Title: Steps Towards a Unified Basis.

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Abstract: A new foundation of quantum mechanics for systems symmetric under a compact symmetry group is proposed. This is given by a link to classical statistics and coupled to the concept of a statistical parameter. A vector \phi of parameters is called an inaccessible c-variable if experiments can be provided for each single parameter, but no experiment can be provided for \phi. This is related to the concept of complementarity in quantum mechanics, but more generally to contrafactual parameters. Using these concepts and some weak assumptions, the Hilbert space of quantum mechanics is constructed. The complete set of axioms for time-independent quantum mechanics is provided by proving Born`s formula under weak assumptions.

1. Helland, I.S. (2006) Extended statistical modeling under symmetry: the link towards quantum mechanics. Ann. Statistics 34, 1, 42-77.
2. Helland, I.S. (2008) Quantum Mechanics from Focusing and Symmetry. Found. Phys. 38, 818-842.
Reference: arXiv 0801.2026. (http://arxiv.org/abs/0801.2026)

Steps Towards a Unified Basis.

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[1] http://folk.uio.no/ingeh

In this small world:

Religion: Different cultures exist across time and space.

Politics: Different cultures exist across time and space.

Ethnic groups: Different cultures exist across time and space.

Art and fashion: Different cultures exist across time and space.

Science: **Different cultures exist across time and space.** Can we have a really objective science on this background? — Perhaps, to the extent that we can translate languages and learn from each other.

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To be discussed and compared here:

Mathematical statistics:

Most of the literature is based on a framework given by parametric classes of probability measures.

Quantum theory:

Most of the literature is based upon a framework where states are represented by unit vectors in a complex Hilbert space.

Each basis has lead to an extremely rich and varied culture. Predictions done from the paradigma of each culture have been verified empirically. But the question remains: Can one find connections between the bases?

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On completeness.

Mathematical statistics has links to very many different empirical sciences.

First question: To which extent can this mathematical basis be considered to be complete?

From mathematical logic:
Gödel's incompleteness theorem:
In every rich enough logical system one can ask questions which cannot be answered within the same system.

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Extensions of the statistical language to be considered:

Focusing: In many investigations it is natural first to introduce an abstract, *conceptually defined set of variables* - called c-variable - ϕ.

The parameters of statistical models θ will be chosen as many-to-one functions of ϕ. *Questions to nature are asked in terms of one such θ.*

Symmetry: For certain given statistical models there is a symmetry group G acting upon the parameters of the model.
- Alternatively: Start with a group on the c-variable space Φ.

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On focusing and symmetry.

These extentions will also later be coupled to quantum mechanics.

By analogy with Gödel the extended concept system is still not complete, but more questions can now be asked and answered within the system.

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Focusing I.

Imagine that a medical patient has
- expected lifetime λ^1 if he gets the treatment 1;
- expected lifetime λ^2 if he gets the treatment 2.

In any case, he receives the treatment, or the treatment begins, at a certain time *t*.
Here, 'expected' should not first be taken to mean in relationship to a probability model, but at the outset related to what the medical expertice expects taken into account the knowledge they have about the patient. The model comes later.

Crucial point: We can estimate from data either λ^1 or λ^2, but we can never estimate the c-variable $\phi = (\lambda^1, \lambda^2)$.

Focusing I'.

In very many cases at the experimental design phase it is natural to define a conceptual variable or c-variable

$$\phi = (\lambda^1, \lambda^2, ..., \lambda^r)$$

at the outset. Only one parameter λ^j is realized by the experiment; the rest are called *counterfactuals*. Counterfactual variables/parameters are also important in other cases than statistical experiments. They have turned out to be essential in causal reasoning; see the book by Pearl and papers by Rubin, Holland and many others.

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Focusing II.

An apparatus for a very special length measurement is so sensitive that it is destroyed after one single measurement. Let μ be the length which is to be measured.

Assume furthermore that the measurement uncertainty σ only can be estimated by destroying the whole apparatus.

Let $\phi = (\mu, \sigma)$. Then it is impossible to estimate the whole c-variable ϕ, only μ or σ can be estimated.
In general, the choice of experimental question is essential.

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Summary of focusing.

