Title: Exact uncertainty, bosonic fields, and interacting classical-quantum systems

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Abstract: The quantum equations for bosonic fields may be derived using an 'exact uncertainty' approach [1]. This method of quantization can be applied to fields with Hamiltonian functionals that are quadratic in the momentum density, such as the electromagnetic and gravitational fields. The approach, when applied to gravity [2], may be described as a Hamilton-Jacobi quantization of the gravitational field. It differs from previous approaches that take the classical Hamilton-Jacobi equation as their starting point in that it incorporates some new elements, in particular the use of a formalism of ensembles on configuration space and the postulate of an exact uncertainty relation. These provide the fundamental elements needed for the transition to the quantum theory. The formalism of ensembles on configuration space is general enough to describe classical, quantum, and interacting classical-quantum systems in a consistent way. This is of some relevance to gravity: although there are many physical arguments in favour of a quantum theory of gravity, it appears that the justification for such a theory does not follow from logical arguments alone [3]. It is therefore of interest to consider the coupling of quantum fields to a classical gravitational field. This leads to a theory that is fundamentally different from standard semiclassical gravity. 1. Michael J W Hall, Kailash Kumar and Marcel Reginatto, Bosonic field equations from an exact uncertainty principle, J. Phys. A 36 (2003) 9779-9794 (http://arxiv.org/abs/hep-th/0307259). 2. M. Reginatto, Exact Uncertainty Principle and Quantization: Implications for the Gravitational Field, Proceedings of DICE2004 in: Braz. J. Phys. 35 (2005) 476-480 (http://arxiv.org/abs/gr-qc/0501030). 3. Mark Albers, Claus Kiefer and Marcel Reginatto, Measurement analysis and quantum gravity, Phys. Rev. D 78 (2008) 064051 (http://arxiv.org/abs/0802.1978)

# Exact uncertainty, bosonic fields and interacting classical-quantum systems

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# Outline

1. Motivation
2. Classical and quantum electromagnetic fields
3. Classical and quantum gravitational fields
4. Coupling of quantum matter to a classical gravitational field
5. Concluding remarks

# Some motivation

- Is it possible to quantize fields using the “exact uncertainty” approach? Which type of fields?
- How does this approach differ from other methods of quantization?
- What about gravity?
- Is it possible to couple quantum fields to a classical gravitational field using ensembles on configuration space?

# The Hamilton-Jacobi equation

- Define $S = \int_{t_1}^{t_2} L(q_k, \dot{q}_k)dt$ and vary $S$,

$$\delta S = \sum_k \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)\right] \delta q_k dt + \left[\sum_k p_k \Delta q_k - H\Delta t\right]_A^B$$

$$\left(\textit{where} \quad \Delta q_k = \delta q_k + \dot{q}_k \Delta t, \quad p_k = \tfrac{\partial L}{\partial \dot{q}_k}, \quad H = \sum \tfrac{\partial L}{\partial \dot{q}_k}\dot{q}_k - L\right).$$

- Endpoint variation leads to

$$\frac{\partial S}{\partial t} + H\left(q_k, \frac{\partial S}{\partial q_k}\right) = 0, \qquad p_k = \frac{\partial S}{\partial q_k}.$$

- $S$ is a function of the *configuration space variables*.

# Electromagnetic field: Hamilton-Jacobi formulation

- Lagrangian density

$$\mathcal{L} = -\frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu} , \qquad F_{\mu\nu} = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu}.$$

- Endpoint variation leads to

$$\frac{\partial S}{\partial t} + \int d^3x \left[ 2\pi \left( \frac{\delta S}{\delta \vec{A}} \right)^2 + \frac{1}{8\pi} \left( \nabla \times \vec{A} \right)^2 \right] = 0, \qquad \frac{\delta S}{\delta A_k} = -\frac{E^k}{4\pi}.$$

- Invariance of $S$ under gauge transformations defined by $\delta A_k = \lambda_{,k}$ requires

$$\delta S = \int d^3x \left( \frac{\delta S}{\delta A_k} \right) \delta A_k = 0 \qquad \Rightarrow \qquad \left( \frac{\delta S}{\delta A_k} \right)_{,k} = 0.$$

