Title: General Relativity for Cosmology - Lecture 3

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Abstract:

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## We notice:

For every (germ of a) function at $p$,

$$f \in \mathcal{F}(p)$$

one naturally obtains an element

$$\text{"}df\text{"} \in T_p(M)^*$$

called the "differential of $f$."

## Namely:

$df : T_p(M) \to \mathbb{R}$ is the linear map:

$$df : \xi \to \xi(f)$$

(Note: thus, we can view "$d$" as a map: $d : \mathcal{F}_p(M) \to T_p(M)^*$ See later...)

15 / 29

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(Note: thus, we can view "$d$" as a map: $d: \mathcal{F}_p(M) \to T_p^*(M)$. See later...)

Concretely, in a cds., i.e. in a chart, $\xi \in T_p(M)$ and $f \in \mathcal{F}(p)$ correspond to some $\eta \in T_q(\mathbb{R}^n)$ and $g \in \mathcal{F}(q)$. Then:

$$dg: T_q(\mathbb{R}^n) \to \mathbb{R}$$

$$dg: \eta \to \eta(g) = \sum_{i=1}^{n} \eta^i \frac{\partial}{\partial x^i}\Big|_{x=q} g(x^1, \dots, x^n)$$

Recall: Since all $\eta \in T_q(\mathbb{R}^n)$ take the form $\eta = \sum_{i=1}^{n} \eta^i \frac{\partial}{\partial x^i}\Big|$

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Recall: Since all $\eta \in T_q(\mathbb{R}^n)$ take the form $\eta = \sum_{i=1}^{n} \eta^i \frac{\partial}{\partial x^i}\Big|_{x=q}$

a basis of $T_q(\mathbb{R}^n)$ is $\left\{ \frac{\partial}{\partial x^i}\Big|_{x=q} \right\}_{i=1}^{n}$

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a basis of $T_q(\mathbb{R}^n)$ is $\left\{ \frac{\partial}{\partial x^i}\Big|_{x=q} \right\}_{i=1}$

Question: What is the dual basis in $T_q(\mathbb{R}^n)^*$?

- Consider the coordinate functions $x^k: \mathbb{R}^n \to \mathbb{R}$.

- Their differentials $dx^k \in T_q(\mathbb{R}^n)^*$ obey:

$$dx^k : T_q(\mathbb{R}^n) \to \mathbb{R}$$

$$dx^k : \frac{\partial}{\partial x^i}\Big|_{x=q} \to \frac{\partial}{\partial x^i} x^k \Big|_{x=q} = \delta_i^k$$

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$$\{dx^k\}_{k=1}^{n}$$

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# Thus:

Every element $\omega \in T_q(\mathbb{R}^n)^*$ takes the form:

$$\omega = \sum_{i=1}^{n} \underbrace{\omega_i}_{\in \mathbb{R}} dx^i$$

and its action is:

$$\omega : T_q(\mathbb{R}^n) \to \mathbb{R}$$

$$\omega : \sum_{j=1}^{n} \eta^j \frac{\partial}{\partial x^j} \to \sum_{i=1}^{n} \omega_i \, dx^i\left( \sum_{j=1}^{n} \eta^j \frac{\partial}{\partial x^j} \right)$$

$$= \sum_{i=1}^{n} \omega_i \sum_{j=1}^{n} \eta^j \underbrace{\frac{\partial}{\partial x^j} x^i}_{= \delta_{ij}}$$

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$$\Rightarrow \omega\left(\sum_{j=1}^{n} \eta^j \frac{\partial}{\partial x^j}\right) = \sum_{i=1}^{n} \omega_i \eta^i \qquad (\mathrm{I})$$

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In particular: For arbitrary $g \in \mathcal{F}(q)$, its differential $dg \in T_q(\mathbb{R}^n)$ must be of the form:

$$dg = \sum_{k=1}^{n} \omega_k \, dx^k \text{ with suitable } \omega_k \in \mathbb{R}.$$

How to calculate them?

