Title: Interaction axiomatics for quantum phenomena.

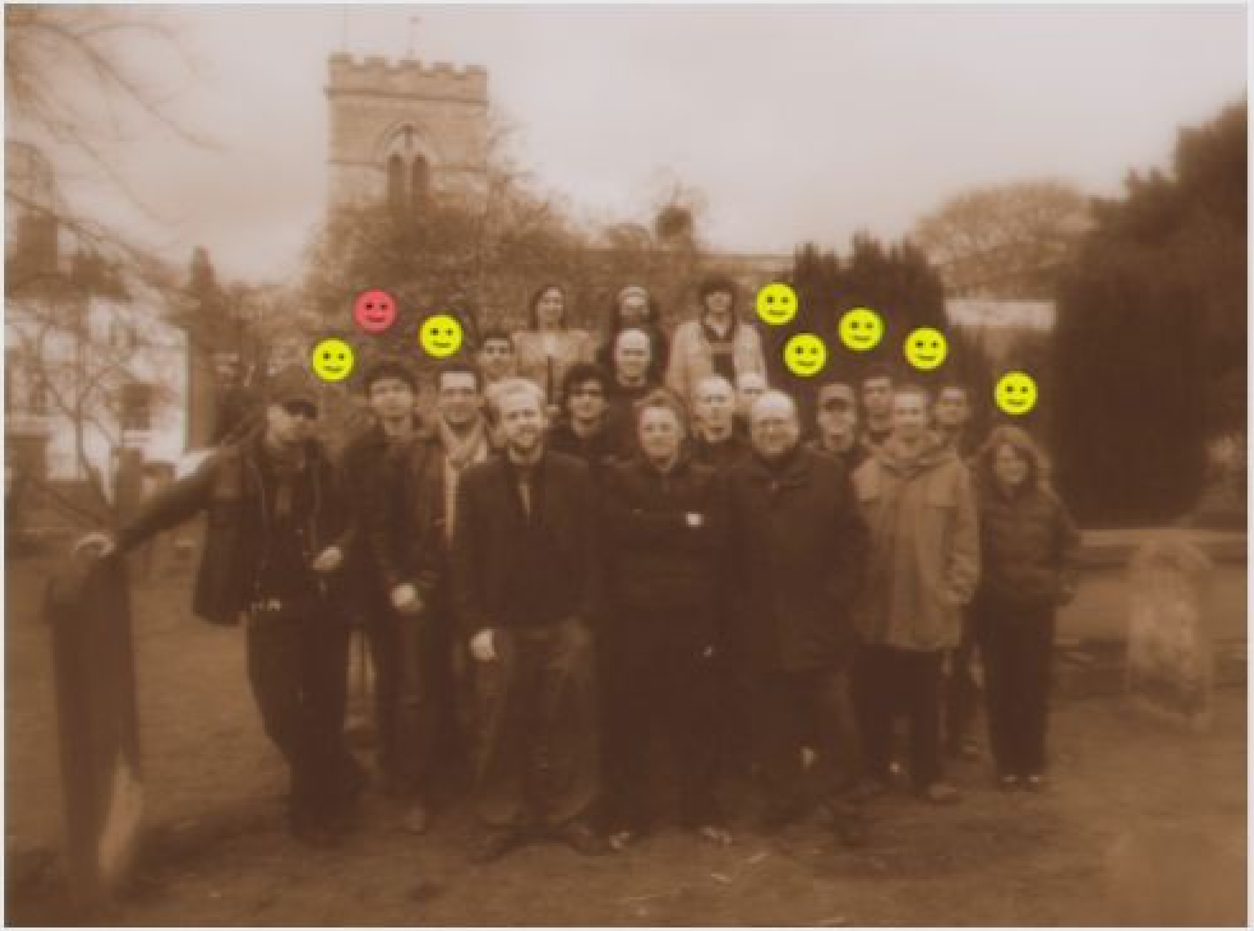
Date: Aug 13, 2009 04:30 PM

URL: http://pirsa.org/09080013

Abstract: In our approach, rather than aiming to recover the 'Hilbert space model' which underpins the orthodox quantum mechanical formalism, we start from a general `pre-operational' framework, and verify how much additional structure we need to be able to describe a range of quantum phenomena. This also enables us to investigate which mathematical models, including more abstract categorical ones, enable one to model quantum theory. Till now, all of our axioms only refer to the particular nature of how compound quantum systems interact, rather that to the particular structure of state-spaces. This is in sharp contrast with other approaches of this kind which aim to recover quantum theory out of a much broader class of theories. A more abstract quantum mechanical model has other many advantages. It elucidates which are the key ingredients that make `the Hilbert space model' work. Since it relies on monoidal categories, it comes with a high-level diagrammatic description (which we think of as `the' mathematical formalism). It moreover removes the dependency on continuous underlying mathematical structures, paving the way for discrete combinatorial models, which might blend better with the other ingredients required for a theory of quantum gravity.

Interaction axiomatics for quantum phenomena

illustrated on Stab*(elizer qm) vs.* Spek*(kens toys)*



strict dress code and diet

Roman Priebe Andrey Akhvlediani Bill Edwards

Not: Recover the 'Hilbert space model' QM

Not: Recover the ‘Hilbert space model’ QM

But: Within ‘pre-operational’ framework, what extra structure is needed to predict qm phenomena?

Not: Recover the 'Hilbert space model' QM

But: Within 'pre-operational' framework, what extra structure is needed to predict qm phenomena? Here

$$\text{structure} \equiv \text{physical capability}$$

Not: Recover the 'Hilbert space model' QM

But: Within 'pre-operational' framework, what extra structure is needed to predict qm phenomena? Here

$$\text{structure} \equiv \text{physical capability}$$

So far: Only interaction (cf. $\otimes$) is relevant (i.e. $\perp$ QL).

Not: Recover the 'Hilbert space model' QM

But: Within 'pre-operational' framework, what extra structure is needed to predict qm phenomena? Here

$$\text{structure} \equiv \text{physical capability}$$

So far: Only interaction (cf. $\otimes$) is relevant (i.e. $\perp$ QL).

Then: Which concrete models are admissible?

Not: Recover the 'Hilbert space model' QM

But: Within 'pre-operational' framework, what extra structure is needed to predict qm phenomena? Here

$$\text{structure} \equiv \text{physical capability}$$

So far: Only interaction (cf. $\otimes$) is relevant (i.e. $\perp$ QL).

Then: Which concrete models are admissible?

Otherwise put:

- Which **FdHilb**-ingredients (cf. LEGO) yield QP?
- Which other **C**'s do this too? ... $\Longrightarrow$ QG

Not: Recover the 'Hilbert space model' QM

But: Within 'pre-operational' framework, what extra structure is needed to predict qm phenomena? Here

$$\text{structure} \equiv \text{physical capability}$$

So far: Only interaction (cf. $\otimes$) is relevant (i.e. $\perp$ QL).

Then: Which concrete models are admissible?

Otherwise put:

- Which **FdHilb**-ingredients (cf. LEGO) yield QP?
- Which other **C**'s do this too? ... $\Longrightarrow$ QG

Today: Hilbert space model *contra* sets and relations.

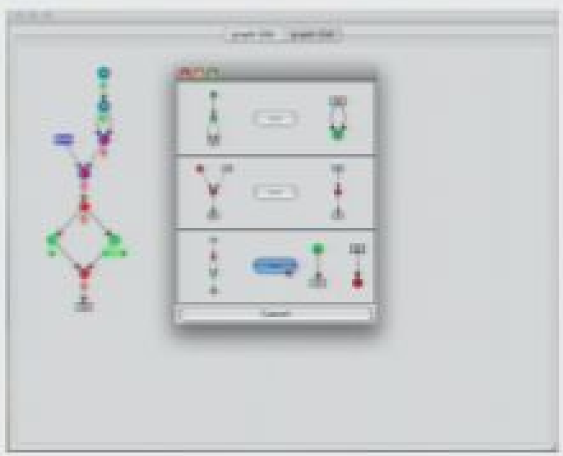
$\pounds s$ €s $\$s$

EPSRC
Engineering and Physical Sciences Research Council

QICS

The Leverhulme Trust

Product: automated q-reasoning software `quantomatic`



—mathematics for compositional situations —

Not: Recover the 'Hilbert space model' QM

But: Within 'pre-operational' framework, what extra <u>structure</u> is needed to predict qm phenomena? Here

$$\text{structure} \equiv \text{physical capability}$$

So far: Only interaction (cf. $\otimes$) is relevant (i.e. $\perp$ QL).

Then: Which concrete models are admissible?

Otherwise put:

- Which **FdHilb**-ingredients (cf. LEGO) yield QP?
- Which other **C**'s do this too? ... $\Longrightarrow$ QG

Today: Hilbert space model *contra* sets and relations.

£s €s $s

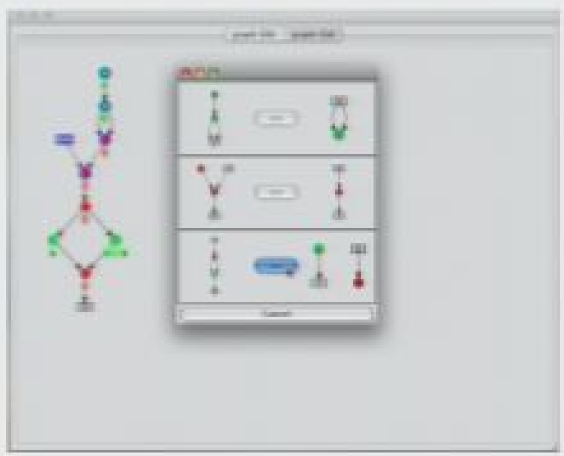
EPSRC
Engineering and Physical Sciences Research Council

QICS

The Leverhulme Trust

fq(x)

Product: automated q-reasoning software `quantomatic`



—mathematics for compositional situations —

—*mathematics for compositional situations* —

1. Let A be a raw potato.

—*mathematics for compositional situations* —

1. Let A be a raw potato.

A admits many *states* e.g. dirty, clean, skinned, ...

—*mathematics for compositional situations* —

1. Let A be a raw potato.

A admits many *states* e.g. dirty, clean, skinned, ...

2. We want to *process* A into cooked potato B.

B admits many *states* e.g. boiled, fried, deep fried, baked with skin, baked without skin, ...

—mathematics for compositional situations —

1. Let A be a raw potato.

A admits many *states* e.g. dirty, clean, skinned, ...

2. We want to *process* A into cooked potato B.

B admits many *states* e.g. boiled, fried, deep fried, baked with skin, baked without skin, ... Let

$$A \xrightarrow{f} B \qquad A \xrightarrow{f'} B \qquad A \xrightarrow{f''} B$$

be boiling, frying, baking.

—mathematics for compositional situations —

1. Let A be a raw potato.

A admits many *states* e.g. dirty, clean, skinned, ...

2. We want to *process* A into cooked potato B.

B admits many *states* e.g. boiled, fried, deep fried, baked with skin, baked without skin, ... Let

$$A \xrightarrow{f} B \qquad A \xrightarrow{f'} B \qquad A \xrightarrow{f''} B$$

be boiling, frying, baking. *States* are *processes*

$$\mathrm{I} := \text{unspecified} \xrightarrow{\psi} A.$$

3. Let

$$A \xrightarrow{g \circ f} C$$

be the *composite process* of first boiling $A \xrightarrow{f} B$ and then salting $B \xrightarrow{g} C$.

—mathematics for compositional situations —

1. Let A be a raw potato.

A admits many *states* e.g. dirty, clean, skinned, ...

2. We want to *process* A into cooked potato B.

B admits many *states* e.g. boiled, fried, deep fried, baked with skin, baked without skin, ... Let

$$A \xrightarrow{f} B \qquad A \xrightarrow{f'} B \qquad A \xrightarrow{f''} B$$

be boiling, frying, baking. *States* are *processes*

$$\mathrm{I} := \text{unspecified} \xrightarrow{\psi} A.$$

3. Let

$$A \xrightarrow{g \circ f} C$$

be the *composite process* of first boiling $A \xrightarrow{f} B$ and then salting $B \xrightarrow{g} C$.

3. Let

$$A \xrightarrow{g \circ f} C$$

be the *composite process* of first boiling $A \xrightarrow{f} B$ and then salting $B \xrightarrow{g} C$. Let

$$X \xrightarrow{\mathbf{1}_X} X$$

be doing nothing. We have $\mathbf{1}_Y \circ \xi = \xi \circ \mathbf{1}_X = \xi$.

4. Let $A \otimes D$ be potato A and carrot D

4. Let $A \otimes D$ be potato A and carrot D and let

$$A \otimes D \xrightarrow{f \otimes h} B \otimes E$$

be boiling potato while frying carrot.

4. Let $A \otimes D$ be potato A and carrot D and let

$$A \otimes D \xrightarrow{f \otimes h} B \otimes E$$

be boiling potato while frying carrot. Let

$$C \otimes F \xrightarrow{x} M$$

be mashing spice-cook-potato and spice-cook-carrot.

5. Total *process*:

$$A\otimes D \xrightarrow{f\otimes h} B\otimes E \xrightarrow{g\otimes k} C\otimes F \xrightarrow{x} M = A\otimes D \xrightarrow{x\circ(g\otimes k)\circ(f\otimes h)} M.$$

5. Total *process*:

$$A \otimes D \xrightarrow{f \otimes h} B \otimes E \xrightarrow{g \otimes k} C \otimes F \xrightarrow{x} M = A \otimes D \xrightarrow{x \circ (g \otimes k) \circ (f \otimes h)} M.$$

6. *Recipe* = *composition structure* on *processes*.

5. Total *process*:

$$A \otimes D \xrightarrow{f \otimes h} B \otimes E \xrightarrow{g \otimes k} C \otimes F \xrightarrow{x} M = A \otimes D \xrightarrow{x \circ (g \otimes k) \circ (f \otimes h)} M.$$

6. *Recipe* = *composition structure* on *processes*.

7. *Law governing recipes*:

$$(\mathbf{1}_B \otimes g) \circ (f \otimes \mathbf{1}_C) = (f \otimes \mathbf{1}_D) \circ (\mathbf{1}_A \otimes g)$$

5. Total *process*:

$$A \otimes D \xrightarrow{f \otimes h} B \otimes E \xrightarrow{g \otimes k} C \otimes F \xrightarrow{x} M = A \otimes D \xrightarrow{x \circ (g \otimes k) \circ (f \otimes h)} M.$$

6. *Recipe* = *composition structure* on *processes*.

7. *Law governing recipes*:

$$(\mathbf{1}_B \otimes g) \circ (f \otimes \mathbf{1}_C) = (f \otimes \mathbf{1}_D) \circ (\mathbf{1}_A \otimes g)$$

i.e.

boil potato then fry carrot = fry carrot then boil potato

7. A more general law on recipes:

$$(g \circ f) \otimes (k \circ h) = (g \otimes k) \circ (f \otimes h)$$

i.e.

boil pot then salt pot, while, fry car then pepper car

||

boil pot while fry car, then, salt pot while pepper car

5. Total *process*:

$$A \otimes D \xrightarrow{f \otimes h} B \otimes E \xrightarrow{g \otimes k} C \otimes F \xrightarrow{x} M = A \otimes D \xrightarrow{x \circ (g \otimes k) \circ (f \otimes h)} M.$$

6. *Recipe* = *composition structure* on *processes*.

7. *Law governing recipes*:

$$(\mathbf{1}_B \otimes g) \circ (f \otimes \mathbf{1}_C) = (f \otimes \mathbf{1}_D) \circ (\mathbf{1}_A \otimes g)$$

i.e.

boil potato then fry carrot = fry carrot then boil potato

7. A more general law on recipes:

$$(g \circ f) \otimes (k \circ h) = (g \otimes k) \circ (f \otimes h)$$

i.e.

boil pot then salt pot, while, fry car then pepper car

||

boil pot while fry car, then, salt pot while pepper car

— (physical) data in monoidal category —

Systems:

$$A \qquad B \qquad C$$

Processes:

$$A \xrightarrow{f} A \qquad A \xrightarrow{g} B \qquad B \xrightarrow{h} C$$

Compound systems:

$$A \otimes B \qquad \mathrm{I} \qquad A \otimes C \xrightarrow{f \otimes g} B \otimes D$$

Temporal composition:

$$A \xrightarrow{h \circ g} C := A \xrightarrow{g} B \xrightarrow{h} C \qquad A \xrightarrow{1_A} A$$

7. A more general law on recipes:

$$(g \circ f) \otimes (k \circ h) = (g \otimes k) \circ (f \otimes h)$$

i.e.

boil pot then salt pot, while, fry car then pepper car

||

boil pot while fry car, then, salt pot while pepper car

— (physical) data in monoidal category —

Systems:

$$A \qquad B \qquad C$$

Processes:

$$A \xrightarrow{f} A \qquad A \xrightarrow{g} B \qquad B \xrightarrow{h} C$$

Compound systems:

$$A \otimes B \qquad \mathrm{I} \qquad A \otimes C \xrightarrow{f \otimes g} B \otimes D$$

Temporal composition:

$$A \xrightarrow{h \circ g} C := A \xrightarrow{g} B \xrightarrow{h} C \qquad A \xrightarrow{1_A} A$$

Thm. *The graphical calculus for symmetric monoidal categories is such that an equational statement between expressions in symmetric monoidal categorical language holds if and only if it is derivable in the graphical calculus*. [Joyal & Street '91]

— *(physical) data in monoidal category* —

Systems:

$$A \qquad B \qquad C$$

Processes:

$$A \xrightarrow{f} A \qquad A \xrightarrow{g} B \qquad B \xrightarrow{h} C$$

Compound systems:

$$A \otimes B \qquad \mathrm{I} \qquad A \otimes C \xrightarrow{f \otimes g} B \otimes D$$

Temporal composition:

$$A \xrightarrow{h \circ g} C \;:=\; A \xrightarrow{g} B \xrightarrow{h} C \qquad A \xrightarrow{1_A} A$$

Thm. *The graphical calculus for symmetric monoidal categories is such that an equational statement between expressions in symmetric monoidal categorical language holds if and only if it is derivable in the graphical calculus*. [Joyal & Street '91]

Thm. *The graphical calculus for symmetric monoidal categories is such that an equational statement between expressions in symmetric monoidal categorical language holds if and only if it is derivable in the graphical calculus*. [Joyal & Street '91]

Similar results hold for categories with more structured objects, for example, dagger compact categories and Frobenius objects therein.

Thm. *The graphical calculus for symmetric monoidal categories is such that an equational statement between expressions in symmetric monoidal categorical language holds if and only if it is derivable in the graphical calculus.* [Joyal & Street '91]

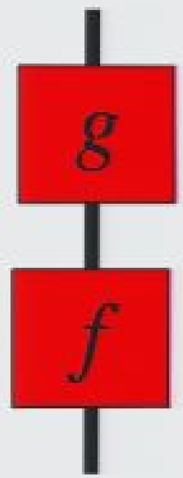
Similar results hold for categories with more structured objects, for example, dagger compact categories and Frobenius objects therein.

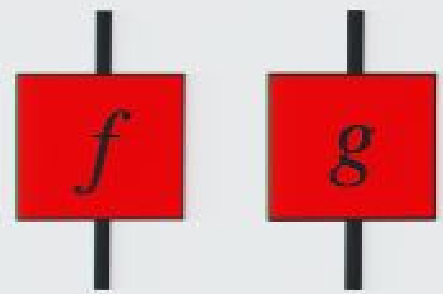
Are those pictures merely a different language or more?

