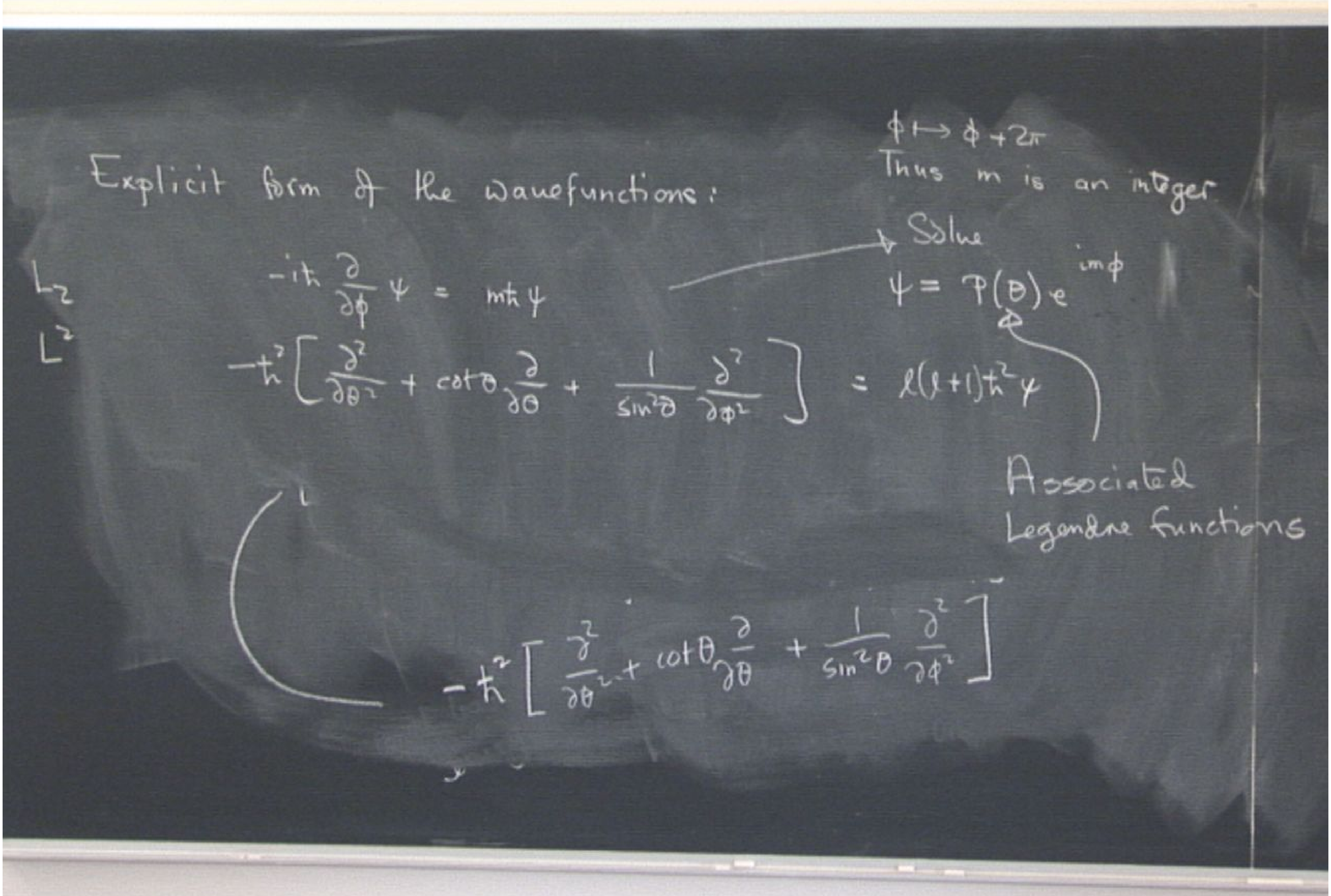


Explicit form of the wavefunctions:

$\phi \mapsto \phi + 2\pi$

Thus m is an integer

L_z

$$-i\hbar \frac{\partial}{\partial\phi}\psi = m\hbar\psi$$

Solve

$$\psi = P(\theta)e^{im\phi}$$

L^2

$$-\hbar^2\left[\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] = \ell(\ell+1)\hbar^2\psi$$

Associated Legendre functions

$$-\hbar^2\left[\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$$

Solve

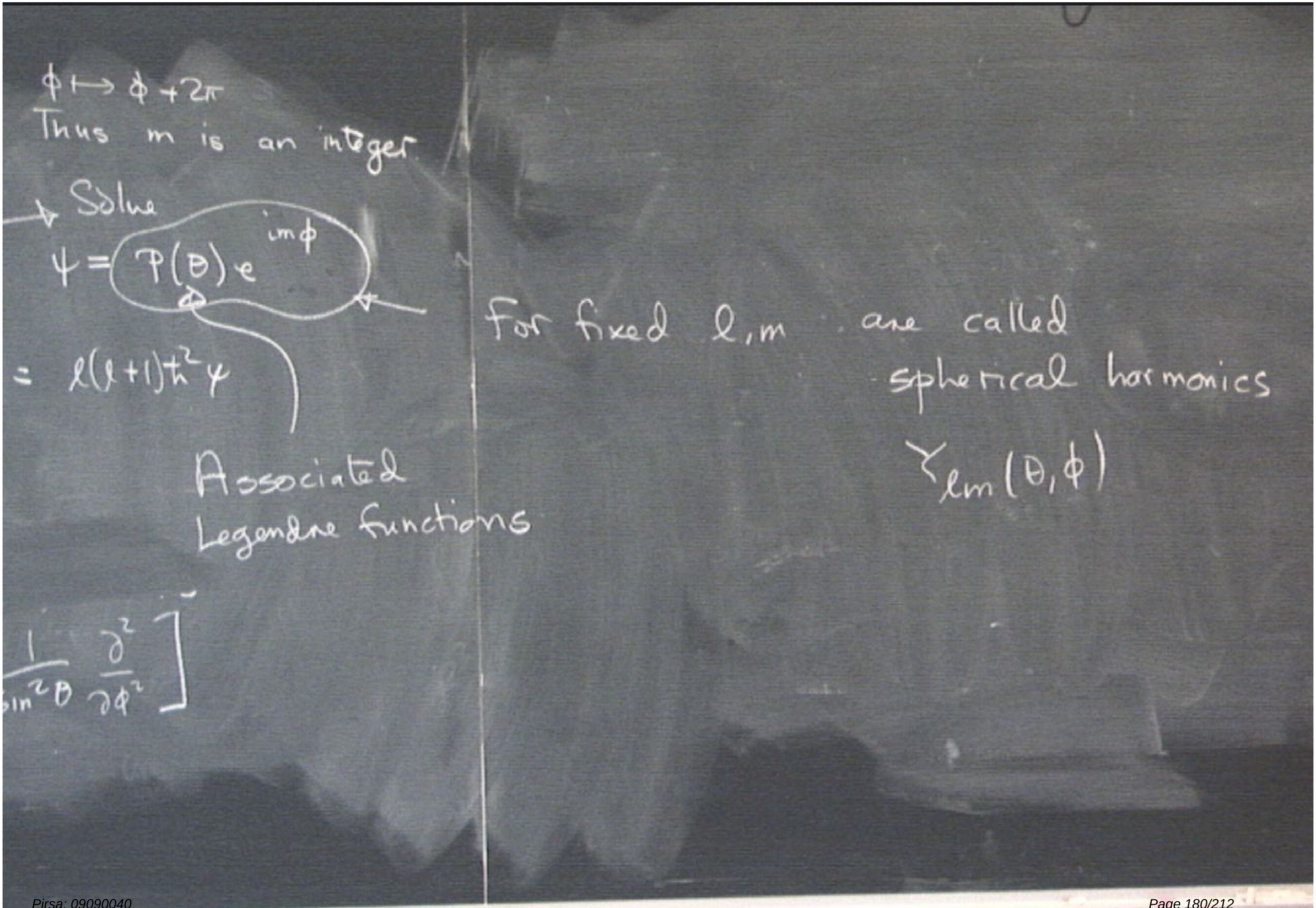
$$\psi = P(\theta)\, e^{im\phi}$$

$$= \ell(\ell+1)\hbar^2 \psi$$

Associated Legendre functions

For fixed ℓ, m

$$\frac{1}{{}^2\theta} \frac{\partial^2}{\partial\phi^2} \Big]$$



$\phi \mapsto \phi + 2\pi$

Thus m is an integer.

