Title: Spacetime can be simultaneously discrete and continuous, in the same way that information can.

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Abstract: TBA

Spacetime could be simultaneously discrete *and* continuous, in the same way that information can.

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Perimeter Institute colloquium, September 16, *2009*

Spacetime at short distances ?

A gedanken experiment (Heisenberg):

Assume we try to resolve a distance more and more precisely

=> increasing momentum uncertainty

momentum gravitates and thus curves space

=> increasing curvature uncertainty

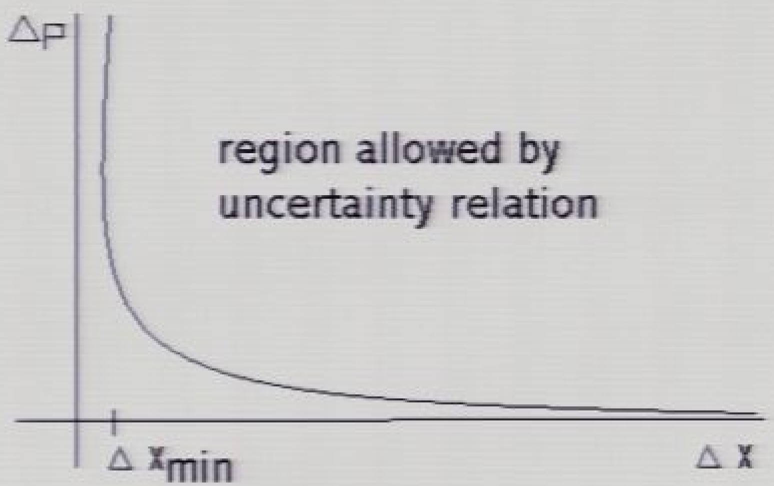
=> increasing distance uncertainties

=> a limit to how precisely distances can be resolved

=> Estimate of "natural ultraviolet cutoff": Planck length, i.e., 10^(-35)m.

Spacetime at the Planck scale ?

General quantum gravity studies (incl. string theory) suggest e.g.:



There could be an underlying structure, below the UV cutoff scale:

- strings, foams, causal sets, emergent, etc...
- several key approaches are being developed and pursued here at PI.
- so far, no experimental access and no consensus.

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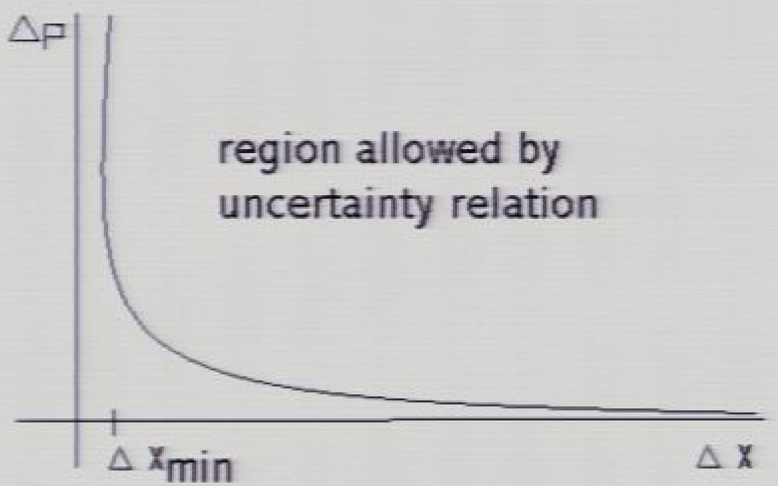

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Spacetime at the Planck scale ?

Problem hard since QFT and GR provide conflicting indications!

General relativity:

- Fields naturally described as living on a differentiable spacetime manifold.

Quantum field theory:

- Fields are quantizable generally only if spacetime described as discrete.

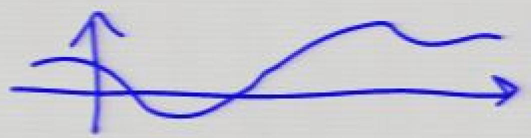
Recall: Information can be both! Let us try this idea:

Spacetime could be both discrete *and* continuous, in the same way that information can.

