Title: Quantum-Bayesian Coherence (or, My Favorite Convex Set)

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Abstract: In a quantum-Bayesian delineation of quantum mechanics, the Born Rule cannot be interpreted as a rule for setting measurement-outcome probabilities from an objective quantum state. (A quantum system has potentially as many quantum states as there are agents considering it.) But what then is the role of the rule? In this paper, we argue that it should be seen as an empirical addition to Bayesian reasoning itself. Particularly, we show how to view the Born Rule as a normative rule in addition to usual Dutch-book coherence. It is a rule that takes into account how one should assign probabilities to the outcomes of various intended measurements on a physical system, but explicitly in terms of prior probabilities for and conditional probabilities consequent upon the imagined outcomes of a special counterfactual reference measurement. This interpretation is seen particularly clearly by representing quantum states in terms of probabilities for the outcomes of a fixed, fiducial symmetric informationally complete (SIC) measurement. We further explore the extent to which the general form of the new normative rule implies the full state-space structure of quantum mechanics. It seems to go some way.

My Favorite Convex Set

(My Favorite Shape)

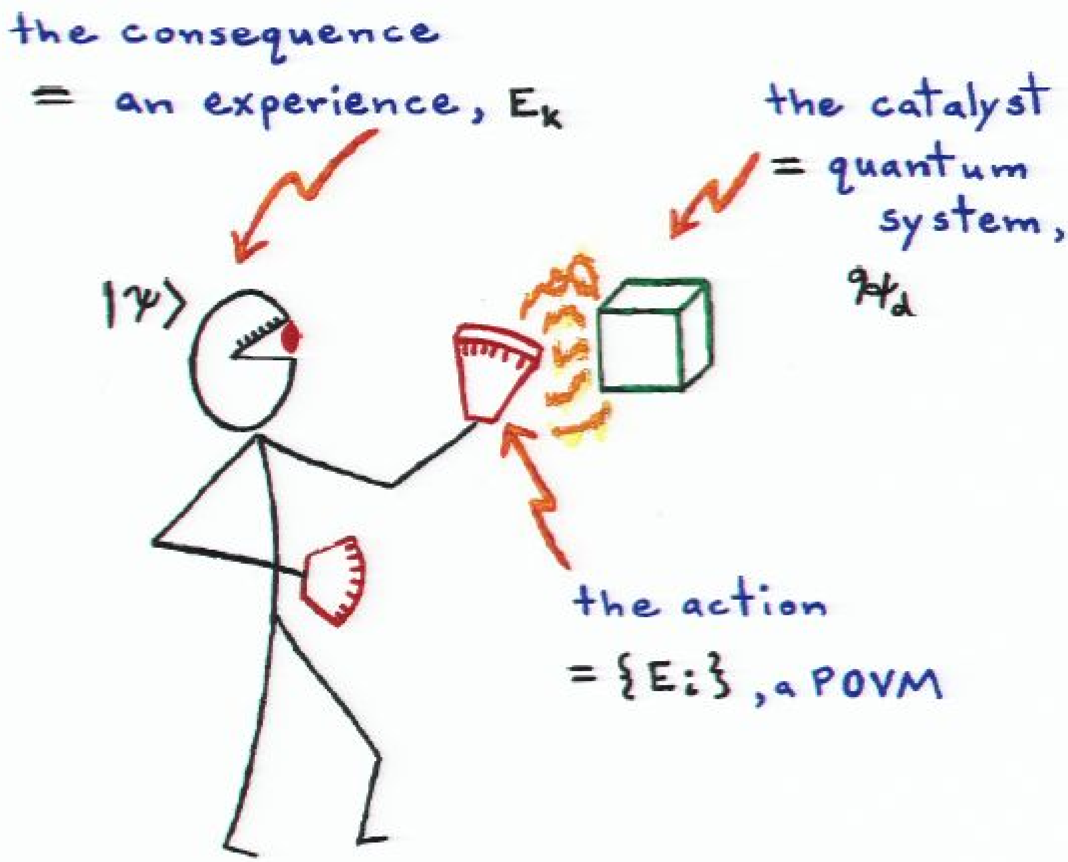
Christopher Fuchs
$\hat{PI}$ - Perimeter Inst.

Work with:

Marcus Appleby
Åsa Ericsson
Rüdiger Schack

arXiv:0906.2187v1 [quant-ph]

"Quantum-Bayesian Coherence"


the consequence
= an experience, E_k
the catalyst
= quantum system,
$|\psi\rangle$
the action
= $\{E_i\}$, a POVM

