Title: Minimal areas on AdS_3 and scattering amplitudes at strong coupling

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Abstract: By using the AdS/CFT duality, the computation of MSYM scattering amplitudes at strong coupling boils down to the computation of minimal areas on AdS_5 with certain boundary conditions. Unfortunately, this seems to be a hard problem. In this talk we show how one can make progress by restricting to AdS_3.

Minimal areas on AdS_3 and scattering amplitudes at strong coupling

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IAS

Perimeter Institute- September 2009

arXiv:0903.4707, arXiv:0904.0663, L.F.A & J. Maldacena

Motivations

We will be interested in gluon scattering amplitudes of planar $\mathcal{N}=4$ super Yang-Mills.

Motivation: It can give non trivial information about more realistic theories but is more tractable.

- Weak coupling: Perturbative computations are easier than in QCD. In the last years a huge technology was developed.
- The strong coupling regime can be studied, by means of the gauge/string duality, through a weakly coupled string sigma model.

Aim of this project

Learn about scattering amplitudes of planar $\mathcal{N} = 4$ super Yang-Mills by means of the AdS/CFT correspondence.

1. Background
   - Gauge theory results
   - String theory set up
   - Explicit example
2. Special kinematical configurations
   - Regular polygons
   - The octagon
3. Conclusions and outlook

Gauge theory amplitudes (Bern, Dixon and Smirnov)

- Focus in gluon scattering amplitudes of $\mathcal{N}=4$ SYM, with $SU(N_c)$ gauge group with N_c large, in the color decomposed form

$$A_n^{L,Full} \sim \sum_\rho Tr(T^{a_{\rho(1)}} \ldots T^{a_{\rho(n)}}) A_n^{(L)}(\rho(1), \ldots, \rho(2))$$

- Leading N_c color ordered $n-$points amplitude at L loops: $A_n^{(L)}$
- The amplitudes are IR divergent.
- Dimensional regularization $D = 4 - 2\epsilon \rightarrow A_n^{(L)}(\epsilon) = 1/\epsilon^{2L} + \ldots$
- Focus on MHV amplitudes and scale out the tree amplitude

$$M_n^{(L)}(\epsilon) = A_n^{(L)}/A_n^{(0)}$$

Based on explicit perturbative computations:

BDS proposal for all loops MHV amplitudes

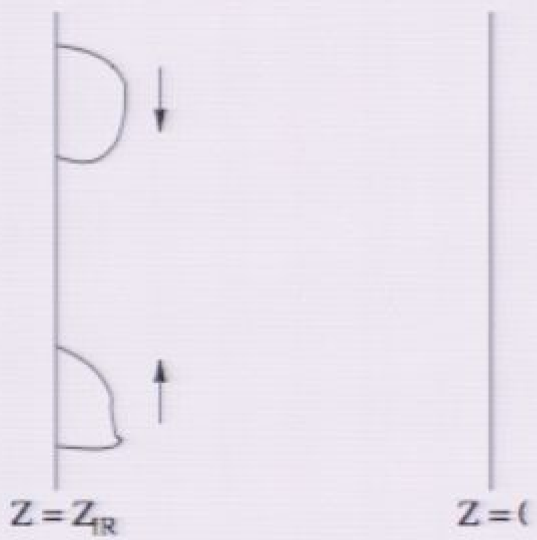
$$\log \mathcal{M}_n = \sum_{i=1}^{n} \left(-\frac{1}{8\epsilon^2} f^{(-2)} \left(\frac{\lambda \mu^{2\epsilon}}{s_{i,i+1}^{\epsilon}} \right) - \frac{1}{\epsilon} g^{(-1)} \left(\frac{\lambda \mu^{2\epsilon}}{s_{i,i+1}^{\epsilon}} \right) \right) + f(\lambda) Fin_n^{(1)}(k)$$

- $f(\lambda)$, $g(\lambda) \rightarrow$ cusp/collinear anomalous dimension.
- Fine for $n = 4, 5$, not fine for $n > 5$.

String theory set up

- Such amplitudes can be computed at strong coupling by considering strings on AdS_5.
- As in the gauge theory, we need to introduce a regulator. Place a D-brane at $z = z_{IR} \gg R$.

$$ds^2 = R^2 \frac{dx_{3+1}^2 + dz^2}{z^2}$$



- The asymptotic states are open strings ending on the D-brane.
- Consider the scattering of these open strings (representing the gluons)

After going to a dual space: $AdS \to A\tilde{d}S$ (*e.g.* $z \to r = 1/z$), the problem reduces to a minimal area problem

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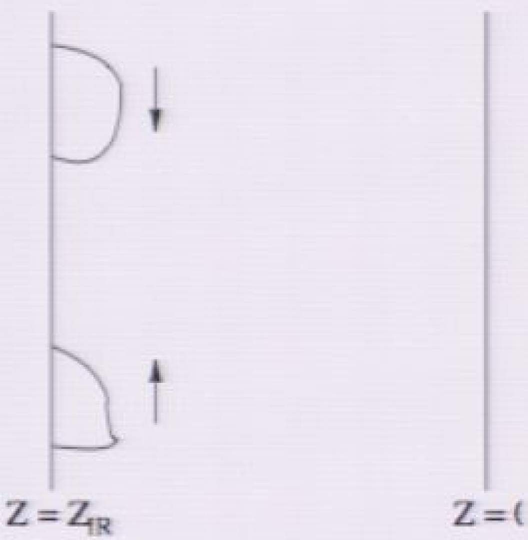

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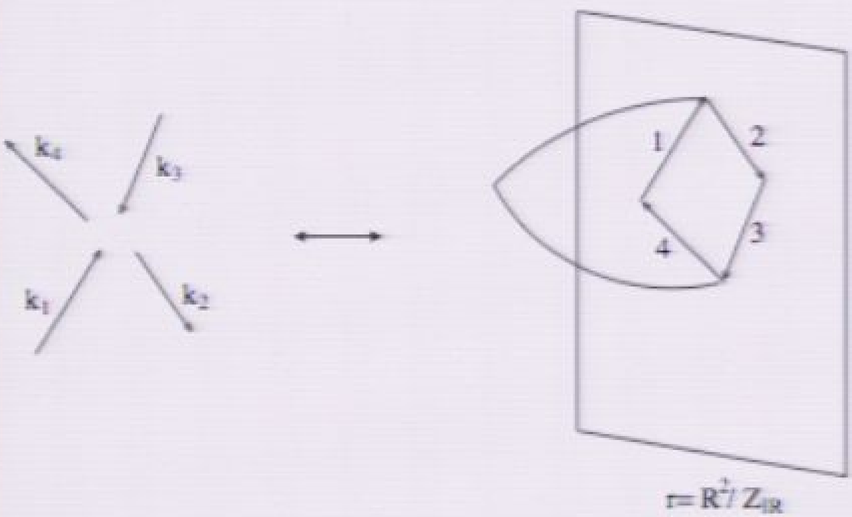

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- As $z_{IR} \rightarrow \infty$ the boundary of the world-sheet moves to $r = 0$.
- Vev of a Wilson-Loop given by a sequence of light-like segments!

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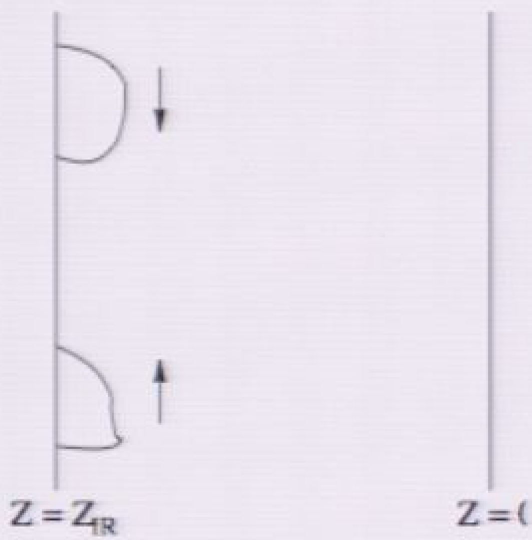

