Title: Quantum Mechanics as a Theory of Systems with Limited Information Content

Date: Aug 09, 2009 11:00 AM

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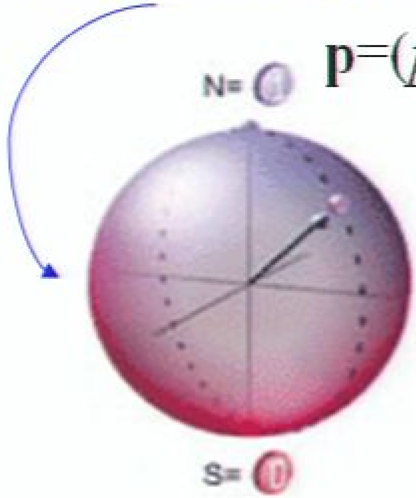
Abstract: I will consider physical theories which describe systems with limited information content. This limit is not due observer's ignorance about some “hidden” properties of the system - the view that would have to be confronted with Bell's theorem - but is of fundamental nature. I will show how the mathematical structure of these theories can be reconstructed from a set of reasonable axioms about probabilities for measurement outcomes. Among others these include the “locality” assumption according to which the global state of a composite system is completely determined by correlations between local measurements. I will demonstrate that quantum mechanics is the only theory from the set in which composite systems can be in entangled (non-separable) states. Within Hardy's approach this feature allows to single out quantum theory from other probabilistic theories without a need to assume the “simplicity” axiom. 1. Borivoje Dakic, Caslav Brukner (in preparation) 2. Caslav Brukner, Anton Zeilinger, Information Invariance and Quantum Probabilities, arXiv:0905.0653 3. Tomasz Paterek, Borivoje Dakic, Caslav Brukner, Theories of systems with limited information content, arXiv:0804.1423

1 Bit Systems (L=2)

d=3 real numbers needed to specify quantum state but only 1 bit can be read out (**Holevo Bound**)

d=1

$\mathbf{p}=(p_x,p_y,p_z)$



1 1

0 0

Bit

Probabilistic Bit p_1 "0"+ p_2 "1"

Qubit

a|0> + b|1>

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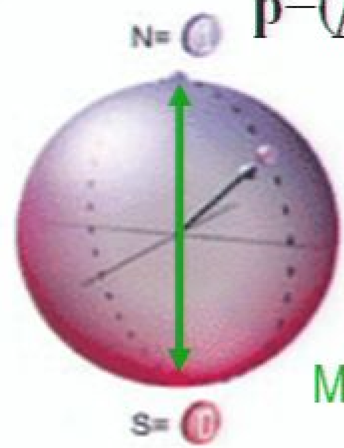

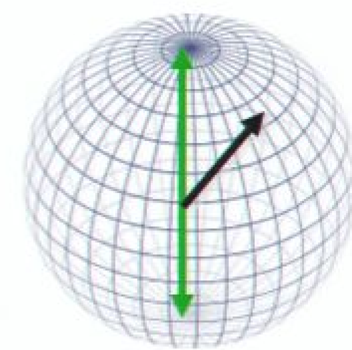

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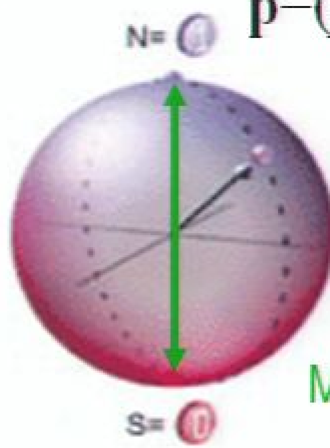

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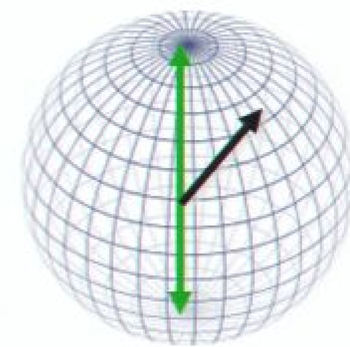

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"The most elementary system contains one bit of information."

(Zeilinger, 1999; Brukner, Zeilinger 2001)

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4. Features of information processing and computation with g-bits ?

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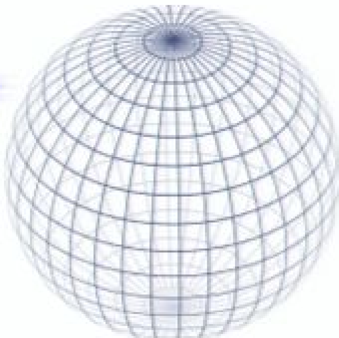
d-1-sphere



Directional degrees of freedom (spin) embedded in d spatial dimensions?

Quantum Gravity: Dimension different from 3 (+1) at "small scales".

"If, as Professor Wheeler has argued, the origin of quantum mechanics' structure is to be sought in a theory of observation and observers and meaning, then we would do well to focus our attention not on amplitudes but on quantities which are more directly observable."

