Title: Reconstructing Quantum Mechanics from the Behavior of Partially Polarized Mixtures

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Abstract: It will be shown that the conventional (i.e. real or complex Hilbert space) model of quantum mechanics can be deduced from the indistinguishability of the simplest types of statistical mixtures. The result does not have the low dimension exclusion of the quantum logic approach.

Reconstructing Quantum Mechanics from the Behavior of Partially Polarized Mixtures

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"Geometry is the archetype of the beauty of the world."

Kepler

Quantum logic axiomatization uses projective geometry.

Limitations of this approach:

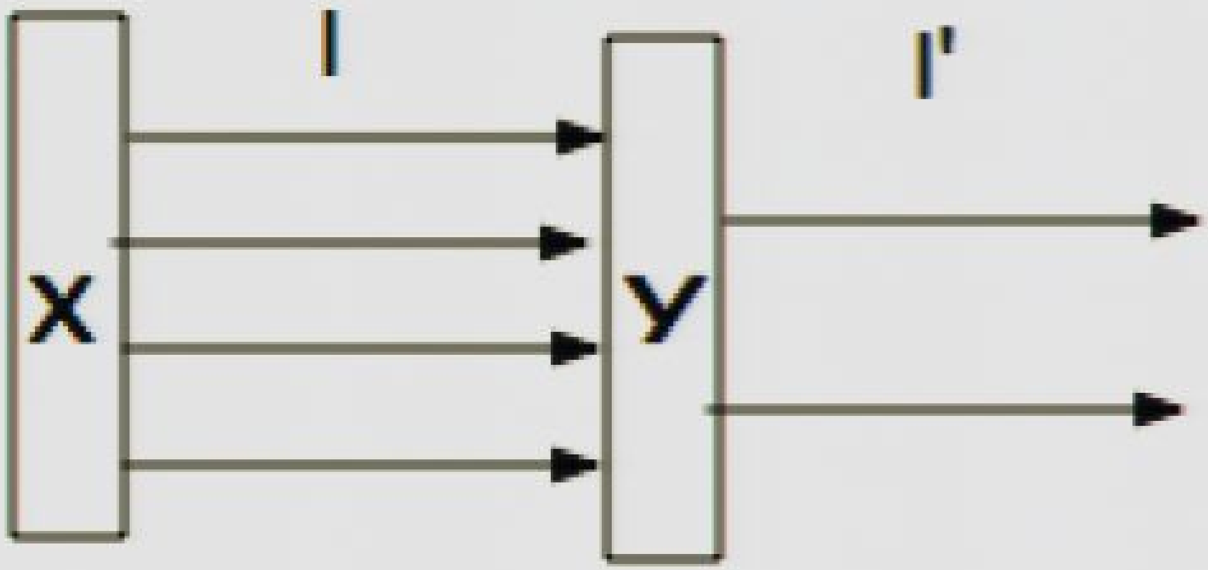
1. Abstract - Not directly connected to experiment.

2. Failure to give a strong representation theorem in low (i.e. one or two) dimensions. No representation for 2 and allows exotic 3 dimensional spaces.

3. Failure to give the Born rule in low (i.e. two) dimensions where Gleason's Theorem is not available. For higher dimensions Hilbert space is prerequisite for Gleason's Theorem.

Gauss taught us that the best way to understand the intrinsic geometry of a surface was to examine its metric as one might determine that the earth is round from a table of the distances between cities. Let us apply this to quantum mechanics.

The set of states is a set $\mathcal{S}$ of labels $x, y, z, \cdots$ for filters. The data is a table $p(x, y)$ of the attenuation function:



$$p(x,y) = I'/I = 1/2$$

Two states are considered identical if the corresponding filters behave identically in every experiment:

$$x = y \implies p(x,z) = p(y,z) \;\; \forall z \in \mathcal{S}.$$

This suggests a metric:

$$d(x,y) = \sup_{z \in \mathcal{S}} |p(x,z) - p(y,z)|$$

The d-metric immediately reveals the impossibility of reproducing the predictions of quantum mechanics with a hidden variable theory:

In quantum mechanics where $p(x,y) = |\langle x|y \rangle|^2$ one obtains

$$d(x,y) = \sqrt{1 - p(x,y)}$$

whereas in hidden variable theories one finds that

$$d(x,y) = 1 - p(x,y).$$

The Bell inequality is the triangle inequality for the second metric.

Conventional model:

The space of states is CP^N with elements represented by homogeneous, complex coordinate vectors. It is equipped with the notions of dimension and subspace. A pair of distinct elements span a CP^2 subspace, i.e. a complex projective line, and a basis can be found in which the first two components of each are the only non-vanishing ones i.e. the points are represented by homogeneous coordinates $\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}$ where the ξ's are complex numbers. Two vectors differing by a complex multiple are identified and the Dirac kets are defined by multiplying by a factor that normalizes the vector. Thus the states are put in one-one correspondence with the points z of the compactified complex plane:

$$|x\rangle = (1+|z|^2)^{-1/2}\begin{pmatrix}1\\ z\end{pmatrix}$$

The complex plane is the image of the Riemann sphere via the stereographic projection

$$z = \tan(\theta/2)e^{i\phi}$$

where θ, ϕ are the zenith and azimuth.