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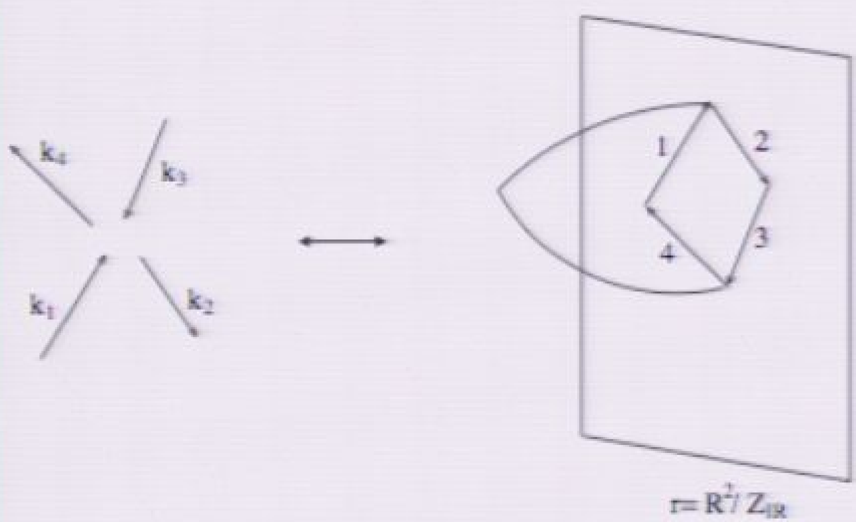

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Prescription

$$\mathcal{A}_n \sim e^{-\frac{\sqrt{\lambda}}{2\pi} A_{min}}$$

- $\mathcal{A}_n$: Leading exponential behavior of the $n-$point scattering amplitude.
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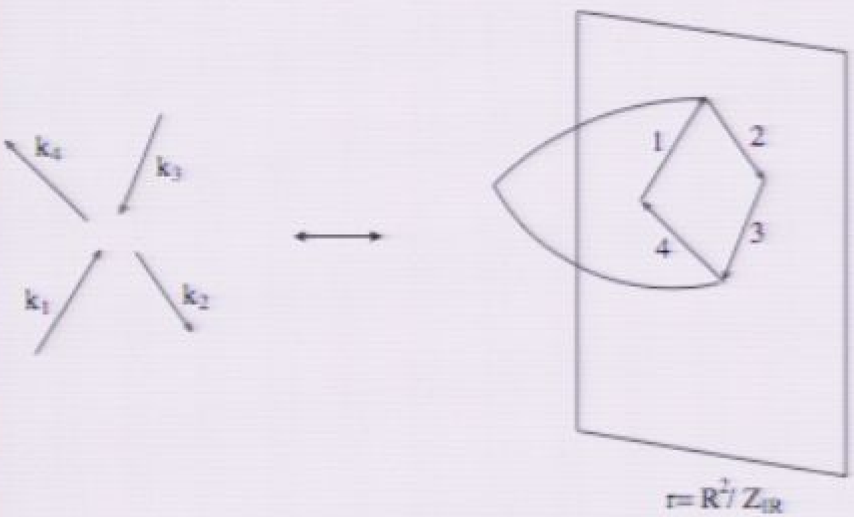

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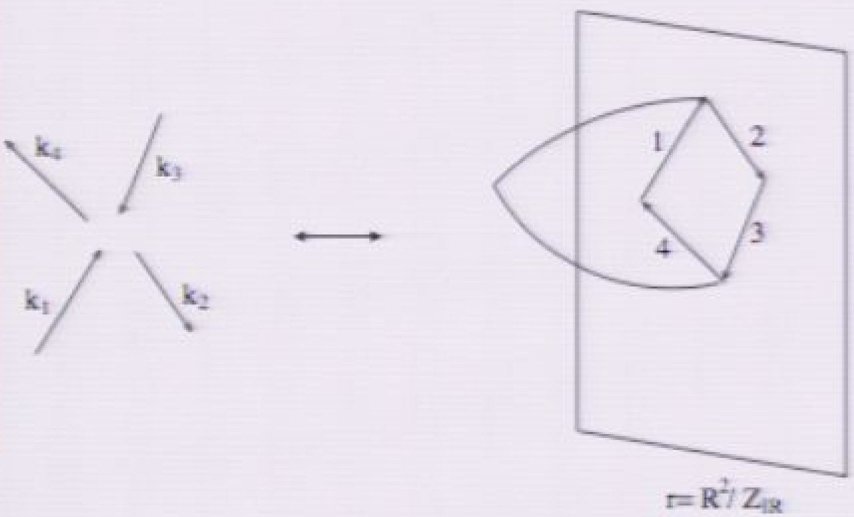

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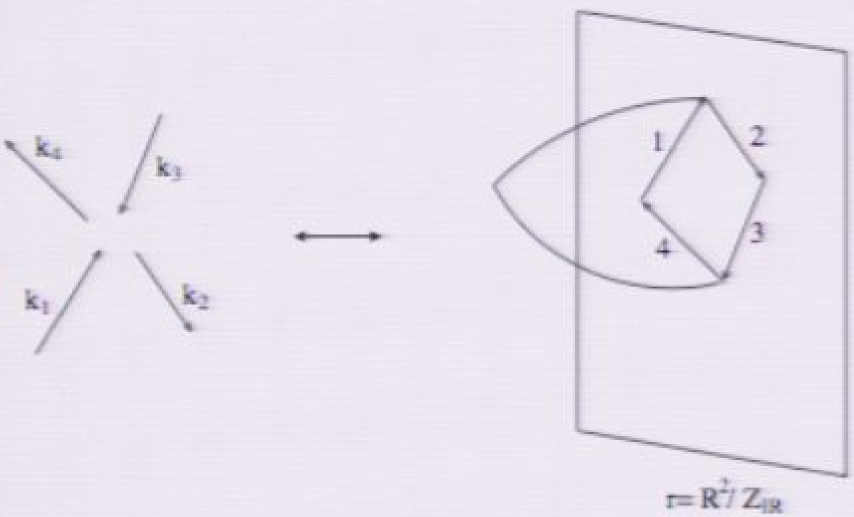

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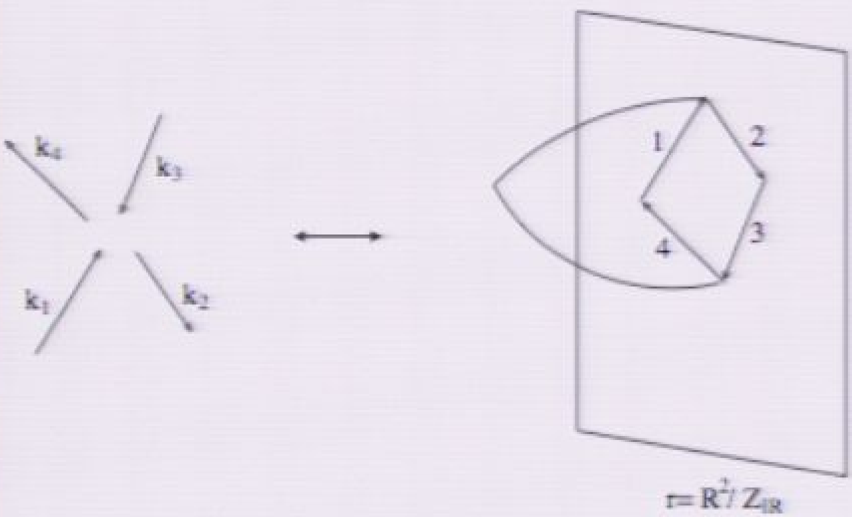

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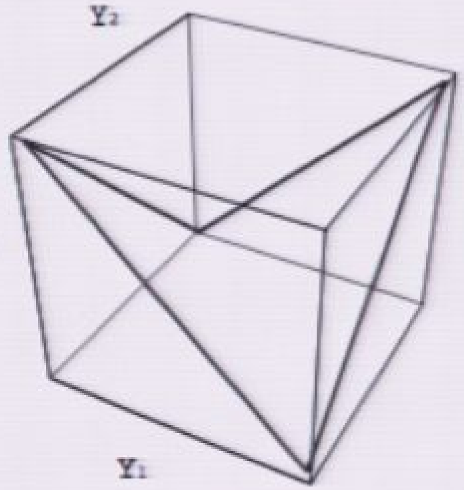
Four point amplitude at strong coupling

Consider $k_1 + k_3 \to k_2 + k_4$

- The simplest case $s = t$.