Solve

$\psi = \boxed{P(\theta) e^{im\phi}}$

$= \ell(\ell+1)\hbar^2 \psi$

Associated Legendre functions

$\frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} \Big]$

For fixed ℓ, m are called spherical harmonics

$Y_{\ell m}(\theta, \phi)$

d ℓ, m are called
spherical harmonics

$$Y_{\ell m}(\theta,\phi)$$

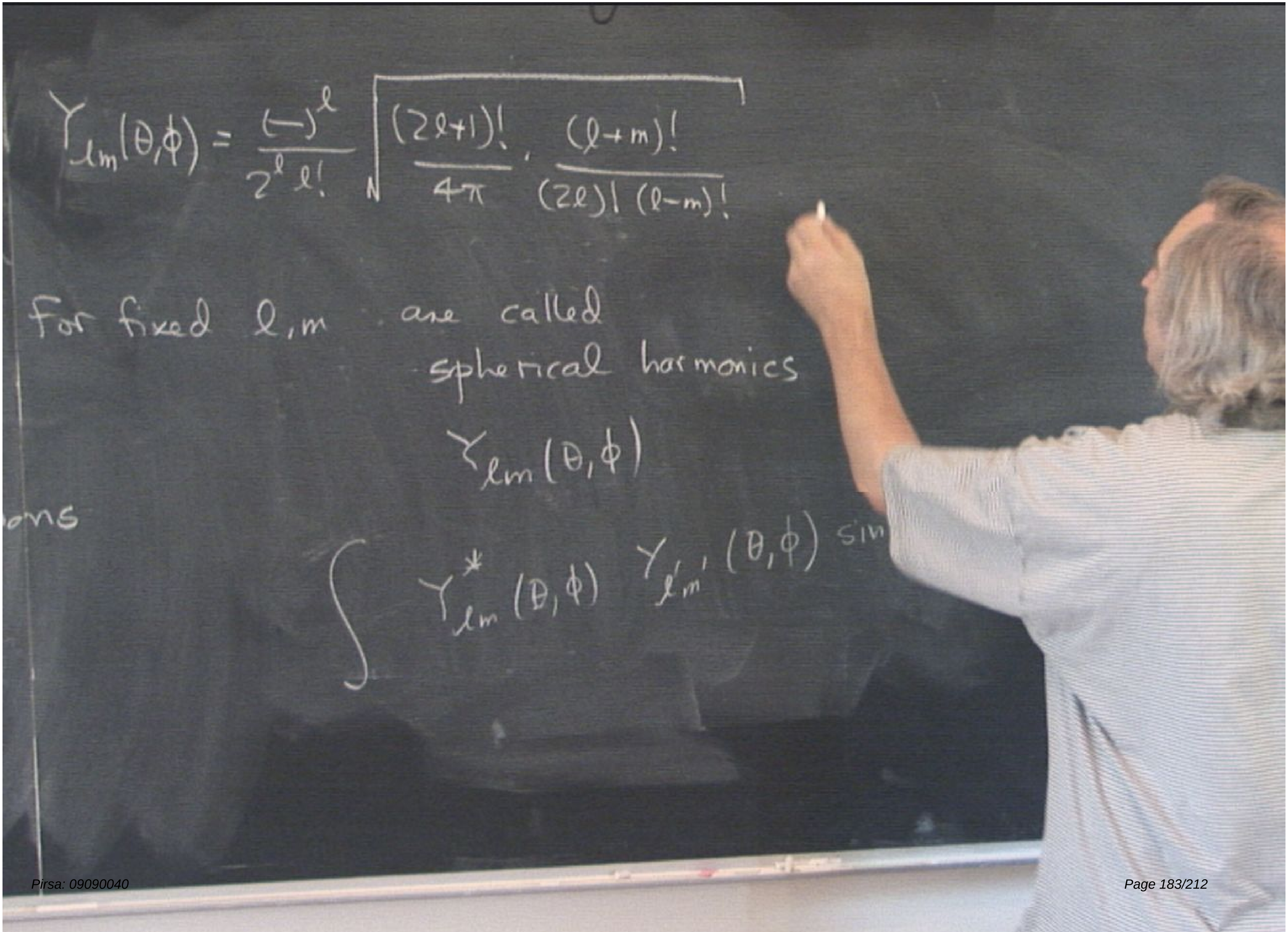
$$\int Y^*_{\ell m}(\theta,\phi)\ Y_{\ell' m'}(\theta,\phi)$$

d l, m are called

spherical harmonics

$Y_{lm}(\theta,\phi)$

$$\int Y^*_{lm}(\theta,\phi)\ Y_{l'm'}(\theta,\phi)\ \sin\theta d\theta d\phi = \delta_{ll'}\delta_{mm'}$$



$$Y_{lm}(\theta,\phi) = \frac{(-)^{l}}{2^{l}\, l!}\sqrt{\frac{(2l+1)!}{4\pi}\cdot\frac{(l+m)!}{(2l)!\,(l-m)!}}$$

For fixed l, m are called spherical harmonics

$$Y_{lm}(\theta,\phi)$$

ons

$$\int Y^{*}_{lm}(\theta,\phi)\; Y_{l'm'}(\theta,\phi)\; \sin$$

$$Y_{\ell m}(\theta,\phi) = \frac{(-)^{\ell}}{2^{\ell}\ell!}\sqrt{\frac{(2\ell+1)!}{4\pi}\cdot\frac{(\ell+m)!}{(2\ell)!\,(\ell-m)!}}\;(\sin\theta)^{-m}\;\frac{d}{d(\cos\theta)}$$

For fixed ℓ, m are called spherical harmonics

$$Y_{\ell m}(\theta,\phi)$$

ons

$$\int Y^{*}_{\ell m}(\theta,\phi)\; Y_{\ell' m'}(\theta,\phi)\,\sin\theta\, d\theta\, d\phi = \delta_{\ell\ell'}\,\delta_{mm'}$$

$$Y_{lm}(\theta,\phi) = \frac{(-)^{l}}{2^{l}\,l!}\sqrt{\frac{(2l+1)!}{4\pi}\cdot\frac{(l+m)!}{(2l)!\,(l-m)!}}\;(\sin\theta)^{-m}\,\frac{d^{\,l-m}}{d(\cos\theta)^{l-m}}$$

For fixed l,m are called spherical harmonics

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$$\int Y^{*}_{lm}(\theta,\phi)\;Y_{l'm'}(\theta,\phi)\,\sin\theta\,d\theta\,d\phi = \delta_{ll'}\,\delta_{mm'}$$

$$Y_{\ell m}(\theta,\phi) = \frac{(-)^{\ell}}{2^{\ell}\,\ell!}\sqrt{\frac{(2\ell+1)!}{4\pi}\cdot\frac{(\ell+m)!}{(2\ell)!\,(\ell-m)!}}\left\{(\sin\theta)^{-m}\frac{d^{\ell-m}}{d(\cos\theta)^{\ell-m}}(\sin\theta)^{2\ell}\right\}e^{im\phi}$$

For fixed ℓ, m are called spherical harmonics

$$Y_{\ell m}(\theta,\phi)$$

ons

$$\int Y^{*}_{\ell m}(\theta,\phi)\; Y_{\ell' m'}(\theta,\phi)\,\sin\theta\, d\theta\, d\phi = \delta_{\ell\ell'}\,\delta_{mm'}$$

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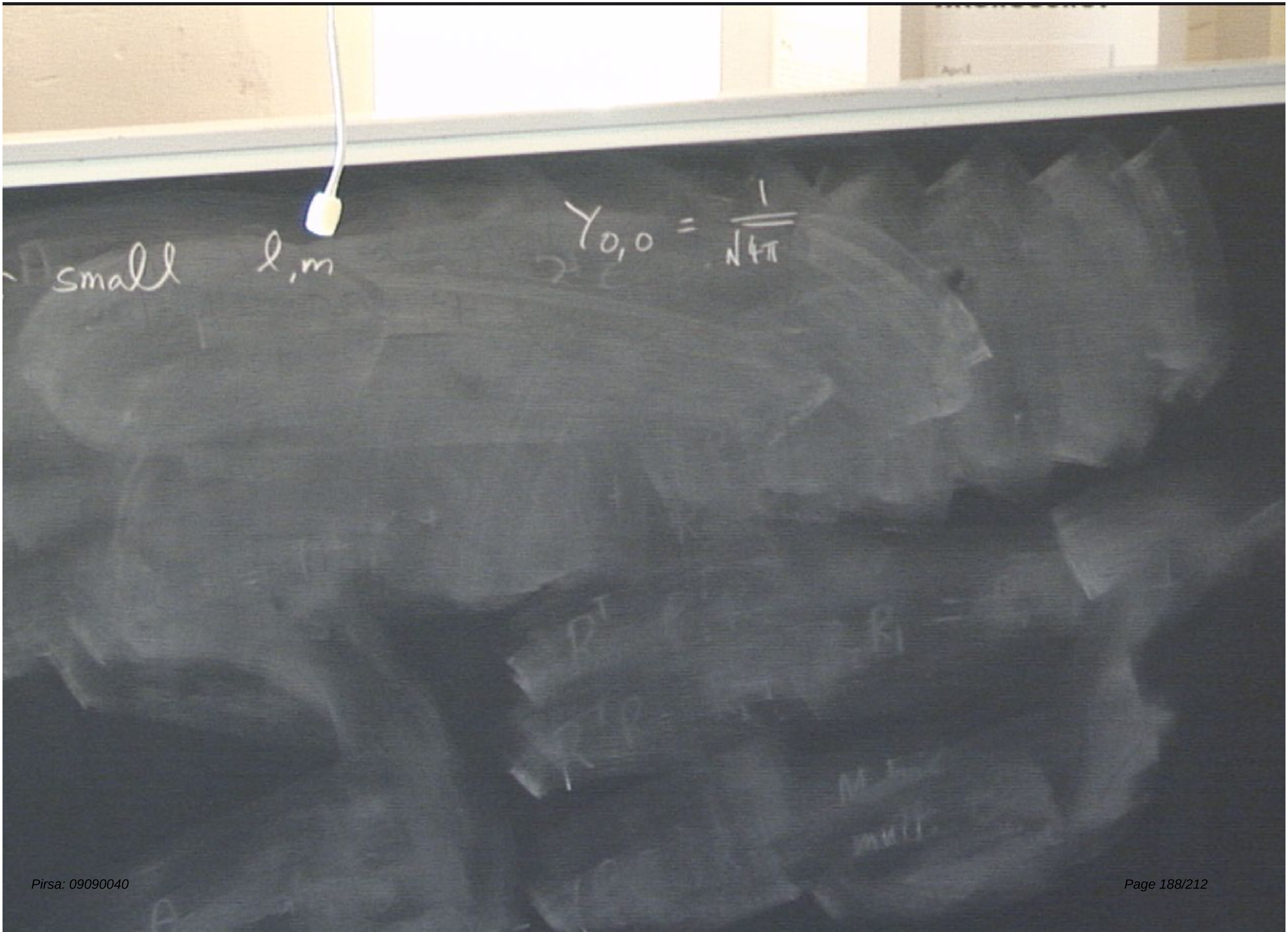
$$\int Y^{*}_{\ell m}(\theta,\phi)\; Y_{\ell' m'}(\theta,\phi)\sin\theta d\theta d\phi = \delta_{\ell\ell'}\delta_{mm'}$$