William Wootters,
Quantum mechanics without probability amplitudes
Foundations of Physics, 1986

Quantum Mechanics based on Reals

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$$\hat{\rho}$$

Density matrix
(L-level system)

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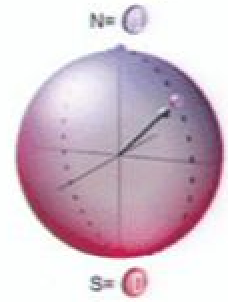

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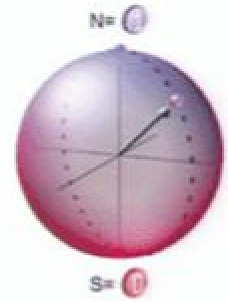

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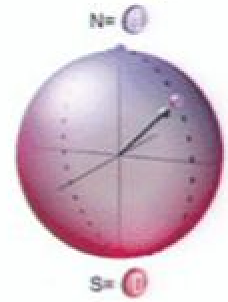

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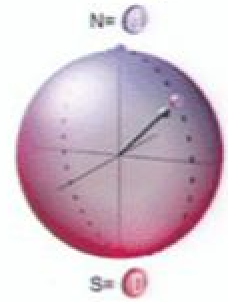

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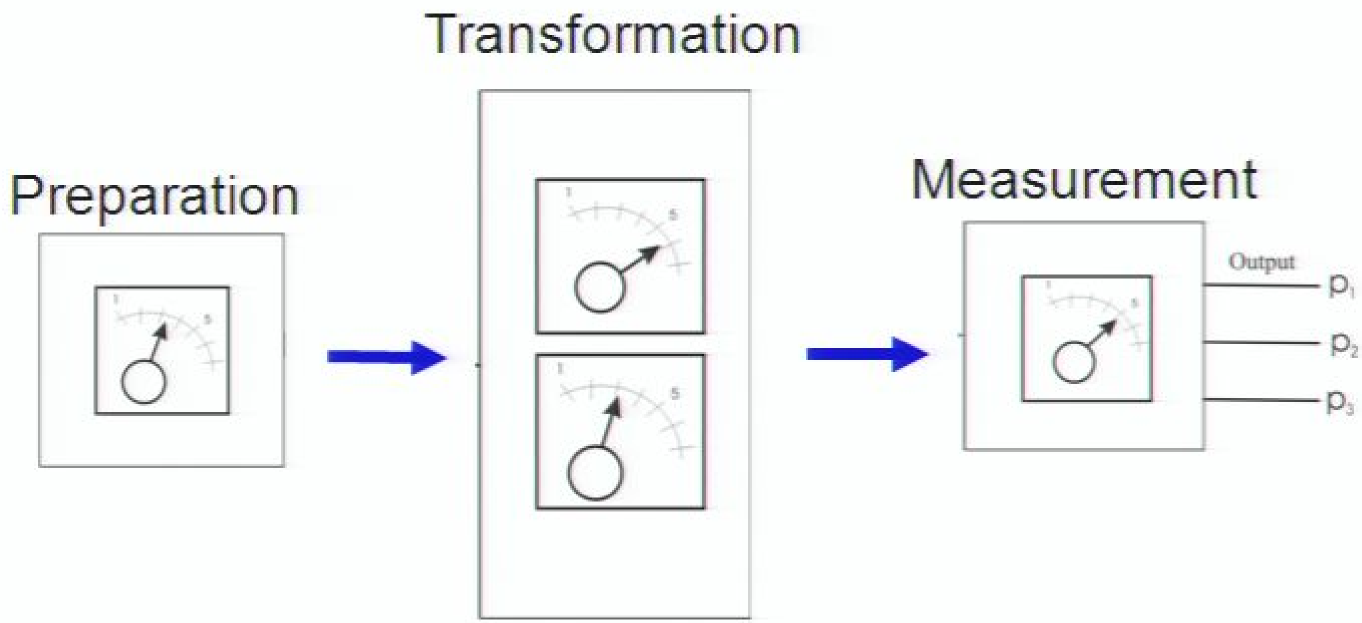
$$|\psi(x)\rangle = \cos\frac{x}{2}|0\rangle|1\rangle + \sin\frac{x}{2}|1\rangle|0\rangle$$