Need to find the minimal surface ending on such sequence of light-like segments

$$r(y_1, y_2) = \sqrt{(1-y_1^2)(1-y_2^2)}$$
$$y_0 = y_1 y_2$$

In embedding coordinates $(-Y_{-1}^2 - Y_0^2 + Y_1^2 + ... + Y_4^2 = -1)$

$$Y_0 Y_{-1} = Y_1 Y_2, \quad Y_3 = Y_4 = 0$$

- "Dual" $SO(2,4)$ isometries $\to$ most general solution ($s \neq t$)

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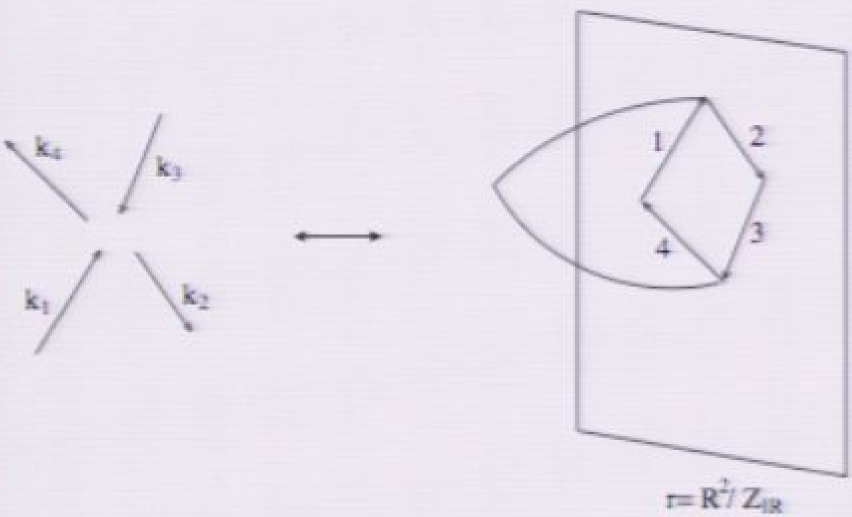

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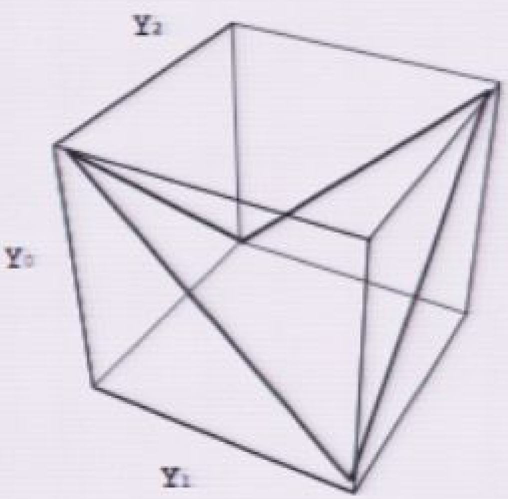

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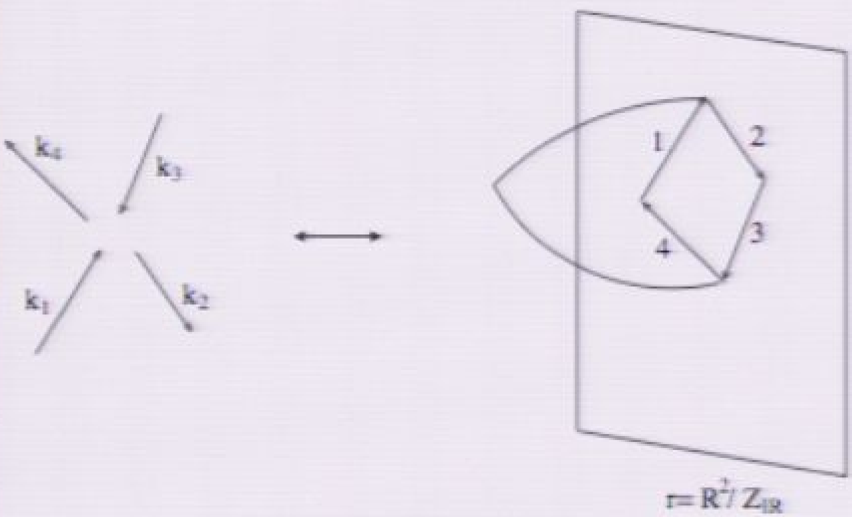

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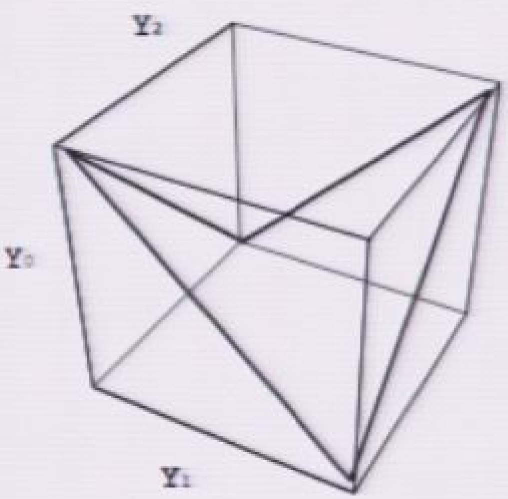

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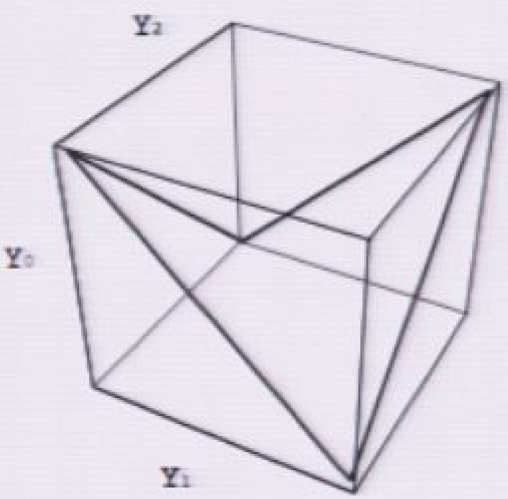

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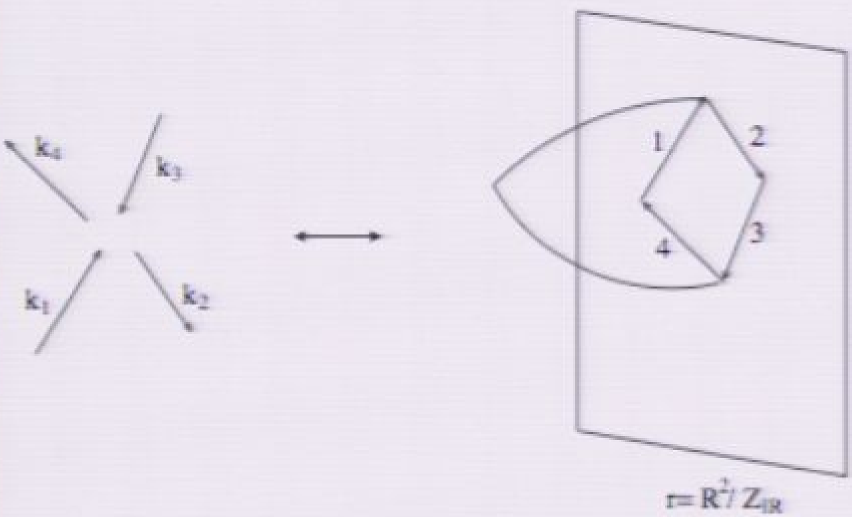

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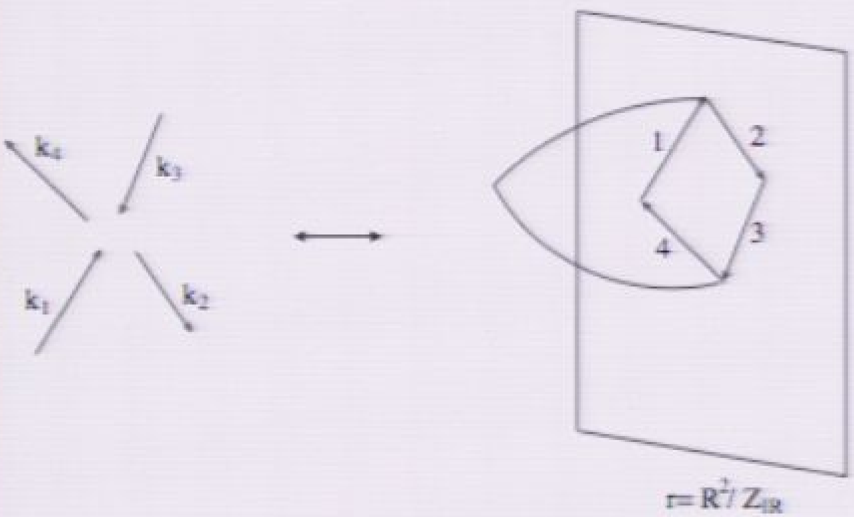

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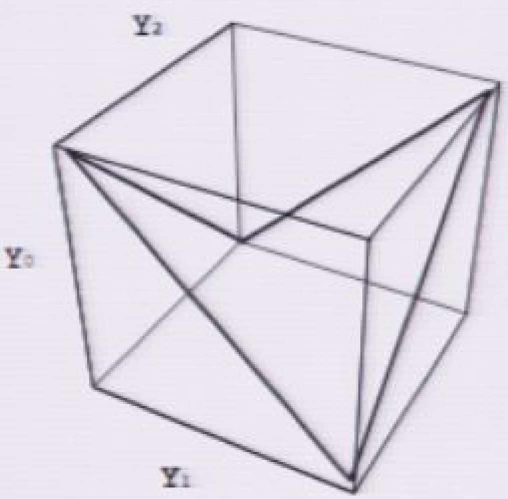

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Let's compute the area...