# Electromagnetic field: classical ensembles

- Hamilton-Jacobi equation and constraint:

$$\frac{\partial S}{\partial t} + 2\pi \left( \frac{\delta S}{\delta \vec{A}} \right)^2 + \frac{1}{8\pi} \left( \nabla \times \vec{A} \right)^2 = 0, \qquad \left( \frac{\delta S}{\delta A_k} \right)_{,k} = 0.$$

- Once you have a Hamilton-Jacobi equation of this form, you can define ensembles on configurations space.

- It requires introducing a probability density $P$ with

$$\left( \frac{\delta P}{\delta A_k} \right)_{,k} = 0$$

and an appropriate ensemble Hamiltonian $\tilde{H}[P, S]$.

# Electromagnetic field: ensemble hamiltonian $\tilde{\mathcal{H}}_C$

- Define the ensemble Hamiltonian

$$\tilde{\mathcal{H}}_C = \int DA \int d^3x \; P \left[ 2\pi \left( \frac{\delta S}{\delta \vec{A}} \right)^2 + \frac{1}{8\pi} \left( \nabla \times \vec{A} \right)^2 \right].$$

- Variation of $\tilde{\mathcal{H}}_C$ leads to the equations

$$\frac{\partial S}{\partial t} + \int d^3x \left[ 2\pi \left( \frac{\delta S}{\delta \vec{A}} \right)^2 + \frac{1}{8\pi} \left( \nabla \times \vec{A} \right)^2 \right] = 0$$

and

$$\frac{\partial P}{\partial t} + 4\pi \int d^3x \left[ \frac{\delta}{\delta A_k} \left( P \frac{\delta S}{\delta A_k} \right) \right] = 0.$$

# Electromagnetic field: quantum theory

- To quantize, introduce non-classical momentum fluctuations $\Rightarrow$ the ensemble Hamiltonian becomes

$$\tilde{\mathcal{H}}_{\mathcal{Q}} = \tilde{\mathcal{H}}_{\mathcal{C}} + \frac{\hbar^2}{8} \int DA \int d^3x \, P \left[ \frac{1}{P^2} \left( \frac{\delta P}{\delta \vec{A}} \right)^2 \right].$$

- If you define the wavefunctional by $\Psi = \sqrt{P} e^{iS/\hbar}$, the equations take the form

$$i\hbar \frac{\partial \Psi}{\partial t} = \int d^3x \left[ -2\pi\hbar^2 \frac{\delta^2 \Psi}{\delta \vec{A}^2} + \frac{1}{8\pi} \left( \nabla \times \vec{A} \right) \cdot \left( \nabla \times \vec{A} \right) \Psi \right].$$

- The constraint on the wavefunction is

$$\left( \frac{\delta \Psi}{\delta A_k} \right)_{,k} = 0.$$

# Remarks

- The approach leads to the functional Schrödinger equation for the fields.

- $\vec{E} = -4\pi \frac{\delta S}{\delta \vec{A}}$ and $\vec{B} = \nabla \times \vec{A}$.

- The exact uncertainty approach corresponds to:

  – adding nonclassical fluctuations to the electric field components of an ensemble of electromagnetic fields,

  – with the fluctuation strength determined by the uncertainty in the magnetic field components.

- No need to go through the canonical formalism of the classical field theory – but further "reconstruction" is needed!

# Technical issues

- Measure over the space of fields.
- Assumptions needed for quantization:
  – Independence,
  – Invariance,
  – Exact uncertainty,
  – First order functional derivatives in $\tilde{\mathcal{H}}_Q$.
- The assumptions regarding the momentum fluctuations lead to an additional nonclassical term in the ensemble Hamiltonian, specified by the *covariance matrix* $\mathrm{Cov}_x(N)$ of the fluctuations at position $x$, where

$$[\mathrm{Cov}_x(N)]^{ab} := \overline{N_x^a N_x^b} \sim (\delta P/\delta f_x^a)(\delta P/\delta f_x^b)/P^2.$$