So the kets are in one-one correspondence with points $X : (\theta, \phi)$ on the Riemann sphere expressed by:

$$|x\rangle = \begin{pmatrix} \cos\theta/2 \\ e^{i\phi}\sin\theta/2 \end{pmatrix}$$

whence by direct calculation

$$|\langle x|y\rangle|^2 = \cos^2(XY/2)$$

in which XY is the angle subtended at the center of the sphere by the great circle arc joining X and Y. This is a mathematical statement which identifies the Fubini-Study

The physics is supplied by the Born rule:

$$p(x, y) = |\langle x|y\rangle|^2$$

In optics the Riemann sphere is called the Poincaré sphere.
The formula $p(x, y) = \cos^2 XY/2$ is called "Malus Law".

Steps in reconstructing the theory from the data:

Let $\mathcal{S}$ be the set of labels $x, y, z, a, b, c \cdots$. I will find a *minimal* set of properties of the p-table which do the following:

(1) Furnish $\mathcal{S}$ with the notions of dimension and subspace ;

(2) Imply that every two-dimensional subspace is a complex projective line CP^2, i.e. the points x are in one-one correspondence with the points X of a Riemann sphere;

(3) Imply that the correspondence is such that $p(x, y) = \cos^2 XY/2$.

(4) Imply that every three-dimensional subspace is a projective plane and use the complex structure of its CP^2 subspaces to show that it is CP^3.

From projective geometry we know that an N dimensional space such that all 3 dimensional subspaces are planes CP^3 is a CP^N space.

We then proceed as in the conventional model to arrive at the formula

$$|\langle x|y\rangle|^2 = \cos^2(XY/2).$$

Now, however, we have established that $p(x,y) = \cos^2(XY/2)$, and so we have derived the Born rule $p(x,y) = |\langle x|y\rangle|^2$.

Some definitions:

1. A statistical mixture with fractions α_j of states x_j is described formally by:

$$\mathcal{M} = \sum_j \alpha_j x_j \text{ with } \sum_j \alpha_j = 1.$$

The attenuation of $\mathcal{M}$ by z is

$$P(\mathcal{M}, z) = \sum_j \alpha_j p(x_j, z).$$

Two mixtures $\mathcal{M}, \mathcal{M}'$ are indistinguishable if

$$P(\mathcal{M}, z) = P(\mathcal{M}', z) \;\; \forall z \in \mathcal{S}.$$

2. Two states x, y are *orthogonal* if $p(x, y) = 0$. A *basis* is a maximal set of mutually orthogonal states. Notice that orthogonal states x, y have $d(x, y) = 1$ and so are maximally separated in the d-metric.

3 A *balanced* mixture of two states consists of equal amounts of two not necessarily orthogonal states. A *normal* mixture of two states consists of not necessarily equal amounts of two orthogonal states.

4. A maximally unpolarized mixture consists of equal amounts of the states belonging to a basis.

The following properties of the p-table and one more given below (needed to eliminate quaternions) suffice to derive the conventional model

1. *Symmetry*: $p(x,x) = 1, \;\; p(x,y) = p(y,x) \;\; \forall x, y \in \mathcal{S}$

2. *Compactness*: There are at most a finite number of elements in any basis. (Equivalently $\mathcal{S}$ is compact in the topology of the metric d.)

3. First indistinguishability property (FIP): Maximally unpolarized statistical mixtures are indistinguishable.

4. Second indistinguishability property (SIP): Every balanced mixture is indistinguishable from some normal mixture.

Consequences of the FIP:

1. *Meaning of dimension.*

All bases of $\mathcal{S}$ have same number (N) of elements called the *dimension.*

Proof:

By the FIP if $\{x\}_N$ and $\{y\}_{N'}$ are bases

$$N^{-1}\sum_{k=1}^{N} p(x_k, z) = N'^{-1}\sum_{j=1}^{N'} p(y_j, z)$$

insert $z = x_n$ and sum over n which gives

$$1 = (N/N')\sum^{N}\sum^{N'} p(y_j, x_n)$$

Similarly insert $z = y_m$ and sum over m to obtain

$$1 = (N'/N) \sum_{m=1}^{N'} \sum_{k=1}^{N} p(x_k, y_m)$$

Since $p(x, y) = p(y, x)$ interchanging gives $N = N'$ $\square$.

2. *Frame function property:*

If $\{x\}_N$ is a basis:

$$\sum_{j=1}^{N} p(x_j, z) = 1 \ \forall z \in \mathcal{S}.$$

Proof:

Set $N = N'$ in the FIP equation . Let the y-basis be obtained by adjoining elements to z to obtain a maximal set. Putting $y_1 = z$

$$\sum_{k=1}^{N} p(x_k, z) = 1 \ \forall z \in \mathcal{S}. \ \square$$

3. *Meaning of subspace:*

The subspace $\mathcal{S}^\star$ of $\mathcal{S}$ *spanned* by the mutually orthogonal set $\{v\}_n$ with $n \leq N$ consists of states w such that

$$\sum_{j=1}^{n} p(v_j, w) = 1.$$

Then

$$\sum_{j=1}^{n} p(v_j, z) = \sum_{j=1}^{n} p(w_j, z) \;\; \forall z \in \mathcal{S}$$

for any mutually orthogonal set $\{w\}_n$ in $\mathcal{S}^\star$, with common value 1 if $z \in \mathcal{S}^\star$.