We know:

$$dg(\eta) = \eta(g) = \sum_{i=1}^{n} \eta^i \underbrace{\frac{\partial}{\partial x^i} g(x)\Big|_{x=q}}_{= \omega^i} \quad (\mathrm{II})$$

Compare I, II $\Rightarrow$ $\omega_i = \frac{\partial}{\partial x^i} g(x)\Big|_{x=q}$

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19 / 29

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**Exercise:** (the "pull back" map)

**Assume** that $\rho \in T_p(M)^*$, under two charts $\alpha, \beta$, as above, corresponds to $\omega \in T_q(\mathbb{R}^n)^*$ and $\mu \in T_s(\mathbb{R}^n)^*$ with:

$$\omega = \sum^n \omega_i \, dx^i \quad \text{and} \quad \mu = \sum_i^n \mu_i \, d\tilde{x}^i$$

**Show** that
$$\mu_i = \sum_{j=1}^n \frac{\partial x^j}{\partial \tilde{x}^i}\bigg|_{\tilde{x}=s} \omega_j$$

Notice that this is the inverse of the Jacobian matrix of $\beta \circ \alpha^{-1}$ at $q$

**Remark:** The physicist's definition of $T_p(M)^*$ uses this

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## Some notation and terminology:

- Elements of $T_p(M)$ are called contravariant vectors
- Elements of $T_p(M)^*$ are called covariant vectors
- One often writes symbolically

$$\xi = \sum_{i=1}^{n} \xi^i \frac{\partial}{\partial x^i}\Big|_p \quad \text{for } \xi \in T_p(M)$$

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## Tensors:



21 / 29

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# Tensors:

**Def:** A tensor, $t$, of rank $(r,s)$ is an element of

$$T_p(M)^r_s := \underbrace{T_p(M)\otimes\ldots\otimes T_p(M)}_{r\text{ factors}}\otimes\underbrace{T_p(M)^*\otimes\ldots\otimes T_p(M)^*}_{s\text{ factors}}$$

**In a chart:**

$$t = \sum_{\substack{i_1,\ldots,i_r\\ j_1,\ldots,j_s=1}}^{n} \underbrace{t^{i_1\ldots i_r}_{j_1\ldots j_s}}_{\in\,\mathbb{R}} \frac{\partial}{\partial x^{i_1}}\otimes\ldots\otimes\frac{\partial}{\partial x^{i_r}}\otimes dx^{j_1}\otimes\ldots\otimes dx^{j_s}$$

**Under chart change:** (physicists, incl. Einstein, defined tensors this way)

$$\tilde{t}^{i_1\ldots i_r}_{j_1\ldots j_s} = \sum^{n} \frac{\partial\tilde{x}^{i_1}}{\partial x^{k_1}}\cdots\frac{\partial\tilde{x}^{i_r}}{\partial x^{k_r}}\frac{\partial x^{l_1}}{\partial\tilde{x}^{j_1}}\cdots\frac{\partial x^{l_s}}{\partial\tilde{x}^{j_s}}\, t^{k_1\ldots k_r}_{l_1\ldots l_s}$$

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## Thus:

$T_p(M) = T_p(M)'$ and $T_p(M)^* = T_p(M)$, i.e.:

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**Thus:** $T_p(M) = T_p(M)'$ and $T_p(M)^* = T_p(M)$, i.e.:

- a tangent vector is a tensor of rank $(1,0)$
- a cotangent vector is a tensor of rank $(0,1)$

**Consider now all $(p, T_p(M))$:**

**Def:** We call $T(M) := \bigcup_{p\in M} (p, T_p(M))$,

viewed as a $\underbrace{\text{diffable manifold}}^{\text{of dimension } 2n}$, the **Tangent bundle**.