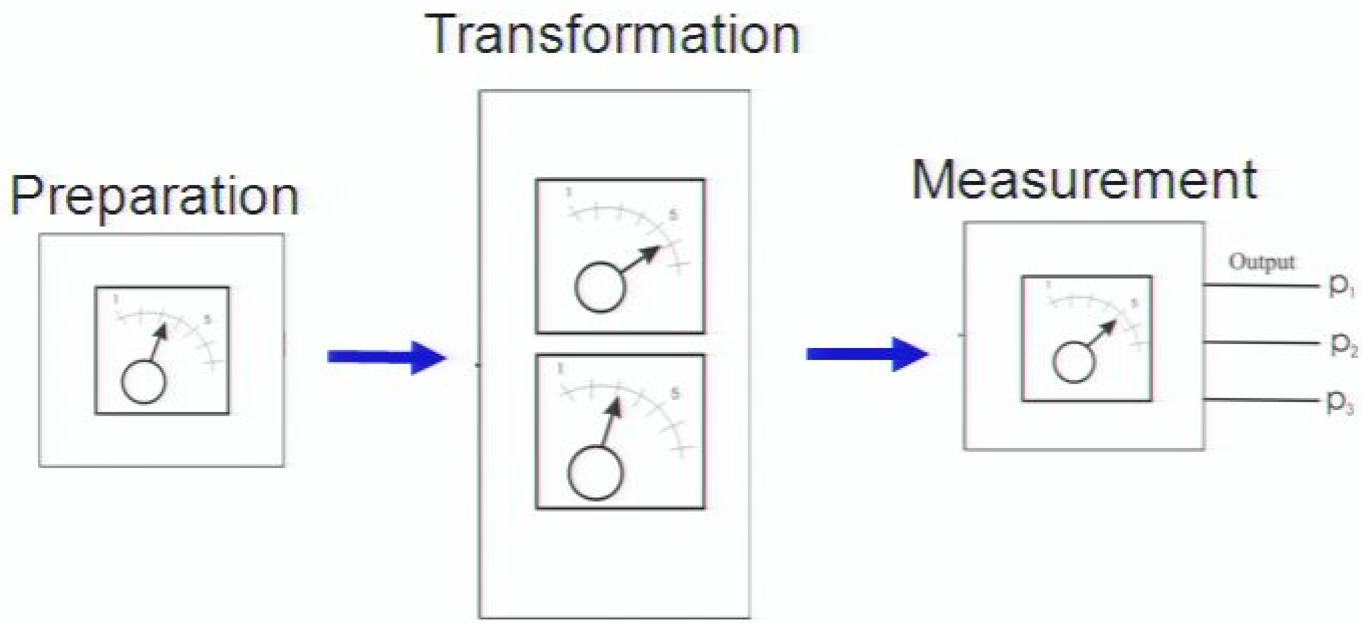
$$\psi(x) = ((0,0,\cos x)^T, (0,0,-\cos x)^T, \mathrm{diag}[\sin x, \sin x, -1])$$

Operational approach



Hardy (2001)

Operational approach
(with some interpretational remarks occasionally*)



*More at PIAF '09 New Perspectives on the Quantum State, Sept. 27 – Oct. 2, 2009

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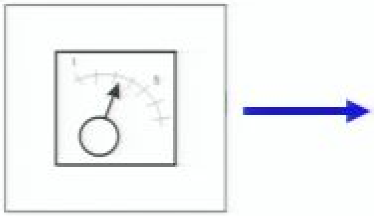
Pure & Mixed States

Preparation



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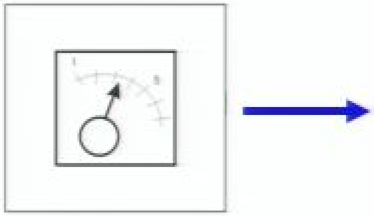

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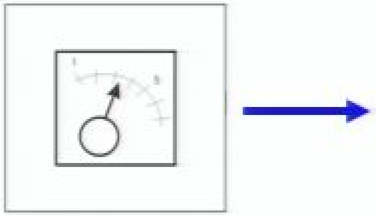

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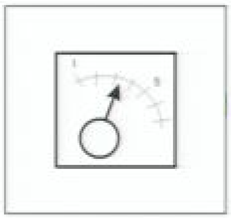

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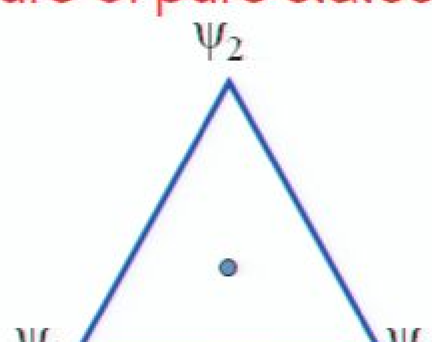

Transformations

Pure & Mixed States

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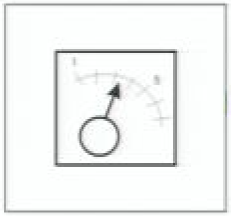

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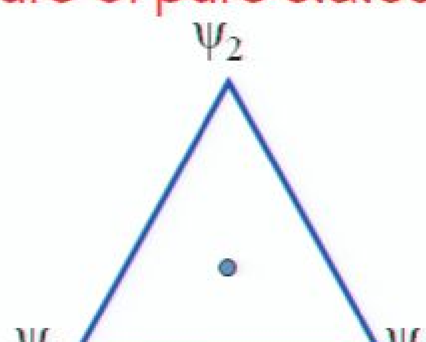