Density Operators

linear operators

$$\rho \in \mathcal{L}(\mathcal{H}_d)$$

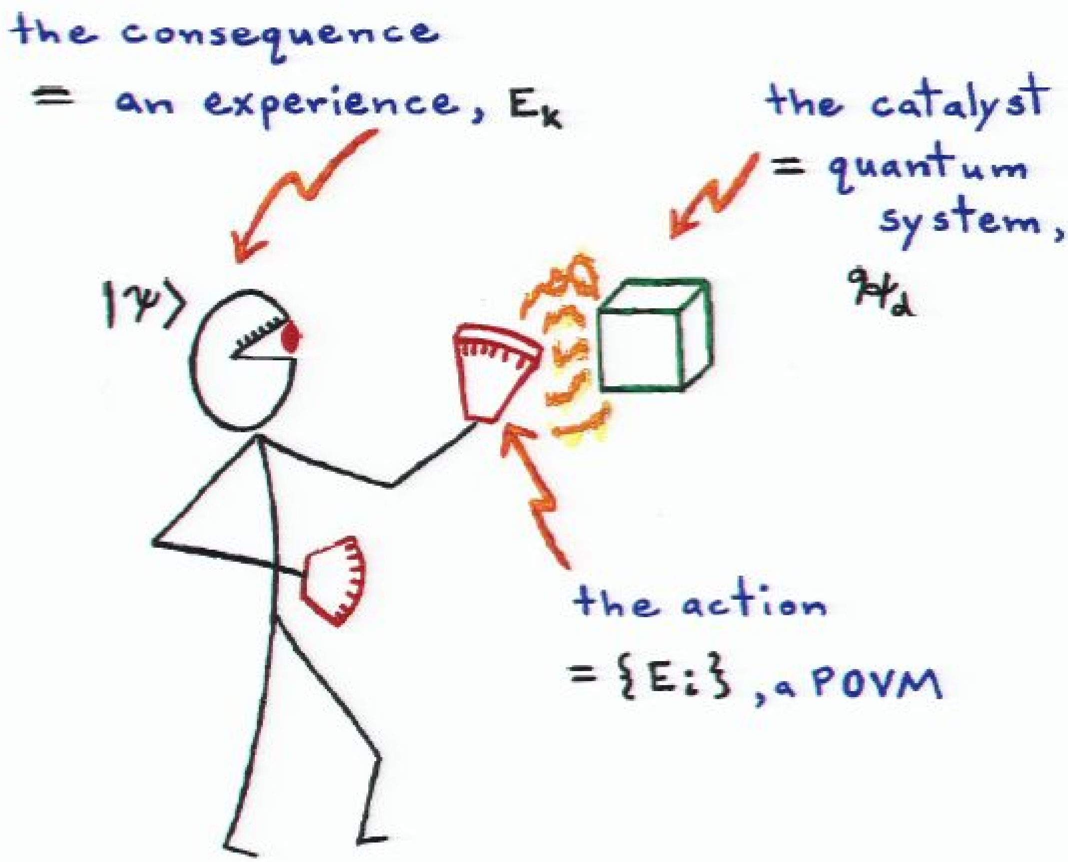
complex vector space

catalog of uncertainties

1) $\rho^\dagger = \rho$
2) $\mathrm{tr}\,\rho = 1$
3) $\lambda_i(\rho) \geq 0$

eigenvalues

convex hull of the set $\{ |\psi\rangle\langle\psi| : |\psi\rangle \in \mathcal{H}_d \}$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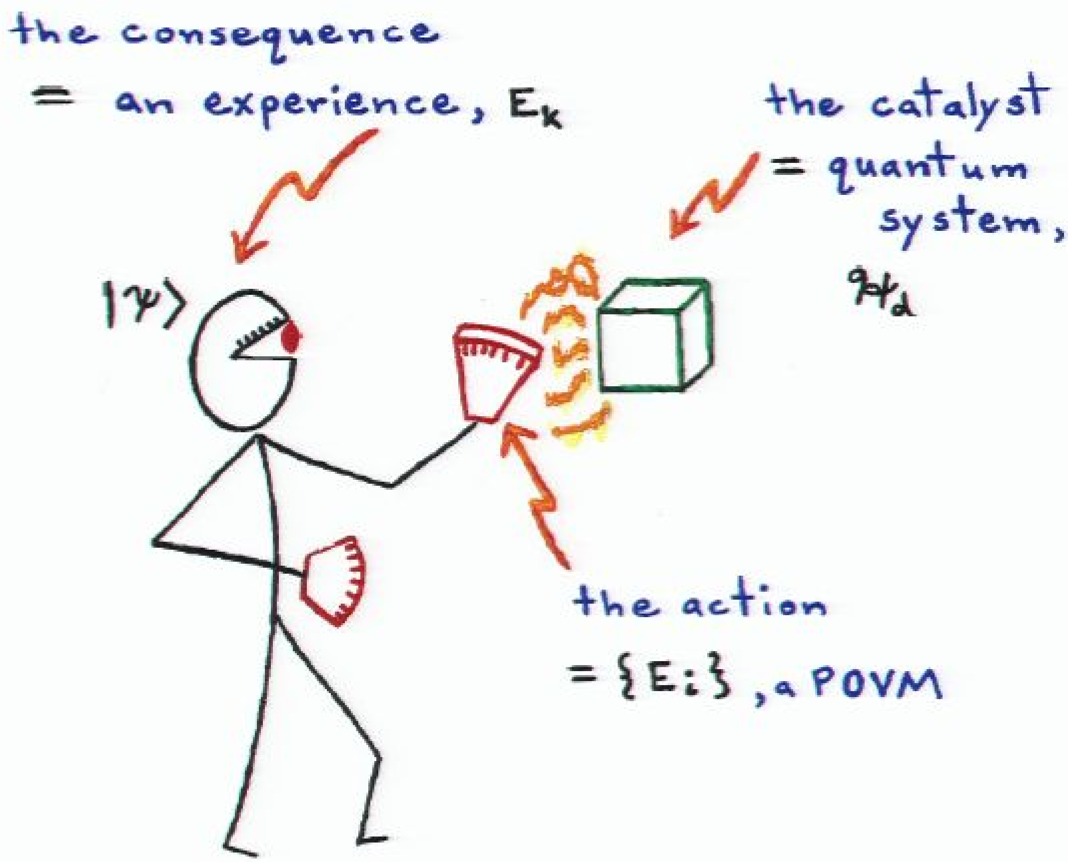
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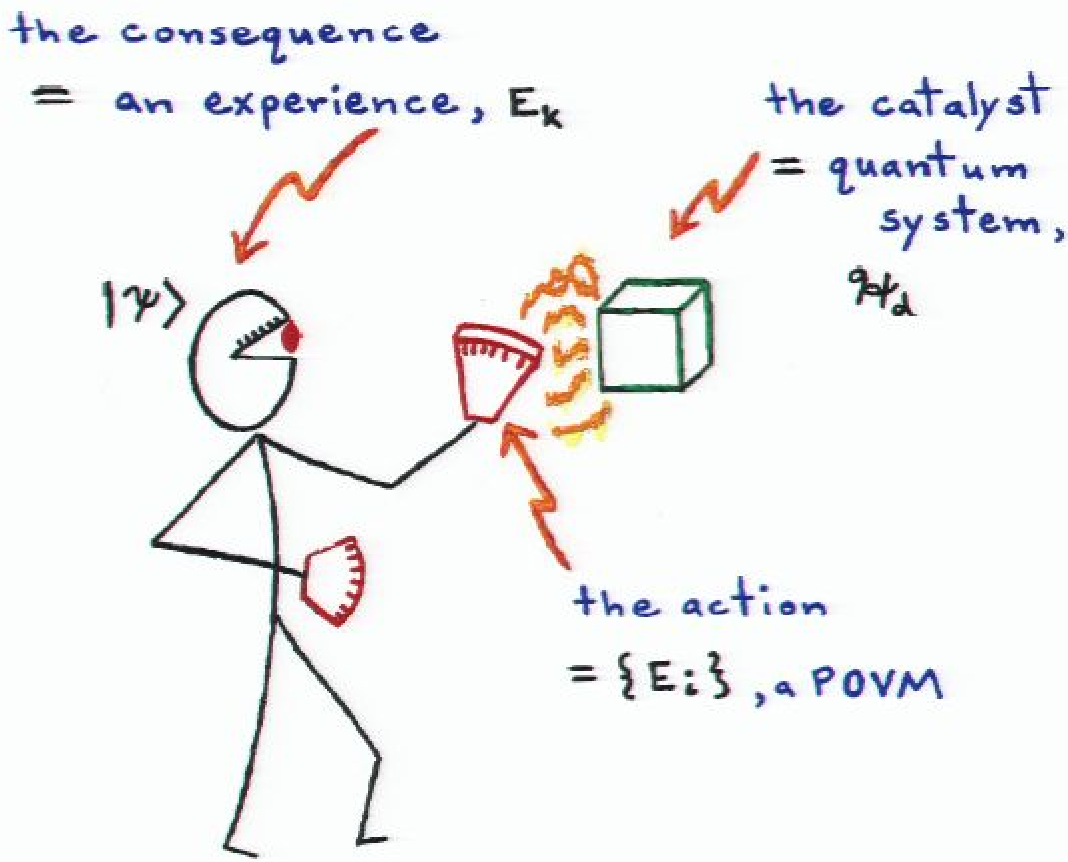
Calculus 1 $\Rightarrow$ Character 1

Calculus 2 $\Rightarrow$ Character 2

Calculus 3 $\Rightarrow$ Character 3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A satisfactory statement about the actual (objective) characteristics of the quantum world should contain no $|\psi\rangle$'s at all.

Really. <u>None</u>!

The Born Rule

Given ρ and $\{E_i\}$,

quantum state

POVM measurement

$$\boxed{p(i) = \operatorname{tr} \rho E_i}$$

"The Born Rule"

The Born Rule

Given ρ and $\{E_i\}$,

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"The Born Rule"

NOT a law of nature.

RATHER something we should strive for.

THE TEN COMMANDMENTS

- Thou shalt not kill.
- Thou shalt not steal.
- Thou shalt not covet thy neighbor's wife.

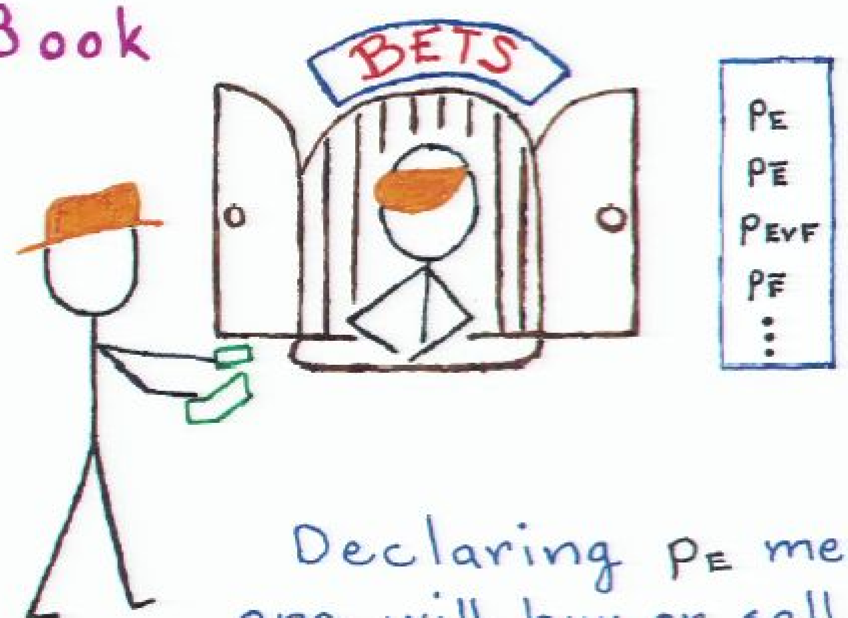
⋮

- The firstling of an ass thou shalt redeem with a lamb.

⋮

Defining Probability

Dutch Book



Declaring p_E means one will buy or sell a lottery ticket

$\boxed{\text{Worth \$1 if } E}$

for $\$p_E$.

Dutch Book

Normative Rule:

Never declare p_E, $p_{\bar{E}}$, $p_{E \vee F}$, etc. that will lead to sure loss.

Example 1:

If $p_E < 0$, bookie will sell ticket for negative money. Sure loss!

Example 2:

If $p_E > 1$, bookie will buy ticket for more than it is worth in best case. Sure loss.

Example 3:

Suppose E and F mutually exclusive.

$\boxed{\text{Worth \$1 if } E \vee F}$

$\boxed{\text{Worth \$1 if } E}$

$\boxed{\text{Worth \$1 if } F}$

buying this is equivalent to buying these two

So must have $p_{E \vee F} = p_E + p_F$.

Example 4:

$\boxed{\text{Worth \$}\frac{m}{n}\text{ if } E}$

Price? $\$\frac{m}{n} p_E$ of course.

Bayes Rule

Consider conditional lotteries:

If $E \wedge F$ give full price, but if $\overline{F}$ return money.