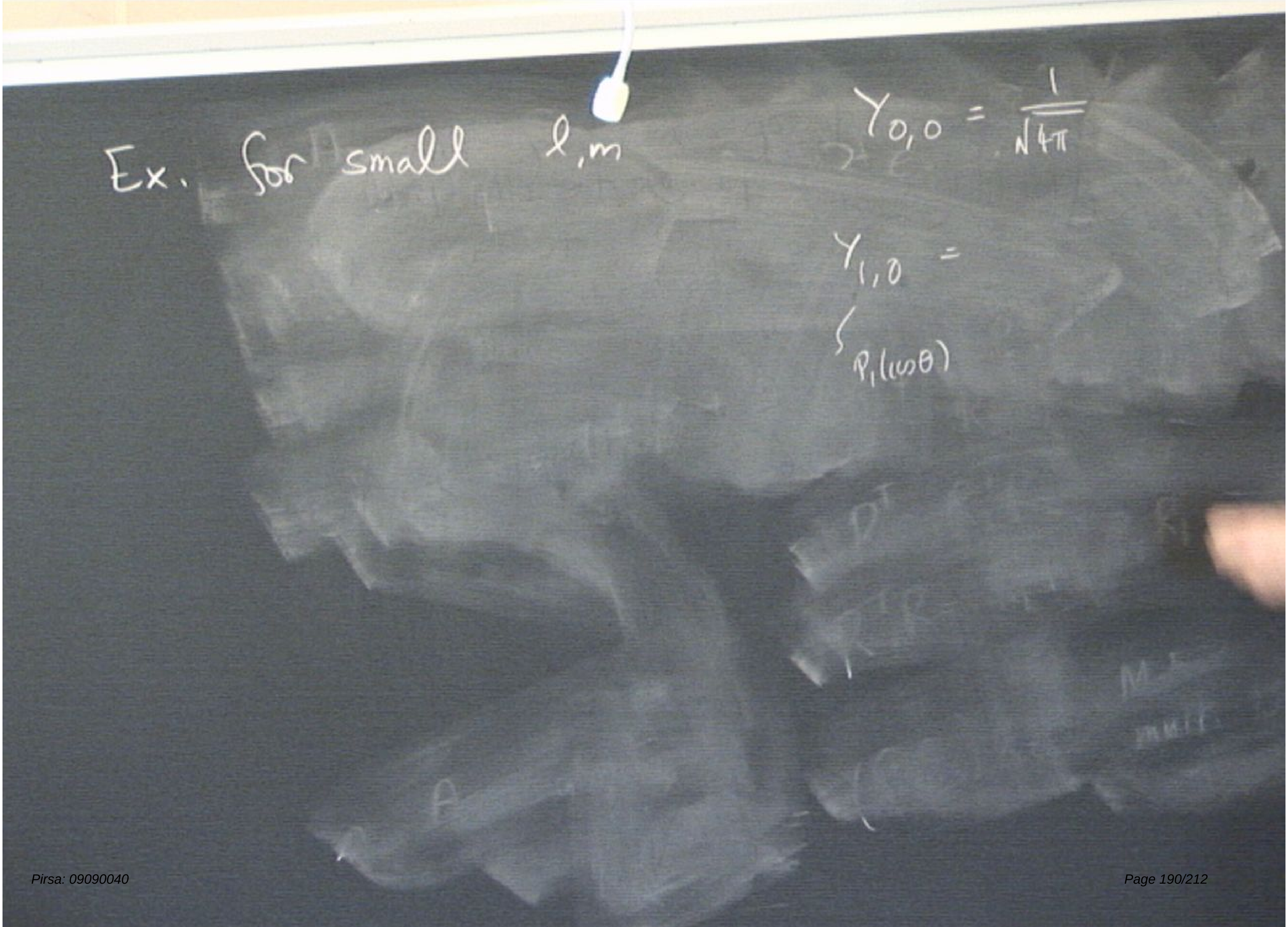
small l,m

$Y_{0,0} = \frac{1}{\sqrt{4\pi}}$

Ex. for small l, m

$$Y_{0,0} = \frac{1}{\sqrt{4\pi}}$$

$$Y_{1,0}$$



Ex. for small l, m

$$Y_{0,0} = \frac{1}{\sqrt{4\pi}}$$

$$Y_{1,0} =$$

$$\sim P_1(\cos\theta)$$

Ex. for small l, m

$$Y_{0,0} = \frac{1}{\sqrt{4\pi}}$$

$$Y_{1,0} = \cos\theta$$

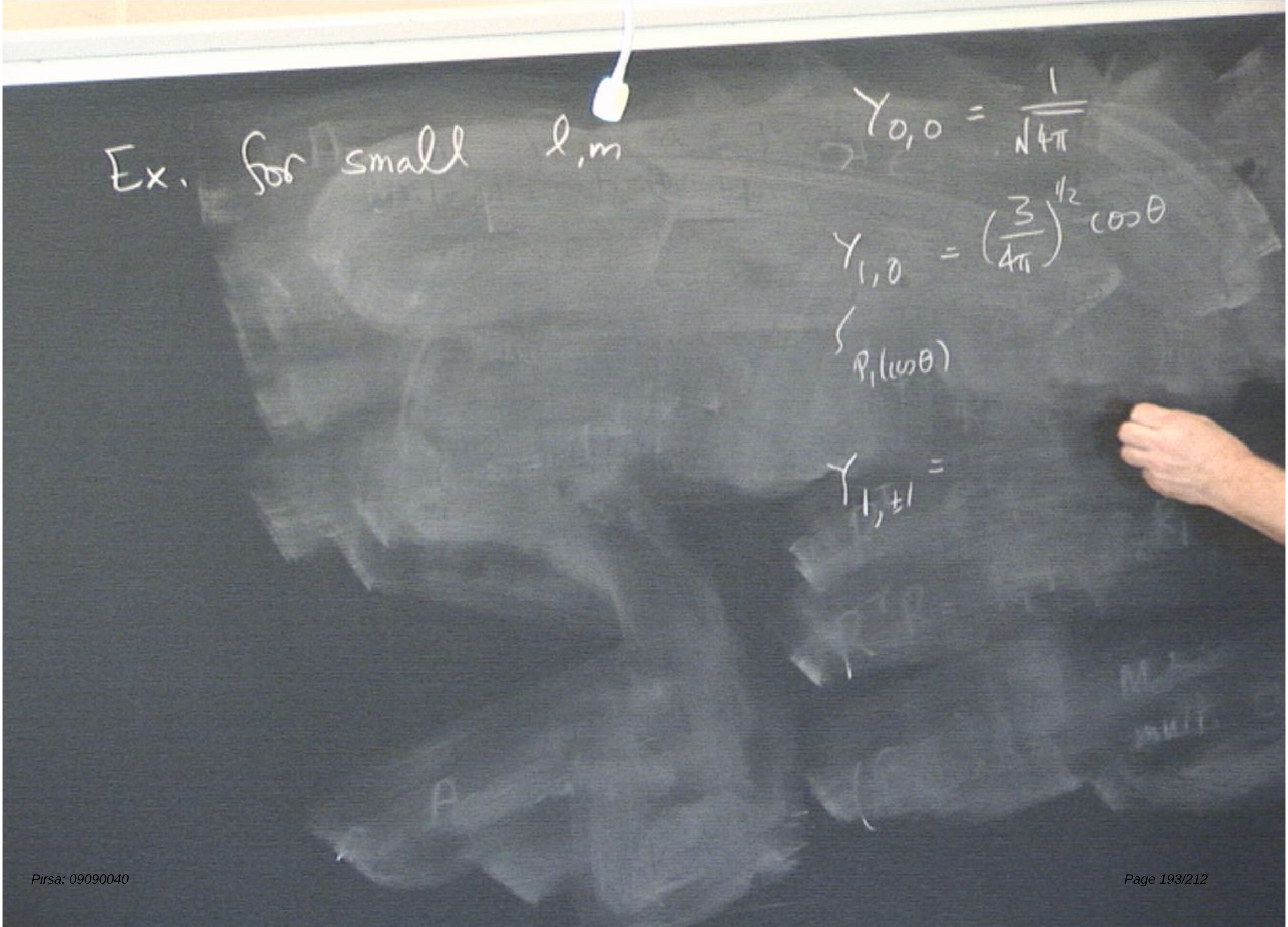
$$\sim P_1(\cos\theta)$$

Ex. for small l, m

$$Y_{0,0} = \frac{1}{\sqrt{4\pi}}$$

$$Y_{1,0} = \left(\frac{3}{4\pi}\right)^{1/2} \cos\theta$$

$P_1(\cos\theta)$



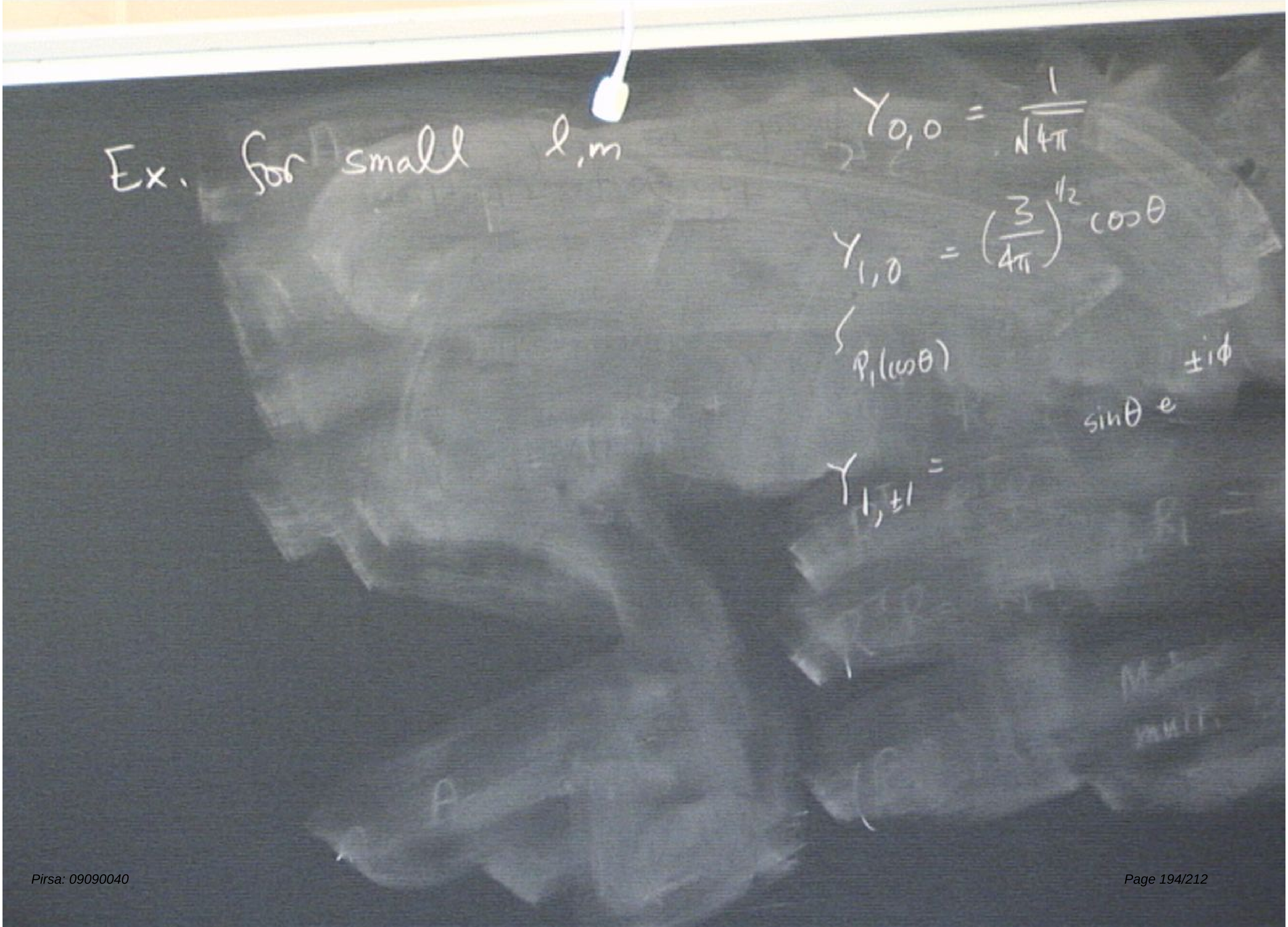
Ex. for small l, m

$$Y_{0,0} = \frac{1}{\sqrt{4\pi}}$$

$$Y_{1,0} = \left(\frac{3}{4\pi}\right)^{1/2} \cos\theta$$