— law from picture —

$f \equiv$ 

— law from picture —

$g \circ f \equiv$ 

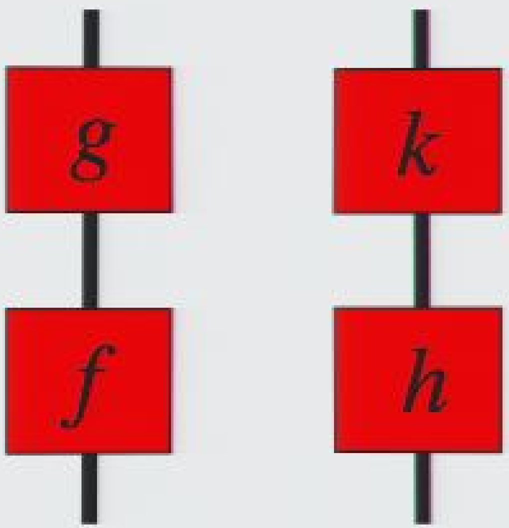
$f \otimes g \equiv$ 

— *law from picture* —

$$(g \circ f) \otimes (k \circ h) = (g \otimes k) \circ (f \otimes h)$$

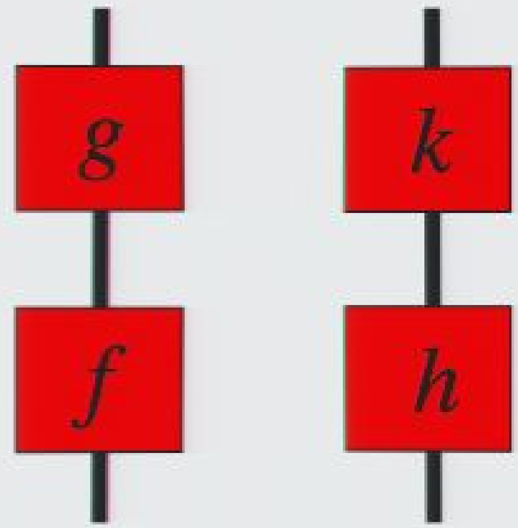
— law from picture —

$$(g \circ f) \otimes (k \circ h)$$



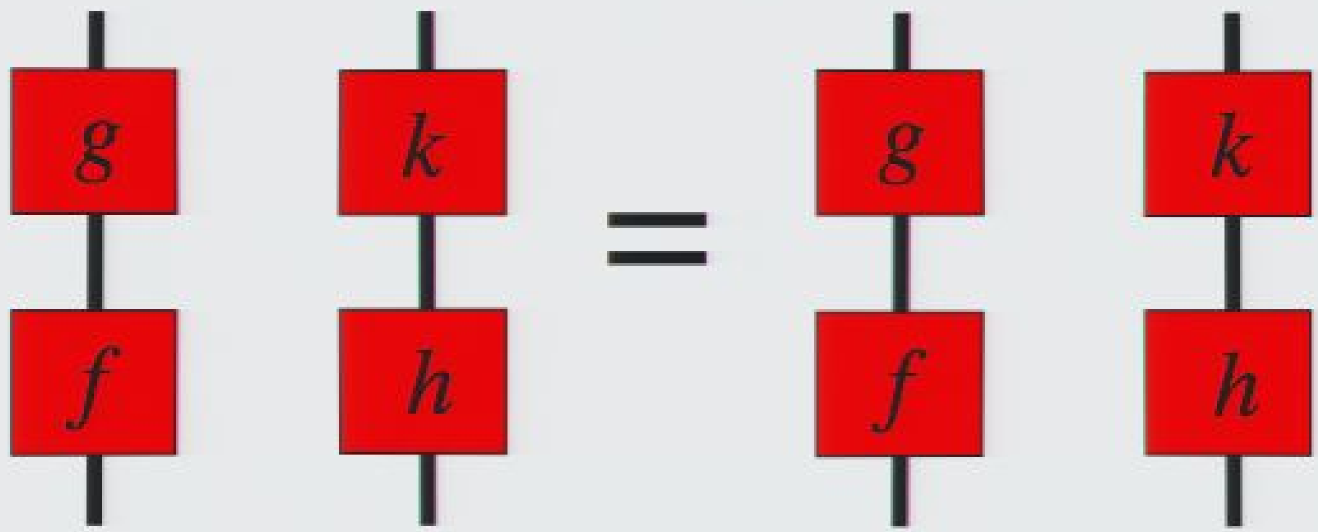
— law from picture —

$$(g \otimes k) \circ (f \otimes h)$$



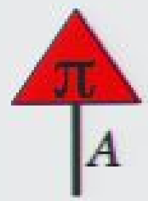
— law from picture —

$$(g \circ f) \otimes (k \circ h) = (g \otimes k) \circ (f \otimes h)$$



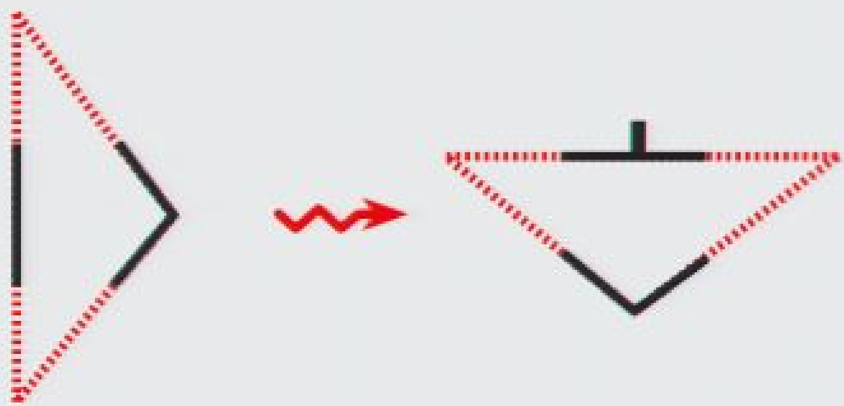
— *states, effects and quantities* —

$\psi : \mathrm{I} \to A$ $\quad\quad$ $\pi : A \to \mathrm{I}$ $\quad\quad$ $\pi \circ \psi : \mathrm{I} \to \mathrm{I}$



— *states, effects and quantities* —

$\psi : \mathrm{I} \rightarrow A$ $\qquad \pi : A \rightarrow \mathrm{I}$ $\qquad \pi \circ \psi : \mathrm{I} \rightarrow \mathrm{I}$



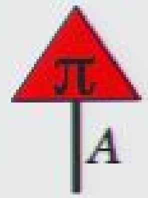
— states, effects and quantities —

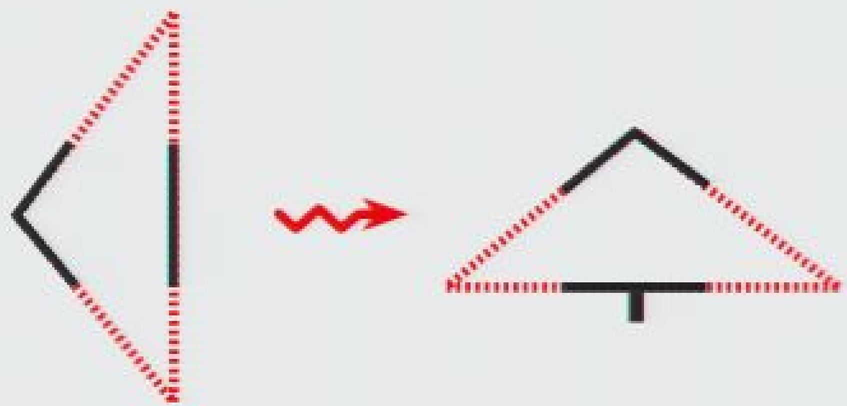
$\psi : \mathrm{I} \to A$ $\qquad$ $\pi : A \to \mathrm{I}$ $\qquad$ $\pi \circ \psi : \mathrm{I} \to \mathrm{I}$

$\langle \quad |$

— states, effects and quantities —

$\psi : \mathrm{I} \to A$ $\qquad$ $\pi : A \to \mathrm{I}$ $\qquad$ $\pi \circ \psi : \mathrm{I} \to \mathrm{I}$





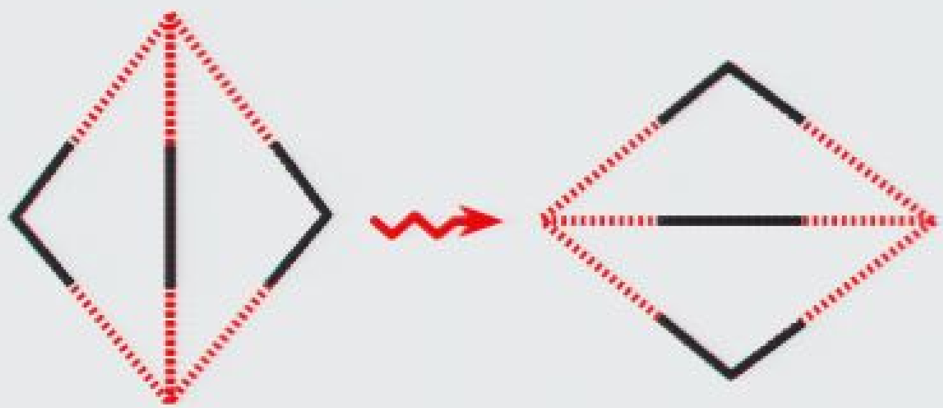
— *states, effects and quantities* —

$\psi : \mathrm{I} \to A$ $\qquad$ $\pi : A \to \mathrm{I}$ $\qquad$ $\pi \circ \psi : \mathrm{I} \to \mathrm{I}$

$$\langle \quad | \quad \rangle$$

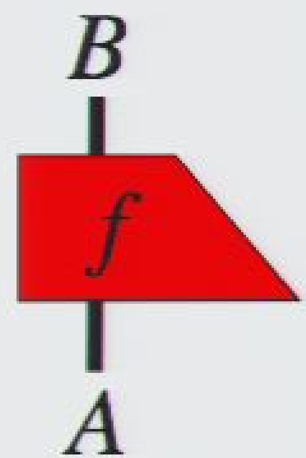
— *states, effects and quantities* —

$\psi : \mathrm{I} \to A$ $\quad\quad$ $\pi : A \to \mathrm{I}$ $\quad\quad$ $\pi \circ \psi : \mathrm{I} \to \mathrm{I}$



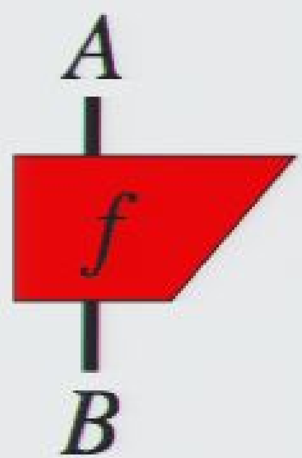
— *adjoint* ⇒ *inner-product* ⇒ *probabilities* —

$$f : A \to B$$



— *adjoint* ⇒ *inner-product* ⇒ *probabilities* —

$$f^{\dagger} : B \to A$$



FHilb :=

- fin. dim. Hilbert spaces
- linear maps
- tensor product
- adjoint

FHilb := Pure quantum states, operations and effects

- fin. dim. Hilbert spaces
- linear maps
- tensor product
- adjoint

FHilb := Pure quantum states, operations and effects

- fin. dim. Hilbert spaces
- linear maps
- tensor product
- adjoint

FRel :=

- fin. sets
- relations
- cartesian product
- converse

FHilb := Pure quantum states, operations and effects

- fin. dim. Hilbert spaces
- linear maps
- tensor product
- adjoint

FRel := Possibilistic states, operations and effects

- fin. sets
- relations
- cartesian product
- converse

QUANTUM STRUCTURE 0:

BELL STATES

Abramsky-Coecke 2004

QUANTUM STRUCTURE 0:

BELL STATES

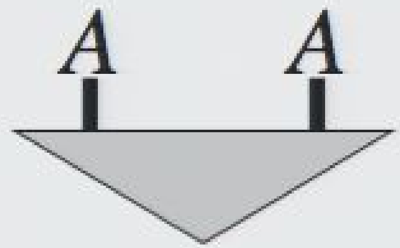
Abramsky-Coecke 2004

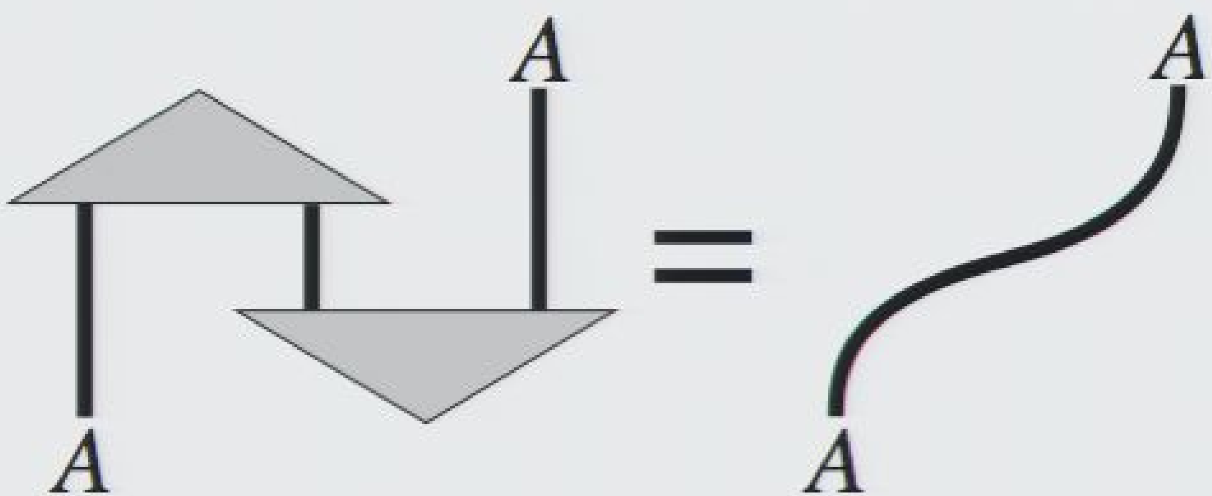
— *Bell structure* —

$$(A\,,\eta : \mathrm{I} \to A \otimes A)$$

$$\begin{array}{ccccc}
A & \xleftarrow{\;\simeq\;} & \mathrm{I} \otimes A & \xleftarrow{\;\eta^{\dagger} \otimes 1_A\;} & (A \otimes A) \otimes A \\
{\scriptstyle 1_A}\uparrow & & & & \uparrow{\scriptstyle \simeq} \\
A & \xrightarrow[\;\simeq\;]{} & A \otimes \mathrm{I} & \xrightarrow[\;1_A \otimes \eta\;]{} & A \otimes (A \otimes A)
\end{array}$$

— Bell structure —



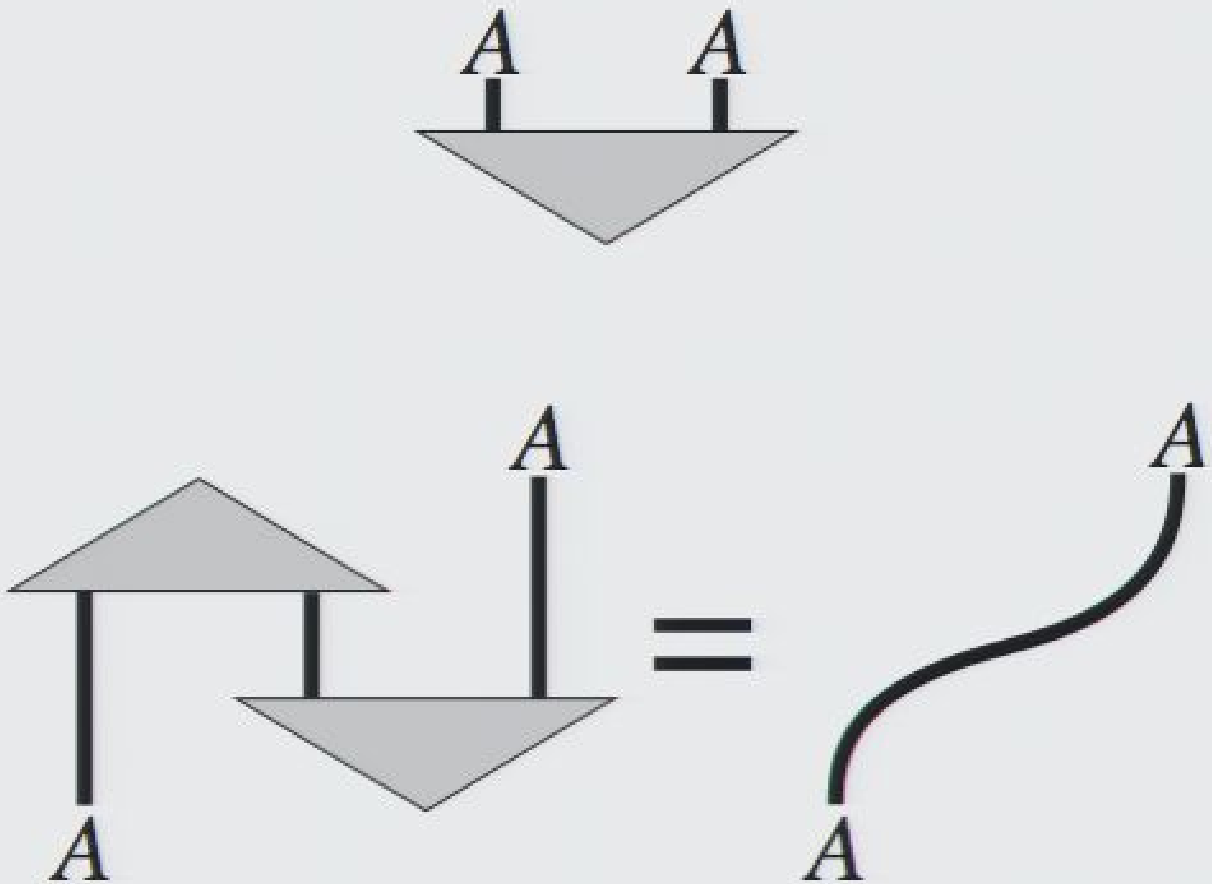


— *Bell structure* —

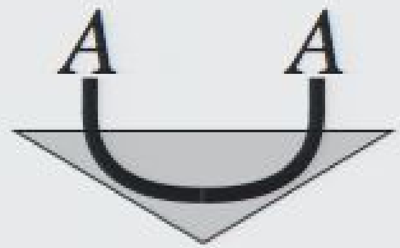
$$(A\,,\eta : \mathrm{I} \to A \otimes A)$$