Thus:

$\boxed{\begin{array}{l}\text{Worth } \$1 \text{ if } E \wedge F; \\ \text{Worth } \$p_{E|F} \text{ if } \overline{F}.\end{array}}$ price $\$p_{E|F}$

But:

$\boxed{\text{Worth } \$1 \text{ if } E \wedge F}$ price $\$p_{E \wedge F}$

$\boxed{\text{Worth } \$p_{E|F} \text{ if } \overline{F}}$ price $\$p_{E|F}\, p_{\overline{F}}$

recall example 4

So must have:

$$p_{E|F} = p_{E \wedge F} + p_{E|F}\, p_{\overline{F}} \Rightarrow \boxed{p_{E \wedge F} = p_F\, p_{E|F}}$$

Example

One contemplates taking

$p(F) = 0.75$

$p(E|F) = 0.50$

$p(E \wedge F) = 0.70$.

One could gamble that way, but it wouldn't be too wise:

Not coherent.

Normative Rule:

Strive for coherence.

Bayes Rule

Consider conditional lotteries:

If $E \wedge F$ give full price, but if $\overline{F}$ return money.

Thus:

Worth \$1 if $E \wedge F$; Worth \$$p_{E\mid F}$ if $\overline{F}$.	price \$$p_{E\mid F}$

But:

Worth \$1 if $E \wedge F$	price \$$p_{E\wedge F}$
Worth \$$p_{E\mid F}$ if $\overline{F}$	price \$$p_{E\mid F}\, p_{\overline{F}}$

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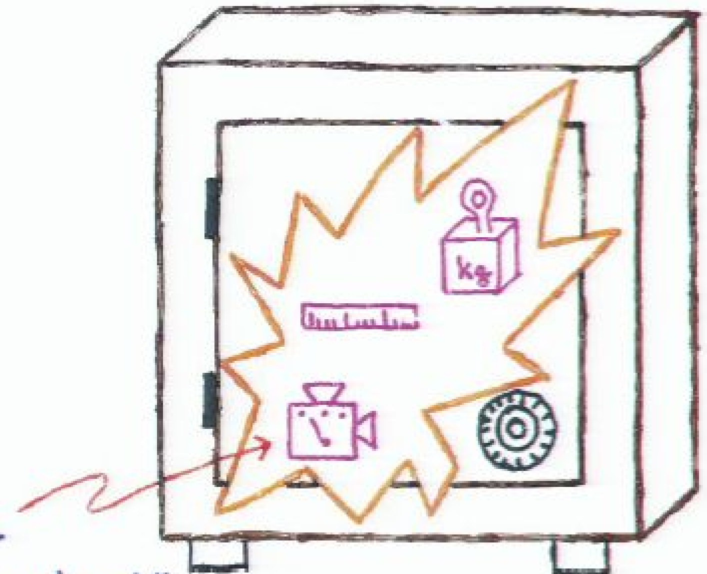
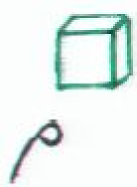
$$\rho \longleftrightarrow p(h)$$

Bureau of Standards



the
"standard"
quantum measurement



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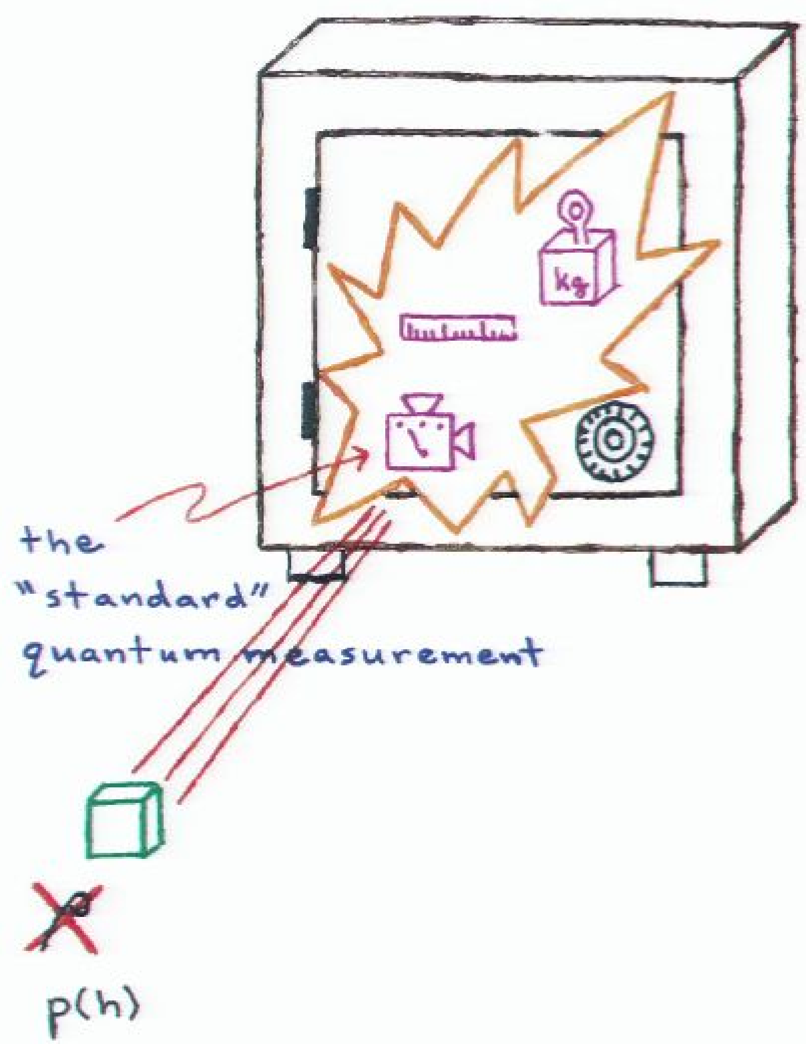


A Very Fundamental Mmt ?

Caves, 1999
Zauner

Suppose d^2 projectors $\Pi_i = |\psi_i\rangle\langle\psi_i|$ satisfying

$$\mathrm{tr}\,\Pi_i \Pi_j = \frac{1}{d+1} \quad , \quad i \neq j$$

exist.

Can prove:

1) the Π_i linearly independent

2) $\sum_i \frac{1}{d} \Pi_i = I$

So good for Bureau of Standards.

Also
$$p(i) = \frac{1}{d} \mathrm{tr}\, \rho \Pi_i$$

$$\boxed{\rho = \sum_i \left[(d+1) p(i) - \frac{1}{d} \right] \Pi_i}$$

Inequivalent SIC Sets

Let $d = 3$, $\omega = e^{\frac{2\pi i}{3}}$.

Set 1

$$\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 0 \\ -\omega \\ \bar{\omega} \end{bmatrix} \quad \begin{bmatrix} 0 \\ -\bar{\omega} \\ \omega \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \\ \bar{\omega} \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \\ -\bar{\omega} \end{bmatrix}$$

$$\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} -1 \\ \omega \\ 0 \end{bmatrix} \quad \begin{bmatrix} -1 \\ \bar{\omega} \\ 0 \end{bmatrix}$$

Set 2

$$\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} -2 \\ \omega \\ \bar{\omega} \end{bmatrix} \quad \begin{bmatrix} -2 \\ \bar{\omega} \\ \omega \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ -2\omega \\ \bar{\omega} \end{bmatrix} \quad \begin{bmatrix} 1 \\ -2\bar{\omega} \\ \omega \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \quad \begin{bmatrix} 1 \\ \omega \\ -2\bar{\omega} \end{bmatrix} \quad \begin{bmatrix} 1 \\ \bar{\omega} \\ -2\omega \end{bmatrix}$$

