

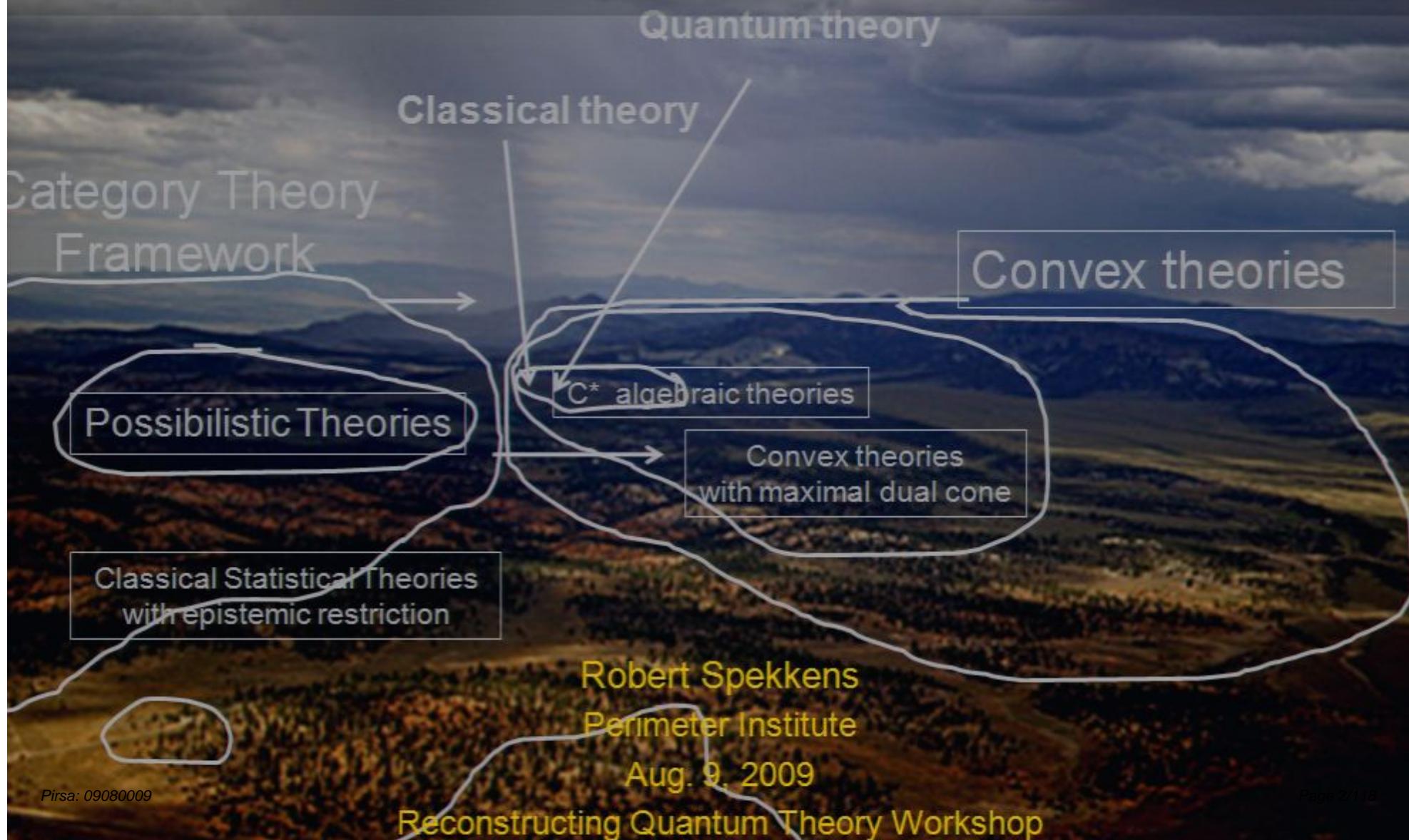
Title: The power of epistemic restrictions in reconstructing quantum theory

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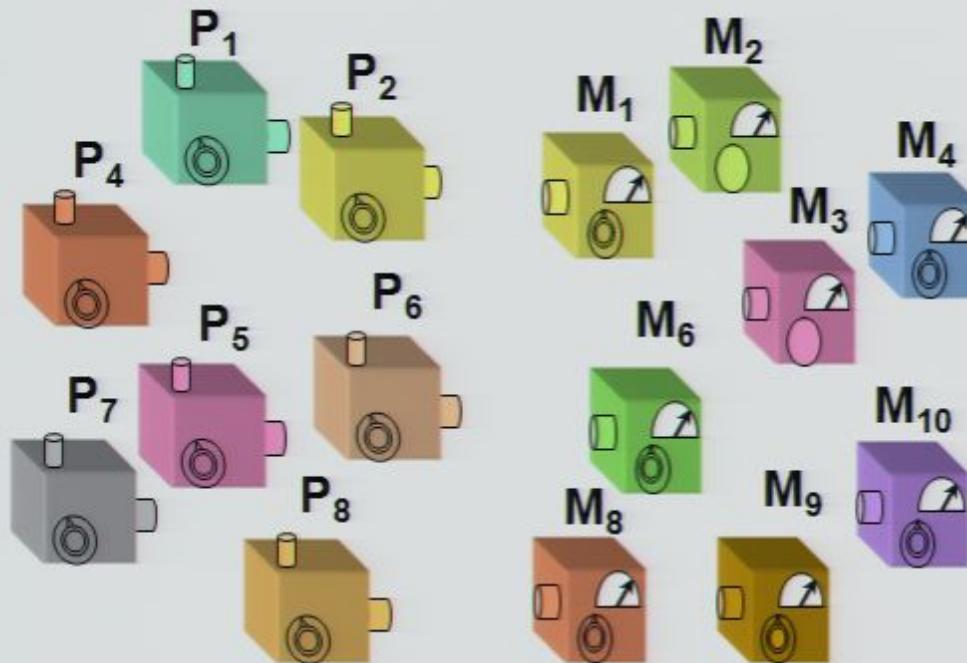
Abstract: <span>A significant part of quantum theory can be obtained from a single innovation relative to classical theories, namely, that there is a fundamental restriction on the sorts of statistical distributions over classical states that can be prepared. &nbsp;(Such a restriction is termed â€epistemicâ• because it implies a fundamental limit on the amount of knowledge that any observer can have about the classical state.) &nbsp;I will support this claim in the particular case of a theory of many classical 3-state systems (trits) where if a particular kind of epistemic restriction is assumed -- one that appeals to the symplectic structure of the classical state space -- it is possible to reproduce the operational predictions of the stabilizer formalism for qutrits. &nbsp;The latter is an interesting subset of the full quantum theory of qutrits, a discrete analogue of Gaussian quantum mechanics. This is joint work with Olaf Schreiber.</span>

# The power of epistemic restrictions in axiomatizing quantum theory



## Two approaches to axiomatization

### Operational approach



Axioms are constraints on experimental statistics  $p(k|M, P)$

### Ontological approach

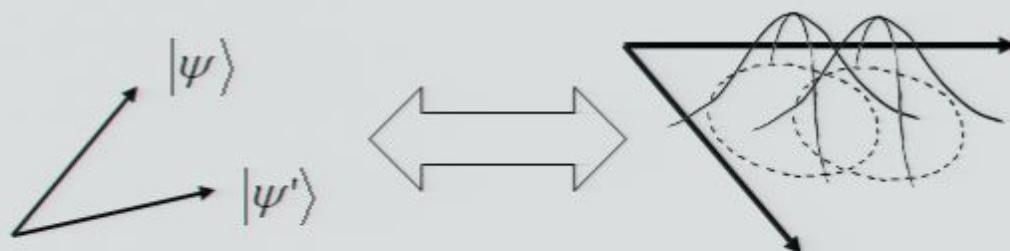


Classical statistical theory  
+  
**fundamental restriction on statistical distributions**  
↓  
A large part of quantum theory

In the sense of reproducing the operational predictions

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In the sense of reproducing the operational predictions



i.e. quantum states emerge as statistical distributions (epistemic states)

## **Classical theory**

Mechanics

## **Statistical theory for the classical theory**

Liouville mechanics

## **Restricted Statistical theory for the classical theory**

Restricted Liouville mechanics  
= Gaussian quantum mechanics

## **Classical theory**

Mechanics

Bits

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Statistical theory of bits

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 $\simeq$  Stabilizer theory for qubits

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These theories include:

- **Most basic quantum phenomena**

e.g. noncommutativity, Interference, coherent superposition, collapse, complementary bases, no-cloning, ...

- **Most quantum information-processing tasks**

e.g. teleportation, key distribution, quantum error correction, improvements in metrology, dense coding, ...

- **A large part of entanglement theory**

e.g. monogamy, distillation, deterministic and probabilistic single copy entanglement transformation, catalysis, ...

- **A large part of the formalism of quantum theory**

e.g. Choi-Jamiolkowski isomorphism, Naimark extension, Stinespring dilation, multiple convex decompositions of states, ...

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## **Classical theory**

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Optics

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Statistical optics

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Restricted statistical optics  
= linear quantum optics

## **Classical theory**

Mechanics

Bits

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Optics

Electrodynamics

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Statistical electrodynamics

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Restricted statistical electrodynamics  
 $=/\simeq$  part of QED?

## **Classical theory**

Mechanics

Bits

Trits

Optics

Electrodynamics

General relativity

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Statistical GR

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 $=/\simeq$  part of QED?

Restricted statistical GR  
 $=/\simeq$  part of quantum gravity?

## Categorizing quantum phenomena

**Those arising in a restricted statistical classical theory**

**Those not arising in a restricted statistical classical theory**

Wave-particle duality

noncommutativity

entanglement

Quantized spectra

Key distribution

Improvements in metrology

**Bell inequality violations  
+ no-signalling**

**Computational speed-up**

Particle statistics

Interference

Teleportation

Coherent superposition

No cloning

Quantum eraser

**Bell-Kochen-Specker theorem**

Pre and post-selection  
“paradoxes”

collapse

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- Pre and post-selection “paradoxes”
- Others...

### Those not arising in a restricted statistical classical theory

- Bell inequality violations + no-signalling
- Computational speed-up (if it exists)
- Bell-Kochen-Specker theorem
- Certain aspects of items on the left
- Others...

Quantized spectra?  
Particle statistics?  
Others...

# Categorizing quantum phenomena

## Those arising in a restricted statistical classical theory

- Interference
- Noncommutativity
- Entanglement
- Collapse
- Wave-particle duality
- Teleportation
- No cloning

***Not so strange after all!***

- Improvements in metrology
- Quantum eraser
- Coherent superposition
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- Others...

## Those not arising in a restricted statistical classical theory

- Bell inequality violations + no-signalling
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- Certain aspects of items on the left
- Others...

***Still surprising!  
Find more!  
Focus on these***

Quantized spectra?  
Particle statistics?  
Others...

## A research program

The success of restricted statistical theories suggests that:

**Quantum theory is best understood as a kind of probability theory**

Hardy, quant-ph/0101012

Barrett, quant-ph/0508211

Leifer, quant-ph/0611233

Etcetera, etcetera

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The success of restricted statistical theories suggests that:

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**Quantum states are states of incomplete knowledge**

Caves and Fuchs, quant-ph/9601025  
Rovelli, quant-ph/9609002  
Hardy, quant-ph/9906123  
Brukner and Zeilinger, quant-ph/0005084  
Kirkpatrick, quant-ph/0106072  
Collins and Popescu, quant-ph/0107082  
Fuchs, quant-ph/0205039  
Emerson, quant-ph/0211035  
etcetera

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Speculative possibility for an axiomatization of quantum theory

Principle 1: There is a fundamental restriction on observers capacities to know and control the systems around them

Principle 2: ??? (Some change to the classical picture of the world)

We need to explore possibilities for principle 2, even if only in toy theories  
Ultimately, we need to derive principle 1 from principle 2

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# Classical complementarity as a statistical restriction with broad applicability

Joint work with Olaf Schreiber

Building upon:

Spekkens, quant-ph/0401052 [Phys. Rev. A 75, 032110 (2007)]

S. van Enk, arxiv:0705.2742 [Found. Phys. 37, 1447 (2007)]

D. Gross, quant-ph/0602001 [J. Math. Phys. 47, 122107 (2006)]

Bartlett, Rudolph, Spekkens, unpublished

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Jointly-measurable observables = a commuting set of observables  
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We begin with classical mechanics...

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Configuration space:  $\mathbb{R}^n \ni (q_1, q_2, \dots, q_n)$

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The linear functionals / canonical variables are:

$$F = a_1 X_1 + b_1 P_1 + \dots + a_n X_n + b_n P_n \quad a_1, b_1, \dots, a_n, b_n \in \mathbb{R}$$

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Poisson bracket of functionals = symplectic inner product of vectors

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The principle of classical complementarity:

An observer can only have knowledge of the values of a commuting set of canonical variables.

