Title: Betting on Quantum Theory

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Abstract: Betting (or gambling) is a useful tool for studying decision-making in the face of [classical] uncertainty. We would like to understand how a quantum "agent" would act when faced with uncertainty about its [quantum] environment. I will present a preliminary construction of a theory of quantum gambling, motivated by roulette and quantum optics. I'll begin by reviewing classical gambling and the Kelly Criterion for optimal betting. Then I'll demonstrate a quantum optical version of roulette, and discuss some of the challenges and pitfalls in designing such analogues. Quantum agents have access to many more strategies than classical agents. Quantum strategies provide no advantage in classical roulette, but I'll show that a quantum agent can outperform a classical agent in quantum roulette.

Grant Salton

Betting on Quantum Theory

Outline

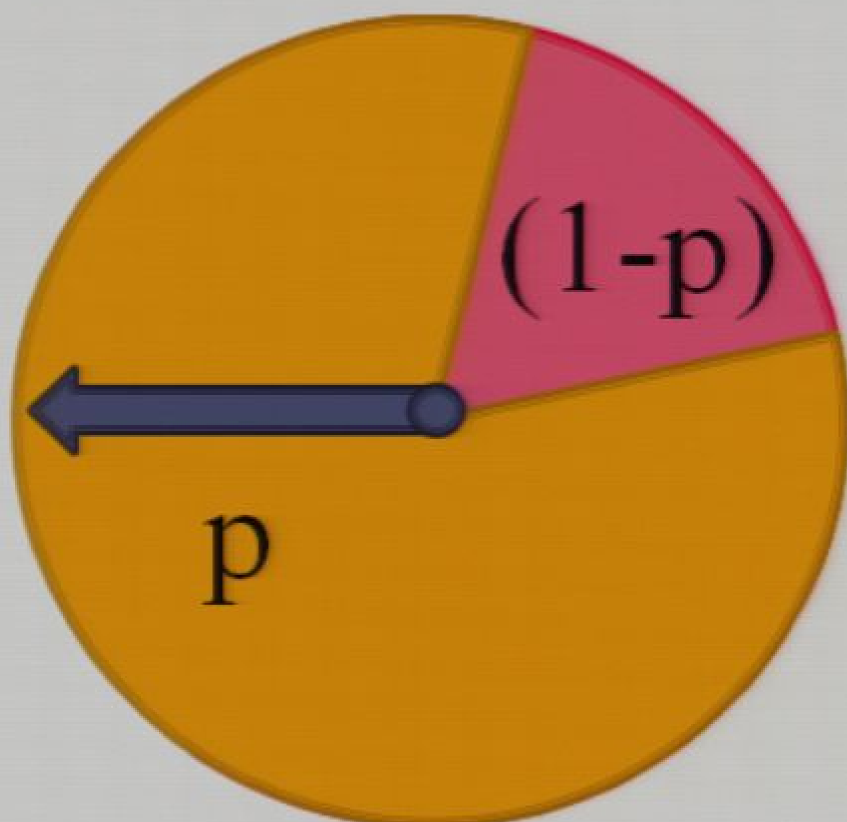
- Motivation
- Classical gambling and the Kelly Criterion
- Model Building
- Five simple quantum scenarios
- Amplitude damping channels
- Amplification basis equivalence
- Solutions to quantum scenarios

Motivation

- We want to study decision-making in the face of uncertainty
- Agents use decision-making rules
- Usually try to optimize something
- Gambling is a paradigm we can use
  - Uncertainty arises from nature of the game
  - Betting strategies are decision making rules
  - Try to optimize winnings, say

Classical Gambling

- Consider generalized roulette

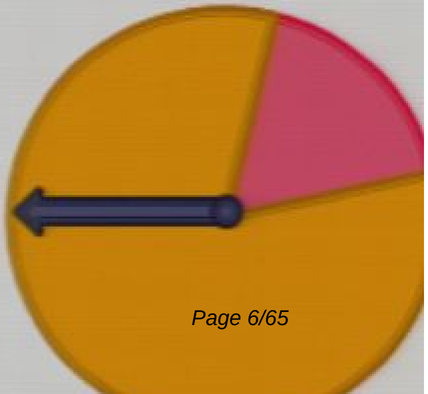


Classical Gambling

- The agent must bet entire bankroll
- Divide bankroll according to the strategy

$$\{x, 1 - x\}$$

- We assume that we do not change strategies between games (i.e., x is a fixed number)
- The goal is to maximize money