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23 / 29

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23 / 29

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**Def:** The map $\pi: T(M) \to M$

$$\pi : (p, T_p(M)) \to p \qquad \left(\text{i.e.: } \pi^{-1}(p) = T_p(M)\right)$$

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24 / 29

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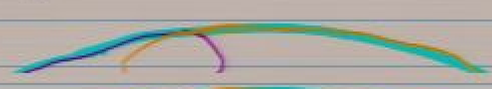

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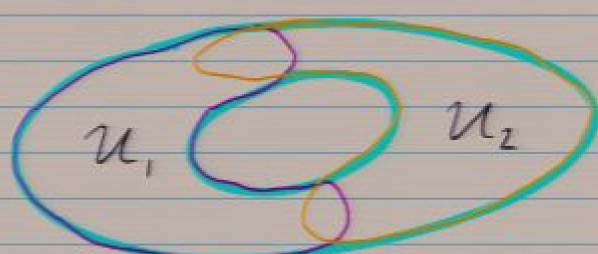


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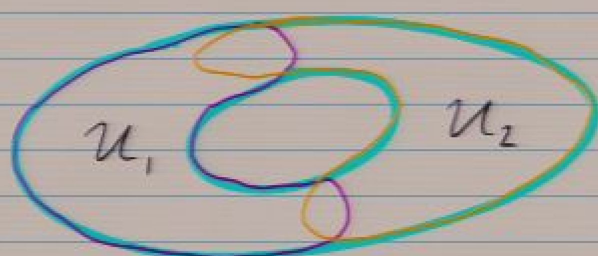


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**Def:** □ <u>A tangent vector field:</u>

is a map $\xi : \underset{M}{\overset{\cup}{p}} \rightarrow \underset{T_p(M)}{\overset{\cup}{\xi_p}}$

In a chart: $\xi = \sum_{i=1}^{n} \xi^i(x) \frac{\partial}{\partial x^i}$

□ <u>A cotangent vector field:</u>

$M$ $\quad T_p(M)^*$

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In a chart: $\omega = \sum_{i=1}^{n} \omega_i(x)\, dx^i$

Similarly, tensor fields:

$t : \underset{M}{p} \rightarrow \underset{T_p(M)^r_s}{t_p}$ , $t = \sum t^{j_1 \cdots j_r}_{i_1 \cdots i_s}(x) \frac{\partial}{\partial x^{j_1}} \cdots \frac{\partial}{\partial x^{j_r}} dx^{i_1} \cdots dx^{i_s}$

Def: Each field is called a section of its fibre bundle.

**Definition:** For the algebra of ($C^\infty$) functions $M \to \mathbb{R}$ we write $\mathcal{F}(M)$.

**Note:** One can show that vector fields are the derivations of the algebra $\mathcal{F}(M)$:

If $\xi$ is a contravariant vector field, then

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Important, because of their special transformation properties under chart changes, are these special tensor fields:

## Differential forms:

- The set $\Lambda_0 := \mathcal{F}(M)$ is called the set of 0-forms.

