Title: General Relativity for Cosmology - Lecture 2

Date: Sep 24, 2009 04:00 PM

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Abstract:

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Finally: Algebraic definition of $T_p(M)$

Recall idea: The elements of $T(p)$ are to be 1st derivatives $\Rightarrow$ recognizable by Leibniz rule.

Definition: The tangent space $T_p(M)$ is the set of "derivations" of $\mathcal{F}(p)$, i.e. the set of linear maps $\xi : \mathcal{F}(p) \to \mathbb{R}$ which obey: (Leibniz rule for differentiable functions g, f)

$$\xi\left(\bar{f}_p \bar{g}_p\right) = \xi\left(\bar{f}_p\right) \cdot \bar{g}_p(p) + \bar{f}_p(p)\, \xi\left(\bar{g}_p\right)$$

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(*) remember this: $\longrightarrow$ $\bar{g}_p(p) = g(p)$, $\bar{f}_p(p) = f(p)$

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Properties of $T_p(M)$:

Simple example: a constant function c:

$c(p) := c$ and c is a constant: $c \in \mathbb{R}$

Then: $\xi(\bar{c}) = 0$ for all $\xi \in T_p(M)$

Leibnit rule

Proof: $\xi(\bar{c}) = c\,\xi(1) = c\,\xi(1\cdot 1) = c\big(\xi(1)\,1 + 1\,\xi(1)\big)$

$= 2c\,\xi(1) \Longrightarrow \xi(\bar{c}) = 0$ ✓

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Example: The case $M = \mathbb{R}^n$

If our definition for $T_p(M)$ is good, we expect that every $\xi \in T_p(M)$ is of the form:

$$\xi = \sum_{i=1}^{n} \xi^i \frac{\partial}{\partial x^i}\Big|_{x=p}$$

Let us check!

- We choose p to have coordinates $x = (0, 0, \dots)$.

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- Notation: $h_{,i}(a^1, \ldots, a^n) := \frac{\partial}{\partial a^i} h(a^1, \ldots, a^n)$

Then:

(Note: these are not 3 numbers! These are 3 function germs, i.e., 3 equivalence classes of functions.)

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$$\xi(\bar{f}(x)) = \xi\big(\overline{f(0)} + \overline{f(x)} - \overline{\underbrace{f(0)}_{\text{a constant function}}}\big)$$

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Notation: $h_{,i}(a^1,\dots,a^n) := \frac{\partial}{\partial a^i} h(a^1,\dots,a^n)$

Then:

(Note: these are not 3 numbers! These are 3 function germs, i.e., 3 equivalence classes of functions.)

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Recall: $\xi : \mathcal{F}(p) \longrightarrow \mathbb{R}$

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$$M = \mathbb{R}^n$$

- How does our definition of $T_p(M)$ work for $M \neq \mathbb{R}^n$, concretely?

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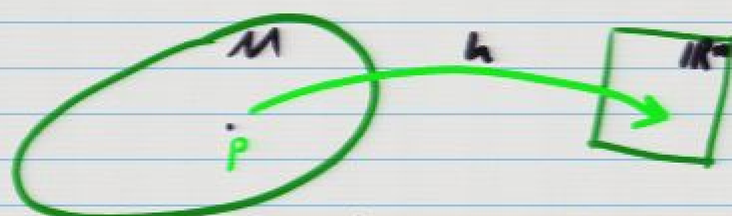

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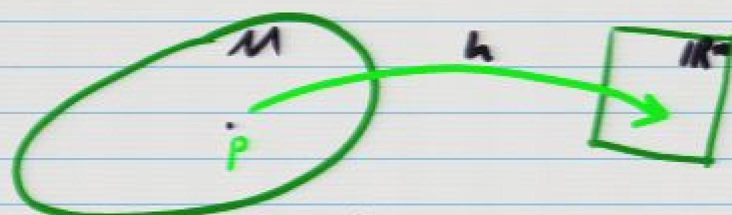

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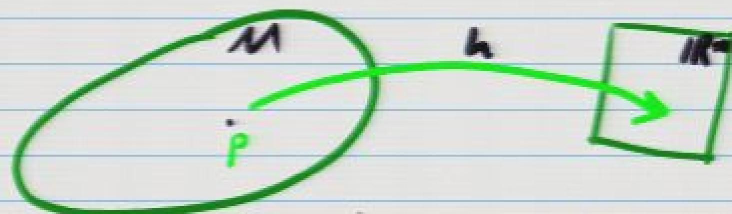

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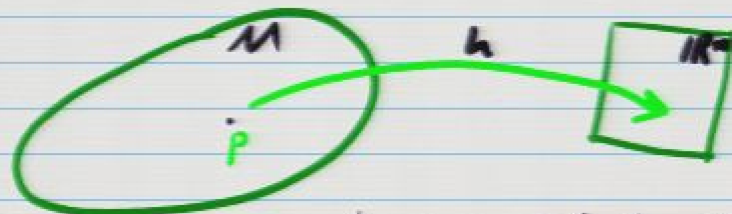

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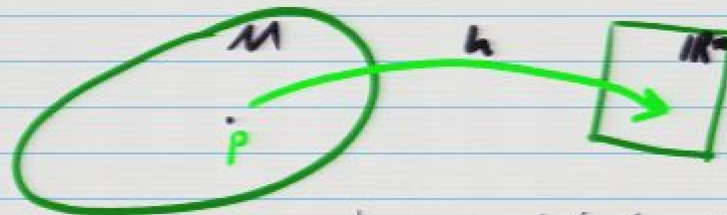

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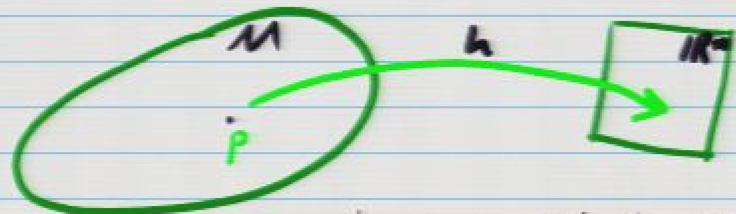

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Consider two diffable manifolds, M and N:

diffeomorphism

N

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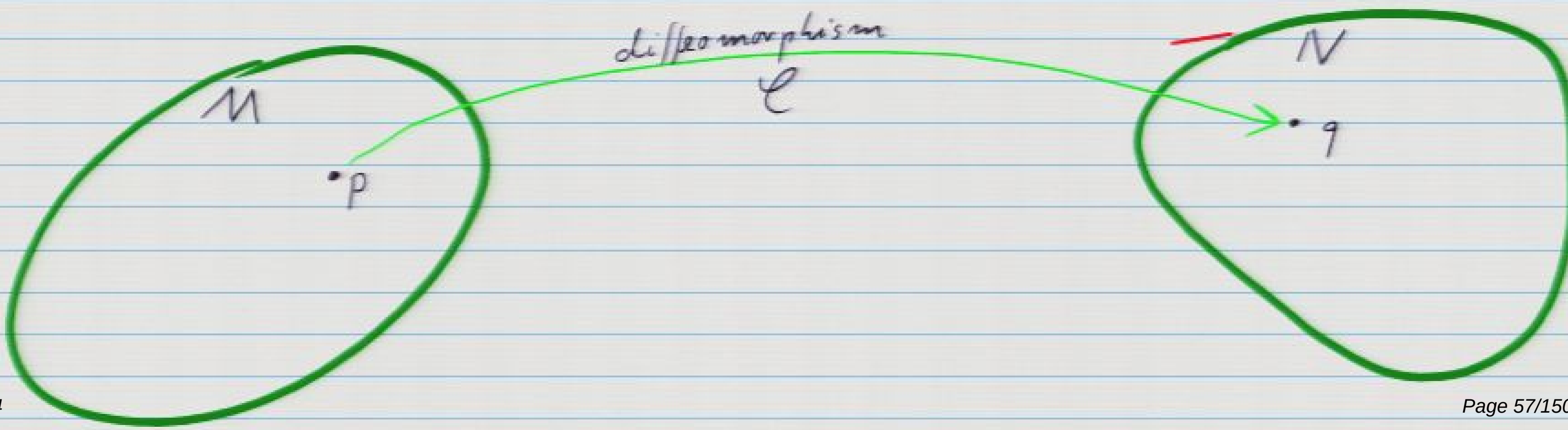

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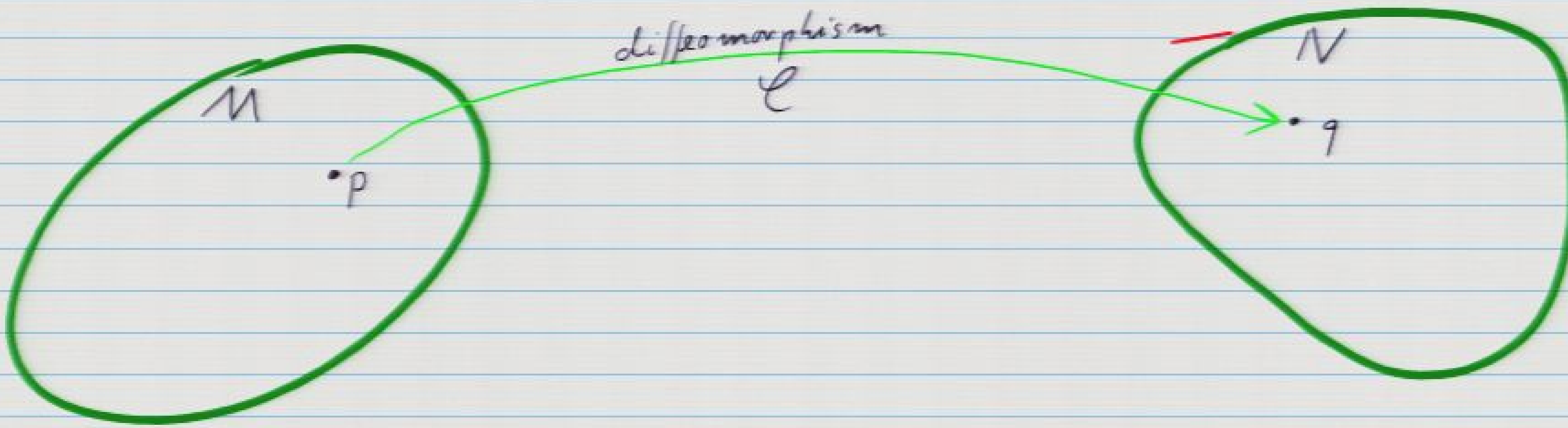

Note: If $N = \mathbb{R}^n$ then φ is a chart.

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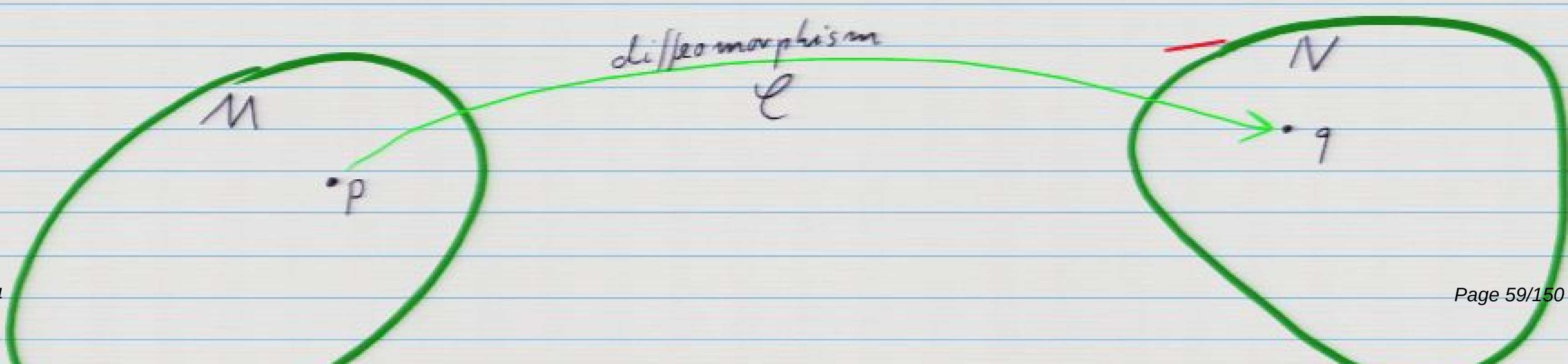

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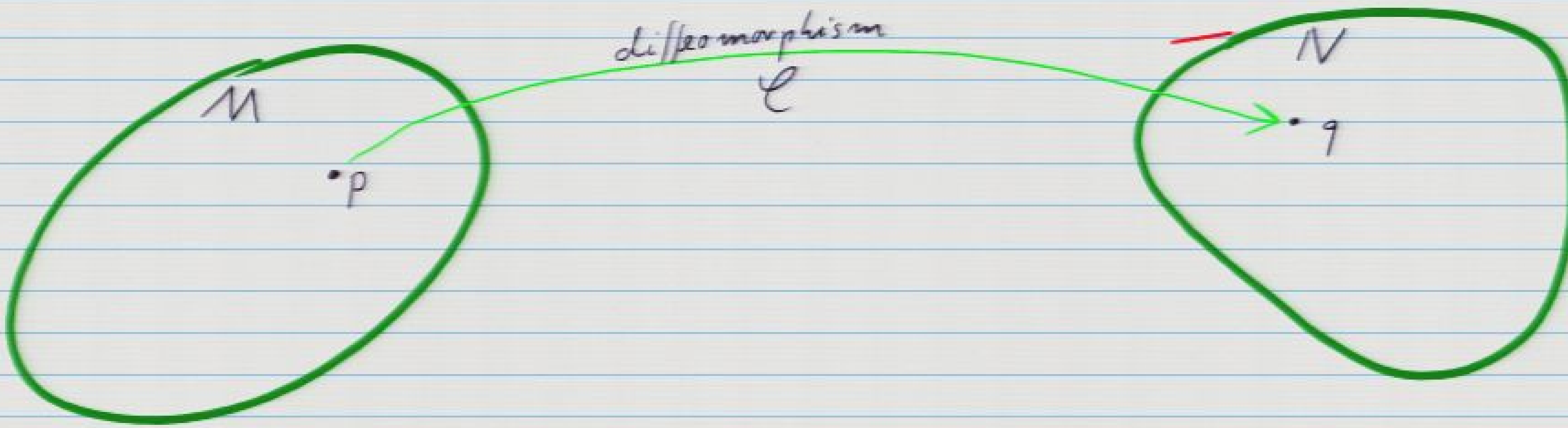