Classical Gambling

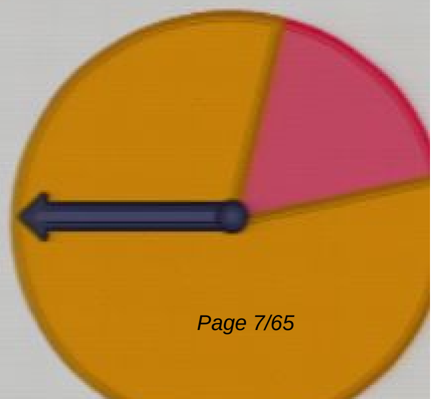
- Guess all or nothing betting $x = \begin{cases} 1, p > 0.5 \\ 0, p < 0.5 \end{cases}$

$$\langle \$ \rangle = (2p)^N \rightarrow \infty \;\; as \;\; N \rightarrow \infty$$

- But this game is geometric, so a single loss would be fatal

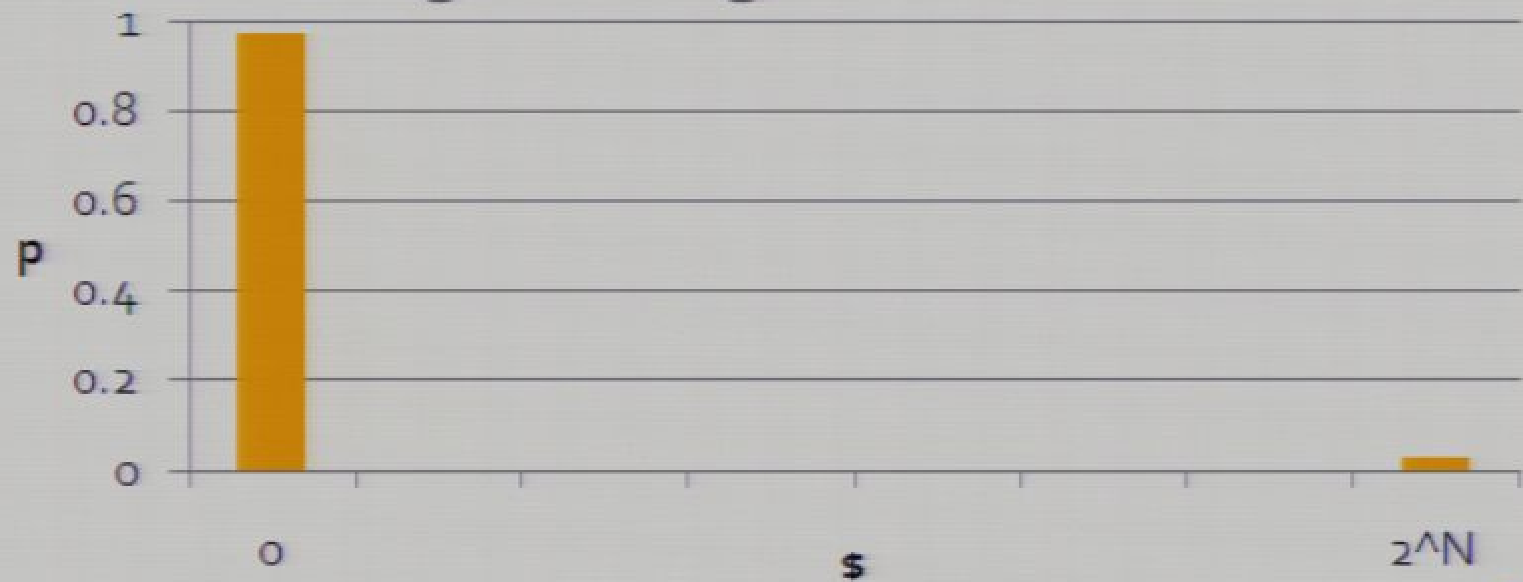
$$\langle \log \$ \rangle = p \log 2 + (1-p) \log 0$$
$$\rightarrow -\infty$$