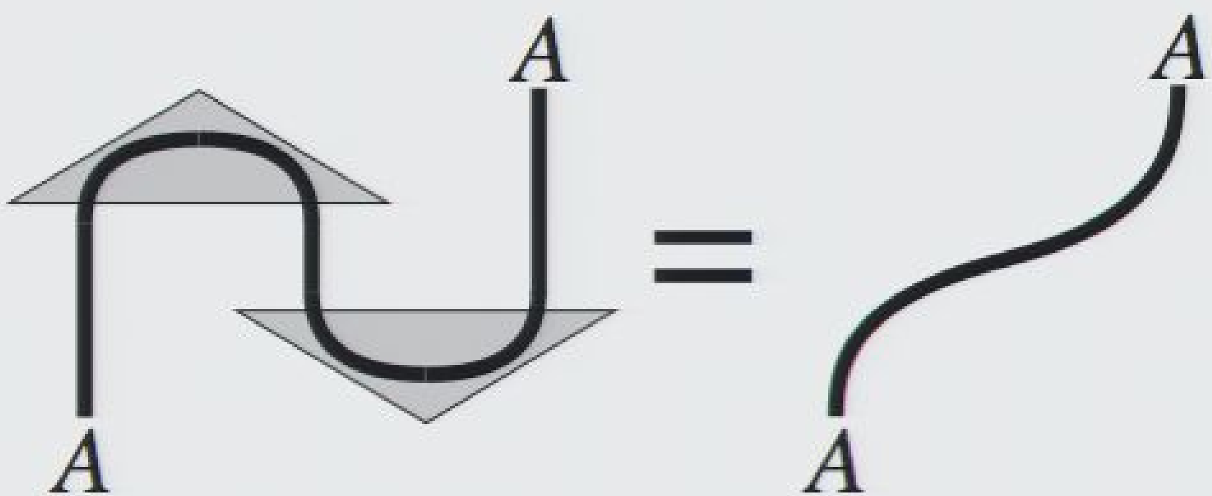
$$\begin{array}{ccccc}
A & \xleftarrow{\ \simeq\ } & \mathrm{I}\otimes A & \xleftarrow{\ \eta^\dagger \otimes 1_A\ } & (A\otimes A)\otimes A \\
{\scriptstyle 1_A}\uparrow & & & & \uparrow{\scriptstyle \simeq} \\
A & \xrightarrow[\ \simeq\ ]{} & A\otimes \mathrm{I} & \xrightarrow[\ 1_A \otimes \eta\ ]{} & A\otimes(A\otimes A)
\end{array}$$

