Title: Four and a Half Axioms for Quantum Mechanics

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Abstract: I will discuss a set of strong, but probabilistically intelligible, axioms from which one can {\em almost} derive the appratus of finite dimensional quantum theory. These require that systems appear completely classical as restricted to a single measurement, that different measurements, and likewise different pure states, be equivalent up to the action of a compact group of symmetries, and that every state be the marginal of a bipartite state perfectly correlating two measurements. This much yields a mathematical representation of measurements, states and symmetries that is already very suggestive of quantum mechanics. One final postulate (a simple minimization principle, still in need of a clear interpretation) forces the theory's state space to be that of a formally real Jordan algebra

4.5 Axioms for QM

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Reconstructing Quantum Theory
Perimeter Institute., August, 2009

1. Motivation
2. Framework
3. Axioms

Two ways to be Puzzled by QM:

(a) as a linear dynamical theory with a familiar mathematical apparatus, but a mysterious probabilistic interpretation.

(b) as a conservative extension of classical probability theory, with a relatively unproblematic interpretation, but a mysterious mathematical apparatus.

Ideally, one would like to have a short list of physically plausible assumptions from which one could deduce [the structure of QM]. Short of this one would like a list from which one could deduce a set of possibilities for the structure ... all but one of which could be show to be inconsistent with suitably planned experiments.
– G. W. Mackey, Mathematical Foundations of QM, 1963

Cones

Definition

A **cone** in a real vector space A is a closed subset $K \subseteq A$ such that

(a) $a \in K, \lambda \geq 0 \Rightarrow \lambda a \in K$

(b) $a, b \in K \Rightarrow a + b \in K$

(c) $K \cap -K = \{0\}$.

Note that K induces a partial order on A: $a \leq b$ iff $b - a \in K$. K is **generating** iff $A = K - K = \{a - b | a, b \in K\}$. In what follows, an *ordered linear space* is a real vector space A with a specified generating cone $K =: A_+$ – and *all spaces are finite-dimensional!*

Self-Duality and Homogeneity

The *dual* of a cone K is

$$K^* := \{f \in A^* | f(a) \geq 0 \ \forall a \in K\}.$$

If A carries an inner product $\langle \ , \ \rangle$, the *internal dual* of K is

$$K^+ := \{b \in A | \langle b, a \rangle \geq 0 \ \forall \in K\} \simeq K^*.$$

K is *self-dual* iff there exists an inner product on A with $K = K^+$.

Example: The cone $\mathscr{L}_+(\mathbf{H})$ of positive operators on $\mathbf{H}$ is self-dual w.r.t. the usual trace inner product.

A_+ is *homogeneous* iff for any pair a, b in the *interior* of A_+, there exists an affine automorphism $\phi : A_+ \to A_+$ taking a to b.

Example: $\mathscr{L}_h(\mathbf{H})$. (Use the spectral theorem.)

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IDEA: If we can motivate homogeneity and self-duality, we come close to motivating QM.

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Definition: A **test space** is a collection $\mathfrak{A}$ of non-empty sets, called *tests*, understood as the outcome-sets of various "measurements".

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- The convex set $\Omega(\mathfrak{A})$ of all probability weights on $\mathfrak{A}$ is the latter's **state space**.

Examples

Example 1: Classical Models. Let $\mathfrak{A} = \{E\}$ where E is a finite set, $\Omega = \Delta(E)$, the set of all probability weights on E.

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Example 2: Quantum Models. Let $\mathfrak{A} = \mathfrak{F}(\mathbf{H})$, the set of orthonormal bases for a Hilbert space $\mathbf{H}$, $\Omega = \Omega(\mathbf{H})$ states of the form $\alpha(x) = \langle \rho x, x \rangle$, ρ a density operator on $\mathbf{H}$. (Gleason's Theorem tells us that all states have this form, if $\dim \mathbf{H} > 2$.)

Linearization

Let $(\mathfrak{A}, \Omega)$ be a model, with total outcome-space $X = \bigcup \mathfrak{A}$. Let $V(\Omega)$ be the span of Ω in $\mathbb{R}^X$, ordered pointwise. Note that Ω is a *base* for this positive cone: every $\mu \geq 0$ in V is a non-negative multiple of a state.