Inaccessible c-variable: In most of the examples above, the c-variable ϕ is such that it is impossible estimate it from the available data. It is then called *inaccessible*.

Questions and answer: To find out something about nature, one must not only look unsystematically for facts, but sometimes focus upon a parameter θ, an accessible part of ϕ, and then look for answer to the question: What is the value of θ?

Complementarity: In quantum mechanics, two questions are called complementary if they are mutually exclusive, that is, part of a common inaccessible c-variable ϕ.

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Symmetry in a statistical modelling situation.

Assume a symmetry group G acting on a parameter space Θ and simultaneously on the sample space by

$$\mathrm{P}^{\theta g}(A) = \mathrm{P}^{\theta}(Ag^{-1}).$$

Examples may be scale change, translation, rotations,...

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Orbits. Transitive group.

- Fix $\theta_0 \in \Theta$, a point in the parameter space. Consider the set $\{\theta_0 g : g \in G\}$, the set of all parameter values that are transforms of θ_0. This is called the *orbit* of G containing θ_0.

- The space Θ is always divided into disjoint orbits.

- An orbit is a minimal invariant set under the group.

- If there is only one orbit, this will consist of the whole space Θ. Then we say that the group is *transitive*. More precisely: G is acting transitively upon Θ.

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Invariant measure.

Under weak conditions there exists a right-invariant measure ρ on the parameter space:

$$\rho(\Gamma g) = \rho(\Gamma) \text{ for } g \in G \text{ and all } \Gamma \subseteq \Theta.$$

- ρ can be taken as a probability measure if Θ is compact.

- ρ is unique if and only if the parameter group is transitive.

- Whenever there is a natural symmetry group acting upon Θ, in particular if it is transitive, there are many arguments for using the invariant measure ρ as a prior in Bayesian data analysis.

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Subparameters.

A subparameter $\lambda = \lambda(\theta)$ is called *permissible* if

$$\lambda(\theta_1) = \lambda(\theta_2) \text{ implies } \lambda(\theta_1 g) = \lambda(\theta_2 g) \text{ for all } g.$$

Then λ transforms under G by

$$\lambda(\theta) \rightarrow (\lambda g)(\theta) = \lambda(\theta g).$$

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Model reduction.

Model reduction is important when you have few data. When there is a symmetry group G acting upon the parameter space, one may impose the following requirement:

The original parameter space is invariant under the group G. Therefore it is natural that the reduced parameter space also should be invariant under G. This implies that

The reduced parameter space should be an orbit/ a set of orbits for G.

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Some summary of the symmetry part.

- In many cases a symmetry group G can be introduced in a natural way to a statistical model. There is a group acting upon the sample space, and a corresponding group acting upon the parameter space.

- In some cases the group is first defined on a larger c-variable space, and then from this induced on the parameter space.

- It is useful to have a group defined, first it is useful for selecting a suitable prior, then in the analysis of subparameters and finally in connection to model reduction.

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'Action at a distance'.

Consider the multivariate latent variable model

$$\boldsymbol{Z} = \boldsymbol{T}\boldsymbol{A} + \boldsymbol{U}\boldsymbol{B}$$

with negligible error. We assume that $\boldsymbol{Z}$ is measured, but the rest is unknown to begin with. Call $\phi = (\boldsymbol{T}, \boldsymbol{A}, \boldsymbol{U}, \boldsymbol{B})$ a c-variable.

Let us now have two distant stations. At station 1 it is possible to measure $\boldsymbol{T}$ or $\boldsymbol{A}$, but not both. At station 2 it is possible to measure $\boldsymbol{U}$ or $\boldsymbol{B}$, but not both.

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Now assume that we measure $\boldsymbol{T}$. Let $\boldsymbol{P} = \boldsymbol{T}(\boldsymbol{T}^t\boldsymbol{T})^{-1}\boldsymbol{T}^t$, and let $\boldsymbol{V} = (\boldsymbol{I} - \boldsymbol{P})\boldsymbol{U}$. Then we know the product

$$\boldsymbol{VB} = (\boldsymbol{I} - \boldsymbol{P})\boldsymbol{Z}.$$

This implies that by using principal component analysis, we can find a considerable proportion of the unknown parameter $\boldsymbol{B}$. On the other hand an essential proportion of the parameter $\boldsymbol{U}$ remains unknown.