Note: This extends the indistinguishability property to unpolarized mixtures of any set of orthogonal states that span the same subspace

Proof:

Extend $\{v\}_n$ to a basis $\{v\}_N$ of $\mathcal{S}$ by adjoining orthogonal elements $v_{n+1}, \cdots, v_N$. From $\sum_{j=1}^{n} p(v_j, u) = 1 \ \forall u \in \mathcal{S}^\star$ the frame function property implies $\sum_{j=n+1}^{N} p(v_j, u) = 0$ whence $p(v_j, u) = 0$ for $j = n+1, \cdots, N$ whence $p(v_j, w_k) = 0$ for the elements w_j of a basis of $\mathcal{S}^\star$. Conversely any w orthogonal to $v_{n+1}, \cdots v_N$ satisfies $\sum_{j=1}^{n} p(v_j, w) = 1$ and hence lies in $S^\star$. Hence if $\{w_1, \cdots, w_k\}$ is a maximal set of mutually orthogonal elements of $\mathcal{S}^\star$, then $w_1, \cdots, w_k, v_{n+1}, \cdots v_N$ is a basis of $\mathcal{S}$. Hence since dim $\mathcal{S}$ is N we must have $k = n$ so that for *any* $z \in \mathcal{S}$ the frame function property implies $\sum_{k=1}^{n} p(w_k, z) + \sum_{j=n+1}^{N} p(v_j, z) = 1 \, \forall z \in \mathcal{S}$. Subtracting $\sum_{j=1}^{N} p(v_j, z) = 1 \, \forall z \in \mathcal{S}$, gives

$$\sum_{j=1}^{n} p(w_j, z) = \sum_{j=1}^{n} p(v_j, z) \, \forall z \in \mathcal{S} \quad \square$$

4. *Orthocomplementation*: For each element x in a two dimensional subspace there is a *unique* element x' called its *antipode* such that x and x' are orthogonal and hence span the space.

Proof: Suppose there is another state x'' with $p(x, x'') = 0$ Then both $\{x, x'\}$ and $\{x, x''\}$ span the space so that

$$p(x, z) + p(x', z) = p(x, z) + p(x'', z) \quad \forall z \in \mathcal{S}$$

whence $p(x', z) = p(x'', z)$ $\forall z \in \mathcal{S}$ so that $x' = x''$. $\square$

5. *Simple criterion for identity of states:*

$p(x, y) = 1 \implies x = y$

Proof:

If $p(x, y) = 1$ then x, y span the same one dimensional subspace whence $p(x, z) = p(y, z)\ \forall z \in \mathcal{S}$ whence $x = y$. $\square$

It is possible to construct hidden variable theories that satisfy the FIP. We shall see that no hidden variable theory can satisfy the SIP:

Consequences of the second indistinguishability property:

For any states a, b there exists a pair of orthogonal states c, c' and a number $0 \leq \lambda \leq 1$ such that:

$$\frac{1}{2}p(a,z) + \frac{1}{2}p(b,z) = \lambda p(c,z) + (1-\lambda)p(c',z) \;\; \forall z \in \mathcal{S}.$$

1. Given any pair of distinct elements a, b there is a unique two dimensional subspace $\mathcal{P}_{ab}$ containing them. Note: This is one of the requirements of a projective geometry, i.e. two distinct points determine a line.

Proof: From the SIP equation a and b are orthogonal to any z which is orthogonal to both c and c' and so belong to the two dimensional subspace $\mathcal{S}^\star$ spanned by c, c'.

Proof of uniqueness:

If a and b are distinct then $\lambda \neq 1$. For if $\lambda = 1$ put $z = c'$ to conclude that a, b are also orthogonal to c' and hence are equal to c and hence to one another. Now suppose that a, b belong to another two dimensional subspace spanned by d, d'. Insert $z = d$ and $z = d'$ in the SIP equation and add the equations. On the left side the sum is unity. The coefficients of λ and $(1 - \lambda)$ on the right must therefore be unity which gives $p(c, d) + p(c, d') = 1$ and a similar equation for c' so that c, c' are in the subspace spanned by d, d' i.e. the two subspaces are identical. $\square$

2. Note that λ and c depend on a, b *not* on z. So we can get information about λ and c by making judicious choices of z. In particular let us assume that z belongs to the two dimensional subspace $\mathcal{P}_{ab}$ spanned by c, c'. Then $p(c, z) + p(c', z) = 1$. Substituting this into the SIP equation it becomes:

$$\frac{1}{2}(p(a, z) + p(b, z)) =$$

$$(2\lambda - 1)p(c, z) + (1 - \lambda) \;\; \forall z \in \mathcal{P}_{ab}.$$

which will be called the "restricted SIP equation". Note that

$$\lambda = 1/2 \iff p(a, b) = 0 \iff b = a'.$$

The case $b = a'$ can be dealt with directly so assume $b \neq a'$ so that $\lambda \neq 1/2$. Apply the restricted SIP equation to $z = a, b,$ or c. and use $p(x, y) = p(y, x)$ and $p(x, x) = 1$.