Evidence for Existence

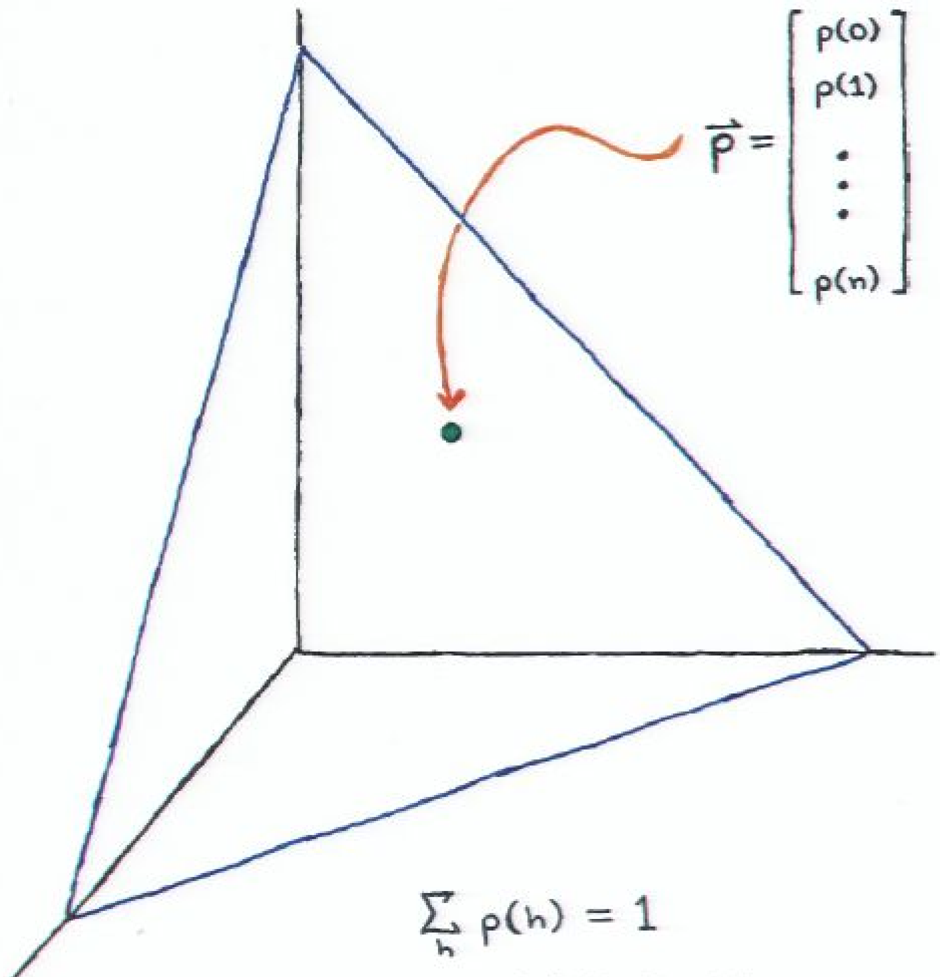
Analytical Constructions

$$d = 2 - 13,\ {}^{14}\ 15,\ 19$$

Numerical $(\epsilon \leq \cancel{10^{-11}})$ 10^{-38} !

$$d = 2 - \cancel{47}\ 67$$

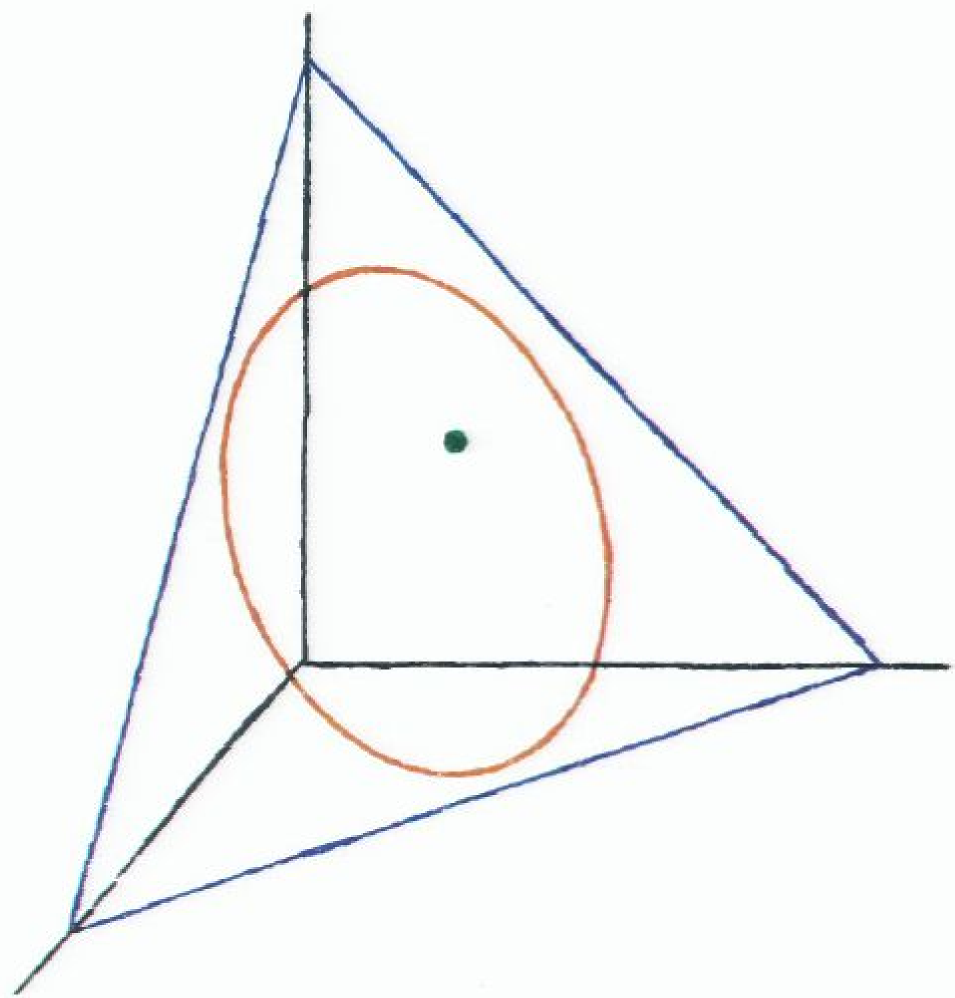
Probability Simplex



$$\vec{p} = \begin{bmatrix} p(0) \\ p(1) \\ \vdots \\ p(n) \end{bmatrix}$$

$$\sum_h p(h) = 1$$

$$p(h) \geq 0 \quad \forall h$$



Remarkable Theorem

Jones & Linden, PRA 71 (2005)
Flammia, (unpub, 2004)

Let A be Hermitian, $A^{\dagger} = A$.

Then, $A = |\psi\rangle\langle\psi|$ if and only if

$$\mathrm{tr}\, A^2 = \mathrm{tr}\, A^3 = 1 .$$

Pure States in SIC Language

Conditions

$$\rho^\dagger = \rho \quad , \quad \mathrm{tr}\,\rho^2 = \mathrm{tr}\,\rho^3 = 1$$

translate to

$$\sum_i p(i)^2 = \frac{2}{d(d+1)}$$

and

$$\sum_{jk\ell} c_{jk\ell}\, p(j)\, p(k)\, p(\ell) = \frac{d+7}{(d+1)^3}$$

where

$$c_{jk\ell} = \mathrm{Re}\;\mathrm{tr}\,\Pi_j \Pi_k \Pi_\ell$$

Could these be independently motivatable physical constants?

Laws of Probability

H_i – various hypotheses one might have

D_j – data values one might gather

Given: $p(D_j|H_i)$ ← expectations for data given hypothesis

$p(H_i)$ ← expectations for hypotheses themselves

Question: What expectations should one have for the D_j?

Answer: $P(D_j) = \sum_i p(H_i)\, p(D_j|H_i)$

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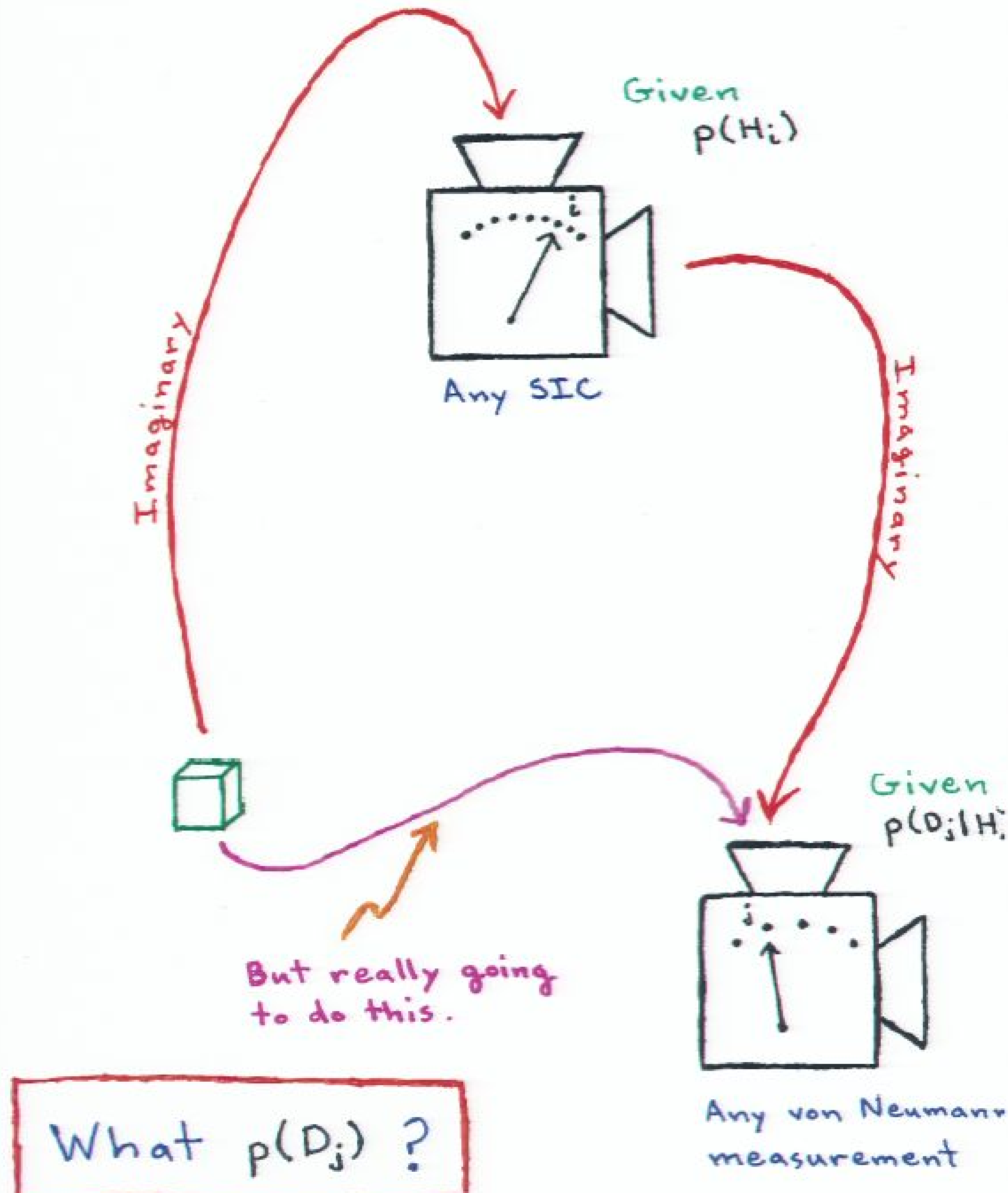

Given
p(Hi)
Any SIC
Imaginary
Imaginary
Given
p(Dj|Hi)
Any von Neumann measurement
But really going to do this.
What p(Dj) ?

$$\underbrace{p(D_j)}_{\text{Quantum}} = (d+1) \underbrace{\sum_i p(H_i)\, p(D_j|H_i)}_{\text{(Usual) Bayesian}} - 1$$

Magic!

