

Title: Interaction axiomatics for quantum phenomena.

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Abstract: In our approach, rather than aiming to recover the 'Hilbert space model' which underpins the orthodox quantum mechanical formalism, we start from a general 'pre-operational' framework, and verify how much additional structure we need to be able to describe a range of quantum phenomena. This also enables us to investigate which mathematical models, including more abstract categorical ones, enable one to model quantum theory. Till now, all of our axioms only refer to the particular nature of how compound quantum systems interact, rather than to the particular structure of state-spaces. This is in sharp contrast with other approaches of this kind which aim to recover quantum theory out of a much broader class of theories. A more abstract quantum mechanical model has other many advantages. It elucidates which are the key ingredients that make 'the Hilbert space model' work. Since it relies on monoidal categories, it comes with a high-level diagrammatic description (which we think of as 'the mathematical formalism'). It moreover removes the dependency on continuous underlying mathematical structures, paving the way for discrete combinatorial models, which might blend better with the other ingredients required for a theory of quantum gravity.

Interaction axiomatics for quantum phenomena *illustrated on Stab(elizer qm) vs. Spek(kens toys)*



strict dress code and diet



Roman Priebe

Andrey Akhvlediani

Bill Edwards

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- Which FdHilb-ingredients (cf. LEGO) yield QP?
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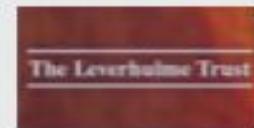
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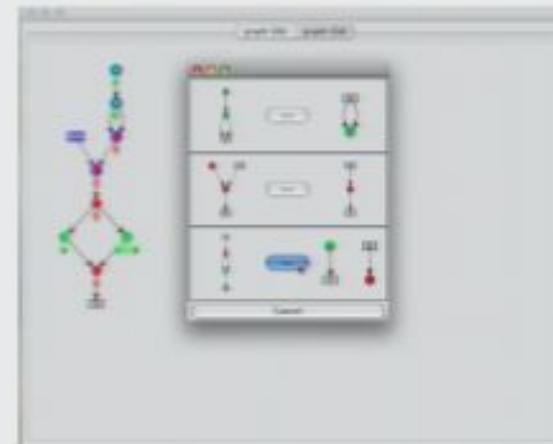
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Today: Hilbert space model *contra* sets and relations.

$\mathcal{L}s$ $\mathbb{E}s$ $\$s$



Product: automated q-reasoning software quantomatic



—mathematics for compositional situations —

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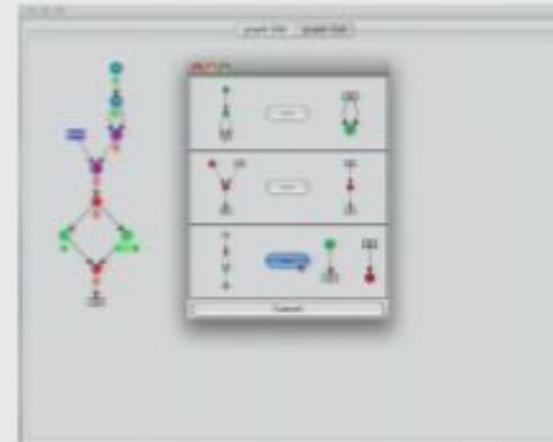
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be boiling, frying, baking. States are processes

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be doing nothing. We have $1_Y \circ \xi = \xi \circ 1_X = \xi$.

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$$C \otimes F \xrightarrow{x} M$$

be mashing spice-cook-potato and spice-cook-carrot.

5. Total *process*:

$$A \otimes D \xrightarrow{f \otimes h} B \otimes E \xrightarrow{g \otimes k} C \otimes F \xrightarrow{x} M = A \otimes D \xrightarrow{x \circ (g \otimes k) \circ (f \otimes h)} M.$$

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$$(g \circ f) \otimes (k \circ h) = (g \otimes k) \circ (f \otimes h)$$

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— (physical) data in monoidal category —

Systems:

$$A \quad B \quad C$$

Processes:

$$A \xrightarrow{f} A \quad A \xrightarrow{g} B \quad B \xrightarrow{h} C$$

Compound systems:

$$A \otimes B \quad I \quad A \otimes C \xrightarrow{f \otimes g} B \otimes D$$

Temporal composition:

$$A \xrightarrow{hog} C := A \xrightarrow{g} B \xrightarrow{h} C \quad A \xrightarrow{1_A} A$$

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Are those pictures merely a different language or more?

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$$f \equiv \begin{array}{c} f \\ \boxed{ } \end{array}$$

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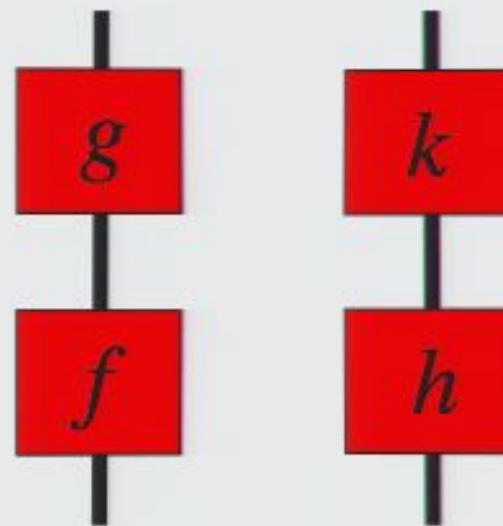
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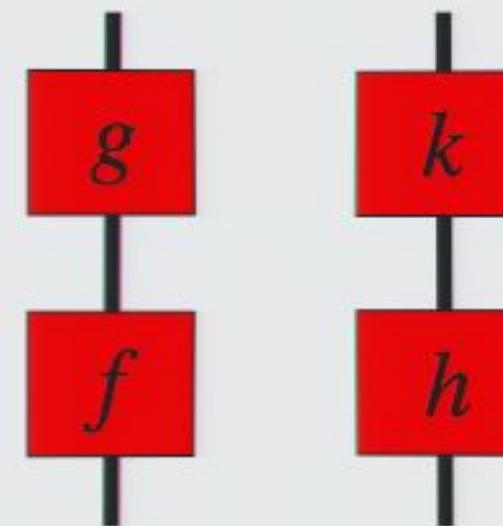
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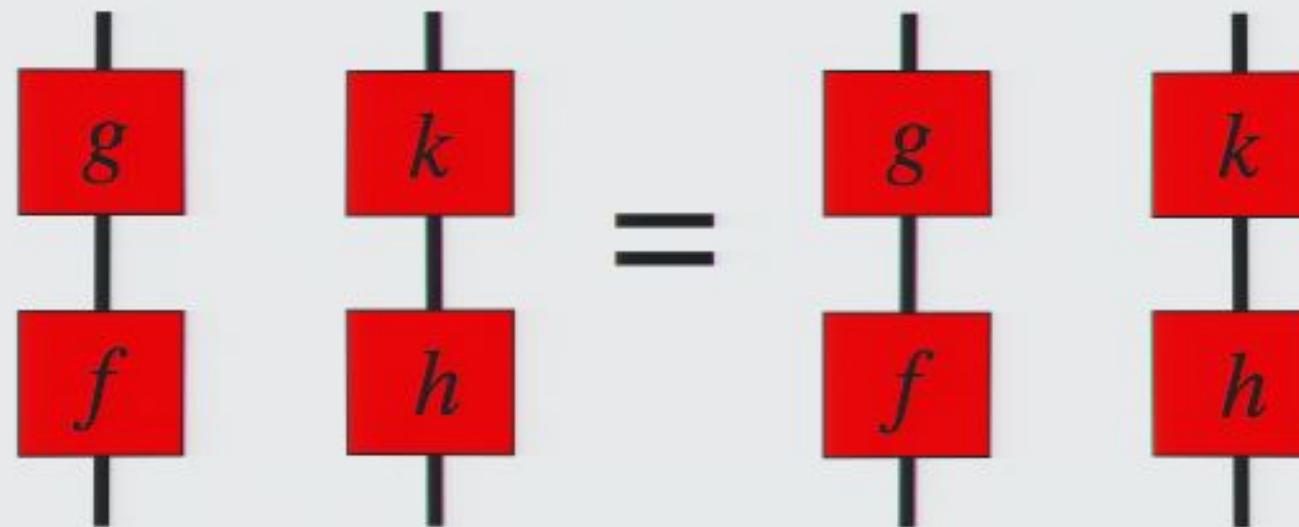
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— states, effects and quantities —

$$\psi : I \rightarrow A$$

$$\pi : A \rightarrow I$$

$$\pi \circ \psi : I \rightarrow I$$

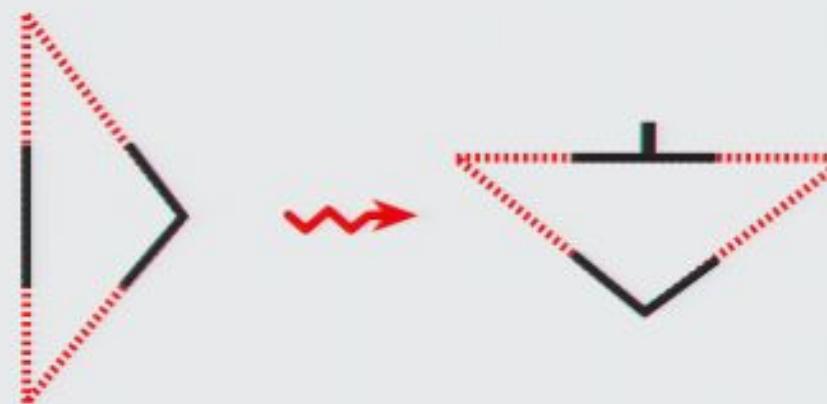


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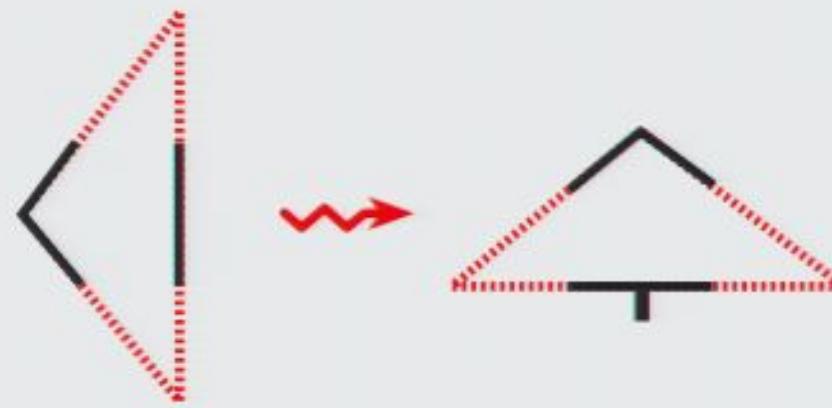
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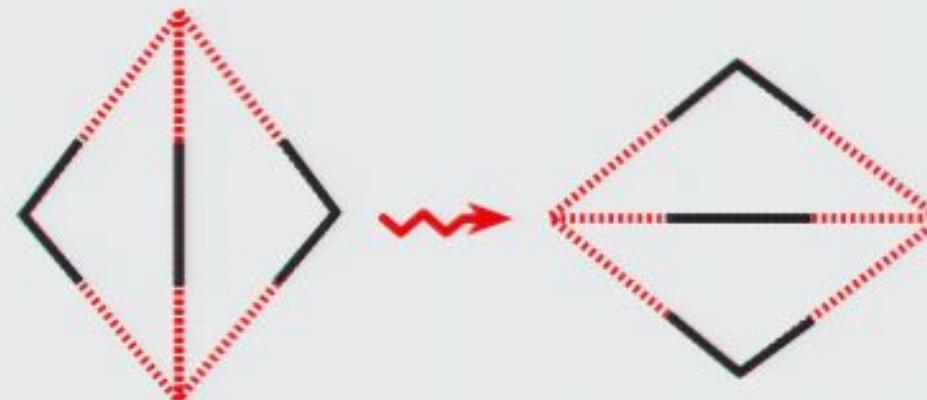
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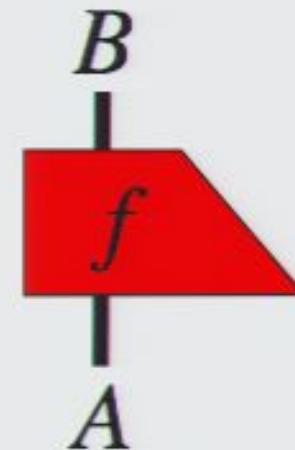


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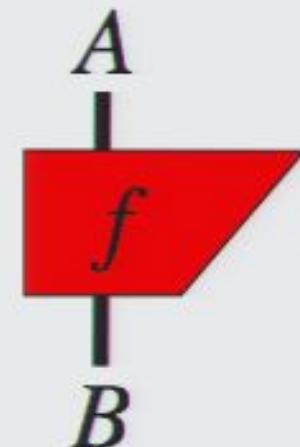
— adjoint \Rightarrow inner-product \Rightarrow probabilities —

$$f : A \rightarrow B$$



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$$f^\dagger: B \rightarrow A$$



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QUANTUM STRUCTURE 0: BELL STATES