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The principle of classical complementarity:

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More precisely, for commuting set S

Values of variables in S are known perfectly

Values of variables not in S are completely unknown

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These are specified by:

A set of known variables  $\mathcal{V}$

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$$v(X_1) = 2, v(P_2) = 2$$

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$$\forall F, G \in \mathcal{V} : \{F, G\} = 0$$

Example:

$$\mathcal{V} = \{X_1, P_2\}$$

A valuation of the known variables

$$v : \mathcal{V} \rightarrow \mathbb{R}/\mathbb{Z}_d$$

$$v(X_1) = 2, v(P_2) = 2$$

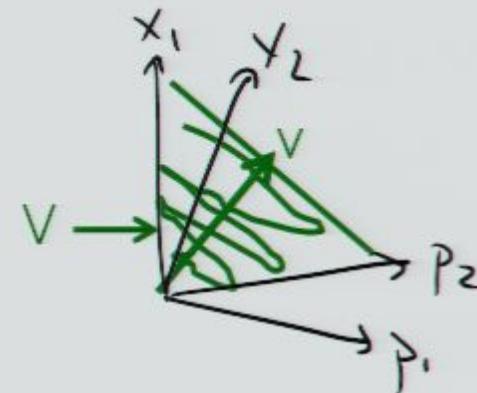
Equivalently,

An isotropic subspace  $V$  of  $\Omega^*$

$$\forall F, G \in V : [F, G] = 0$$

A valuation vector  $v \in V$

$$v : \forall F \in V, F^T v = v(F)$$



The ontic states consistent with the epistemic state  $(V, v)$  are  
 $\{m \in \Omega \mid \forall F \in \mathcal{V} : F(m) = v(F)\}$

## Valid epistemic states:

These are specified by:

A set of known variables  $\mathcal{V}$

$$\forall F, G \in \mathcal{V} : \{F, G\} = 0$$

Example:

$$\mathcal{V} = \{X_1, P_2\}$$

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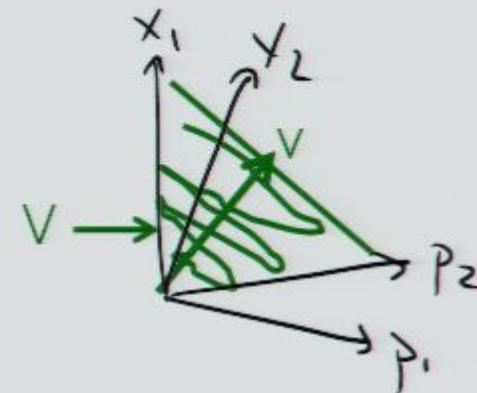
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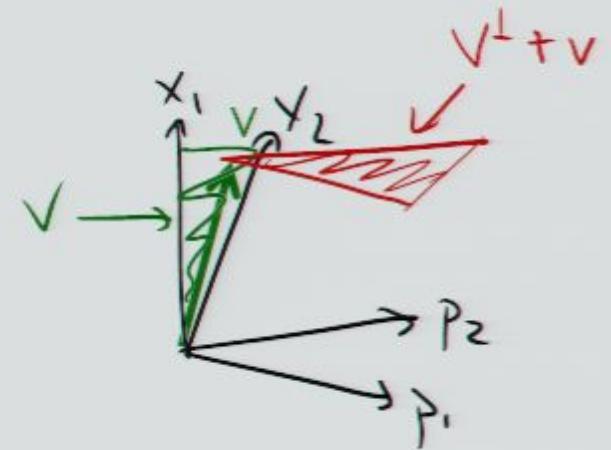
$$\begin{aligned} & \{m \in \Omega \mid \forall F \in \mathcal{V} : F(m) = v(F)\} \\ &= \{m \in \Omega \mid \forall F \in V : F^T m = F^T v\} \end{aligned}$$

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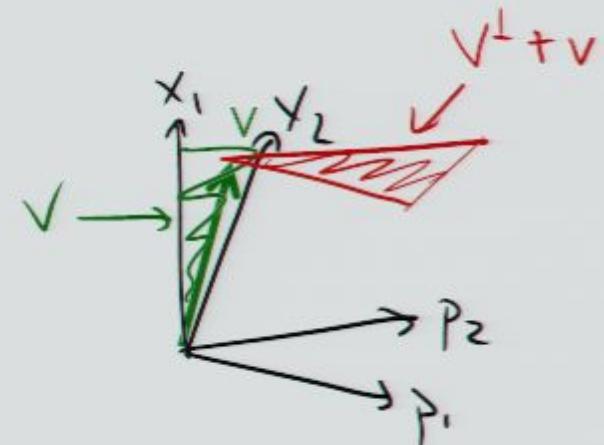
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(Dirac-delta / Kronecker delta)

The associated distribution is

$$p_{V,v}(m) = \frac{1}{N} \delta_{V^\perp + v}(m)$$



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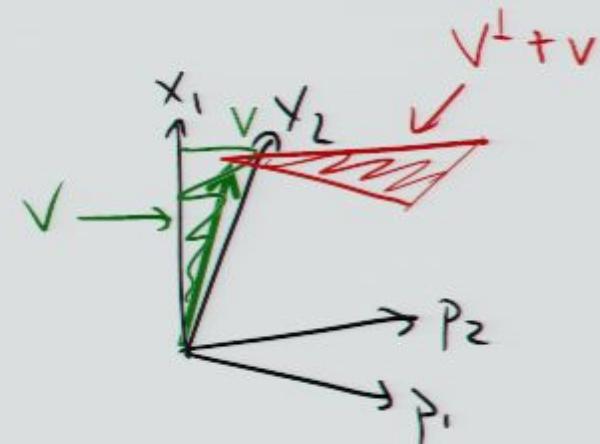
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Example

$$\mathcal{V} = \{X_1, X_2\}$$

$$v(X_1) = 1, v(X_2) = 2$$

$$\begin{aligned} V^\perp + v &= \{m \in \Omega \mid X_1(m) = 0, X_2(m) = 2\} \\ &= \{(0, s, 2, t) \mid s, t \in \mathbb{R}\} \end{aligned}$$



The ontic states consistent with the epistemic state  $(V, v)$  are

$$\begin{aligned} & \{m \in \Omega \mid \forall F \in \mathcal{V} : F(m) = v(F)\} \\ &= \{m \in \Omega \mid \forall F \in V : F^T m = F^T v\} \\ &= \{m \in \Omega \mid P_V m = v\} \\ &\equiv V^\perp + v \end{aligned}$$

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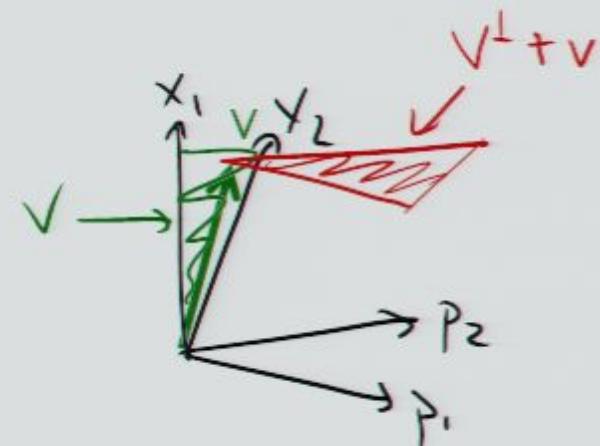
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Valid reversible transformations:

Those that preserve the Poisson bracket / symplectic inner product:

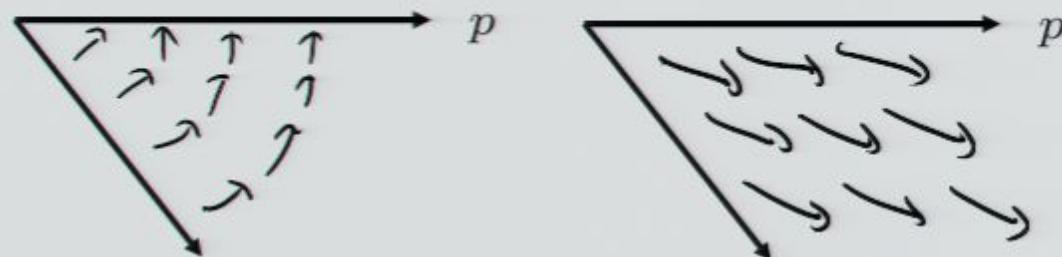
The group of symplectic affine transformations (Clifford group)

for  $m \in \Omega$

$$m \mapsto Sm + a$$

where  $[Su, Sv] = [u, v]$  Symplectic

and  $a \in \Omega$  Affine (Heisenberg-Weyl)

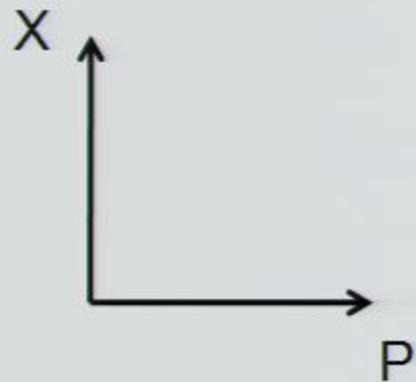


**Valid reproducible measurements:**

Any commuting set of canonical variables

## Restricted Liouville mechanics

$$\Omega = \mathbb{R}^{2n}$$

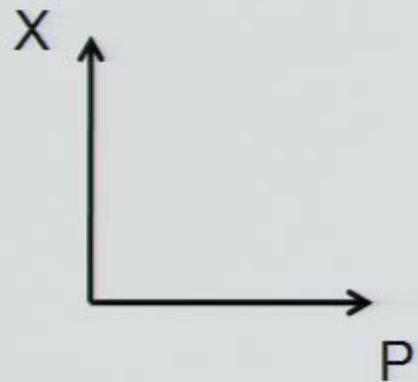


Valid reproducible measurements:

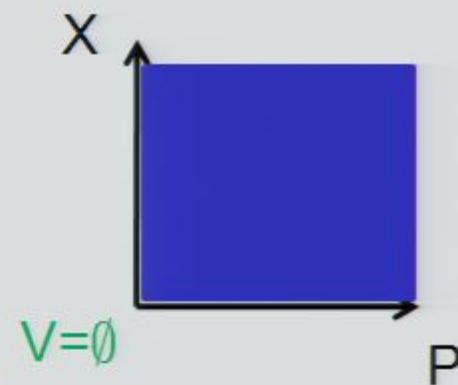
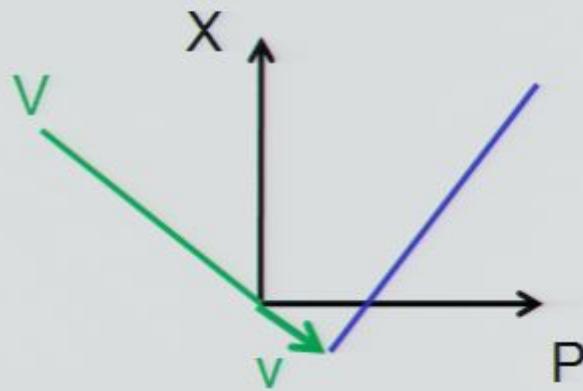
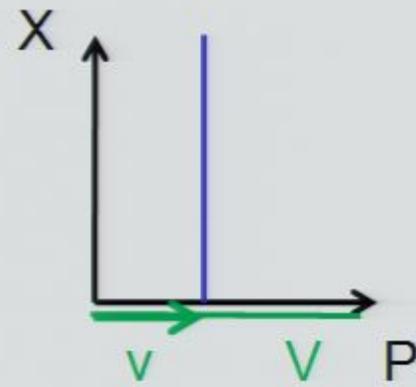
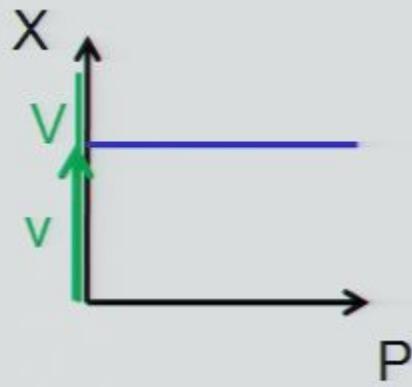
Any commuting set of canonical variables