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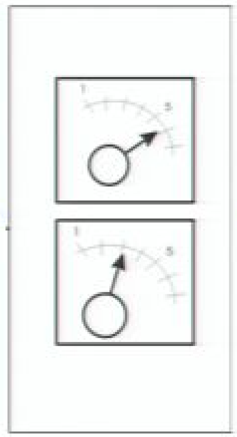

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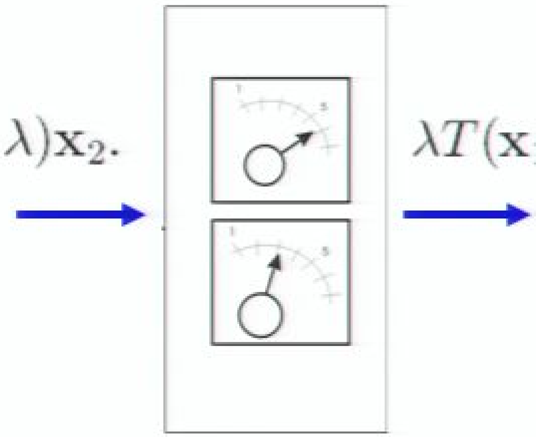

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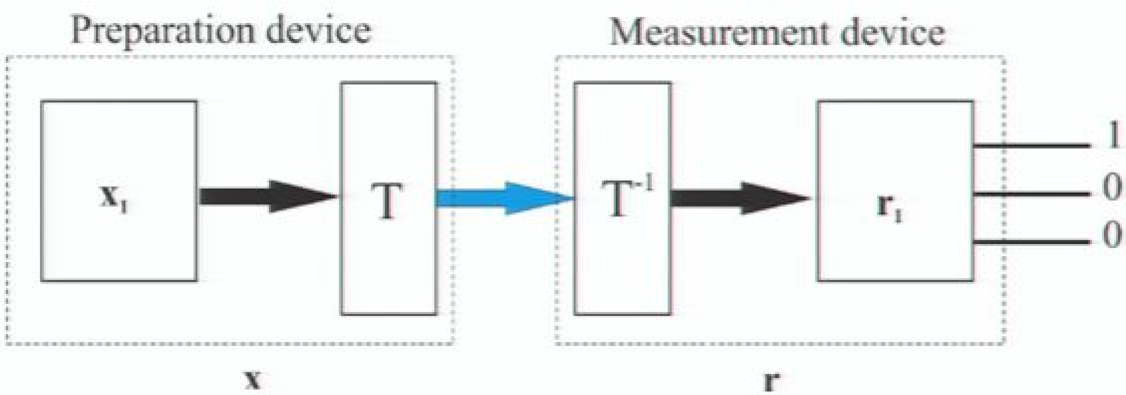

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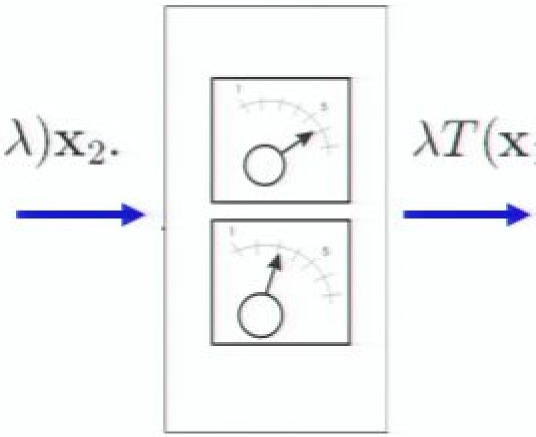

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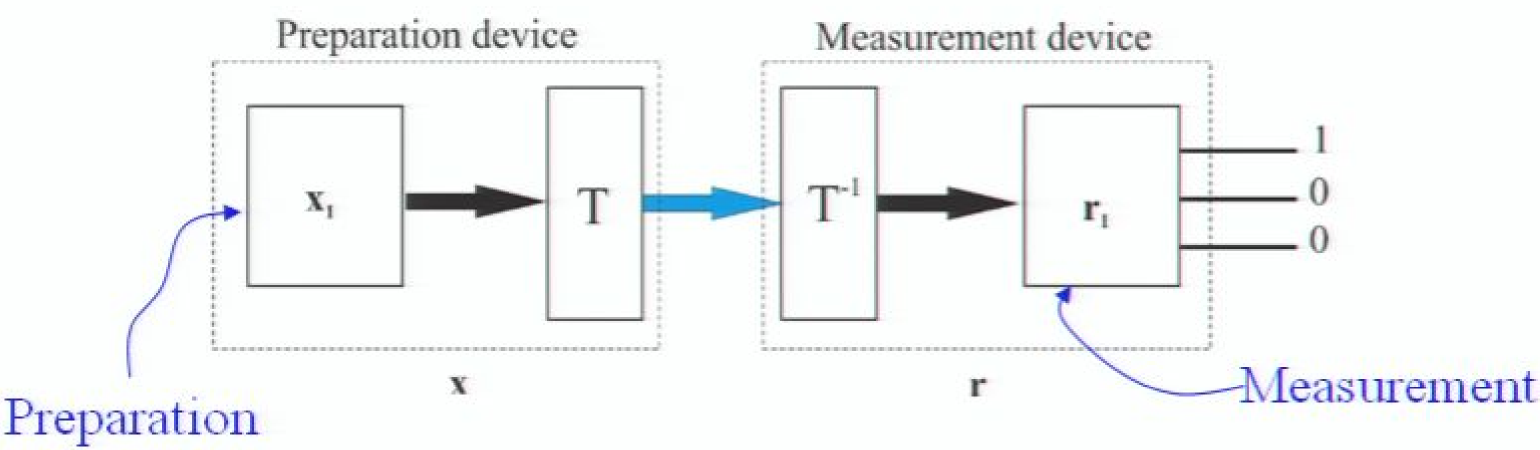