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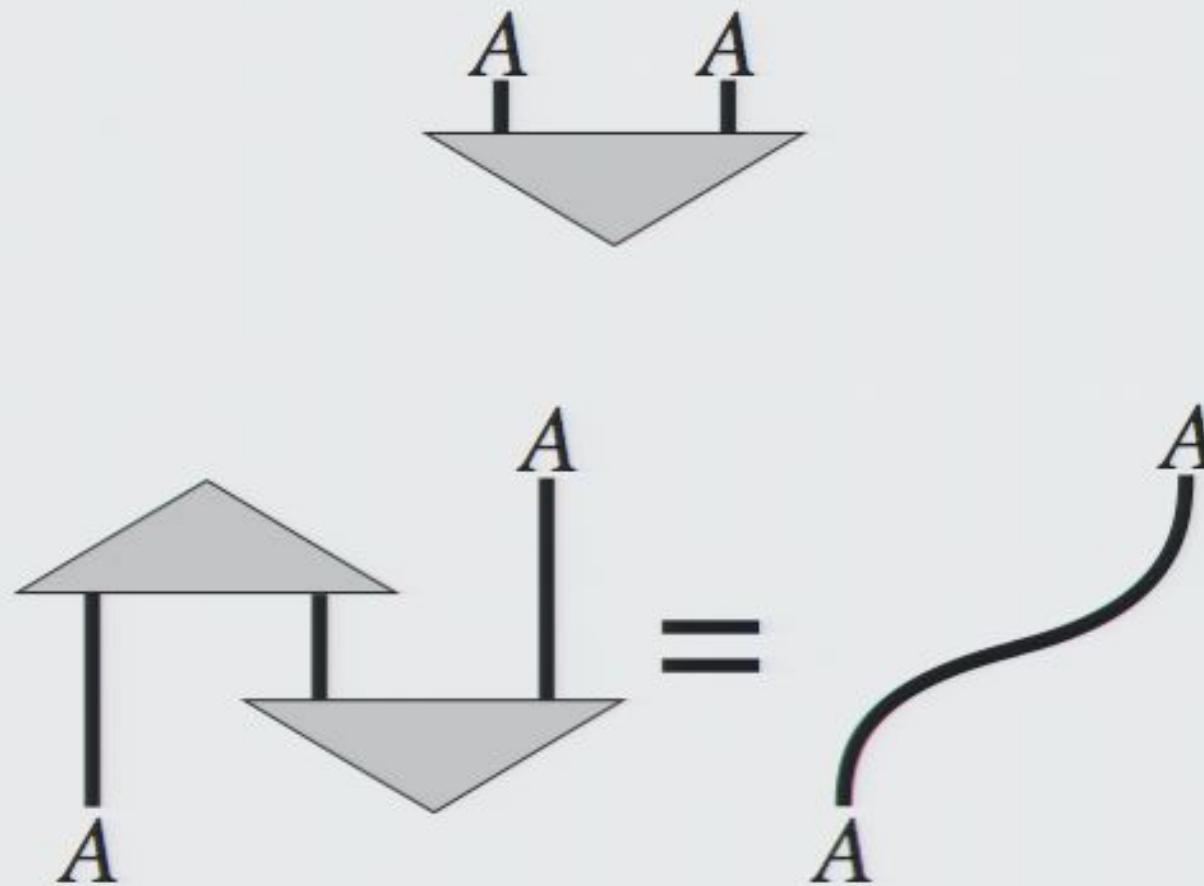
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— Bell structure —

$$(A, \eta : I \rightarrow A \otimes A)$$

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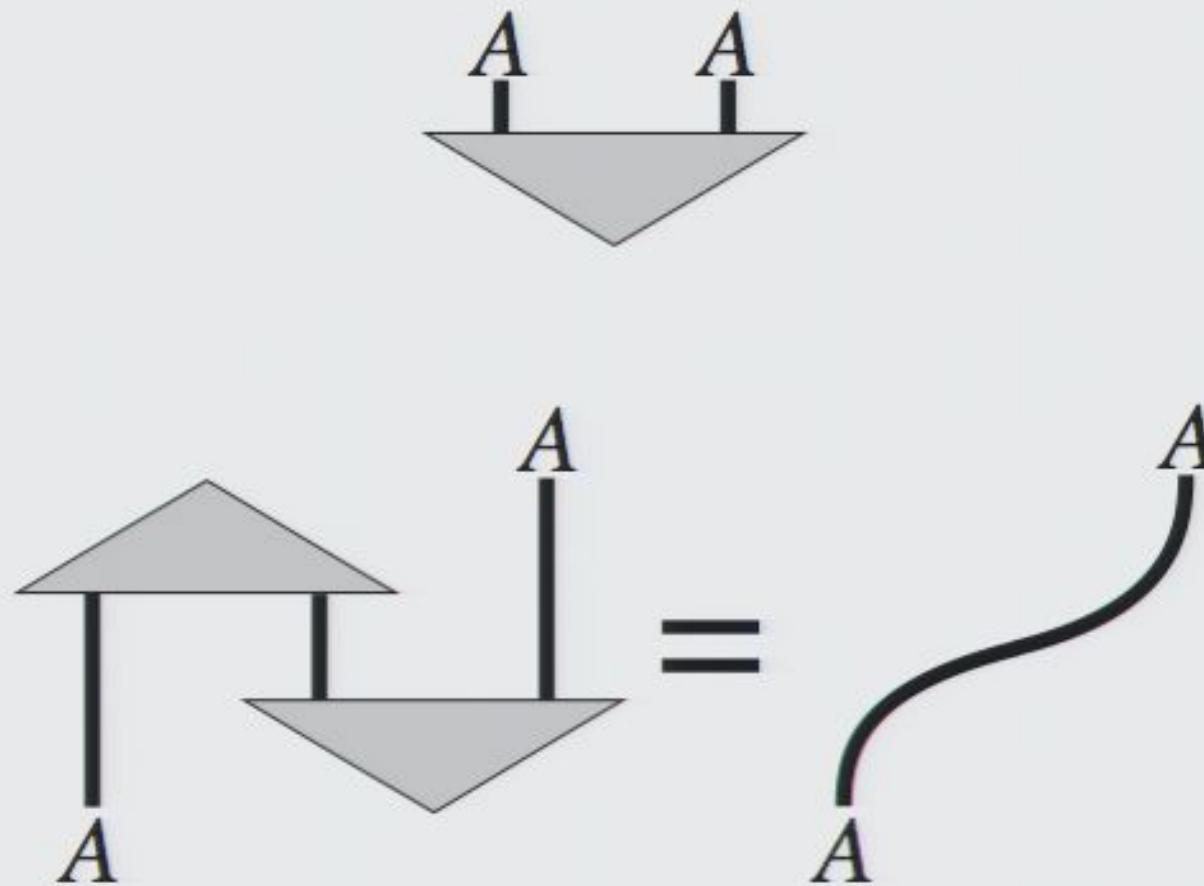