— Bell structure —

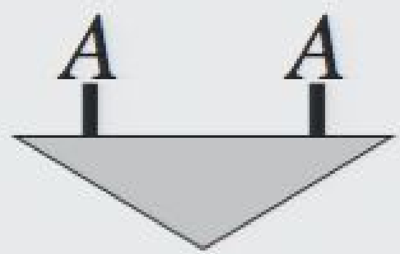


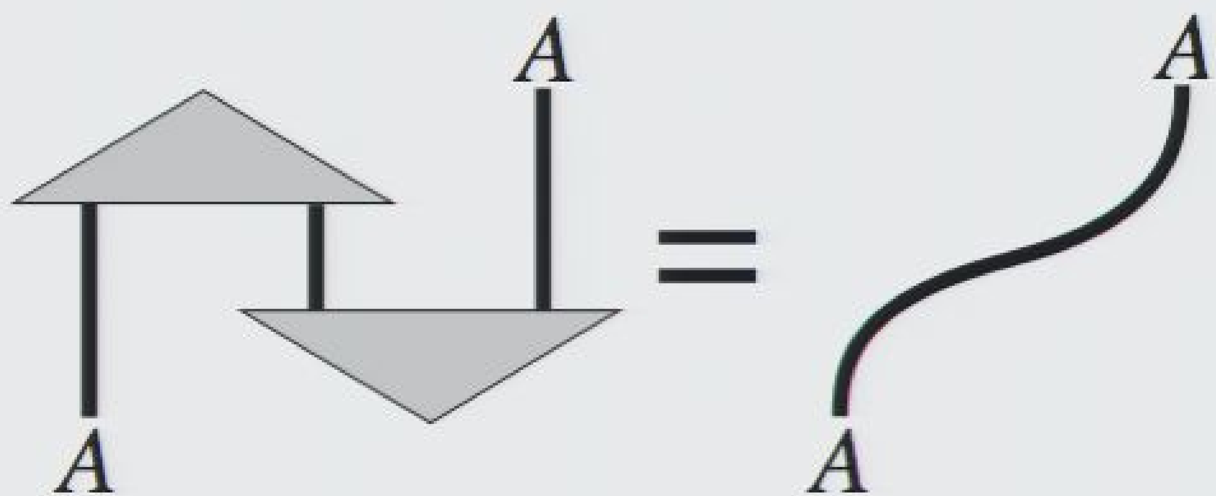
— Bell structure —



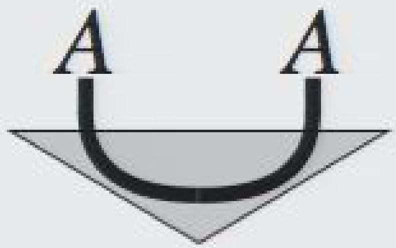


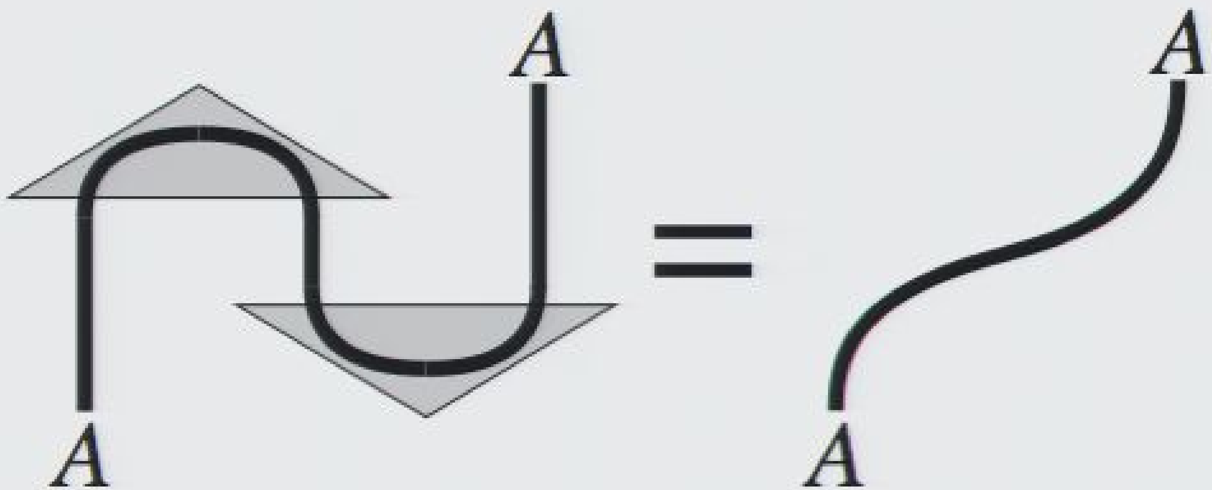
— Bell structure —



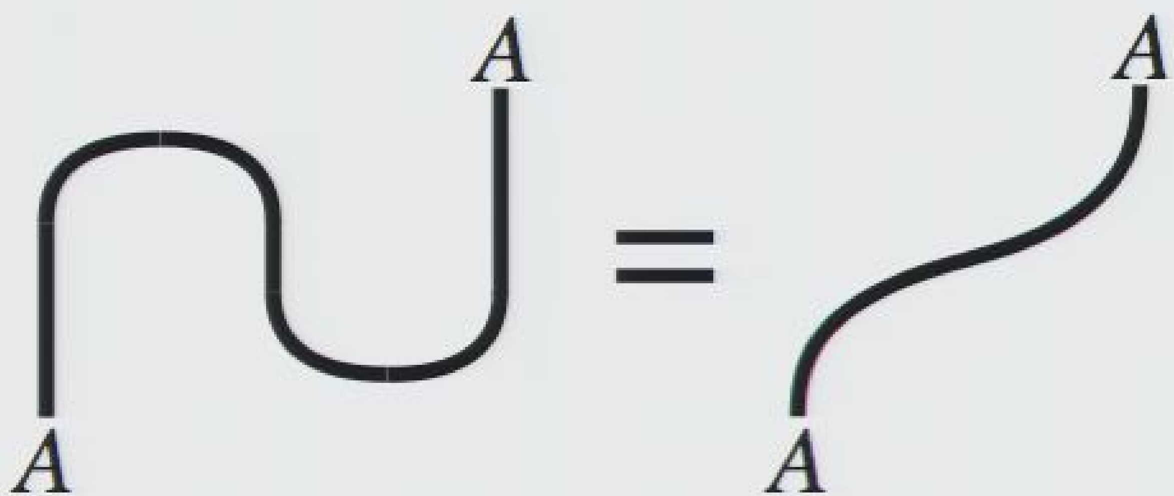


— Bell structure —



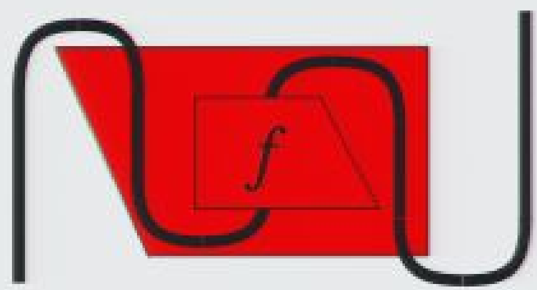


— Bell structure —

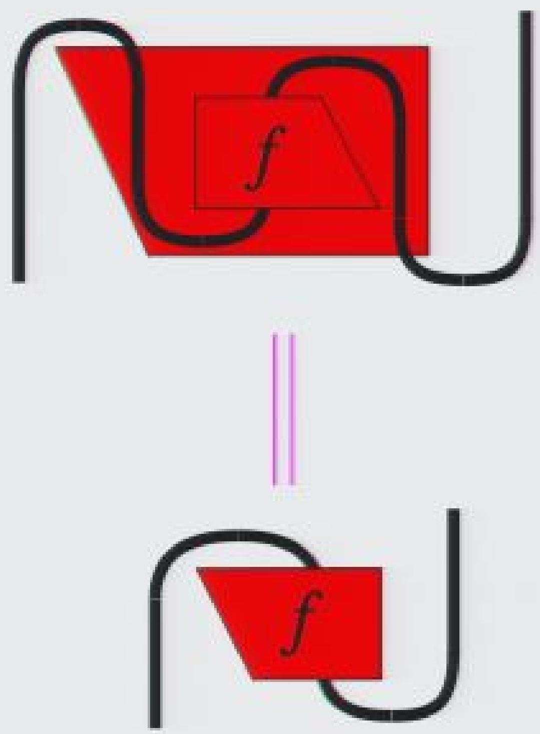


— ‘sliding’ —

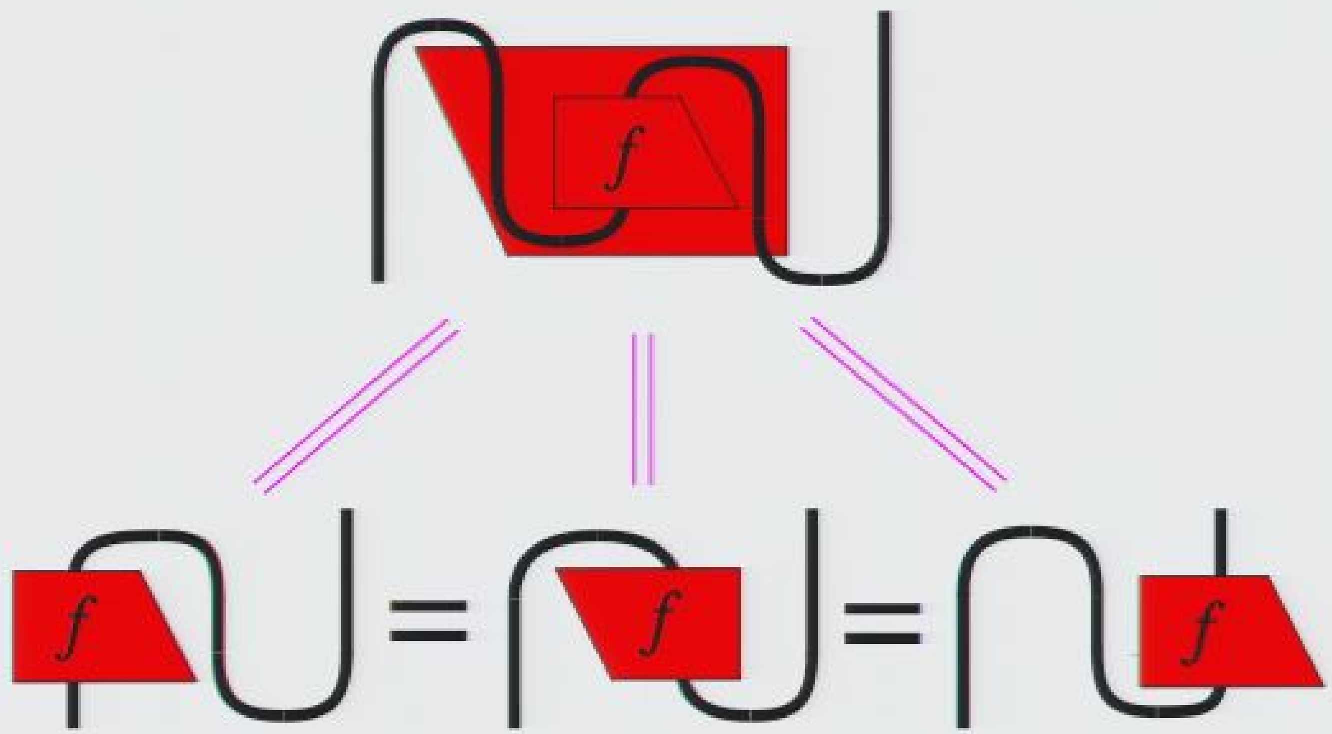
— ‘sliding’ —

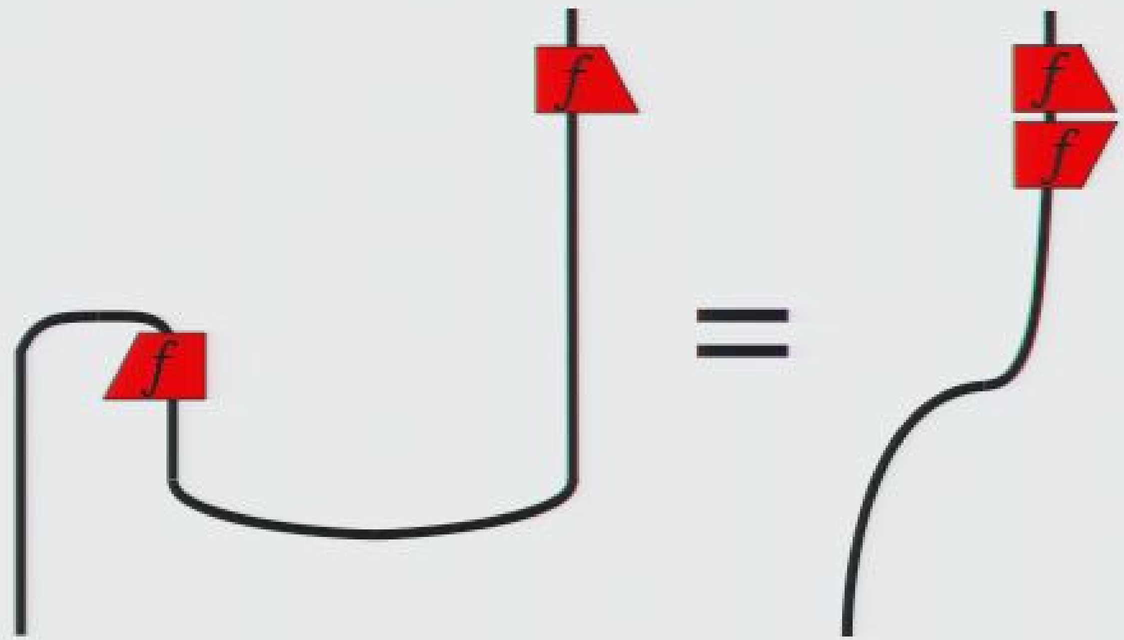


— *'sliding'* —

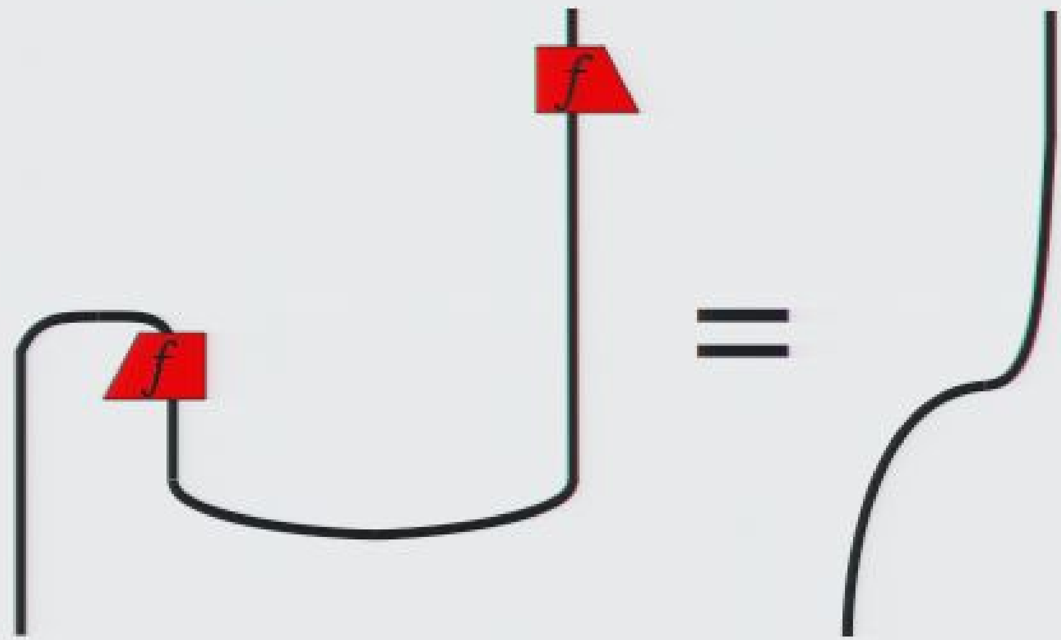


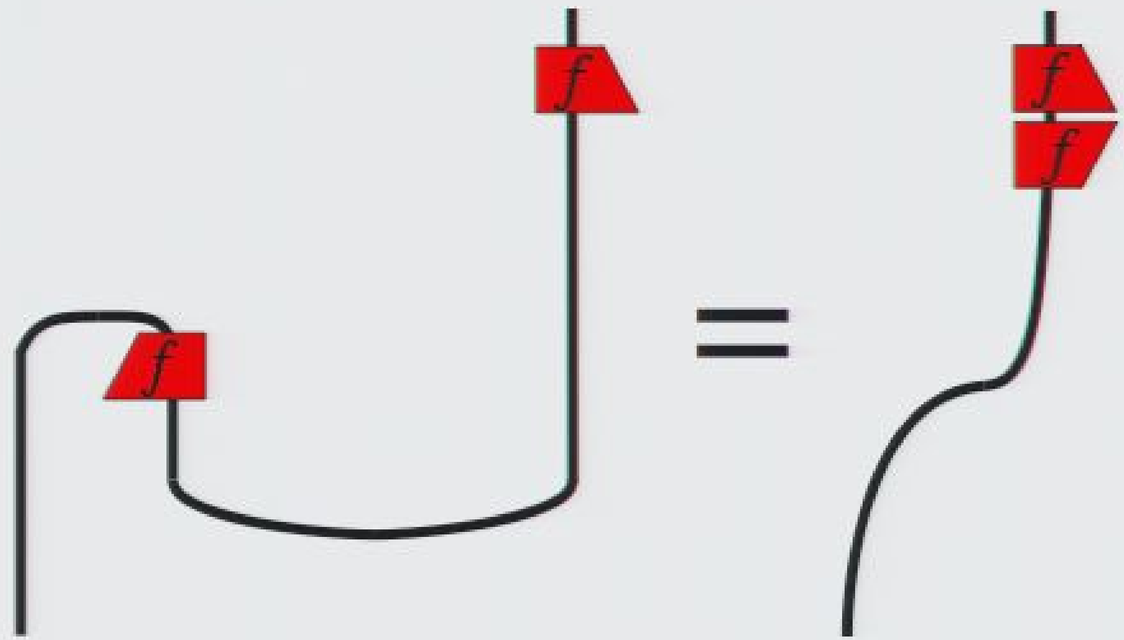
— *'sliding'* —

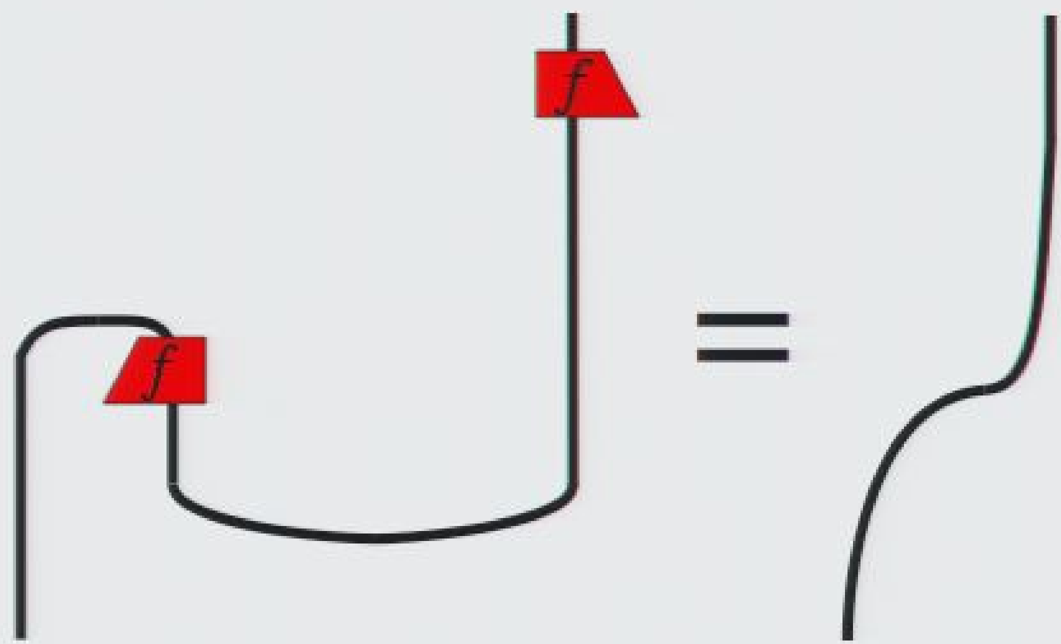




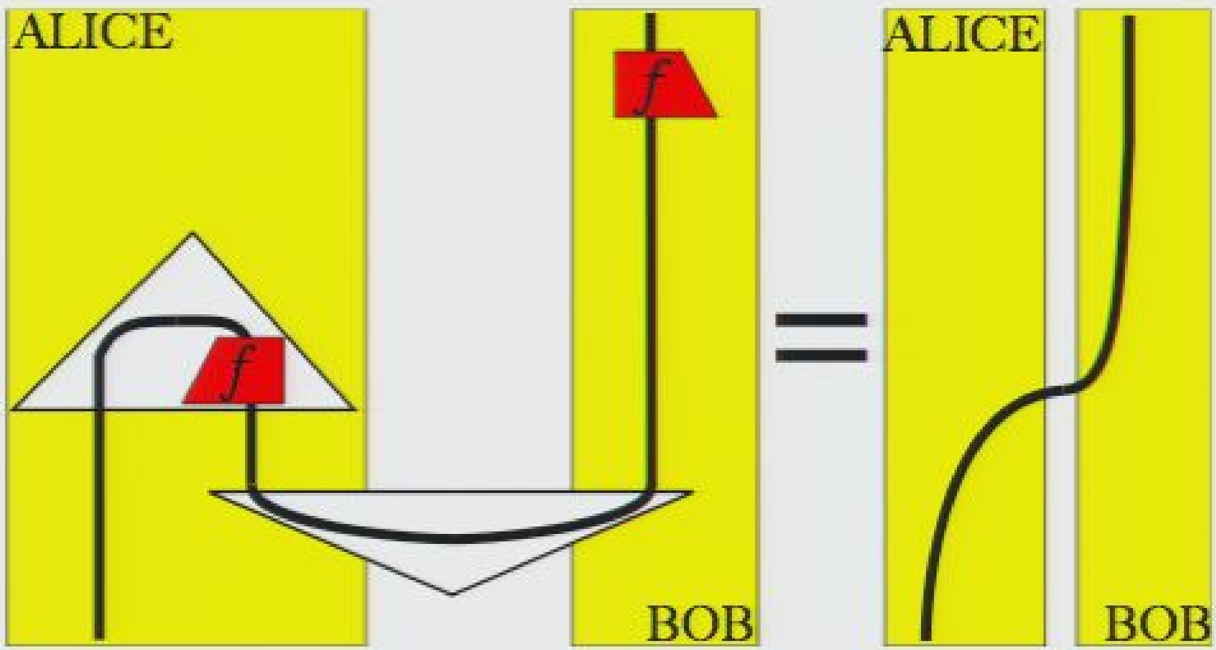
f
f
f
=


f
f
=


f
f
f
=



f
f
=



⇒ **quantum teleportation**

Example categories:

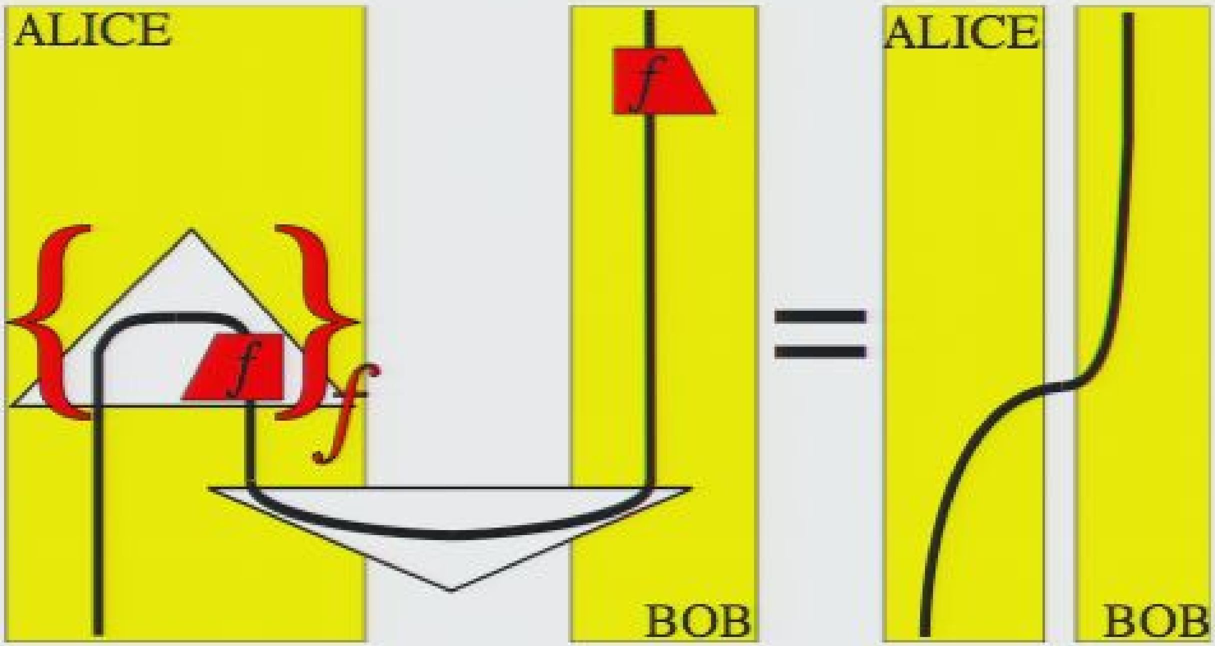
FHilb :

$$\eta_{\mathcal{H}} : \mathbb{C} \to \mathcal{H} \otimes \mathcal{H} :: 1 \mapsto \sum_i |ii\rangle$$

Rel :

$$\eta_X := \{(*, (x, x)) | x \in X\}$$

Which effects?



Example categories:

FHilb :

$$\eta_{\mathcal{H}} : \mathbb{C} \to \mathcal{H} \otimes \mathcal{H} :: 1 \mapsto \sum_i |ii\rangle$$

Rel :

$$\eta_X := \{(*, (x, x)) | x \in X\}$$

Other results in this language:

CP maps and Choi-Jamiolkowski – *Selinger 2005* .

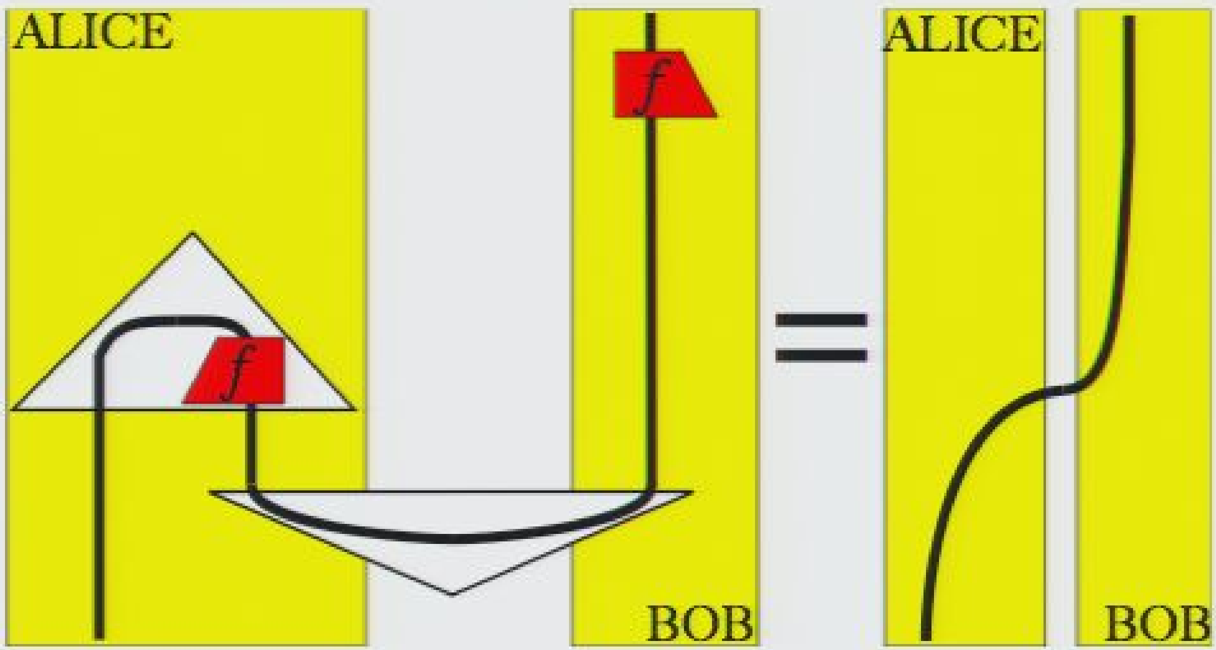
hence this structure and cartesian structure are incompatible

FHilb is complete for †-compact cats – *Selinger 2008*.

hence **FHilb** *is not just 'an' example, but 'a key' example*

NoCloning/Broadcasting – *Abramsky / Coecke 2009*.

hence this structure and cartesian structure are incompatible



⇒ **quantum teleportation**

Example categories:

FHilb :

$$\eta_{\mathcal{H}} : \mathbb{C} \to \mathcal{H} \otimes \mathcal{H} :: 1 \mapsto \sum_i |ii\rangle$$

Rel :

$$\eta_X := \{(*, (x, x)) | x \in X\}$$

Other results in this language:

CP maps and Choi-Jamiolkowski – *Selinger 2005* .

hence this structure and cartesian structure are incompatible

FHilb is complete for †-compact cats – *Selinger 2008*.

hence **FHilb** *is not just 'an' example, but 'a key' example*

NoCloning/Broadcasting – *Abramsky / Coecke 2009*.

hence this structure and cartesian structure are incompatible

Example categories:

FHilb :

$$\eta_{\mathcal{H}} : \mathbb{C} \to \mathcal{H} \otimes \mathcal{H} :: 1 \mapsto \sum_i |ii\rangle$$

Rel :

$$\eta_X := \{(*, (x, x)) | x \in X\}$$

Other results in this language:

CP maps and Choi-Jamiolkowski – *Selinger 2005* .

hence this structure and cartesian structure are incompatible

FHilb is complete for †-compact cats – *Selinger 2008*.

hence **FHilb** *is not just 'an' example, but 'a key' example*

NoCloning/Broadcasting – *Abramsky / Coecke 2009*.

hence this structure and cartesian structure are incompatible

Other results in this language:

CP maps and Choi-Jamiolkowski – *Selinger 2005* .

hence this structure and cartesian structure are incompatible

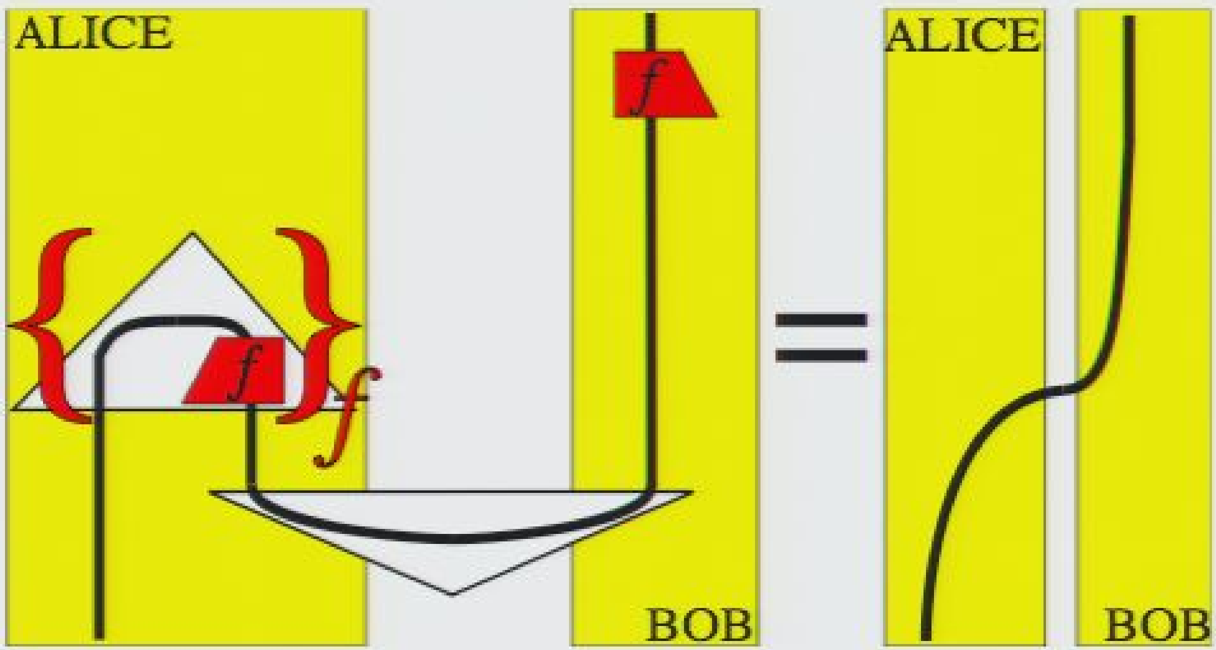
FHilb is complete for †-compact cats – *Selinger 2008*.

hence **FHilb** *is not just 'an' example, but 'a key' example*

NoCloning/Broadcasting – *Abramsky / Coecke 2009*.

hence this structure and cartesian structure are incompatible

Which effects?



QUANTUM STRUCTURE I:

OBSERVABLES

Coecke, Pavlovic, Vicary 2006

— *observable* / *classical* := *copying* + *deleting* —

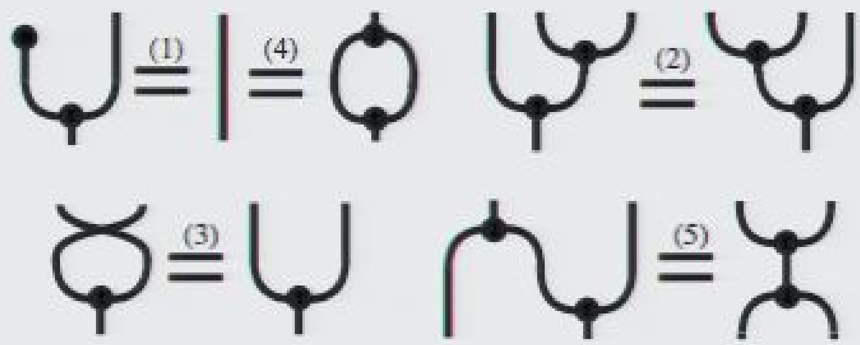
— *observable* / *classical* := *copying* + *deleting* —

$A \xrightarrow{\delta} A \otimes A =$  $A \xrightarrow{\epsilon} \mathrm{I} =$ 

— *observable* / *classical* := *copying* + *deleting* —

$$A \xrightarrow{\delta} A \otimes A =$$  $$A \xrightarrow{\epsilon} \mathrm{I} =$$ 

such that:

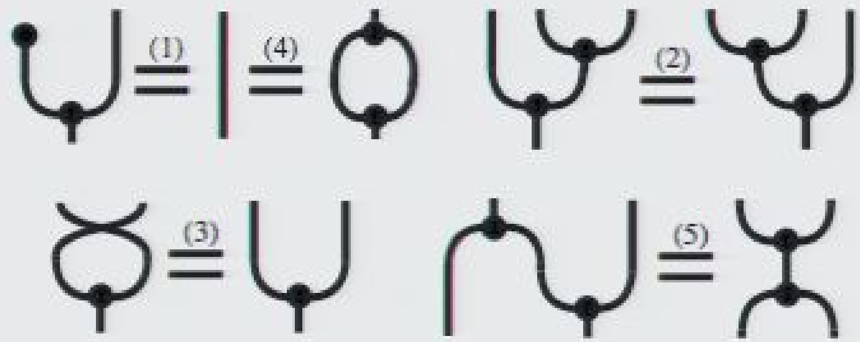


$\Rightarrow$ it is a *commutative special* †*-Frobenius algebra*.

— *observable* / *classical* := *copying* + *deleting* —

$A \xrightarrow{\delta} A \otimes A =$  $\quad A \xrightarrow{\epsilon} \mathrm{I} =$ 

such that:



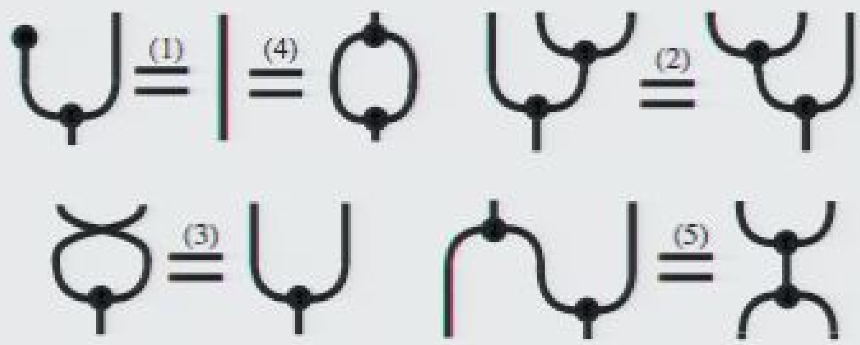
$\Rightarrow$ it is a *commutative special †-Frobenius algebra*.