- This is gambler's ruin

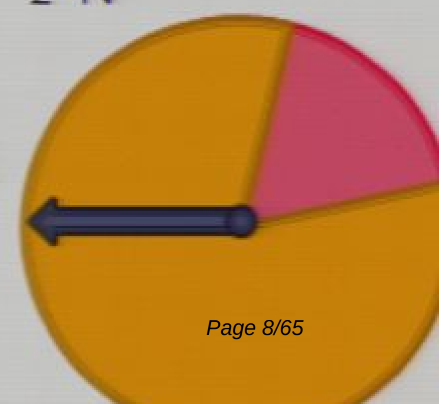


Classical Gambling

- All or nothing betting:



- log$ follows a binomial random walk
  - Median follows mean



The Kelly Criterion

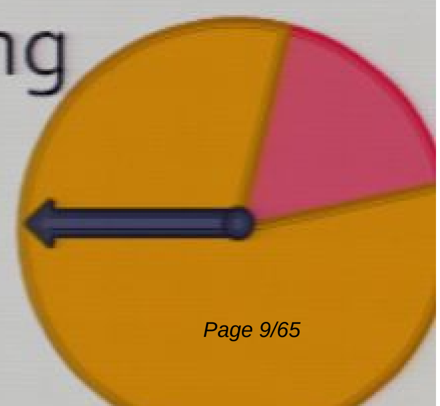
- We actually want to maximize

$$\langle \log \$ \rangle = p \log 2x + (1-p) \log 2(1-x)$$

- For maximum growth, we find

$$x = p$$

- This is the **Kelly Criterion** for gambling (John Kelly, 1956)



Model Building

- We studied different (unsuccessful) models
  - Carefully constructed *physical* models
  - Harmonic Oscillators
- These models suffer from problems
  - E.g., oscillator model requires knowledge of agent's phase space state

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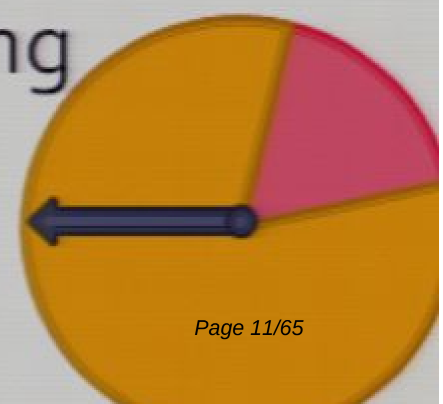

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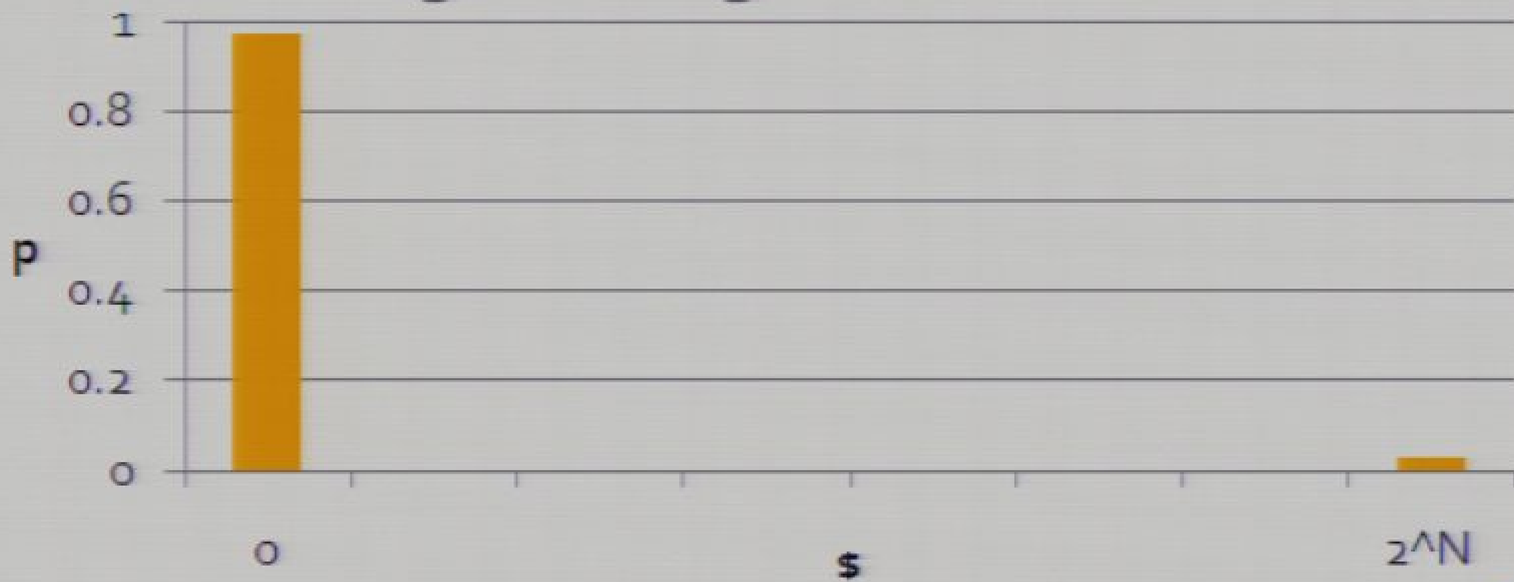