Discrete vs. continuous in information theory

Information can be:

- continuous (music, speech, etc):



- discrete (letters, digits, etc): R72SB

Unified in 1949 by Shannon, through: Sampling theory

Applications are ubiquitous in all of:

- analog / digital conversion
- communication engineering
- signal processing and analysis
- scientific data taking

The basic Shannon sampling theorem

- Assume f is "bandlimited", i.e:

$$f(x) = \int_{-\omega_{\max}}^{\omega_{\max}} \tilde{f}(\omega)\, e^{-2\pi i \omega x}\, d\omega$$

- Take samples of f(x) with spacing:

$$x_{n+1} - x_n = (2\omega_{\max})^{-1}$$

- **Then, exact reconstruction is possible:**

samples

$$f(x) = \sum_n f(x_n) \frac{\sin[2\pi(x-x_n)\omega_{\max}]}{\pi(x-x_n)\omega_{\max}}$$

No magic required

Traditional proof:

- uses Fourier theory
- but Fourier theory would not be useful on curved spaces

New method of proof:

1) Notice: Trivial for any N-dimensional function space

- Consider a function space specified by N given basis functions.
- Any function, f, is determined by its N coefficients in that basis.
- The values of f at N points yield N equations to determine the N coefficients.

=> we obtain f in a basis, thus we know f completely, i.e., everywhere.

2) For suitable function spaces, can take the limit N -> infinity.

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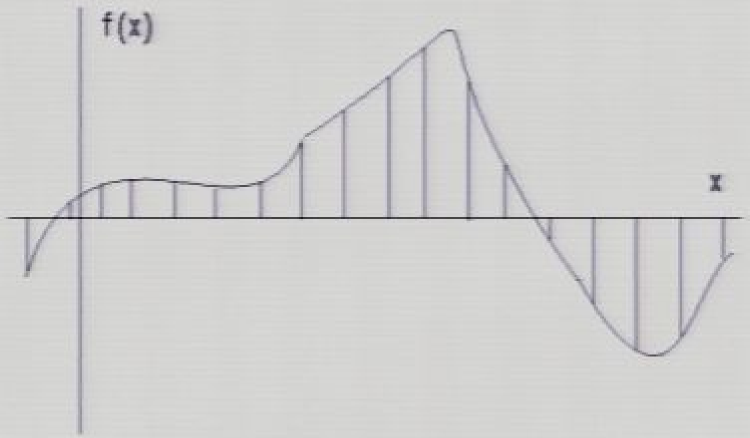

- **Then, exact reconstruction is possible:**

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$$f(x) = \sum_n f(x_n)\, \frac{\sin[2\pi(x-x_n)\omega_{max}]}{\pi(x-x_n)\omega_{max}}$$

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Fun with bandlimited calculus

- **Differential operators are also finite difference operators.**
- **Differential equations are also finite difference equations.**
- **Integrals are also series:**

$$\int_{-\infty}^{\infty} f(x)^* g(x)\, dx = \frac{1}{2\omega_{\max}} \sum_{n=-\infty}^{\infty} f(x_n)^* g(x_n)$$

Notice:

Useful also as a summation tool for series
(traditionally used, e.g., in analytic number theory)

<u>Sampling theory for physical fields ?</u>

Let us ask: (for now only in the <u>euclidean signature</u> case)

Could there be a natural UV cutoff in QFT such that

fields possess a suitable "finite spatial bandwidth" ?

Consequences, if yes:

- **Fields, actions and equations of motion possess equivalent representations:**
  - **on a spacetime manifold**
    (shows preservation of external symmetries incl. Killing vector fields)
  - **on any lattice of sufficiently dense spacing**
    (showing UV finiteness)

"Bandlimited" physical fields

Generalize "bandlimitation":

- Cut off spectrum of the Laplacian at Λ, e.g., at the Planck scale.
- Thus, the space of fields, F, that is summed over in QFT path integral
  is spanned only by the eigenfunctions with eigenvalues $\lambda_i < \Lambda$.
- Cutoff is covariant since Laplacian is scalar.

Observation: On any finite-volume region of space, F is finite dimensional.

$\Rightarrow$ **Thus, we have a sampling theorem for any finite region of space !**

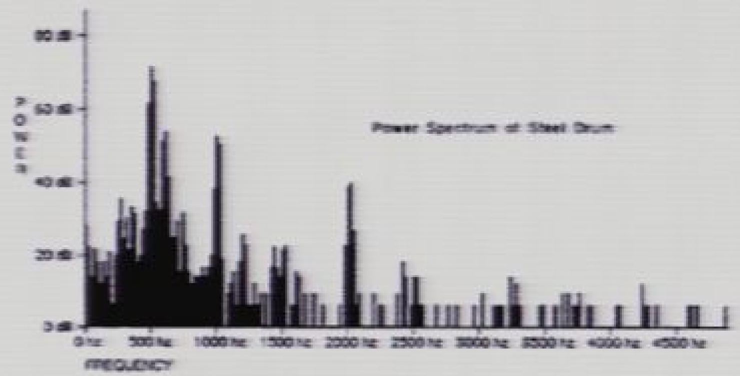
Q: Does (# samples/Volume) stay finite as (Volume -> infinty) ?