Thm. Observables in **FHilb** exactly correspond with orthonormal bases on the underlying Hilbert space.

— *observable* / *classical* := *copying* + *deleting* —

$A \xrightarrow{\delta} A \otimes A =$  $\quad A \xrightarrow{\epsilon} \mathrm{I} =$ 

such that:

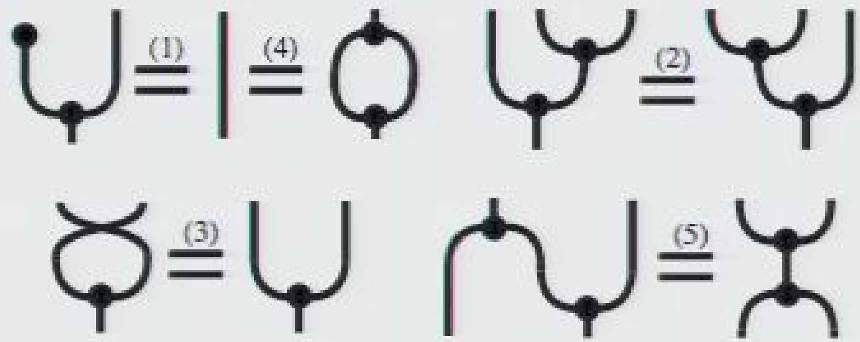


$\Rightarrow$ it is a *commutative special †-Frobenius algebra*.

— *observable* / *classical* := *copying* + *deleting* —

$A \xrightarrow{\delta} A \otimes A =$  $\quad A \xrightarrow{\epsilon} \mathrm{I} =$ 

such that:



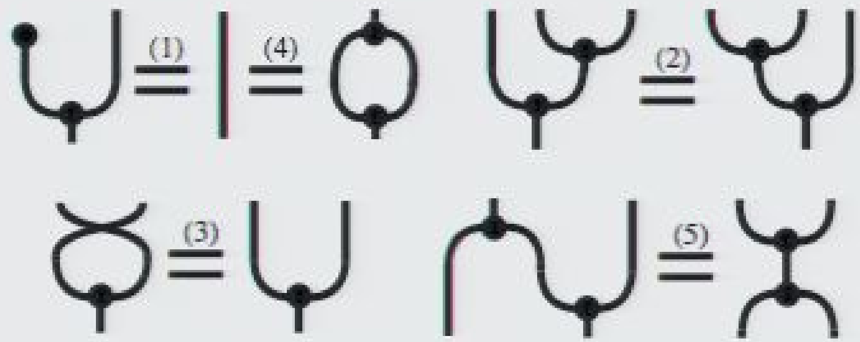
$\Rightarrow$ it is a *commutative special* †*-Frobenius algebra*.

Thm. Observables in **FHilb** exactly correspond with orthonormal bases on the underlying Hilbert space.

— *observable* / *classical* := *copying* + *deleting* —

$A \xrightarrow{\delta} A \otimes A =$  $A \xrightarrow{\epsilon} \mathrm{I} =$ 

such that:



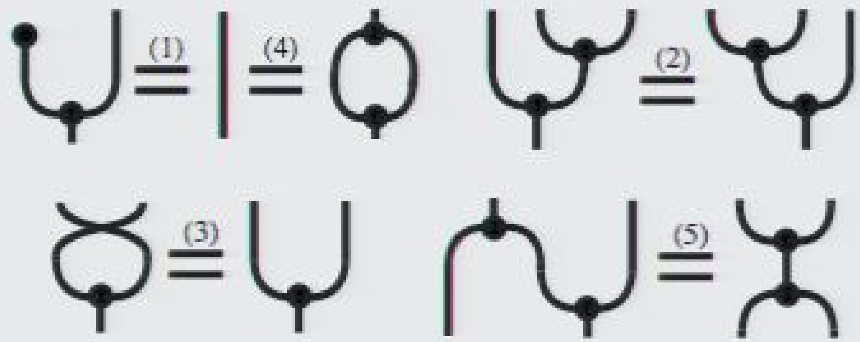
$\Rightarrow$ it is a *commutative special †-Frobenius algebra*.

Thm. Observables in **FHilb** exactly correspond with orthonormal bases on the underlying Hilbert space.

— *observable* / *classical* := *copying* + *deleting* —

$A \xrightarrow{\delta} A \otimes A =$  $A \xrightarrow{\epsilon} \mathrm{I} =$ 

such that:

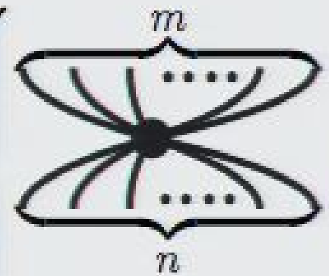


$\Rightarrow$ it is a *commutative special †-Frobenius algebra*.

Thm. Observables in **FHilb** exactly correspond with orthonormal bases on the underlying Hilbert space.

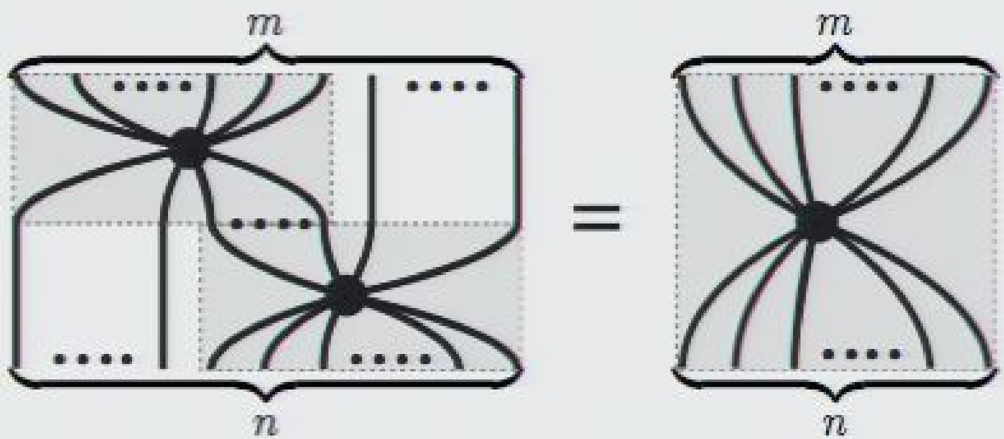
— ***observable* / *classical*** := ***copying* + *deleting*** —

Diagrammatic calculus with ‘spiders’:



$\mid n, m \in \mathbb{N}$

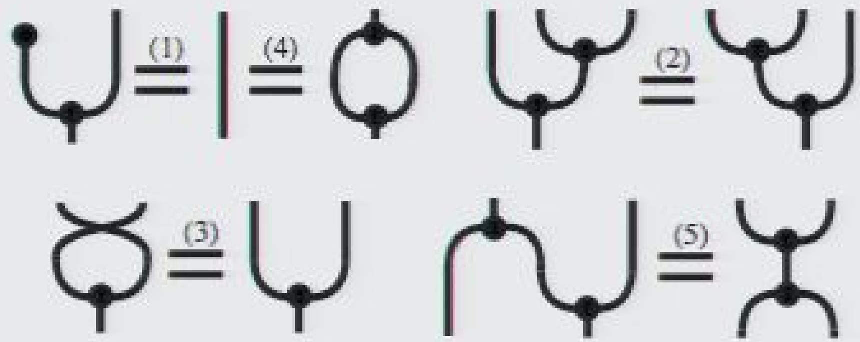
which are such that:



— *observable* / *classical* := *copying* + *deleting* —

$A \xrightarrow{\delta} A \otimes A =$  $\quad A \xrightarrow{\epsilon} \mathrm{I} =$ 

such that:



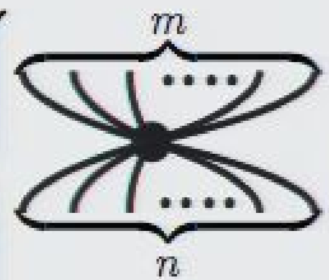
$\Rightarrow$ it is a *commutative special †-Frobenius algebra*.

Thm. Observables in **FHilb** exactly correspond with orthonormal bases on the underlying Hilbert space.

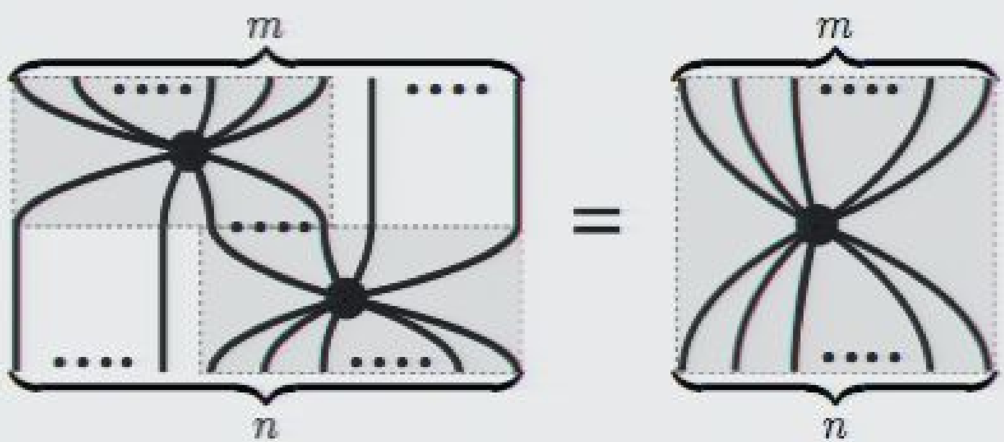
— *observable* / *classical* := *copying* + *deleting* —

Diagrammatic calculus with 'spiders':

$$\left\{ \text{spider} \;\middle|\; n, m \in \mathbb{N} \right\}$$

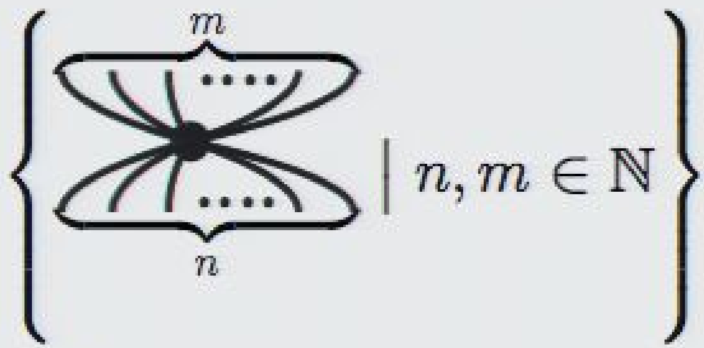


which are such that:

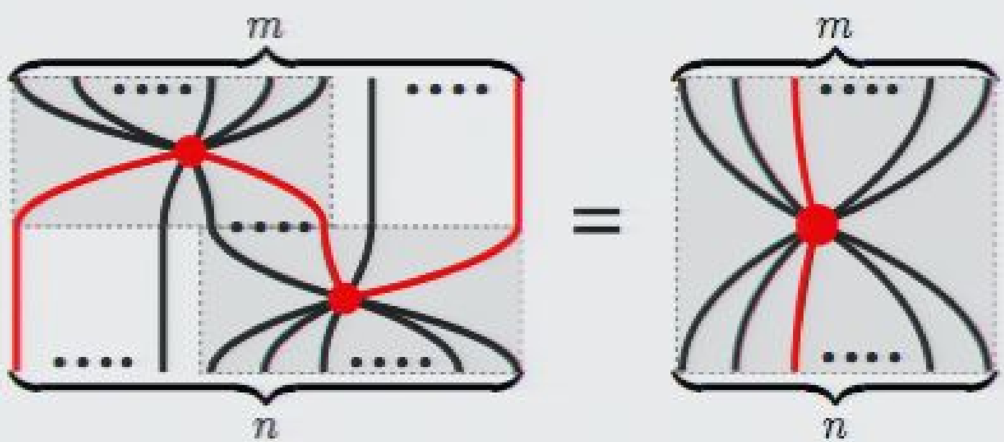


— *observable* / *classical* := *copying* + *deleting* —

Diagrammatic calculus with 'spiders':



which are such that:



Example categories:

FHilb :

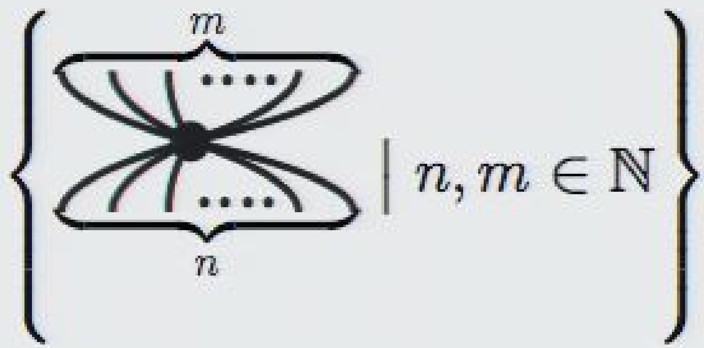
$$\begin{cases} \delta_{\mathcal{H}} : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H} :: |i\rangle \mapsto |ii\rangle \\ \epsilon_{\mathcal{H}} : \mathcal{H} \to \mathbb{C} :: |i\rangle \mapsto 1 \end{cases}$$

Rel :

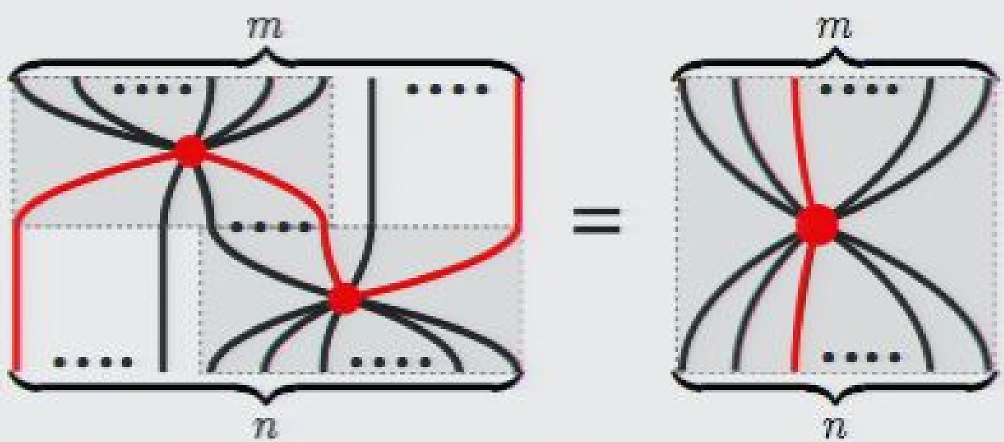
$$\begin{cases} \delta_X := \{(x, (x, x)) | x \in X\} \\ \epsilon_X := \{(x, *) | x \in X\} \end{cases}$$

— *observable* / *classical* := *copying* + *deleting* —

Diagrammatic calculus with 'spiders':



which are such that:



Example categories:

FHilb :

$$\begin{cases} \delta_{\mathcal{H}} : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H} :: |i\rangle \mapsto |ii\rangle \\ \epsilon_{\mathcal{H}} : \mathcal{H} \to \mathbb{C} :: |i\rangle \mapsto 1 \end{cases}$$

Rel :

$$\begin{cases} \delta_X := \{(x,(x,x)) | x \in X\} \\ \epsilon_X := \{(x,*) | x \in X\} \end{cases}$$

Other results in this language:

C., Paquette & Pavlovic: *Classical and quantum structuralism.* arXiv:0904.1997.

Vicary: *Categorical formulation of quantum algebras.* arXiv:0805.0432.

Many connections with representation theory, knot theory, statistical physics, topological quantum field theory, ... see John Baez' blogs.

QUANTUM STRUCTURE II:

COMPLEMENTARITY

Coecke-Duncan 2008

Other results in this language:

C., Paquette & Pavlovic: *Classical and quantum structuralism.* **arXiv:0904.1997.**

Vicary: *Categorical formulation of quantum algebras.* **arXiv:0805.0432.**

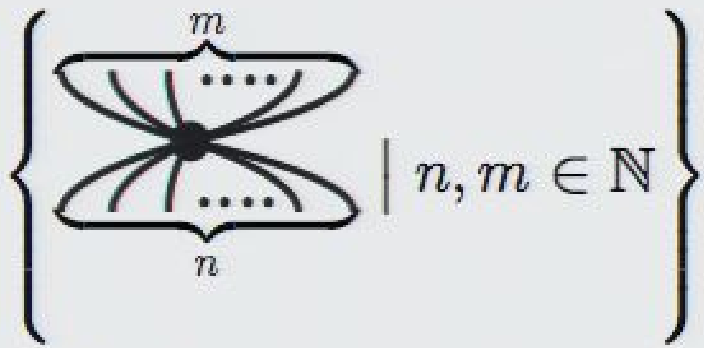
Many connections with representation theory, knot theory, statistical physics, topological quantum field theory, ... see John Baez' blogs.

QUANTUM STRUCTURE II:
COMPLEMENTARITY

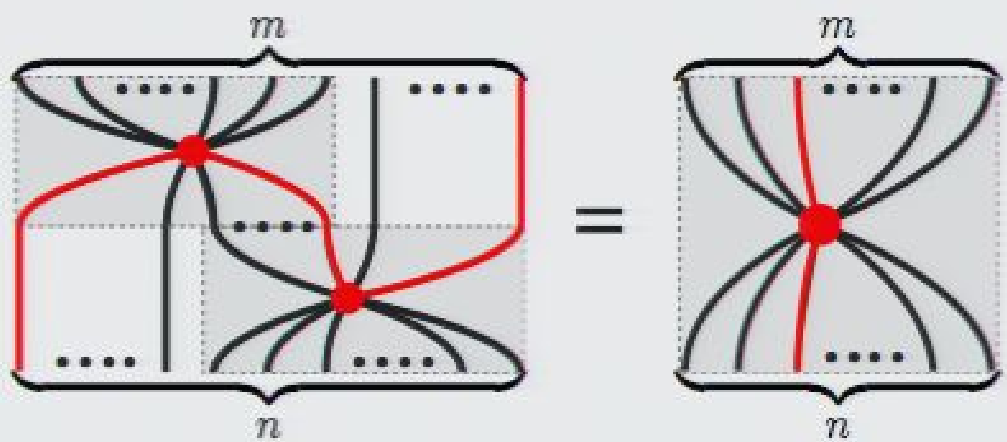
Coecke-Duncan 2008

— *observable* / *classical* := *copying* + *deleting* —

Diagrammatic calculus with 'spiders':



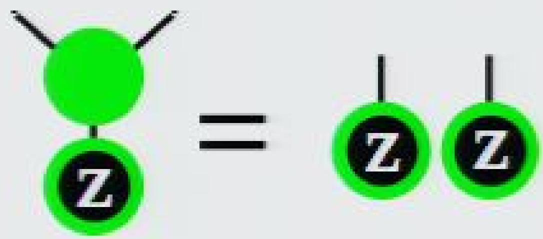
which are such that:

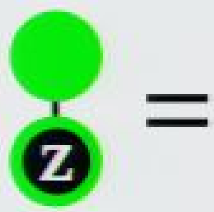


Def. Two observables are complementary (=unbiased) if the classical states for one are unbiased for the other.

Def. Two observables are complementary (=unbiased) if the classical states for one are unbiased for the other.