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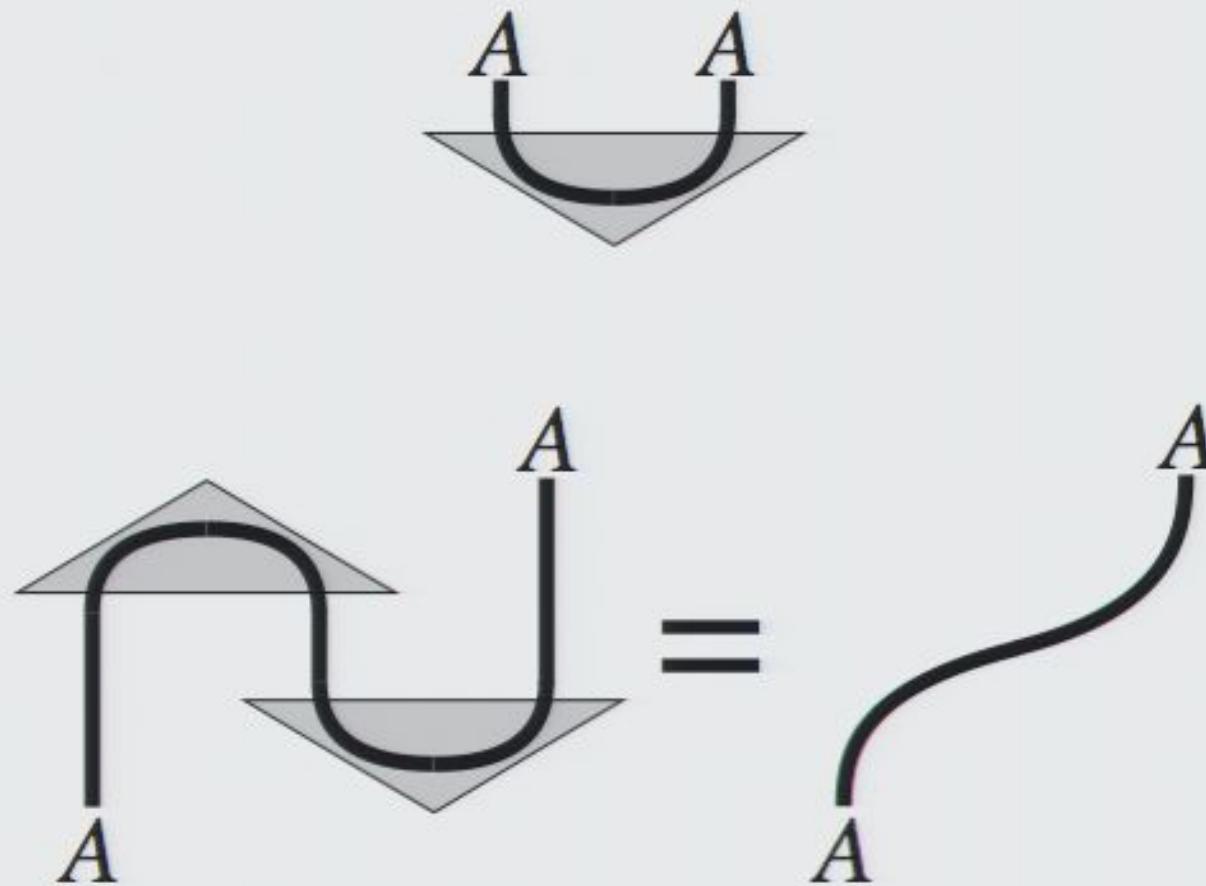
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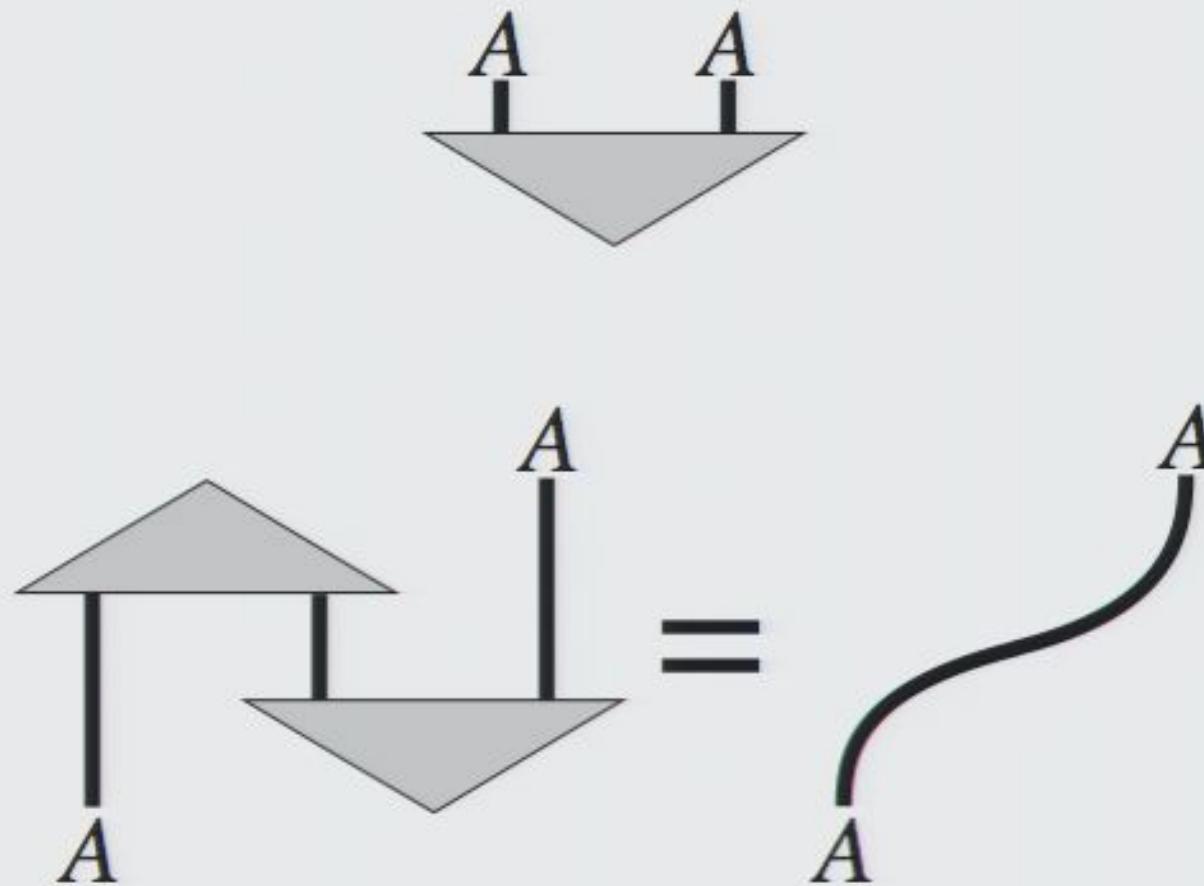
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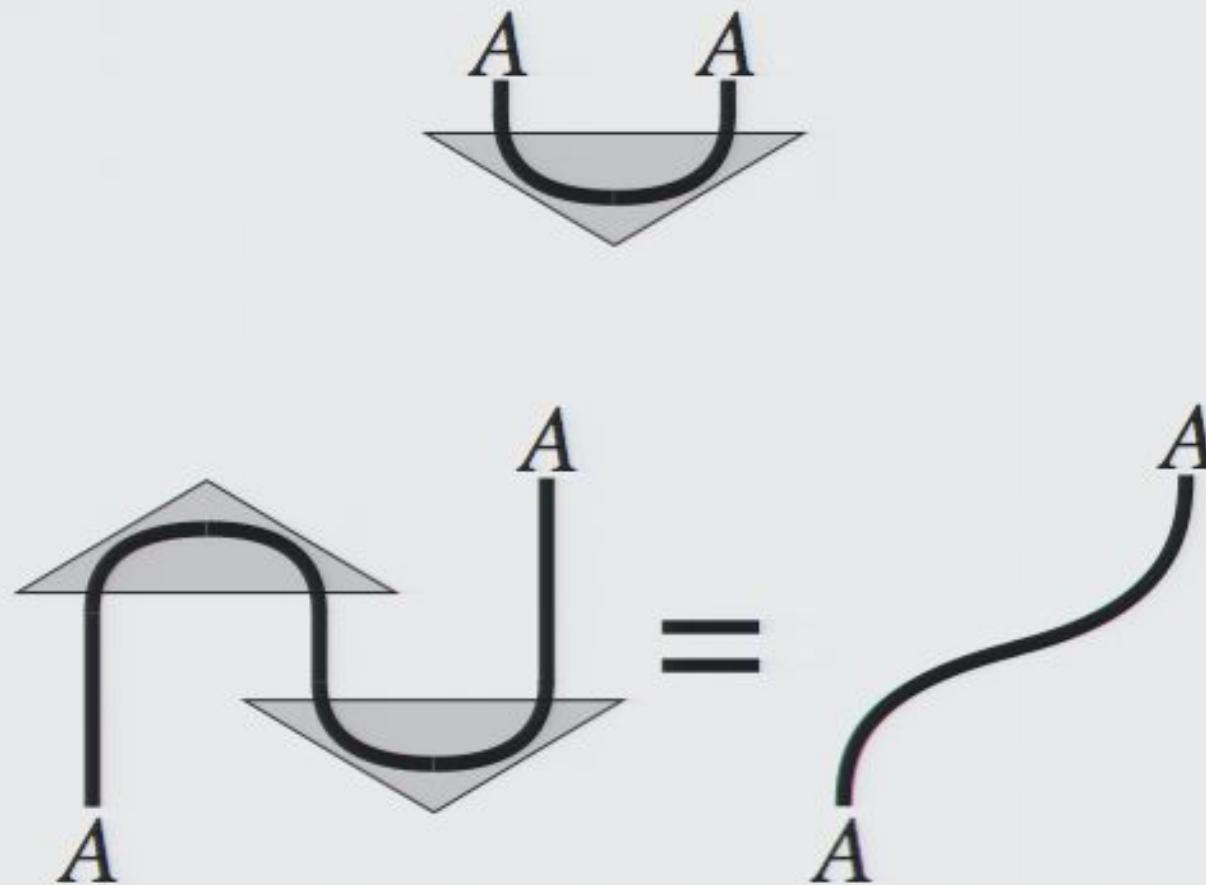
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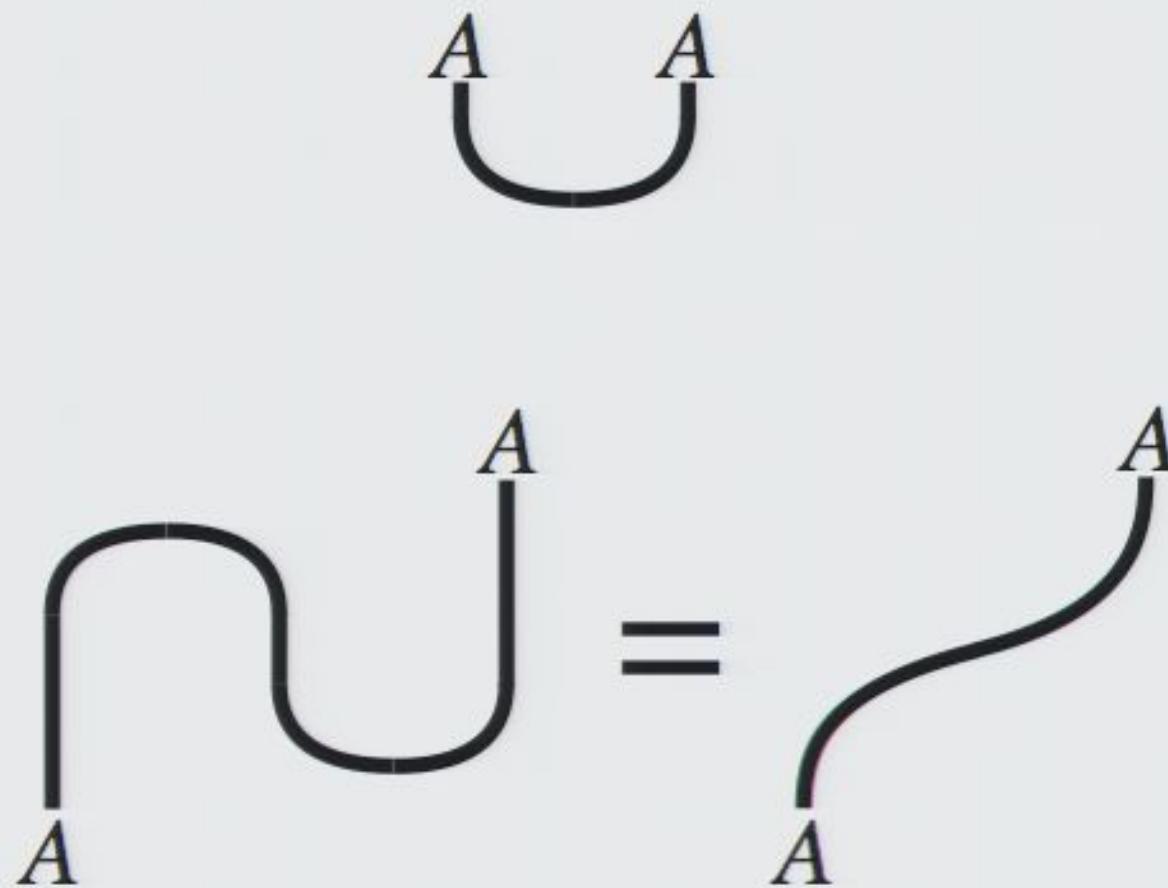
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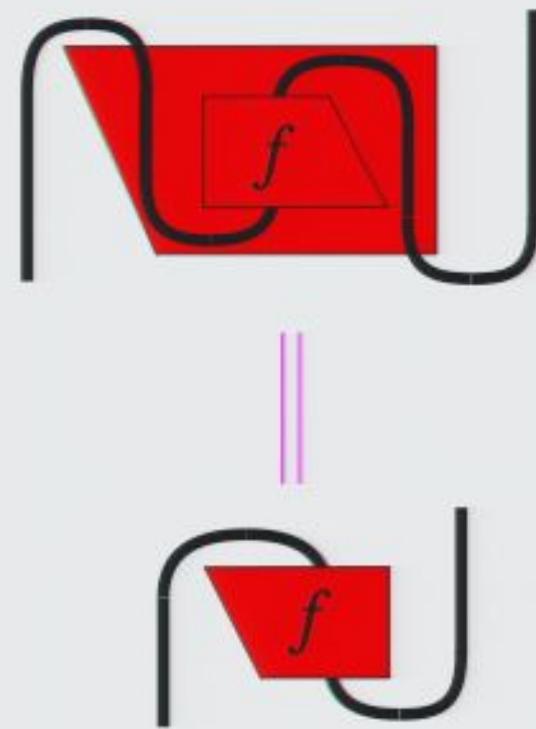
— ‘sliding’ —



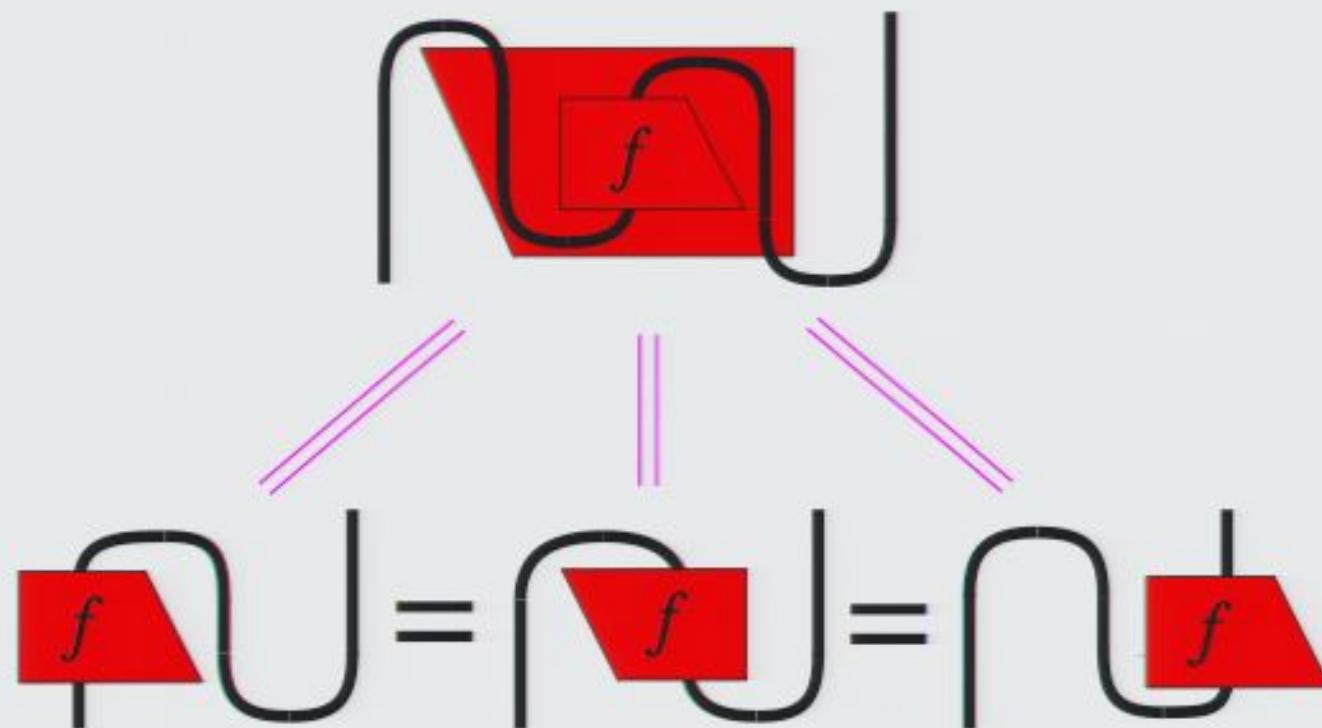
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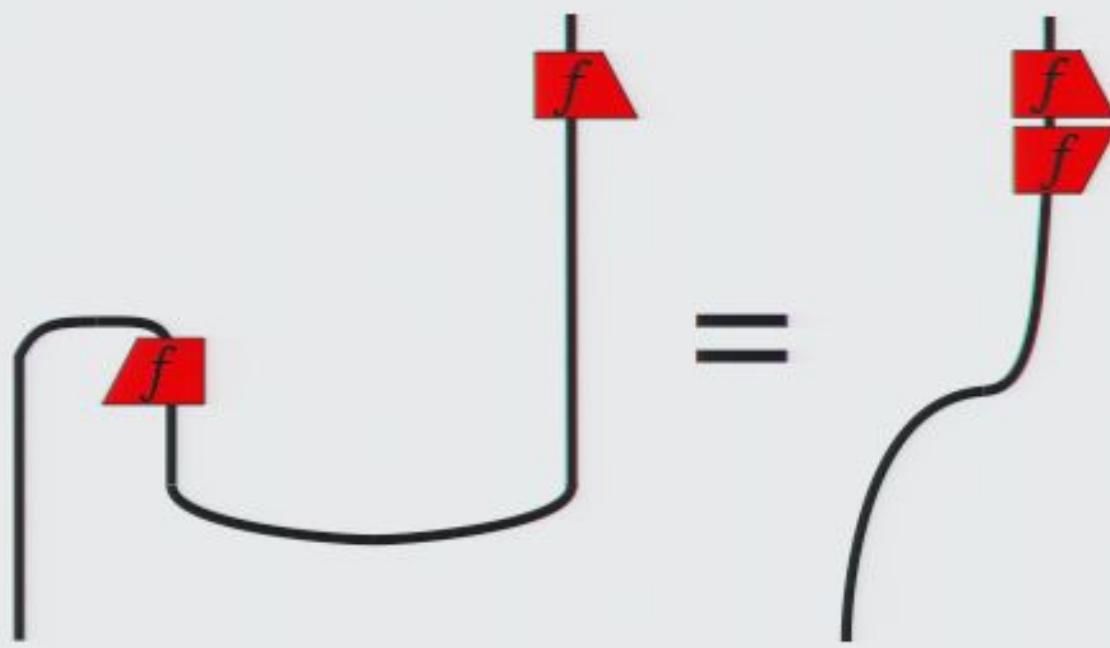