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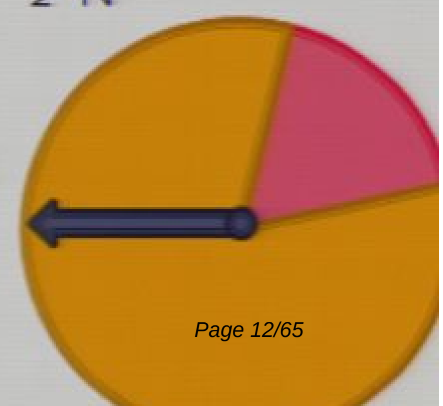

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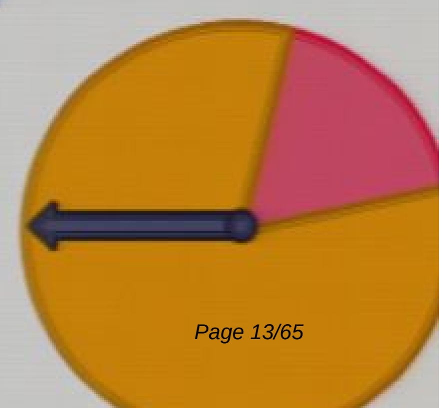

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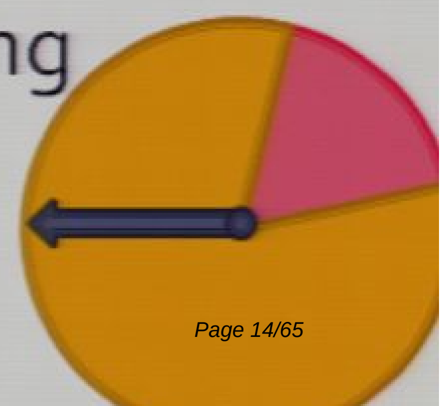

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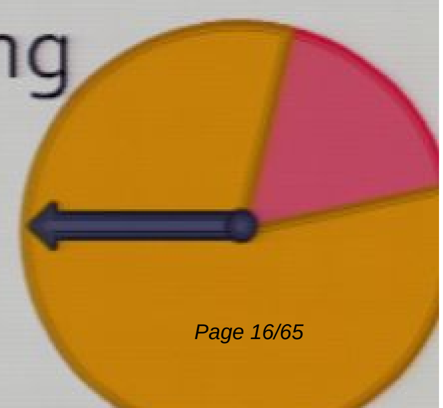

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Model Building

- Let's start from scratch
- Let's keep it very abstract

Model Building

- Our simplified model:
- Two registers represented by a product state

  e.g., $|\psi\rangle = |n\rangle|m\rangle$

- This model has the same Hilbert space as a Bosonic field with two modes

- Agents can perform any CP map on states (betting), which does not change $n + m$

Model Building

- The 'casino' performs two operations:
  - Filtration (erasing one register)
  - Doubling (or Amplification)
- The game consists of two alternating parts
  - First the agent bets (using approved CP maps)
  - Then the casino filters and doubles

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- The game is played unobserved many times
- At the end, observers measure the agent's success (perhaps $\widehat{N}$ or $\log \widehat{N}$)
- We want to address the measurement problem
  - Try not to meddle with the agent – see what he can do while we're not looking

Model Building

- Now draw an analogy with quantum optics
  - Excitation of the field are photons
  - Modes correspond to Horizontal and Vertical polarization
- We can use the Fock Space representation

e.g., $|H\rangle = |1,0\rangle$ $\quad \frac{|HV\rangle + |VH\rangle}{\sqrt{2}} = |1,1\rangle$