Generalizations

When measurement on the ground is any other SIC:

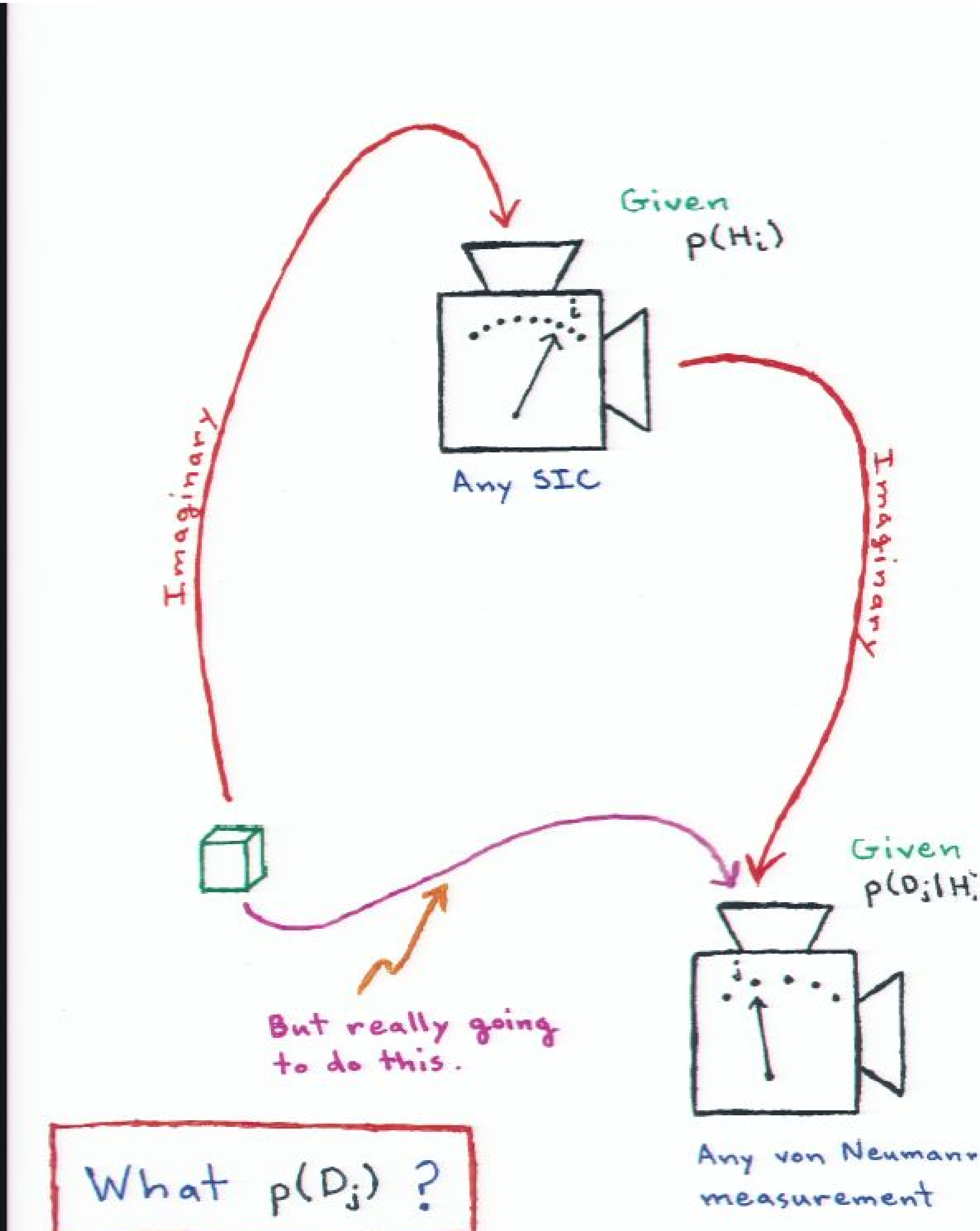
$$p(D_j) = (d+1)\sum_i p(H_i)\,p(D_j|H_i) - \frac{1}{d}$$

(Compare to unitary evolution.)

And

When measurement on the ground is a completely general POVM $\{D_j\}$, $j=1,\ldots,m$,

$$\boxed{p(D_j) = (d+1)\sum_i p(H_i)\,p(D_j|H_i) - \frac{1}{d}\sum_i p(D_j|H_i)}$$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Bayesian Perspective

No logical reason why situation with conditional lotteries should be commensurate with situation without conditional lotteries.

$$p(D_j) \neq \sum_i p(H_i)\, p(D_j|H_i)$$

(Need better notation, though.)

Quantum Perspective

Nonetheless, there may be empirical reasons for adopting a relation. — normative!

This is the content of the Born Rule.