- In order for the area to converge we need to introduce a regulator.
- Dimensional reduction scheme: Start with $\mathcal{N}=1$ in D=10 and go down to $D=4-2\epsilon$.
- For integer D this is exactly the low energy theory living on Dp–branes ($p=D-1$)

Regularized supergravity background

$$ds^2=\sqrt{\lambda_D c_D}\left(\frac{dy_D^2+dr^2}{r^{2+\epsilon}}\right)\rightarrow S_\epsilon=\frac{\sqrt{\lambda_D c_D}}{2\pi}\int\frac{\mathcal{L}_{\epsilon=0}}{r^\epsilon}$$

- The regularized area can be computed and it agrees precisely with the BDS ansatz!
- What about other cases with $n>4$?
- for all n $SO(2,4)\rightarrow A_{strong}=A_{1-loop}+F(\frac{x_{ij}x_{kl}}{x_{ik}x_{jl}})$ (Drummond et. al.)

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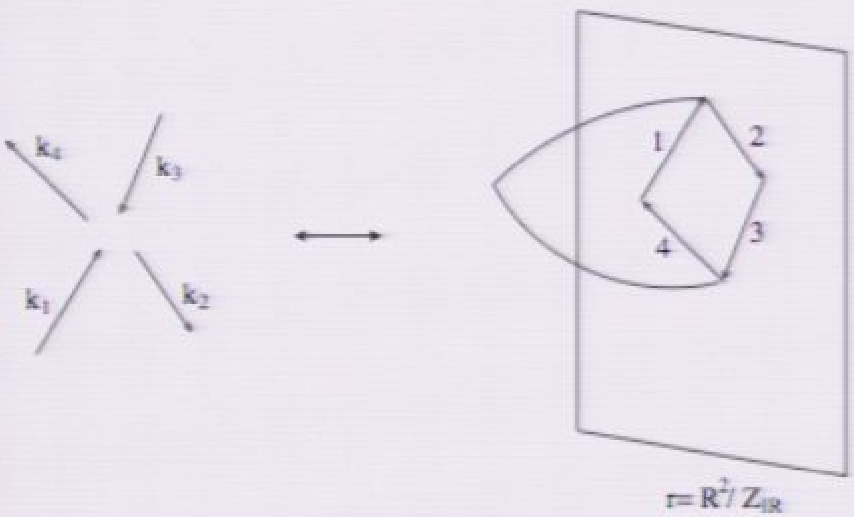

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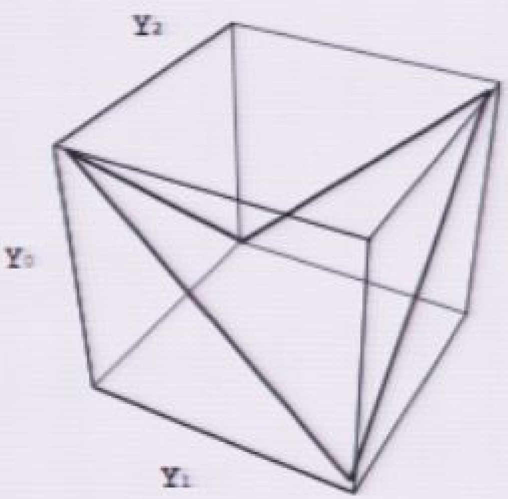

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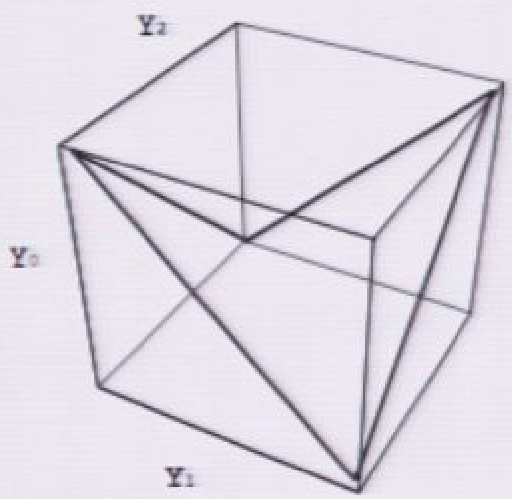

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Based on explicit perturbative computations:

BDS proposal for all loops MHV amplitudes

$$\log \mathcal{M}_n = \sum_{i=1}^{n} \left(-\frac{1}{8\epsilon^2} f^{(-2)} \left(\frac{\lambda \mu^{2\epsilon}}{s_{i,i+1}^{\epsilon}} \right) - \frac{1}{\epsilon} g^{(-1)} \left(\frac{\lambda \mu^{2\epsilon}}{s_{i,i+1}^{\epsilon}} \right) \right) + f(\lambda) Fin_n^{(1)}(k)$$

- $f(\lambda)$, $g(\lambda) \rightarrow$ cusp/collinear anomalous dimension.
- Fine for $n = 4, 5$, not fine for $n > 5$.

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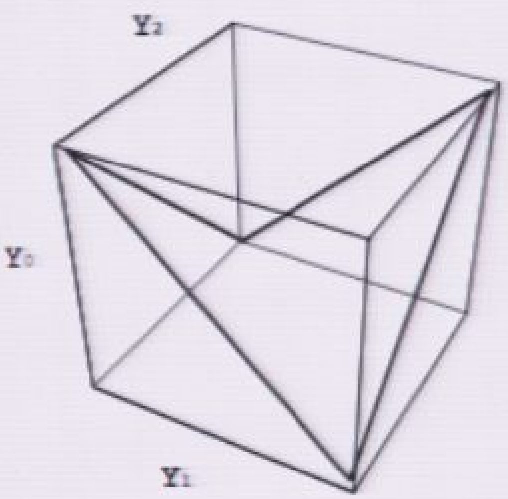

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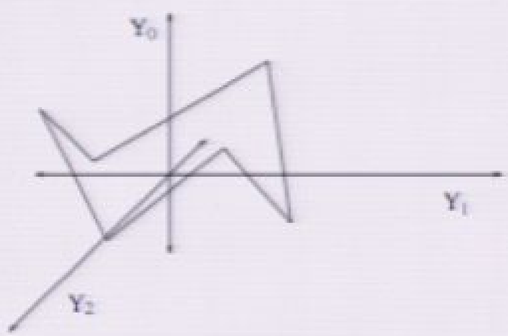

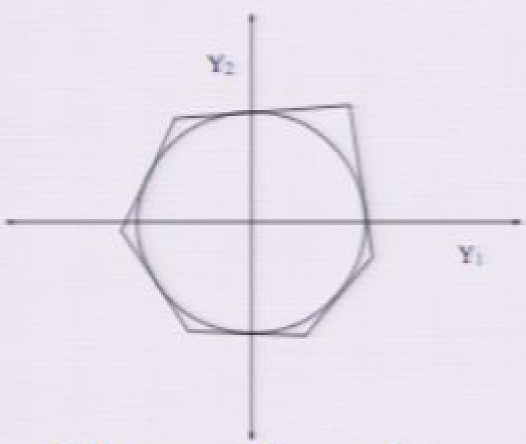

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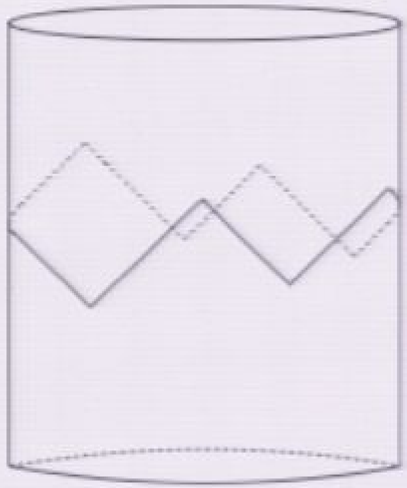