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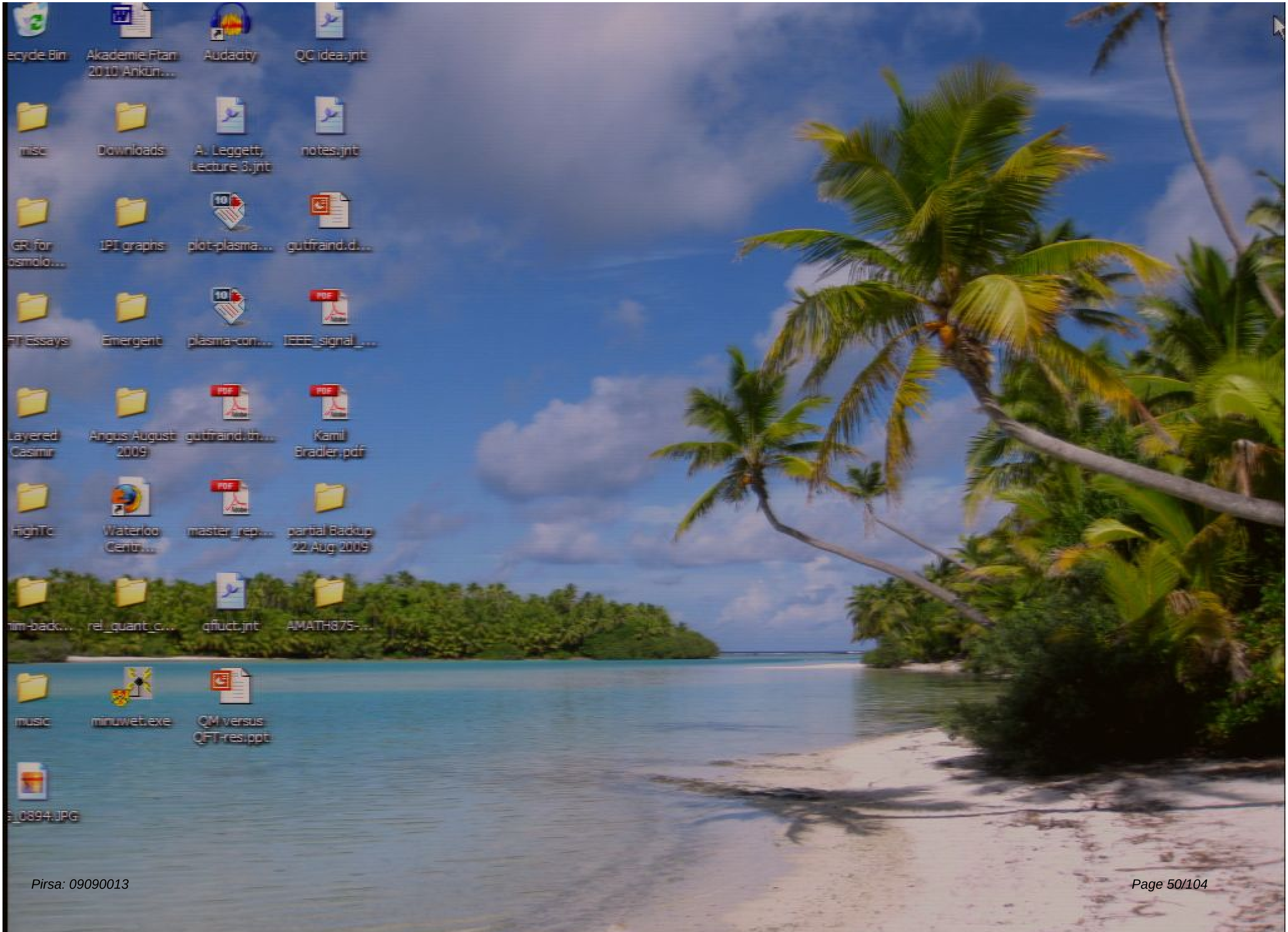
□ For $r = 2, 3, \ldots$ the set, $\Lambda_r$, of $r$-forms is defined to be the set of totally anti-symmetric tensor fields of rank $(0, r)$.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**Definition:** If $r > 1$ and $v \in T_p(M)_r$, then we define the "anti-symmetric part of $v$" as the image $\tilde{v} = A(v)$ of the linear antisymmetrization map $A$:

$$\tilde{v}(\xi_1, \dots, \xi_r) = A(v)(\xi_1, \dots, \xi_r)$$

$$:= \frac{1}{r!} \sum_{\sigma \in S_r} \operatorname{sgn}(\sigma)\, v(\xi_{\sigma(1)}, \dots, \xi_{\sigma(r)})$$

the sign ($\pm 1$) of the permutation $\sigma$

group of all $r!$ permutations of $(1, 2, \dots, r)$

**Why consider these?**

They will be key for integration! Only antisym. cov. tensors transform under chart change so as to match the Jacobian determinant arising in integrals when changing charts.