# Hamilton-Jacobi formulation of gravity (in the metric representation)

- Einstein-Hamilton-Jacobi equation

$$\int d^3x \, N \left[ \frac{1}{2} G_{ijkl} \frac{\delta S}{\delta h_{ij}} \frac{\delta S}{\delta h_{kl}} - \sqrt{h} R \right] = 0.$$

- Momentum constraints (invariance under spatial coordinate transformations)

$$D_j \left( h_{ik} \frac{\delta S}{\delta h_{kj}} \right) = 0.$$

*Notation*: $R$ is the curvature scalar and $D_j$ the covariant derivative on a three-dimensional spatial hypersurface with (positive definite) metric $h_{kl}$, and

$$G_{ijkl} = \frac{1}{\sqrt{h}} \left( h_{ik} h_{jl} + h_{il} h_{jk} - h_{ij} h_{kl} \right).$$

## Gravitational field: ensemble Hamiltonian $\tilde{H}_c$

- An appropriate ensemble Hamiltonian for the gravitational field is given by

$$\tilde{H}_c = \int Dh \int d^3x \, P \, N \left[ \frac{1}{2} G_{ijkl} \frac{\delta S}{\delta h_{ij}} \frac{\delta S}{\delta h_{kl}} - \sqrt{h} R \right].$$

- Variation of $\tilde{H}_c$ (set $\frac{\partial S}{\partial t} = \frac{\partial P}{\partial t} = 0$) leads to the equations

$$\int d^3x \, N \left[ \frac{1}{2} G_{ijkl} \frac{\delta S}{\delta h_{ij}} \frac{\delta S}{\delta h_{kl}} - \sqrt{h} R \right] = 0$$

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$$\int d^3x \, N \frac{\delta}{\delta h_{ij}} \left( P G_{ijkl} \frac{\delta S}{\delta h_{kl}} \right) = 0.$$

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# Remarks

- The interpretation of the equation

$$\int d^3x \, N \frac{\delta}{\delta h_{ij}} \left( P G_{ijkl} \frac{\delta S}{\delta h_{kl}} \right) = 0$$

as a continuity equation is only possible if $\frac{\partial h_{kl}}{\partial t}$ is linear in $G_{ijkl} \frac{\delta S}{\delta h_{kl}}$.

- Then, you can write

$$\delta h_{ij} = \left( N G_{ijkl} \frac{\delta S}{\delta h_{kl}} + D_i N_j + D_j N_i \right) \delta t$$

where the gauge transformation $(D_k N_l + D_l N_k) = \delta_\epsilon h_{kl}$ has been included in the expression for $\delta h_{ij}$.

- In this way, you get the remaining six Einstein equations.

# Gravitational field: quantum theory

- To quantize, introduce non-classical momentum fluctuations $\Rightarrow$ the ensemble Hamiltonian becomes

$$\tilde{H}_q = \tilde{H}_c + \frac{\hbar}{8} \int Dh \int d^3x \, \frac{1}{P} G_{ijkl} \frac{\delta P}{\delta h_{ij}} \frac{\delta P}{\delta h_{kl}}.$$

- If you define the wavefunctional by $\Psi = \sqrt{P} e^{iS/\hbar}$ , you get the Wheeler-DeWitt equation

$$\left[ -\frac{\hbar^2}{2} \frac{\delta}{\delta h_{ij}} G_{ijkl} \frac{\delta}{\delta h_{kl}} - \sqrt{h} R \right] \Psi = 0.$$

- Notice that the exact uncertainty approach specifies a particular operator ordering for the Wheeler-DeWitt equation.

# Remarks

- The extrinsic curvature tensor can be written as $K_{ij} = \frac{1}{2} G_{ijkl} \frac{\delta S}{\delta h_{kl}}$.
- The exact uncertainty approach corresponds to

  – adding nonclassical fluctuations to the extrinsic curvature tensor,

  – with the fluctuation strength determined by the uncertainty in the spatial metric.
- The approach can also be used in the connection representation of gravity, because in this formulation the Hamilton-Jacobi equation is also quadratic in the field momenta.