## Restricted Liouville mechanics

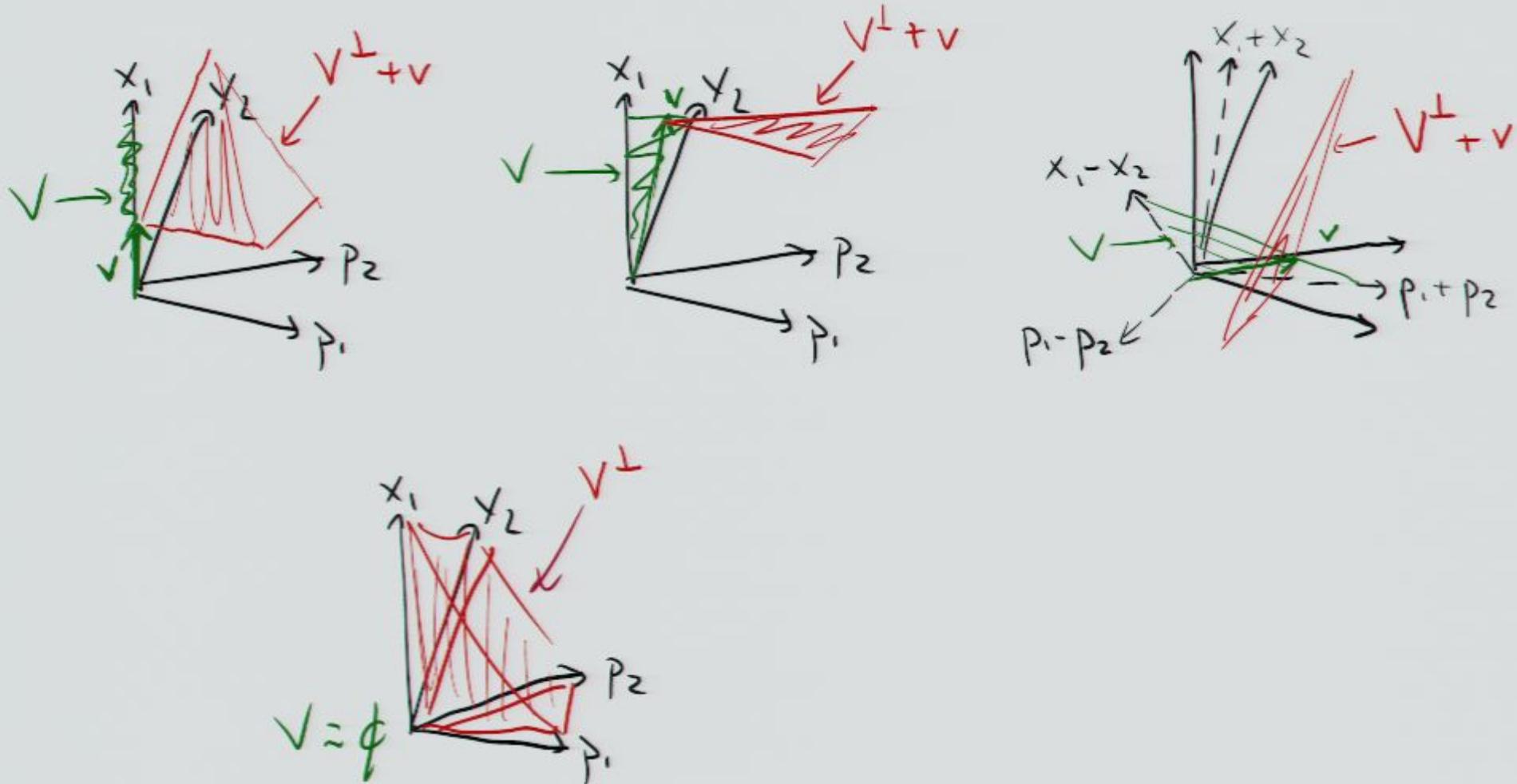
$$\Omega = \mathbb{R}^{2n}$$



## Valid epistemic states for a single degree of freedom

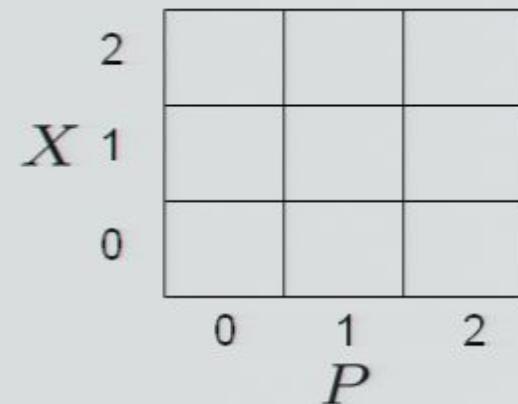


## Valid epistemic states for a pair of degrees of freedom



## Restricted statistical theory of trits

$$\Omega = (\mathbb{Z}_3)^{2n}$$



## Valid epistemic states for a single trit

Canonical variables  $aX + bP$

$X, P, X + P, X - P (= X + 2P)$

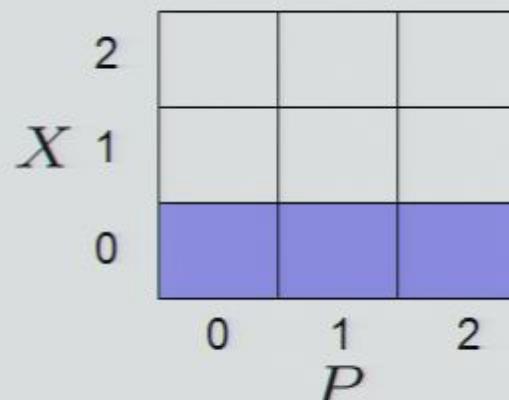
**Commuting** sets:

The singleton sets

the empty set

Suppose  $X$  is known to be 0

$$\begin{aligned}\mu(X, P) &= \frac{1}{3} \text{ if } X = 0 \\ &= 0 \text{ otherwise}\end{aligned}$$



## Epistemic states of maximal knowledge

$X$  known

	2		
$X$	1		
	0		
	0	1	2
		$P$	

$P$  known

	2		
$X$	1		
	0		
	0	1	2
		$P$	

$X + P$  known

	2		
$X$	1		
	0		
	0	1	2
		$P$	

$X - P$  known

	2		
$X$	1		
	0		
	0	1	2
		$P$	

	2		
$X$	1		
	0		
	0	1	2
		$P$	

	2		
$X$	1		
	0		
	0	1	2
		$P$	

	2		
$X$	1		
	0		
	0	1	2
		$P$	

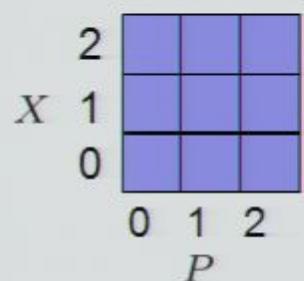
	2		
$X$	1		
	0		
	0	1	2
		$P$	

	2		
$X$	1		
	0		
	0	1	2
		$P$	

	2		
$X$	1		
	0		
	0	1	2
		$P$	

## Epistemic states of non-maximal knowledge

Nothing known



## Valid epistemic states for a pair of trits

Canonical variables  $a_1 X_1 + b_1 P_1 + a_2 X_2 + b_2 P_2 \quad a_1, b_1, a_2, b_2 \in \mathbb{Z}_3$

1<sup>st</sup> system variables  $X_1, P_1, X_1 + P_1, X_1 - P_1$

2<sup>nd</sup> system variables  $X_2, P_2, X_2 + P_2, X_2 - P_2$

Joint variables  $X_1 + X_2, X_1 + P_2, X_1 + (X_2 + P_2), \dots$

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Joint variables  $X_1 + X_2, X_1 + P_2, X_1 + (X_2 + P_2), \dots$

**Commuting** sets :

2 variables known:

$\{X_1, X_2\}, \{X_1, P_2\}, \{X_1, X_2 + P_2\}, \dots$

$\{X_1 - X_2, P_1 + P_2\}, \{X_1 + X_2, P_1 - P_2\}, \dots$

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1 variable known

The singleton sets

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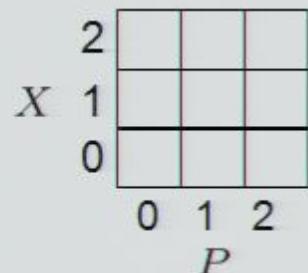
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1 variable known

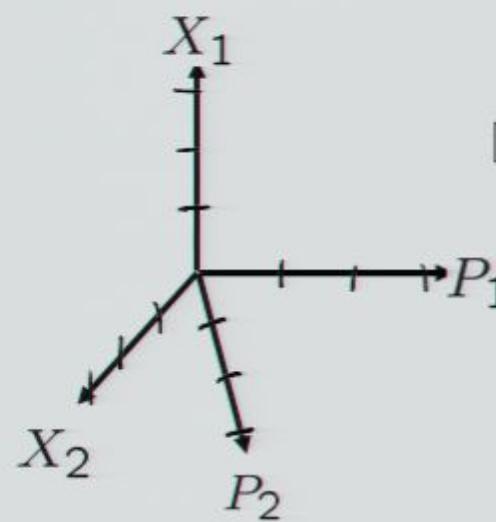
The singleton sets

Nothing known

How to represent this graphically



00	01	02	10	11	12	20	21
( $X, P$ )							
22							

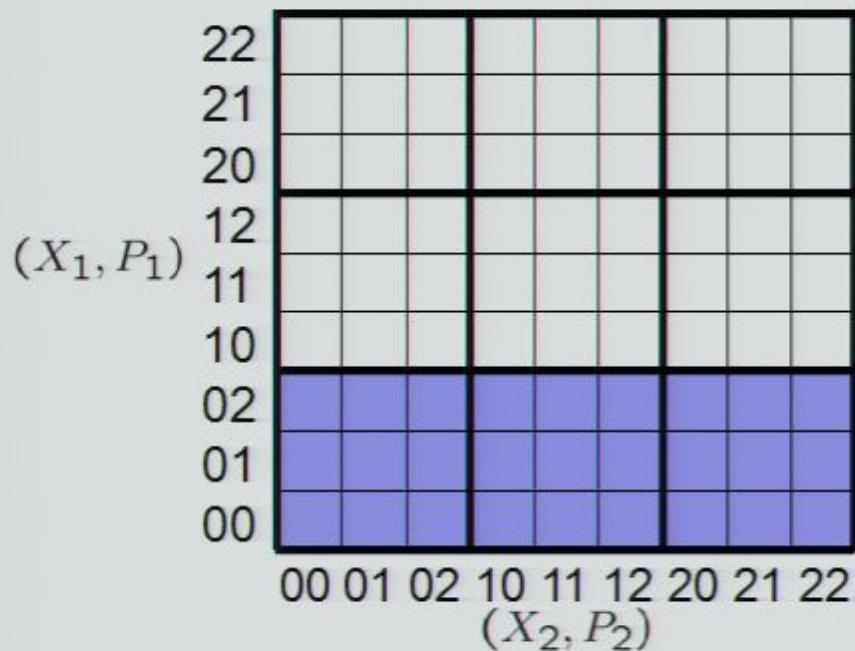


$(X_1, P_1)$

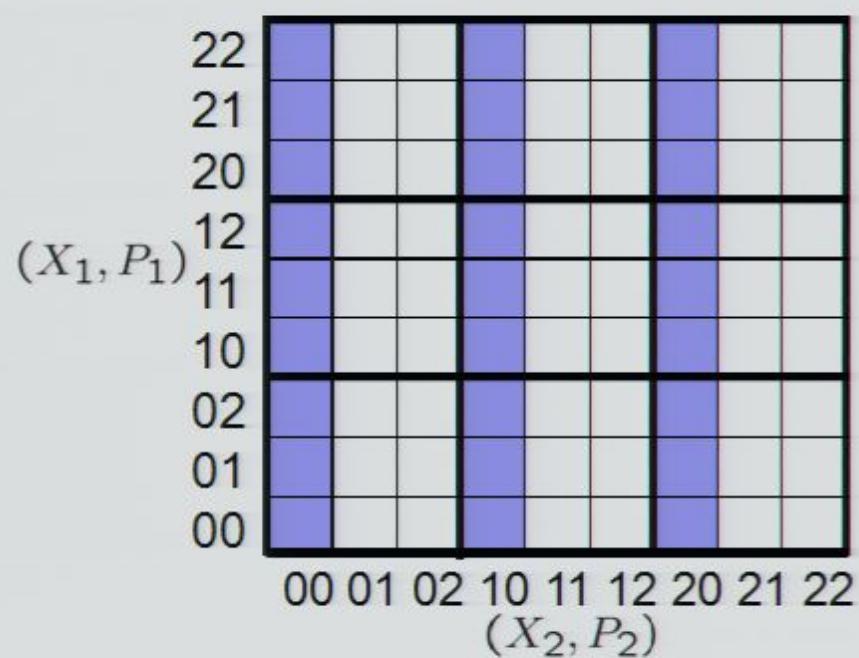
22							
21							
20							
12							
11							
10							
02							
01							
00							
00	01	02	10	11	12	20	21
( $X_2, P_2$ )							