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$$A \circ A = A$$

(i.e. it is a self-adj... lin. map with the ...)

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5 / 30

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Definition:

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For $r > 1$ we define the space of differential $r$-forms (or 'exterior' $r$-forms) $\Lambda_r(p)$ at $p \in M$ as the subspace of totally anti-symmetric tensors of rank $(0, r)$:

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Def: If $\omega \in \Lambda_s(p)$, $v \in \Lambda_r(p)$ then the wedge product $\wedge$ yields a new form:

$$\wedge : \Lambda_r(p) \times \Lambda_s(p) \rightarrow \Lambda_{r+s}(p)$$

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- associative:

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- "graded" commutative:

$$\omega\wedge\nu = (-1)^{rs}\,\nu\wedge\omega \quad \text{if } \omega\in\Lambda_r,\ \nu\in\Lambda_s$$

8 / 30

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(E.g. for $dx^i\in\Lambda_1(p)$: $dx^i\wedge dx^j = -dx^j\wedge dx^i$ since $r=s=1$.)

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Page Width

# Properties of $\wedge$:

- **bi-linear:**

$$(\omega + \nu)\wedge\eta = \omega\wedge\eta + \nu\wedge\eta$$

$$(a\omega)\wedge\nu = \omega\wedge(a\nu) = a(\omega\wedge\nu) \quad \text{if } a \in \mathbb{R}$$

- **associative:**

$$(\omega\wedge\nu)\wedge\eta = \omega\wedge(\nu\wedge\eta)$$

- **"graded" commutative:**

$$\omega\wedge\nu = (-1)^{rs}\,\nu\wedge\omega \quad \text{if } \omega\in\Lambda_r,\ \nu\in\Lambda_s$$

(E.g. for $dx^i \in \Lambda_1(p)$: $dx^i\wedge dx^j = -dx^j\wedge dx^i$ since $r=s=1$.)

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Page Width

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- We can use $\wedge$ to build bases of $\Lambda_r(p)$:

<u>Assume</u>: $\{\theta^i\}_{i=1}^{n}$ is a basis of $\Lambda_1 = T_p(M)^*$.

(for example $\theta^i = dx^i$)

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**Then**: $\{\theta^{i_1} \wedge \theta^{i_2} \wedge \ldots \wedge \theta^{i_r}\}_{1 \leq i_1 < i_2 \leq \ldots \leq i_r \leq n}$

is a basis of $\Lambda_r(p)$ for $r > 1$.

**Therefore**:

$$\dim(\Lambda_r(p)) = \binom{n}{r}$$

$\Rightarrow$ no diff. forms of degree $r > n$!



Example: For $p \in M = \mathbb{R}^3$ we have bases:

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**Example:** For $p \in M = \mathbb{R}^3$ we have bases:

$$\mathrm{span}(dx^1, dx^2, dx^3) = \Lambda_1$$

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**Definition:**

$(\dim(\Lambda) = 2^n)$

$\Lambda(p) := \bigoplus_{i=0}^{n} \Lambda_i(p)$ equipped with the multiplication $\wedge$, is an associative algebra, called the exterior algebra or the Grassmann algebra over $T_p(M)$.



Generalization to fields:

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Page Width

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Generalization to fields:

- A differential form field is a mapping that associates to each $p \in M$ an element:

$$\omega(p) \in \Lambda(p)$$

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Page Width

# Generalization to fields:

- A differential form field is a mapping that associates to each $p \in M$ an element:

$$\omega(p) \in \Lambda(p)$$

It is usually also called simply a differential form and denoted $\omega$.

- They form the Grassmann algebra of differential forms, $\Lambda(M)$.



12 / 30

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Recall:

Given an algebra, it is often useful to consider derivations of the algebra, i.e. to consider maps that obey the Leibniz rule.