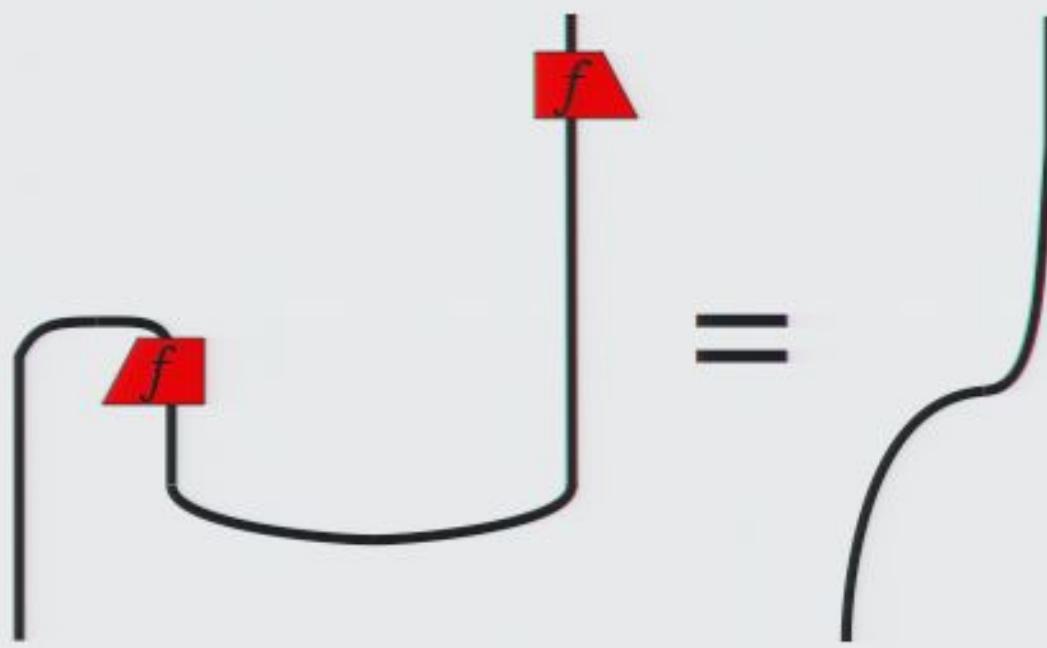
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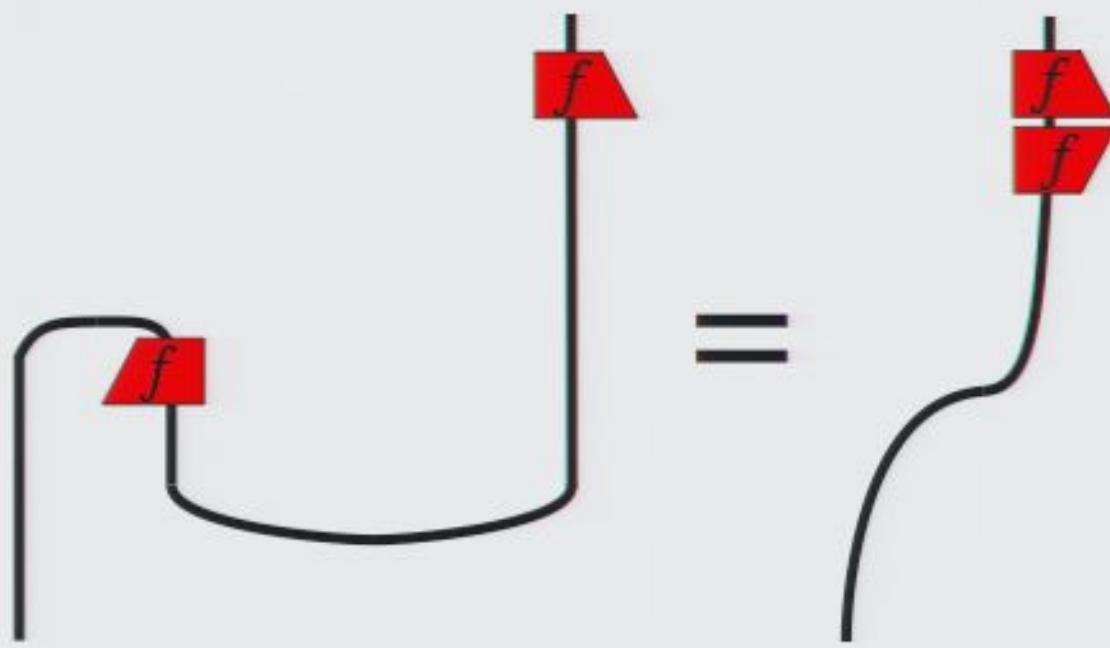


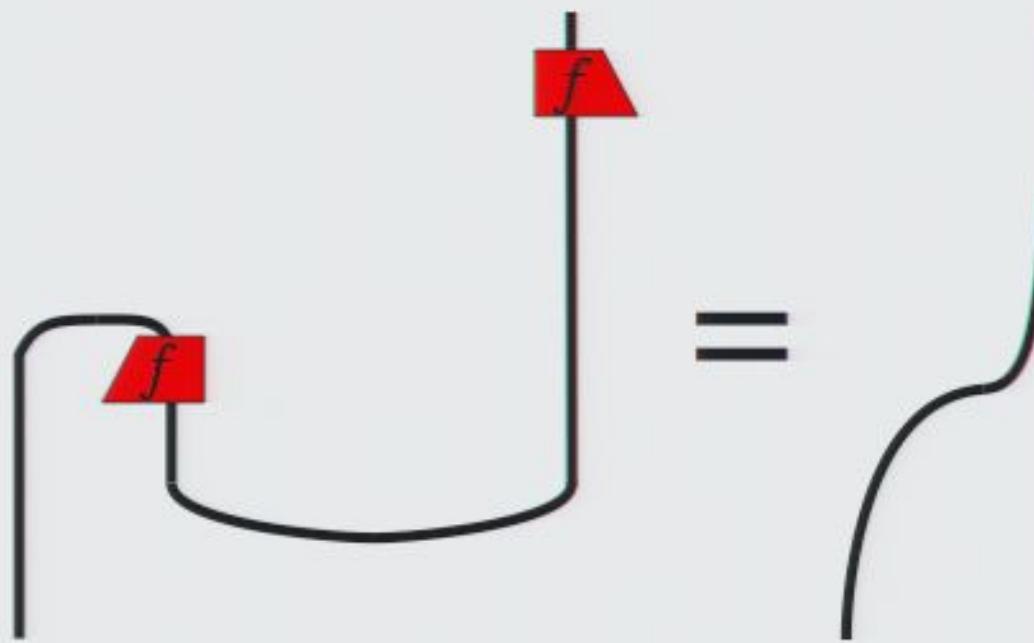
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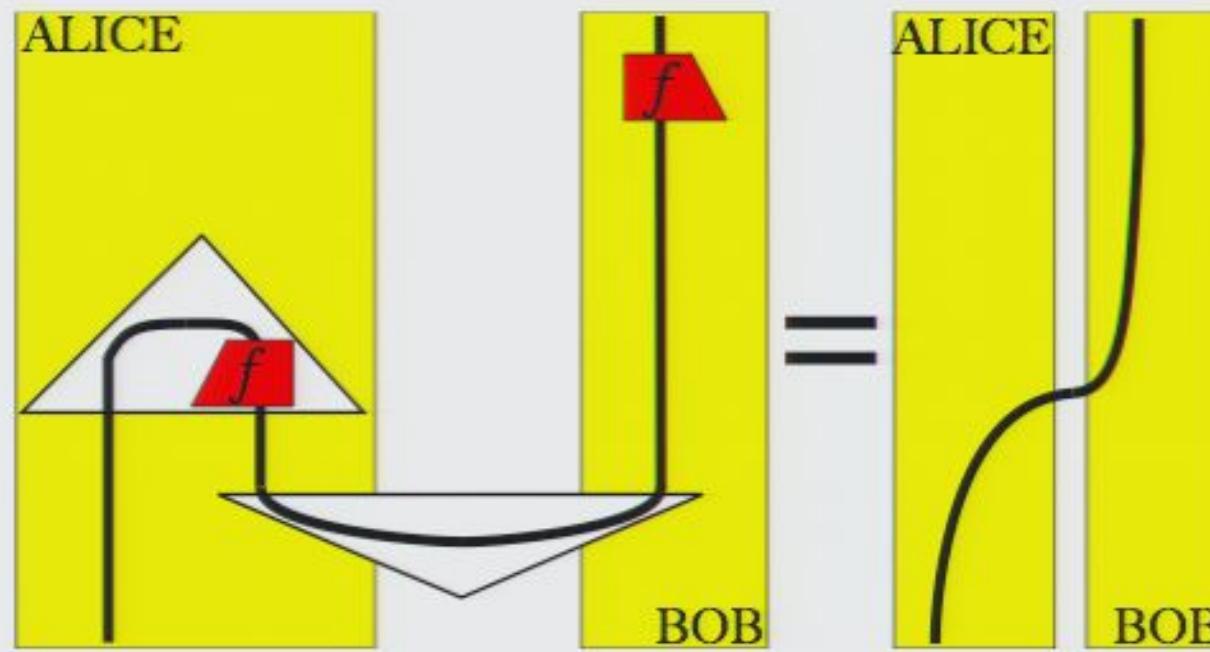












⇒ quantum teleportation

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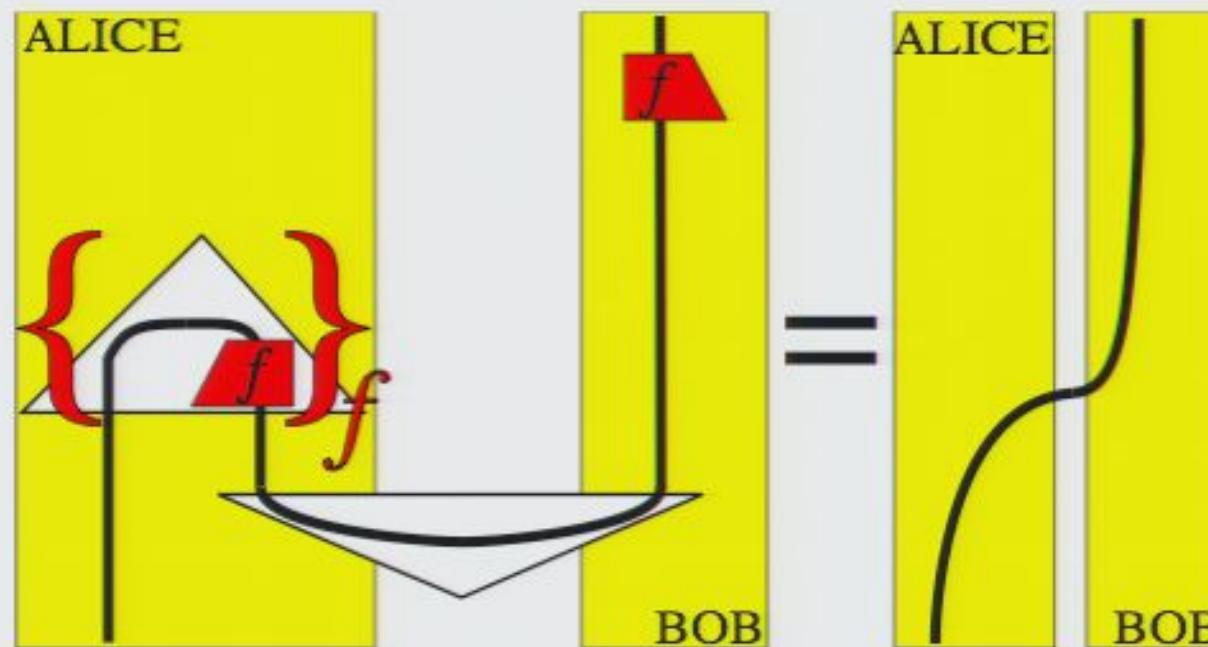
FHilb :

$$\eta_{\mathcal{H}} : \mathbb{C} \rightarrow \mathcal{H} \otimes \mathcal{H} :: 1 \mapsto \sum_i |ii\rangle$$

Rel :

$$\eta_X := \{(*, (x, x)) | x \in X\}$$

Which effects?



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Other results in this language:

CP maps and Choi-Jamiolkowski – *Selinger 2005*.