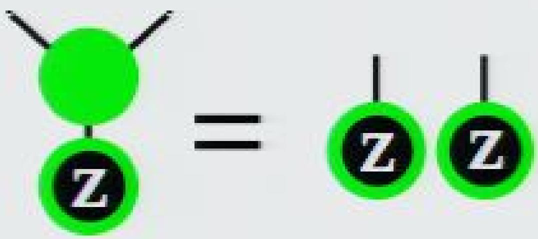
Def. Classicality:

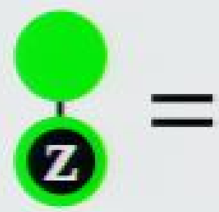




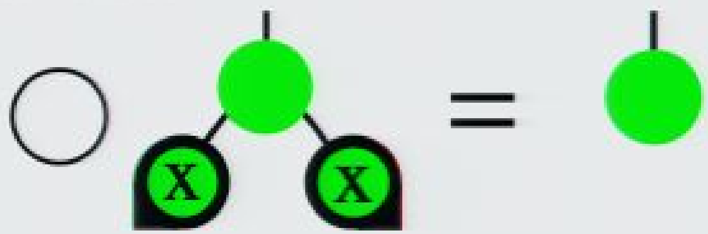
Def. Two observables are complementary (=unbiased) if the classical states for one are unbiased for the other.

Def. Classicality:





Def. Unbiasedness:



Example categories:

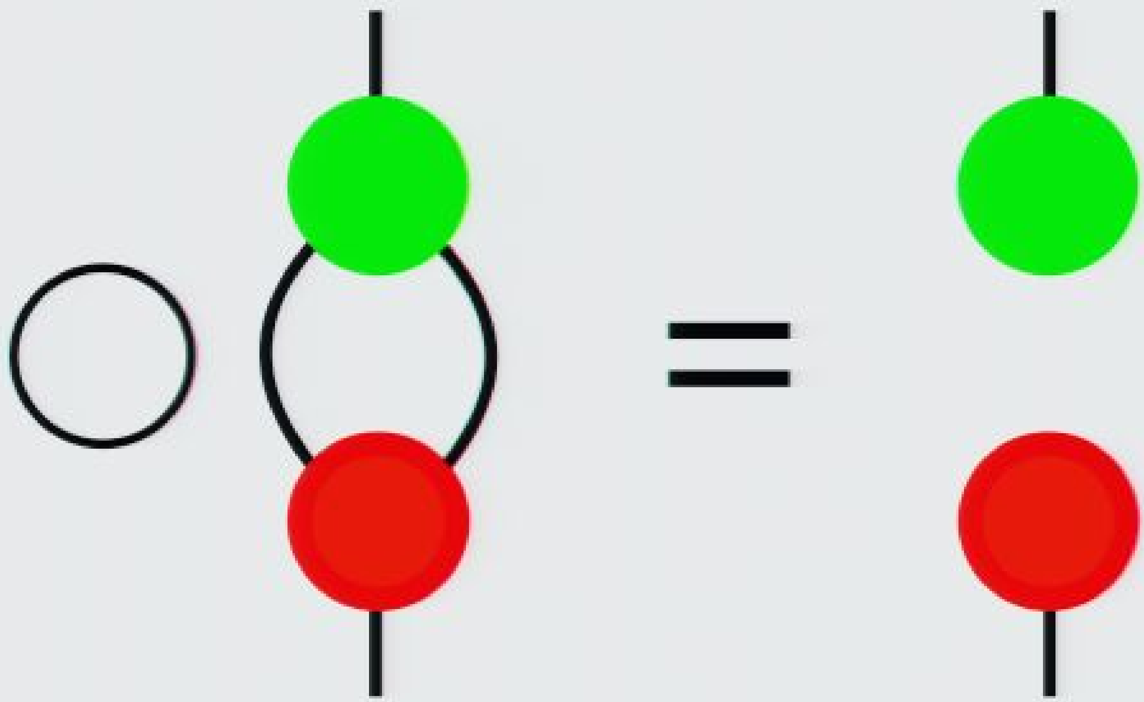
FHilb :

$$\begin{cases} \delta_{\mathcal{H}} : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H} :: |i\rangle \mapsto |ii\rangle \\ \epsilon_{\mathcal{H}} : \mathcal{H} \to \mathbb{C} :: |i\rangle \mapsto 1 \end{cases}$$

Rel :

$$\begin{cases} \delta_X := \{(x,(x,x))|x \in X\} \\ \epsilon_X := \{(x,*)|x \in X\} \end{cases}$$

Thm. Observables are complementary if and only if:



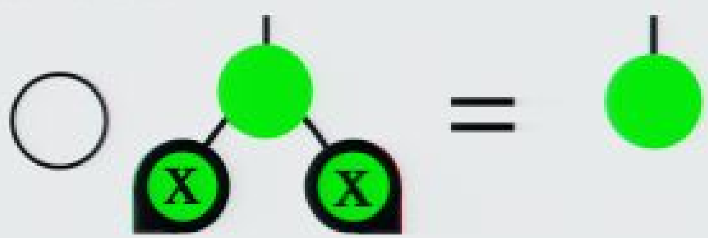
Def. Two observables are complementary (=unbiased) if the classical states for one are unbiased for the other.

Def. Classicality:

Def. Two observables are complementary (=unbiased) if the classical states for one are unbiased for the other.

Def. Classicality:

Def. Unbiasedness:

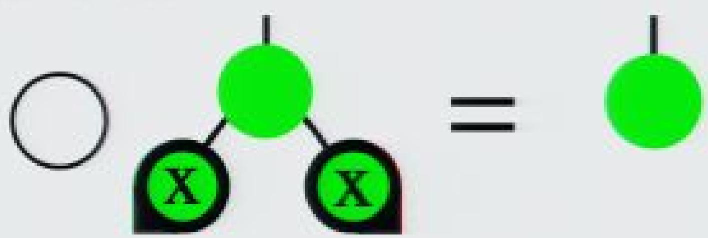


Def. Two observables are complementary (=unbiased) if the classical states for one are unbiased for the other.

Def. Two observables are complementary (=unbiased) if the classical states for one are unbiased for the other.

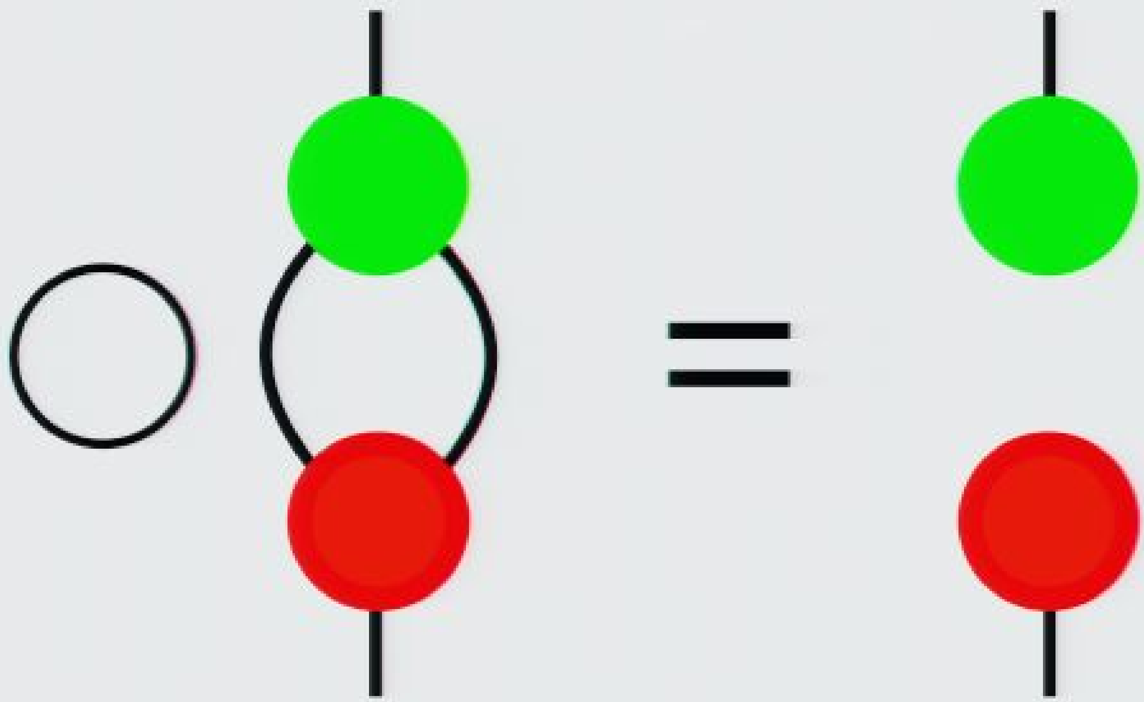
Def. Classicality:

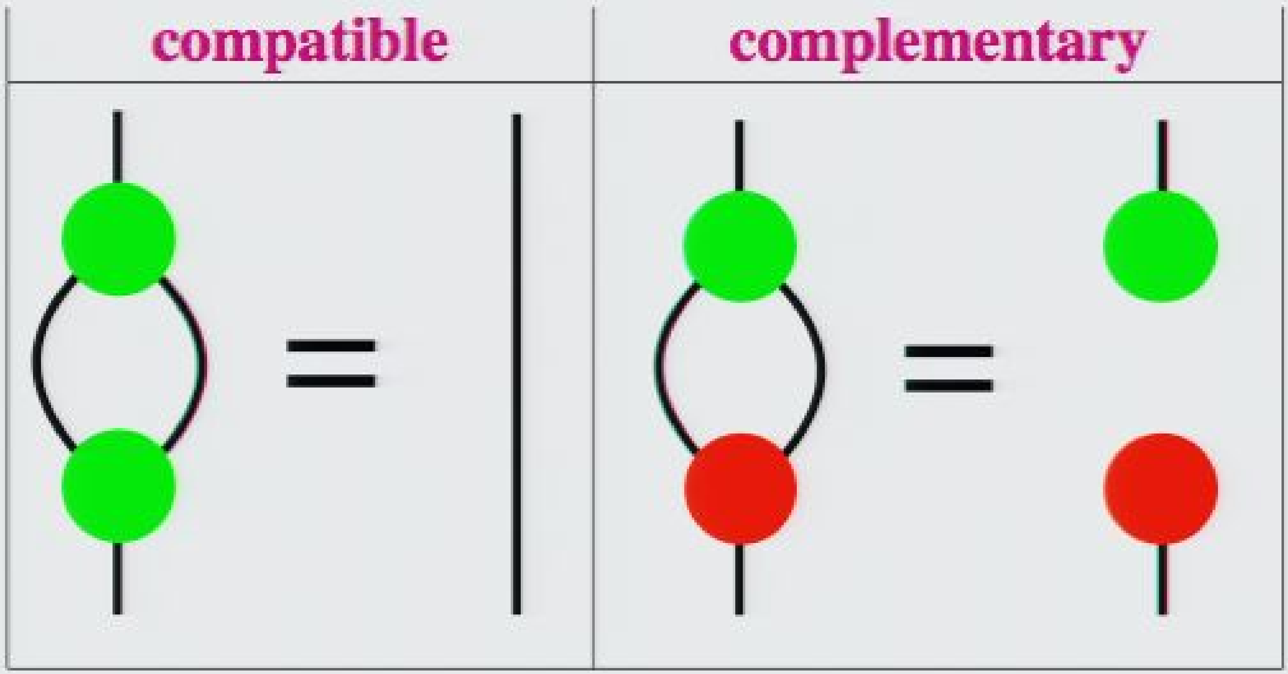
Def. Unbiasedness:



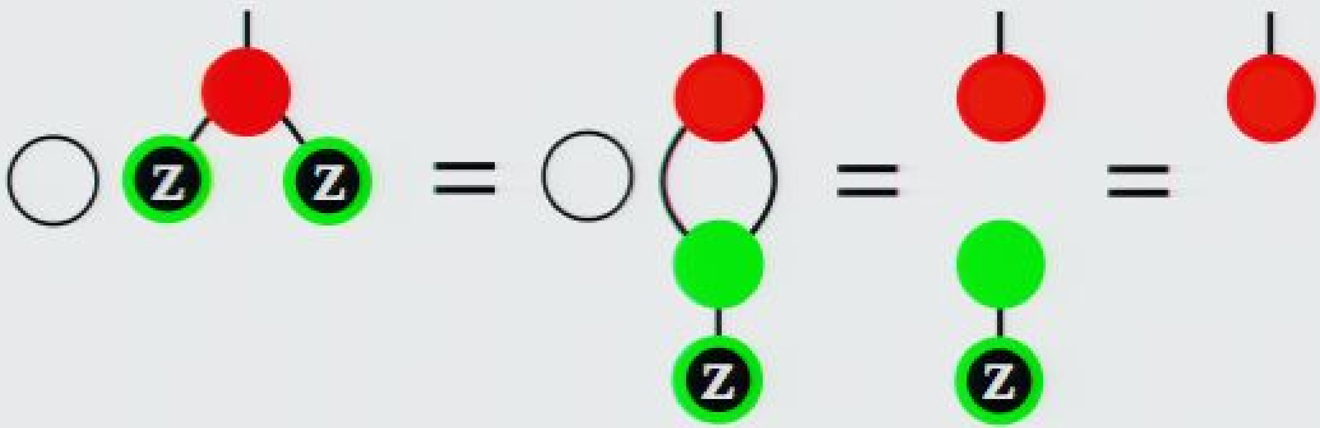
Def. Two observables are complementary (=unbiased) if the classical states for one are unbiased for the other.

Thm. Observables are complementary if and only if:




compatible
complementary
=
=

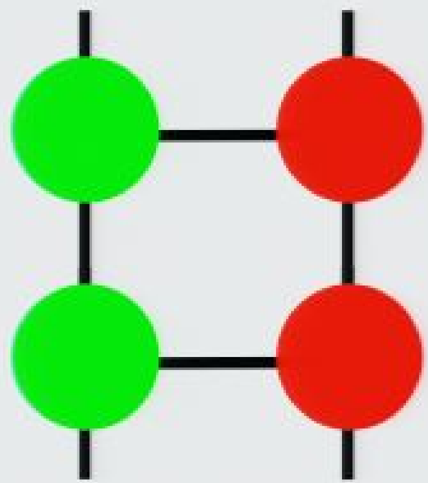
Proof. Hopf law $\Longrightarrow$ [class $\Rightarrow$ unbiased]



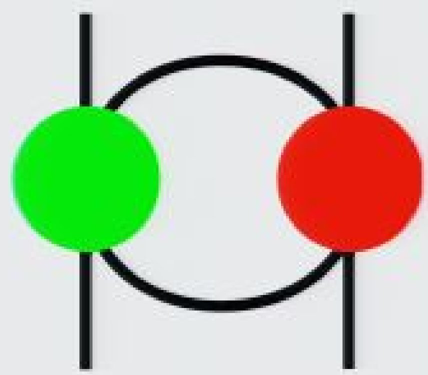
— *two* CX *gates* —

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \circ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = ?$$

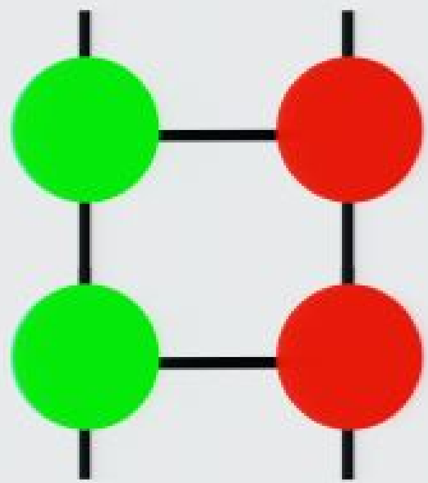
— *two* CX *gates* —



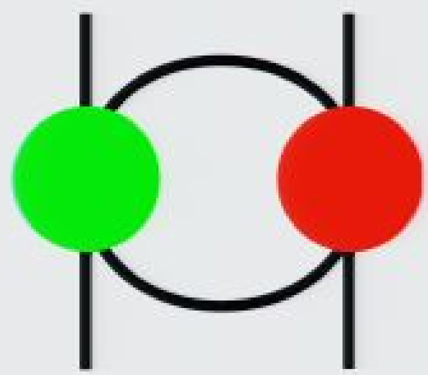
— *two* CX *gates* —



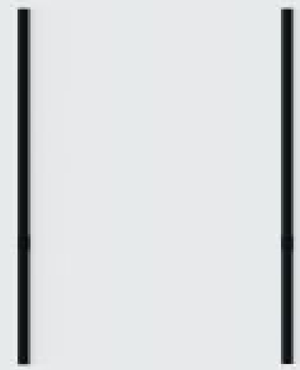
— *two* CX *gates* —



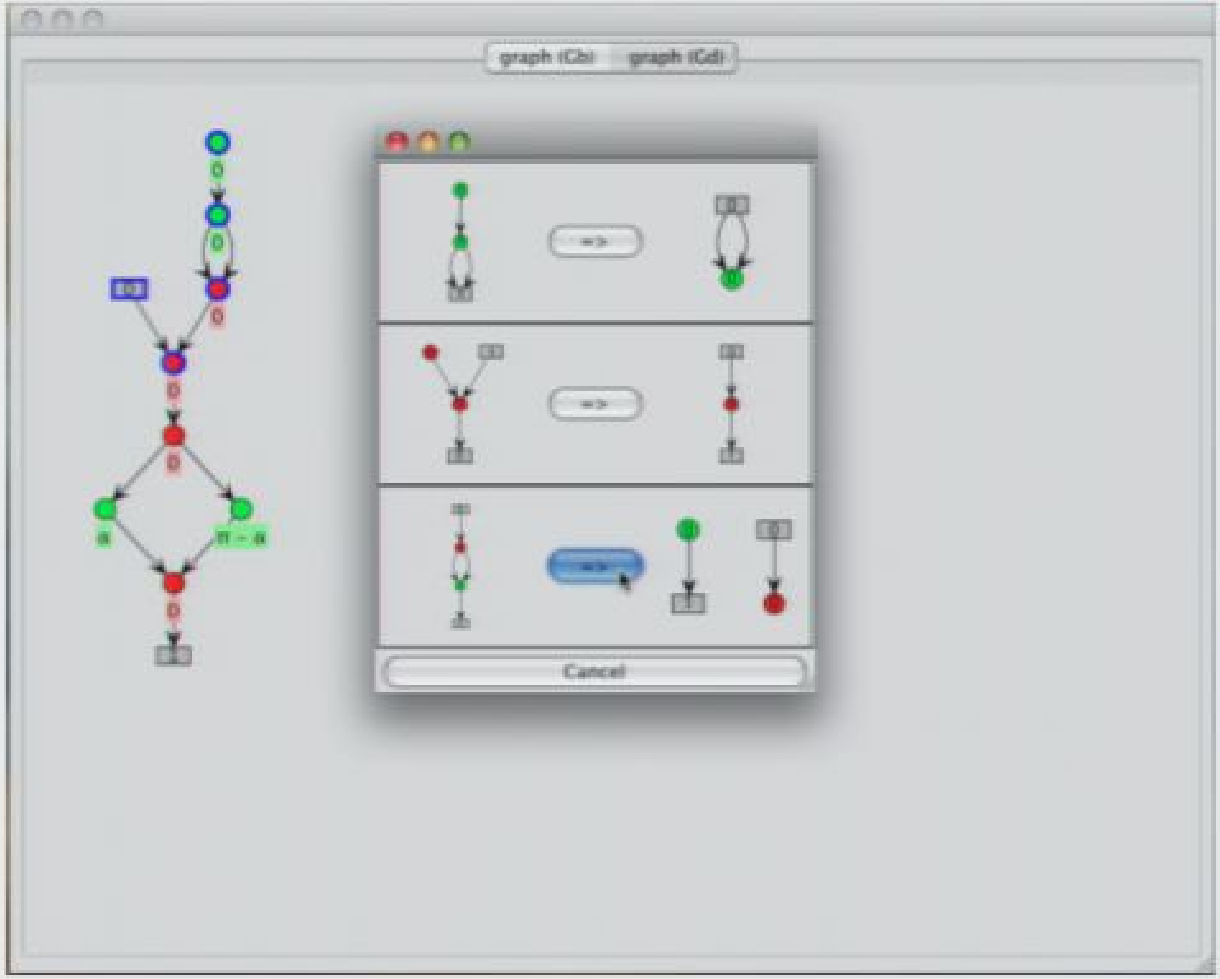
— *two* CX *gates* —



— *two* CX *gates* —



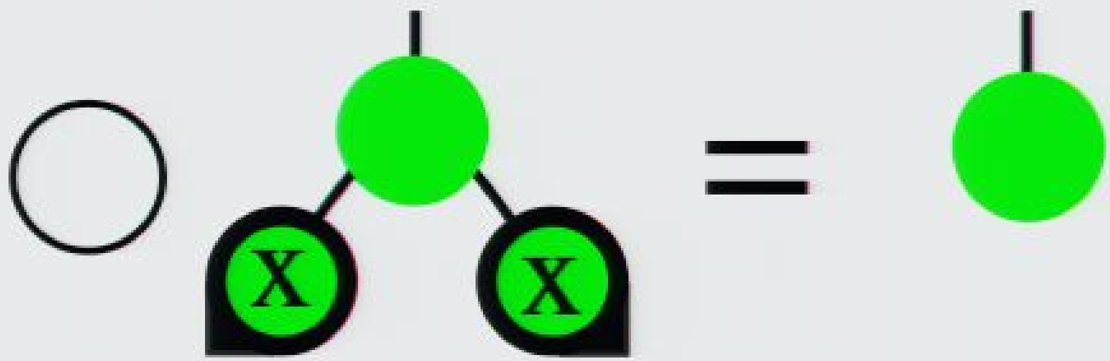
quantomatic – *Dixon / Duncan / Kissinger*