## 1 variable known

$X_1$  known

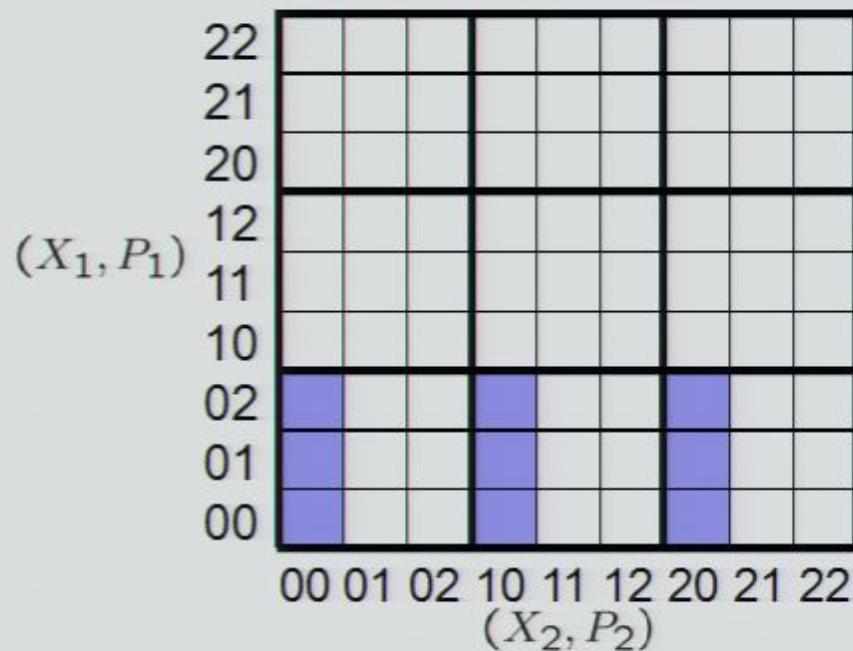


$P_2$  known



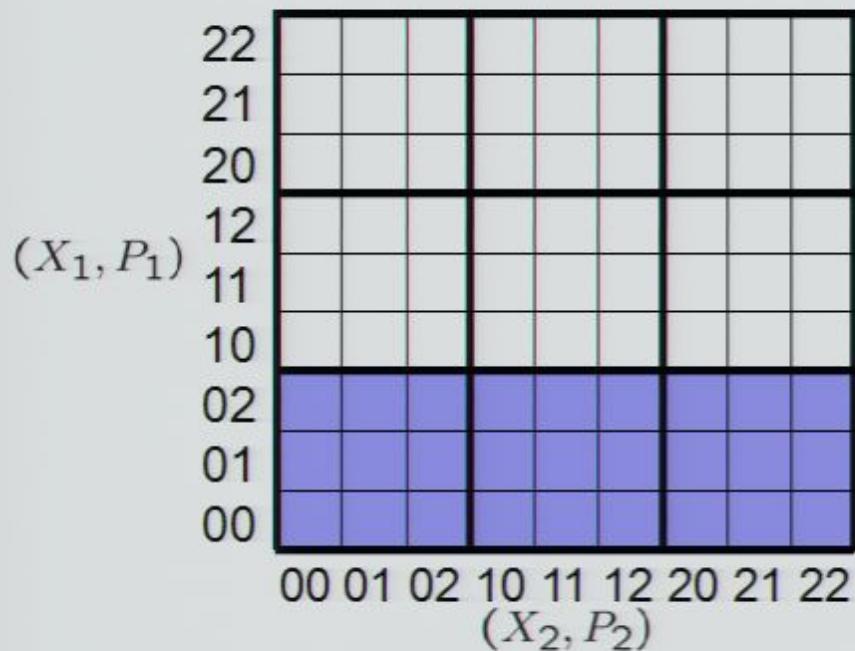
2 variables known

$X_1$  and  $P_2$  known

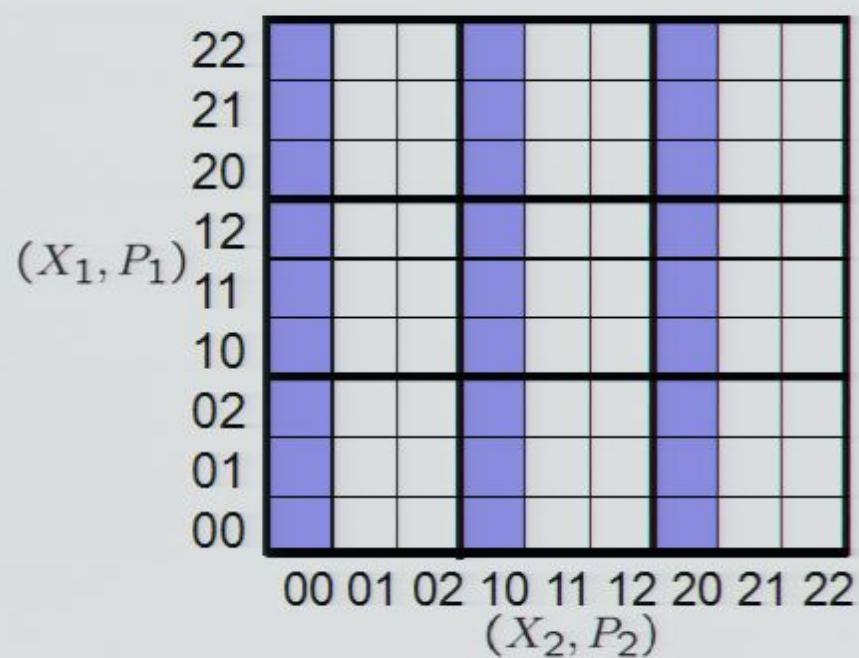


## 1 variable known

$X_1$  known

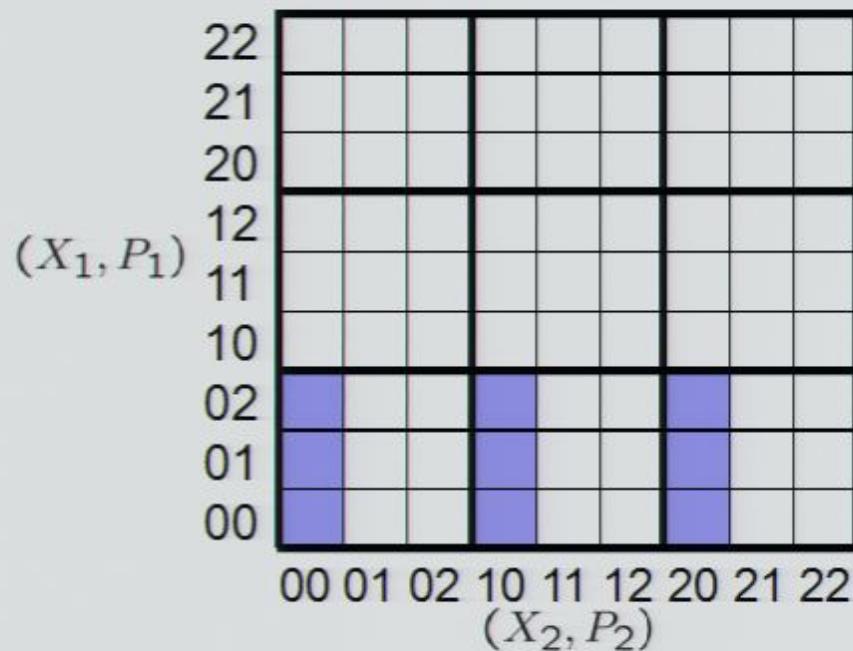


$P_2$  known

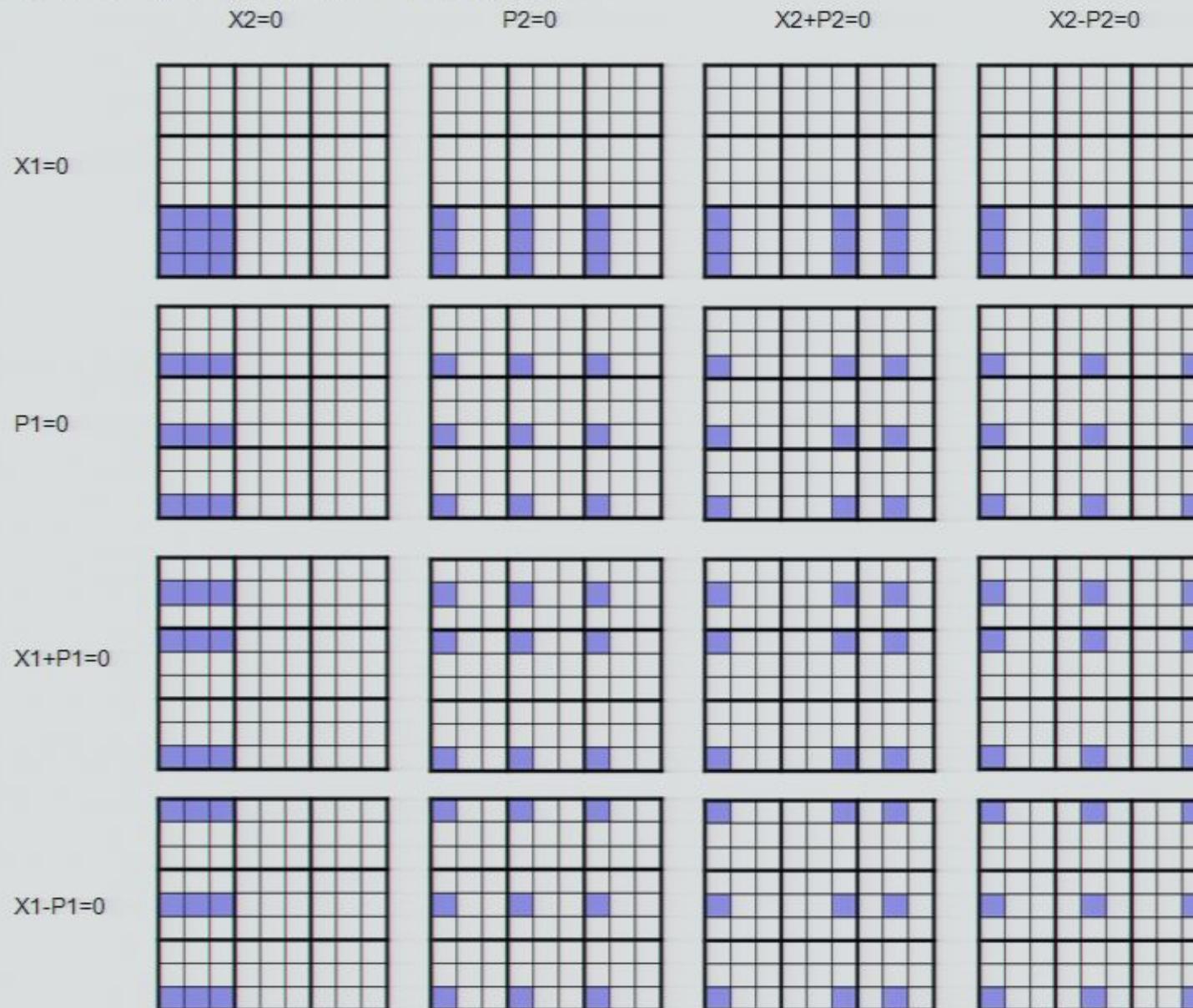


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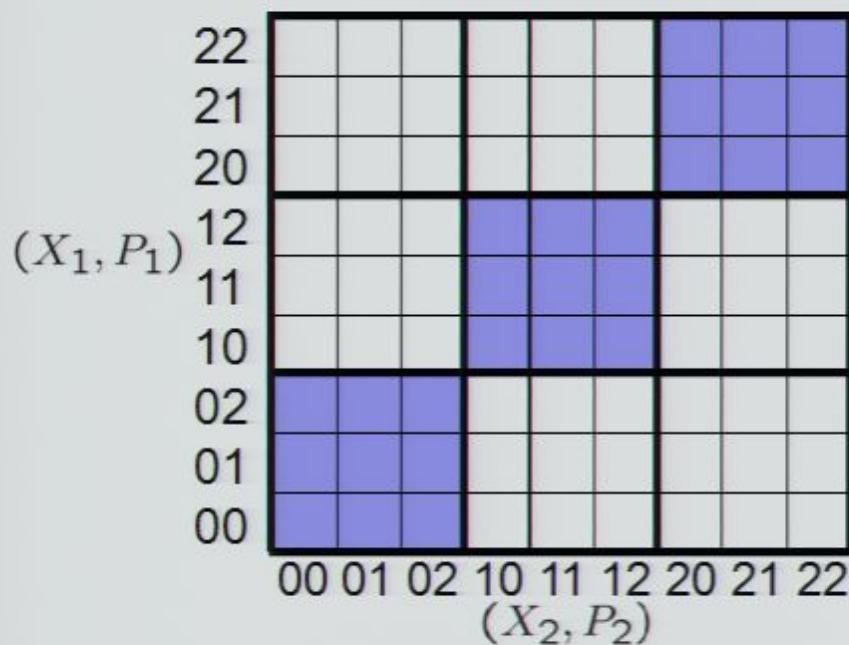