Putting $z = a$ get

$$\frac{1}{2}(1 + p(b,a)) = (2\lambda - 1)p(c,a) + (1 - \lambda)$$

and $z = b$ to get

$$\frac{1}{2}(1 + p(a,b)) = (2\lambda - 1)p(c,b) + (1 - \lambda)$$

Since $p(a,b) = p(b,a)$ and $\lambda \neq \frac{1}{2}$ there follows

$$p(c,a) = p(c,b)$$

. Now put $z = c$.

$$\frac{1}{2}(p(a,c) + p(b,c)) = (2\lambda - 1) + (1 - \lambda)$$

whence $p(a,c) = \lambda$. Hence

$$\frac{1}{2}(1 + p(a,b)) = (2\lambda - 1)p(c,b) + (1 - \lambda) = 2\lambda^2 - 2\lambda + 1$$

whence $p(a,b) = (2\lambda - 1)^2$ i.e.

$$\lambda = \frac{1}{2}(1 + \sqrt{p(a,b)})$$

Defining $\theta(x, y)$ by

$$p(x, y) = \cos^2(\theta(x, y)/2)$$

we then obtain:

$$\theta(a, c) = \theta(c, b) = \frac{1}{2}\theta(a, b).$$

$$\lambda = \cos^2(\theta(a, b)/4)$$

Putting $z = a$ get

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3. At this point it is useful to check that we are on the right track to reproduce quantum mechanics by showing that we can reproduce the quantum mechanical formula relating $d(x,y)$ and $p(x,y)$ which as we have seen rules out hidden variables.:

$$d(x,y) = \sqrt{1 - p(x,y)}.$$

Proof: Let b' be the antipode of b in $\mathcal{P}_{ab}$. Then for any $z \in \mathcal{S}$ we have $p(b,z) + p(b',z) = p(c,z) + p(c',z)$. Using this to replace $p(b,z)$ with $p(c,z) + p(c',z) - p(b',z)$ in the SIP equation, we obtain

$$p(a,z) - p^{(}b',z) = (2\lambda - 1)(p(c,z) - p(c',z)).$$

Taking the supremum of the absolute values on both sides over all $z \in \mathcal{S}$ we obtain

$$d(a,b') = |2\lambda{-}1| d(c,c') = |2\lambda{-}1| = \sqrt{p(a,b)} = \sqrt{1 - p(a,b')}$$

Interchanging b and b' in the argument the assertion follows. □

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Interchanging b and b' in the argument the assertion follows. □

4. Expressing p in terms of θ in the restricted SIP equation it (magically) becomes:

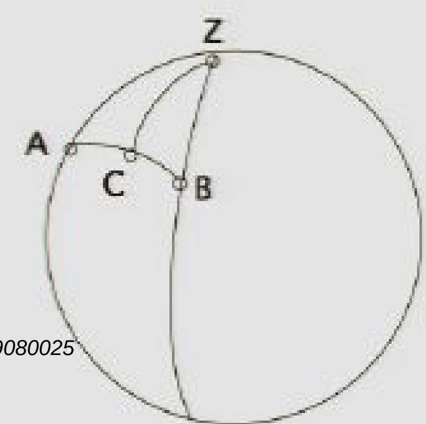
$$\cos\theta(a,z) + \cos\theta(b,z) = 2\cos\tfrac{1}{2}\theta(a,b)\cos\theta(c,z)$$

which looks just like the following theorem of spherical trig:

Let A,B,Z be the vertices of a spherical triangle and let XY be the angle subtended at the center by the great circle arc joining X,Y. Then

$$\cos AZ + \cos BZ = 2\cos\tfrac{1}{2}AB\cos CZ$$

where C is the midpoint of the great circle arc joining A and B.



Let us say that the states x, y are *properly mapped to a sphere* if the images X, Y are joined by a great circle arc which subtends the angle

$$XY = \theta(x, y)$$

at the center.

A set of states will be said to be properly mapped to the sphere if every pair of elements is properly mapped.

A state will be said to be *properly adjoined* to a properly mapped set if the resulting set remains properly mapped.

One checks that if a belongs to a properly mapped set its antipode a' can be properly adjoined by mapping it to the antipode A' of A on the sphere.

Suppose that the set $\{a, b, z\} \in \mathcal{P}_{ab}$ is properly mapped. Then the restricted SIP equation becomes

$$\tfrac{1}{2}(\cos AZ + \cos BZ) = \cos \tfrac{1}{2}AB \cos\theta(c, z)$$

Comparing with the spherical triangle formula we have:

$$\theta(c, z) = CZ.$$

We also have from previous calculation:

$$\theta(c, a) = \theta(c, b) = \tfrac{1}{2}\theta(a, b)$$

and since a, b are properly mapped we have

$$\theta(a, b) = AB \text{ whence}$$

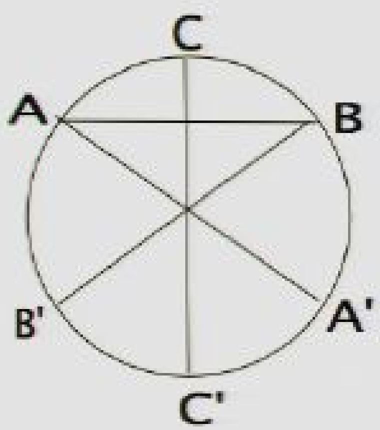
$$\theta(c, a) = CA, \;\; \theta(c, b) = CB.$$

Important conclusion:

If the set of states $\{a, b, z\}$ with $b \neq a'$ is properly mapped to the vertices of a spherical triangle ABZ, there is a state c which can be properly adjoined by mapping it to the midpoint of AB.