$$\sim P_1(\cos\theta)$$

$$Y_{1,\pm 1} =$$



Ex. for small l,m

$$Y_{0,0} = \frac{1}{\sqrt{4\pi}}$$

$$Y_{1,0} = \left(\frac{3}{4\pi}\right)^{1/2} \cos\theta$$

$P_1(\cos\theta)$

$$Y_{1,\pm 1} = \quad \sin\theta \, e^{\pm i\phi}$$

Ex. for small l, m

$$Y_{0,0} = \frac{1}{\sqrt{4\pi}}$$

$$Y_{1,0} = \left(\frac{3}{4\pi}\right)^{1/2} \cos\theta$$

$P_1(\cos\theta)$

$$Y_{1,\pm1} = \left(\frac{3}{8\pi}\right)^{1/2} \sin\theta \, e^{\pm i\phi}$$

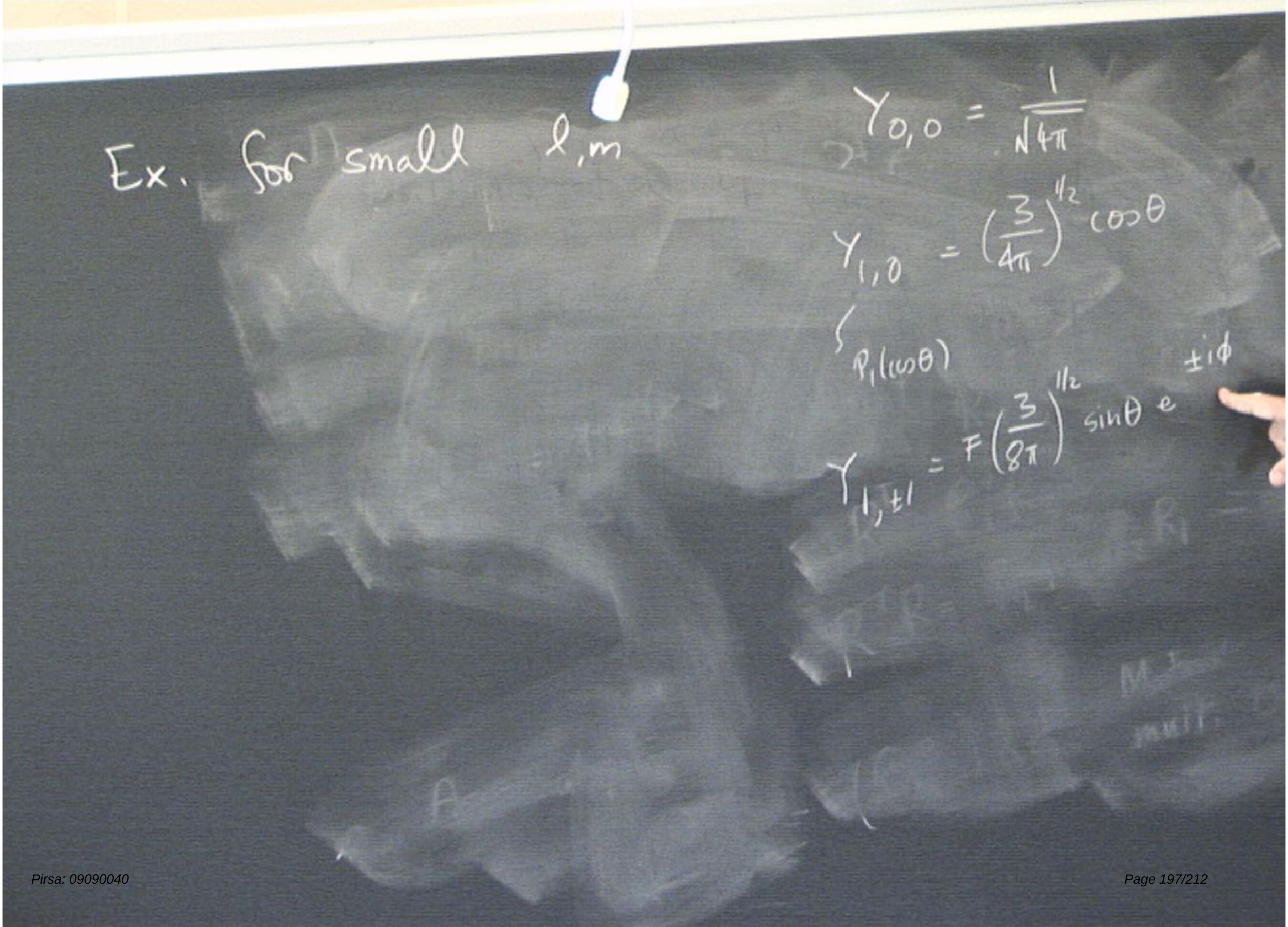
Ex. for small l, m

$$Y_{0,0} = \frac{1}{\sqrt{4\pi}}$$

$$Y_{1,0} = \left(\frac{3}{4\pi}\right)^{1/2} \cos\theta$$

$P_1(\cos\theta)$

$$Y_{1,\pm 1} = \mp\left(\frac{3}{8\pi}\right)^{1/2} \sin\theta \, e^{\pm i\phi}$$