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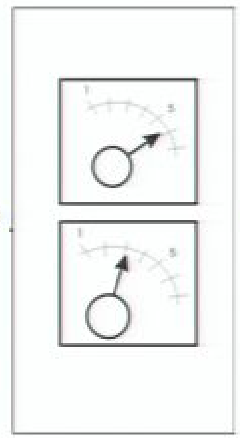

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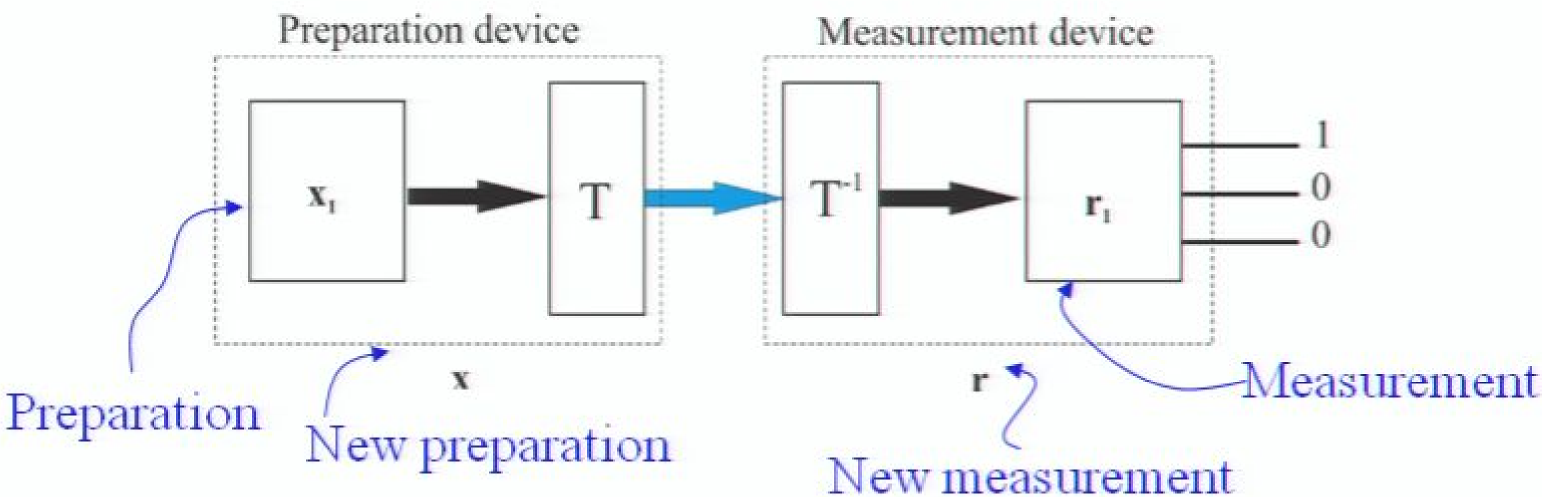

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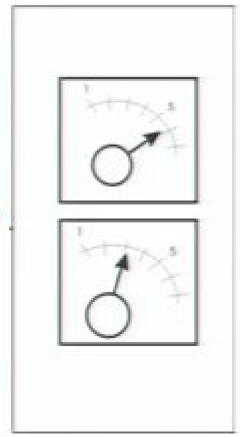