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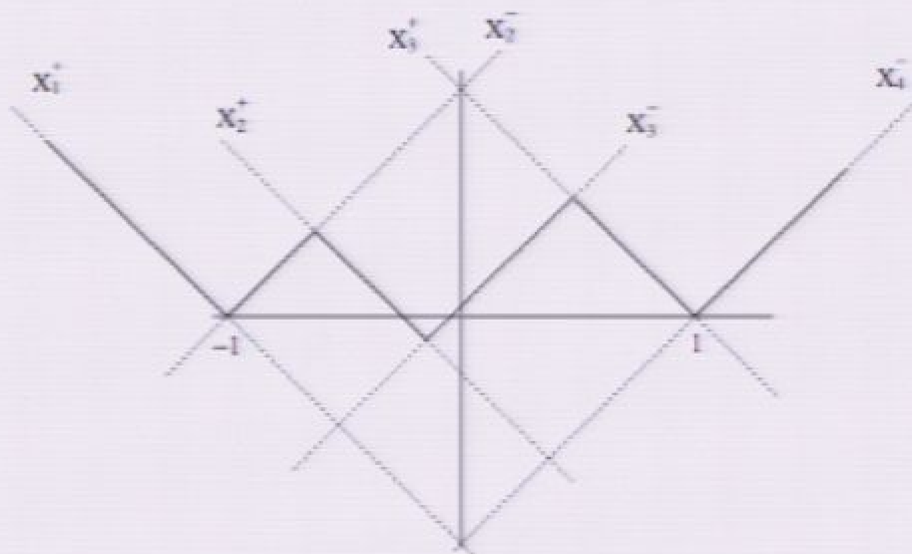

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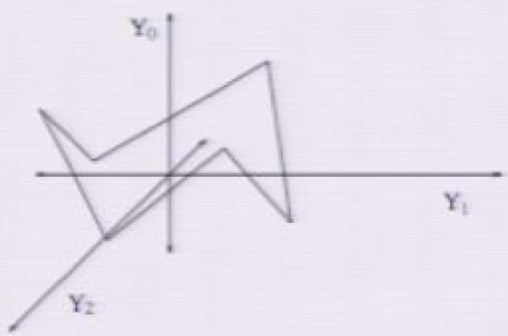

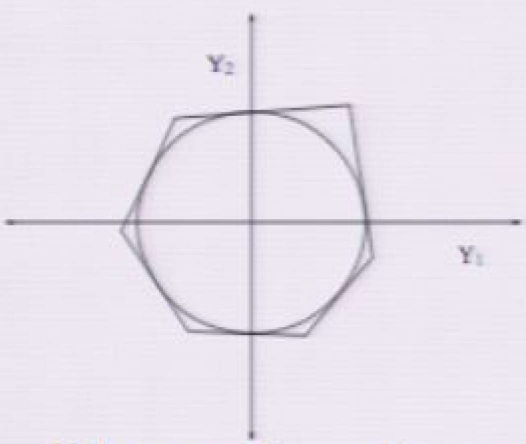

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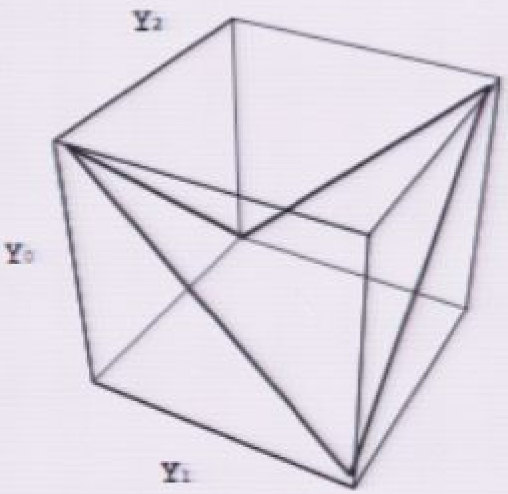

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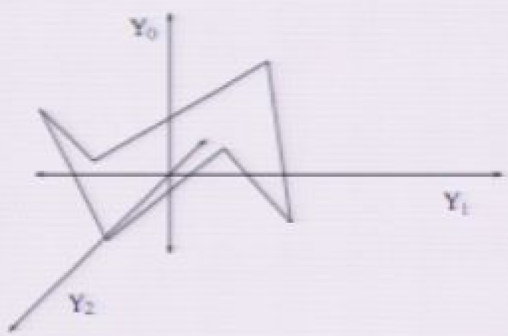

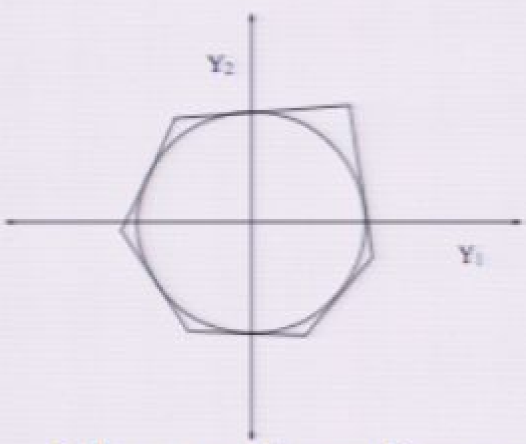

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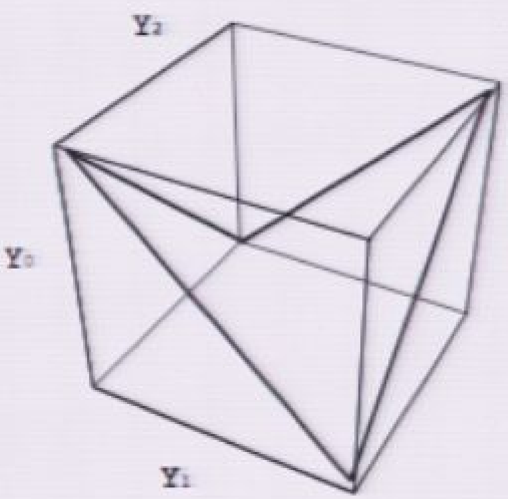

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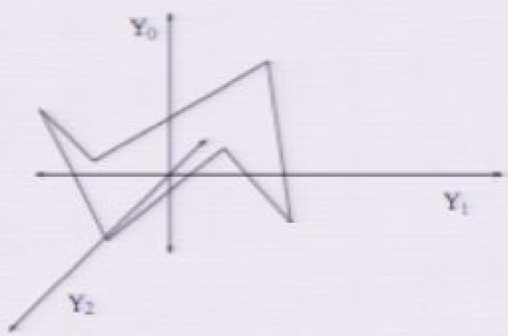

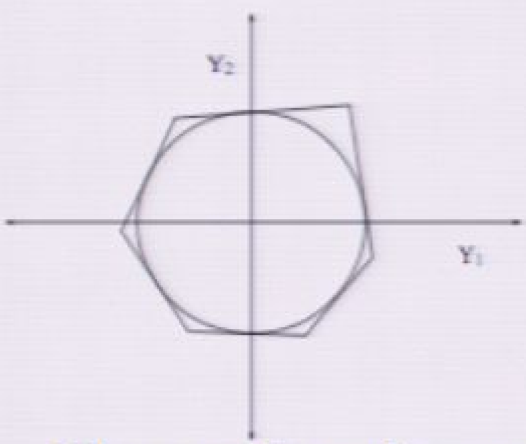

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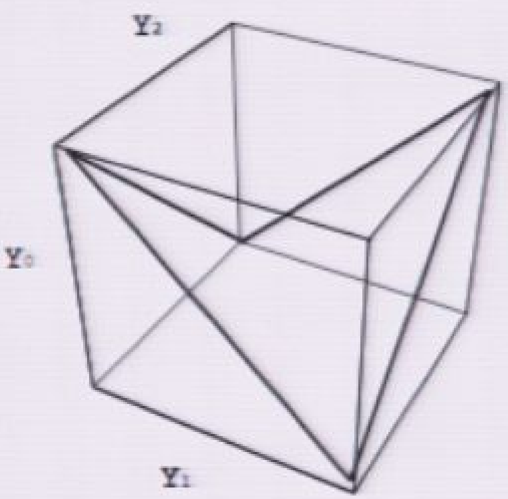

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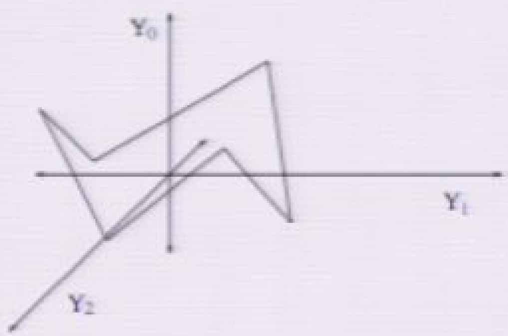