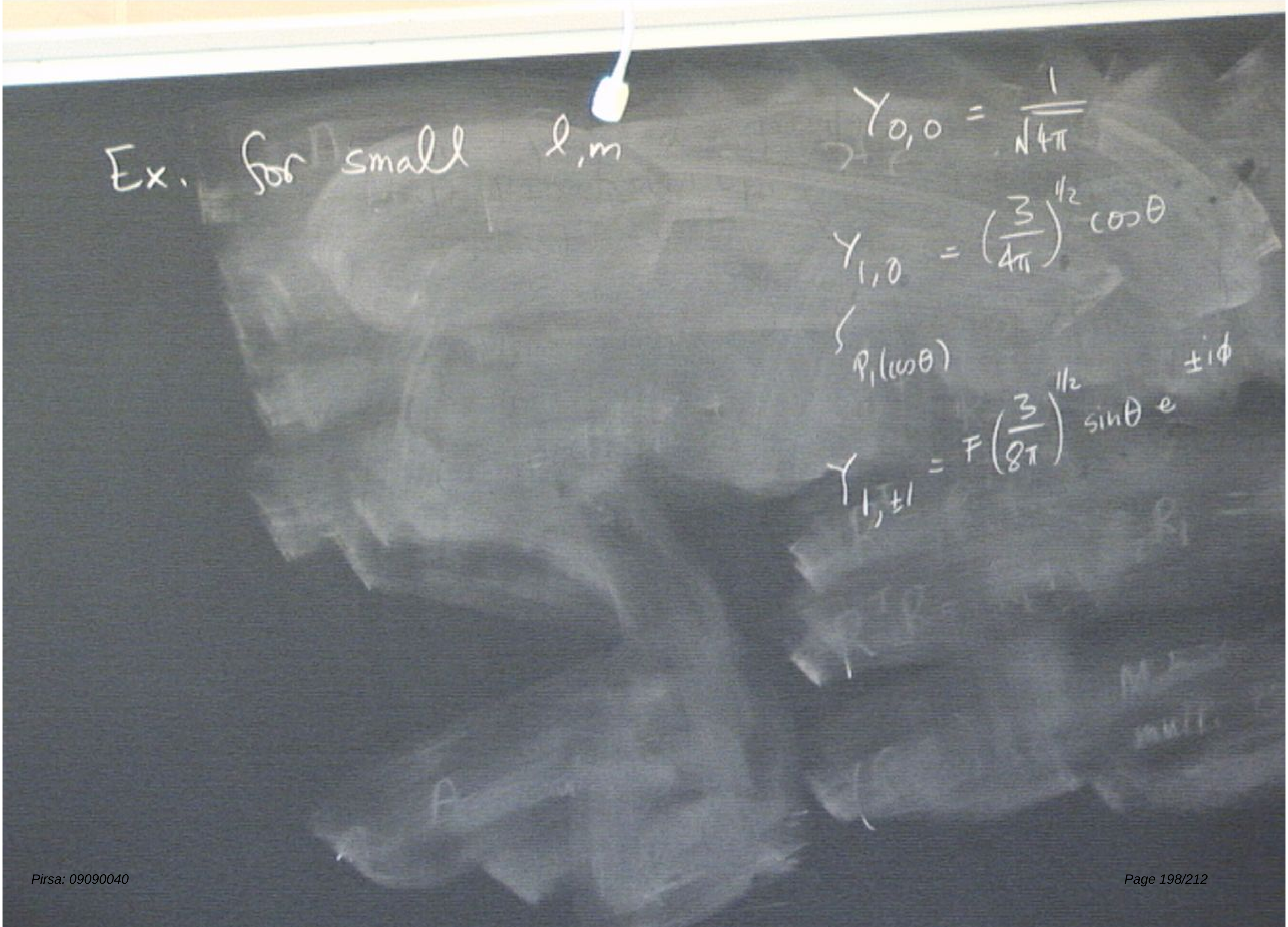
Ex. for small l,m

$$Y_{0,0} = \frac{1}{\sqrt{4\pi}}$$

$$Y_{1,0} = \left(\frac{3}{4\pi}\right)^{1/2} \cos\theta$$

$$\sim P_1(\cos\theta)$$

$$Y_{1,\pm1} = \mp\left(\frac{3}{8\pi}\right)^{1/2} \sin\theta \, e^{\pm i\phi}$$



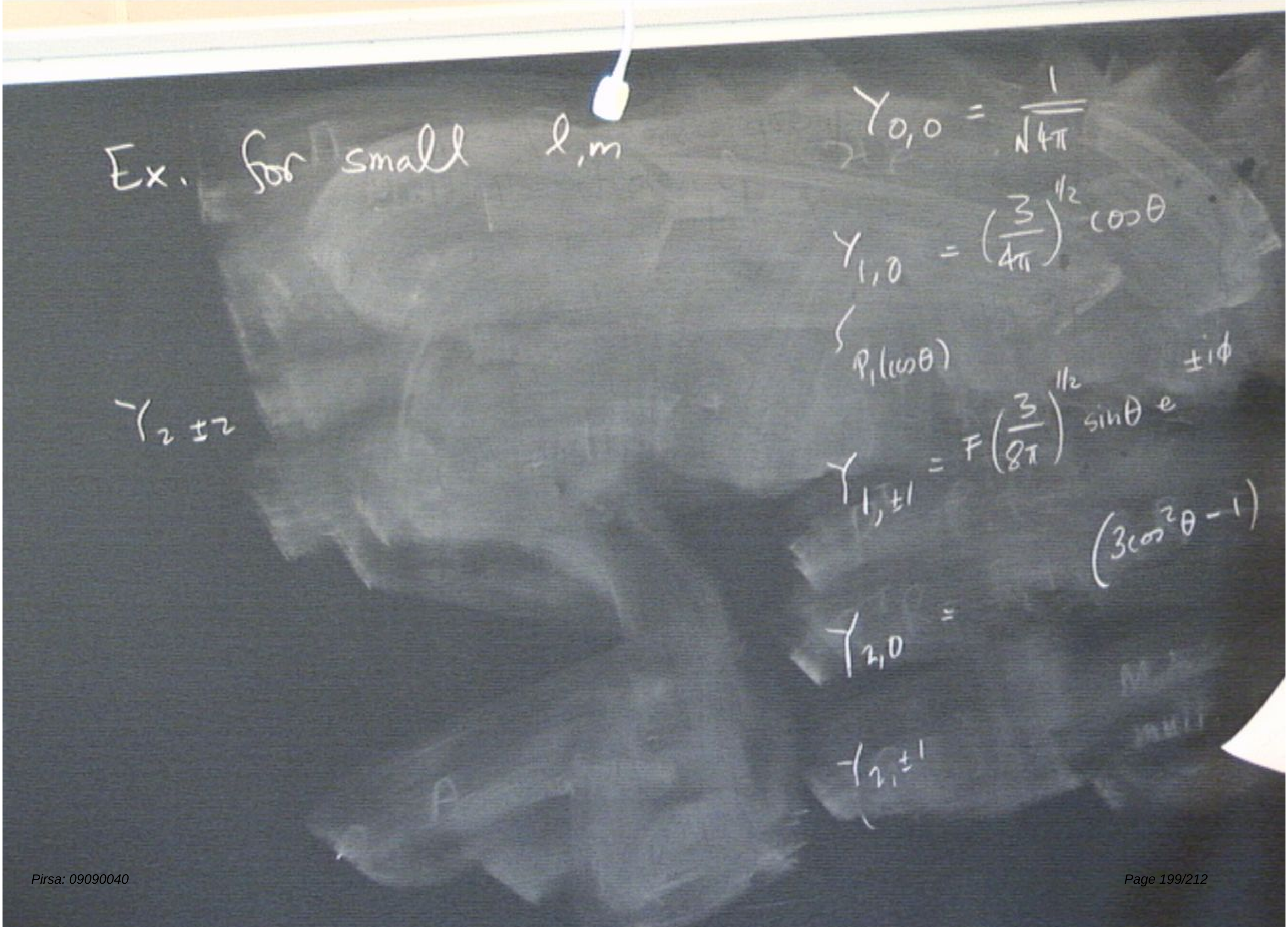
Ex. for small ℓ, m

$$Y_{0,0} = \frac{1}{\sqrt{4\pi}}$$

$$Y_{1,0} = \left(\frac{3}{4\pi}\right)^{1/2} \cos\theta$$

$P_1(\cos\theta)$

$$Y_{1,\pm 1} = \mp \left(\frac{3}{8\pi}\right)^{1/2} \sin\theta \, e^{\pm i\phi}$$