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$$\int d^3x\, N \left[\frac{1}{2} G_{ijkl} \frac{\delta S}{\delta h_{ij}} \frac{\delta S}{\delta h_{kl}} - \sqrt{h} R\right] = 0$$

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- The approach can also be used in the connection representation of gravity, because in this formulation the Hamilton-Jacobi equation is also quadratic in the field momenta.

# Remarks

- Stochastic rate equations? Comparison to classical ensembles suggests the *stochastic* rate equation

$$\frac{\partial h_{ij}}{\partial t} \rightarrow NG_{ijkl}\left(\frac{\delta S}{\delta h_{kl}} + \phi^{kl}\right) + D_i N_j + D_j N_i$$

  where $\phi^{kl}$ is a stochastic field.

- The quantization procedure described here amounts to a "Hamilton-Jacobi quantization of gravity" – i.e., without going first to a canonical formulation of classical gravity. Has the problem of "Dirac consistency" been avoided with this approach?

- The usual difficulties of the Wheeler-DeWitt equation remain!

# Quantum matter fields and classical gravity

- To what extent can a mixed classical/quantum system provide a consistent, satisfactory description of matter and gravitation?
- The study of such systems can provide clues that may help in the search for a full quantum theory of gravity.
- Dyson has argued that it might be impossible in principle to observe the existence of individual gravitons, and this has lead him to the conjecture that "the gravitational field described by Einstein's theory of general relativity is a purely classical field without any quantum behaviour." If his conjecture is correct, mixed classical/quantum systems become unavoidable.

# Coupling of a quantized scalar field to a classical gravitational field

- The ensemble Hamiltonian is the sum of two parts,

$$\tilde{H}_{\phi h} = \int d^3x \int Dh\, P\, N \left[\mathcal{H}_{\phi h} + F_\phi\right].$$

- $\mathcal{H}_{\phi h}$ is the contribution to the ensemble Hamiltonian that describes classical gravity with a scalar field,

$$\mathcal{H}_{\phi h} = \mathcal{H}_h + \frac{1}{2\sqrt{h}}\left(\frac{\delta S}{\delta \phi}\right)^2 + \sqrt{h}\left[\frac{1}{2}h^{ij}\frac{\partial \phi}{\partial x^i}\frac{\partial \phi}{\partial x^j} + V(\phi)\right].$$

- $F_\phi$ is a contribution that results from the quantization of the scalar field,

$$F_\phi = \frac{\hbar^2}{8}\frac{1}{\sqrt{h}}\left(\frac{1}{P}\frac{\delta P}{\delta \phi}\right)^2.$$

# Remarks

- The gravitational field remains classical, but this does not mean that we can not have states that are subject to uncertainty.
- The approach is one in which non-classical momentum fluctuations are added to $\frac{\delta S}{\delta \phi}$, but no fluctuations are added to $\frac{\delta S}{\delta h_{kl}}$.
- The result is fundamentally different from semiclassical gravity, where the expectation value of the energy momentum of the quantum field couples to a classical gravitational field.
- The equations are non-linear functional differential equations.

# A "hybrid" model in spherically symmetric gravity

- Consider spherically symmetric gravity, with the line element

$$g_{\mu\nu} dx^{\mu} dx^{\nu} = -N^2 dt^2 + \Lambda^2 \left(dr + N^r dt\right)^2 + R^2 d\Omega^2.$$

For example, the line element that describes the Einstein universe can be written as

$$N = 1, \qquad N^r = 0, \qquad \Lambda = a, \qquad R = a \sin r.$$

- An ensemble Hamiltonian for classical gravity with a quantized scalar field is of the form

$$\tilde{H}_{\phi\Lambda R} = \int dr \int Dh\, P\, N \left[\mathcal{H}_{\phi\Lambda R} + F_{\phi}\right].$$