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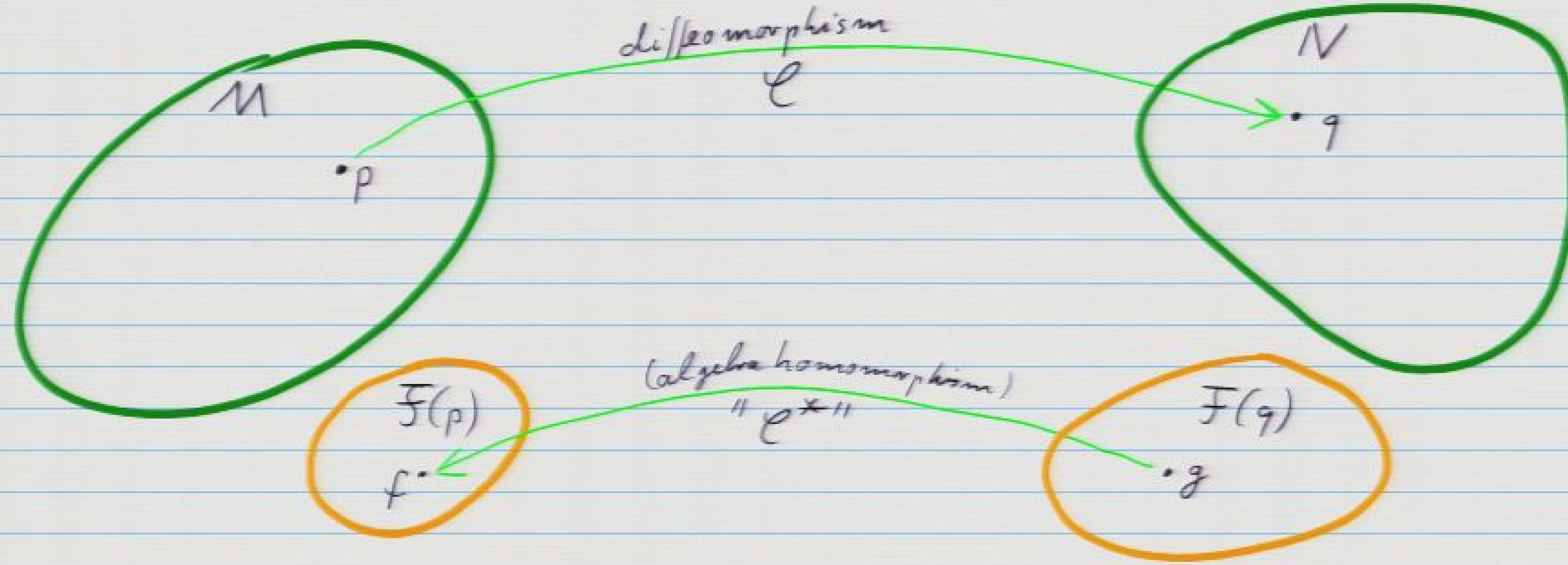
(that's the case we'll need but it's easy



Here: ▪ $\mathcal{F}(q)$ and $\mathcal{F}(p)$ are algebras of function (germs).

▪ Given φ we obtain a map $\varphi^*: \mathcal{F}(q) \to \mathcal{F}(p)$

$$\varphi^*: g \to f = \varphi^*(g) \quad \text{with } f(x) = g(\varphi(x)) \ \forall x \in M$$

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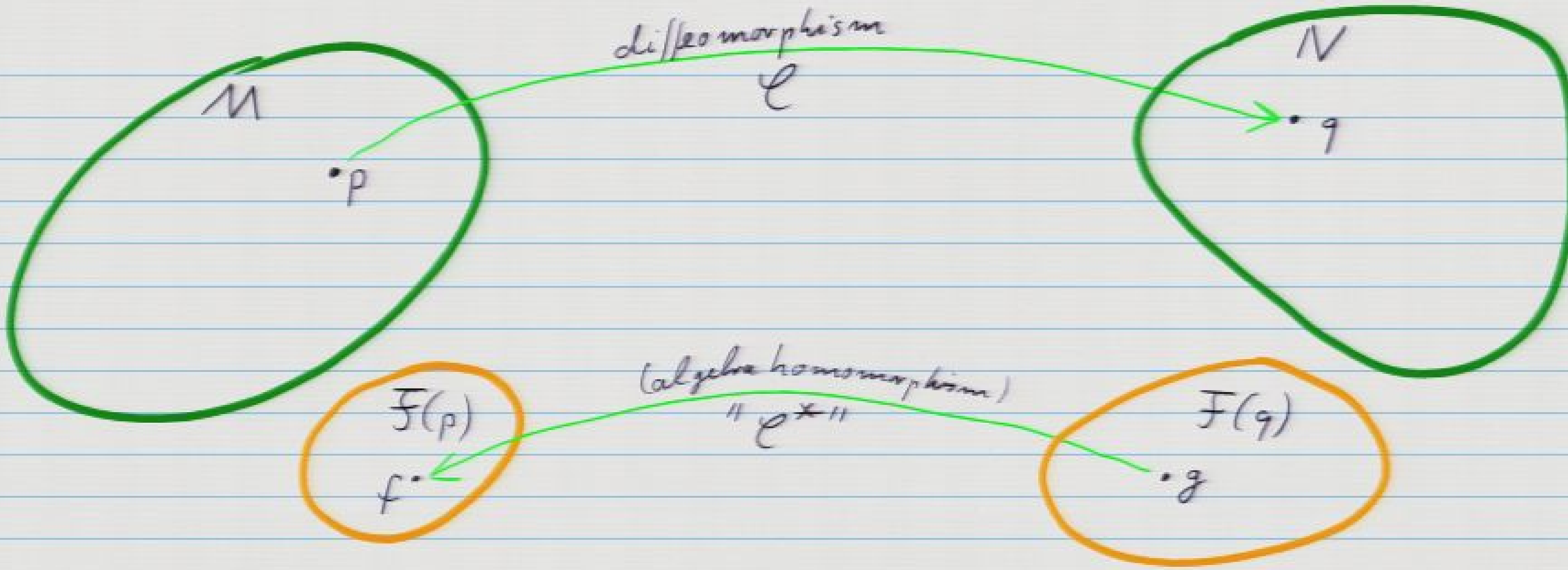

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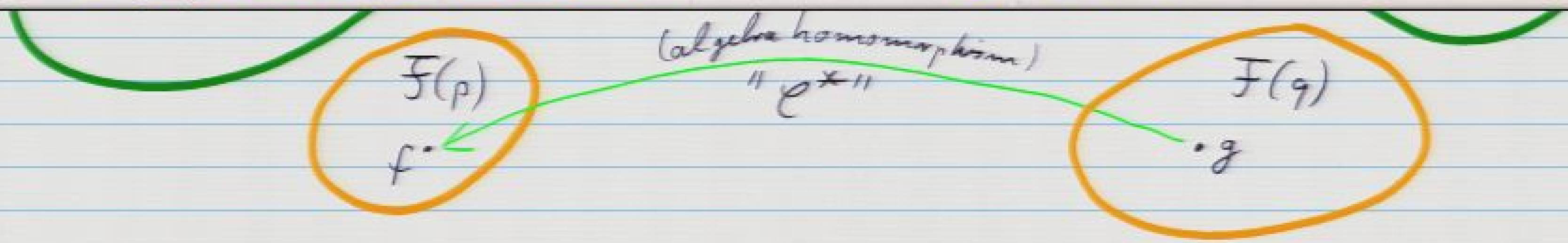

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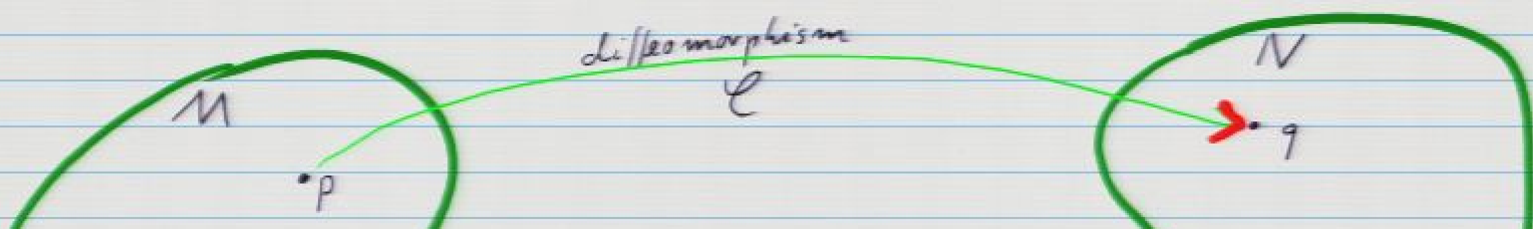

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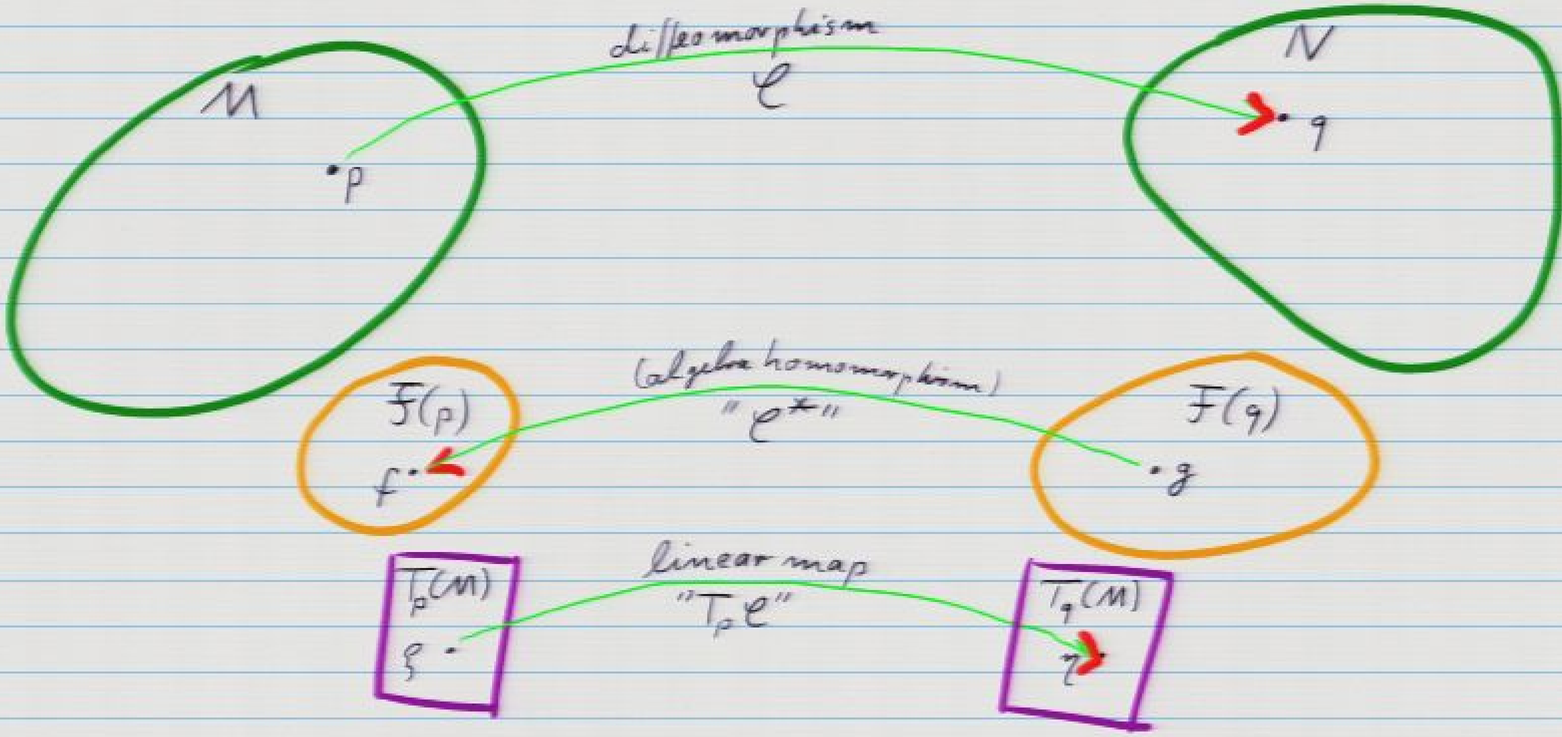

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Namely: $\eta = \xi \circ \varphi^*$

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From (+) $\Rightarrow$

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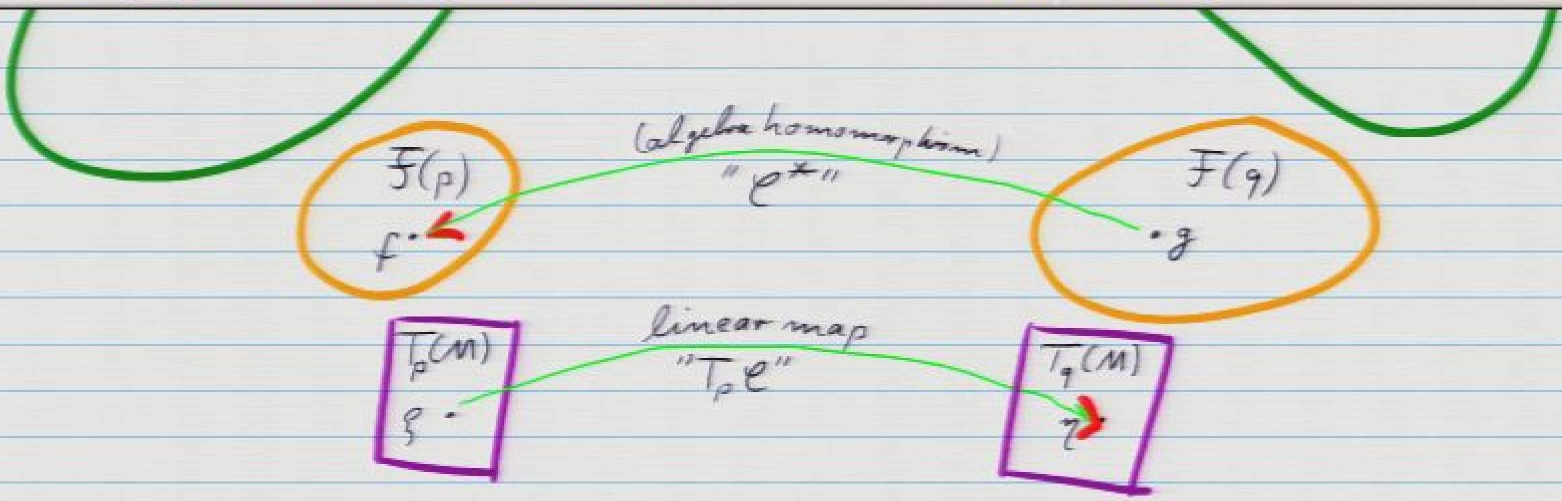
The crucial special case:



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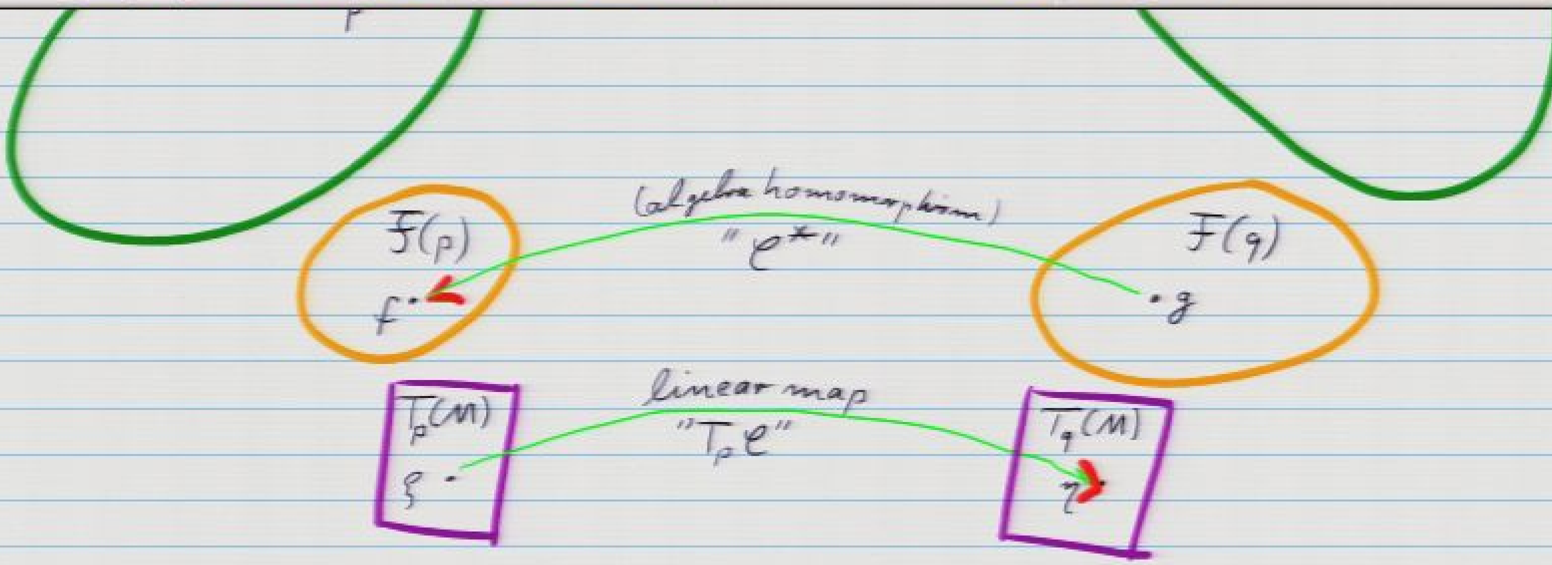

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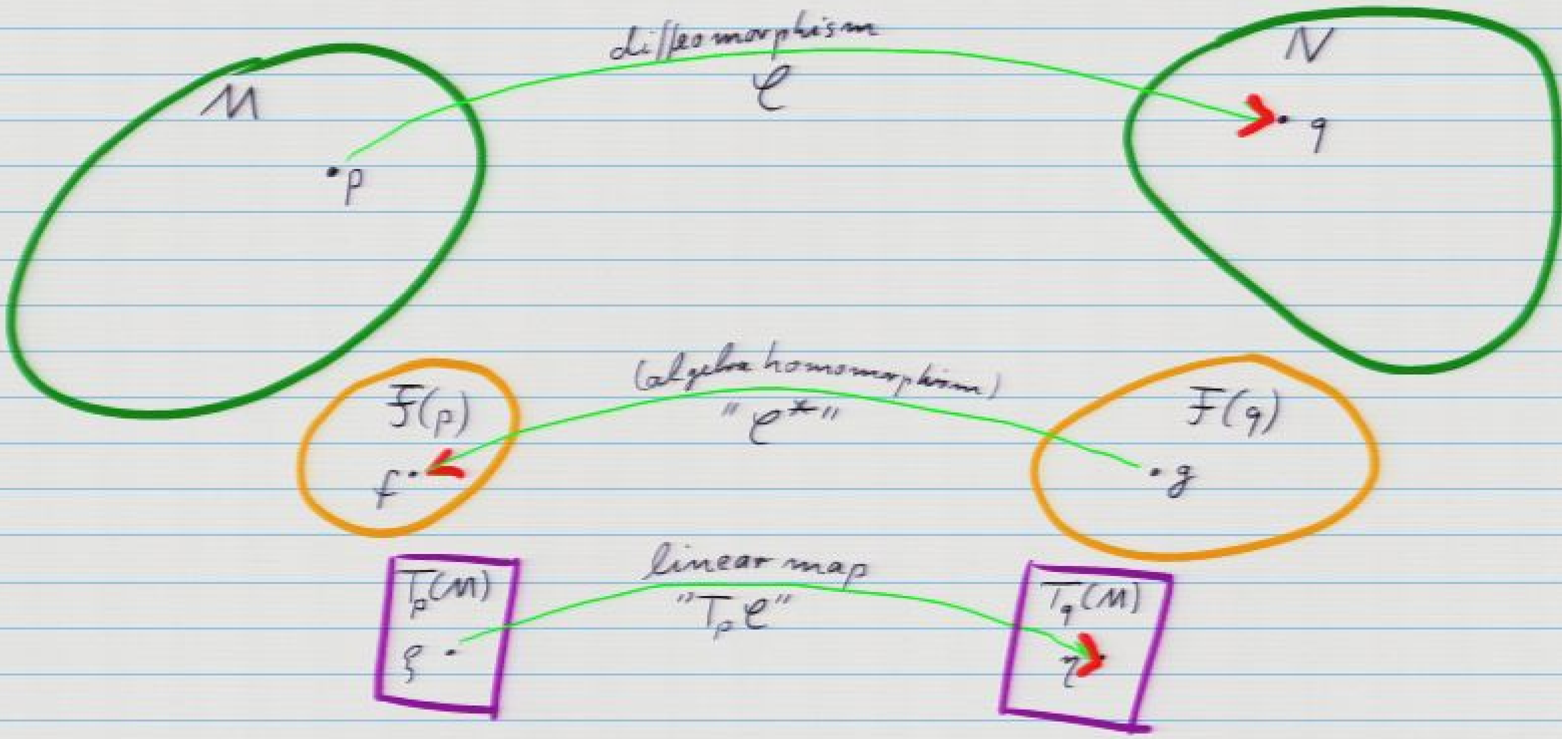

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- ($\Rightarrow \varphi^*$ is algebra isomorphism)
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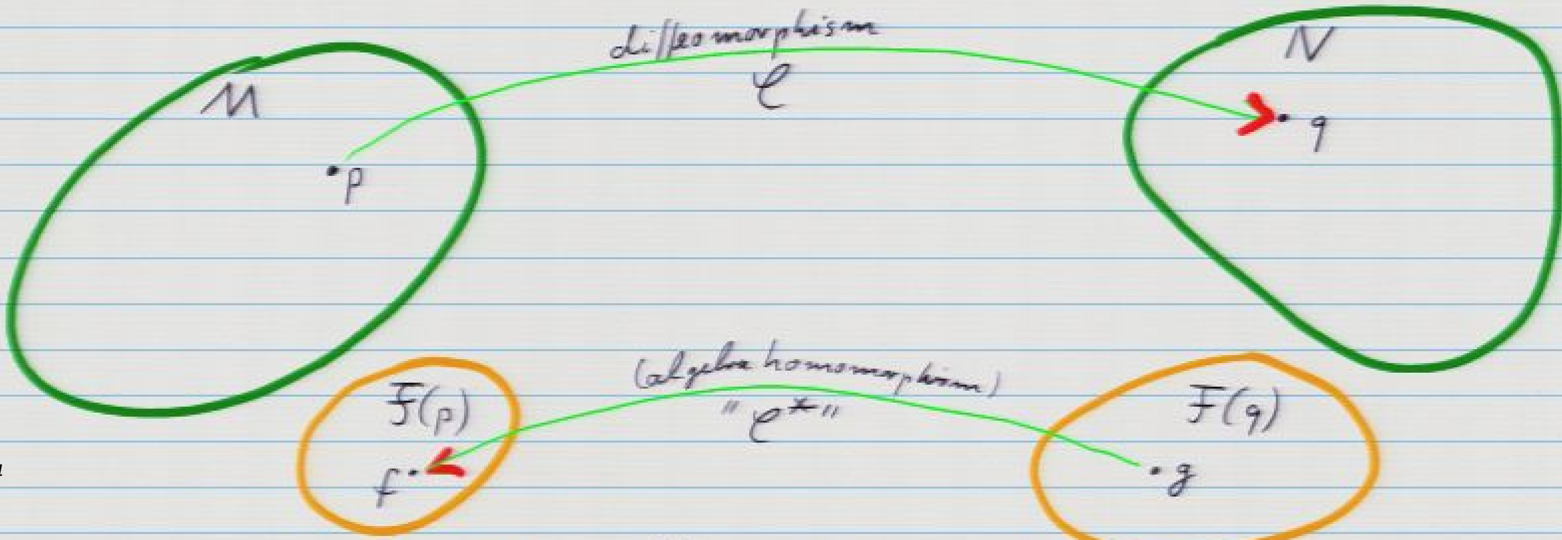

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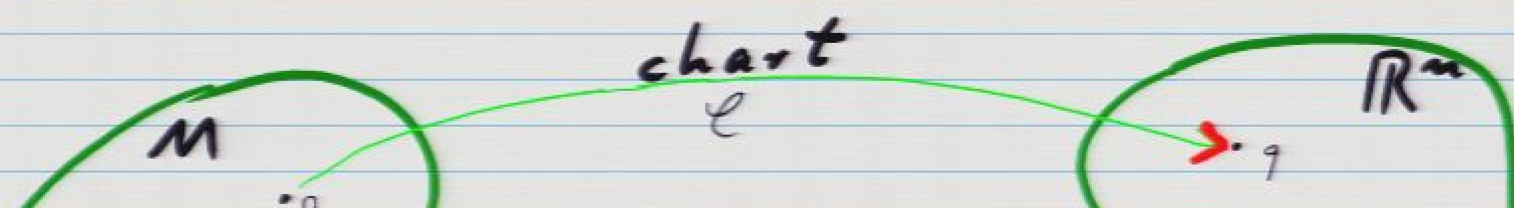

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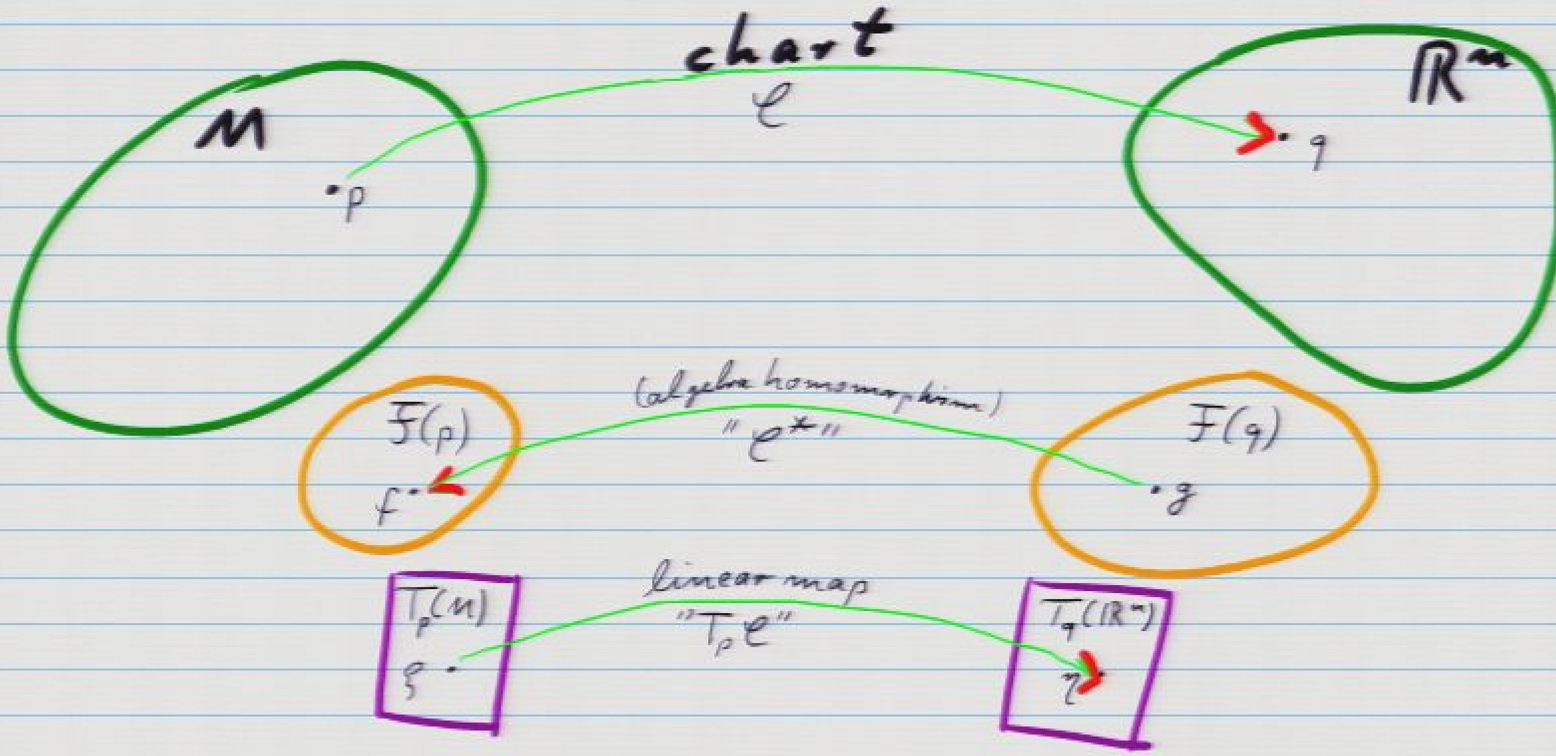