## Uncorrelated pure epistemic states

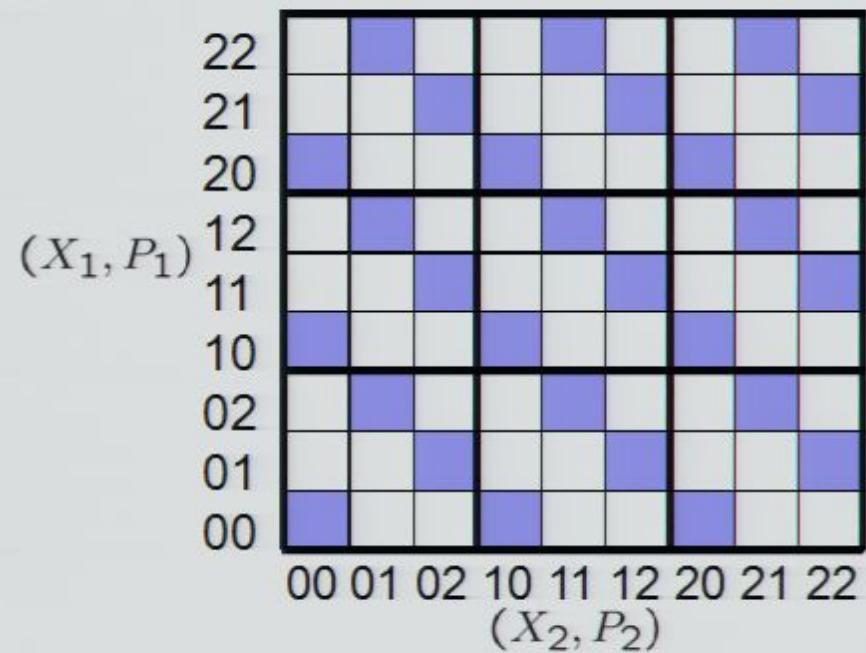


## 1 variable known

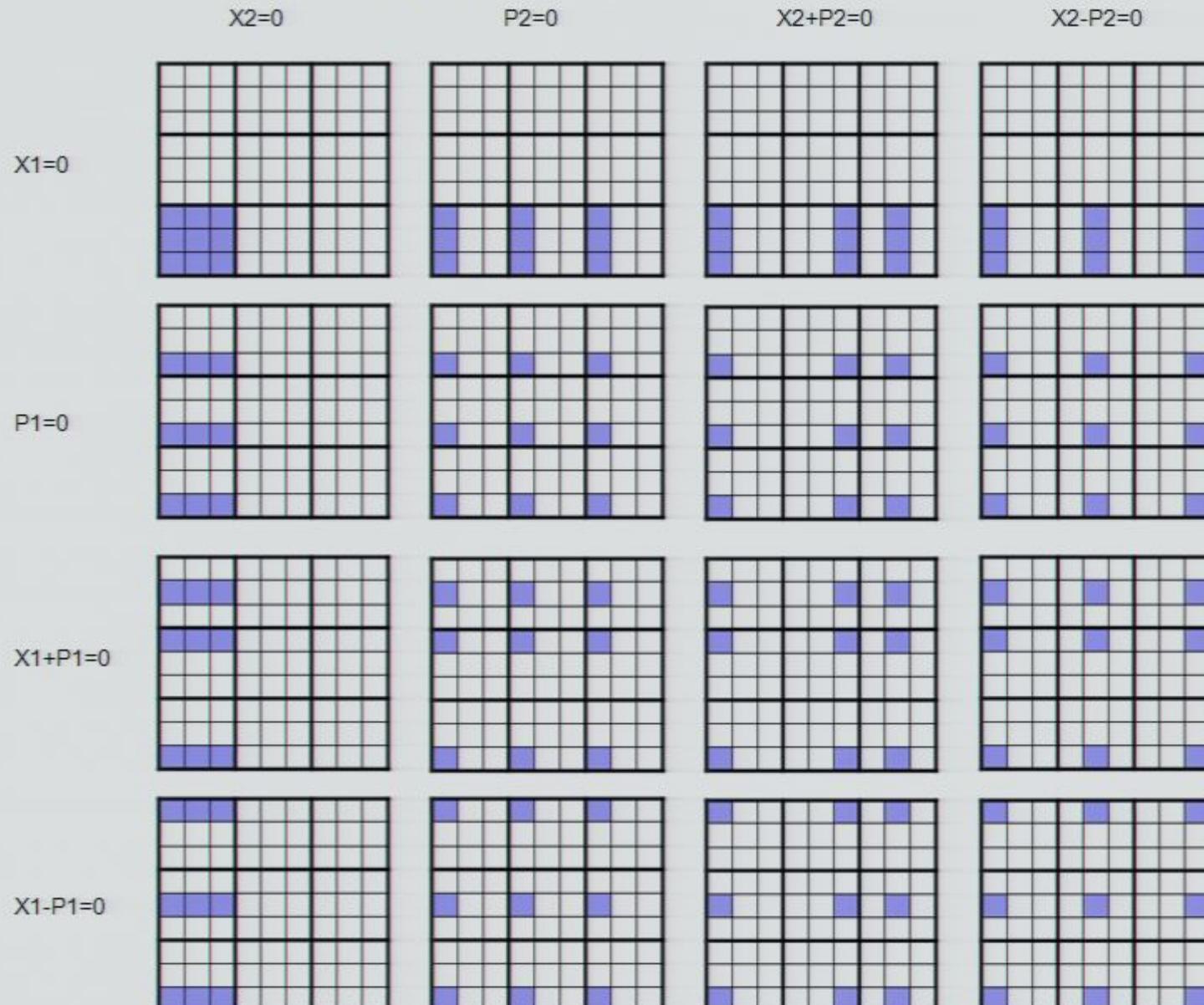
$X_1 - X_2$  known



$P_1 + P_2$  known

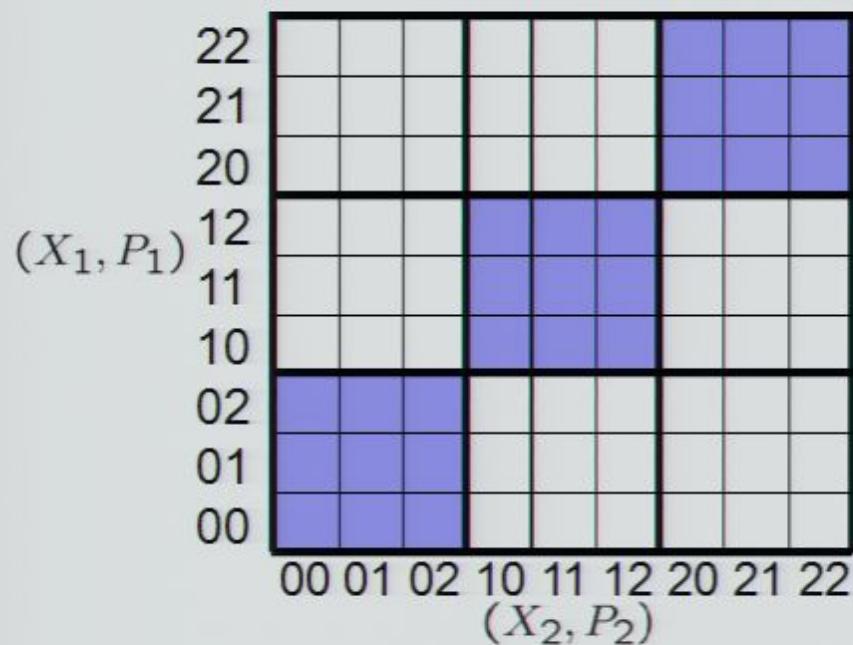


## Uncorrelated pure epistemic states

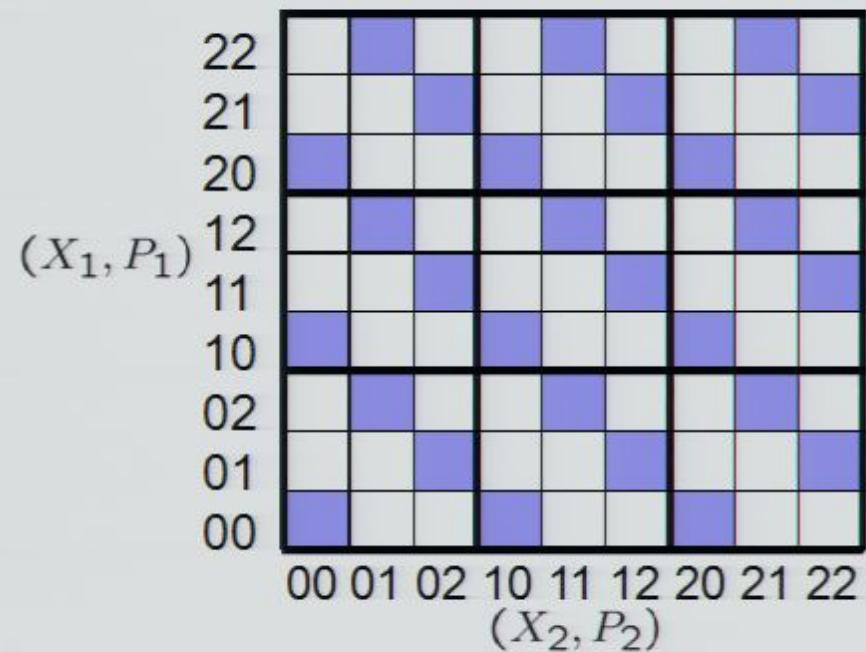


## 1 variable known

$X_1 - X_2$  known

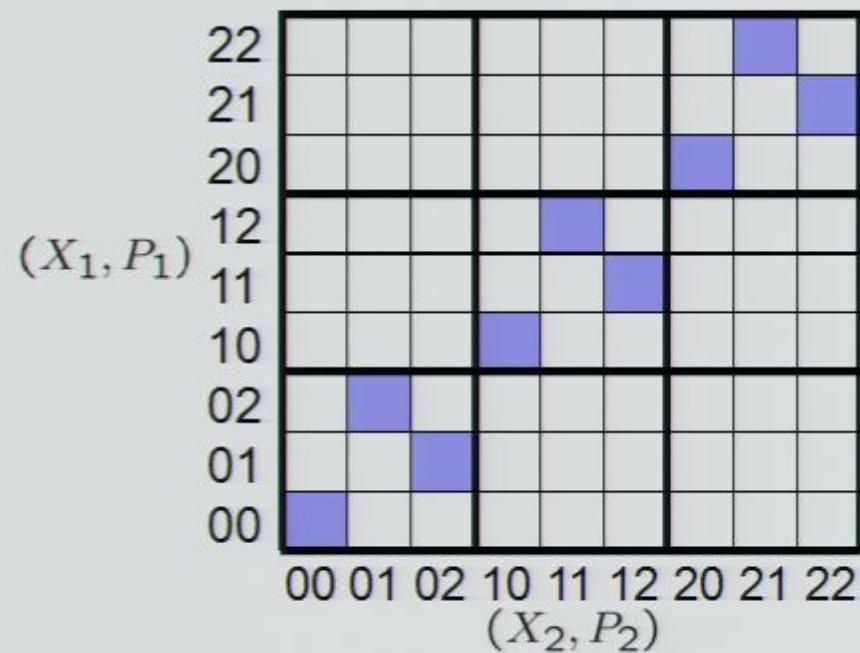


$P_1 + P_2$  known

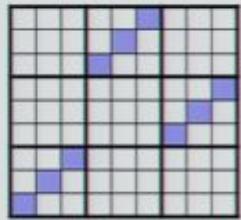


2 variables known

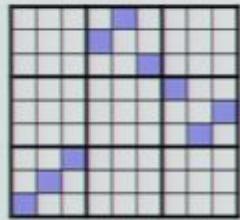
$X_1 - X_2$  and  $P_1 + P_2$  known



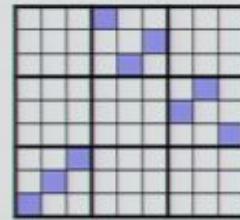
## Correlated pure epistemic states



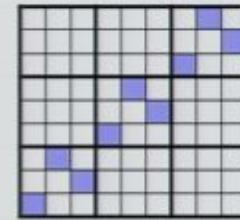
$$\begin{aligned} X_1 + X_2 &= 0 \\ P_1 - P_2 &= 0 \end{aligned}$$



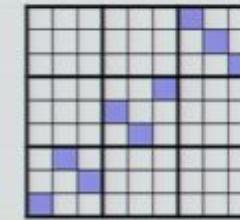
$$\begin{aligned} X_1 + X_2 &= 0 \\ X_1 + P_1 - P_2 &= 0 \end{aligned}$$



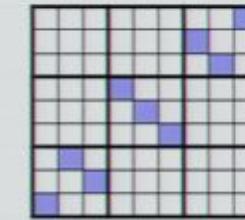
$$\begin{aligned} X_1 + X_2 &= 0 \\ X_1 - P_1 + P_2 &= 0 \end{aligned}$$



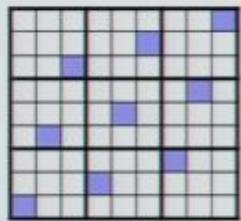
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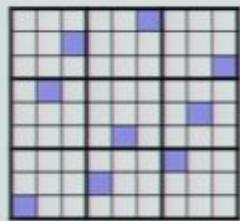
$$\begin{aligned} X_1 - X_2 &= 0 \\ X_1 - P_1 - P_2 &= 0 \end{aligned}$$