Ex. for small l, m

$$Y_{0,0} = \frac{1}{\sqrt{4\pi}}$$

$$Y_{1,0} = \left(\frac{3}{4\pi}\right)^{1/2} \cos\theta$$

$$\sim P_1(\cos\theta)$$

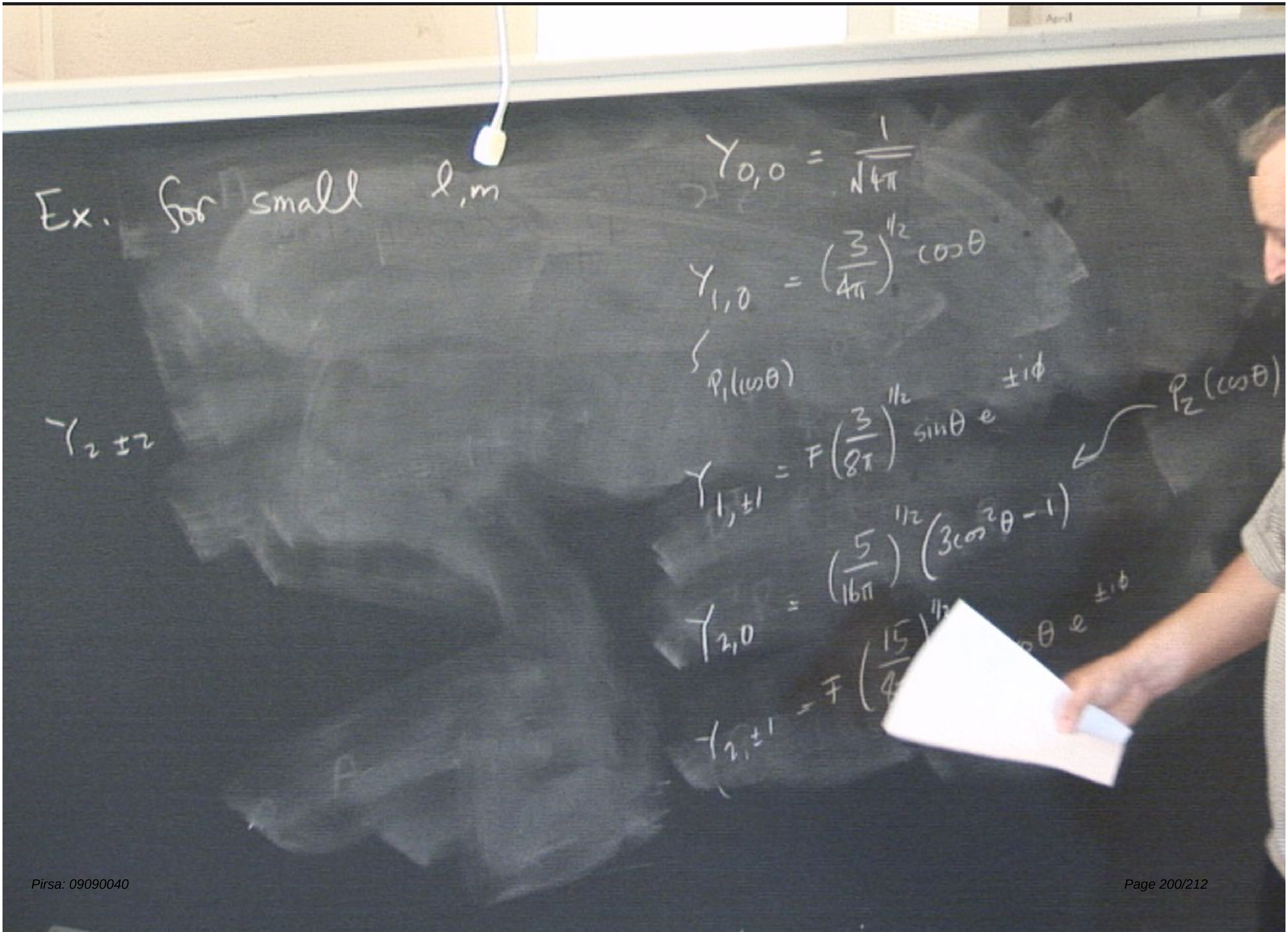
$$Y_{1,\pm1} = \mp\left(\frac{3}{8\pi}\right)^{1/2} \sin\theta \, e^{\pm i\phi}$$

$$(3\cos^2\theta - 1)$$

$$Y_{2,0} =$$

$$Y_{2,\pm1}$$

$$Y_{2\pm2}$$



Ex. for small l, m

$$Y_{0,0} = \frac{1}{\sqrt{4\pi}}$$

$$Y_{1,0} = \left(\frac{3}{4\pi}\right)^{1/2} \cos\theta$$

$P_1(\cos\theta)$

$$Y_{1,\pm 1} = \mp\left(\frac{3}{8\pi}\right)^{1/2} \sin\theta \, e^{\pm i\phi}$$

$P_2(\cos\theta)$

$$Y_{2,0} = \left(\frac{5}{16\pi}\right)^{1/2} \left(3\cos^2\theta - 1\right)$$

$Y_{2,\pm 1} = \mp\Big(\frac{15}{4}$ [illegible] $\theta \, e^{\pm i\phi}$

$Y_{2\,\pm 2}$

Ex. for small l, m

$$Y_{0,0} = \frac{1}{\sqrt{4\pi}}$$

$$Y_{1,0} = \left(\frac{3}{4\pi}\right)^{1/2}\cos\theta$$

$$\sim P_1(\cos\theta)$$

$$Y_{1,\pm1} = \mp\left(\frac{3}{8\pi}\right)^{1/2}\sin\theta\, e^{\pm i\phi}$$

$$Y_{2,0} = \left(\frac{5}{16\pi}\right)^{1/2}\left(3\cos^2\theta - 1\right) \longleftarrow P_2(\cos\theta)$$

$$Y_{2,\pm1} = \mp\left(\frac{15}{8\pi}\right)^{1/2}\sin\theta\cos\theta\, e^{\pm i\phi}$$

$$Y_{2\pm2} = \left(\frac{15}{32\pi}\right)^{1/2}\sin^2\theta\, e^{\pm 2i\phi}$$

Ex. for small l, m

$$Y_{2 \pm 2} = \left(\frac{15}{32\pi}\right)^{1/2} \sin^2\theta \, e^{\pm 2i\phi}$$

$$Y_{0,0} = \frac{1}{\sqrt{4\pi}}$$

$$Y_{1,0} = \left(\frac{3}{4\pi}\right)^{1/2} \cos\theta$$

$P_1(\cos\theta)$

$$Y_{1,\pm 1} = \mp\left(\frac{3}{8\pi}\right)^{1/2} \sin\theta \, e^{\pm i\phi}$$

$P_2(\cos\theta)$

$$Y_{2,0} = \left(\frac{5}{16\pi}\right)^{1/2} \left(3\cos^2\theta - 1\right)$$

$$Y_{2,\pm 1} = \mp\left(\frac{15}{8\pi}\right)^{1/2} \sin\theta\cos\theta \, e^{\pm i\phi}$$

Ex. for small l, m

$$Y_{2\pm2} = \left(\frac{15}{32\pi}\right)^{1/2} \sin^2\theta \, e^{\pm 2i\phi}$$

$$Y_{lm}(\theta, \phi) = \langle x | lm \rangle$$

$$Y_{0,0} = \frac{1}{\sqrt{4\pi}}$$

$$Y_{1,0} = \left(\frac{3}{4\pi}\right)^{1/2} \cos\theta$$

$$\wr$$

$$P_1(\cos\theta)$$

$$Y_{1,\pm1} = \mp\left(\frac{3}{8\pi}\right)^{1/2} \sin\theta \, e^{\pm i\phi}$$

$$Y_{2,0} = \left(\frac{5}{16\pi}\right)^{1/2} (3\cos^2\theta - 1) \leftarrow P_2(\cos\theta)$$

$$Y_{2,\pm1} = \mp\left(\frac{15}{8\pi}\right)^{1/2} \sin\theta\cos\theta \, e^{\pm i\phi}$$

Ex. for small l, m

$$Y_{2\,\pm 2} = \left(\frac{15}{32\pi}\right)^{1/2} \sin^2\theta\, e^{\pm 2i\phi}$$