Example: For the quantum model $(\mathfrak{F}(\mathbf{H}), \Omega(\mathbf{H}))$, $V = \mathcal{L}_h(\mathbf{H})$ with the usual ordering.

Definition: Let V^* be the dual space of V, ordered pointwise on Ω, i.e., $f \in V_+^*$ iff $f(\alpha) \geq 0$ for every $\alpha \in \Omega$. Note that there is a natural mapping $X \to V^*$ given by $x \mapsto \hat{x}$ where $\hat{x}(\alpha) = \alpha(x)$ for all $\alpha \in V$. Note, too, that for every $E \in \mathfrak{A}$, we have

$$\sum_{x \in E} \hat{x} = u$$

where $u(\alpha) \equiv 1$.

Example: For the quantum model, we again have $V \simeq V^* = \mathcal{L}_h(\mathbf{H})$.

Axiom 1: Lots of States

Definition: A set Ω of states on $\mathfrak{A}$ is **separating** iff $x \mapsto \hat{x}$ is injective. Ω is **unital** iff for every $x \in X$, there exists at least one $\alpha \in \Omega$ with $\alpha(x) = 1$.

Axiom 1: Ω is separating and unital.

Note that separating is an axiom of convenience: if not, simply identify x with $\hat{x}$. I'll do this in any case!

Symmetry

Definition: Let G be a group. A G-**test space** is a test space $\mathfrak{A}$ such that $X = \bigcup \mathfrak{A}$ carries a G action, with $gE \in \mathfrak{A}$ for all $g \in G$, $E \in \mathfrak{A}$. A G-test space $\mathfrak{A}$ is **fully symmetric** iff

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Example: The test space $\mathfrak{F}(\mathbf{H})$ of frames of $\mathbf{H}$ is fully symmetric under $\mathbf{H}$'s unitary group $U(\mathbf{H})$.

Axiom 2: Lots of Symmetry

Note that G acts also on Ω by $g(\alpha)(x) = \alpha(g^{-1}(x))$, $x \in X$. If Ω is invariant under this action, we say that G acts on the model $(\mathfrak{A}, \Omega)$.

> Axiom 2: There is a compact group G acting continuously on $(\mathfrak{A}, \Omega)$, in such a way that
> (i) G acts fully symmetrically on $\mathfrak{A}$
> (ii) G acts transitively on pure states. Ω_{ext}.

A classical test space $\mathfrak{A} = \{E\}$ satisfies Axiom 2 trivially with $G = S(E)$, the symmetric group on E. A quantum test space $(\mathfrak{F}(\mathbf{H}), \Omega_{\mathbf{H}})$ satisfies Axiom 2 with $G = U(\mathbf{H})$, the unitary group of $\mathbf{H}$.

Call an inner product on V^* **positive** iff $\langle a, b \rangle \geq 0$ for all $a, b \in V^*_+$. Note that the trace inner product on $V^* = \mathscr{L}_h(\mathbf{H})$ is positive in this sense.

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Representation I

Lemma

Subject to Axiom 2, there exists a positive, G-invariant inner product on V^*.

Proof: Represent Ω_{ext} as G/K with K the stabilizer of some (any) pure state α_o. Any $f \in V^*$ gives rise to a continuous function $G \to \mathbb{R}$ by $g \mapsto f(g\alpha_o)$. This gives an embedding of V^* as a G-invariant real subspace of the algebra $C(G)$ of continuous complex-valued functions on G. The restriction of the natural inner product on the latter to V^* is a real, G-invariant inner product, and is positive, simply because the convolution of positive functions on G is positive. □

Denote the inner product arising from $C(G)$ by $\langle\ ,\ \rangle_G$.

Representation II

Lemma

Let $\langle , \rangle$ be any positive, G-invariant inner product on V^*. There is an embedding $x \mapsto v_x$ of the outcome-space X into the unit sphere of V^* with $\langle v_x, v_y \rangle = 0$ for all $x \perp y \in X$.

Proof: For each $x \in X$, set $q_x = x - \langle x, u \rangle u$. Then $\langle q_x, u \rangle = 0$ and $\sum_{x \in E} q_x = 0$ for any $E \in \mathfrak{A}$. As G acts 2-transitively on X, $\exists$ constants $r > 0$ and s_q with

$$\|q_x\| \equiv r \ \forall x \in X \ \text{ and } \ \langle q_x, q_y \rangle \equiv s_q \ \forall x \perp y$$

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Lemma

Let $s \equiv \langle x, y \rangle$ where $x \perp y$. Then, for all outcomes x and y in X, $\langle v_x, v_y \rangle = \langle x, y \rangle - s$.