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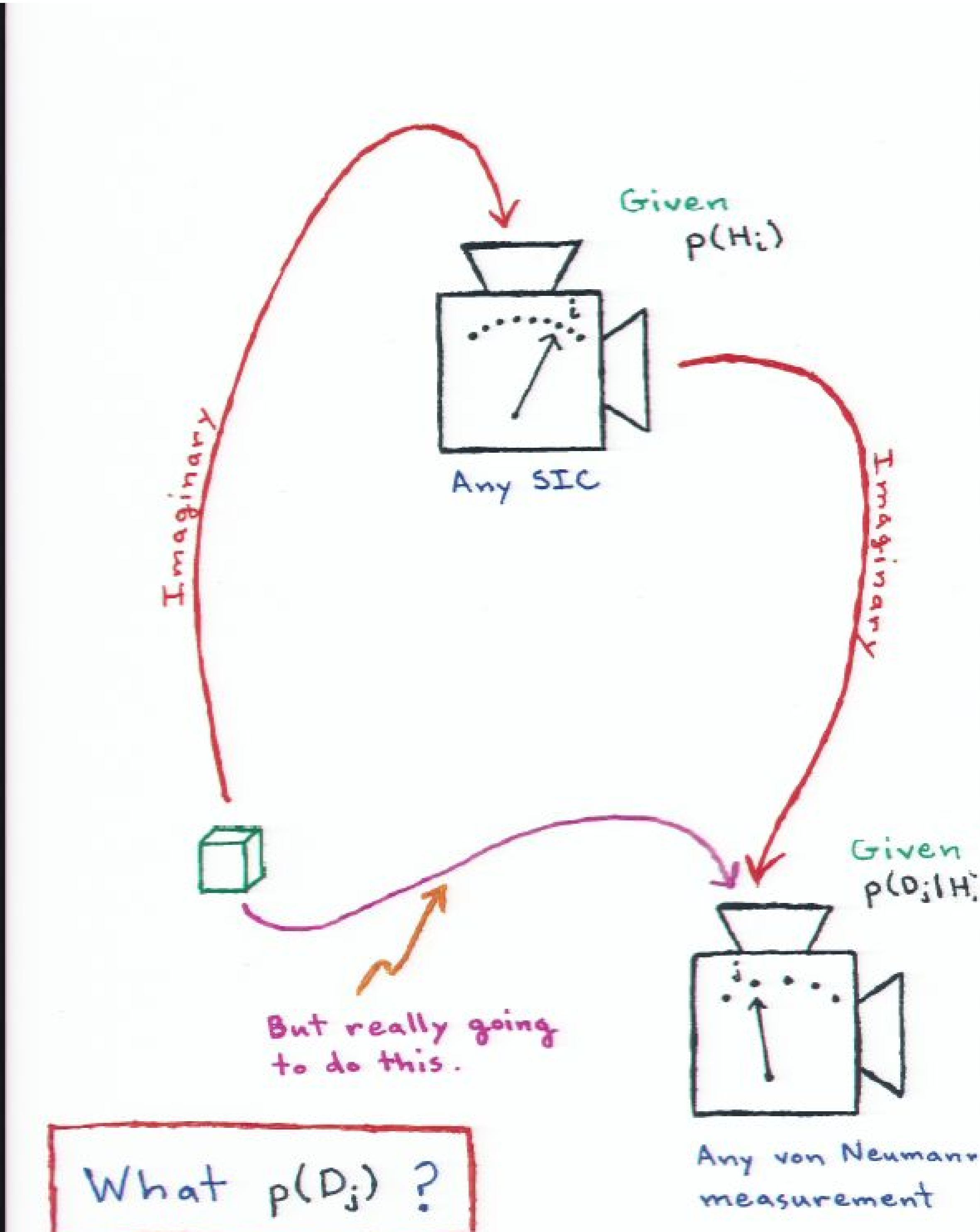
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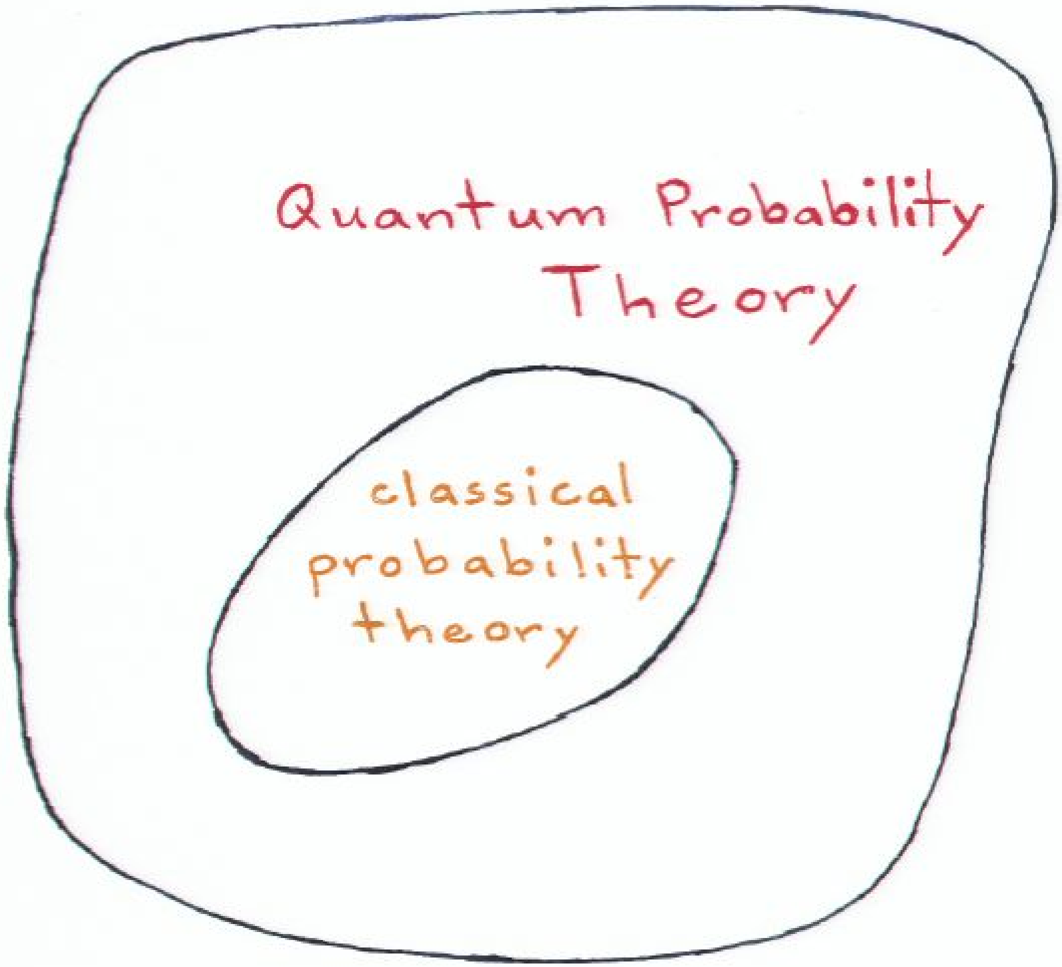
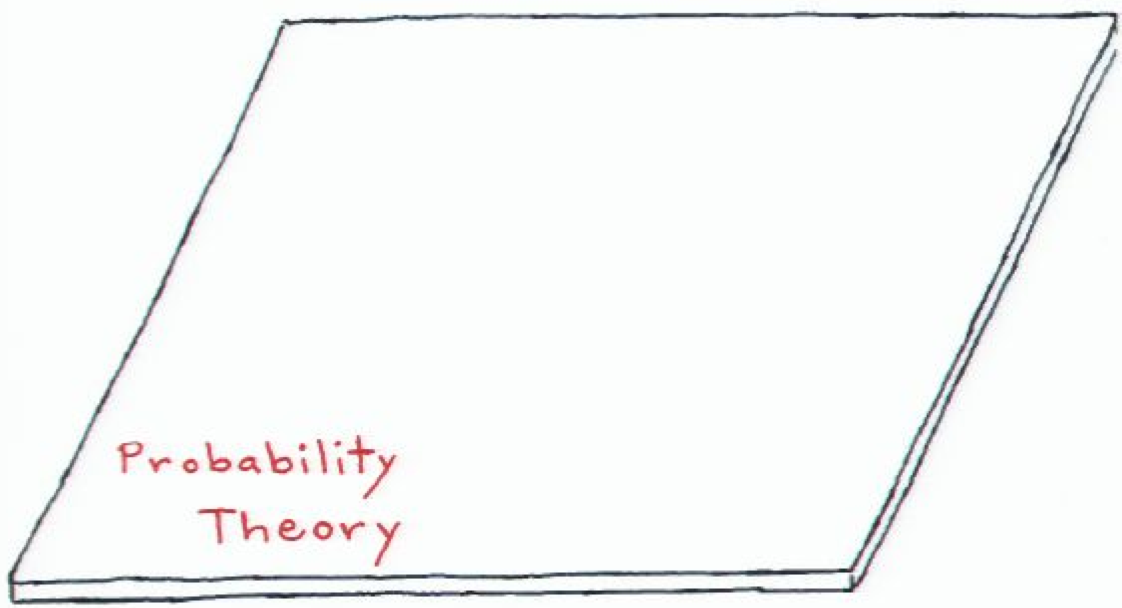
This is the content of the Born Rule.

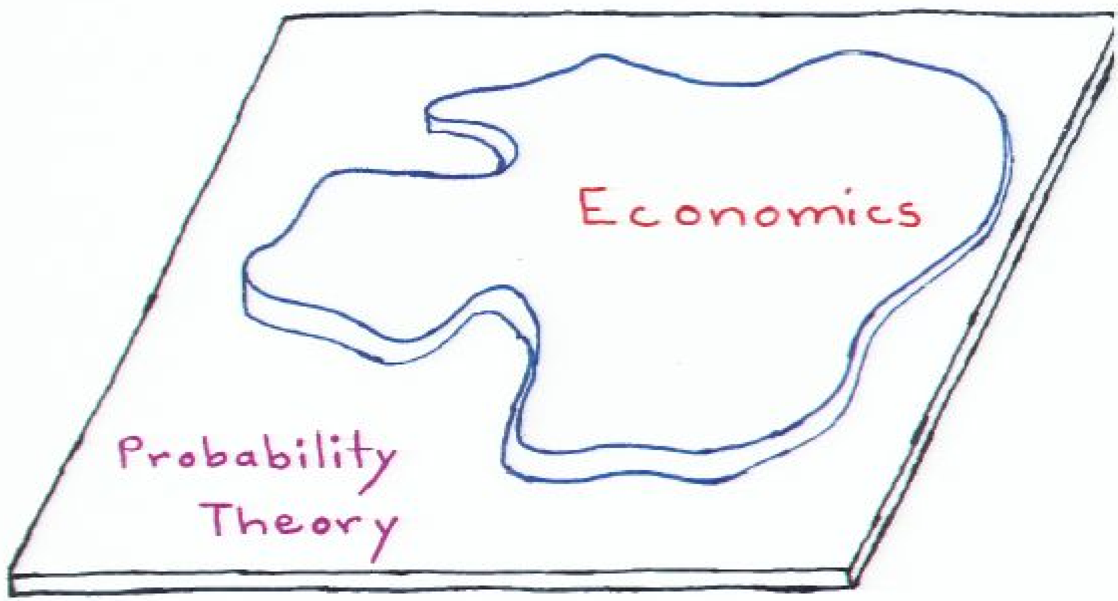
But big question:

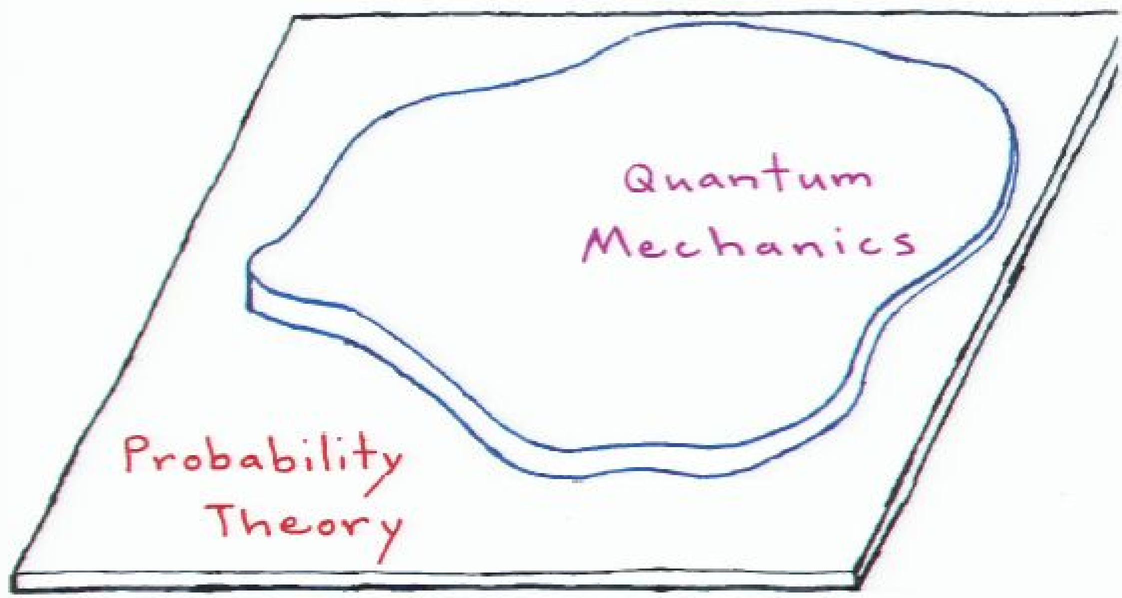
If one's probabilities do not satisfy this relation, what bad thing can happen in the single case?

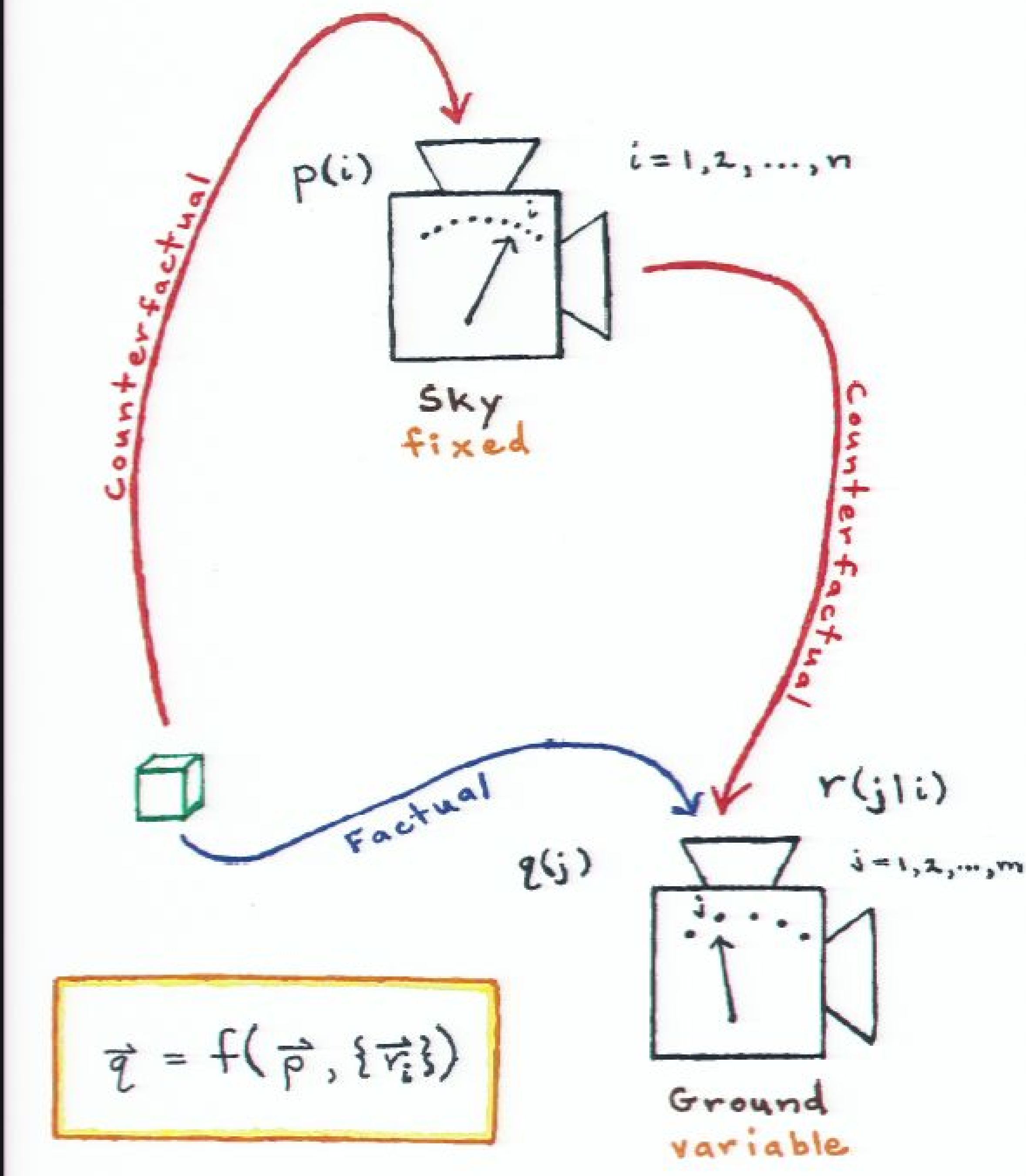


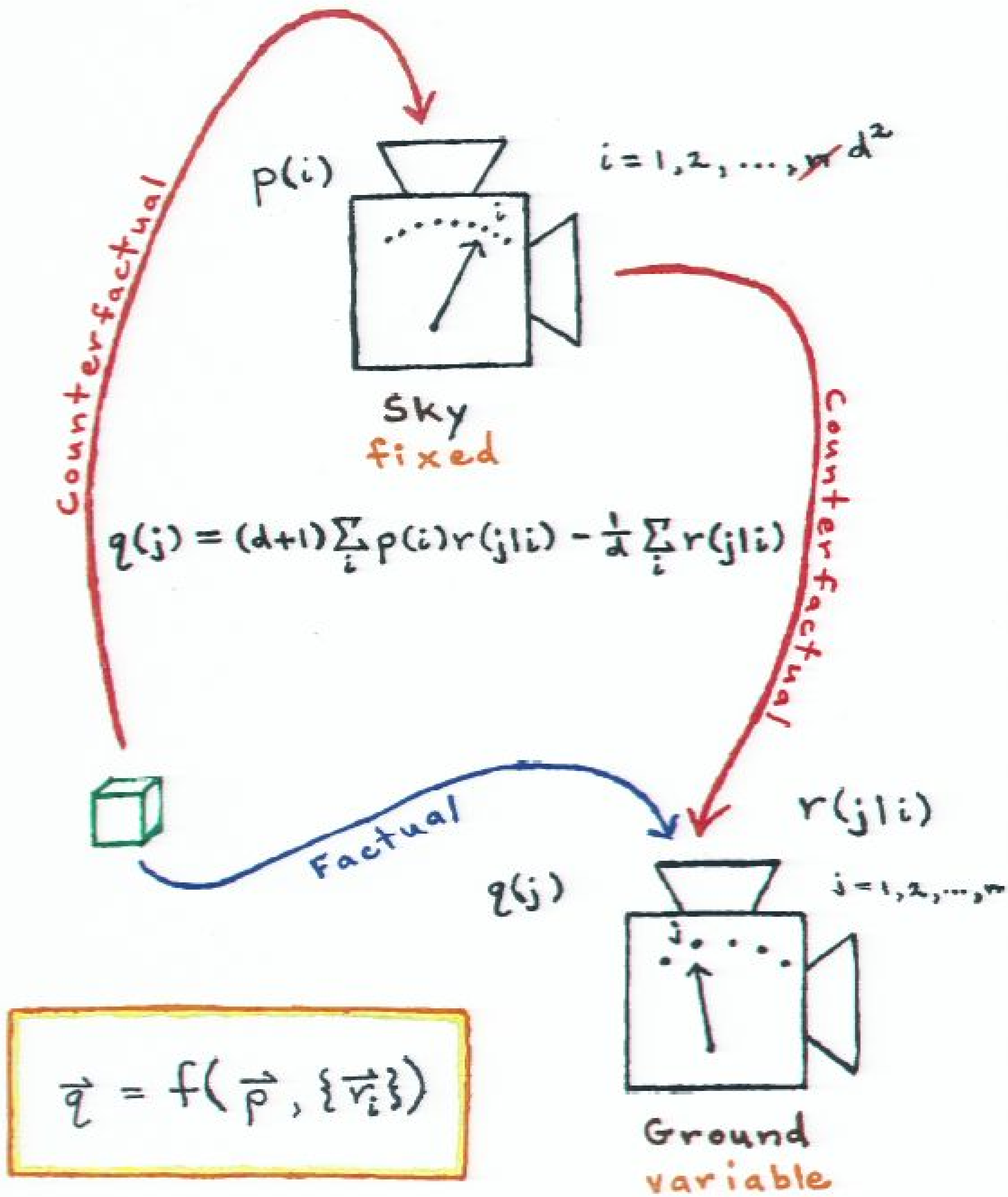
Classical probability is "just" the commutative case.