$$Y_{lm}(\theta, \phi) = \langle x | \underline{lm} \rangle$$

$$Y_{0,0} = \frac{1}{\sqrt{4\pi}}$$

$$Y_{1,0} = \left(\frac{3}{4\pi}\right)^{1/2} \cos\theta$$

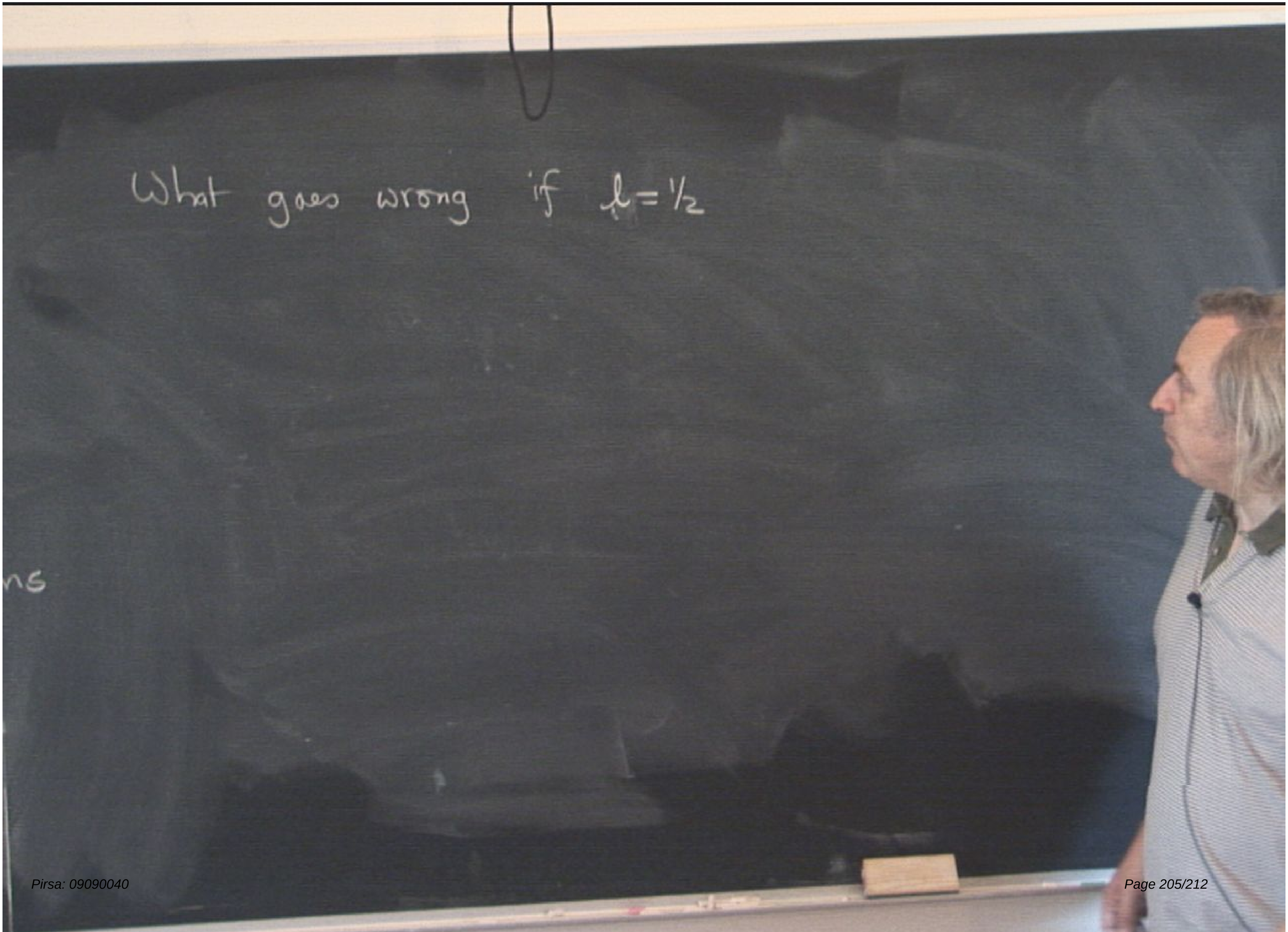
$P_1(\cos\theta)$

$$Y_{1,\pm 1} = \mp\left(\frac{3}{8\pi}\right)^{1/2} \sin\theta\, e^{\pm i\phi}$$

$P_2(\cos\theta)$

$$Y_{2,0} = \left(\frac{5}{16\pi}\right)^{1/2} (3\cos^2\theta - 1)$$

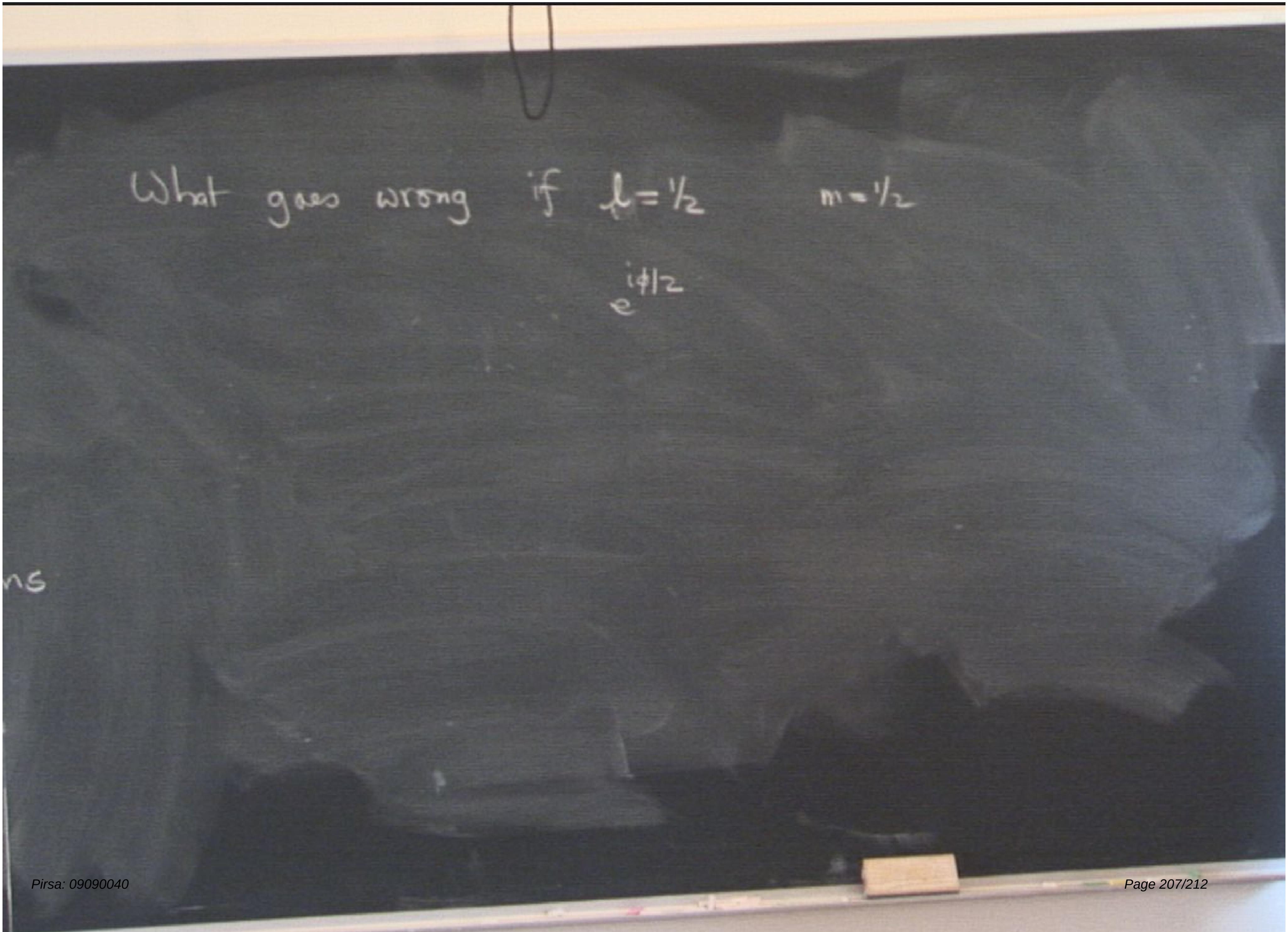
$$Y_{2,\pm 1} = \mp\left(\frac{15}{8\pi}\right)^{1/2} \sin\theta\cos\theta\, e^{\pm i\phi}$$