PHASES

Coecke-Duncan 2008

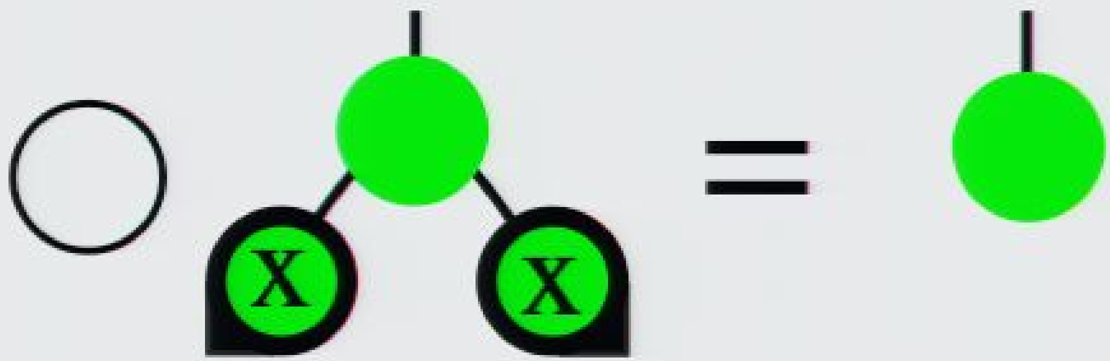
Def. Unbiasedness:



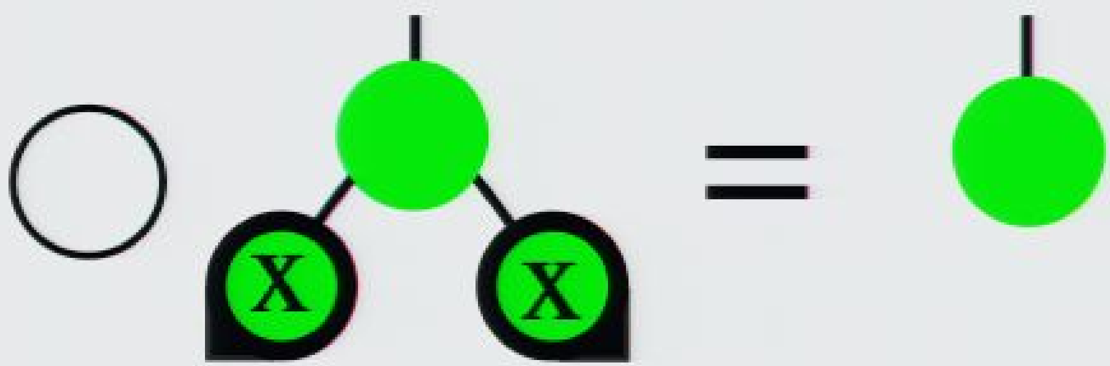
PHASES

Coecke-Duncan 2008

Def. Unbiasedness:

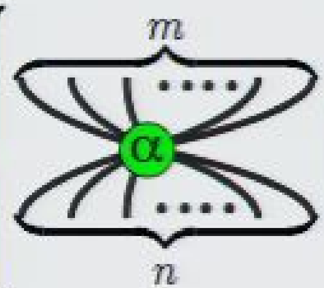


Def. Unbiasedness:



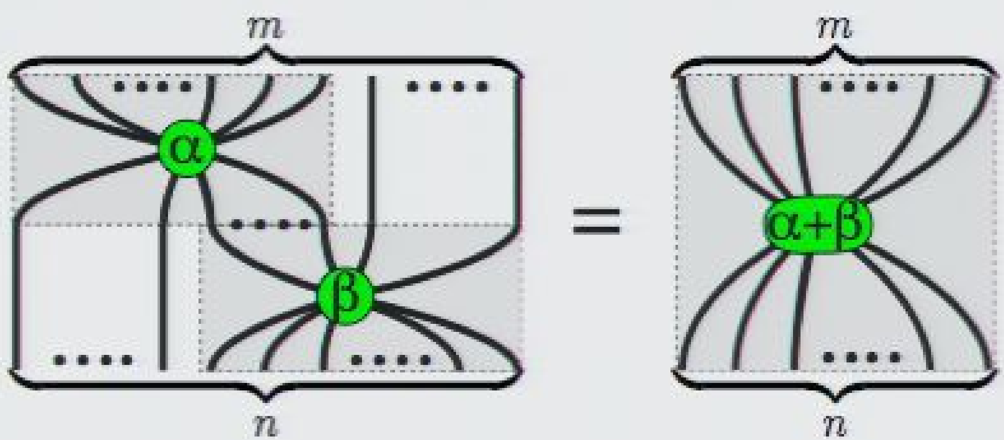
Thm. Unbiased states of an observable always constitute an Abelian group with conjugates as inverses.

Thm. Unbiased states of an observable always constitute an Abelian group with conjugates as inverses.

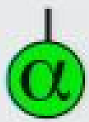
$$\left\{ \right.$$ 

$$\left. \middle| \; n, m \in \mathbb{N}_0, \alpha \in G \right\}$$

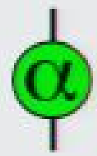
invariant under flipping and swapping, and such that:



For qubits in **FHilb** with green $\equiv \{|0\rangle, |1\rangle\} \equiv Z$:



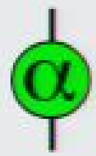
$$= \begin{pmatrix} 1 \\ e^{i\alpha} \end{pmatrix}_Z$$



$$= Z_\alpha = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix}_Z$$

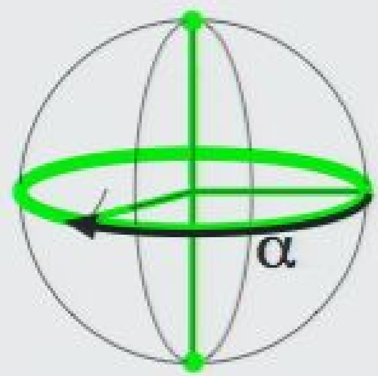
For qubits in **FHilb** with green $\equiv \{|0\rangle, |1\rangle\} \equiv Z$:

$= \begin{pmatrix} 1 \\ e^{i\alpha} \end{pmatrix}_Z$

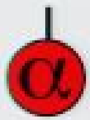


$= Z_\alpha = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix}_Z$

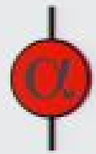
These are relative phases for Z, hence in X-Y:



For qubits in **FHilb** with red $\equiv \{|+\rangle, |-\rangle\} \equiv X$:

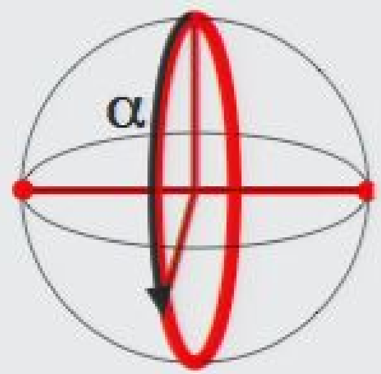


$$= \begin{pmatrix} 1 \\ e^{i\alpha} \end{pmatrix}_X$$



$$= X_\alpha = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix}_X$$

These are relative phases for X, hence in Z-Y:



Example category $\mathbf{FHilb}_2$

Thm. Every linear map in $\mathbf{FHilb}_2$ can be expressed in the language of a pair of complementary observables and the corresponding phases, that is, it can be written down using only red and green decorated spiders.

Example category $\mathbf{FHilb}_2$

Thm. Every linear map in $\mathbf{FHilb}_2$ can be expressed in the language of a pair of complementary observables and the corresponding phases, that is, it can be written down using only red and green decorated spiders.

Example category $\mathbf{FRel}_2$

- Copying-Deleting canonically:

$$\delta_Z :: \begin{cases} |0\rangle \mapsto |00\rangle \\ |1\rangle \mapsto |11\rangle \end{cases} \qquad \epsilon_Z :: \begin{cases} |0\rangle \mapsto 1 \\ |1\rangle \mapsto 1 \end{cases}$$

Example category $\mathbf{FRel}_2$

- Copying-Deleting canonically:

$$\delta_Z :: \begin{cases} |0\rangle \mapsto |00\rangle \\ |1\rangle \mapsto |11\rangle \end{cases} \qquad \epsilon_Z :: \begin{cases} |0\rangle \mapsto 1 \\ |1\rangle \mapsto 1 \end{cases}$$

- Copying-Deleting 'magically':

$$\delta_X :: \begin{cases} |0\rangle \mapsto |00\rangle + |11\rangle \\ |1\rangle \mapsto |01\rangle + |10\rangle \end{cases} \qquad \epsilon_X :: \begin{cases} |0\rangle \mapsto 1 \\ |1\rangle \mapsto 0 \end{cases}$$

$\Rightarrow$ **genuine complementarity**

Coecke & Edwards. Toy quantum categories. arXiv:0808.1037

THE ORIGIN NON-LOCALITY

Coecke, Edwards, Spekkens 2009

Example category $\mathbf{FRel}_2$

- Copying-Deleting canonically:

$$\delta_Z :: \begin{cases} |0\rangle \mapsto |00\rangle \\ |1\rangle \mapsto |11\rangle \end{cases} \qquad \epsilon_Z :: \begin{cases} |0\rangle \mapsto 1 \\ |1\rangle \mapsto 1 \end{cases}$$

- Copying-Deleting 'magically':

$$\delta_X :: \begin{cases} |0\rangle \mapsto |00\rangle + |11\rangle \\ |1\rangle \mapsto |01\rangle + |10\rangle \end{cases} \qquad \epsilon_X :: \begin{cases} |0\rangle \mapsto 1 \\ |1\rangle \mapsto 0 \end{cases}$$

$\Rightarrow$ **genuine complementarity**

Coecke & Edwards. Toy quantum categories. arXiv:0808.1037

THE ORIGIN NON-LOCALITY

Coecke, Edwards, Spekkens 2009

What is non-locality?

What is non-locality?

Name: "It's <u>not</u> locality"

Tech defn: "There is <u>no</u> local hidden variable theory"

What is non-locality?

Name: "It's <u>not</u> locality"

Tech defn: "There is <u>no</u> local hidden variable theory"

Bell's thm: "Some inequality is <u>not</u> satisfied"

What is non-locality?

Name: "It's not locality"

Tech defn: "There is no local hidden variable theory"

Bell's thm: "Some inequality is not satisfied"

GHZ's thm: "Some correlations cannot be obtained"

What is non-locality?

Name: "It's <u>not</u> locality"

Tech defn: "There is <u>no</u> local hidden variable theory"

Bell's thm: "Some inequality is <u>not</u> satisfied"

GHZ's thm: "Some correlations can<u>not</u> be obtained"

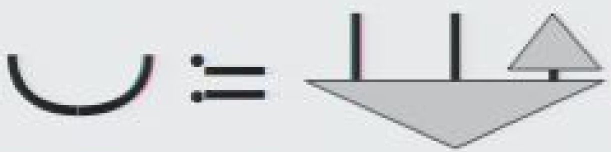
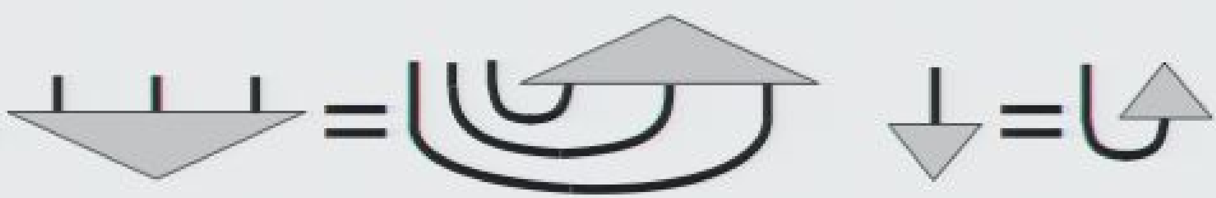
This tells us what it is not, ... but what IS it?

— *non-locality a la GHZ* —

Defn. A GHZ-state is a triple

$$(X\ ,\ \Psi : \mathrm{I} \to X \otimes X \otimes X\ ,\ \epsilon : X \to \mathrm{I})$$

with

- 
-  *such that* 
- 

What is non-locality?

Name: "It's <u>not</u> locality"

Tech defn: "There is <u>no</u> local hidden variable theory"

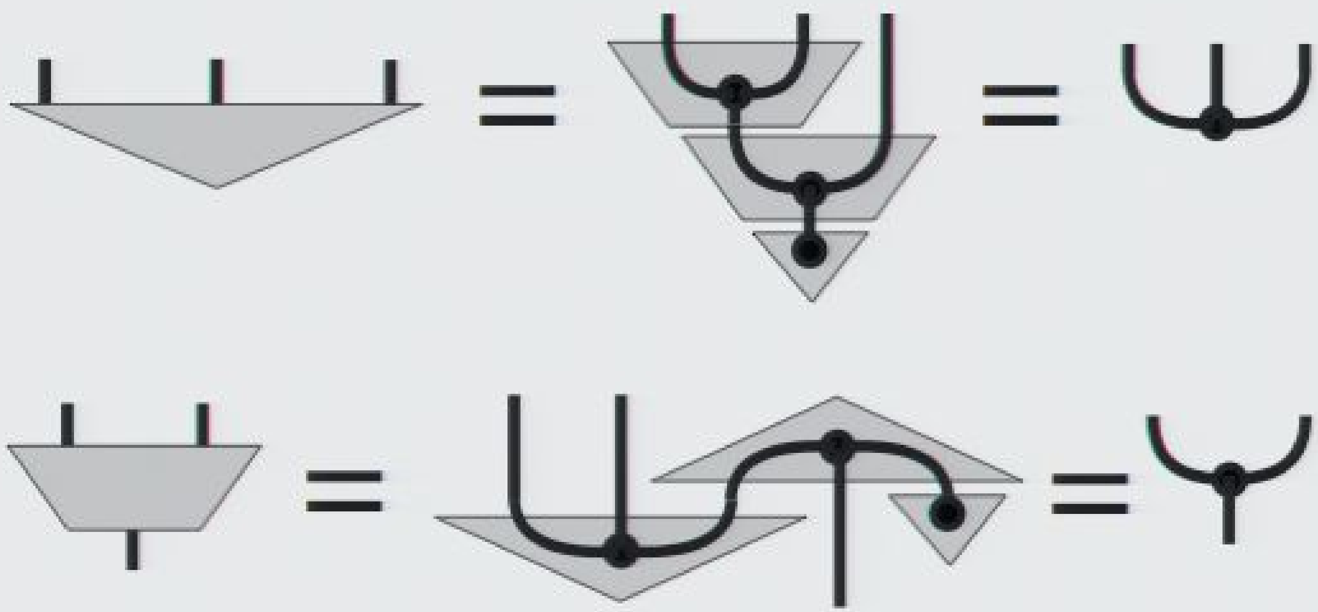
Bell's thm: "Some inequality is <u>not</u> satisfied"

GHZ's thm: "Some correlations can<u>not</u> be obtained"

This tells us what it is not, ... but what IS it?