Proof: Let m = the (constant) value of $\langle x, u \rangle$. Set $q_x = x - mu$ as in the proof of Lemma 2, so that $\langle q_x, u \rangle = 0 \; \forall x \in X$. Recall that $v_x = q_x + cu$ where $-c^2 = s_q$ = constant value of $\langle q_x, q_y \rangle$ for $x \perp y$. Thus,

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$$\langle q_x, q_y \rangle = \langle x - mu, y - mu \rangle = \langle x, y \rangle - m\langle x, u \rangle - m\langle u, y \rangle + m^2 = \langle x, y \rangle - m^2.$$

Considering the case where $x \perp y$, this yields

$$s_q = s - m^2.$$

Hence,

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$$\langle q_x, q_y \rangle = \langle x - mu, y - mu \rangle = \langle x, y \rangle - m\langle x, u \rangle - m\langle u, y \rangle + m^2 = \langle x, y \rangle - m^2.$$

Considering the case where $x \perp y$, this yields

$$s_q = s - m^2.$$

Hence,

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as promised. □

Provisional Axiom 3

Definition: A G-invariant, positive inner product on V^* is **minimizing** iff the constant s of Lemma 3 is the minimum value of $\langle x, y \rangle$ on $X \times X$.

True for the trace inner product on $\mathscr{L}_h(\mathbf{H})$, where $s = 0$!

Lemma

For a minimizing inner product, the vectors v_x of Lemma 2 lie in the positive cone of V^*. Moreover, $\varepsilon_x := \langle v_x | \cdot \rangle$ is the unique state with $\varepsilon_x(x) = 1$.

Proof: Immediate from Lemma 3. □

> Provisional Axiom 3 (Minimization): There exists a minimizing G-invariant, positive inner product on V^*.

Fact: all positive inner products on $V^* = \mathscr{L}_h(\mathbf{H})$ invariant under the unitary group of $\mathbf{H}$, are in fact minimizing.

Lemma

Let $s \equiv \langle x, y \rangle$ where $x \perp y$. Then, for all outcomes x and y in X, $\langle v_x, v_y \rangle = \langle x, y \rangle - s$.

Proof: Let $m =$ the (constant) value of $\langle x, u \rangle$. Set $q_x = x - mu$ as in the proof of Lemma 2, so that $\langle q_x, u \rangle = 0 \; \forall x \in X$. Recall that $v_x = q_x + cu$ where $-c^2 = s_q =$ constant value of $\langle q_x, q_y \rangle$ for $x \perp y$. Thus,

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Self-Duality

Proposition

Subject to Axioms 1, 2 and Provisional axiom 3, V_+^* is self-dual.

Proof: Let $\langle , \rangle$ be a minimizing, G-invariant positive inner product. Positivity gives us $V_+^* \subseteq V^{*+} \simeq V_+$. Letting v_x be defined as in Lemma 2, Lemma 7 tells us that $\alpha_x(y) := \langle v_x, v_y \rangle$ defines a state making x certain (since $\langle v_x, v_x \rangle = \| v_x \| = 1$). By Axiom 3, this is the pure state $< x|$. It follows from Axiom 2 that every pure state has the form $g < x| =< gx|$ for some $g \in G$. Thus, every pure state is represented in the cone V_+^*, so that $V^{*+} \subseteq V_+^*$. $\square$

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Digression: Coupled Systems

If $\mathfrak{A}$ and $\mathfrak{B}$ are test spaces, let $\mathfrak{A} \times \mathfrak{B} = \{E \times F | E \in \mathfrak{A}, F \in \mathfrak{B}\}$. A state ω on $\mathfrak{A} \times \mathfrak{B}$ is *non-signaling* if it has well-defined *marginal states*

$$\omega_1(x) := \sum_{y \in F} \omega(x,y) \text{ and } \omega_2(y) := \sum_{x \in E} \omega(x,y)$$

(independent of $E \in \mathfrak{A}$ and $F \in \mathfrak{B}$, respectively).