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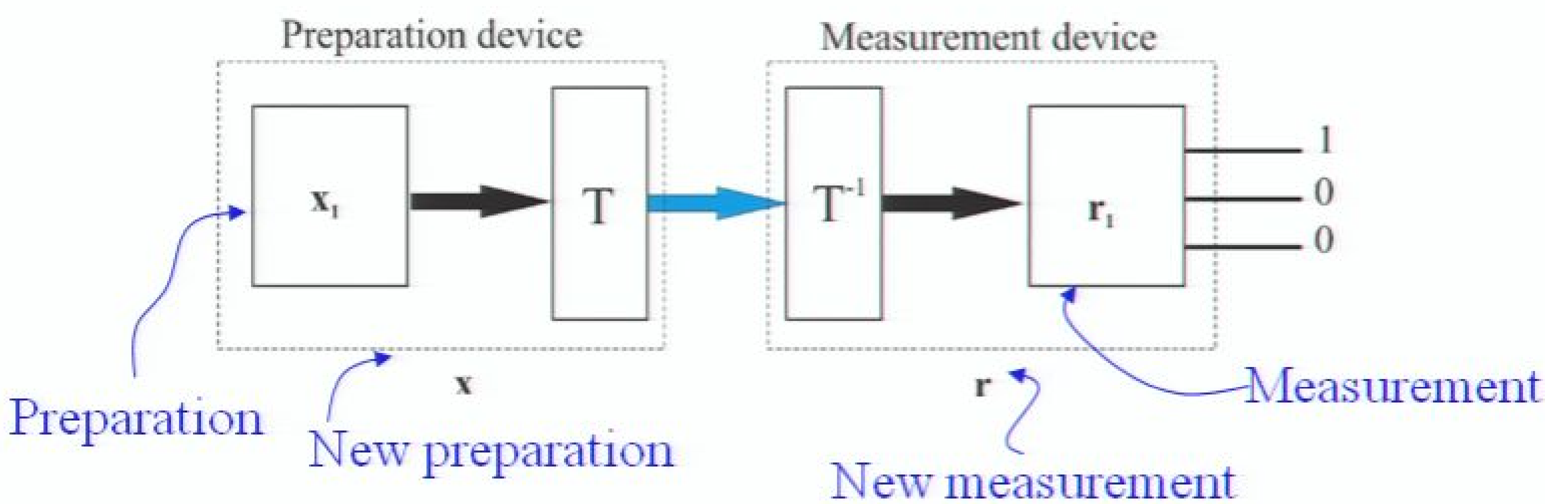

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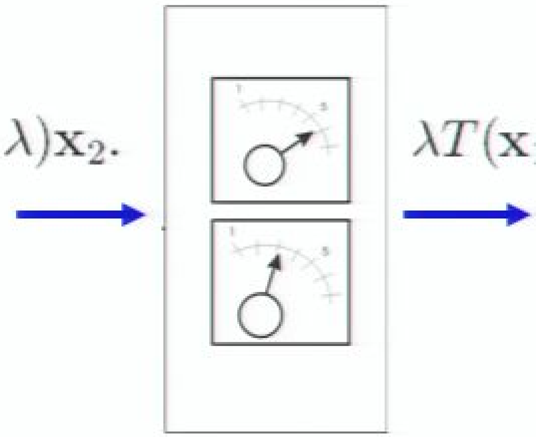

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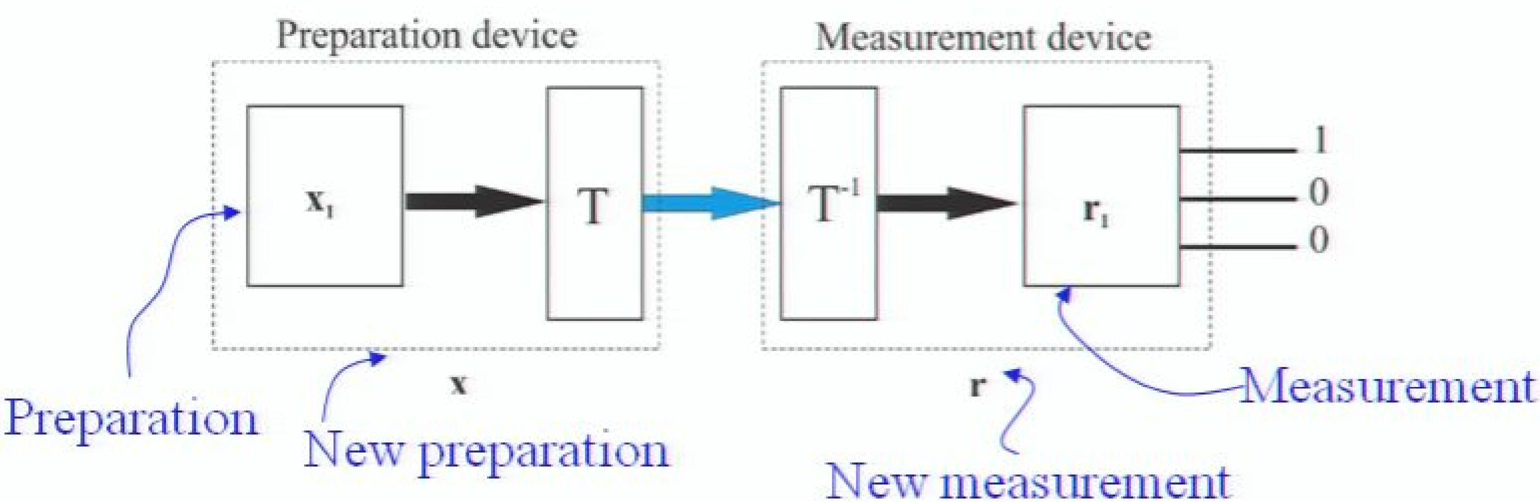