$|V\rangle = |0,1\rangle$

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Model Building

- In optics, the filtration is done with a polarizing filter
- Doubling can be done using an amplifier
- Example:

  state (photon): $|\psi\rangle = |H\rangle$

  filter: aligned horizontally

  amplifier: aligned horizontally

  Once the game is played:

$$|\psi'\rangle\langle\psi'| = |HH\rangle\langle HH| = |2,0\rangle\langle 2,0|$$

Model Setup

- Agent is given initial pure state with **well defined *N***

  $N \equiv$ excitation number - e.g., $\hat{N}|n,m\rangle = \overbrace{(n+m)}^{N}|n,m\rangle$
- Agent is free to allocate the *N* excitations using any CP map which commutes with $\hat{N}$
- The prepared state is then filtered through some operation $\mathcal{F}$ and doubled by another operation $\mathcal{D}$

Five Simple Quantum Scenarios

1) A probabilistic filter in H/V, doubling in H/V
2) A deterministic filter in $\psi = \alpha H + \beta V$, doubling in H/V
3) A probabilistic filter in $\psi/\bar{\psi}$, doubling in H/V
4) A probabilistic filter in $\psi_1, \psi_2, \psi_3, \ldots$, doubling in H/V
5) Filter in $\psi/\bar{\psi}$, double in $\phi/\bar{\phi}$, agent only knows about H/V (classical agent)

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Scenario #1 › Prob. filter in H/V › Double in H/V

- The agent is free to allocate *N*, e.g.,

$$|\psi\rangle = \sum_{n=0}^{N} c_n \, |n, N-n\rangle$$

$$|\psi\rangle\langle\psi| = \sum_{j=0}^{N}\sum_{k=0}^{N} c_k^* c_j \, |j, N-j\rangle\langle k, N-k|$$

- $\mathcal{F}$ filters out V with probability *p*

  H with probability (1-*p*)

$$\mathcal{F}[|\psi\rangle\langle\psi|] = ?$$

Amplitude Damping Channel

- We need a map which takes (for example)

$$|n,m\rangle\langle n,m| \rightarrow |n,0\rangle\langle n,0|$$

- One might expect

$$\hat{F}[|\psi\rangle] = \left(\sum_n |n,0\rangle\langle n,0|\right)|\psi\rangle$$

- However, if $|\psi\rangle$ has only terms with non-zero V modes (e.g., $|\psi\rangle = \frac{|1,1\rangle + |0,2\rangle}{\sqrt{2}}$), $\hat{F}|\psi\rangle = 0|\psi\rangle$

- The map is not unitary

Amplitude Damping Channel

- Use the Stinespring Dilation Theorem
  - Introduce an ancilla
  - Perform a unitary SWAP
  - Throw away the ancilla

$$|n_A, m_B, 0_C\rangle \rightarrow |n_A, 0_B, m_C\rangle$$

$$|n_A, 0_B, m_C\rangle\langle n_A, 0_B, m_C| \rightarrow Tr_C|n_A, 0_B, m_C\rangle\langle n_A, 0_B, m_C|$$

- Here, n_A and m_B are correlated, since the total N is **well defined** $(n + m = N)$

Amplitude Damping Channel

- Tracing out the ancilla removes information
  - $\Rightarrow$ Decoherence
- Represent the operation with superoperator:

$$\mathcal{F}[|i,j\rangle\langle n,m|] = \langle j|m\rangle |i,0\rangle\langle n,0|$$

- The inner product removes coherences between terms with different numbers of V
- N well defined $\Rightarrow$ $\mathcal{F}$ removes all coherences
- If N is not well defined the result doesn't hold

Scenario #1

Prob. filter in H/V → Double in H/V

- Return to scenario 1. We now find

$$\mathcal{F}[|\psi\rangle\langle\psi|] = p\sum_{n=0}^{N}|c_n|^2\,|n,0\rangle\langle n,0| + (1-p)\sum_{n=0}^{N}|c_n|^2\,|0,N-n\rangle\langle 0,N-n|$$

- We now apply the doubling operation
  - In the H/V basis:

$$\mathcal{D}[|n,m\rangle\langle n,m|] = |2n,2m\rangle\langle 2n,2m|$$

- Now find $\langle \log \widehat{N} \rangle = Tr\left[\log \widehat{N}\,\rho\right]$ and maximize