Degenerate case: If the set $\{a, b\}$ with $b \neq a'$ is properly mapped to points A,B on a circle then there is a state c which can be properly adjoined by mapping it to the mid point C of AB.

We can properly adjoin a', b', c' as shown in the figure. We refer to the process of adding midpoints as "midpoint adjunction".



Repeating midpoint adjunction to properly adjoin the inverse images of AC, BC, AC' etc. ad infinitum generates *a set of states* $\mathcal{C}_{ab}$ *which are properly mapped to a dense set of points on the circle.*

In order to start the adjunction process we need a pair a, b that are properly mapped to a circle. If $p(x, y) = 1/2$ we can map x to one point X on the circle and y to a point Y at right angles to it so that $\cos^2(XY/2) = 1/2$, i.e. they are properly mapped.

It is possible that this process exhausts $\mathcal{P}_{ab}$. *In this case we conclude that there is a map from every two dimensional subset of the set states $\mathcal{S}$ to the points of a circle such that*

$$p(x, y) = \cos^2(XY/2).$$

Note that there are precisely two points e and its antipode e' on $\mathcal{C}_{ab}$ such that $p(a, e) = p(a, e') = 1/2$. Thus the "equator" opposite any state consists of one point and its antipode so

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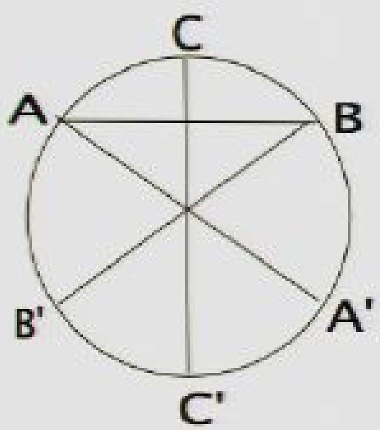

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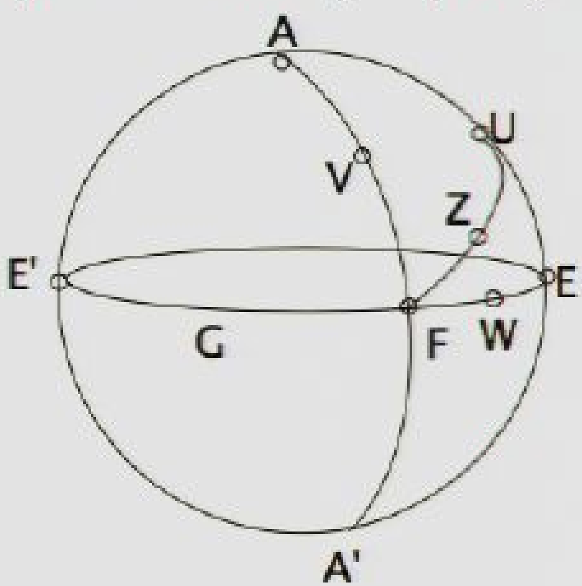
Suppose next that there are three states a, e, f such that $p(a, e) = p(a, f) = p(e, f) = 1/2$. If we map a to the north pole of a sphere and e, f to two points on the equator at right angles to one another, one readily checks that the set $\{a, e, f\}$ is properly mapped to a spherical triangle on the sphere. We may then use the midpoint adjunction process to fill up the sphere with properly mapped states.



If the only states g for which $p(a,g) = \frac{1}{2}$ lie on the equator $\mathcal{C}_{ef}$ the midpoint adjunction process will exhaust $\mathcal{P}_{ab}$.

In that case we will have established that for every two dimensional subspace there is a map from its elements to a Riemann sphere such that

$$\theta(x,y) = XY.$$

Hence

$$p(x,y) = \cos^2(XY/2).$$

The points X of the sphere may now be mapped to the complex plane by stereographic projection as described earlier which demonstrates that the two-dimensional subspaces are in CP^2 and can be represented by kets so that $|\langle x|y\rangle|^2 = \cos^2(XY/2)$

We have thus demonstrated that the Born rule applies to every two dimensional subspace.