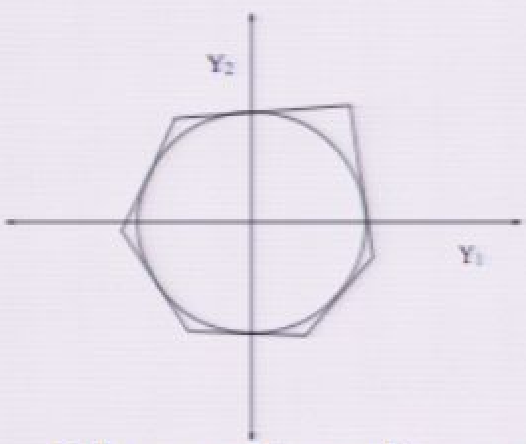

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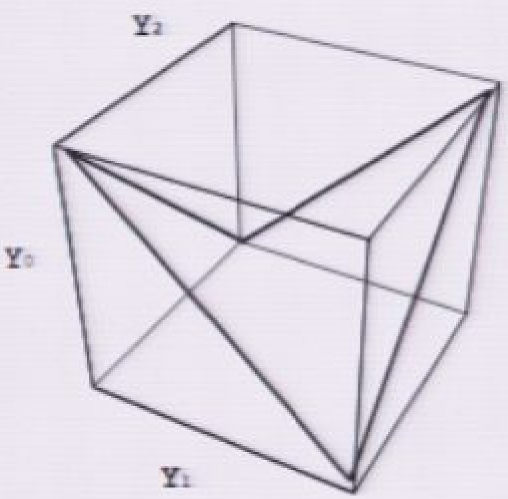

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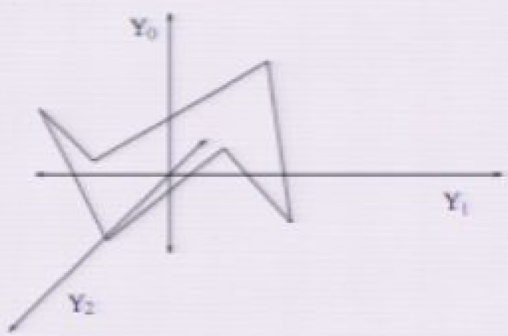

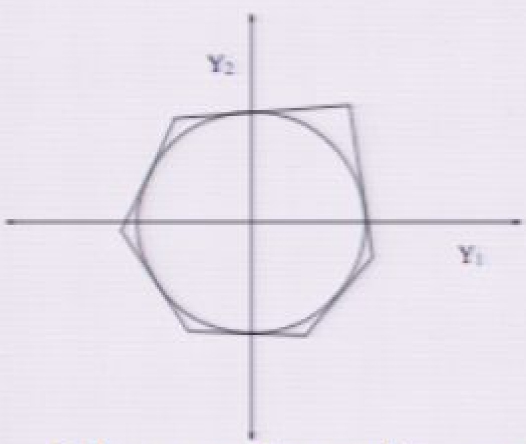

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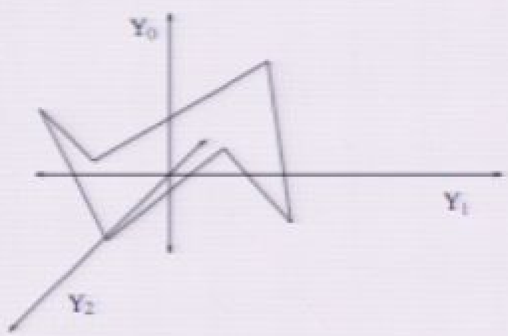

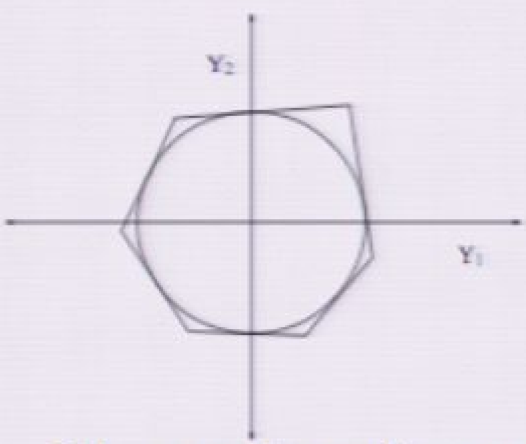

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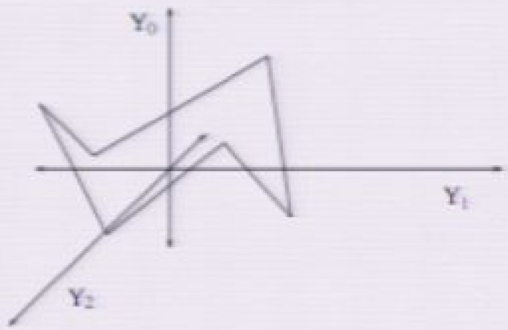

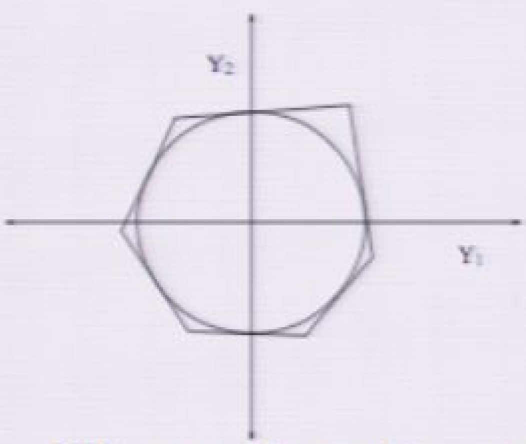

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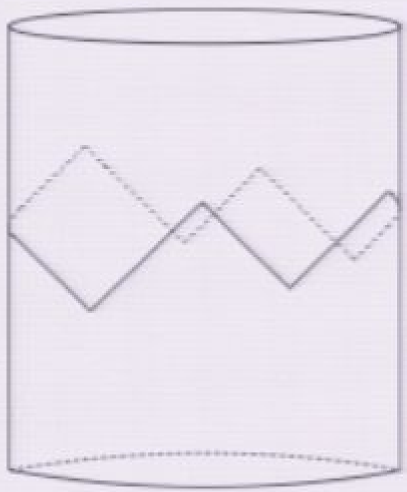

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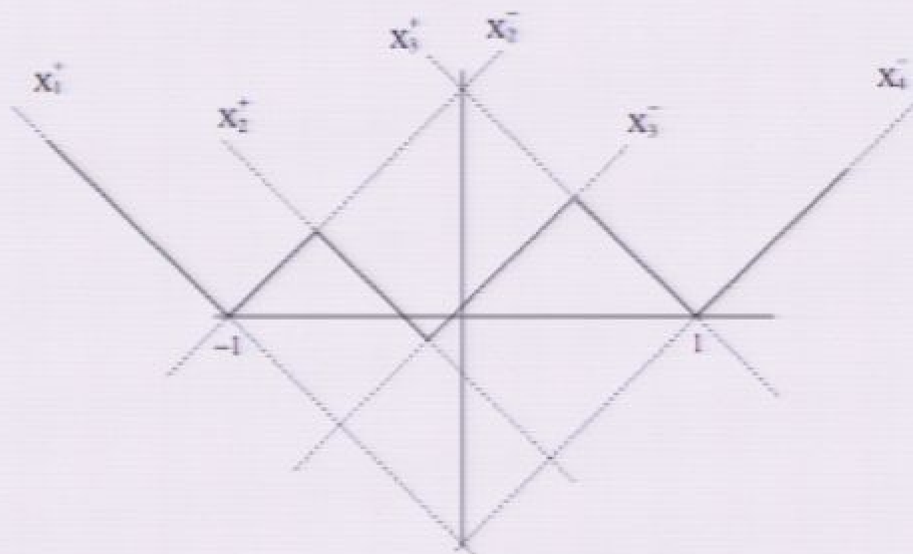

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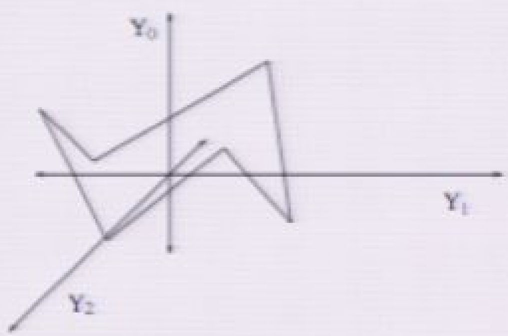