Probability
Theory


Economics
Probability
Theory


Quantum
Mechanics
Probability
Theory


Counterfactual
p(i)
i=1,2,...,n
i
Sky
fixed
Counterfactual
Factual
r(j|i)
q(j)
j=1,2,...,m
j
Ground
variable
$\vec{q} = f(\vec{p}, \{\vec{r}_i\})$



Counterfactual
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$i = 1, 2, \ldots, \cancel{n}\, d^2$
Sky
fixed
$q(j) = (d+1)\sum_i p(i) r(j|i) - \frac{1}{d}\sum_i r(j|i)$
Counterfactual
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$\vec{q} = f(\vec{p}, \{\vec{r}_i\})$
Ground
variable

Property of QM

Suppose initial $\rho = \frac{1}{d} I$, and actually follow path in sky arriving with click j for POVM $\{G_j\}$ on ground.

Bayes' Rule gives us a posterior for which click i occured in sky:

$$\mathrm{Prob}(i|j) = \frac{p(i) r(j|i)}{\sum_k p(k) r(j|k)} = \frac{\mathrm{tr}\, G_j \Pi_i}{d\, \mathrm{tr}\, G_j}$$

$$= \frac{1}{d} \mathrm{tr}\, \rho_j \Pi_i$$

But that is just the SIC representation of $\rho_j = \frac{G_j}{\mathrm{tr}\, G_j}$

ANY ρ_j can be gotten this way by suitable choice of $\{G_j\}$.

Reciprocity Axiom

- Starting from a state of maximal uncertainty for the sky, one can use the posterior supplied by Bayes' rule

$$\mathrm{Prob}(i|j) = \frac{r(j|i)}{\sum_k r(j|k)}$$

as a valid prior. Moreover all valid priors can be generated in this way.

Consequence: Rewriting

$$q(j) = \left(\sum_k r(j|k)\right)\left[(d+1)\sum_i p(i)\,\mathrm{Prob}(i|j) - \frac{1}{d}\right]$$

for any two valid priors $p(i)$ and $s(i)$,

$$\vec{p}\cdot\vec{s} = \sum_i p(i)s(i) \geq \frac{1}{d(d+1)} .$$

Basis States

Consider case where

$$\text{ground} = \text{sky} .$$

Consistency requires for any valid $\vec{p}$,

$$p(j) = (d+1)\sum_i p(i) r(j|i) - \frac{1}{d} .$$

Consequently,

$$r(j|i) = \frac{1}{d+1}\left(\delta_{ij} + \frac{1}{d}\right)$$

and by Reciprocity Axiom, all $\vec{p}$ of the form

$$\vec{e}_k = \left[\frac{1}{d(d+1)}, \ldots, \frac{1}{d}, \ldots, \frac{1}{d(d+1)}\right]$$

must be valid priors.

Note: $\vec{e}_k \cdot \vec{e}_k = \frac{2}{d(d+1)}$.

Homework

containing the $\vec{e}_k$

Call a set $\mathcal{S} \subseteq \Delta_{d^2}$ within the probability simplex

a) consistent if for any $\vec{p}, \vec{q} \in \mathcal{S}$

$$\frac{1}{d(d+1)} \leq \vec{p} \cdot \vec{q} \leq \frac{2}{d(d+1)} ,$$

b) maximal if adding any further $\vec{p} \in \Delta_{d^2}$ makes it inconsistent

Example: If $\mathcal{S}$ is set of quantum states, it is consistent & maximal.

Problem: Characterize all such $\mathcal{S}$; compare to quantum.

Examples

1) Take $\vec{q} = \vec{p}$. Consequently must have

$$\vec{p} \cdot \vec{p} \leq \frac{2}{d(d+1)}$$

Same as quantum.

2) Consider a subset $\{\vec{p}_k\} \subseteq \mathcal{S}$ with $k = 1, \ldots, m$ such that

$$\vec{p}_k \cdot \vec{p}_k = \frac{2}{d(d+1)}$$

$$\vec{p}_k \cdot \vec{p}_\ell = \frac{1}{d(d+1)} \qquad k \neq \ell .$$

How large can m be?

Answer: d, same as quantum

Property of QM

Suppose initial $\rho = \frac{1}{d} I$, and actually follow path in sky arriving with click j for POVM $\{G_j\}$ on ground.

Bayes' Rule gives us a posterior for which click i occured in sky:

$$\mathrm{Prob}(i|j) = \frac{p(i)\,r(j|i)}{\sum_k p(k)\,r(j|k)} = \frac{\mathrm{tr}\, G_j \Pi_i}{d\, \mathrm{tr}\, G_j}$$

$$= \frac{1}{d}\, \mathrm{tr}\, \rho_j \Pi_i$$

But that is just the SIC representation of $\rho_j = \frac{G_j}{\mathrm{tr}\, G_j}$

ANY ρ_j can be gotten this way by suitable choice of $\{G_j\}$.

Basis States

Consider case where

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Challenge

What further postulates must be made to recover precisely quantum state space?

I.e. the convex hull of

1) $\sum_i p(i)^2 = \frac{2}{d(d+1)}$

2) $\sum_{ijk} c_{ijk}\, p(i) p(j) p(k) = \frac{d+7}{(d+1)^3}$

with c_{ijk} possessing correct properties

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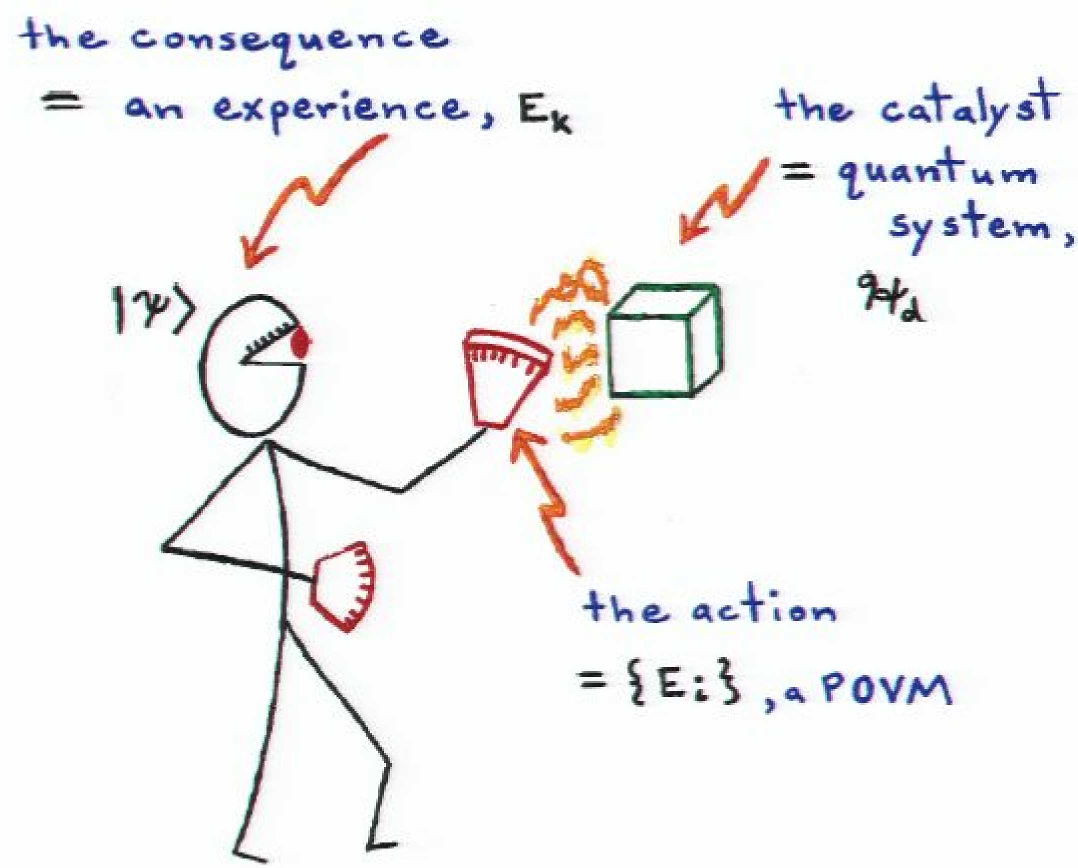
Think SIC thoughts!

... and maybe by way of it we'll come to understand quantum mechanics a little better.



the consequence
= an experience, E_k
the catalyst
= quantum system,
$\mathcal{H}_d$
$|\psi\rangle$
the action
= $\{E_i\}$, a POVM