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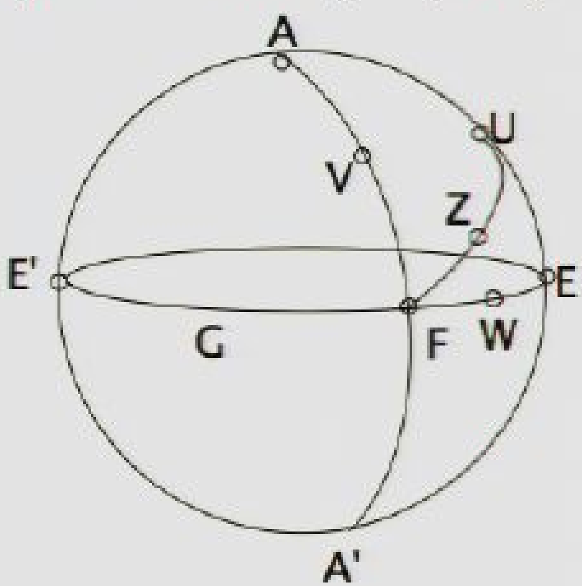

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The points X of the sphere may now be mapped to the complex plane by stereographic projection as described earlier which demonstrates that the two-dimensional subspaces are in CP^2 and can be represented by kets so that $|\langle x|y\rangle|^2 = \cos^2(XY/2)$

We have thus demonstrated that the Born rule applies to every two dimensional subspace.

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The equator will then no longer be a circle and we arrive at the place where quaternionic quantum mechanics rears its head!

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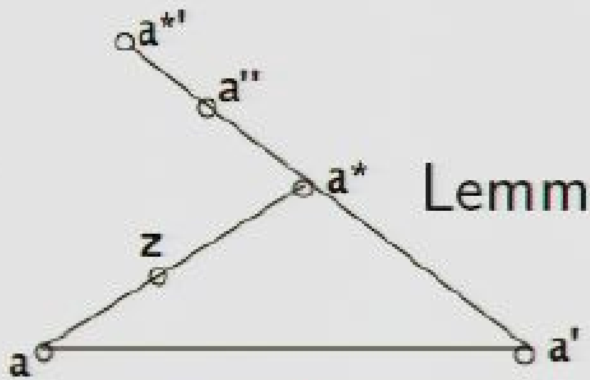
Thus we have an *isometry* between the d- metric on the space of states and the chord metric on the sphere.

To complete the reconstruction we must show that that the results above for 2-dimensional subspaces imply the conventional model for all dimensions:

Theorem: Three non-colinear states define a projective plane.

Proof



Lemma 1: Given any point $z \notin \mathcal{P}_{aa'}$ there is a state a'' orthogonal to both a and a' such that z is in the three dimensional subspace spanned by a, a', a''.

Proof: On $\mathcal{P}_{az}$ there is an $a^* \neq a'$ which is orthogonal to a. On $\mathcal{P}_{a'a^*}$ there is an a'' orthogonal to a' and an $a^{*\prime}$ orthogonal to a^*. Since a is orthogonal to both a' and a^* it is orthogonal to every point on on $\mathcal{P}_{a'a^*}$ and hence both $\{a, a^*, a^{*\prime}\}$ and $\{a, a', a''\}$ are orthogonal sets. Since $p(a, z) + p(a^*, z) = 1$ it follows that $p(a^{*\prime}, z) = 0$. But since both $\{a', a''\}$ and

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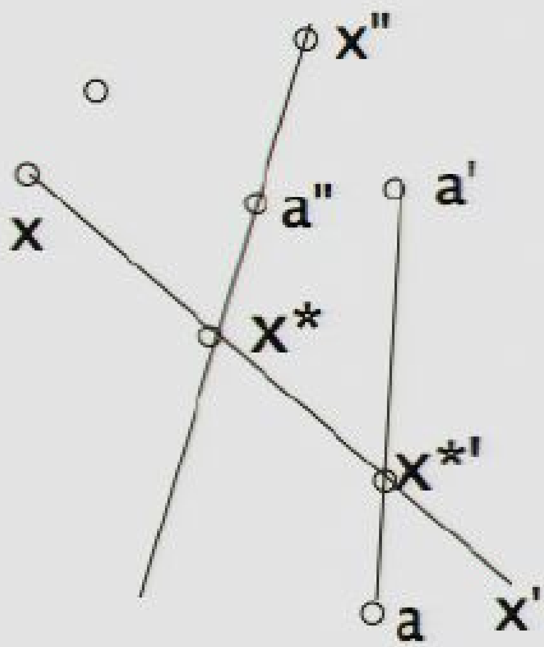
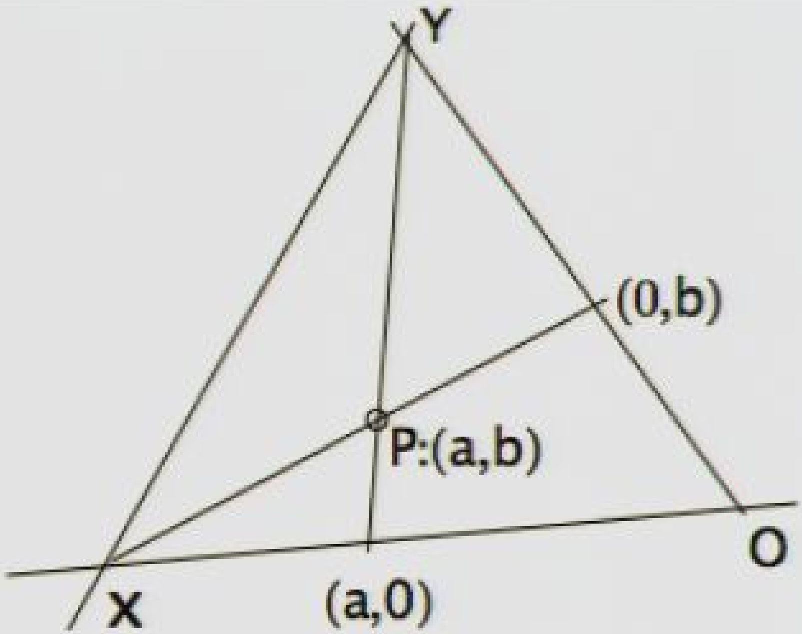
Lemma 2: In every three dimensional subspace of $\mathcal{S}$ every pair of lines intersect once and only once. Thus it is a projective plane:



Proof: Let $\mathcal{P}_{xy}$ be another line. Identify $\mathcal{P}_{ab}$ with $\mathcal{P}_{aa'}$ and $\mathcal{P}_{xy}$ with $\mathcal{P}_{xx'}$. Since $N = 3$ there are unique elements a'' and x'' such that a, a', a'' and x, x', x'' are bases of $\mathcal{S}$. The line $\mathcal{P}_{x''a''}$ contains an element $x^\star$ which is orthogonal to x'' and hence lies in $\mathcal{P}_{xx'}$. It's antipode $x^{\star\prime}$ is orthogonal to both $x^\star$ and x'' and hence is orthogonal to $\mathcal{P}_{x^\star x''}$ which is identical to $\mathcal{P}_{x''a''}$. Hence it is orthogonal to a'' so that $x^{\star\prime}$ lies in $\mathcal{P}_{aa'}$ as well as in $\mathcal{P}_{xx'}$. We have thus established that two lines determine a point. If the system is non-classical there will be four points no three of which are colinear which is the remaining requirement for a projective plane.

We can now apply the standard procedure for coordinatizing a projective plane to any plane in $\mathcal{S}$.



Since the lines in this plane are complex projective lines the coordinates will be complex numbers and hence are a field.

It is a theorem of projective geometry that if one plane in a projective space is coordinatized with a field (i.e. is "Pappian") then so also is the whole space. This eliminates quaternions.

Once we have coordinatized the states with complex numbers any pair of states will span a two dimensional subspace to which the Born rule has been shown to apply.

Remarks on the quaternion issue:

In my old paper I pointed out that one can also get rid of quaternions if one requires that dynamical processes that leave a basis invariant form an abelian group. Thus in two dimensions if we place the two states of that basis at the north and south poles the group of transformations will transform the equator into itself. If it is a circle the group will be abelian but if it is a sphere as it would be for quaternihe hons it will not be. if that group is not abelian I shall have to read Steve Adler's book to find out what is going on

Summary:

I have shown that the p-table reveals the necessity for the conventional model of quantum mechanics (i.e. the Hilbert space struture and the Born rule) through the indistinguishability of cetain simple mixtures and a limit to three in the membership of mutually equatorial sets,

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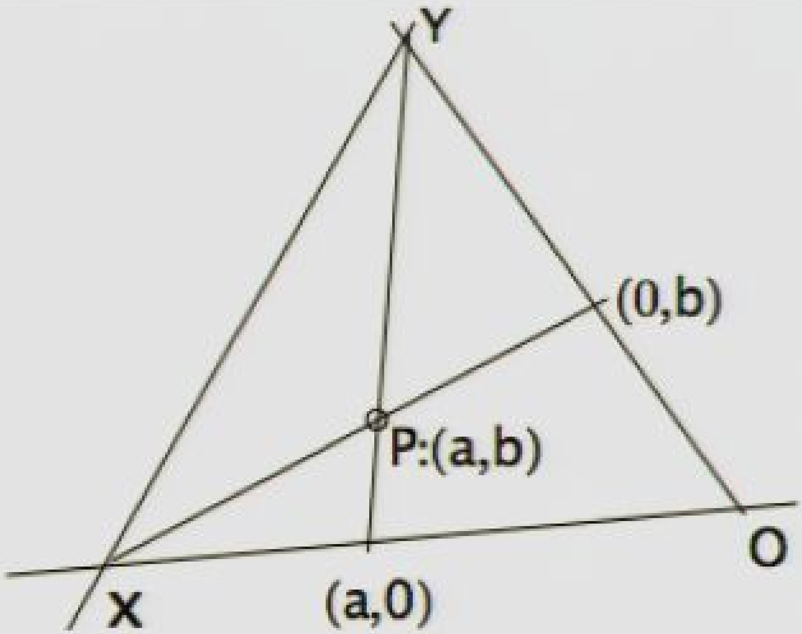