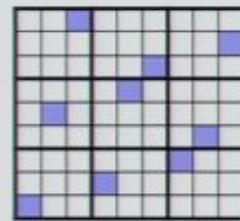
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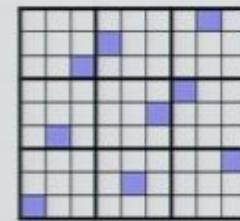
$$\begin{aligned} X_1 - P_2 &= 0 \\ P_1 - X_2 &= 0 \end{aligned}$$



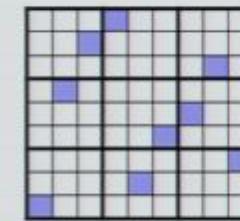
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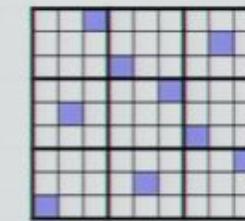
$$\begin{aligned} X_1 - P_2 &= 0 \\ P_1 - X_2 - P_2 &= 0 \end{aligned}$$



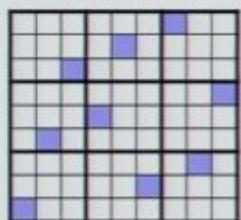
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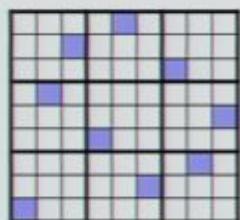
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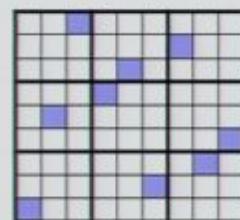
$$\begin{aligned} P_1 - P_2 &= 0 \\ X_1 - P_1 + X_2 &= 0 \end{aligned}$$



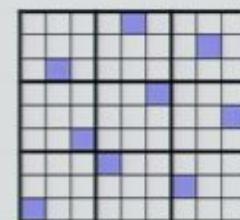
$$\begin{aligned} P_1 - X_2 &= 0 \\ X_1 - P_1 - P_2 &= 0 \end{aligned}$$



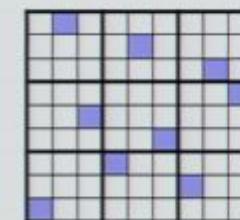
$$\begin{aligned} X_1 + P_1 - X_2 &= 0 \\ X_1 - P_1 - X_2 - P_2 &= 0 \end{aligned}$$



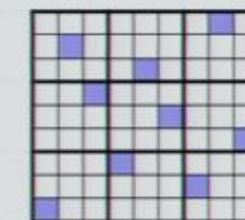
$$\begin{aligned} X_1 - X_2 - P_2 &= 0 \\ X_1 - P_1 + X_2 &= 0 \end{aligned}$$



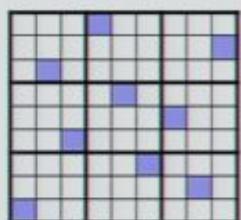
$$\begin{aligned} X_1 + P_2 &= 0 \\ P_1 + X_2 &= 0 \end{aligned}$$



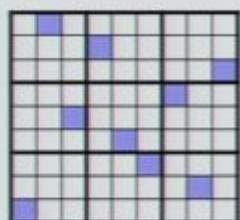
$$\begin{aligned} X_1 + P_2 &= 0 \\ P_1 + X_2 + P_2 &= 0 \end{aligned}$$



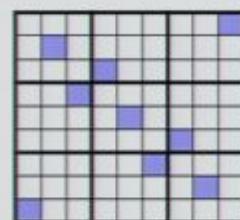
$$\begin{aligned} X_1 + P_2 &= 0 \\ P_1 + X_2 - P_2 &= 0 \end{aligned}$$



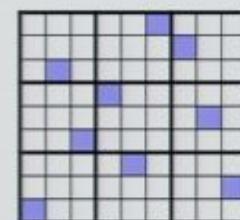
$$\begin{aligned} P_1 + X_2 &= 0 \\ X_1 + X_2 + P_2 &= 0 \end{aligned}$$



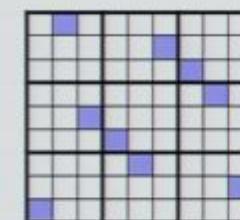
$$\begin{aligned} X_1 + P_1 - P_2 &= 0 \\ P_1 - X_2 + P_2 &= 0 \end{aligned}$$



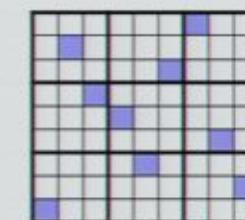
$$\begin{aligned} X_1 + P_1 + P_2 &= 0 \\ P_1 - P_2 &= 0 \end{aligned}$$



$$\begin{aligned} P_1 + X_2 &= 0 \\ X_1 + P_1 + P_2 &= 0 \end{aligned}$$



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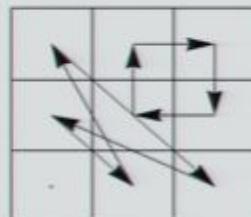
$$\begin{aligned} X_1 - P_1 - P_2 &= 0 \\ P_1 - X_2 - P_2 &= 0 \end{aligned}$$

## Valid reversible transformations

1 trit example:

$$X \mapsto P$$

$$P \mapsto -X$$



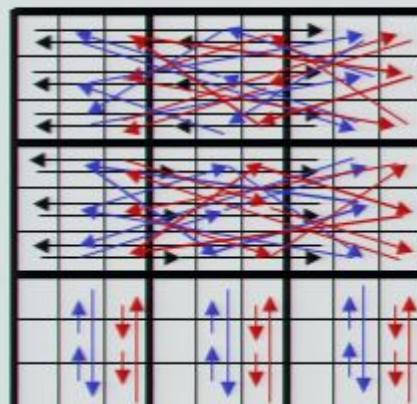
2 trit example:

$$X_1 \mapsto X_1$$

$$P_1 \mapsto P_1 - P_2$$

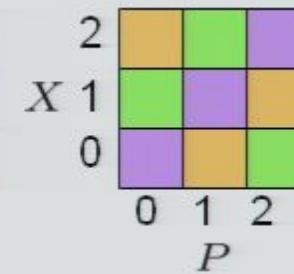
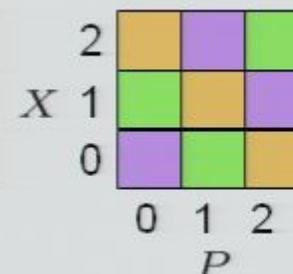
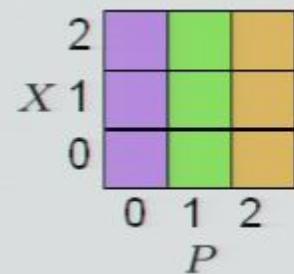
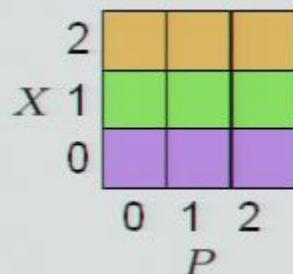
$$X_2 \mapsto X_1 + X_2$$

$$P_2 \mapsto P_2$$

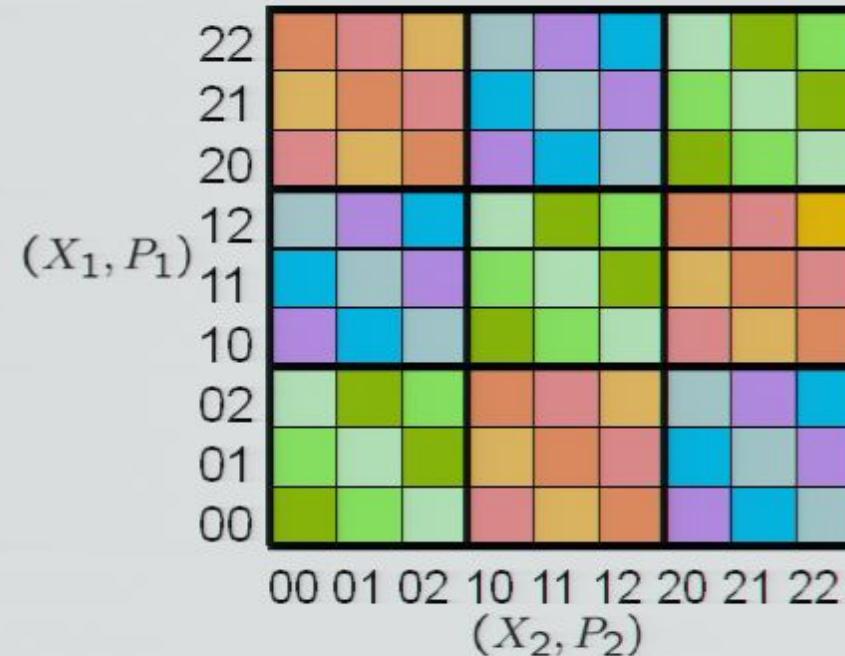
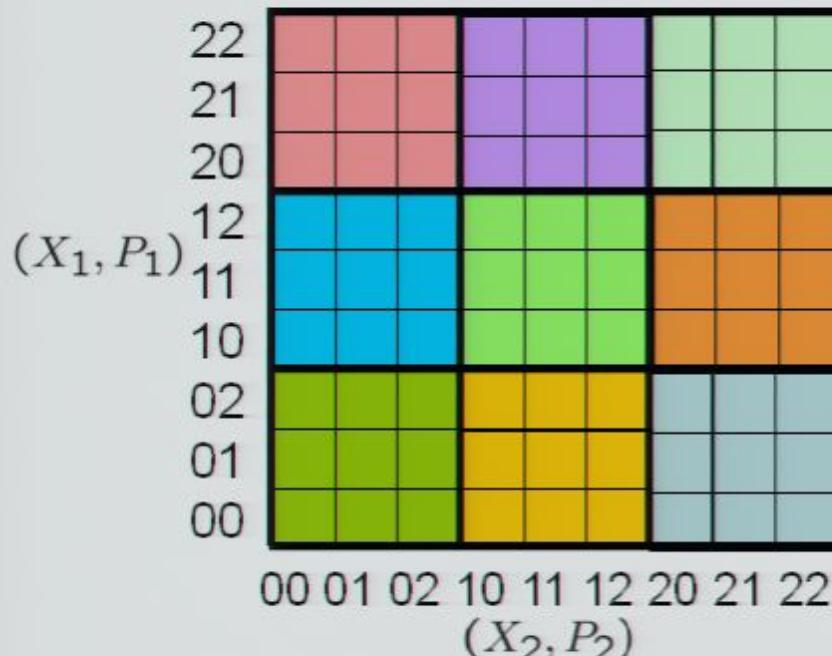


## Valid reproducible measurements

On a single trit



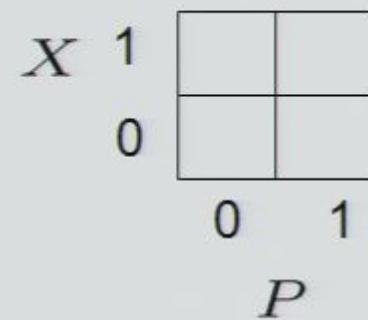
On a pair of trits



etc.