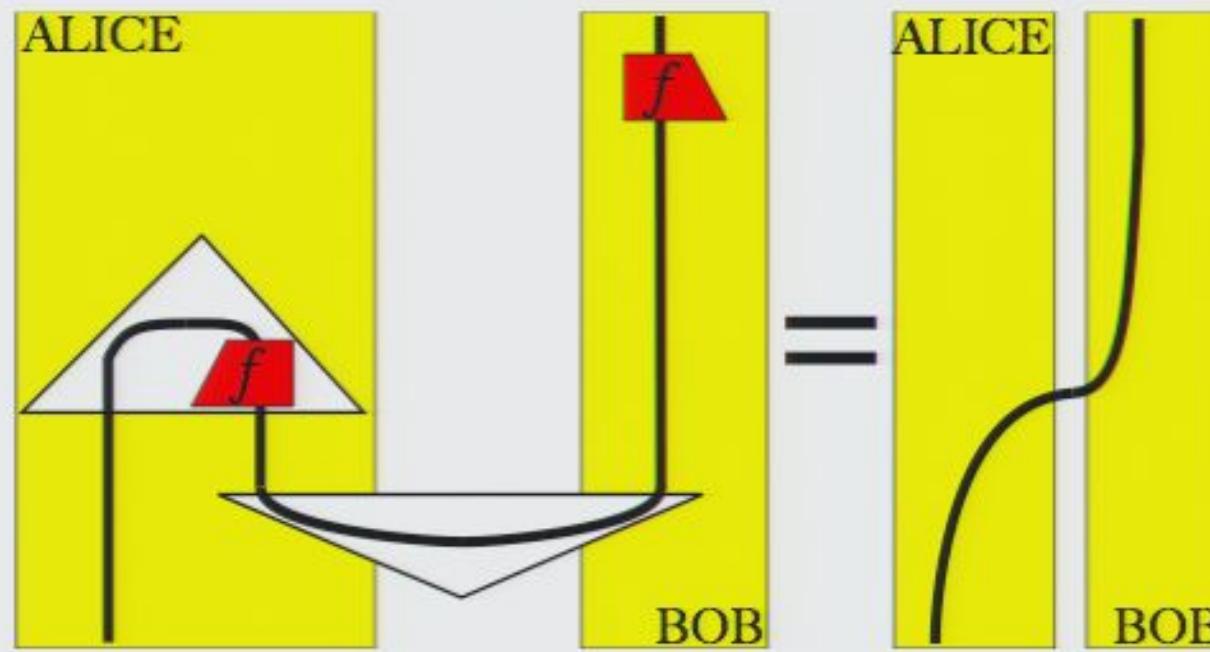
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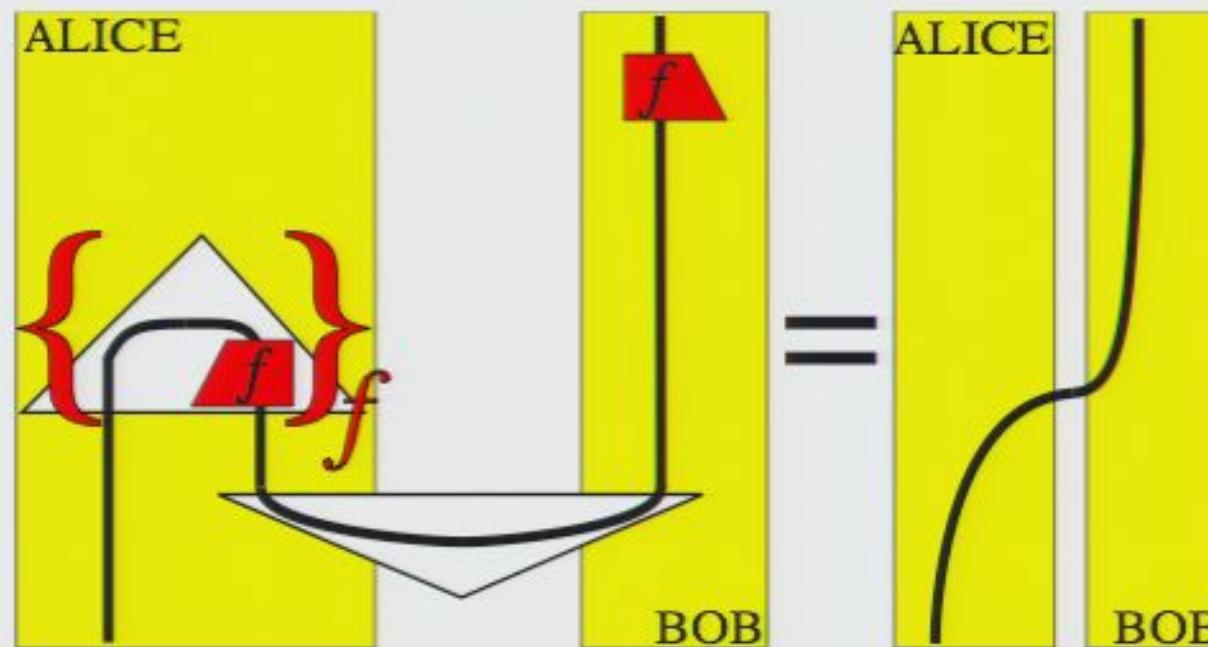
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QUANTUM STRUCTURE I: OBSERVABLES

Coecke, Pavlovic, Vicary 2006

— *observable / classical* := *copying + deleting* —

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$$A \xrightarrow{\delta} A \otimes A = \text{█} \quad A \xrightarrow{\epsilon} I = \text{█}$$

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$$A \xrightarrow{\delta} A \otimes A = \text{Diagram of a cup-like shape with two legs meeting at a central dot.} \quad A \xrightarrow{\epsilon} I = \text{Diagram of a keyhole shape with one leg ending in a circle.}$$

such that:

$$\begin{array}{ccc} \text{Diagram of a cup with a vertical line segment between legs, labeled } (1) & \text{Diagram of a circle with a dot, labeled } (4) & \text{Diagram of two separate U-shaped legs, labeled } (2) \\ \text{Diagram of a circle with a dot, labeled } (3) & \text{Diagram of two legs joined at their bases, labeled } (5) & \end{array}$$

⇒ it is a *commutative special \dagger -Frobenius algebra*.

— *observable / classical* := *copying + deleting* —

$$A \xrightarrow{\delta} A \otimes A = \text{Diagram of a cup-and-cap tensor product symbol} \quad A \xrightarrow{\epsilon} I = \text{Diagram of a single point}$$

such that:

$$\begin{array}{ccc} \text{Diagram 1: } & \text{Diagram 2: } & \\ \text{Diagram 3: } & \text{Diagram 4: } & \text{Diagram 5: } \\ \text{Diagram 6: } & & \end{array}$$

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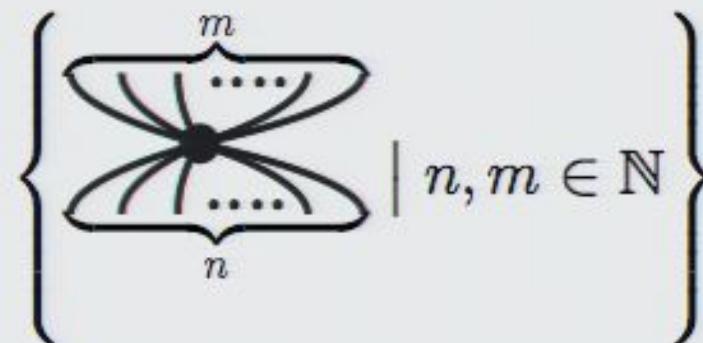
The diagram consists of five string relations labeled (1) through (5).
Relation (1): A cup symbol is equal to a vertical line segment, which is itself equal to an empty circle.
Relation (2): Two cups joined at their bases are equal to a single cup with two strands entering from the bottom.
Relation (3): A cap symbol is equal to an empty circle.
Relation (4): A vertical line segment is equal to a cap symbol.
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Diagrammatic calculus with ‘spiders’:



which are such that:

A diagrammatic equation showing the distributive property of spiders. On the left, a large spider with m legs at the top and n legs at the bottom is shown. It has a central body with two smaller spiders attached by dashed lines. The top spider has m legs at the top and n legs at the bottom. The bottom spider has n legs at the top and m legs at the bottom. A horizontal line connects the two spiders. An equals sign follows, and on the right is a single spider with m legs at the top and n legs at the bottom.

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$$A \xrightarrow{\delta} A \otimes A = \text{Diagram of a coproduct (Y-shaped junction)} \quad A \xrightarrow{\epsilon} I = \text{Diagram of an identity element (vertical line with dot)}$$

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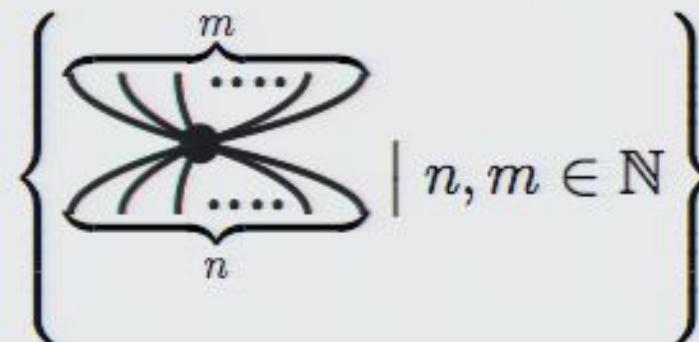
$$\begin{array}{ccc} \text{Diagram 1: } \cup & \stackrel{(1)}{=} & \text{Diagram 2: } | \\ & \stackrel{(4)}{=} & \text{Diagram 3: } \circ \\ \text{Diagram 4: } \circ & & \text{Diagram 5: } \cup \\ & \stackrel{(2)}{=} & \text{Diagram 6: } \cup \\ & & \text{Diagram 7: } \cup \\ \text{Diagram 8: } \circ & \stackrel{(3)}{=} & \text{Diagram 9: } \cup \\ & & \text{Diagram 10: } \cup \\ & & \stackrel{(5)}{=} \text{Diagram 11: } \cup \end{array}$$

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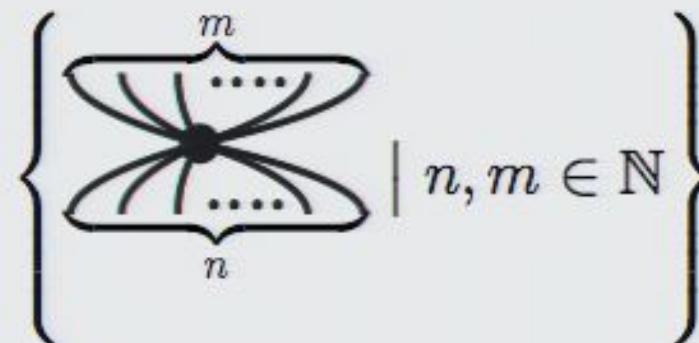
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which are such that:

$$\text{Diagram with legs } m \text{ and } n \text{ in a dashed frame} = \text{Diagram with legs } m \text{ and } n \text{ in a dashed frame}$$

Example categories:

FHilb :

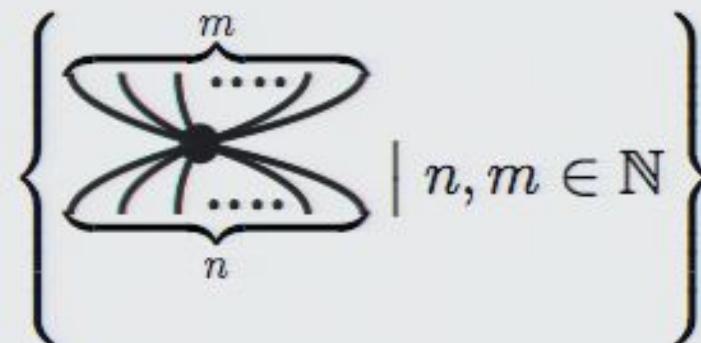
$$\begin{cases} \delta_{\mathcal{H}} : \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H} :: |i\rangle \mapsto |ii\rangle \\ \epsilon_{\mathcal{H}} : \mathcal{H} \rightarrow \mathbb{C} :: |i\rangle \mapsto 1 \end{cases}$$

Rel :

$$\begin{cases} \delta_X := \{(x, (x, x)) | x \in X\} \\ \epsilon_X := \{(x, *) | x \in X\} \end{cases}$$

— *observable / classical* := *copying + deleting* —

Diagrammatic calculus with ‘spiders’:



which are such that:

A diagrammatic equation showing the equivalence of two configurations of spiders. On the left, a red rectangle encloses a complex web of black lines forming a spider with m legs at the top and n legs at the bottom. A red line connects the center of the web to the center of a second, smaller spider below it. On the right, a red rectangle encloses a simplified configuration where the red line from the top spider connects directly to the center of the bottom spider, which has a single red line extending downwards.