A: Yes: Can be shown via Weyl's asymptotic formula of *spectral geometry*.

(appeared in PRL 2008)

Recall: Spectral Geometry

- M. Kac 1966: Can one hear the shape of a drum ?



- Spectral geometry: Basically yes!

except in a few subtle cases (but with UV cutoff a coarser equivalence will suffice)

Shape of Riemannian manifold $\cong$ *Spectrum of Laplacian*

Sampling theory of spacetime itself ?

Idea: Nontrivial shape <=> nontrivial distance relations <=> nontrivial 2-point correlators

Method: to sample and reconstruct a piece of a euclidean signature spacetime:
(see arxiv:0908.3061)

– At **N** points, sample propagator matrix elements, e.g., $< x_a | 1/\Delta | x_b >$

– From matrix elements, $< x_a | 1/\Delta | x_b >$, calculate lowest eigenvalues of Δ .

– Use spectral geometry to reconstruct the piece of spacetime, up to UV cutoff.

– **A piece of (euclidean signature) spacetime with this UV cutoff**

= an equivalence class of mflds with same spectrum up to the UV cutoff.

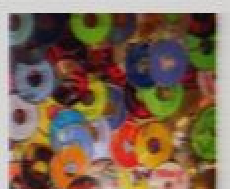
(a spacetime is an equivalence class of manifolds differing only by sub-Planckian ripples).

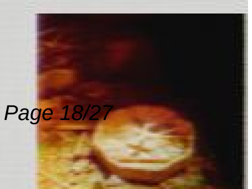
Consistent: Sub-Planckian ripples in space cannot be resolved as fields are bandlimited.

Representation-theoretic view

- **In general relativity:**
  - A choice of coordinates is merely a choice of representation of an underlying Riemannian manifold.

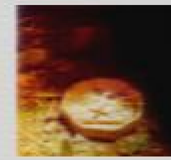
- **If there is also this UV cutoff, one has, further:**
  - Even the choice of Riemannian manifold is merely a choice of representation of an underlying spacetime with UV cutoff.
    (the spacetime is specified by a spectrum up to Λ).
  - One can represent a spacetime as any manifold whose Laplacian's spectrum is as specified, up to Λ.





Dynamics of spacetime ?

- Consider a simple action:



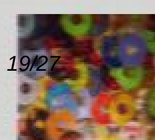
$$S_{matter} = \int d^n x \sqrt{|g|}\ \tfrac{1}{2}\phi(x)(\Delta + m^2)\phi(x)$$

$$= \sum_{i=1}^{N} \tfrac{1}{2}\phi_i(\lambda_i + m^2)\phi_i$$

$$= Tr\left(\tfrac{1}{2}(\Delta + m^2)|\phi)(\phi| \right)$$

- In contains the degrees of freedom of the field and the spacetime, except for N.

- **Simplest possible action for N?** $S_{Size} = \alpha\, N = \alpha\, Tr(1)$



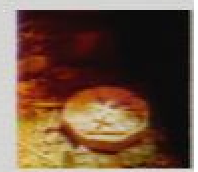
Tr(1) = Einstein action

A result of spectral geometry (Gilkey 1975, but anticipated via Sakharov mechanism 1969):

$$N = \frac{1}{16\pi}\int d^4x\sqrt{|g|}\ \left\{\frac{\Lambda^2}{2} + \frac{\Lambda}{6}R + O(R^2,\Lambda^{-1})\right\}$$

=> Einstein action emerges when choosing $\alpha = 6\,\Lambda$.

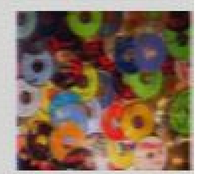
(Note: an analogous result for the Dirac operator has been used in NC geometry by Rovelli, Landi, Connes et al)



Here, obtain new interpretation of role of cosmological constant and curvature:

Without curvature, the density of samples (i.e. of degrees of freedom) is:

$$\frac{N}{V} = \frac{\Lambda^2}{32\pi}$$ (i.e., around the Planck density)



=> Curvature can be viewed as local perturbation of density of degrees of freedom.