## Restricted statistical theory of bits

$$\Omega = (\mathbb{Z}_2)^{2n}$$



## A single bit

Canonical variables  $aX + bP$        $a, b \in \mathbb{Z}_2$   
 $X, P, X + P (= X - P)$

Epistemic states of maximal knowledge

$X$  known

$X$	1	■■■■
	0	■■■■
	0	1

$P$  known

$X$	1	■■■■
	0	■■■■
	0	1

$X + P$  known

$X$	1	■■■■
	0	■■■■
	0	1

$X$	1	■■■■
	0	■■■■
	0	1

$X$	1	■■■■
	0	■■■■
	0	1

$X$	1	■■■■
	0	■■■■
	0	1

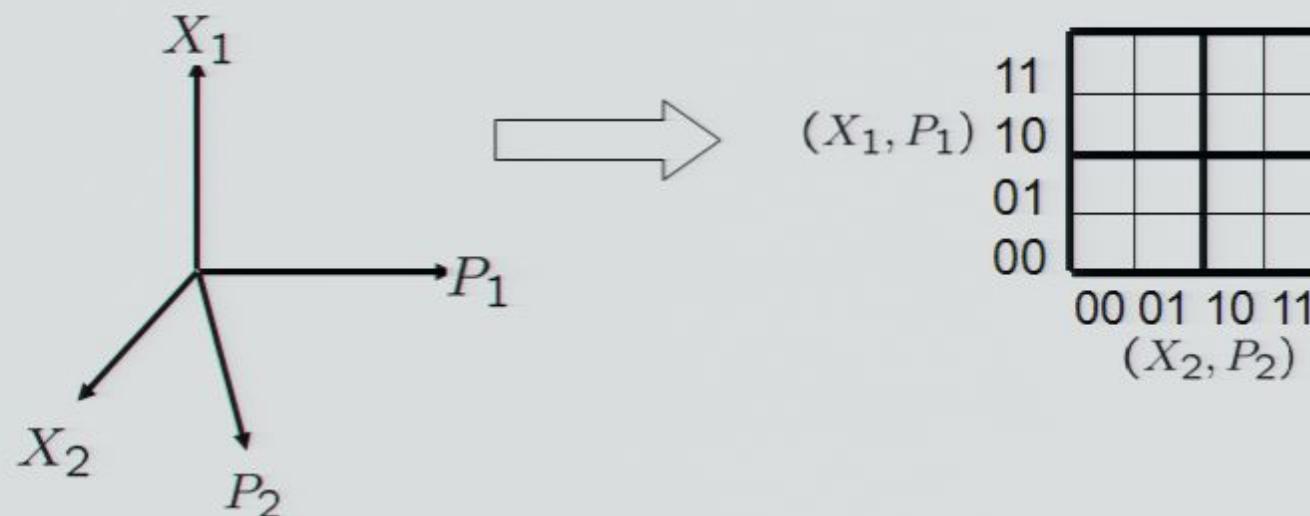
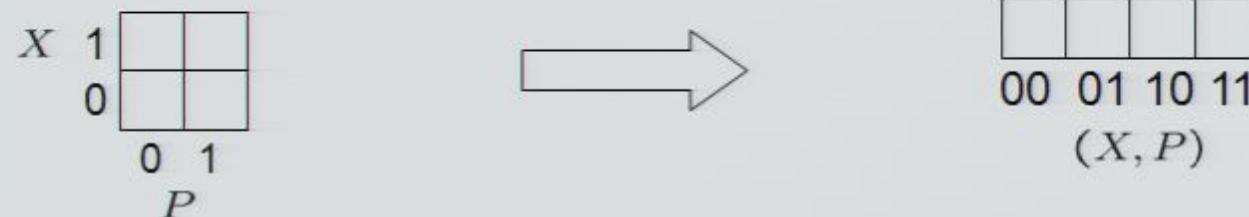
Epistemic states of non-maximal knowledge

Nothing known

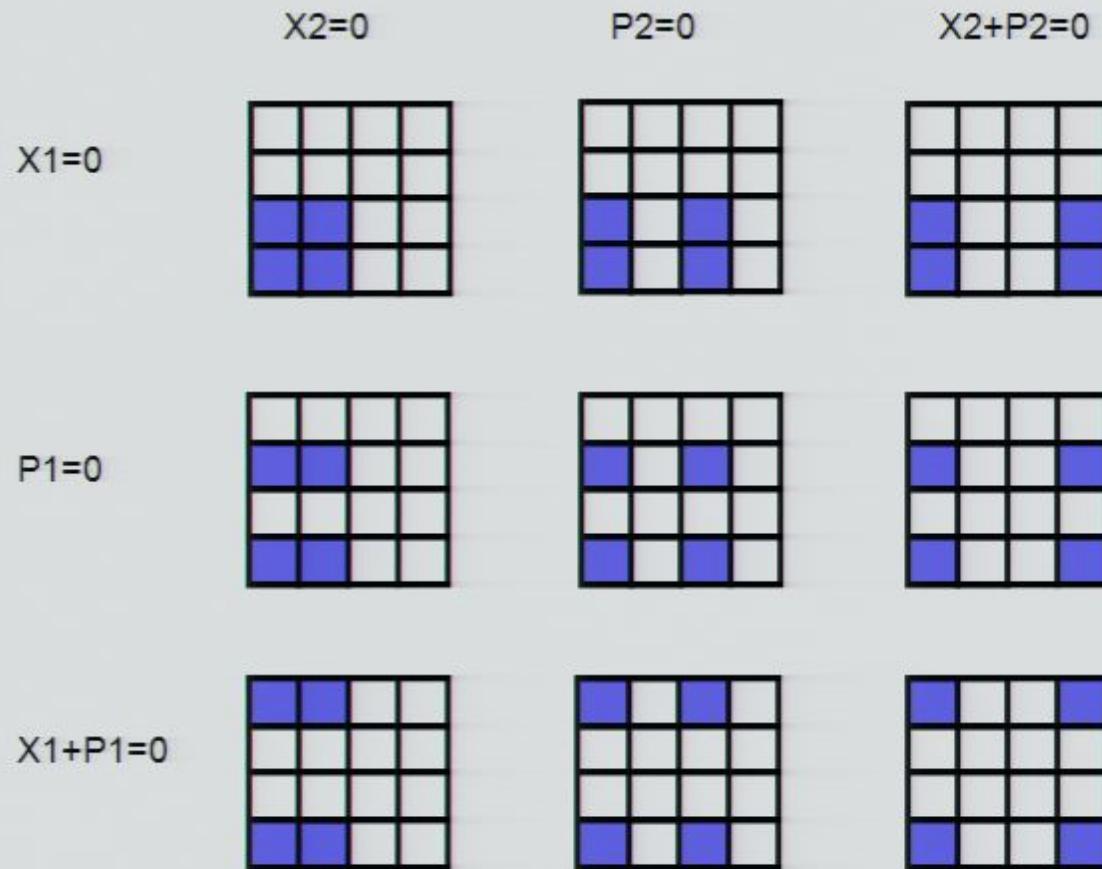
$X$	1	■■■■
	0	■■■■
	0	1

## A pair of bits

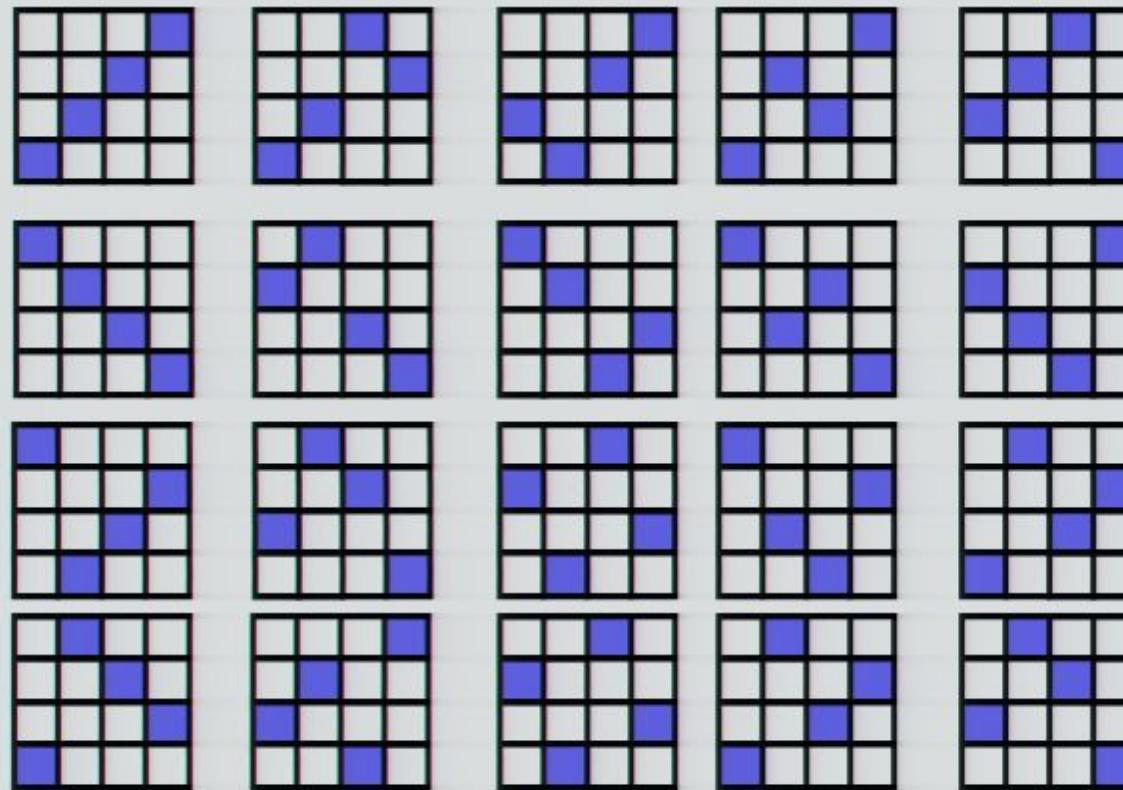
Canonical variables  $a_1 X_1 + b_1 P_1 + a_2 X_2 + b_2 P_2 \quad a_1, b_1, a_2, b_2 \in \mathbb{Z}_2$

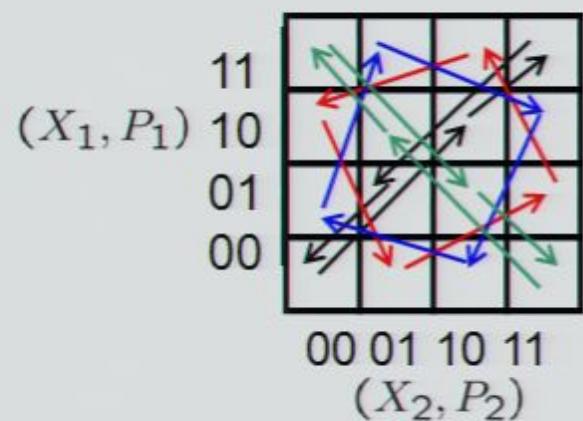
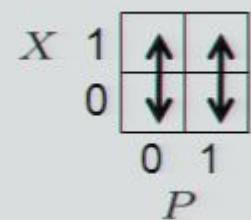


## Uncorrelated pure epistemic states



## Correlated pure epistemic states

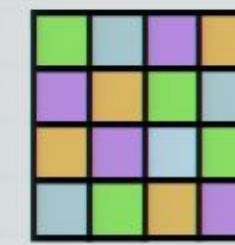
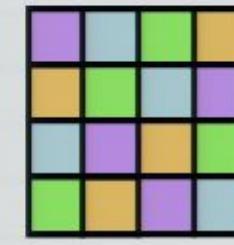
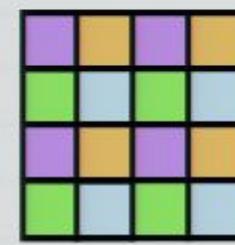
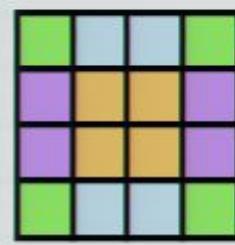
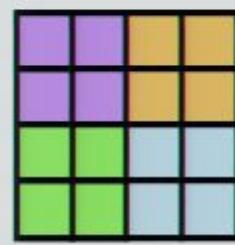




$$X \begin{matrix} 1 \\ 0 \end{matrix} \begin{matrix} \textcolor{red}{\square} & \textcolor{red}{\square} \\ \textcolor{green}{\square} & \textcolor{green}{\square} \end{matrix} \quad P \begin{matrix} 0 & 1 \end{matrix}$$

$$X \begin{matrix} 1 \\ 0 \end{matrix} \begin{matrix} \textcolor{green}{\square} & \textcolor{red}{\square} \\ \textcolor{green}{\square} & \textcolor{red}{\square} \end{matrix} \quad P \begin{matrix} 0 & 1 \end{matrix}$$

$$X \begin{matrix} 1 \\ 0 \end{matrix} \begin{matrix} \textcolor{red}{\square} & \textcolor{green}{\square} \\ \textcolor{green}{\square} & \textcolor{red}{\square} \end{matrix} \quad P \begin{matrix} 0 & 1 \end{matrix}$$

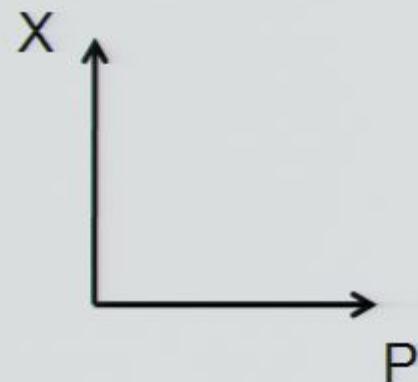


## Equivalence of these restricted statistical theories to “subtheories” of quantum theory

Look to a representation of quantum theory on phase space  
– the Wigner representation

Restricted Liouville mechanics  
= Quadrature Quantum Mechanics

$$\Omega = \mathbb{R}^{2n}$$



## Quadrature quantum mechanics

Hermitian operators:  $\hat{F} : \mathcal{L}^2(\mathbb{R}^n) \rightarrow \mathcal{L}^2(\mathbb{R}^n)$

Commutator:

$$[\hat{F}, \hat{G}] \equiv \hat{F}\hat{G} - \hat{G}\hat{F}$$

The quadrature operators are:

$$\hat{F} = a_1\hat{X}_1 + b_1\hat{P}_1 + \cdots + a_n\hat{X}_n + b_n\hat{P}_n \quad a_1, b_1, \dots, a_n, b_n \in \mathbb{R}$$