# A “Hybrid” model in spherically symmetric gravity

- The explicit form of the ensemble Hamiltonian is

$$
\begin{aligned}
\tilde{H}_{\phi\Lambda R} &= \int dr \int Dh\, P\, N \left[\mathcal{H}_{\phi\Lambda R} + F_\phi,\right] \\
\mathcal{H}_{\phi\Lambda R} &= \mathcal{H}_{\Lambda R} + \frac{1}{2\Lambda R^2}\left(\frac{\delta S}{\delta\phi}\right)^2 + \frac{R^2}{2\Lambda}\phi'^2 + \frac{\Lambda R^2 m^2}{2}\phi^2, \\
\mathcal{H}_{\Lambda R} &= -\frac{1}{R}\frac{\delta S}{\delta R}\frac{\delta S}{\delta\Lambda} + \frac{1\Lambda}{2R^2}\left(\frac{\delta S}{\delta\Lambda}\right)^2 + \lambda\frac{\Lambda R^2}{2} + V, \\
V &= \frac{RR''}{\Lambda} - \frac{RR'\Lambda'}{\Lambda^2} + \frac{R'^2}{2\Lambda} - \frac{\Lambda}{2}, \\
F_\phi &= \frac{1}{8\Lambda R^2}\left(\frac{1}{P}\frac{\delta P}{\delta\phi}\right)^2.
\end{aligned}
$$

# A particular solution for the "hybrid" model

- Assume that

  (1) The foliation is of spaces of constant positive curvature and the lapse function $N$ is constant,

  (2) The functionals $S$ and $P$ are of the form

  $S\,[R, \Lambda, \phi] = 0,$

  $P\,[R, \Lambda, \phi] \sim$ ground state Gaussian functional of $\phi$.

- In this particular case, the problem reduces to that of solving for the the ground state of a quantized scalar field in an Einstein universe.

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- In this particular case, the problem reduces to that of solving for the the ground state of a quantized scalar field in an Einstein universe.

# Remarks (some preliminary results)

- The solutions depend on the value of the mass $m$ of the field, and the space of solutions is non-trivial due to the renormalization of the quantum field.
- The equation that describes the scale factor $a$ is of the form
$$\lambda a^4 - 3a^2 + \alpha(ma) = 0,$$
where $\alpha(ma)$ is a complicated function.
- For certain values of the mass, $\alpha(ma) > 0$, and the cosmological constant $\lambda$ may be set to zero.
- For a given mass, the value of the radius $a$ depends on the state of the quantized scalar field with the result that $a$ takes discrete values.

# A “Hybrid” model in spherically symmetric gravity

- The explicit form of the ensemble Hamiltonian is

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\mathcal{H}_{\Lambda R} &= -\frac{1}{R}\frac{\delta S}{\delta R}\frac{\delta S}{\delta\Lambda} + \frac{1\Lambda}{2R^2}\left(\frac{\delta S}{\delta\Lambda}\right)^2 + \lambda\frac{\Lambda R^2}{2} + V, \\
V &= \frac{RR''}{\Lambda} - \frac{RR'\Lambda'}{\Lambda^2} + \frac{R'^2}{2\Lambda} - \frac{\Lambda}{2}, \\
F_{\phi} &= \frac{1}{8\Lambda R^2}\left(\frac{1}{P}\frac{\delta P}{\delta\phi}\right)^2 .
\end{aligned}
$$

# A particular solution for the "hybrid" model

- Assume that

  (1) The foliation is of spaces of constant positive curvature and the lapse function $N$ is constant,

  (2) The functionals $S$ and $P$ are of the form

  $S[R, \Lambda, \phi] = 0,$

  $P[R, \Lambda, \phi] \sim$ ground state Gaussian functional of $\phi$.

- In this particular case, the problem reduces to that of solving for the the ground state of a quantized scalar field in an Einstein universe.

# Remarks (some preliminary results)

- The solutions depend on the value of the mass $m$ of the field, and the space of solutions is non-trivial due to the renormalization of the quantum field.

- The equation that describes the scale factor $a$ is of the form
$$\lambda a^4 - 3a^2 + \alpha(ma) = 0,$$
where $\alpha(ma)$ is a complicated function.

- For certain values of the mass, $\alpha(ma) > 0$, and the cosmological constant $\lambda$ may be set to zero.