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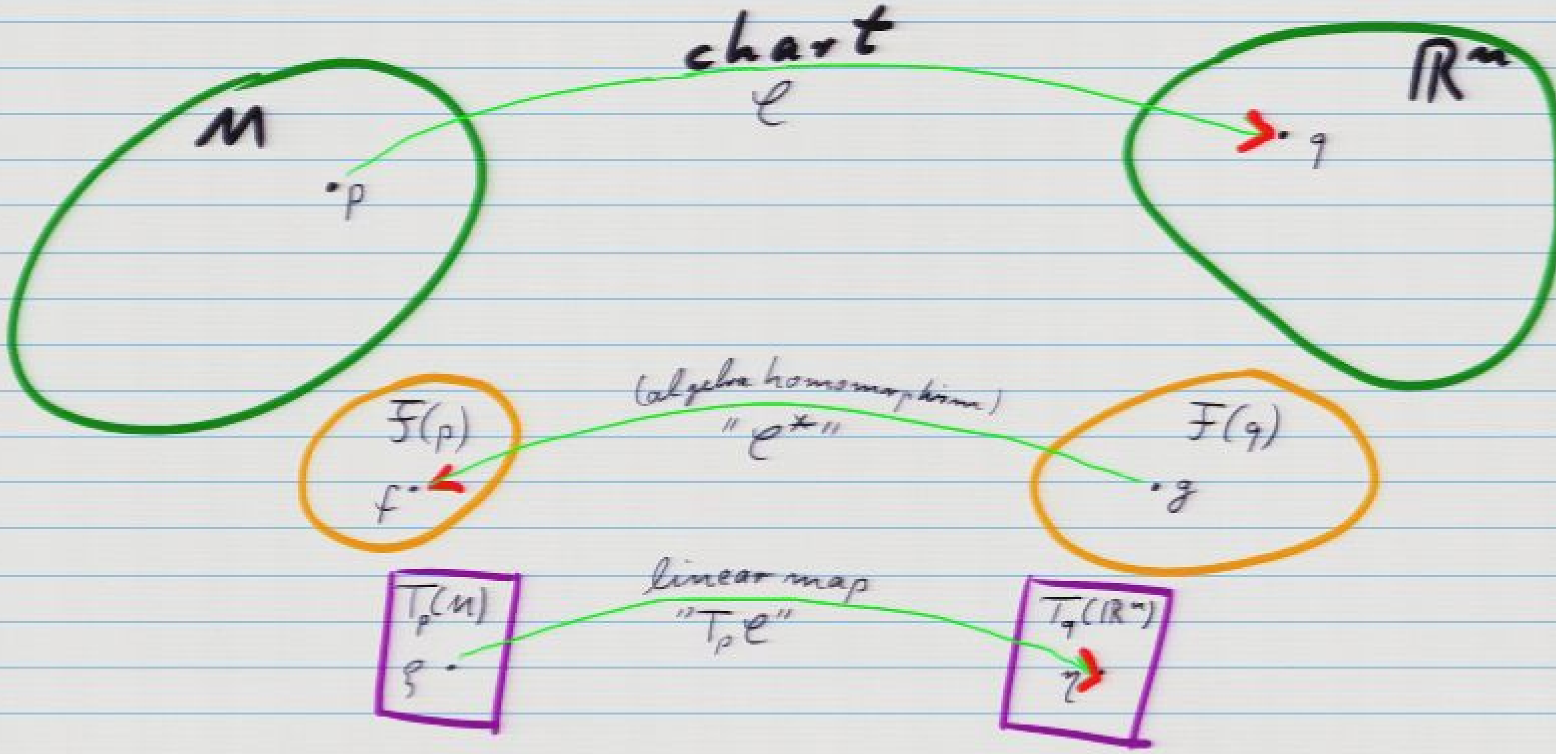

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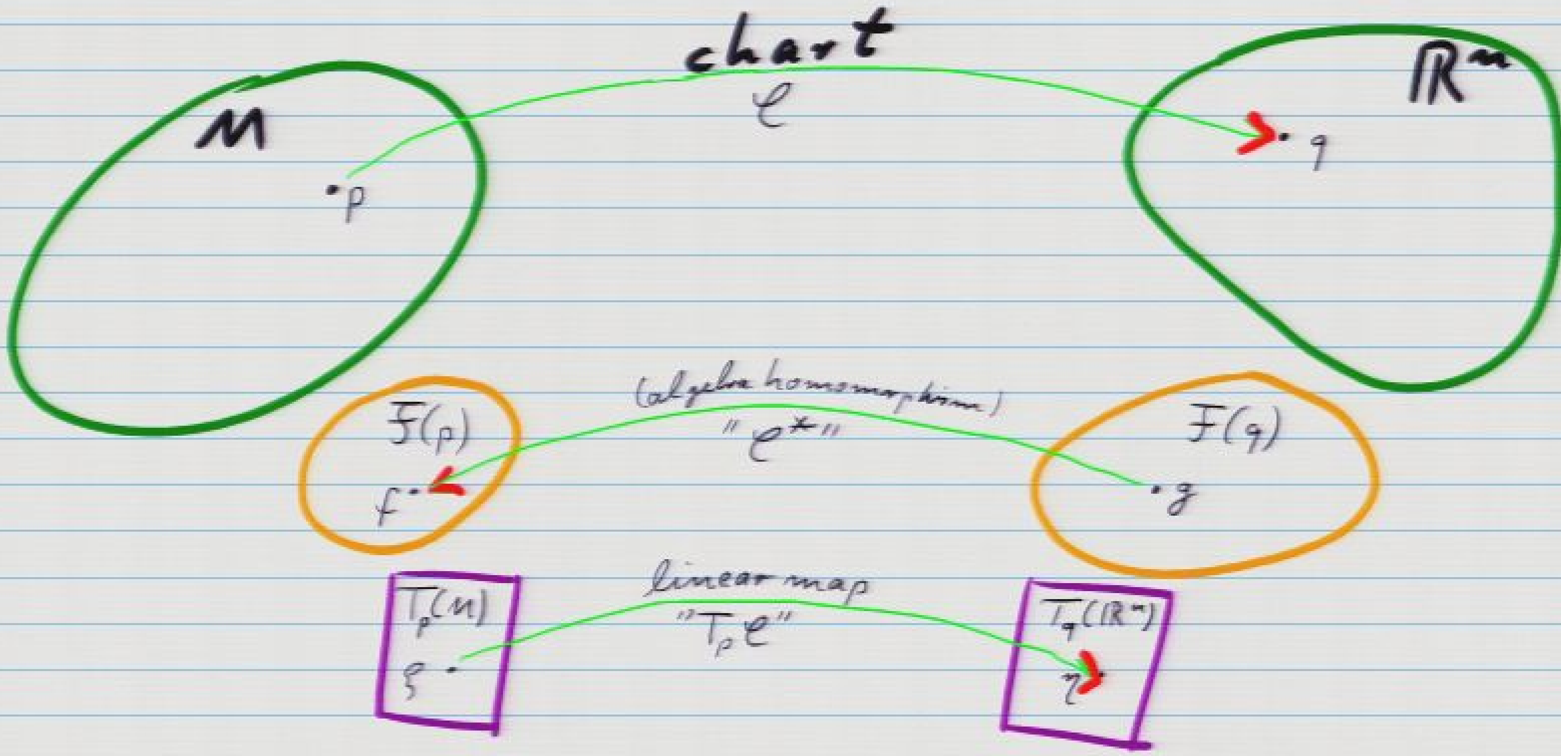

Namely:

Given a chart ℓ, every abstract

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Namely:

- Given a chart $\mathcal{C}$, every abstract point $p \in M$ has a concrete image $\mathcal{C}(p) \in \mathbb{R}^n$, <u>and</u>:

- Every abstract vector $\xi \in T_p(M)$ has a concrete image

$$\underbrace{T_p\mathcal{C}(\xi)}_{=\eta} \in T_{\underbrace{\mathcal{C}(p)}_{=q}}(\mathbb{R}^n)$$

- The image η is concrete because η is tangent vector to a point $q \in \mathbb{R}^n$

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Conversely: (and very conveniently)

- Assuming a fixed φ, any choice of a $q = (x^1, \ldots, x^n)$ denotes a $p \in M$ and any choice of a $(\eta^1, \ldots, \eta^n)$ denotes a $\xi \in T_p(M)$.

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- E.g. $\eta = \frac{\partial}{\partial x^i}\Big|_{x=q}$ is the image of some abstract $\xi \in T_p(M)$, for fixed φ.

Notation: $\xi = \text{"}\frac{\partial}{\partial x^i}\Big|_{x=p}\text{"}$

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Conversely: (and very conveniently)

- Assuming a fixed φ, any choice of a $q = (x^1, \ldots, x^n)$ denotes a $p \in M$ and any choice of a $(\eta^1, \ldots, \eta^n)$ denotes a $\xi \in T_p(M)$.

  some numbers

- E.g. $\eta = \frac{\partial}{\partial x^i}\Big|_{x=q}$ is the image of some abstract $\xi \in T_p(M)$, for fixed φ.

Notation: $\xi = {}^{\prime\prime}\frac{\partial}{\partial x^i}\Big|_{x=p}{}^{\prime\prime}$

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symbolic notation

Question:

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↑ symbolic notation

Question:

If we hold p and $\xi \in T_p(M)$ fixed, how do the numbers $(x^1, \ldots, x^n)$

and $(\ldots$

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$\uparrow$ symbolic notation

Question:

If we hold p and $\xi \in T_p(M)$ fixed, how do the numbers $(x^1, \ldots, x^n)$ and $(\eta^1, \ldots, \eta^n)$ change when we change the chart?

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change the chart?

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and $(q^1, \dots, q^n)$ change when we

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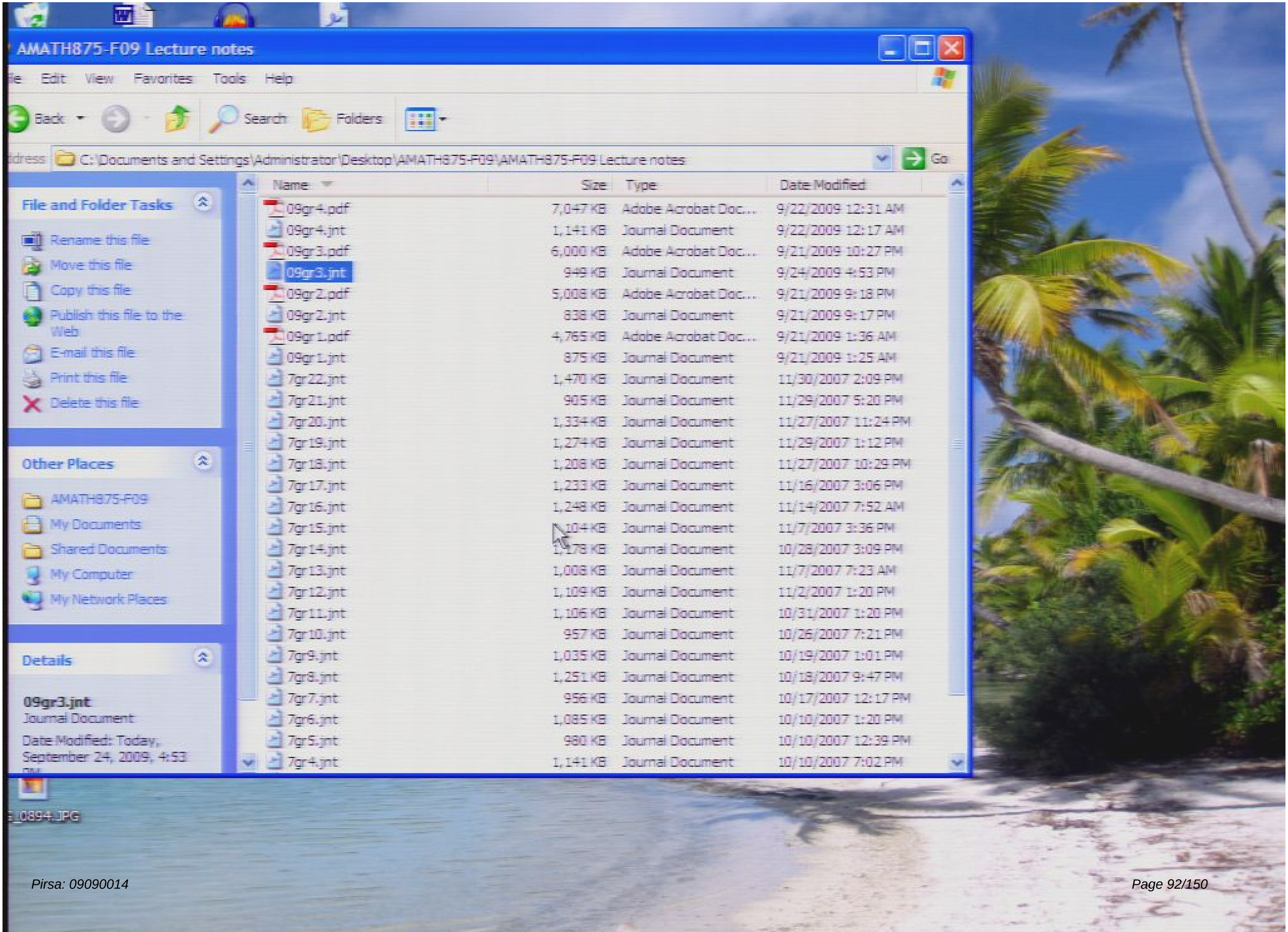
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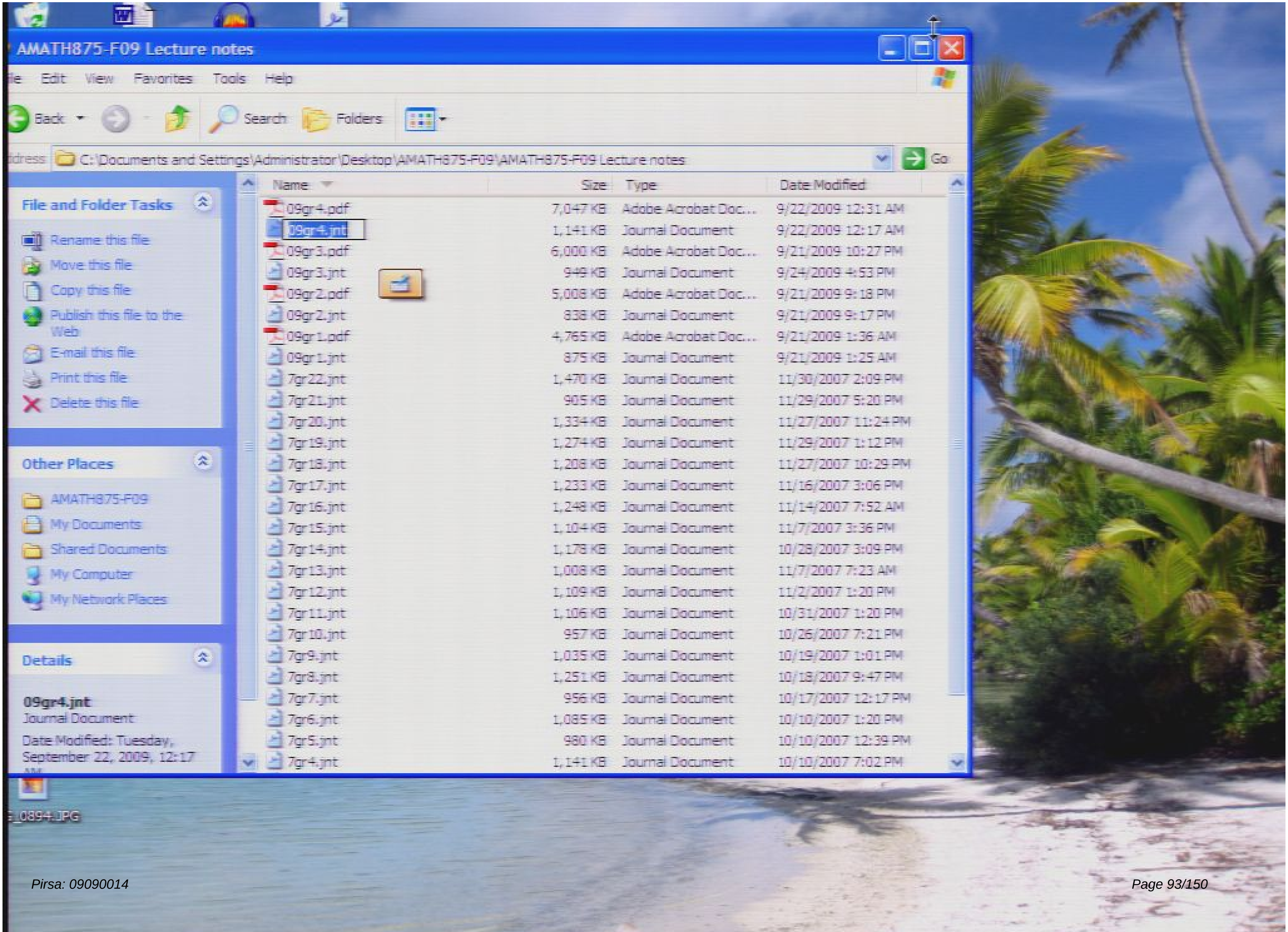
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Def: A tangent vector $\xi^{(phys)} \in T_p^{(phys)}(M)$ is a map that assigns to each (germ of a) chart a coefficient vector $\in \mathbb{R}^n$, <u>so that if</u>