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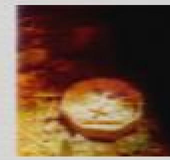

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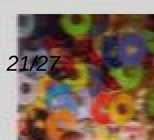

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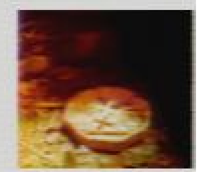

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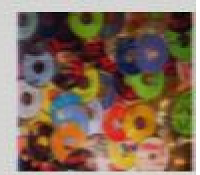

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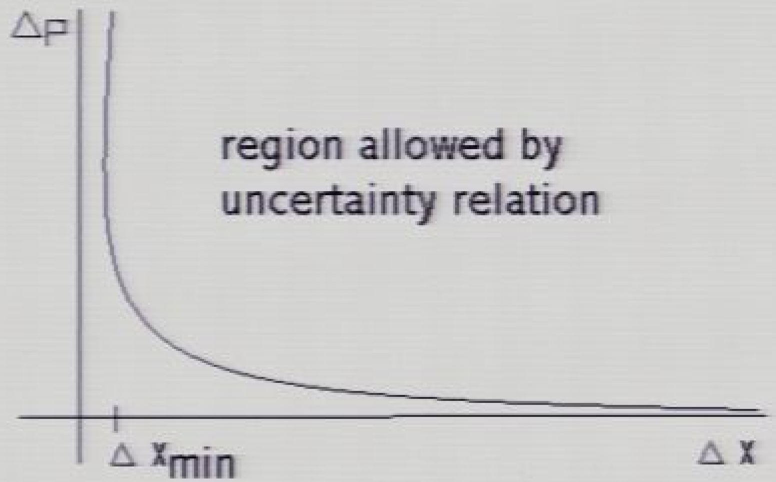
Any theoretical evidence ?

- **Recall quantum gravity studies:**

  => there may exist a minimum length uncertainty $\Delta x_{\min}$:



- **Functional analysis shows:**

  Functions (wave functions or fields etc) in the domain of any operator X with lower uncertainty bound necessarily possess the sampling property.

Experimental evidence?

- **Applications to inflationary cosmology:**
  - Sampling theoretic UV cutoff is easily applied to FRW spacetimes.
  - Multiple groups have worked out predictions for CMB.

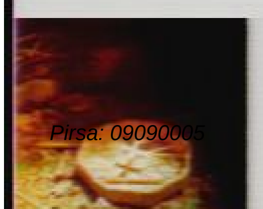
=> Predicted experimental signatures:

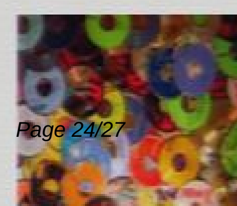
- **Characteristic, O(10^(-5)) modulations in CMB power spectrum.**

  However:
  Modulations could have many causes, e.g., suitable inflaton potential.

- **Characteristic deviation from scalar/tensor consistency relation in B-polarization data.**

  However:
  B-polarization is being sought but is very hard to measure.





Summary

Spacetime can be discrete & continuous, in same way information can.

- combines mathematical languages of QT and GR
- expresses minimum length uncertainty principle

- **Fields, actions and equations of motion possess equivalent representations**
  - **on a spacetime manifold**
  - **on any lattice of sufficiently dense spacing**

- **Describes gravity via eigenvalues of the Laplacian:**
  - **representational degrees of freedom are modded out**
  - **Einstein action is automatically induced**

Outlook:

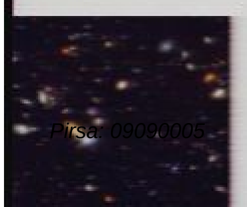
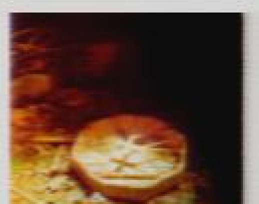
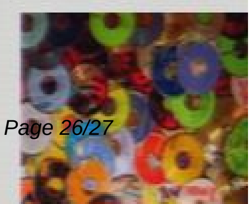
- **develop Lorentz covariant framework further** (see e.g. PRL 2004, new soon).
- **study spacetime as a channel with fundamental quantum noise** (see arxiv:0908.3144)

If not a fundamental property of nature?

- Any new sampling theoretic methods could be useful in communication engineering and signal processing (2 patents)

- Covariant sampling theory yields tools that should be useful in various quantum gravity theories that describe substructure (as in number theory):

  - For any discrete theory can define an equivalent continuum theory
  - Use, e.g., to express finite difference equations as differential equations.
  - Use to define and stabilize dimension of lattices.
  - Replace continuum limit with reconciling discrete and continuum formulations.

Lorentzian signature sampling

- Hard because Lorentzian spectral geometry less developed.
- Results on fully covariant sampling theory exist for:
  - Minkowski space
    [AK, Phys.Rev.Lett., 92, 221301 (2004)]
  - The FRW cases of de Sitter space, and power law inflation
    [work with R.T. Martin, soon to be posted]
- Key feature of Lorentz covariant sampling:
  - Spatial modes obey temporal sampling theorem, and vice versa
  - This way, Lorentz contraction and time dilatation ensure covariance.