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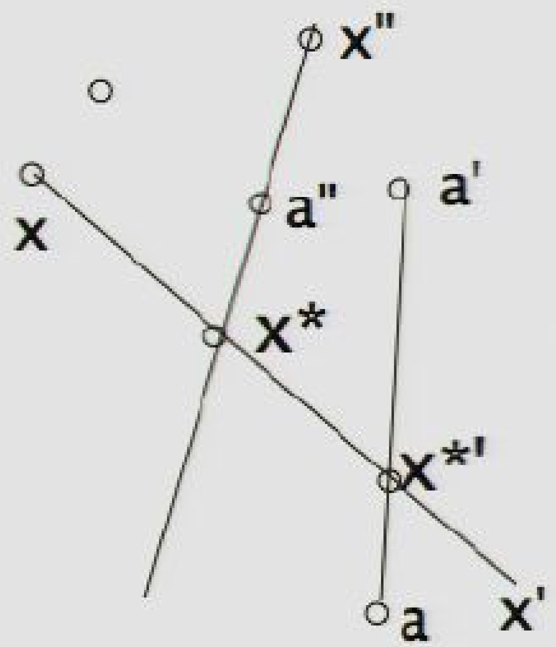

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The following properties of the p-table and one more given below (needed to eliminate quaternions) suffice to derive the conventional model

1. *Symmetry*: $p(x,x) = 1, \;\; p(x,y) = p(y,x) \;\; \forall x, y \in \mathcal{S}$

2. *Compactness*: There are at most a finite number of elements in any basis. (Equivalently $\mathcal{S}$ is compact in the topology of the metric d.)

3. First indistinguishability property (FIP): Maximally unpolarized statistical mixtures are indistinguishable.

4. Second indistinguishability property (SIP): Every balanced mixture is indistinguishable from some normal mixture.

Reconstructing Quantum Mechanics from the Behavior of Partially Polarized Mixtures

D. Fivel

fivel@physics.umd.edu

August 13, 2009

Conventional model:

The space of states is CP^N with elements represented by homogeneous, complex coordinate vectors. It is equipped with the notions of dimension and subspace. A pair of distinct elements span a CP^2 subspace, i.e. a complex projective line, and a basis can be found in which the first two components of each are the only non-vanishing ones i.e. the points are represented by homogeneous coordinates $\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}$ where the ξ's are complex numbers. Two vectors differing by a complex multiple are identified and the Dirac kets are defined by multiplying by a factor that normalizes the vector. Thus the states are put in one-one correspondence with the points z of the compactified complex plane:

$$|x\rangle = (1+|z|^2)^{-1/2}\begin{pmatrix}1\\ z\end{pmatrix}$$

The complex plane is the image of the Riemann sphere via the stereographic projection

$$z = \tan(\theta/2)e^{i\phi}$$

where θ, ϕ are the zenith and azimuth.

So the kets are in one-one correspondence with points $X : (\theta, \phi)$ on the Riemann sphere expressed by:

$$|x\rangle = \begin{pmatrix} \cos\theta/2 \\ e^{i\phi}\sin\theta/2 \end{pmatrix}$$

whence by direct calculation

$$|\langle x|y\rangle|^2 = \cos^2(XY/2)$$

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The physics is supplied by the Born rule:

$$p(x, y) = |\langle x|y\rangle|^2$$

In optics the Riemann sphere is called the Poincaré sphere.
The formula $p(x, y) = \cos^2 XY/2$ is called "Malus Law".

Steps in reconstructing the theory from the data:

Let $\mathcal{S}$ be the set of labels $x, y, z, a, b, c \cdots$. I will find a *minimal* set of properties of the p-table which do the following:

(1) Furnish $\mathcal{S}$ with the notions of dimension and subspace ;

(2) Imply that every two-dimensional subspace is a complex projective line CP^2, i.e. the points x are in one-one correspondence with the points X of a Riemann sphere;

(3) Imply that the correspondence is such that $p(x, y) = \cos^2 XY/2$.

(4) Imply that every three-dimensional subspace is a projective plane and use the complex structure of its CP^2 subspaces to show that it is CP^3.

From projective geometry we know that an N dimensional space such that all 3 dimensional subspaces are planes CP^3 is

Some definitions:

1. A statistical mixture with fractions α_j of states x_j is described formally by:

$$\mathcal{M} = \sum_j \alpha_j x_j \text{ with } \sum_j \alpha_j = 1.$$

The attenuation of $\mathcal{M}$ by z is

$$P(\mathcal{M}, z) = \sum_j \alpha_j p(x_j, z).$$

Two mixtures $\mathcal{M}, \mathcal{M}'$ are indistinguishable if

$$P(\mathcal{M}, z) = P(\mathcal{M}', z) \;\; \forall z \in \mathcal{S}.$$

Suppose that the set $\{a, b, z\} \in \mathcal{P}_{ab}$ is properly mapped. Then the restricted SIP equation becomes

$$\tfrac{1}{2}(\cos AZ + \cos BZ) = \cos \tfrac{1}{2}AB \cos\theta(c, z)$$

Comparing with the spherical triangle formula we have:

$$\theta(c, z) = CZ.$$

We also have from previous calculation:

$$\theta(c, a) = \theta(c, b) = \tfrac{1}{2}\theta(a, b)$$

and since a, b are properly mapped we have

$$\theta(a, b) = AB \text{ whence}$$

$$\theta(c, a) = CA, \;\; \theta(c, b) = CB.$$

Important conclusion:

If the set of states $\{a, b, z\}$ *with* $b \neq a'$ *is properly mapped to the vertices of a spherical triangle* ABZ*, there is a state* c *which can be properly adjoined by mapping it to the midpoint of* AB*.*

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