By doing the complementary experiment at station 1, namely measuring $\boldsymbol{A}$, we obtain considerable information about $\boldsymbol{U}$, while $\boldsymbol{B}$ remains largely unknown.

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There is no direct action at a distance here, but by taking a decision on what to measure at station 1, we determine what parameter to get information on at station 2.

In my view this simple thought experiment bears some relationship with the EPR experiment, which throughout the years has caused much discussion in the quantummechanical literature.

The focus is not upon what the values of $\boldsymbol{U}$ and $\boldsymbol{B}$ are, but upon what information we can get about $\boldsymbol{U}$ and $\boldsymbol{B}$. The corresponding general view in quantum theory is called epistemic.

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Further discussion.

Much of the discussion so far - on focusing and on symmetry - has been in terms of statistical parameters, or more generally c-variables. These entities are not in themselves connected to position in space, although they may be so in special cases.

Historically, quantum theory has developed from classical mechanics, where position in space is crucial. This has lead to long discussions about locality in situations where the latter concept may not be relevant if the foundation is formulated in a more statistical language.

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A statistical approach to electron spin.

Start by modelling angular momentum by a vector, a c-variable ϕ.
Focus upon the angular momentum component θ^a in some direction $\boldsymbol{a}$.
Let ϕ be given a rotation group. The largest subgroup G^a with respect to which θ^a is permissible, is given by rotations around the axis $\boldsymbol{a}$ together with a reflection in a plane perpendicular to $\boldsymbol{a}$.
Finally, introduce a model reduction: The orbits of G^a as acting on θ^a are given by two-point sets $\{\pm\kappa\}$ together with the single point 0. A maximal model reduction is to one such orbit, say, the nontrivial orbit $\{\pm 1\}$. Let λ^a be the reduced parameter.

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State:

1) Focused question: Given *a*, what is the value of λ^a?
2) Answer: $\lambda^a = +1$ or $\lambda^a = -1$.

This is equivalent to a Bloch sphere vector, which is equivalent to an ordinary state vector for electron spin.

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On Bell's inequality.

Assume that spin components λ^a and λ^b are measured in the directions given by unit vectors a and b on two particles at distant sites A and B. The measured values $\hat{\lambda}^a$ and $\hat{\mu}^b$ are each assumed to take values ± 1. Let this be repeated 4 times: Two settings a, a' at site A combined with two settings b, b' at site B.

A combinatorial argument then shows

$$\hat{\lambda}^a \hat{\mu}^b \leq \hat{\lambda}^a \hat{\mu}^{b'} + \hat{\lambda}^{a'} \hat{\mu}^b + \hat{\lambda}^{a'} \hat{\mu}^{b'} + 2.$$

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Seemingly the CHSH version of Bell's inequality follows:

$$E(\hat{\lambda}^a \hat{\mu}^b) \leq E(\hat{\lambda}^a \hat{\mu}^{b'}) + E(\hat{\lambda}^{a'} \hat{\mu}^b) + E(\hat{\lambda}^{a'} \hat{\mu}^{b'}) + 2,$$

but this is known to be broken by quantum mechanics and by experiments.

There is a large literature om Bell's inequality. It si often argued that the implication leading to the inequality follows from 'local realism', a concept inherited from classical mechanics.

Here we are more interested in concepts inherited from statistical theory. We note that there are 4 experiments involved in these inequalities.

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Bell's inequality from a statistical point of view.

Going from the inaccessible c-variable ϕ to the observations there are really three steps involved at each node: The component $\theta(\phi)$ is selected, there is a model reduction $\lambda = \lambda(\theta)$, and finally there is the observation $\hat{\lambda}$.

According to the *conditionality principle*, a principle on which there seems to be a fair amount of concensus among statisticians, inference in each experiment should always be conditional upon the experiment actually performed. Hence different expectation operators should be used in each term above, and the derivation of the Bell inequality is not valid.

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General construction of the quantum space I.