What goes wrong if $\ell = 1/2$

What goes wrong if $\ell = 1/2$ $m = 1/2$

ns



What goes wrong if $\ell = 1/2$ $m = 1/2$

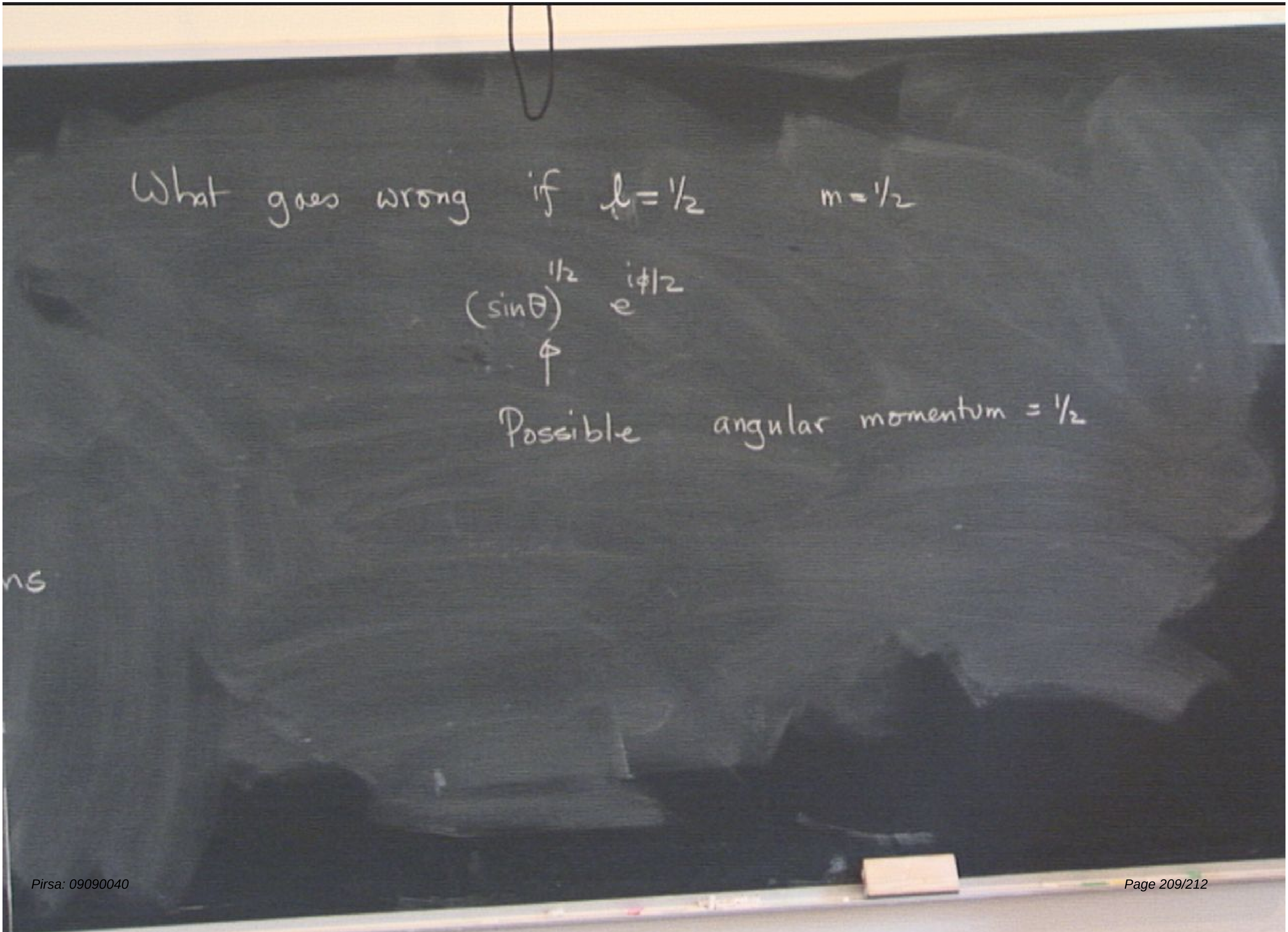
$e^{i\phi/2}$

ns

What goes wrong if $\ell = 1/2$ $m = 1/2$

$$(\sin\theta)^{1/2} \, e^{i\phi/2}$$

ns



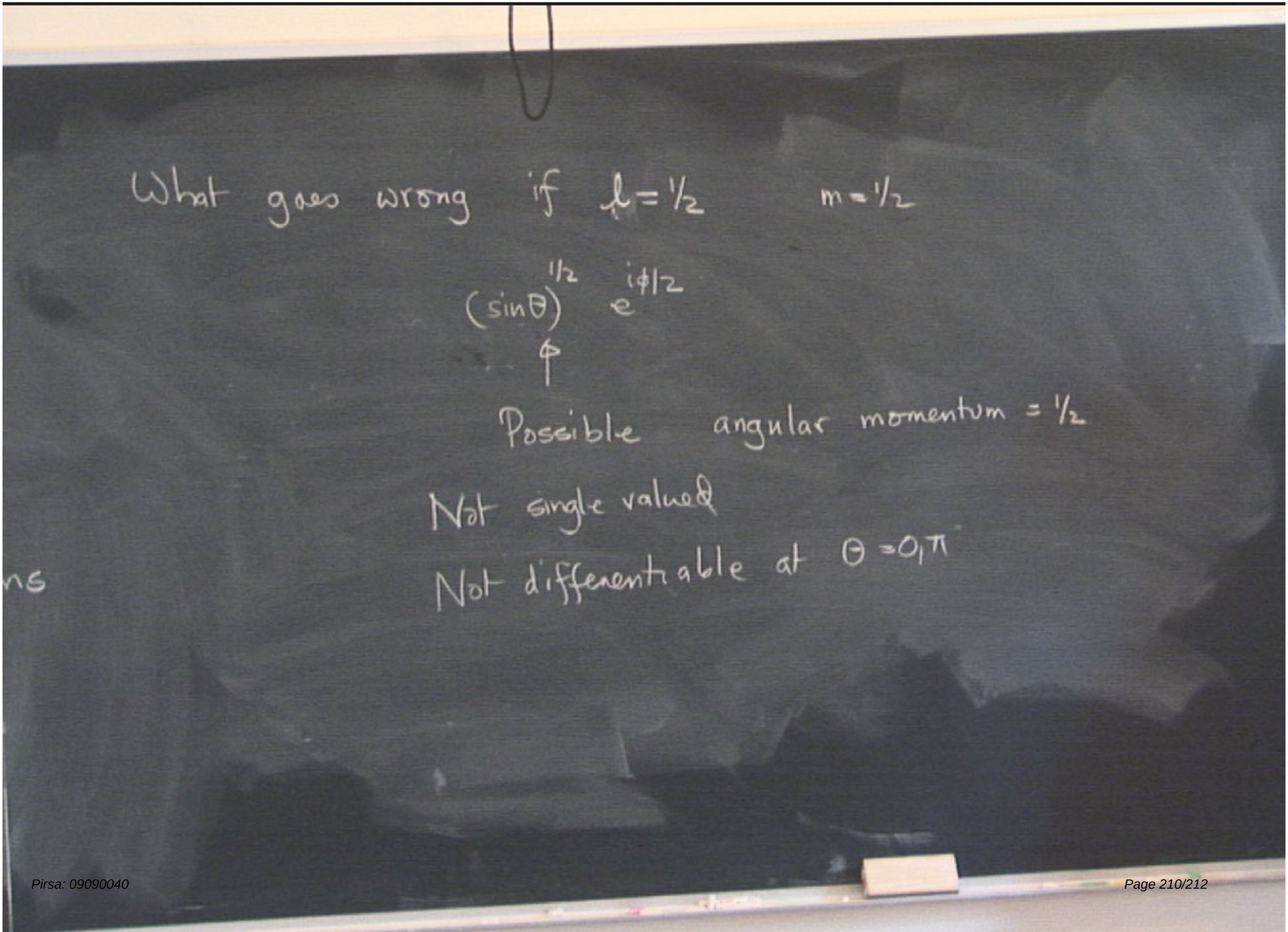
What goes wrong if $\ell = 1/2$ $m = 1/2$

$$(\sin\theta)^{1/2} \, e^{i\phi/2}$$

$$\uparrow$$

Possible angular momentum $= 1/2$

ns



What goes wrong if $\ell = 1/2$ $m = 1/2$

$$(\sin\theta)^{1/2} e^{i\phi/2}$$

$$\uparrow$$

Possible angular momentum = 1/2

Not single valued

Not diffenentiable at $\theta = 0, \pi$

ns

What goes wrong if $\ell = 1/2$ $m = 1/2$

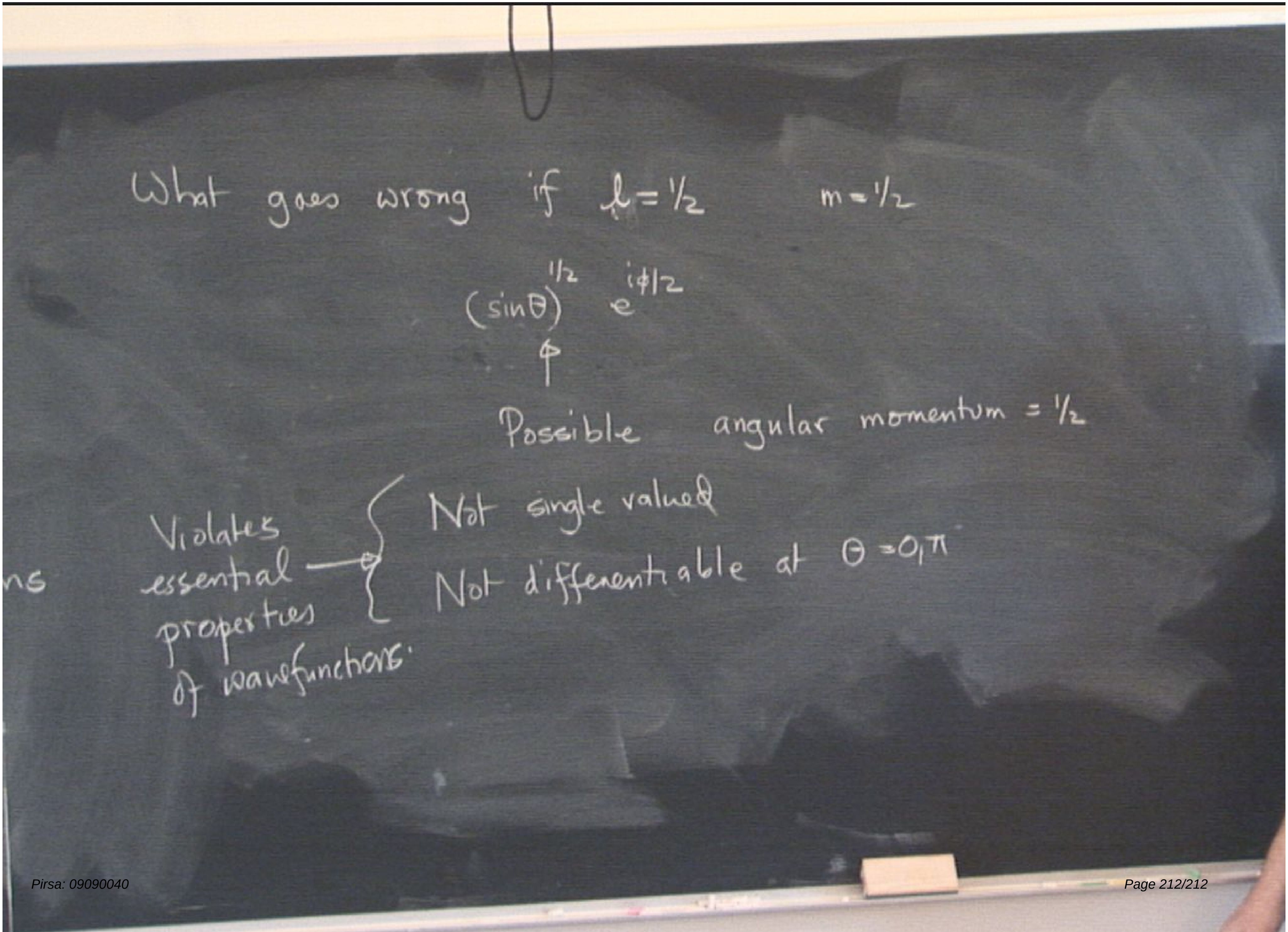
$$(\sin\theta)^{1/2} \; e^{i\phi/2}$$

$\uparrow$

Possible angular momentum = $1/2$

ns

Violates essential properties of wavefunctions $\longrightarrow$
$\left\{ \begin{array}{l} \text{Not single valued} \\ \text{Not differentiable at } \theta = 0, \pi \end{array} \right.$



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