- $(\eta^1, \dots, \eta^n)$ is coefficient vector w. resp. to chart α
- $(\nu^1, \dots, \nu^n)$ is coefficient vector w. resp. to chart β

<u>then:</u>

$$\nu^i = \sum_{j=1}^{n} \left.\frac{\partial \tilde{x}^i}{\partial x^j}\right|_{x=\beta(p)} \eta^j \qquad \text{with } \tilde{x} = \phi(x), \quad \phi = \beta \circ \alpha^{-1}$$

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Given $\xi \in T_p(M)$, its images in charts α, β, namely $\eta = \sum_{i=1}^{n} \eta^i \frac{\partial}{\partial x^i}$ and $\nu = \sum_{i=1}^{n} \nu^i \frac{\partial}{\partial \tilde{x}^i}$, are related by

Jacobian matrix $D\phi$

$$\underbrace{\nu^i = \sum_{j=1}^{n} \frac{\partial \tilde{x}^i}{\partial x^j}\Big|_{x=q} \eta^j}= \sum_{j=1}^{n} \frac{\partial \phi^i(x^1, \ldots, x^n)}{\partial x^j}\Big|_{x=q} \eta^j$$

This transformation property can also be used as the starting point for a definition of tangent vectors!

with: $\tilde{x}^i = \phi^i(x^1, \ldots, x^n)$

The "physicist's definition of $T_p(M)$":

Def: A tangent vector $\xi^{(phys)} \in T_p^{(phys)}(M)$ is a map

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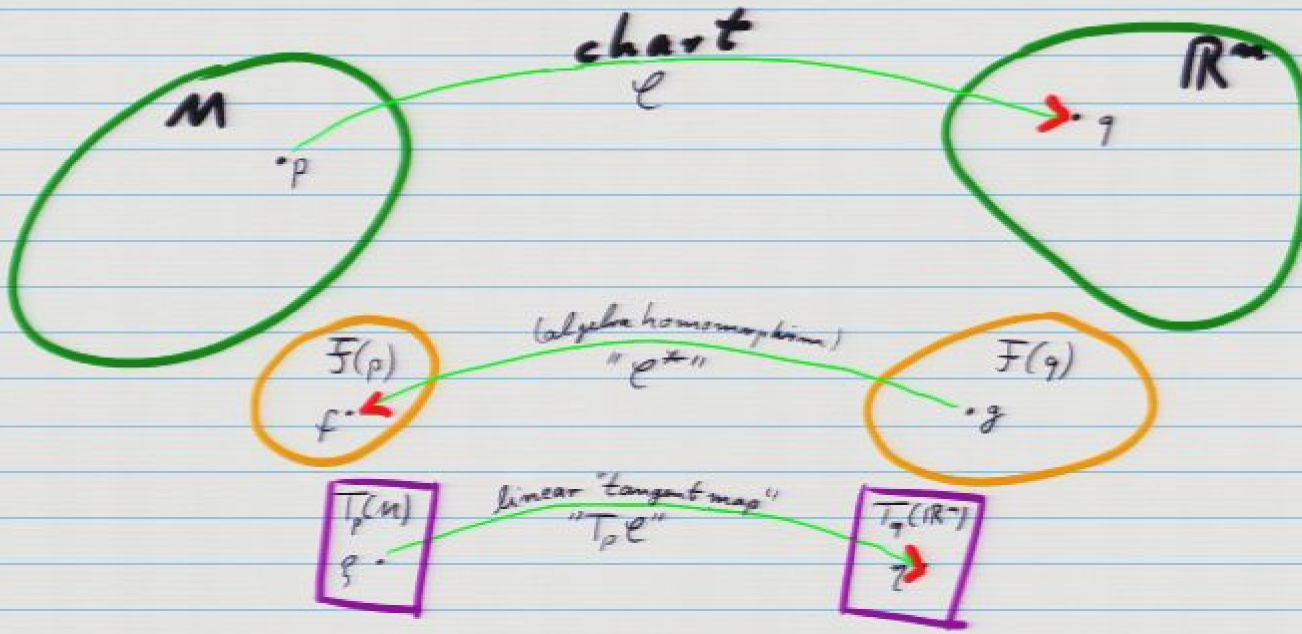
by using a chart $\varphi: M \to \mathbb{R}^n$:



Note: $T_p \varphi$ is isomorphism of vector spaces

Namely:

□ Each $p \in M$ has now concrete image $q \in \mathbb{R}^n$

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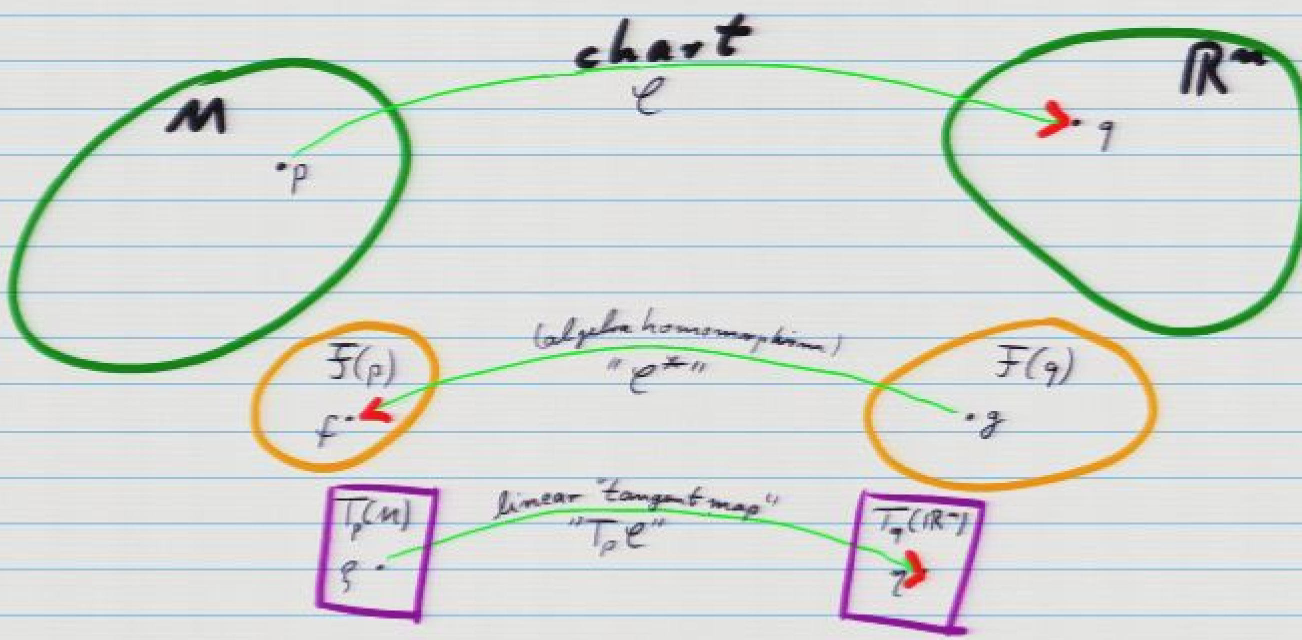
GR for Cosmology, Achim Kempf, Fall 09, Lecture 4

Note Title 9/24/2005

Recall: Get concrete handle on the abstract $p \in \mathcal{M}$ and $f \in \mathcal{F}(p)$ and $\xi \in T_p(\mathcal{M})$ by using a chart $\varphi: \mathcal{M} \to \mathbb{R}^n$:



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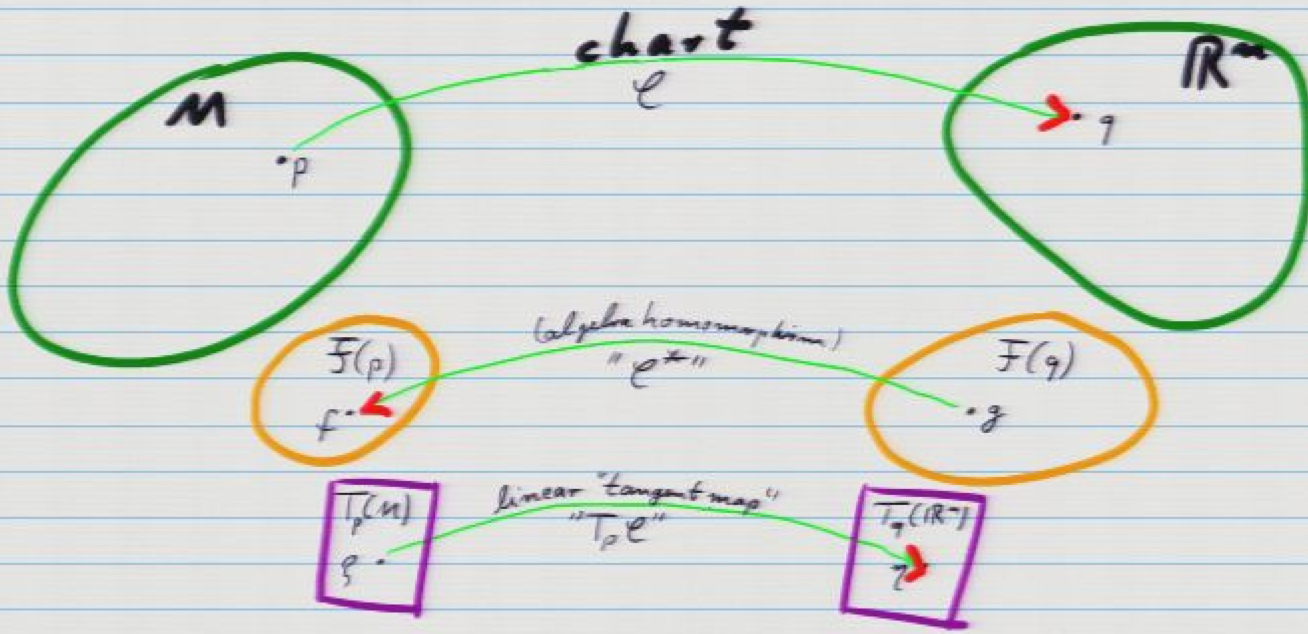

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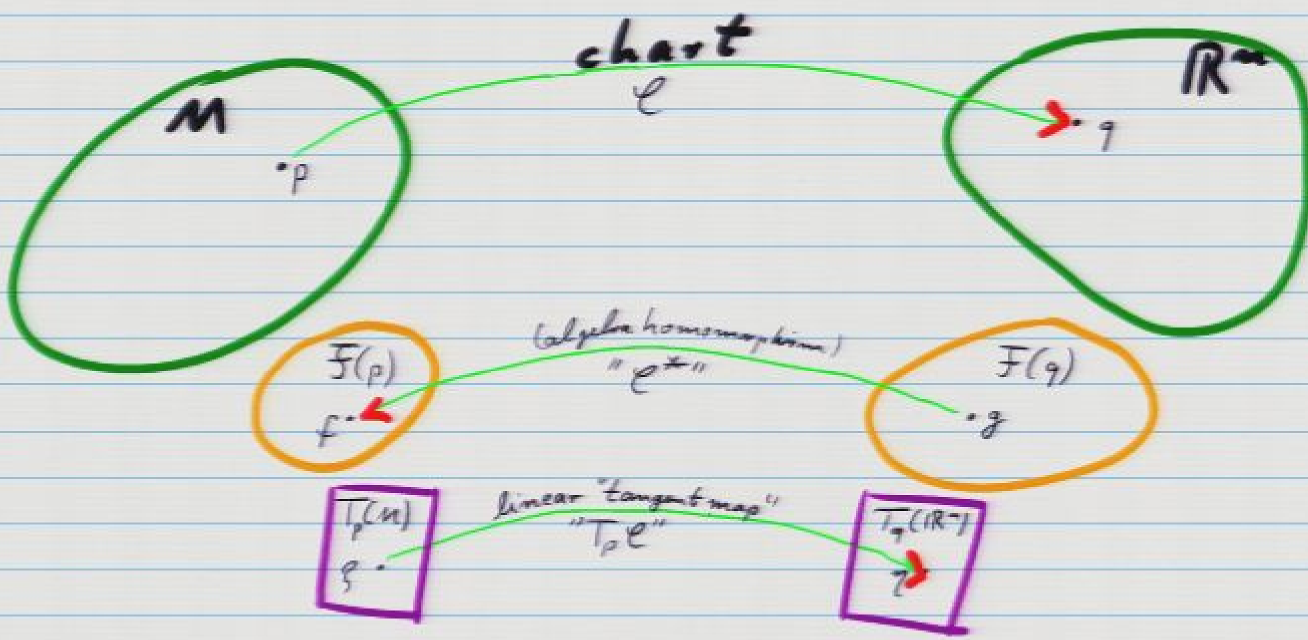

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Namely:

- Each $p \in M$ has now concrete image $q \in \mathbb{R}^n$ i.e. it has 'coordinates'.