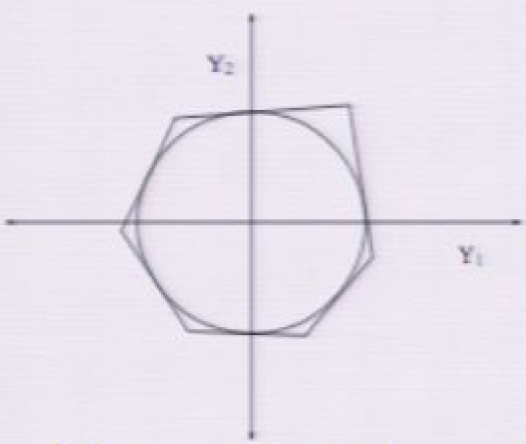

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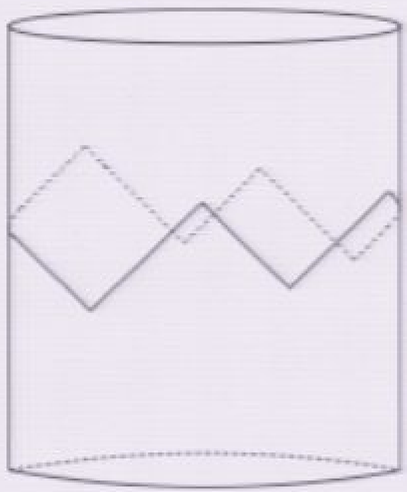

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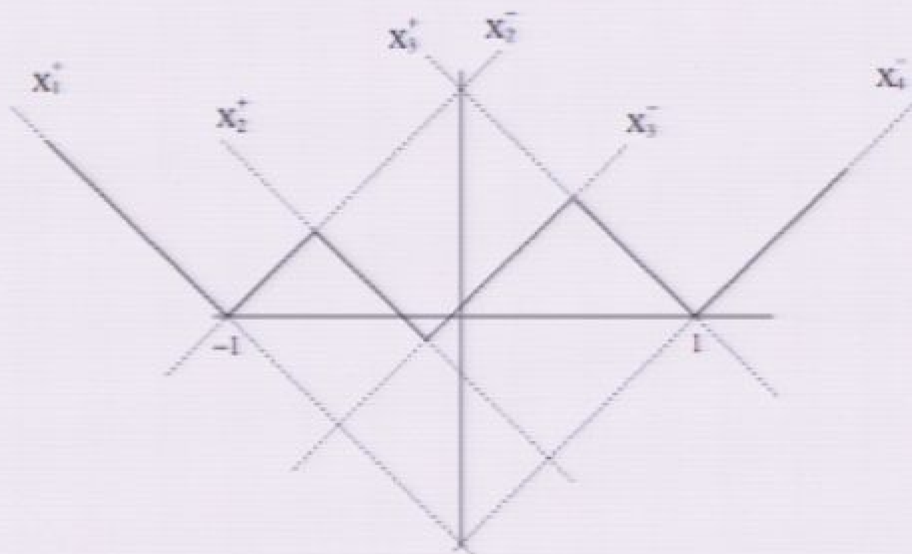

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Polhmeyer kind of reduction → generalized Sinh-Gordon

$$\alpha(z,\bar{z}) = \log(\partial\vec{Y}.\bar{\partial}\vec{Y}), \quad p = -e^{-\alpha}\epsilon_{abcd}\partial^2 Y^a Y^b \partial Y^c \bar{\partial} Y^d$$

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Generalized Sinh-Gordon → Strings on AdS_3?

- From α, p construct flat connections $B_{L,R}$ and solve two linear auxiliary problems.

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One can check that Y constructed that way has all the correct properties.

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Relation to Hitchin equations

Consider self-dual YM in 4d reduced to 2d

- $A_{1,2} \to A_{1,2}$: 2d gauge field, $A_{3,4} \to \Phi, \Phi^*$: Higgs field.

Hitchin equations

$$F^{(4)} = *F^{(4)} \quad \to \quad \begin{array}{l} D_{\bar{z}}\Phi = D_z\Phi^* = 0 \\ F_{z\bar{z}} + [\Phi, \Phi^*] = 0 \end{array}$$

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$$dw = \sqrt{p(z)}dz, \quad \hat{\alpha} = \alpha - \frac{1}{4}\log p\bar{p} \rightarrow \partial_w \bar{\partial}_{\bar{w}} \hat{\alpha} = sinh2\hat{\alpha}$$

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$n = 2$ "square" solution: $p(z) = 1, \quad \hat{\alpha} = 0$

- For the solutions relevant to scattering amplitudes we require $p(z)$ to be a polynomial and $\hat{\alpha}$ to decay at infinity.

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Consider a generic polynomial of degree $n-2$

$$p(z) = z^{n-2} + c_{n-4}z^{n-4} + \ldots + c_1 z + c_0$$

- We have used translations and re-scalings in order to fix the first two coefficients to one and zero.
- For a polynomial of degree $n-2$ we are left with $2n-6$ (real) variables.
- This is exactly the number of invariant cross-ratios in two dimensions for the scattering of $2n$ gluons!

Null Wilsons loops of $2n$ sides $\Leftrightarrow P^{n-2}(z)$ and $\alpha(z,\bar{z})$

Regular polygons

- Simplest case: $p(z) = z^{n-2} \;\rightarrow\; \alpha(z,\bar{z}) = \alpha(\rho)$

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$$\hat{\alpha}''(\rho) + \frac{\hat{\alpha}'(\rho)}{\rho} = \frac{1}{2}\sinh(2\hat{\alpha}(\rho))$$

- Solved in terms of Painleve transcendentals, well studied in the literature.

Another interesting feature

- $w = z^{n/2}$: As we go once around the $z-$plane, we go around the $w-$plane $n/2$ times.

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- The solution has a Z_n symmetry and indeed corresponds to the regular polygon! (each quadrant in the $w-$plane corresponds to a cusp.)
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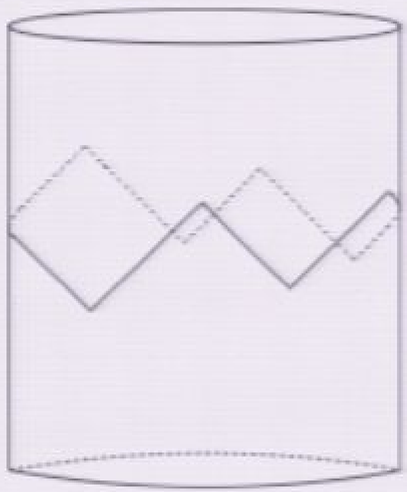
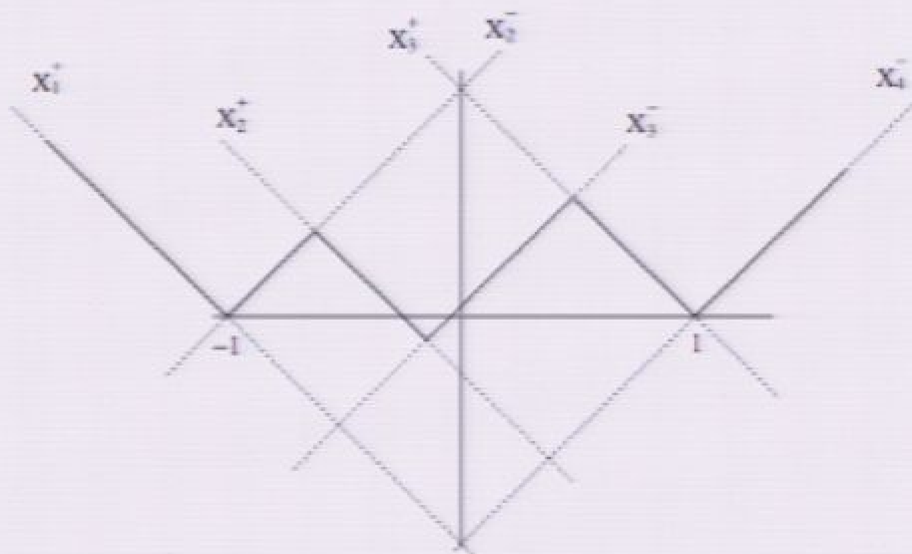
The scattering is equivalent to a $2D$ scattering, *e.g.* in the cylinder.



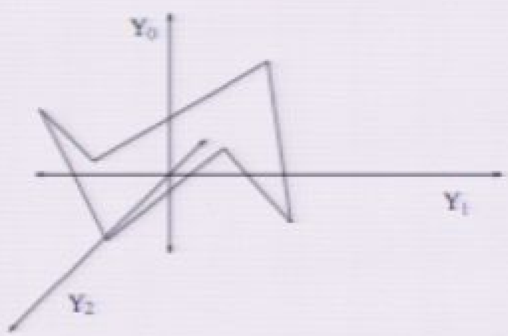
- Consider a zig-zagged Wilson loop of $2n$ sides
- Parametrized by n X_i^+ coordinates and n X_i^- coordinates.
- We can build $2n-6$ invariant cross ratios.