Consider ϕ, the group G, assumed compact and transitive, and the invariant measure ρ. Focus on a choice a and a reduced parameter $\lambda^a = \lambda^a(\phi)$ taking a discrete set of values $\{\lambda_k\}$. Let G^a be the maximal subgroup under which the parameter λ^a is permissible.
Let the Hilbert space $\mathbf{L}^a$ consist of all functions in $\mathrm{L}^2(\Phi, \rho)$ which are of the form

$$f(\phi) = \tilde{f}(\lambda^a(\phi)).$$

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General construction of the quantum space II.

By a unitary transformation W, let $\mathbf{H}^a = W\mathbf{L}^a$, similarly $\mathbf{H}^b = W\mathbf{L}^b$ etc.

Aim through a series of theorems: Obtain coinciding Hilbert spaces $\mathbf{H}^a$, $\mathbf{H}^b$ etc.

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Assumptions:

i) For all a and b there exists $g_{ab} \in G$ such that

$$\lambda^b(\phi) = \lambda^a(\phi g_{ab}).$$

Consequence: The groups G^a and G^b can be connected by the group element g_{ab}:

$$g^b = g_{ab} g^a g_{ab}^{-1}.$$

ii) The reduced groups G^a, G^b, ... generate the whole group G.

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The right regular representation of G.

Let $U(g)f(\phi) = f(\phi g)$. Then $f_k^b = U(g_{ab})f_k^a$, where $f_k^a(\phi) = I(\lambda^a(\phi) = \lambda_k)$ are the basis vectors of $\mathbf{L}^a$ etc.. Hence $\mathbf{L}^b = U(g_{ab})\mathbf{L}^a$, and therefore $\mathbf{H}^b = V(g_{ab})\mathbf{H}^a$ with $V(g) = WU(g)W^\dagger$.

Theorem 1. For a compact group, every irreducible unitary representation $V(g)$ can be written as $V(g) = WU(g)W^\dagger$ for some W, with $U(g)$ being a subrepresentation of the right regular representation.

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General construction of the Hilbert space III.

Theorem 2. Fixing some preliminary W_0, the fixed Hilbert space $\mathbf{H}^a$ is an invariant space for some abstract group representation V (possibly multivalued) of the whole group G.

Sketch of proof:

Prove $V(g_1 g_2 g_3) = V(g_1)V(g_2)V(g_3)$
for $g_1 \in G^a$, $g_2 \in G^b$ and $g_3 \in G^c$ etc.

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General construction of the Hilbert space IV.

Theorem 3. Then, from $\mathbf{H}^b = V(g_{ab})\mathbf{H}^a$ etc., we have $\mathbf{H}^a = \mathbf{H}^b = \mathbf{H}^c = \ldots$, and this can be taken as the quantum mechanical space $\mathbf{H}$.

Example: $SU(2)$ gives a twodimensional invariant space for electron spin, coupled to the rotation group.

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States in Hilbert space.

State vector: $v_k^a = Wf_k^a$ with $f_k^a(\phi) = I(\lambda^a(\phi) = \lambda_k)$.
Interpretation: i) Focused question: What is λ^a? ii) Answer: $\lambda^a = \lambda_k$.

Selfadjoint operator: $T^a = WS^a W^\dagger$ with $S^a f(\phi) = \lambda^a(\phi) f(\phi)$.
T^a has eigenvectors v_k^a and eigenvalues λ_k.
T^a corresponds to a perfect experiment seeking answer of the question: What is λ^a?
Assuming non-denerate eigenvalues corresponds to assuming that λ^a is accessible - can be estimated - and that it is maximally so under the ordering $\mu \ll \lambda$ if $\mu = h(\lambda)$.
Note: $T^a = \sum_k \lambda_k v_k^a v_k^{a\dagger}$,

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Auxiliary quantity I: Density operator: $\sigma = \sum_k \pi_k v_k^a v_k^{a\dagger}$

when a prior or posterior probability π_k over λ^a is given.
In the quantum mechanical tradition, a density operator may be defined as any non-negative selvadjoint operator with trace 1.