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- Mixtures of product states $(\alpha \otimes \beta)(x,y) := \alpha(x)\beta(y)$ are obviously influence-free.
- Unless $\Omega(\mathfrak{A})$ or $\Omega(\mathfrak{B})$ is a simplex, there exist **entangled** non-signaling states *not* mixtures of product states.

If ω is non-signaling, every $\mathfrak{A}$-outcome x yields a **conditional state** $\omega_{2|x} \in \Omega(\mathfrak{B})$:

$$\omega_{2|x}(y) := \frac{\omega(x,y)}{\omega_1(x)}$$

if $\omega_1(x) > 0$; otherwise $\omega_{2|x} \equiv 0$. Likewise, every $\mathfrak{B}$-outcome yields a conditional state $\omega_{1|y}$ on $\mathfrak{A}$. We have obvious *laws of total probability*: for any tests $E \in \mathfrak{A}$, $F \in \mathfrak{B}$,

$$\omega_1 = \sum_{y \in F} \omega_2(y)\omega_{1|y} \text{ and } \omega_2 = \sum_{x \in E} \omega_1(x)\omega_{2|x}.$$

Correlation

Definition: A non-signaling state ω on $\mathfrak{A} \times \mathfrak{A}$ is **correlating** iff, for some tests $E, F \in \mathfrak{A}$, and some bijection $f : E \to F$, $\omega(xy) = 0$ for all $(x, y) \in E \times F$ with $y \neq f(x)$.

Axiom 3 (Correlation): Every state is the marginal of a correlating bipartite state.

This is satisfied by both classical and quantum systems: trivially in the first case, and by the Schmidt decomposition in the second.

Spectrality

Lemma

Every state on $\mathfrak{A}$ is spectral, i.e., for every $\alpha \in \Omega$ there exists a test E with

$$\alpha = \sum_{x \in E} \alpha(x)\varepsilon_x.$$

Proof: For every $E \in \mathfrak{A}$, we have

$$\omega_2(y) = \sum_{x \in E} \omega_1(x)\omega_{2|x},$$

where $\omega_{2|x}$ is the conditional state on $\mathfrak{A}$ given outcome $x \in E$. If ω correlates E with F via $f : E \to F$, then we have, for all $x \in E$ with $\omega_1(x) > 0$, that $\omega_{2|x}(y) = 1$ if $y = f(x) \in F$; hence, $\omega_{2|x} = \varepsilon_{f(x)}$. Thus, $\omega_2 = \sum_{y \in F} \omega_1(f^{-1}(y))\varepsilon_y$. Evaluating both sides at any $y \in F$, we have $\omega_1(f^{-1}(y)) = \omega_2(y)$, so

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Filtering

Axiom 5 (Filtering): For every test E and every $f : E \to (0,1]$, there exists an automorphism $\phi : V^* \to V^*$ with $\phi(x) = f(x)x$.

This says that the outcomes of a test can simultaneously and independently be attenuated by any (non-zero) factors we like, by a reversible physical process. In this respect, we insist that the theory, as restricted to the single measurement E, "look completely classical."'

Proposition

Subject to Axioms 1-5, the cone V_+^* is homogeneous.

Proof: Let a, b be interior points of V_+^*. By Proposition 1, $\langle a|$ and $\langle b|$ are (un-normalized) states. By Lemma 4, they are spectral – say

$$\langle a| = \sum_{x \in E} \langle a, x \rangle \langle x| \text{ and } \langle b| = \sum_{y \in F} \langle b, y \rangle \langle y|.$$

Hence, we have $a = \sum_{x \in E} \langle a, x \rangle x$, and similarly for b. Since a and b are interior points, $\langle a, x \rangle$ and $\langle b, y \rangle$ are non-zero for all x, y. Let $g \in G$ define a bijection $E \to F$, and set

$$t(x) = \langle b, gx \rangle / \langle a, x \rangle > 0$$

for every $x \in E$. By Axiom 5, there is an order-automorphism $\phi : V^* \to V^*$ with $\phi : x \mapsto t(x)x$ for every $x \in E$. Hence,

$$\phi(a) = \sum_{x \in E} \langle a, x \rangle \phi(x) = \sum_{x \in E} \langle a, x \rangle t(x) x = \sum_{x \in E} \langle b, gx \rangle x.$$

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