— *non-locality a la GHZ* —

Thm. **observables ≡ GHZ-states**

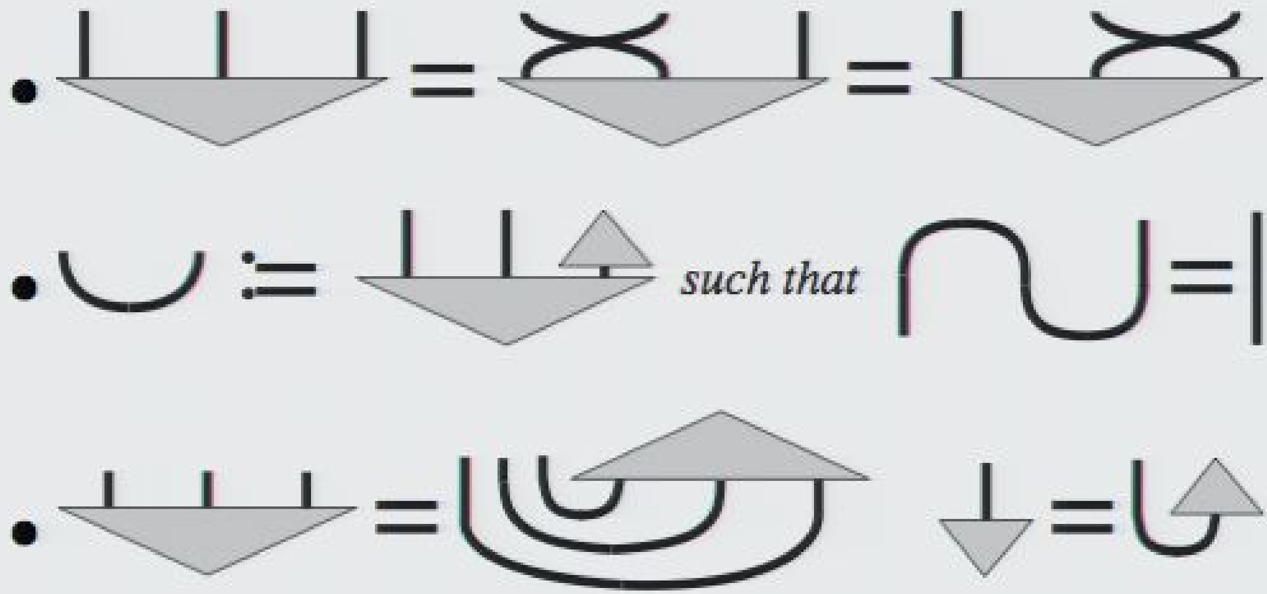


— *non-locality a la GHZ* —

Defn. A GHZ-state is a triple

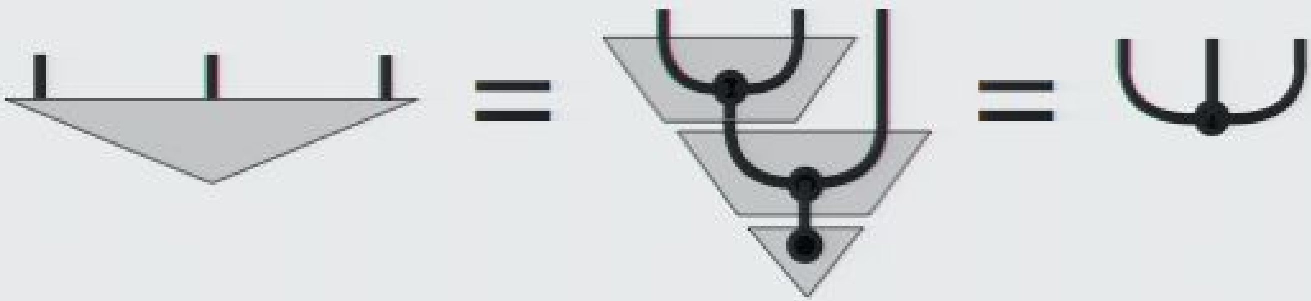
$$(X\,,\ \Psi : \mathrm{I} \to X \otimes X \otimes X\,,\ \epsilon : X \to \mathrm{I})$$

with



— *non-locality a la GHZ* —

Thm. **observables ≡ GHZ-states**

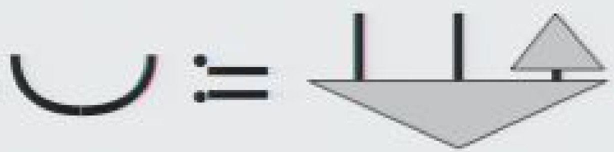
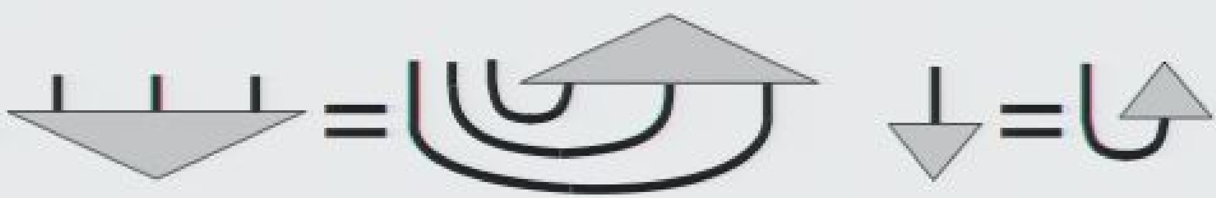


— non-locality a la GHZ —

Defn. A GHZ-state is a triple

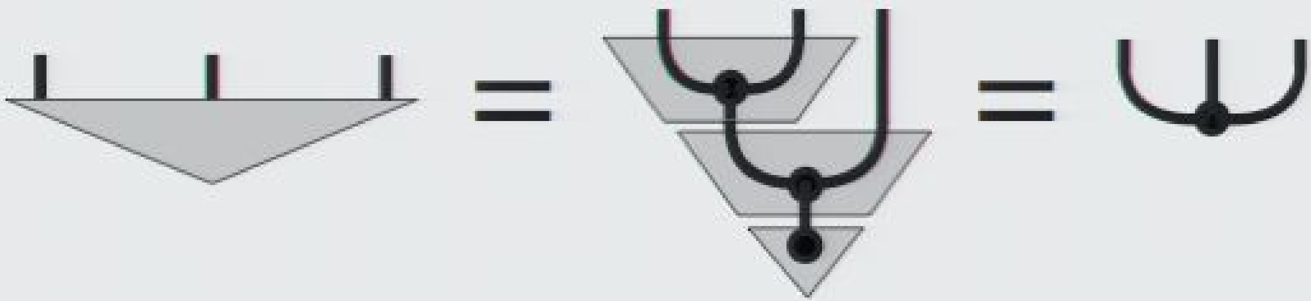
$$(X\ ,\ \Psi : \mathrm{I} \to X \otimes X \otimes X\ ,\ \epsilon : X \to \mathrm{I})$$

with

- 
-  *such that* 
- 

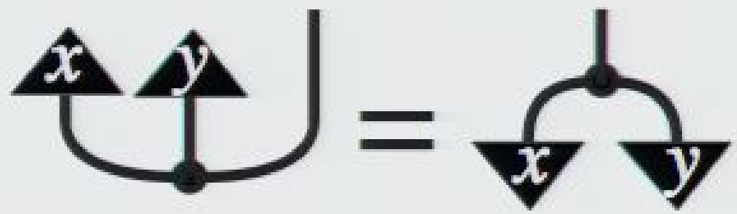
— *non-locality a la GHZ* —

Thm. **observables ≡ GHZ-states**



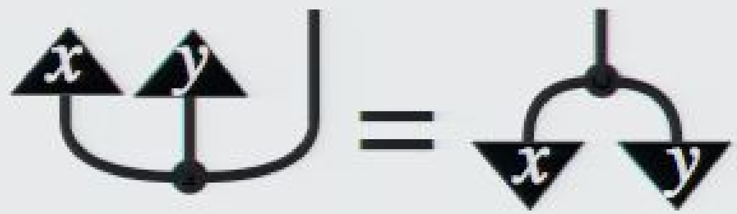
— *non-locality a la GHZ* —

Dfn. GHZ-correlations:



— *non-locality a la GHZ* —

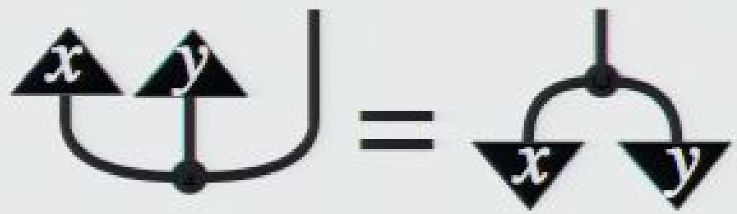
Dfn. GHZ-correlations:



Dfn. Mutually unbiassed (sub-)theory: states are either eigenstates or unbiassed states for observables.

— *non-locality a la GHZ* —

Dfn. GHZ-correlations:

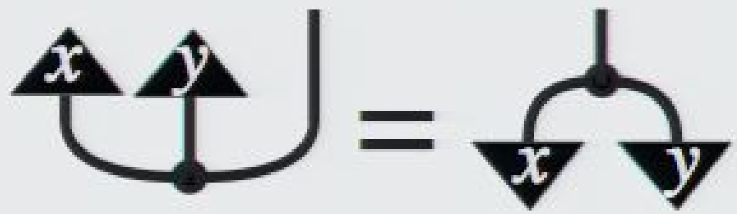


Dfn. Mutually unbiassed (sub-)theory: states are either eigenstates or unbiassed states for observables.

Thm. In mutually unbiassed theories GHZ-correlations are completely determined by the phase group.

— *non-locality a la GHZ* —

Dfn. GHZ-correlations:

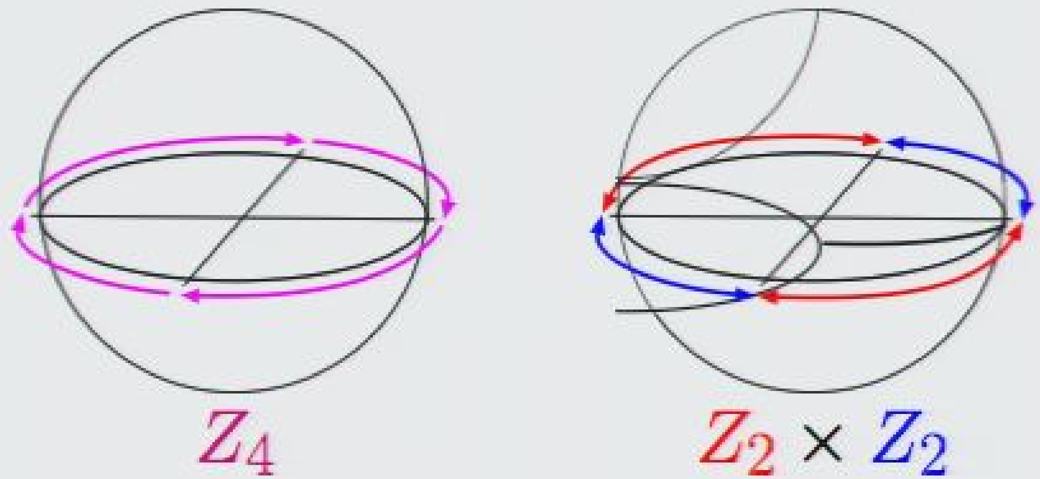


Dfn. Mutually unbiassed (sub-)theory: states are either eigenstates or unbiassed states for observables.

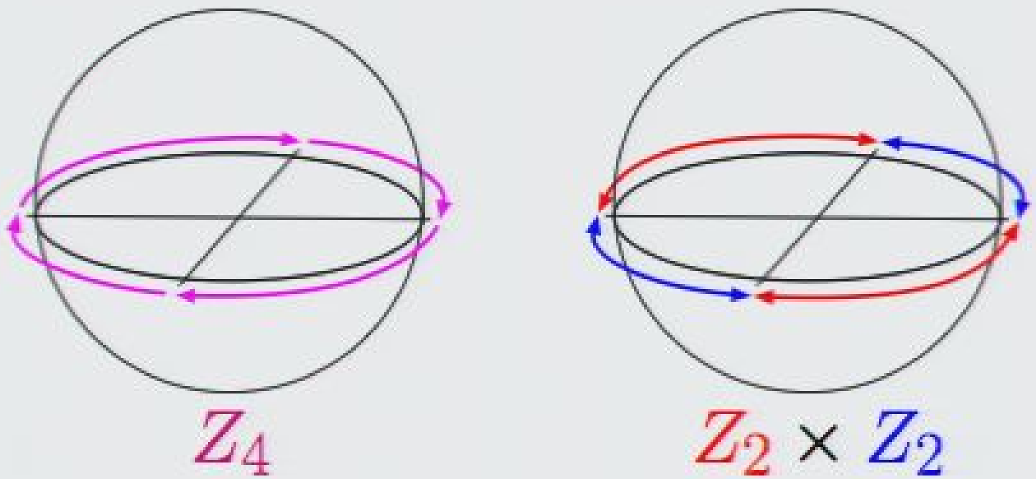
Thm. In mutually unbiassed theories GHZ-correlations are completely determined by the phase group.

⇒ Abelian groups classify non-local behaviours!

Recall that there are only two four-element groups:



Recall that there are only two four-element groups:

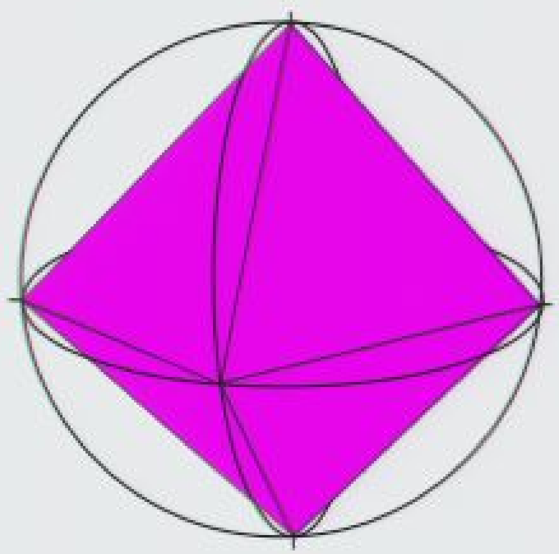


These correspond to:

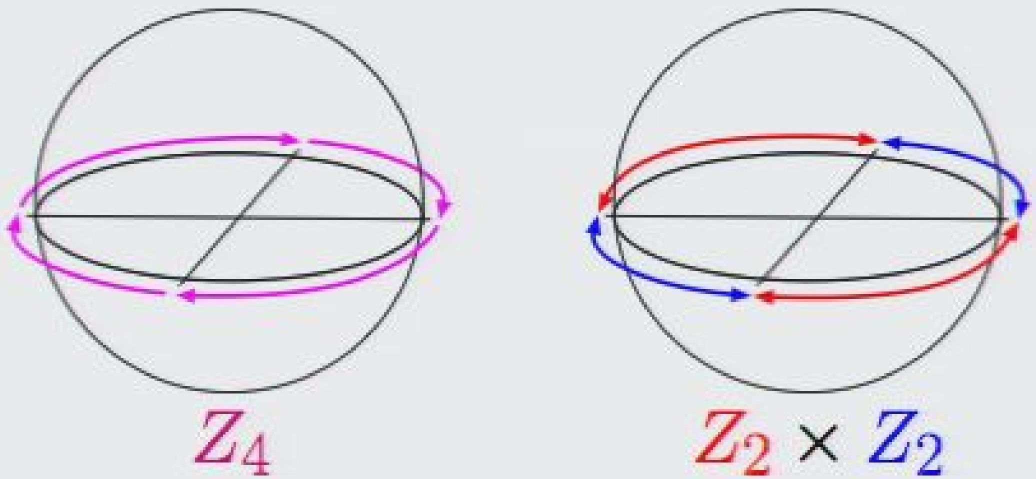
- Z_4 ~ **qubit stabiliser QM** ~ *non-local*
- $Z_2 \times Z_2$ ~ **Spekkens' toy theory** ~ *local*

Stab := sub-†-SMC of **FHilb** generated by

- nth powers of qubits $\mathcal{Q}$
- unitaries on $\mathcal{Q}$ $\bigcap$ symmetries Bloch-octahedron
- $$\mathcal{Q} \to \mathcal{Q} \otimes \mathcal{Q} :: \begin{cases} |0\rangle \mapsto |00\rangle \\ |1\rangle \mapsto |11\rangle \end{cases} \quad + \quad \textit{its unit}$$



Recall that there are only two four-element groups:



These correspond to:

- Z_4 ~ **qubit stabiliser QM** ~ *non-local*
- $Z_2 \times Z_2$ ~ **Spekkens' toy theory** ~ *local*

Stab := sub-†-SMC of **FHilb** generated by

- nth powers of qubits $\mathcal{Q}$
- unitaries on $\mathcal{Q}$ $\bigcap$ symmetries Bloch-octahedron
- $$\mathcal{Q} \to \mathcal{Q} \otimes \mathcal{Q} :: \begin{cases} |0\rangle \mapsto |00\rangle \\ |1\rangle \mapsto |11\rangle \end{cases} \quad + \quad \textit{its unit}$$

Spek := sub-†-SMC of **FRel** generated by

- nth powers of quads $\mathbb{IV}$
- all permutations on $\mathbb{IV}$
- $$\mathbb{IV} \to \mathbb{IV} \times \mathbb{IV} :: \begin{cases} 1 \mapsto \{(1,1),(2,2)\} \\ 2 \mapsto \{(1,2),(2,1)\} \\ 3 \mapsto \{(3,3),(4,4)\} \\ 4 \mapsto \{(3,4),(4,3)\} \end{cases} \quad + \quad \textit{its unit}$$

‘Z-, X- and Y-spin’ in **Spek**:

$$\mathrm{IV} \to \mathrm{IV} \times \mathrm{IV} :: \begin{cases} 1 \mapsto \{(1,1),(2,2)\} \\ 2 \mapsto \{(1,2),(2,1)\} \\ 3 \mapsto \{(3,3),(4,4)\} \\ 4 \mapsto \{(3,4),(4,3)\} \end{cases} \quad + \quad \textit{its unit}$$

$$\mathrm{IV} \to \mathrm{IV} \times \mathrm{IV} :: \begin{cases} 1 \mapsto \{(1,1),(3,3)\} \\ 3 \mapsto \{(1,3),(3,1)\} \\ 2 \mapsto \{(2,2),(4,4)\} \\ 4 \mapsto \{(2,4),(4,2)\} \end{cases} \quad + \quad \textit{its unit}$$

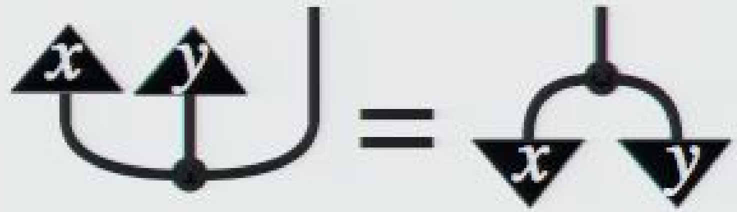
$$\mathrm{IV} \to \mathrm{IV} \times \mathrm{IV} :: \begin{cases} 1 \mapsto \{(1,1),(4,4)\} \\ 4 \mapsto \{(1,4),(4,1)\} \\ 3 \mapsto \{(3,3),(2,2)\} \\ 2 \mapsto \{(3,2),(2,3)\} \end{cases} \quad + \quad \textit{its unit}$$

Both **Stab** and **Spek** are generated by:

- nth powers of elementary system A
- group $G \simeq S(4)$ of isos on A
- 'copying' $\delta : A \to A \otimes A$ + *its unit*

Only difference is the interaction between copying and local unitaries. These are encoded in the phase group.

GHZ-correlations:



i.e. state of the third system given the outcomes of measurements on the first two systems, hence:

GHZ-correlations ≡ group multiplication (*)

Claim: The Z_4 phase group structure, via (*), suffices for an argument forbidding local hidden variables.

For $|+\rangle$ the unit and $|-\rangle$ the involutive we have:

$$|+\rangle \odot |+\rangle = |+\rangle \quad |+\rangle \odot |-\rangle = |-\rangle \quad |-\rangle \odot |-\rangle = |+\rangle$$

i.e. even occurrences of $|-\rangle$ in correlations.

For remaining elements $|=\rangle$ and $|\sharp\rangle$ we have:

$$|\sharp\rangle \odot |=\rangle = |+\rangle \quad |=\rangle \odot |=\rangle = |-\rangle \quad |\sharp\rangle \odot |\sharp\rangle = |-\rangle$$

i.e. odd occurrences of $\{|-\rangle, |=\rangle\}$ in correlations.

$$\{|+\rangle, |-\rangle\} \times \{|+\rangle, |-\rangle\} \times \{|+\rangle, |-\rangle\}$$

$$\{|+\rangle, |-\rangle\} \times \{|\sharp\rangle, |=\rangle\} \times \{|\sharp\rangle, |=\rangle\}$$

$$\{|\sharp\rangle, |=\rangle\} \times \{|+\rangle, |-\rangle\} \times \{|\sharp\rangle, |=\rangle\}$$

$$\{|\sharp\rangle, |=\rangle\} \times \{|\sharp\rangle, |=\rangle\} \times \{|+\rangle, |-\rangle\}$$

Diagrammatic QM introductions:

Appetizer: *Kindergarten Quantum Mechanics.* **arXiv:quant-ph/0510032.**

Survey: *Quantum picturalism.* **arXiv:0908.1787. (New)**

Categories for physicists:

Appetizer: *Introducing categories to the practicing physicist.* **arXiv:0808.1032.**

Tutorial: *Categories for the practicing physicist.* **arXiv:0905.3010. (New)**

Some technical papers:

C., Edwards and Spekkens: *The group theoretic origin of non-locality for qubits.* **web.comlab.ox.ac.uk/publications/publication3026-abstract.html.**

C. & Duncan: *Interacting quantum observables.* **arXiv:0906.4725.**

C., Paquette & Pavlovic: *Classical and quantum structuralism.* **arXiv:0904.1997.**