- Each $f \in \mathcal{F}(p)$ is image of a concrete function germ $g \in \mathcal{F}(q)$.

- Each $\xi \in T_p(M)$ has now concrete image

$$\eta \in T_q(\mathbb{R}^n)$$

which we know has the form:

$$\eta = \sum_{i=1}^{n} \overbrace{\eta_i}^{\text{coefficients } \in \mathbb{R}} \frac{\partial}{\partial x^i}\Big|_{x=q}$$

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Recall: Get concrete handle on the abstract

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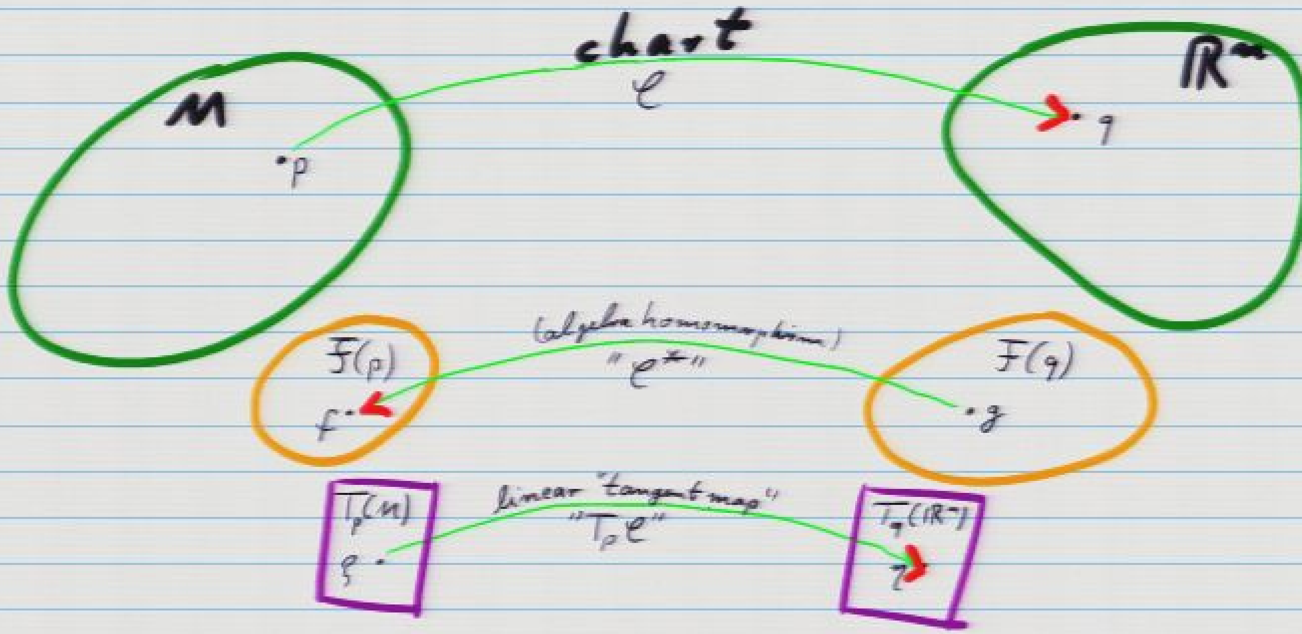

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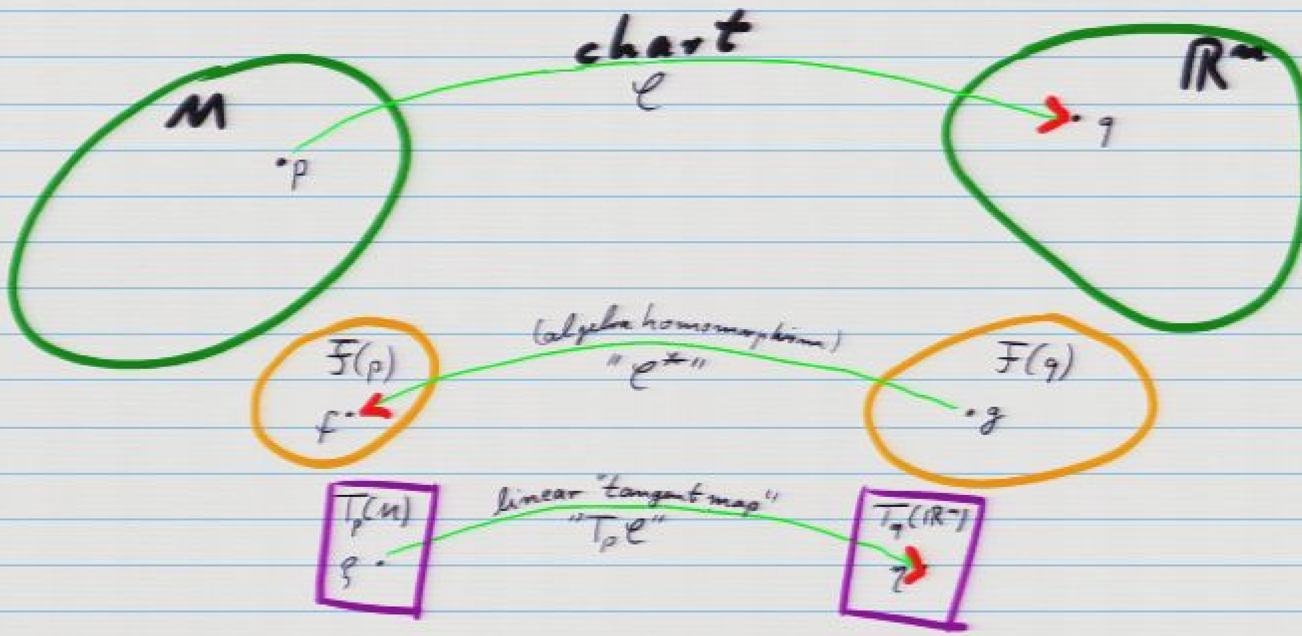

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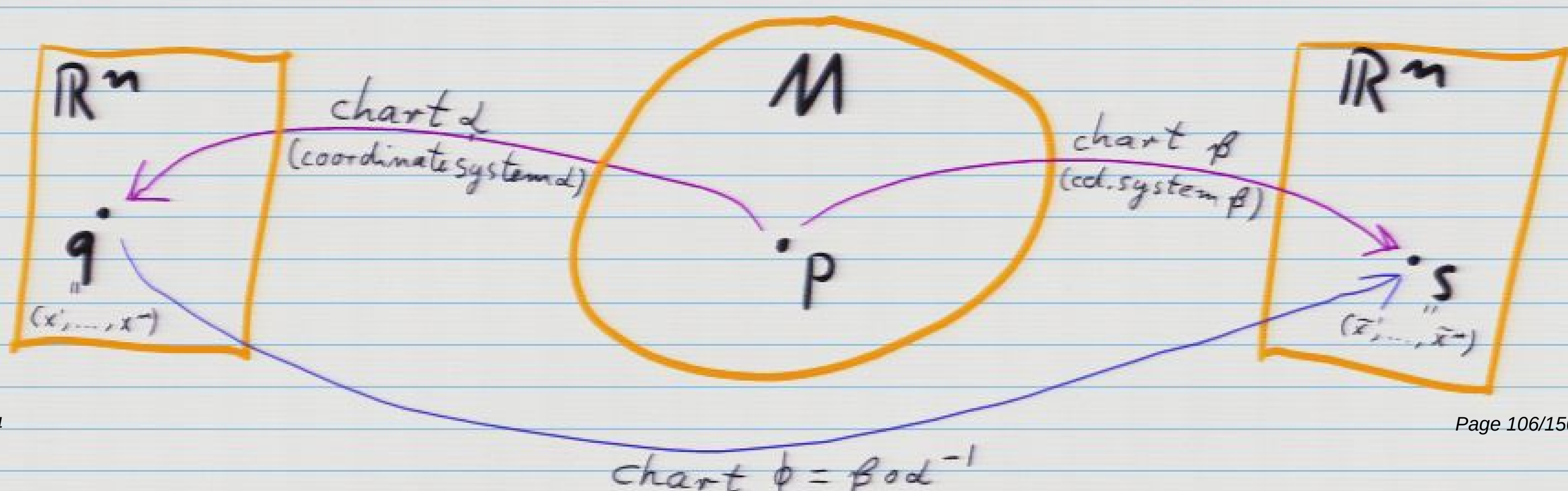
Given a $p \in M$ and a $\xi \in T_p(M)$, how do their coordinates and coefficients change under a change of charts?



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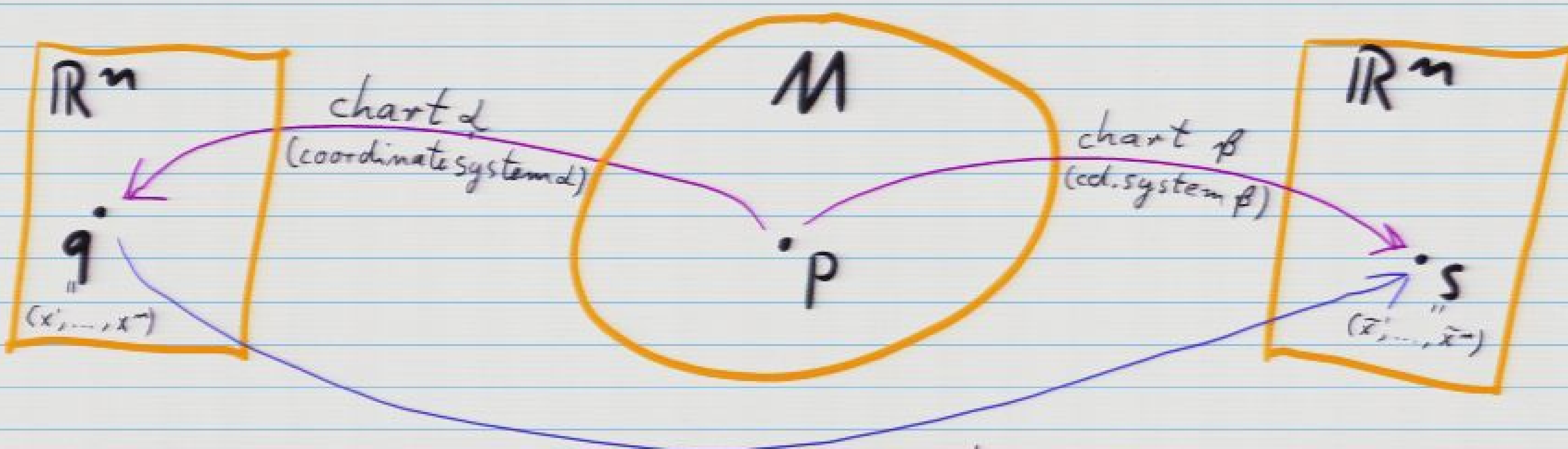

chart $\phi = \beta \circ \alpha^{-1}$

i.e.: $\phi : \mathbb{R}^n \to \mathbb{R}^n$

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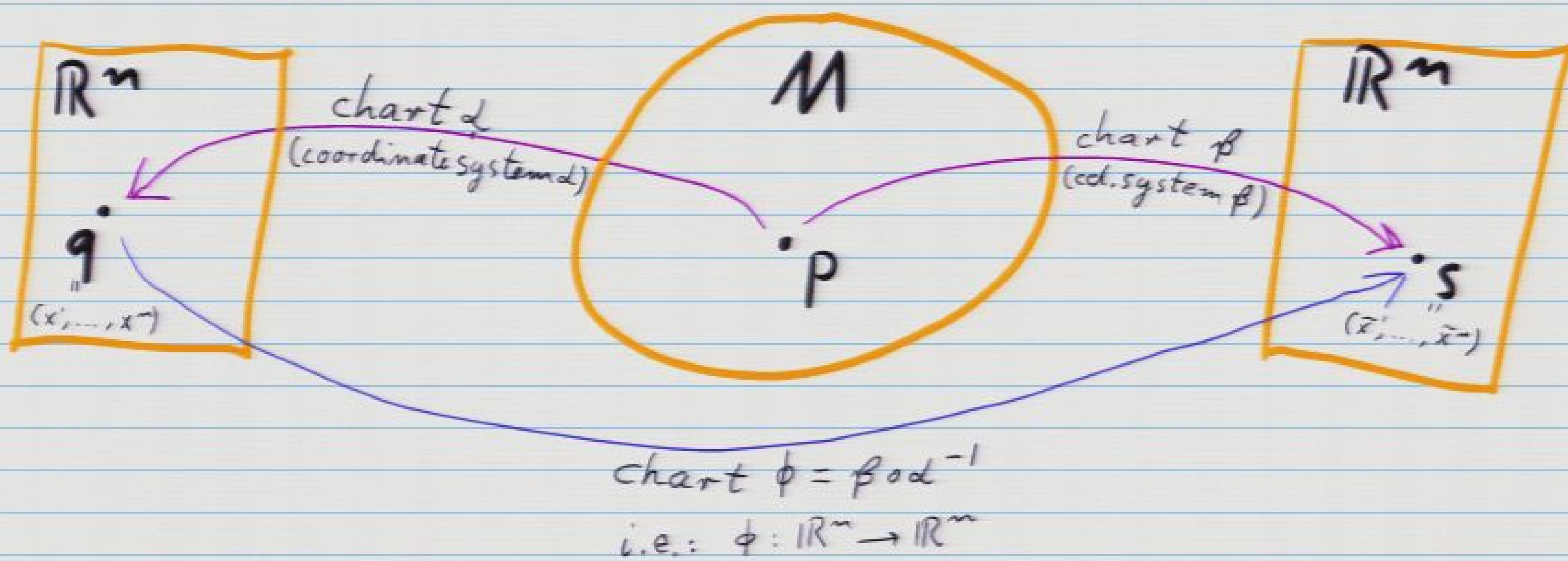

$\Rightarrow$ When changing from cds. α to cds. β:

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$\Rightarrow$ When changing from cds. α to cds. β:

1. The coordinates of p change from

$q = (x^1, \dots, x^n)$ to $s = (\tilde{x}^1, \dots, \tilde{x}^n)$:

$$(\tilde{x}^1, \dots, \tilde{x}^n) = \phi(x^1, \dots, x^n)$$

where $\tilde{x}^i = \phi^i(x^1, \dots, x^n)$.