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Scenario #1

Prob. filter in H/V → Double in H/V

- After maximizing $\langle \log \widehat{N} \rangle$, we find
  - Strategies with allocations such as $|n, 0\rangle$ or $|0, m\rangle$ lead to gambler's ruin

$$|\psi\rangle = \sum_{n=0}^{N} c_n |n, N-n\rangle$$

- We know that a superposition of strategies (or allocations) is not optimal:
  - Filtration removes coherences, leaving a mixture
  - We know that *mixtures* of strategies are not optimal

Scenario #1

Prob. filter in H/V → Double in H/V

- So our strategy is of the form $|n, N-n\rangle$
- Find optimal strategy is: **Kelly!**

$$|\psi\rangle = |Np, N(1-p)\rangle$$

- Quantum agents have no advantage in a classical game

Scenario #2

Deterministic filter in ψ → Double in H/V

- The filter is now characterized by

$$\psi = \alpha H + \beta V$$

- With sufficient classical knowledge (α & β) 100% transmission through filter
- Doubling is basis specific:
  - Amplification of an arbitrary state:

$$|\psi\rangle \rightarrow |\psi\rangle \otimes |\psi\rangle$$

  violates the No-Cloning theorem

Doubling Basis Equivalence

Alice	Bob
Filter in HV	Filter in HV
Double in HV	Double in $\phi/\bar{\phi}$

$$|\phi\rangle = \alpha|H\rangle + \beta|V\rangle$$

$$|\bar{\phi}\rangle = \beta|H\rangle - \alpha|V\rangle$$

Doubling Basis Equivalence

- We assume they have well defined N
- To change basis:

$$|n,0\rangle\langle n,0|_{HV} = \sum_{k=0}^{n}\sum_{j=0}^{n}\left[\sqrt{\binom{n}{k}\binom{n}{j}}\frac{\alpha^{2n-k-j}\beta^{k+j}}{(\alpha^2+\beta^2)^n}|n-k,k\rangle\langle n-j,j|_{\phi\bar{\phi}}\right]$$

- Alice doubles in HV, then changes to $\phi/\bar{\phi}$
- Bob changes to $\phi/\bar{\phi}$ then doubles
- We compare their final density matrices

Doubling Basis Equivalence

- After filtering and doubling, Alice has

$$\mathcal{D}_A[\rho_A] = \sum_{n=0}^{N} |c_n|^2 \sum_{k=0}^{2n} \sum_{j=0}^{2n} \left[\sqrt{\binom{2n}{k}\binom{2n}{j}} \frac{\alpha^{4n-k-j}\beta^{k+j}}{(\alpha^2+\beta^2)^{2n}} |2n-k, k\rangle\langle 2n-j, j|_{\phi\bar{\phi}} \right]$$

- And Bob has

$$\mathcal{D}_B[\rho_B] = \sum_{n=0}^{N} |c_n|^2 \sum_{k=0}^{n} \sum_{j=0}^{n} \left[\sqrt{\binom{n}{k}\binom{n}{j}} \frac{\alpha^{2n-k-j}\beta^{k+j}}{(\alpha^2+\beta^2)^{n}} |2(n-k), 2k\rangle\langle 2(n-j), 2j|_{\phi\bar{\phi}} \right]$$

- The structure is block diagonal corresponding to fixed values of *N*

Doubling Basis Equivalence

- Trace of the blocks is invariant:

$$Tr[\mathcal{D}_A[\rho_A]]_N = Tr[\mathcal{D}_B[\rho_B]]_N = |c_n|^2$$

- The agent can apply any betting CP-map

- $\Rightarrow$ Choice of doubling basis is irrelevant

Scenario #2

Deterministic filter in ψ → Double in H/V

- Filter characterization is unitarily equivalent to H
- Doubling basis is unitarily equivalent to $\psi/\bar{\psi}$
- Actually, doubling basis is irrelevant
  - Filter in HV (just H), double in HV

- We find that scenario #2 is identical to scenario #1 with p=1 for a quantum agent

Scenario #2

Deterministic filter in ψ → Double in H/V

- Then optimal strategy is: $|N,0\rangle_{\psi/\bar{\psi}}$
- This is **Quantum Kelly.**
  - $|N,0\rangle_{\psi/\bar{\psi}}$ is a superposition strategy (when written in the HV basis)
  - Classically, this strategy is not accessible – it is a quantum strategy

Scenario #3 — Prob. filter in ψ — Double in H/V

- Similarly to scenario #2, this can be transformed into scenario #1.
- Unitary equivalence to prob. filter in HV, and double in $\psi/\bar{\psi}$
- Double basis irrelevant $\Rightarrow$ scenario #3 is identical to scenario #1.