(we defined tangent vectors as derivations of $F_r(M)$)

Here: For the algebra $\Lambda(M)$, let us consider the exterior and the inner derivations:

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Page Width

# Definition:

A linear map $\Phi : \Lambda(M) \to \Lambda(M)$ is called a derivation of degree $k$, if:

$$\Phi : \Lambda_s(M) \to \Lambda_{s+k}(M) \quad \text{for all } s$$

$$\Phi : \alpha \wedge \beta \to \Phi(\alpha) \wedge \beta + \alpha \wedge \Phi(\beta)$$

for all $\alpha, \beta \in \Lambda(M)$.

Leibniz rule

Also:



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13 / 30

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Page Width

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A linear map $\Phi : \Lambda(M) \to \Lambda(M)$ is called an anti-derivation of degree k, if:

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$$\Phi : \quad \alpha \wedge \beta \to \Phi(\alpha)\wedge\beta + (-1)^q \alpha \wedge \Phi(\beta)$$

for all $\alpha \in \Lambda^q(M)$, $\beta \in \Lambda(M)$.

Anti-Leibniz rule

**Proposition:**



15 / 30

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## Proposition:

(Because of the Leibniz rule and linearity) any (anti-) derivation

$$\Phi : \Lambda(M) \to \Lambda(M)$$

is already fully determined by its action only on $\Lambda_0(M)$ and on a basis of $\Lambda_1(M)$.

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Page Width

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## The exterior derivative

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# The exterior derivative

The exterior derivative,

$$d: \Lambda(M) \to \Lambda(M)$$

is the anti-derivation of degree $K=1$ defined through:

$$\left.\begin{aligned} d &: \Lambda_0(M) \to \Lambda_1(M) \\ d &: \quad f \to df \end{aligned}\right\} \text{action of } d \text{ on } \Lambda_0(M)$$

$d: dx^i \to 0$ for all $i$. $\Big\}$ action of $d$ on a basis of $\Lambda_1(M)$



In a chart:

17 / 30

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Page Width

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We had: $d: f(x) \to df(x) = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} dx^i$

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Page Width

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$$\beta = \sum_{i_1 < \ldots < i_s} \underbrace{\overbrace{\beta_{i_1,\ldots,i_s}(x)}^{\in \Lambda_0(M)}}_{\Rightarrow \text{like } f(x) \text{ above}} \wedge \overbrace{dx^{i_1} \wedge \ldots \wedge dx^{i_s}}^{\in \Lambda_s(M)} \in \Lambda_s(M)$$

$$d: \beta \to d\beta$$

namely, by applying the anti-Leibnitz rule:

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Page Width

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$+$ terms of the form $d(dx^{i_1} \wedge \dots \wedge dx^{i_s})$

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Page Width

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18 / 30

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**Proposition:** $d: \Lambda(M) \to \Lambda(M)$ obeys:

$$\boxed{d \circ d = 0}$$

**Proof:**

$$d\circ d(\beta) = \sum_{\substack{i_1<\dots<i_s \\ j,k}} \underbrace{\frac{\partial^2 \beta_{i_1,\dots,i_s}(x)}{\partial x^j \partial x^k}}_{\text{sym. in } j,k} \underbrace{dx^k \wedge dx^j}_{\substack{dx^k\wedge dx^j = -dx^j\wedge dx^k \\ \text{i.e. antisym. in } j,k}} \wedge dx^{i_1} \wedge \dots \wedge dx^{i_s}$$

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## Example:

e.g.: electric potential

- For $M = \mathbb{R}^3$ and $f \in \mathcal{F}(M)$ we have:

$$df = \sum_{i=1}^{3} \frac{\partial f}{\partial x^i} dx^i$$

- **Notice:** $\left(\frac{\partial f}{\partial x^1}, \frac{\partial f}{\partial x^2}, \frac{\partial f}{\partial x^3}\right)$ is the "Gradient field $\nabla f$ of $f$"

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- Now assume $\gamma \in \Lambda_1(M)$ is an