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QUANTUM STRUCTURE II: COMPLEMENTARITY

Coecke-Duncan 2008

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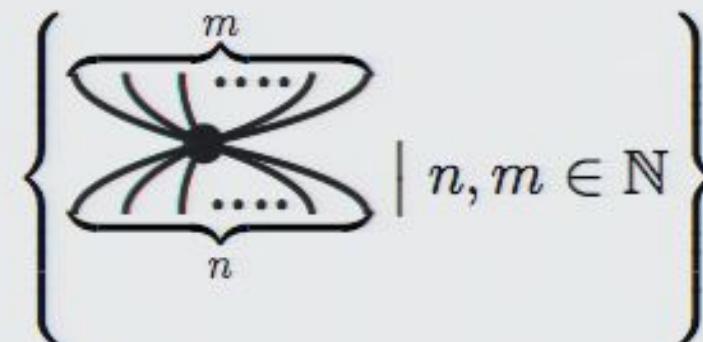
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Diagrammatic calculus with ‘spiders’:



which are such that:

A diagrammatic equation showing the equivalence of two configurations of a spider web. On the left, a red rectangle encloses a spider web with m strands and n legs. A red line connects the top and bottom vertices of the rectangle. On the right, a red rectangle encloses a spider web with m strands and n legs, where the strands are now concentrated into a single vertical strand passing through the center of the rectangle.

$$\text{Diagrammatic equation showing the equivalence of two configurations of a spider web.}$$

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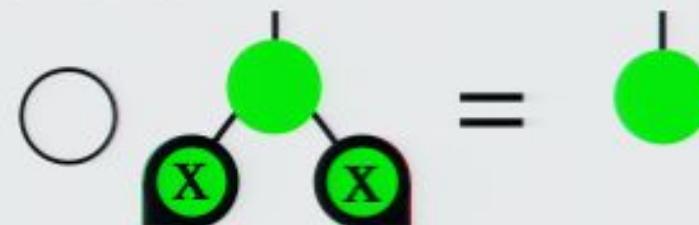
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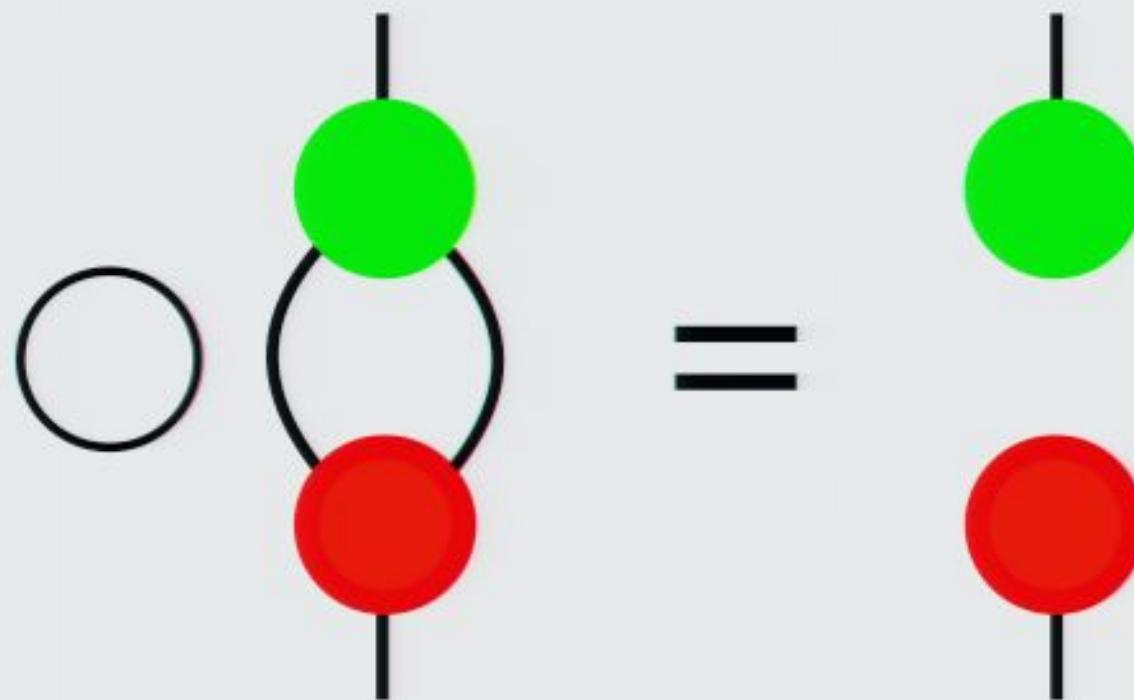
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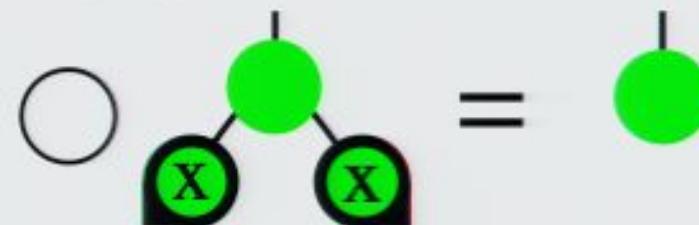
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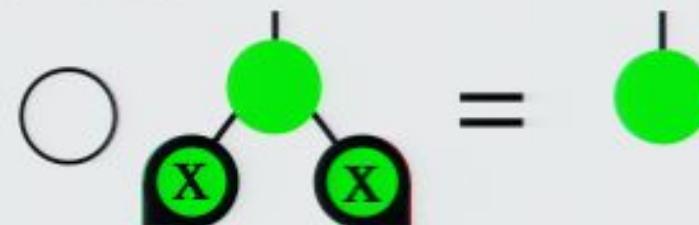


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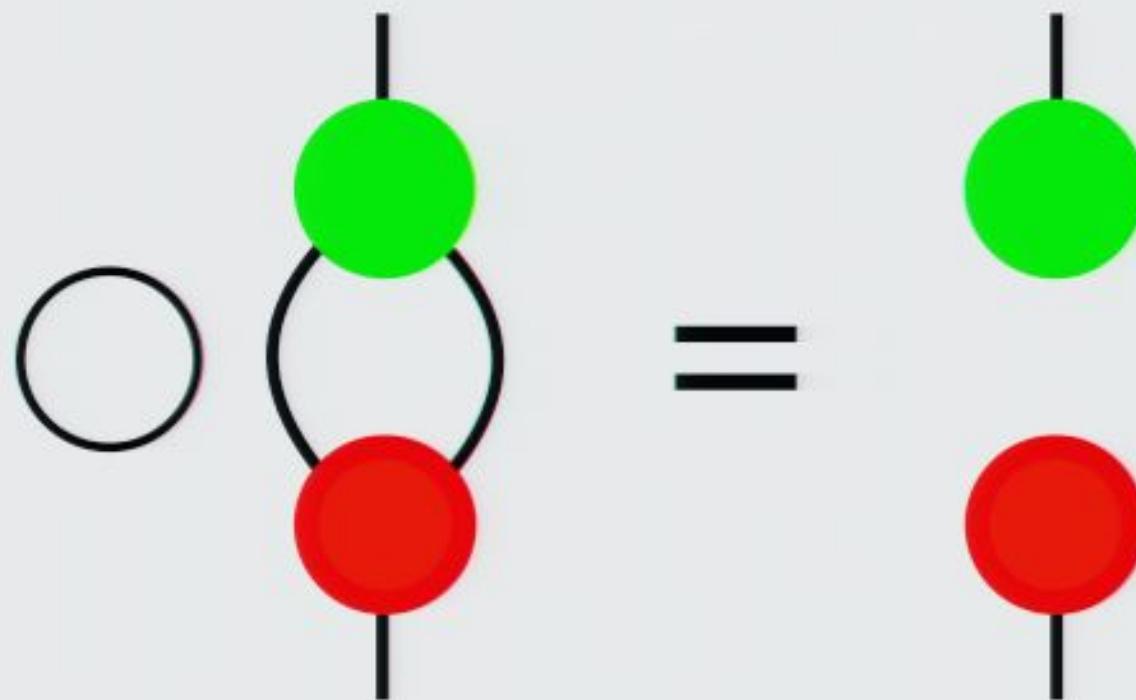
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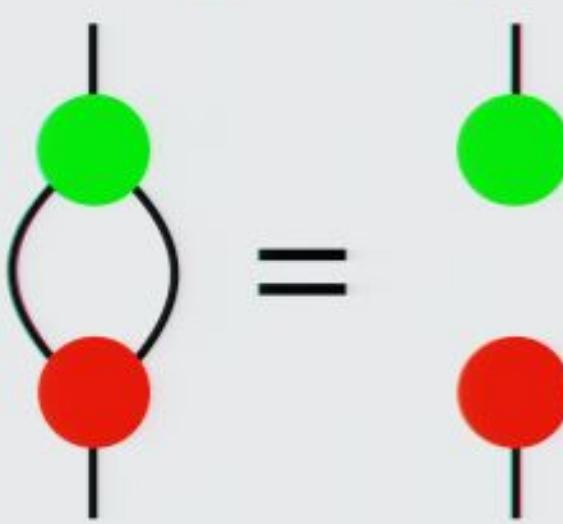
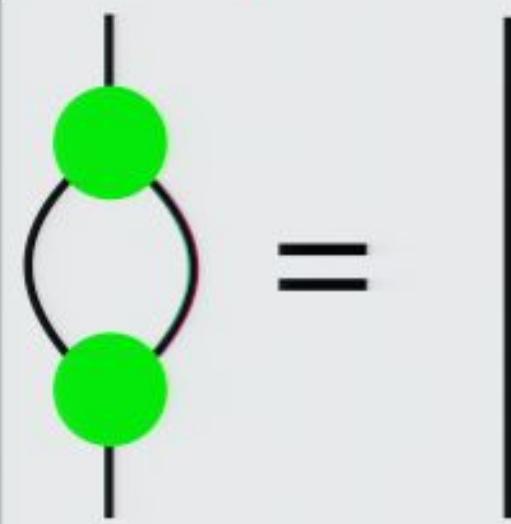


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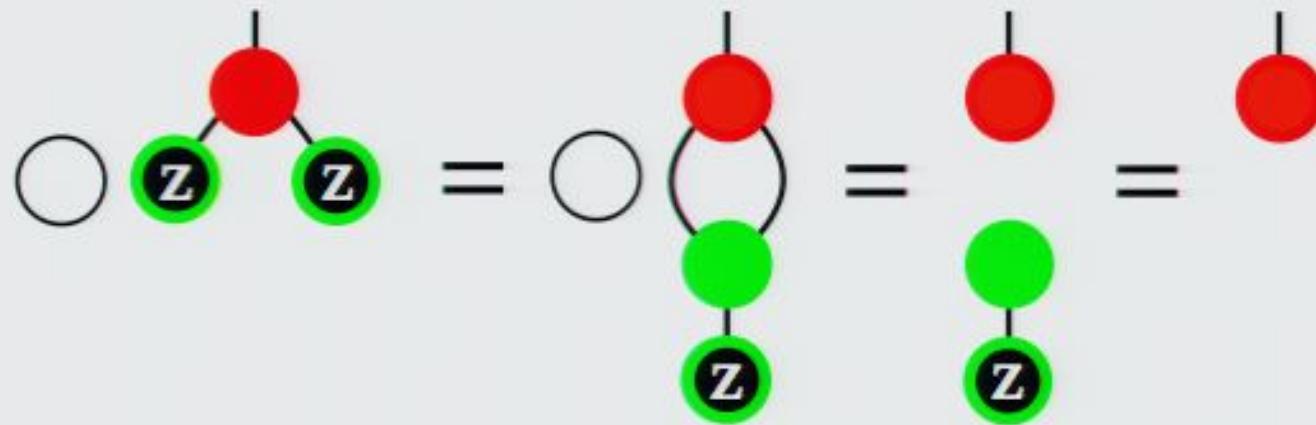


compatible

complementary



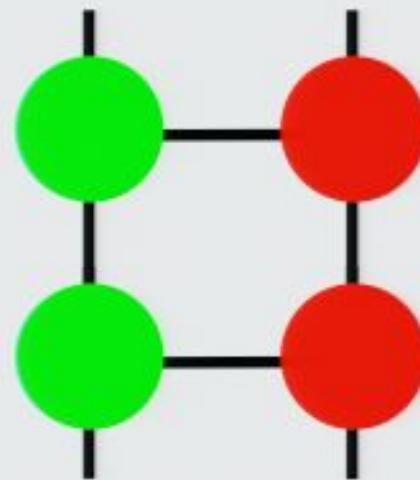
Proof. Hopf law \Rightarrow [class \Rightarrow unbiased]



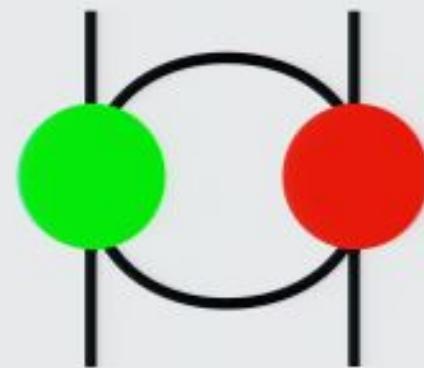
— two CX gates —

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \circ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = ?$$