- For well defined *N*, quantum games become classical, and **Kelly** is still optimal

Scenario #3

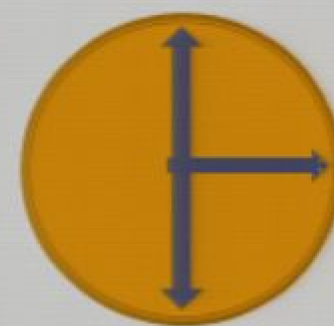
Prob. filter in ψ › Double in H/V

- Note that the Kelly strategy here is again **quantum Kelly**

- $|Np, N(1-p)\rangle_{\psi/\bar{\psi}}$ is a superposition strategy
  - A superposition of different allocations in HV

Scenario #4

Prob. filter in $\psi_1/\psi_2/\ldots$ | Double in H/V



- The filter acts probabilistically in non-orthogonal 'directions'
  - Filter mode $|\psi_i\rangle$ with probability p_i

- We wish to characterize such a filter
- Perhaps the filter is completely described by

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$$

Scenario #4

Prob. filter in $\psi_1/\psi_2/\ldots$ | Double in H/V

- It turns out: *No!* Why?
- Consider 50/50 filtering in HV

$$\rho_{HV} = \frac{1}{2}|HH\rangle\langle HH| + \frac{1}{2}|VV\rangle\langle VV| = \frac{\mathbb{I}}{2}$$

- and 50/50 filtering in AD

$$\rho_{AD} = \frac{1}{2}|DD\rangle\langle DD| + \frac{1}{2}|AA\rangle\langle AA| = \frac{\mathbb{I}}{2}$$

- But in these cases, the optimal strategies are different: $\left|\frac{N}{2},\frac{N}{2}\right\rangle_{HV} \neq \left|\frac{N}{2},\frac{N}{2}\right\rangle_{AD}$

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Scenario #5

Prob. filter in ψ → Double in φ

- The quantum agent changes from $\psi/\bar{\psi}$ to HV
- Then doubling basis is equivalent to HV
  - Quantum agent has transformed to scenario #1

- Classical agent cannot change the system and must work in HV
- No concrete results yet – brainstorming!

Scenario #4

Prob. filter in $\psi_1/\psi_2/...$ | Double in H/V

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Next Steps…

- Complete analysis of scenarios 4 and 5
- Consider the case of indeterminate *N*
- Different betting strategies outside of those considered (e.g., superposition **of** strategies)
- Agent does not have perfect knowledge of the system

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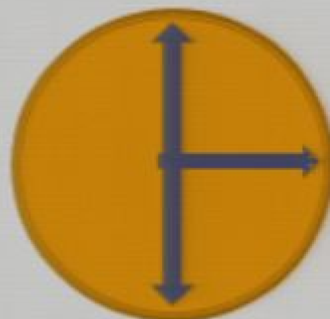

- We wish to characterize such a filter
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$$\rho_{HV} = \frac{1}{2}|HH\rangle\langle HH| + \frac{1}{2}|VV\rangle\langle VV| = \frac{\mathbb{I}}{2}$$

- and 50/50 filtering in AD

$$\rho_{AD} = \frac{1}{2}|DD\rangle\langle DD| + \frac{1}{2}|AA\rangle\langle AA| = \frac{\mathbb{I}}{2}$$

- But in these cases, the optimal strategies are different: $\left|\frac{N}{2},\frac{N}{2}\right\rangle_{HV} \neq \left|\frac{N}{2},\frac{N}{2}\right\rangle_{AD}$

Scenario #1

Prob. filter in H/V › Double in H/V

- The agent is free to allocate N, e.g.,

$$|\psi\rangle = \sum_{n=0}^{N} c_n \, |n, N-n\rangle$$

$$|\psi\rangle\langle\psi| = \sum_{j=0}^{N}\sum_{k=0}^{N} c_k^* c_j \, |j, N-j\rangle\langle k, N-k|$$

- $\mathcal{F}$ filters out V with probability p

  H with probability $(1-p)$

$$\mathcal{F}[|\psi\rangle\langle\psi|] = ?$$