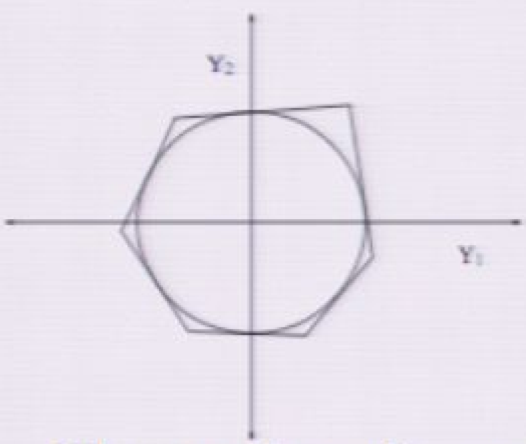


- Consider classical strings on AdS_3.

Special kinematical configurations

- Unfortunately its very hard to find classical solutions...
- Consider a special kinematical configuration





- Projection of the world-sheet to the (y_1, y_2) plane is a polygon which circumscribes the unit circle.
- Eom's and boundary conditions are consistent with $Y_3 = Y_4 = 0$.

The surface lives effectively in a AdS_3 subspace!

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In order to compute A_{div} use dimensional regularization...

$$A_{div}(\epsilon) = \int \frac{1}{r^\epsilon} d^2 w \approx \frac{n}{(\sin\frac{\pi}{2n})^\epsilon \epsilon^2}$$

- It has the expected IR behavior!
- The integrand of A_{sinh} was studied by Zamolodchikov!

$$A_{sinh} = \frac{\pi}{4n}(3n^2 - 8n + 4),$$

- When $n \to \infty$, the regular Wilson loop approaches the circular Wilson loop.
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First non trivial case: $p(z) = z^2 - m$, the "octagon"

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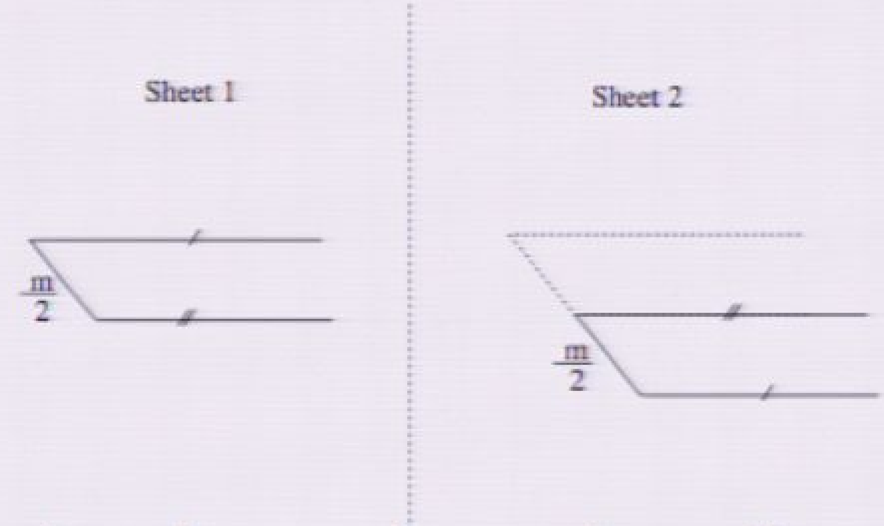

- The information of m survives at large distances and we can compute cross-ratios vs. $m = m_r + i m_i$.

$$e^{m_r} = \frac{(x_4^+ - x_1^+)(x_3^+ - x_2^+)}{(x_4^+ - x_3^+)(x_2^+ - x_1^+)}, \quad e^{m_i} = \frac{(x_4^- - x_1^-)(x_3^- - x_2^-)}{(x_4^- - x_3^-)(x_2^- - x_1^-)}$$

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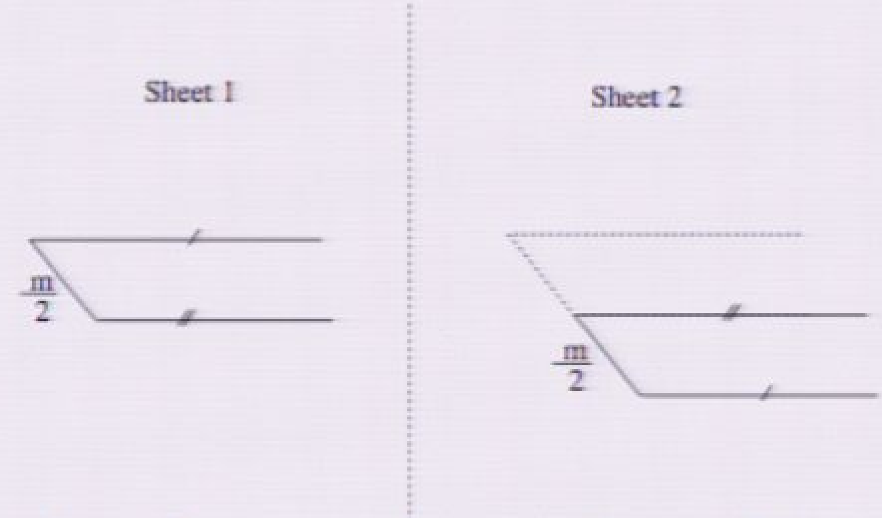

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Gathering all the terms and working a little bit...

Eight sided Wilson loop at strong coupling

$$A_{sinh} + A_{extra} = \frac{1}{2}\int dt \frac{\bar{m}e^{t} - me^{-t}}{\tanh 2t} \log\left(1 + e^{-\pi(\bar{m}e^{t} + me^{-t})}\right)$$

- This is the remainder function for the scattering of eight gluons (for this particular configuration)
- Correct limits as $|m| \to 0$ and $|m| \to \infty$.
- Correct analytic structure.

What have we done?

- We have given a further small step towards the computation of classical solutions relevant to scattering amplitudes at strong coupling.
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Prescription

$$\mathcal{A}_n \sim e^{-\frac{\sqrt{\lambda}}{2\pi} A_{min}}$$

- $\mathcal{A}_n$: Leading exponential behavior of the $n-$point scattering amplitude.
- $A_{min}(k_1^\mu, k_2^\mu, ..., k_n^\mu)$: Area of a minimal surface that ends on a sequence of light-like segments on the boundary.

Based on explicit perturbative computations:

BDS proposal for all loops MHV amplitudes

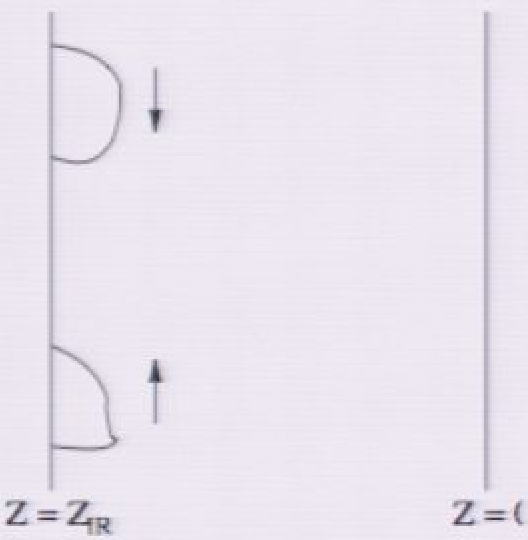
$$\log \mathcal{M}_n = \sum_{i=1}^{n} \left(-\frac{1}{8\epsilon^2} f^{(-2)} \left(\frac{\lambda \mu^{2\epsilon}}{s_{i,i+1}^{\epsilon}} \right) - \frac{1}{\epsilon} g^{(-1)} \left(\frac{\lambda \mu^{2\epsilon}}{s_{i,i+1}^{\epsilon}} \right) \right) + f(\lambda) Fin_n^{(1)}(k)$$

- $f(\lambda)$, $g(\lambda) \rightarrow$ cusp/collinear anomalous dimension.
- Fine for $n = 4, 5$, not fine for $n > 5$.

String theory set up

- Such amplitudes can be computed at strong coupling by considering strings on AdS_5.
- As in the gauge theory, we need to introduce a regulator. Place a D-brane at $z = z_{IR} \gg R$.

$$ds^2 = R^2 \frac{dx_{3+1}^2 + dz^2}{z^2}$$



- The asymptotic states are open strings ending on the D-brane.
- Consider the scattering of these open strings (representing the gluons)

After going to a dual space: $AdS \to \tilde{AdS}$ (*e.g.* $z \to r = 1/z$), the problem reduces to a minimal area problem