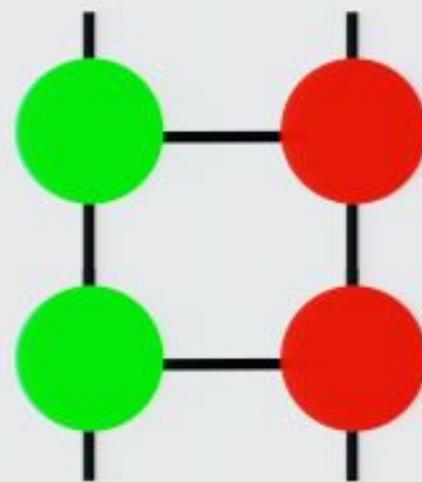
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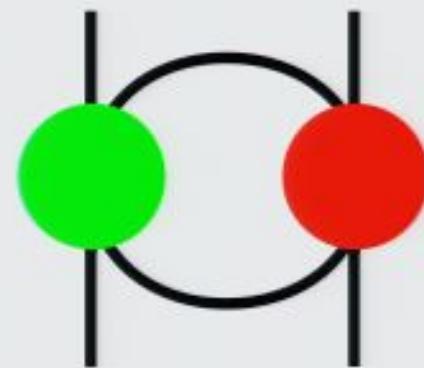
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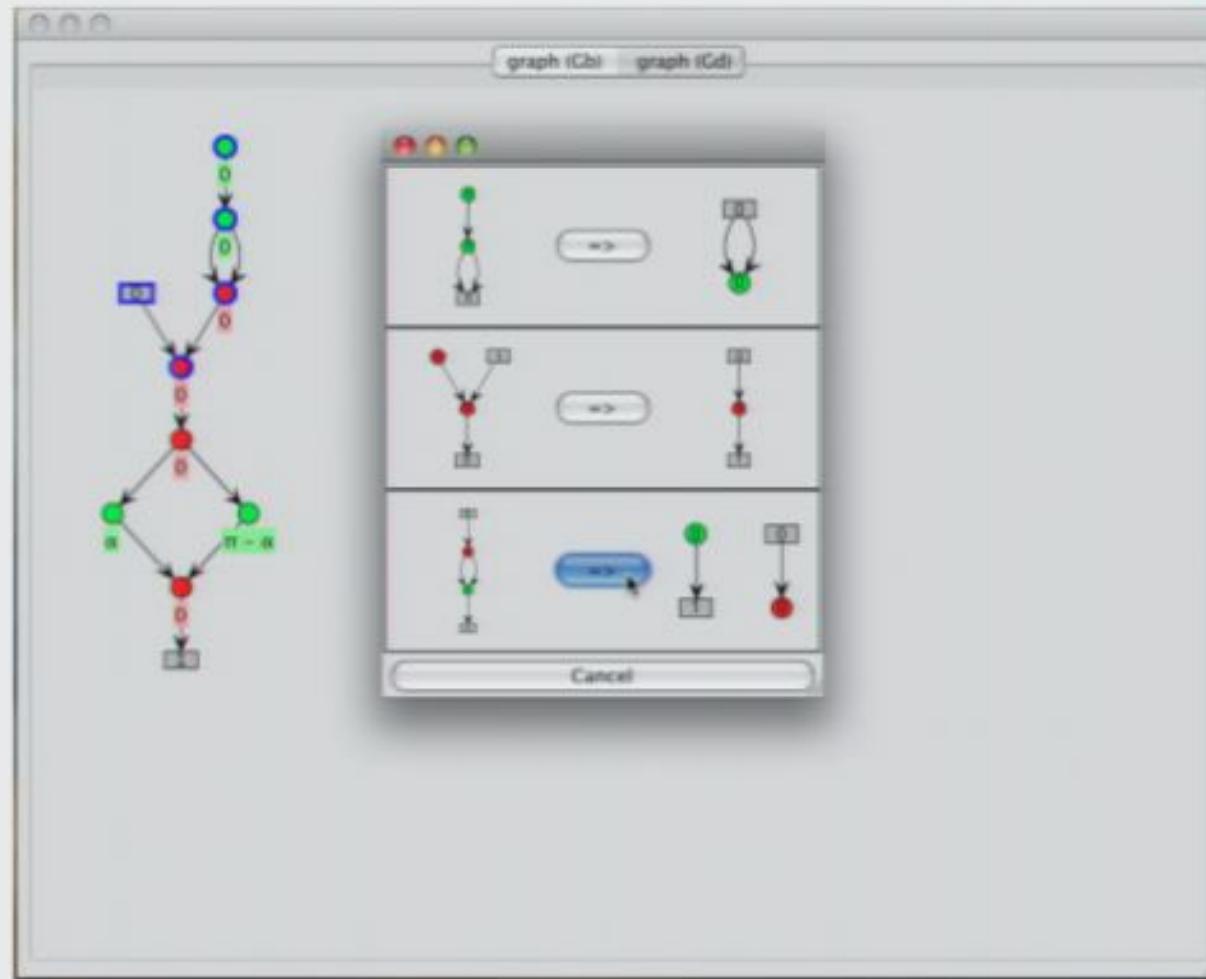
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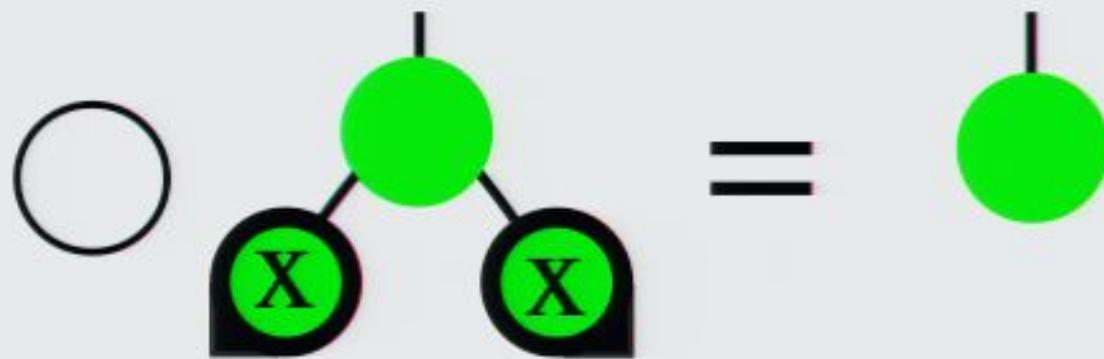
quantomatic – Dixon / Duncan / Kissinger



PHASES

Coecke-Duncan 2008

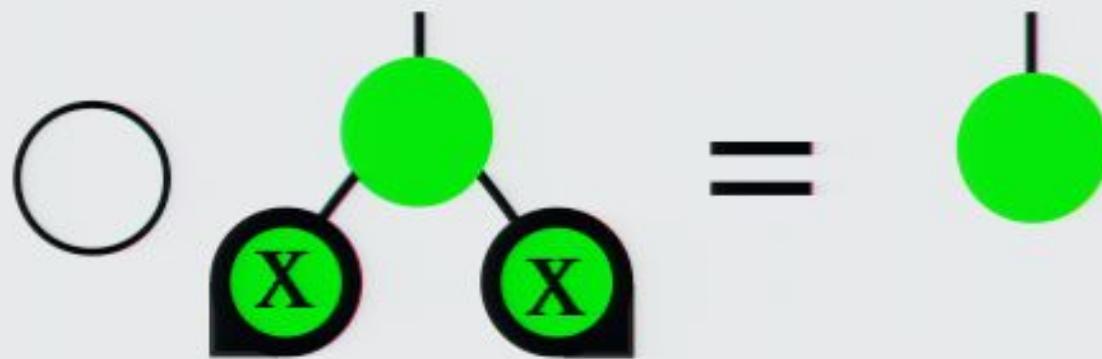
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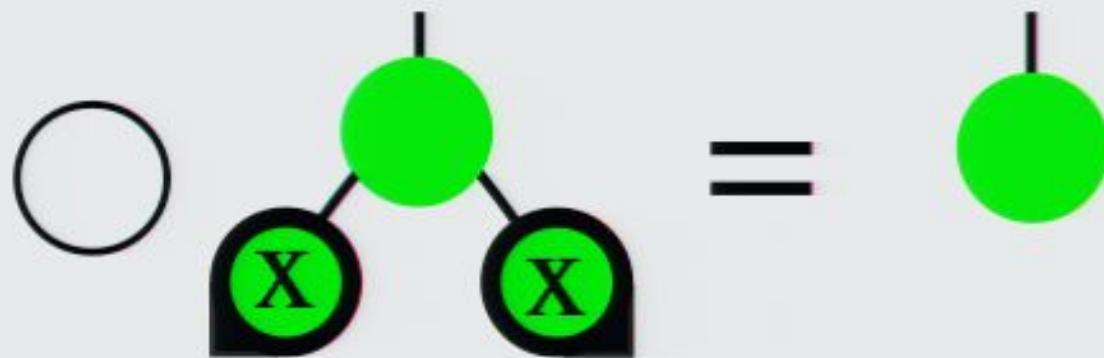
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$$\left\{ \begin{array}{c} \text{Diagram of a butterfly-shaped network with a green dot labeled } \alpha \text{ at the center, connected to } m \text{ strands above and } n \text{ strands below.} \\ | \quad n, m \in \mathbb{N}_0, \alpha \in G \end{array} \right\}$$

invariant under flipping and swapping, and such that:

$$\begin{array}{ccc} \text{Diagram showing two butterfly networks side-by-side, each with a green dot labeled } \alpha \text{ and } \beta \text{ respectively, with strands labeled } m \text{ and } n. & = & \text{Diagram showing a single butterfly network with a green dot labeled } \alpha + \beta \text{ at the center, with strands labeled } m \text{ and } n. \end{array}$$

For qubits in **FHilb** with $\text{green} \equiv \{|0\rangle, |1\rangle\} \equiv Z$:

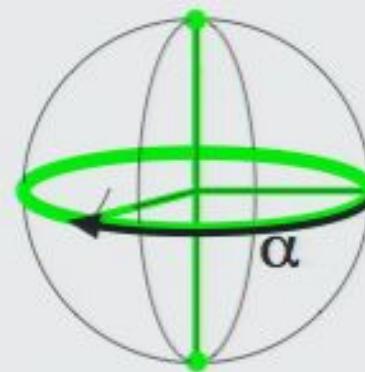
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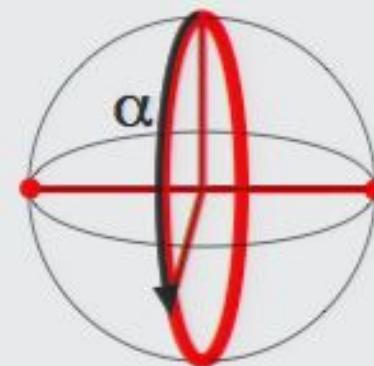
These are relative phases for Z , hence in X - Y :



For qubits in **FHilb** with $\text{red} \equiv \{|+\rangle, |-\rangle\} \equiv X$:

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Example category FHilb_2

Thm. Every linear map in FHilb_2 can be expressed in the language of a pair of complementary observables and the corresponding phases, that is, it can be written down using only red and green decorated spiders.

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Example category \mathbf{FRel}_2

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⇒ **genuine complementarity**

Coecke & Edwards. Toy quantum categories. arXiv:0808.1037

THE ORIGIN NON-LOCALITY

Coecke, Edwards, Spekkens 2009

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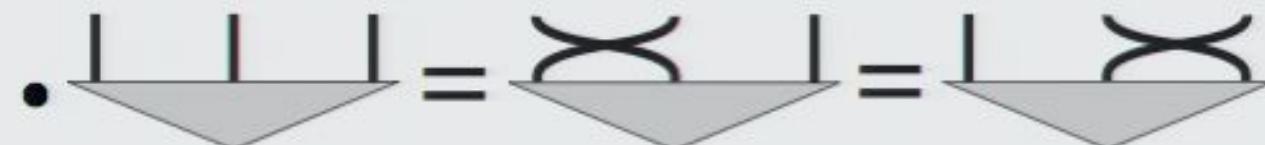
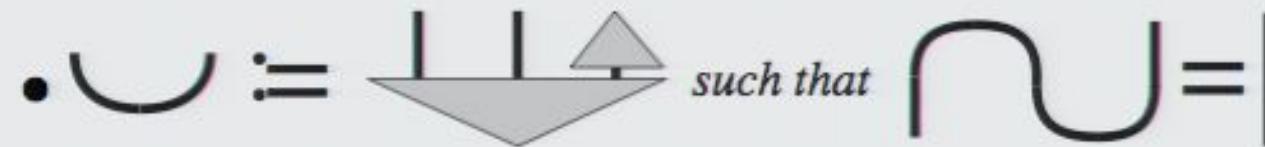
This tells us what it is not, ... but what IS it?