2. A function (germ) $f \in \mathcal{F}_p(M)$ has images $g \in \mathcal{F}_q(\mathbb{R}^n)$ and $h \in \mathcal{F}_s(\mathbb{R}^n)$, related by:

$$f(p) = g(q) = h(s) \quad (\in \mathbb{R})$$

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and $h(\tilde{x}^1, \dots, \tilde{x}^m) = g(x^1, \dots, x^m)$ $(*)$

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3. A tangent vector $\xi \in T_p(M)$ now has two images, $\eta \in T_q(\mathbb{R}^m)$ and $\nu \in T_s(\mathbb{R}^m)$.

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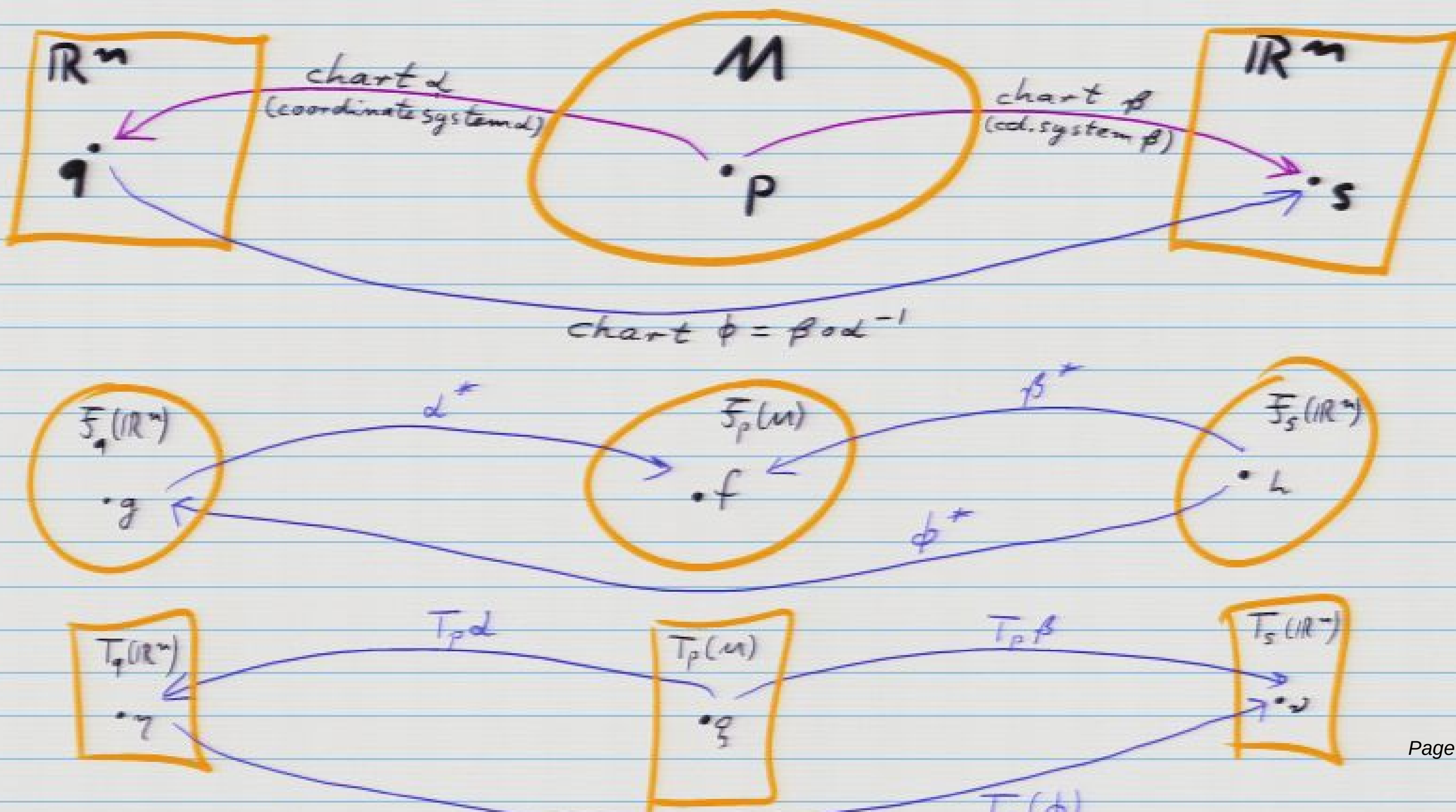
3. A tangent vector $\xi \in T_p(M)$ now has two images, $\eta \in T_q(\mathbb{R}^n)$ and $\nu \in T_s(\mathbb{R}^n)$.



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By construction:

$$\xi(f) = \eta(g) = \nu(h) \quad (\in \mathbb{R})$$

$\Rightarrow$ in particular:

$$\sum_{i=1}^{n} \eta^i \frac{\partial}{\partial x^i} g(x^1, \dots, x^n)\Big|_{x=q} = \sum_{j=1}^{n} \nu^j \frac{\partial}{\partial \tilde{x}^j} \underbrace{h(\tilde{x}^1, \dots, \tilde{x}^n)}_{\overset{\text{by } (*)}{=} g(x^1, \dots, x^n)}\Big|_{\tilde{x}=s}$$

$$= \sum_{\substack{j=1 \\ k=1}}^{n} \nu^j \frac{\partial x^k}{\partial \tilde{x}^j}\Big|_{\tilde{x}=s} \frac{\partial}{\partial x^k} g(x^1, \dots, x^n)\Big|_{x=q}$$

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$\Rightarrow$ in particular:

$$\sum_{i=1}^{n} \eta^i \frac{\partial}{\partial x^i} g(x^1,\dots,x^n)\Big|_{x=q} = \sum_{j=1}^{n} v^j \frac{\partial}{\partial \tilde{x}^j} \underbrace{h(\tilde{x}^1,\dots,\tilde{x}^n)}_{\underset{\text{by } (*)}{=} g(x^1,\dots,x^n)}\Big|_{\tilde{x}=s}$$

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Must be true for all g!

$\Rightarrow$ $$\sum_{i=1}^{n} \eta^i \frac{\partial}{\partial x^i} = \sum_{j=1}^{n} v^j \frac{\partial x^i}{\partial \tilde{x}^j}\Big| \frac{\partial}{\partial x^i}$$

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Must be true for all q!

$$\Rightarrow \quad \sum_{i=1}^{n} \eta^i \frac{\partial}{\partial x^i} = \sum_{\substack{j=1 \\ i=1}}^{n} v^j \left.\frac{\partial x^i}{\partial \tilde{x}^j}\right|_{\tilde{x}=s} \frac{\partial}{\partial x^i}$$

Jacobian matrix $D\phi^{-1}$ of ϕ^{-1} at s.

$$\Rightarrow \quad \eta^i = \sum_{j=1}^{n} \left.\frac{\partial x^i}{\partial \tilde{x}^j}\right|_{\tilde{x}=s} v^j$$

Jacobian matrix $D\phi$ of ϕ at q.

$\Rightarrow$ conversely: $$v^i = \sum_{j=1}^{n} \left.\frac{\partial \tilde{x}^i}{\partial x^j}\right|_{x=q} \eta^j$$

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chart $\phi = \beta \circ \alpha^{-1}$

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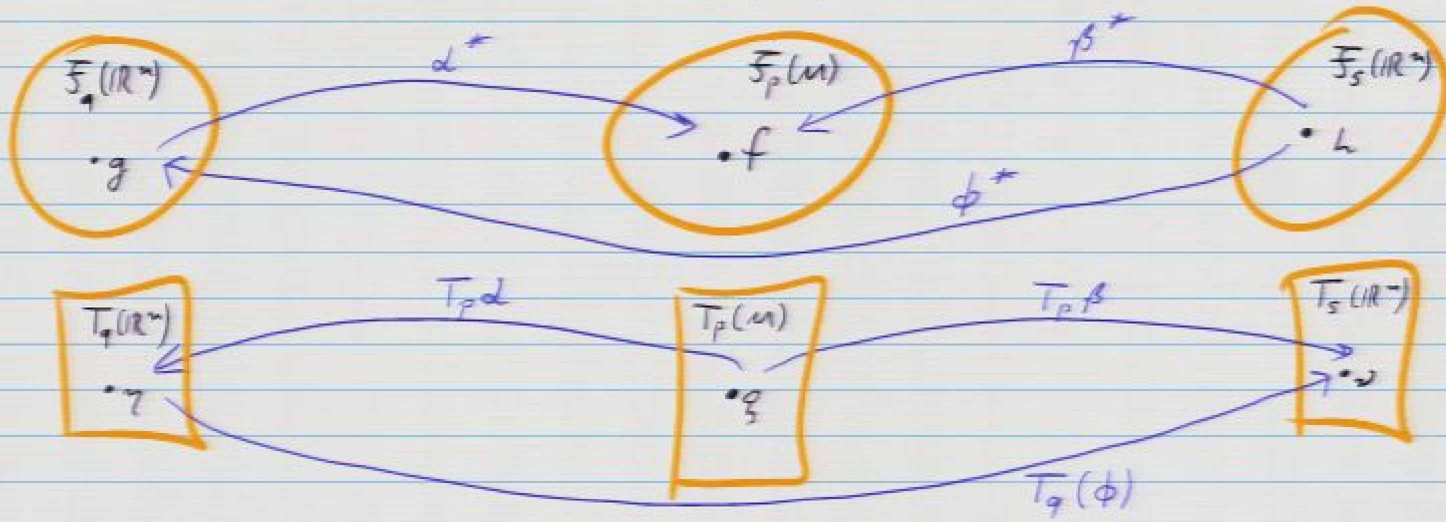

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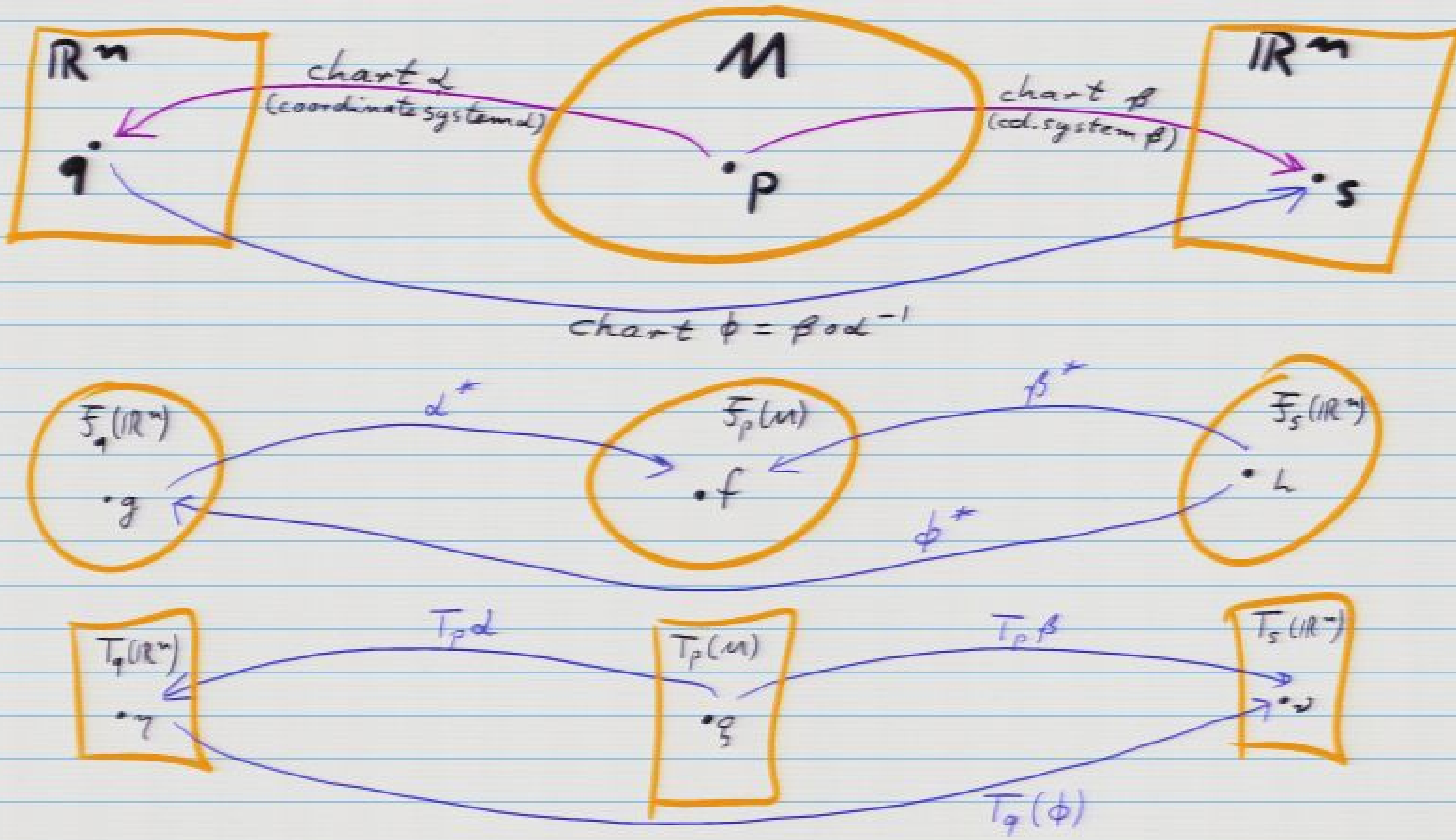