Quadrature states are eigenstates of a commuting set of quadrature operators

Specified by an isotropic subspace  $V$  and a valuation vector  $v \in V$

(Quadrature transformations and measurements take quadrature states to quadrature states)

## Wigner representation of quantum mechanics

Weyl operator  $\hat{w}(m) = e^{-i \sum_i q_i \hat{P}_i + p_i \hat{X}_i}$

Quantum state  $\rho$

Characteristic function  $\chi_\rho(m) = \text{Tr}(\rho \hat{w}(m)^\dagger)$

Wigner function  $W_\rho(m) = \sum_a e^{-i[m,a]} \chi_\rho(m)$

For quadrature state associated with  $V, v$

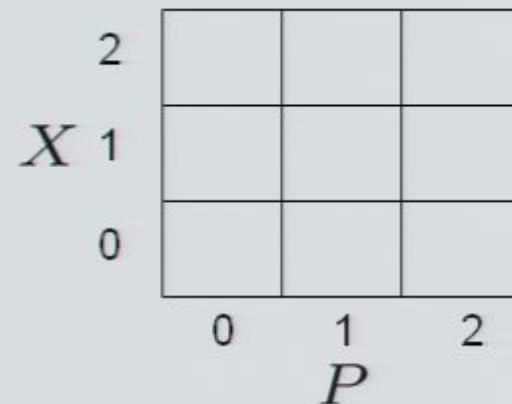
$$W_{V,v}(m) = \frac{1}{N} \delta_{V^\perp + v}(m)$$

Equivalence of states implies equivalence of measurements and transformations  
Therefore

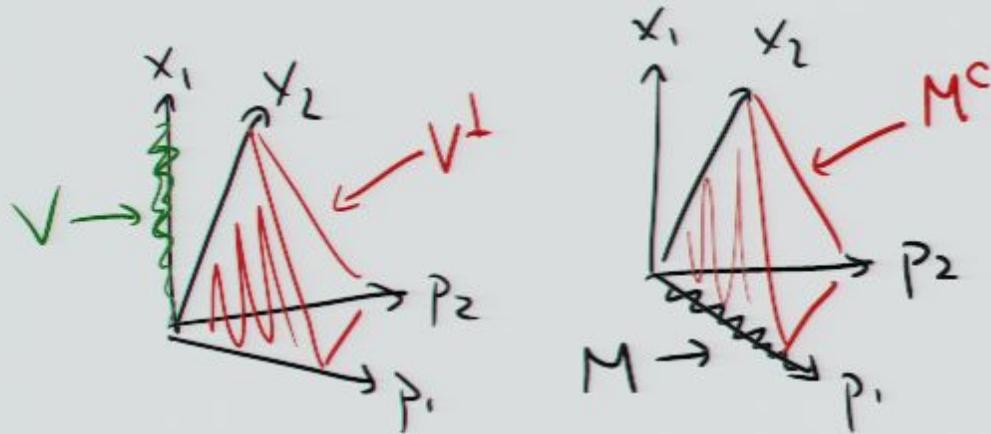
**Theorem:** Restricted statistical Liouville mechanics is empirically equivalent to quadrature quantum mechanics

Restricted statistical theory of trits  
= Stabilizer theory for qutrits

$$\Omega = (\mathbb{Z}_3)^{2n}$$



$$p_{V,V}(a) = \frac{1}{N} S_{V^\perp + V}(a)$$



$$\text{Let } M = JV$$

Note:  $\forall m \in M : p_{V,V}(a+m) = p_{V,V}(a)$  stabilized

$$\begin{aligned} V^\perp &= \{ b \in \Omega \mid b^T a = 0 \quad \forall a \in V \} \\ &= \{ b \in \Omega \mid b^T J a = 0 \quad \forall a \in M \} \\ &= M^c \quad \text{symplectic complement of } M \end{aligned}$$

$$p_{M,V}(a) = \frac{1}{N} S_{M^c + V}(a)$$

## Discrete phase space formalism

basis  $|x\rangle \quad x \in \mathbb{Z}_3$  for single qutrit  $\mathbb{C}_3$

shift operator  $\hat{S}(q)|x\rangle = |x+q\rangle \quad q \in \mathbb{Z}_3$

boost operator  $\hat{B}(p)|x\rangle = e^{\frac{2\pi i}{3} p x} |x\rangle \quad p \in \mathbb{Z}_3$

Weyl operator for single qutrit

$$\hat{\omega}(m) = e^{-\frac{2\pi i}{3} \cdot 2pq} \hat{B}(p) \hat{S}(q)$$

$$\text{where } m = (q, p) \in (\mathbb{Z}_3)^2$$

Weyl operator for  $n$  qutrits

$$\hat{\omega}(m) = e^{-\frac{2\pi i}{3} \cdot 2 \sum_i p_i q_i} \prod_i \hat{B}(p_i) \hat{S}(q_i)$$

$$\text{where } m = (q_1, p_1, \dots, q_n, p_n) \in (\mathbb{Z}_3)^n$$

## Stabilizer states

$$\rho_{M,v} = \frac{1}{|M|} \sum_{m \in M} e^{\frac{2\pi i}{3} [v, m]} \hat{\omega}(m)$$

where  $M$  is an isotropic subspace  
 $v$  is a vector

$$\hat{\omega}(m) = e^{-\frac{2\pi i}{3} \cdot 2 \sum_i p_i q_i} \prod_i \hat{B}_i(p_i) \hat{S}_i(q_i)$$

$\rho_{M,v}$  is the normalized projector onto  
the joint eigenspace of  $\{\hat{\omega}(m) | m \in M\}$   
w/ eigenvalues  $\{e^{-\frac{2\pi i}{3} [v, m]} | m \in M\}$

## A discrete Wigner function

David Gross, quant-ph/0602001

building on Gibbons, Hoffman & Wootters, q-p/0401155

Characteristic fn' for  $\rho$  of  $n$  qubits

$$X_\rho(b) = \frac{1}{3^n} \text{Tr}(w^\dagger(b)\rho)$$

Wigner fn' = symplectic Fourier transform  
of characteristic fn'

$$W_\rho(a) = \frac{1}{3^n} \sum_{b \in \mathbb{Z}_3^{2n}} e^{\frac{2\pi i}{3} [a, b]} X_\rho(b)$$

Wigner functions of stabilizer states

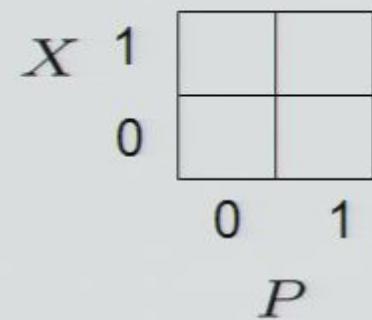
$$W_{\rho_{M,V}}(a) = \frac{1}{N} S_{M+V}(a)$$

Equivalence of states implies equivalence of measurements and transformations  
Therefore

**Theorem:** The restricted statistical theory of trits is empirically equivalent to the Stabilizer theory for qutrits

Restricted statistical theory of bits  
 $\simeq$  Stabilizer theory for qubits

$$\Omega = (\mathbb{Z}_2)^{2n}$$



Analogously to what we did for trits, one can:

Define stabilizer theory for qubits

Define Gross' discrete Wigner function for qubits

Find: Wigner function can be negative for qubit stabilizer states

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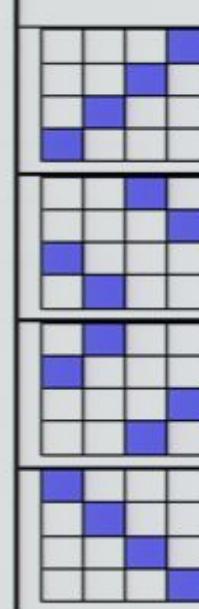
The restricted statistical theory of bits is not equivalent but very close to the Stabilizer theory for qubits

## Why the restricted statistical theory of bits Is **not** equivalent to qubit stabilizer theory

	$\{ 0\rangle,  1\rangle\}$	$\{ +\rangle,  -\rangle\}$	$\{ +i\rangle,  -i\rangle\}$
$ \Phi^+\rangle$	C	C	A
$ \Phi^-\rangle$	C	A	C
$ \Psi^+\rangle$	A	C	C
$ \Psi^-\rangle$	A	A	A

Even number of correlations

	I I II II	I II I II	I II II I
	C	C	C
	C	A	A
	A	C	A
	A	A	C



Odd number of correlations

Qubit stabilizer theory is nonlocal and contextual (e.g. GHZ)  
Restricted statistical theory of bits is local and noncontextual

Stay tuned for Bob Coecke's talk...

Beyond classical complementarity: could a different statistical restriction get us closer to quantum theory?

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NO for discrete degrees of freedom

Supplementing the unitary representation of the Clifford group with a single non-Clifford unitary yields all unitaries

## Beyond classical complementarity: could a different statistical restriction get us closer to quantum theory?

NO for discrete degrees of freedom

Supplementing the unitary representation of the Clifford group with a single non-Clifford unitary yields all unitaries

YES for continuous degrees of freedom

In addition to rotations and displacements in phase space, one can add squeezing – one gets all the quadratic Hamiltonians

(Bartlett, Rudolph, Spekkens, unpublished)

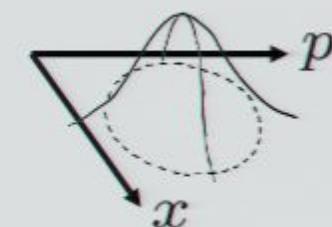
The classical uncertainty principle:

The only Liouville distributions that can be prepared are those satisfying

$$\gamma(\mu) + i\hbar J \geq 0$$

and that have maximal entropy for a given set of second-order moments.

$$\gamma(\mu) = 2 \begin{pmatrix} \Delta^2 x_1 & C_{x_1, p_1} & C_{x_1, x_2} & C_{x_1, p_2} & \dots \\ C_{p_1, x_1} & \Delta^2 p_1 & C_{p_1, x_2} & C_{p_1, p_2} & \\ C_{x_2, x_1} & C_{x_2, p_1} & \Delta^2 x_2 & C_{x_2, p_2} & \\ C_{p_2, x_1} & C_{p_2, p_1} & C_{p_2, x_2} & \Delta^2 p_2 & \\ \vdots & & & & \dots \\ 0 & -1 & & & \dots \\ 1 & 0 & & & \\ & & 0 & -1 & \\ & & 1 & 0 & \\ \vdots & & & & \dots \end{pmatrix}$$



$$\mu(x_1, p_1, x_2, p_2, \dots)$$

## Knowledge balance vs. classical complementarity

Contrast:

The principle of classical complementarity:

An observer can only have knowledge of the values of a commuting set of canonical variables.

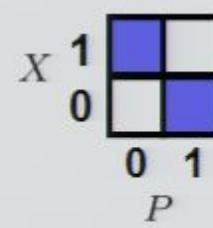
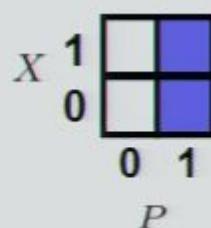
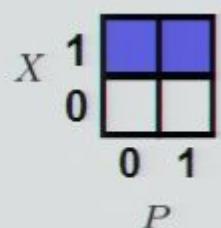
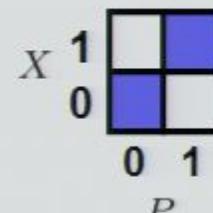
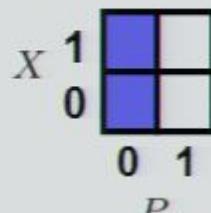
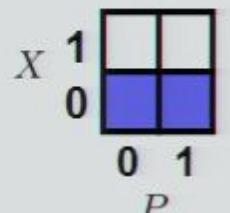
The knowledge-balance principle:

The only distributions that can be prepared are those that correspond to knowing at most half the information

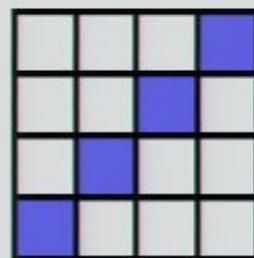
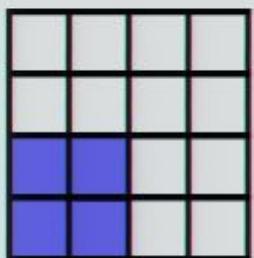
From: Spekkens, quant-ph/0401052 [Phys. Rev. A 75, 032110 (2007)]

According to Knowledge-Balance

Valid epistemic states for a single system



Valid epistemic states for a pair of systems



Plus permutations of rows and columns

## Knowledge balance vs. classical complementarity

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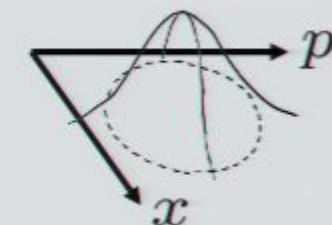
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$$\mu(x_1, p_1, x_2, p_2, \dots)$$