- For a given mass, the value of the radius $a$ depends on the state of the quantized scalar field with the result that $a$ takes discrete values.

## Concluding remarks

- The “exact uncertainty” approach for particles can be generalized and used to derive bosonic field equations.
- Non-classical fluctuations are added to $\sim \frac{\delta S}{\delta F}$.
- It is not necessary to use a classical canonical theory as the starting point (“Hamilton-Jacobi quantization" ).
- A theory of interacting classical and quantum fields can be formulated using ensembles on configuration space.
- In particular, it is possible to study systems where quantum matter fields couple to a classical spacetime. The theory that you get is fundamentally different from semiclassical gravity.

# THANKS FOR THE ATTENTION!!!

# Coupling of a quantized scalar field to a classical gravitational field

- The ensemble Hamiltonian is the sum of two parts,

$$\tilde{H}_{\phi h} = \int d^3x \int Dh\, P\, N \left[\mathcal{H}_{\phi h} + F_\phi\right].$$

- $\mathcal{H}_{\phi h}$ is the contribution to the ensemble Hamiltonian that describes classical gravity with a scalar field,

$$\mathcal{H}_{\phi h} = \mathcal{H}_h + \frac{1}{2\sqrt{h}}\left(\frac{\delta S}{\delta \phi}\right)^2 + \sqrt{h}\left[\frac{1}{2}h^{ij}\frac{\partial \phi}{\partial x^i}\frac{\partial \phi}{\partial x^j} + V(\phi)\right].$$

- $F_\phi$ is a contribution that results from the quantization of the scalar field,

$$F_\phi = \frac{\hbar^2}{8}\frac{1}{\sqrt{h}}\left(\frac{1}{P}\frac{\delta P}{\delta \phi}\right)^2.$$

# Gravitational field: quantum theory

- To quantize, introduce non-classical momentum fluctuations $\Rightarrow$ the ensemble Hamiltonian becomes

$$\tilde{H}_q = \tilde{H}_c + \frac{\hbar}{8}\int Dh \int d^3x \, \frac{1}{P} G_{ijkl} \frac{\delta P}{\delta h_{ij}} \frac{\delta P}{\delta h_{kl}}.$$

- If you define the wavefunctional by $\Psi = \sqrt{P} e^{iS/\hbar}$ , you get the Wheeler-DeWitt equation

$$\left[ -\frac{\hbar^2}{2} \frac{\delta}{\delta h_{ij}} G_{ijkl} \frac{\delta}{\delta h_{kl}} - \sqrt{h} R \right] \Psi = 0.$$

- Notice that the exact uncertainty approach specifies a particular operator ordering for the Wheeler-DeWitt equation.

# Gravitational field: ensemble Hamiltonian $\tilde{H}_c$

- An appropriate ensemble Hamiltonian for the gravitational field is given by

$$\tilde{H}_c = \int Dh \int d^3x\, P\, N \left[\frac{1}{2} G_{ijkl} \frac{\delta S}{\delta h_{ij}} \frac{\delta S}{\delta h_{kl}} - \sqrt{h} R\right].$$

- Variation of $\tilde{H}_c$ (set $\frac{\partial S}{\partial t} = \frac{\partial P}{\partial t} = 0$) leads to the equations

$$\int d^3x\, N \left[\frac{1}{2} G_{ijkl} \frac{\delta S}{\delta h_{ij}} \frac{\delta S}{\delta h_{kl}} - \sqrt{h} R\right] = 0$$

and

$$\int d^3x\, N \frac{\delta}{\delta h_{ij}} \left(P G_{ijkl} \frac{\delta S}{\delta h_{kl}}\right) = 0.$$

# Hamilton-Jacobi formulation of gravity (in the metric representation)

- Einstein-Hamilton-Jacobi equation

$$\int d^3x\, N \left[\frac{1}{2} G_{ijkl} \frac{\delta S}{\delta h_{ij}} \frac{\delta S}{\delta h_{kl}} - \sqrt{h} R\right] = 0.$$