— non-locality a la GHZ —

Defn. A GHZ-state is a triple

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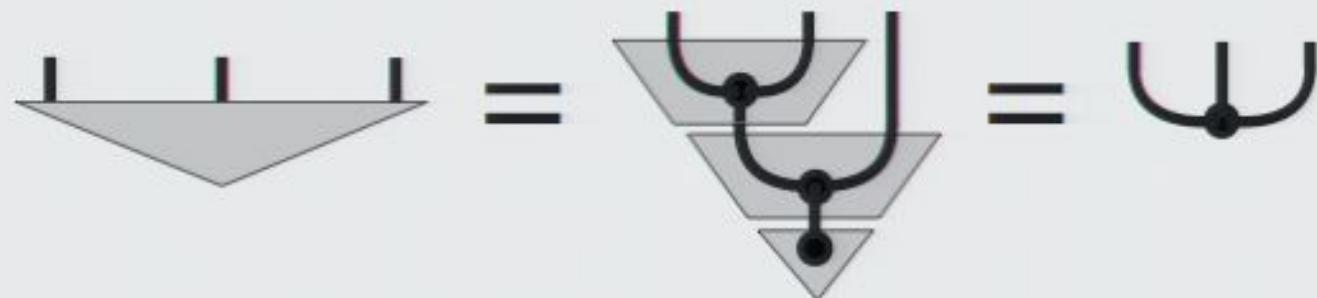
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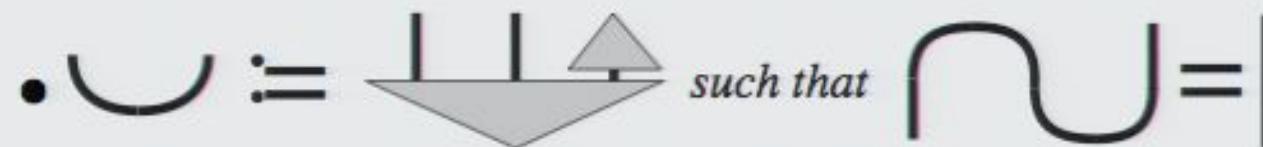
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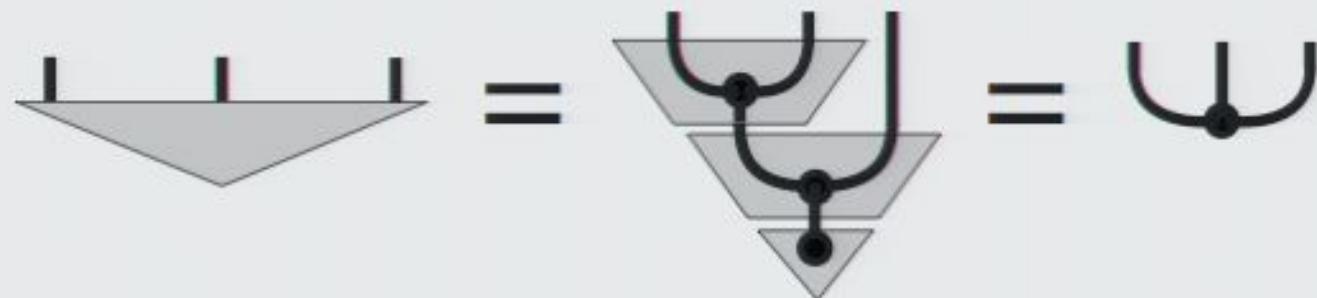
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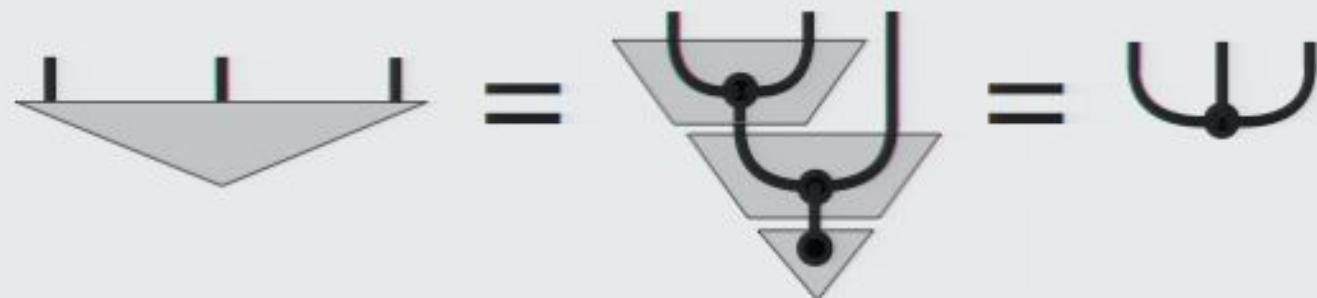
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— *non-locality a la GHZ* —

Dfn. GHZ-correlations:

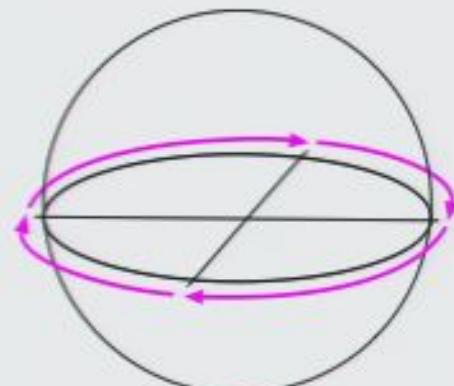
$$\begin{array}{c} x \\ \nearrow \\ y \end{array} \cup = \begin{array}{c} \nearrow \\ x \\ \cup \\ y \end{array}$$

Dfn. Mutually unbiased (sub-)theory: states are either eigenstates or unbiased states for observables.

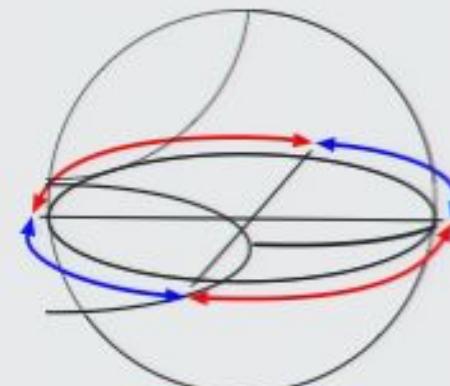
Thm. In mutually unbiased theories GHZ-correlations are completely determined by the phase group.

⇒ **Abelian groups classify non-local behaviours!**

Recall that there are only two four-element groups:

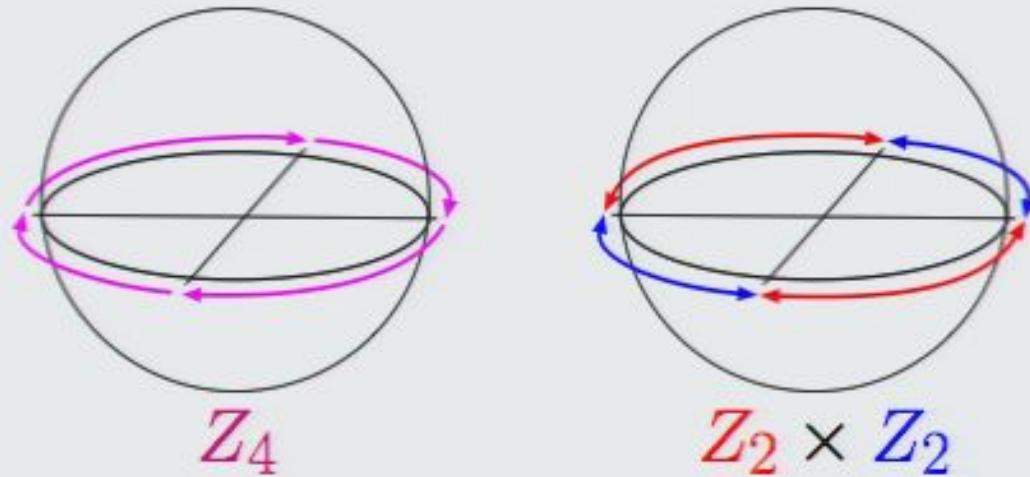


$$\mathbb{Z}_4$$



$$\mathbb{Z}_2 \times \mathbb{Z}_2$$

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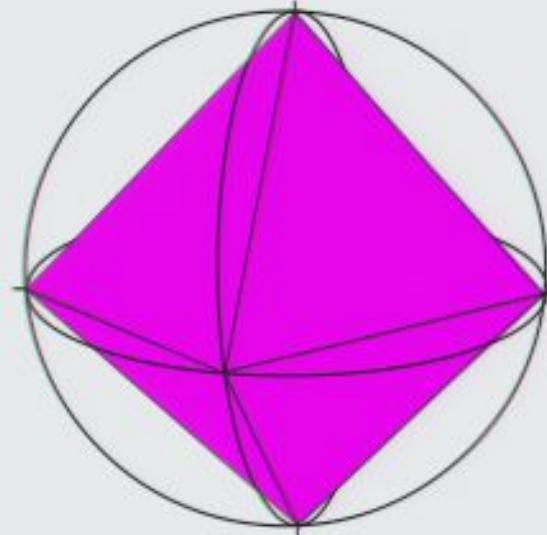


These correspond to:

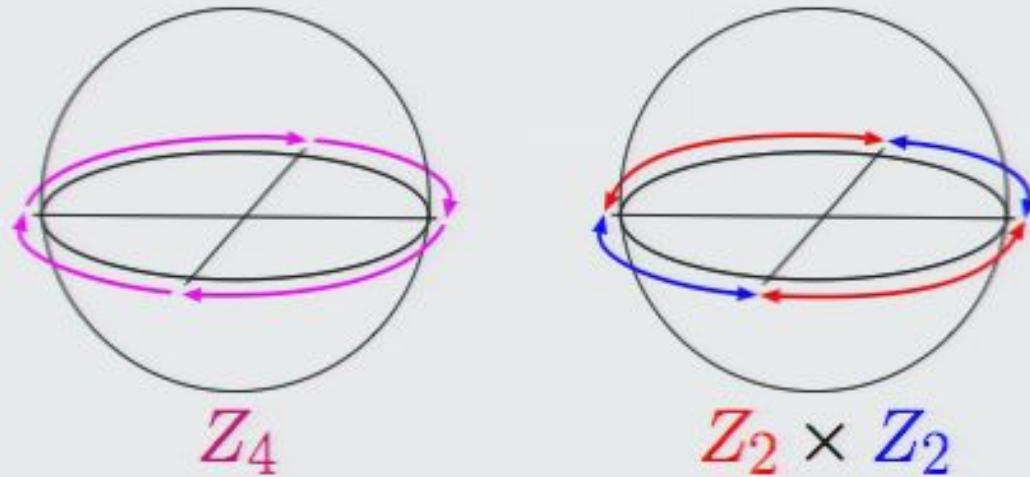
- $Z_4 \sim \text{qubit stabiliser QM} \sim \text{non-local}$
- $Z_2 \times Z_2 \sim \text{Spekkens' toy theory} \sim \text{local}$

Stab := sub- \dagger -SMC of **FHilb** generated by

- n th powers of qubits \mathcal{Q}
- unitaries on $\mathcal{Q} \cap$ symmetries Bloch-octahedron
- $\mathcal{Q} \rightarrow \mathcal{Q} \otimes \mathcal{Q} :: \begin{cases} |0\rangle \mapsto |00\rangle \\ |1\rangle \mapsto |11\rangle \end{cases} + \text{its unit}$



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-

Spek := sub- \dagger -SMC of **FRel** generated by

- *n*th powers of quads IV
- all permutations on IV
- $IV \rightarrow IV \times IV :: \begin{cases} 1 \mapsto \{(1, 1), (2, 2)\} \\ 2 \mapsto \{(1, 2), (2, 1)\} \\ 3 \mapsto \{(3, 3), (4, 4)\} \\ 4 \mapsto \{(3, 4), (4, 3)\} \end{cases} + \text{its unit}$

‘Z-, X- and Y-spin’ in Spek:

$$\text{IV} \rightarrow \text{IV} \times \text{IV} :: \begin{cases} 1 \mapsto \{(1, 1), (2, 2)\} \\ 2 \mapsto \{(1, 2), (2, 1)\} \\ 3 \mapsto \{(3, 3), (4, 4)\} \\ 4 \mapsto \{(3, 4), (4, 3)\} \end{cases} + \textit{its unit}$$

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Both Stab and Spek are generated by:

- *n*th powers of elementary system A
- group $G \simeq S(4)$ of isos on A
- ‘copying’ $\delta : A \rightarrow A \otimes A$ + its unit

Only difference is the interaction between copying and local unitaries. These are encoded in the phase group.

GHZ-correlations:



i.e. state of the third system given the outcomes of measurements on the first two systems, hence:

GHZ-correlations \equiv group multiplication (*)

Claim: The Z_4 phase group structure, via (*), suffices for an argument forbidding local hidden variables.

For $|+\rangle$ the unit and $|-\rangle$ the involutive we have:

$$|+\rangle \odot |+\rangle = |+\rangle \quad |+\rangle \odot |-\rangle = |-\rangle \quad |-\rangle \odot |-\rangle = |+\rangle$$

i.e. even occurrences of $|-\rangle$ in correlations.

For remaining elements $|=\rangle$ and $|\sharp\rangle$ we have:

$$|\sharp\rangle \odot |=\rangle = |+\rangle \quad |=\rangle \odot |=\rangle = |-\rangle \quad |\sharp\rangle \odot |\sharp\rangle = |-\rangle$$

i.e. odd occurrences of $\{|-\rangle, |=\rangle\}$ in correlations.

$$\{|+\rangle, |-\rangle\} \times \{|+\rangle, |-\rangle\} \times \{|+\rangle, |-\rangle\}$$

$$\{|+\rangle, |-\rangle\} \times \{| \# \rangle, | = \rangle\} \times \{| \# \rangle, | = \rangle\}$$

$$\{| \# \rangle, | = \rangle\} \times \{|+\rangle, |-\rangle\} \times \{| \# \rangle, | = \rangle\}$$

$$\{| \# \rangle, | = \rangle\} \times \{| \# \rangle, | = \rangle\} \times \{|+\rangle, |-\rangle\}$$

Diagrammatic QM introductions:

Appetizer: *Kindergarten Quantum Mechanics*. arXiv:quant-ph/0510032.

Survey: *Quantum picturalism*. arXiv:0908.1787. (New)

Categories for physicists:

Appetizer: *Introducing categories to the practicing physicist*. arXiv:0808.1032.

Tutorial: *Categories for the practicing physicist*. arXiv:0905.3010. (New)

Some technical papers:

C., Edwards and Spekkens: *The group theoretic origin of non-locality for qubits*. web.comlab.ox.ac.uk/publications/publication3026-abstract.html.

C. & Duncan: *Interacting quantum observables*. arXiv:0906.4725.

C., Paquette & Pavlovic: *Classical and quantum structuralism*. arXiv:0904.1997.