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Auxiliary quantity II: Effect: $\mathcal{E} = \sum_k p_k(t) v_k^a v_k^{a\dagger}$

when an experiment with likelihood $p_k(t)$ and parameter λ^a is given.
In the quantum mechanical tradition, an effect may be defined as any selfadjont operator with eigenvalues between 0 and 1.

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Born's formula

Linking different experiments:

$$P(\lambda^b = \lambda_j | \lambda^a = \lambda_k) = |v_j^{b\dagger} v_k^a|^2.$$

Proved under different assumptions by Deutsch, Gill, Wallace, Saunders, Aerts and Zurek.

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In our setting the following assumptions suffice together with assumptions made earlier:

i) The transition probabilities exist.
ii) The transition probability to an identical state is 1
iii) All $\mu(\phi) = \lambda^a(\phi g_{bc})$ are valid parameters.
iv) $P(\lambda^b(\phi) = \lambda_i | \lambda^a(\phi) = \lambda_k) = P(\lambda^b(\phi g_{bc}) = \lambda_i | \lambda^a(\phi g_{bc}) = \lambda_k)$ always.

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Tool for proving Born's formula:

The Busch-Gleason theorem

Consider the set of effects on a Hilbert space **H**. Assume that there is a generalized probability measure π on these effects, i.e., a set function satifying

- $\pi(\mathcal{E}) \geq 0$ for all $\mathcal{E}$,
- $\pi(\mathcal{I}) = 1$,
- $\sum_i \pi(\mathcal{E}_i) = \pi(\mathcal{E})$ for effects $\mathcal{E}_i$ whose sum is an effect $\mathcal{E}$.

Then π is necessarily of the form $\pi(\mathcal{E}) = \mathrm{tr}(\rho\mathcal{E})$ for some positive, selfadjoint, trace 1 operator ρ.

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Steps in proving Born's formula:

1) For $\mathcal{E} = \sum_i p_i(t) v_i^b v_i^{b\dagger}$, define $\pi(\mathcal{E}) = \sum_i p_i(t) P(\lambda^b = \lambda_i | \lambda^a = \lambda_k)$.

2) Show, by considering the experiments, that $\pi(\mathcal{E}_1 + \mathcal{E}_2) = \pi(\mathcal{E}_1) + \pi(\mathcal{E}_2)$ when $\mathcal{E}_1$, $\mathcal{E}_2$ and $\mathcal{E}_1 + \mathcal{E}_2$ all are effects.

3) From this, the conditions of the Busch-Gleason theorem are satisfied, in particular for $\pi_{a,k}(\mathcal{E})$ being equal to the transition probability from v_k^a to the effect $\mathcal{E} = vv^\dagger$ for an arbitrary state vector v.

4) It follows that $\pi_{a,k}(vv^\dagger) = v^\dagger \rho v$ for some $\rho = \sum_j c_j u_j u_j^\dagger$ with $c_j \geq 0$.

5) Specializing to $v = v_k^a$, implies that $\rho = v_k^a v_k^{a\dagger}$, hence the transition probability is $\pi_{a,k} = |v^\dagger v_k^a|^2$.

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Consequences of Born's formula:

I. The expected result of a perfect measurement:

$$E(\lambda|v_k^a) = \sum_j \lambda_j P(\lambda^b = \lambda_j|\lambda^a = \lambda_k) = v_k^{a\dagger} T v_k^a \text{ with } T = \sum_j \lambda_j v_j^b v_j^{b\dagger}.$$

II. The probability distribution of the result of experiment *b*:

$$P(dy|\lambda^a = \lambda_k) = v_k^{a\dagger} \mathcal{M}(dy) v_k^a \text{ with } \mathcal{M}(dy) = \sum_j p_j(y) v_j^b v_j^{b\dagger} dy.$$

Conclusion:

The whole formalism of time-independent quantum theory follows.

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Summary.

Extentions needed for the transition from statistics to quantum mechanics: *Focusing and symmetry.*

Specific additional restrictions: Assumptions coupled to the symmetry group.

Empirical extension: Born's formula gives transition probability between differently focused ideal experiments.

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So far: Mostly parameters/ ideal experiments.
Further development: Coupling to data/ measurement theory.

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Progress in science
owes more to the clash of ideas
than to the steady accumulation of facts.

John Arcibald Wheeler

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