- Momentum constraints (invariance under spatial coordinate transformations)

$$D_j \left( h_{ik} \frac{\delta S}{\delta h_{kj}} \right) = 0.$$

*Notation*: $R$ is the curvature scalar and $D_j$ the covariant derivative on a three-dimensional spatial hypersurface with (positive definite) metric $h_{kl}$, and

$$G_{ijkl} = \frac{1}{\sqrt{h}} \left( h_{ik} h_{jl} + h_{il} h_{jk} - h_{ij} h_{kl} \right).$$

# Electromagnetic field: quantum theory

- To quantize, introduce non-classical momentum fluctuations $\Rightarrow$ the ensemble Hamiltonian becomes

$$\tilde{\mathcal{H}}_{\mathcal{Q}} = \tilde{\mathcal{H}}_{\mathcal{C}} + \frac{\hbar^2}{8} \int DA \int d^3x\, P \left[ \frac{1}{P^2} \left( \frac{\delta P}{\delta \vec{A}} \right)^2 \right].$$

- If you define the wavefunctional by $\Psi = \sqrt{P} e^{iS/\hbar}$, the equations take the form

$$i\hbar \frac{\partial \Psi}{\partial t} = \int d^3x \left[ -2\pi\hbar^2 \frac{\delta^2 \Psi}{\delta \vec{A}^2} + \frac{1}{8\pi} \left( \nabla \times \vec{A} \right) \cdot \left( \nabla \times \vec{A} \right) \Psi \right].$$

- The constraint on the wavefunction is

$$\left( \frac{\delta \Psi}{\delta A_k} \right)_{,k} = 0.$$

# The Hamilton-Jacobi equation

- Define $S = \int_{t_1}^{t_2} L(q_k, \dot{q}_k) dt$ and vary $S$,

$$\delta S = \sum_k \int_{t_1}^{t_2} \left[ \frac{\partial L}{\partial q_k} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_k} \right) \right] \delta q_k dt + \left[ \sum_k p_k \Delta q_k - H \Delta t \right]_A^B$$

$$\left( where \quad \Delta q_k = \delta q_k + \dot{q}_k \Delta t, \quad p_k = \tfrac{\partial L}{\partial \dot{q}_k}, \quad H = \sum \tfrac{\partial L}{\partial \dot{q}_k} \dot{q}_k - L \right).$$

- Endpoint variation leads to

$$\frac{\partial S}{\partial t} + H \left( q_k, \frac{\partial S}{\partial q_k} \right) = 0, \qquad p_k = \frac{\partial S}{\partial q_k}.$$

- $S$ is a function of the *configuration space variables*.

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and

$$\int d^3x\, N\,\frac{\delta}{\delta h_{ij}}\left(PG_{ijkl}\frac{\delta S}{\delta h_{kl}}\right) = 0.$$

# Remarks

- The interpretation of the equation

$$\int d^3x\, N \frac{\delta}{\delta h_{ij}} \left( P G_{ijkl} \frac{\delta S}{\delta h_{kl}} \right) = 0$$

as a continuity equation is only possible if $\frac{\partial h_{kl}}{\partial t}$ is linear in $G_{ijkl} \frac{\delta S}{\delta h_{kl}}$.

- Then, you can write

$$\delta h_{ij} = \left( N G_{ijkl} \frac{\delta S}{\delta h_{kl}} + D_i N_j + D_j N_i \right) \delta t$$

where the gauge transformation $(D_k N_l + D_l N_k) = \delta_\epsilon h_{kl}$ has been included in the expression for $\delta h_{ij}$.

- In this way, you get the remaining six Einstein equations.

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# Remarks

- The extrinsic curvature tensor can be written as $K_{ij} = \frac{1}{2} G_{ijkl} \frac{\delta S}{\delta h_{kl}}$.
- The exact uncertainty approach corresponds to

  – adding nonclassical fluctuations to the extrinsic curvature tensor,

  – with the fluctuation strength determined by the uncertainty in the spatial metric.
- The approach can also be used in the connection representation of gravity, because in this formulation the Hamilton-Jacobi equation is also quadratic in the field momenta.

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