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Must be true for all q!

$$\Rightarrow \quad \sum_{i=1}^{m} \eta^i \frac{\partial}{\partial x^i} = \sum_{\substack{j=1 \\ i=1}}^{m} v^j \left.\frac{\partial x^i}{\partial \tilde{x}^j}\right|_{\tilde{x}=s} \frac{\partial}{\partial x^i}$$

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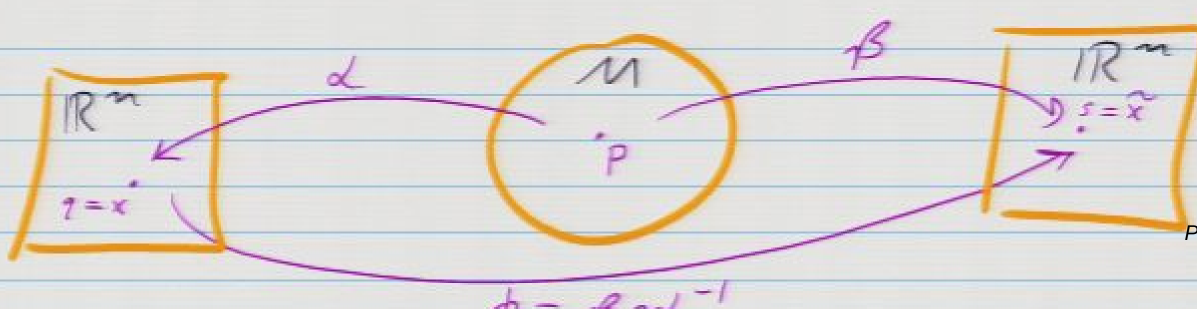

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$$\Rightarrow \qquad \eta^i = \sum_{j=1}^{n} \left.\frac{\partial x^i}{\partial \tilde{x}^j}\right|_{\tilde{x}=s} v^j$$

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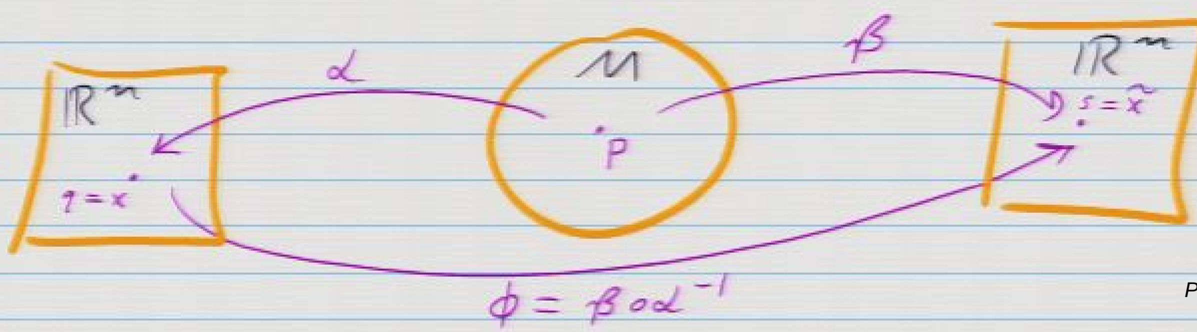

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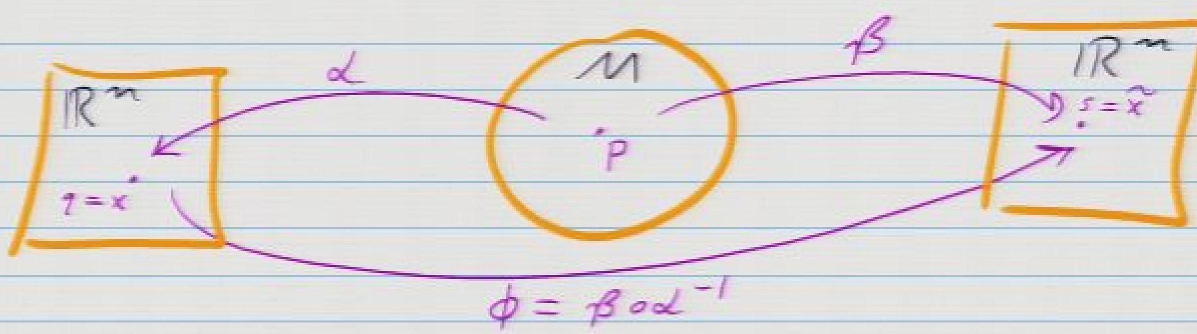

Given $\xi \in T_p(M)$, its images in charts α, β, namely $\eta = \sum_{i=1}^{n} \eta^i \frac{\partial}{\partial x^i}$ and $\nu = \sum_{i=1}^{n} \nu^i \frac{\partial}{\partial \tilde{x}^i}$, are related by

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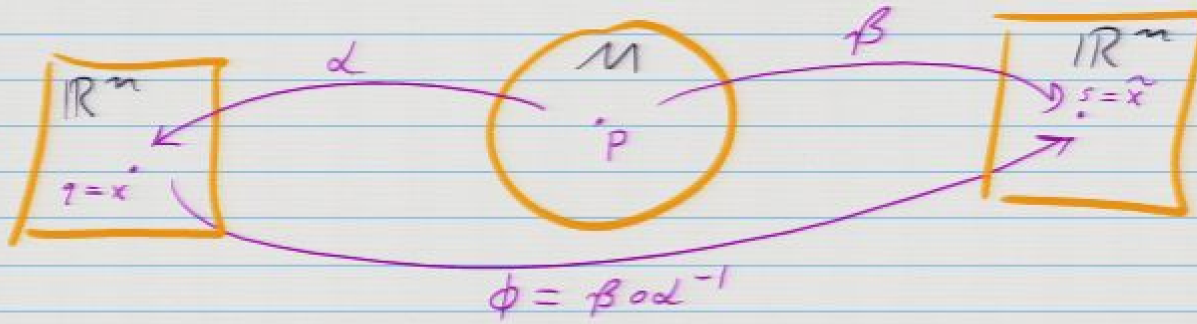

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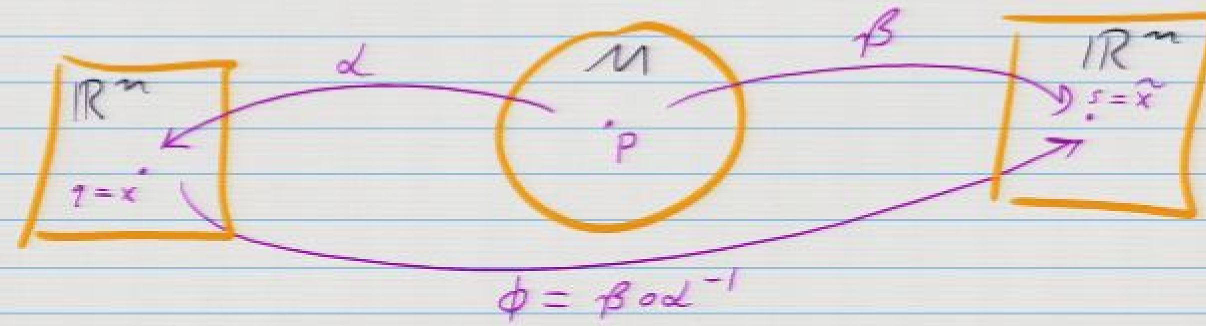

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This transformation property can also be used as the starting point for definition of tangent vectors!

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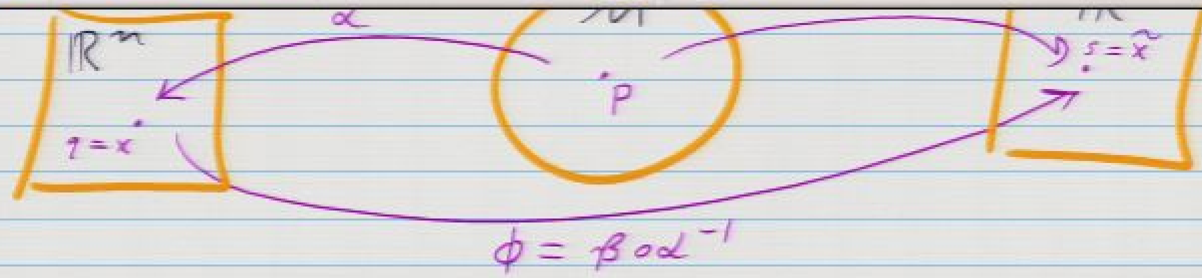

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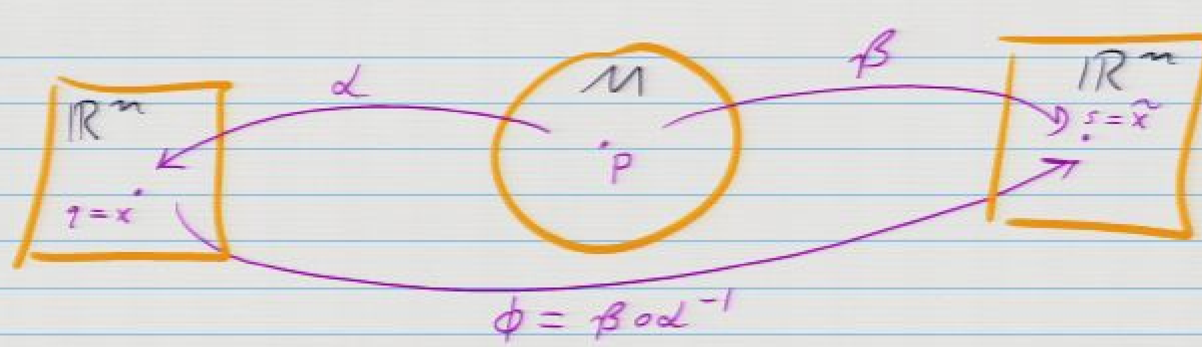

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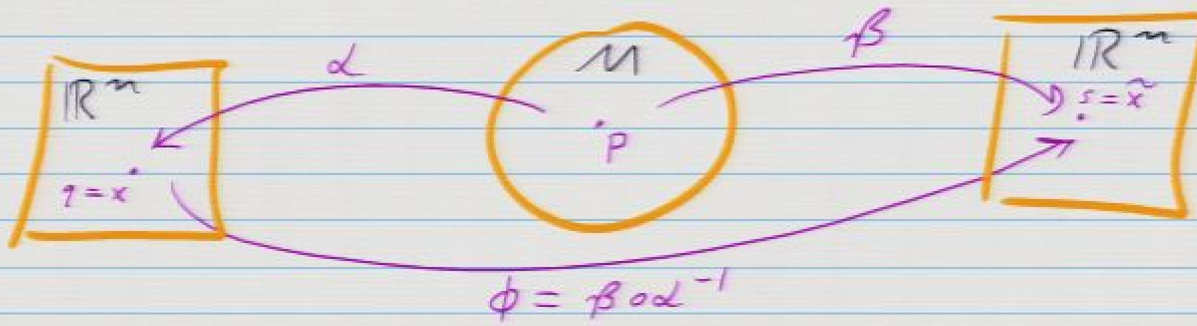

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The "physicist's definition of $T_p(M)$":

Def: A tangent vector $\xi^{(phys)} \in T_p^{(phys)}(M)$ is a map that assigns to each (germ of a) chart a coefficient vector $\in \mathbb{R}^n$, so that if

- $(\eta^1, \dots, \eta^n)$ is coefficient vector w. resp. to chart α
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So far, 2 equiv. defs. of $T_p(M)$:

In a chart, α, a tangent vector, $\xi \in T_p(M)$ is:

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  i.e. it is a directional derivative.

  Defining property: Leibniz rule.

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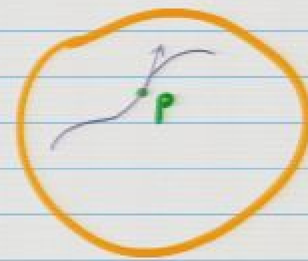

Consider paths in M that pass through p:

$$\gamma : \mathbb{R} \to M$$
$$\gamma(0) = p$$

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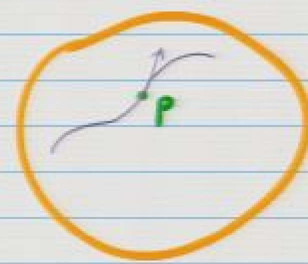

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really: each equivalence class of diffable paths through p

Yes: Each path γ defines a linear map $\overline{\gamma}$:

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$$= \frac{d}{dt} f(\gamma(t))\Big|\; g(\underbrace{\gamma(0)}_{=p}) + f(\underbrace{\gamma(0)}_{=p}) \frac{d}{dt} g(\gamma(t))\Big|_{\dots}$$

$$= \bar{\gamma}(f)\, g + f\, \bar{\gamma}(g) \quad (\text{alg})$$

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The "Cotangent Space" $T(M)^*$:

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The "Cotangent Space" $T_p(M)^*$:

Recall:

Given an n-dimensional vector space V, the set of linear maps $\omega: V \to \mathbb{R}$ forms also an n-dim. vector space. It is called the "dual space", and denoted V^*.

Definition:

The dual vector space to $T_p(M)$ is called the Cotangent Space, and denoted $T_p(M)^*$.

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We notice:

For every (germ of a) function at p,

$$f \in \mathcal{F}(p)$$

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called the "<u>differential</u> of f."

<u>Namely:</u>

$df : T_p(M) \to \mathbb{R}$ is the linear map:

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(Note: thus, we can view "d" as a map: $d : \mathcal{F}_p(M) \to T_p(M)^*$ See later...)

Concretely, in a cds., i.e. in a chart, $\xi \in T_p(M)$

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$\eta \in T_q(\mathbb{R}^n)$ and $g \in \mathcal{F}(q)$. **Then:**

$dg : T_q(\mathbb{R}^n) \to \mathbb{R}$

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Recall: Since all $\eta \in T_q(\mathbb{R}^n)$ take the form $\eta = \sum_{i=1}^{n} \eta^i \frac{\partial}{\partial x^i}\Big|$