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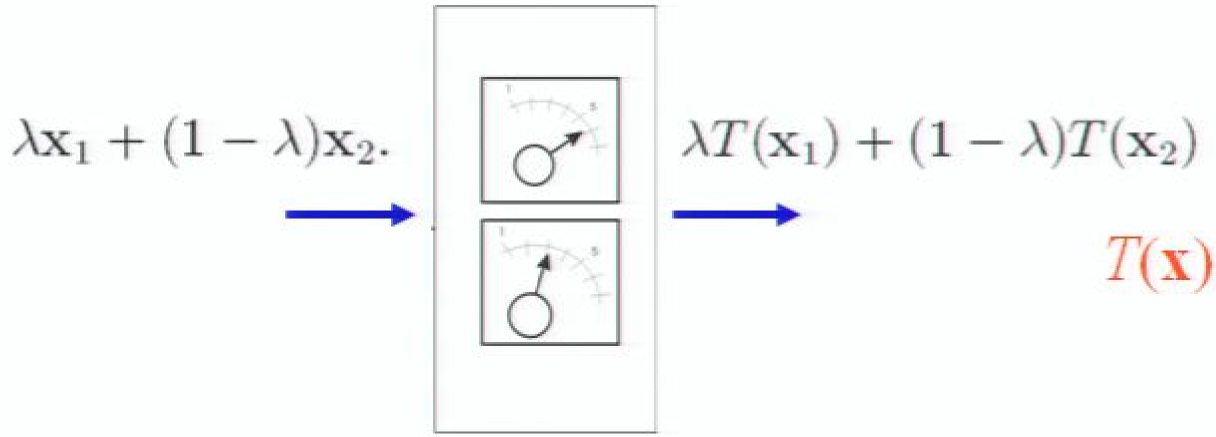

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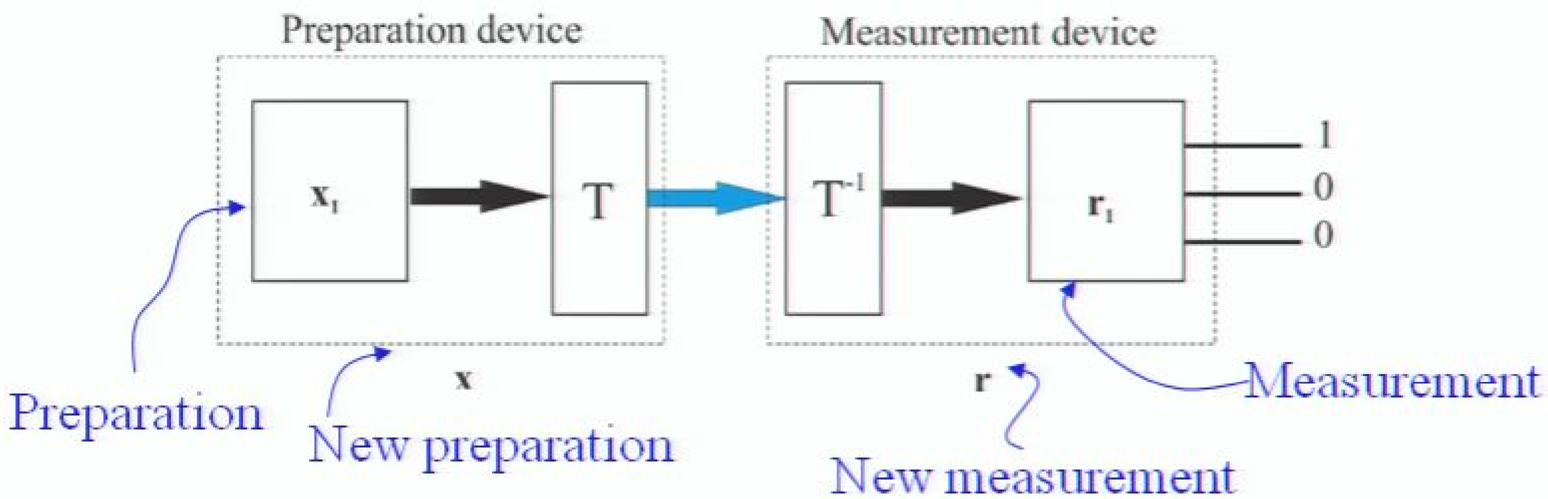

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→ All measurements return the same probability **1/L** → $\mathbf{E} = \vec{0}$ → $T(\vec{0}) = \vec{0}$

T represented by d x d invertable matrix

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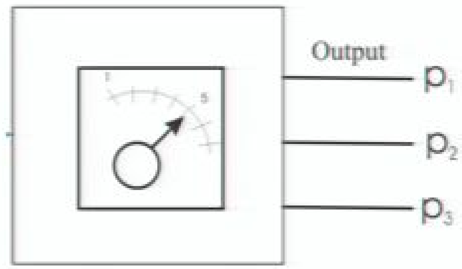
Included:

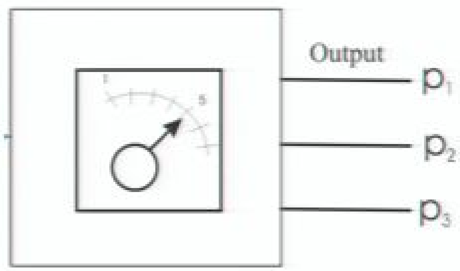
- finite groups
- continuous compact Lie groups

Measurements



$$P_{\mathrm{meas}} = f(\mathbf{x})$$

Measurements

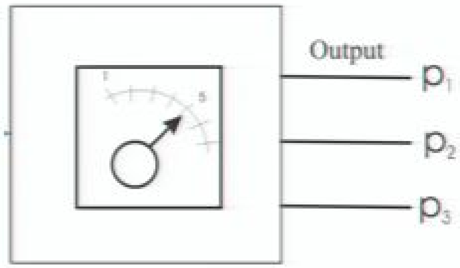


$$P_{\text{meas}} = f(\mathbf{x})$$

Mixing coefficients are unchanged:

$$f(\lambda \mathbf{x}_1 + (1-\lambda)\mathbf{x}_2) = \lambda f(\mathbf{x}_1) + (1-\lambda) f(\mathbf{x}_2) \quad \& \quad f(\vec{0}) = \frac{1}{L}$$

Measurements



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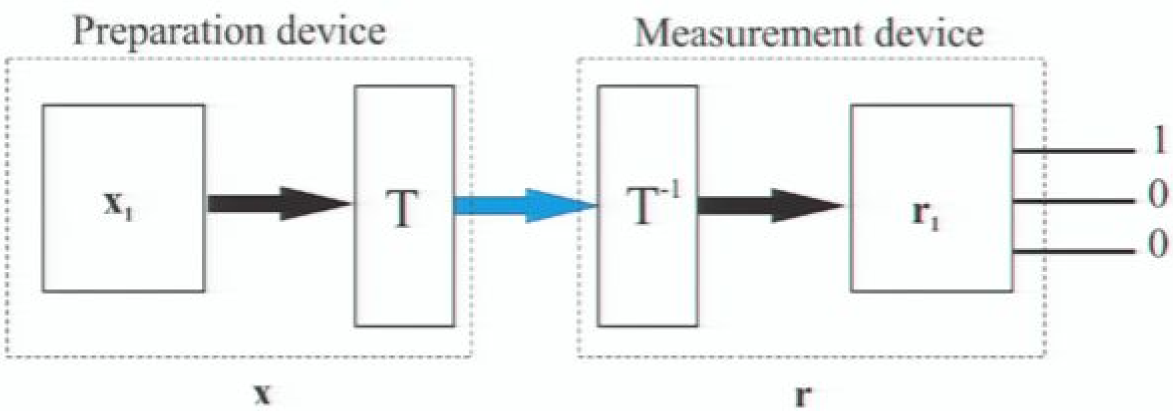
Linear affine transformation:

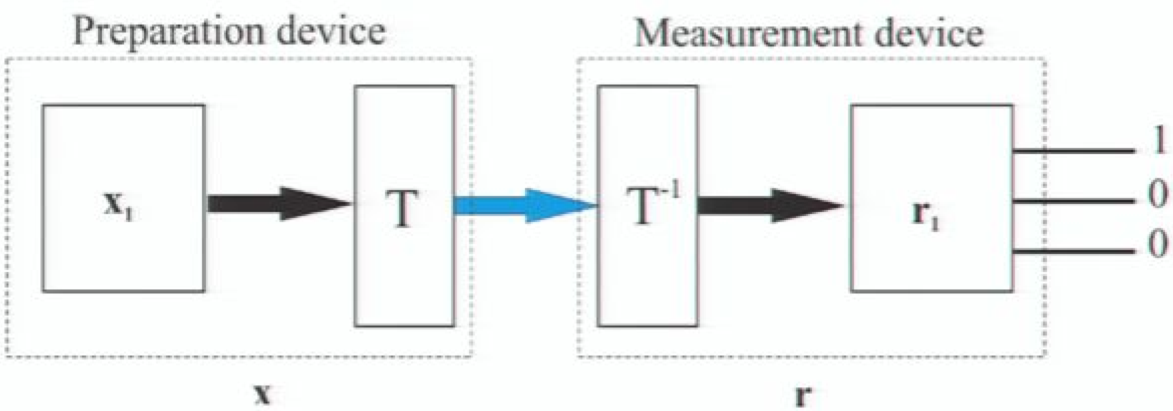
$$\longrightarrow \; P_{\text{meas}} = \frac{1}{L}(1 + (L-1)\mathbf{r}^T \mathbf{x})$$ **Probability rule**

The vector **r** represent the outcome for the given measurement setting, e.g. vector (1, 0, 0, …) represents one of the outcomes for the first measurement.

The set of measurement vectors



The set of measurement vectors

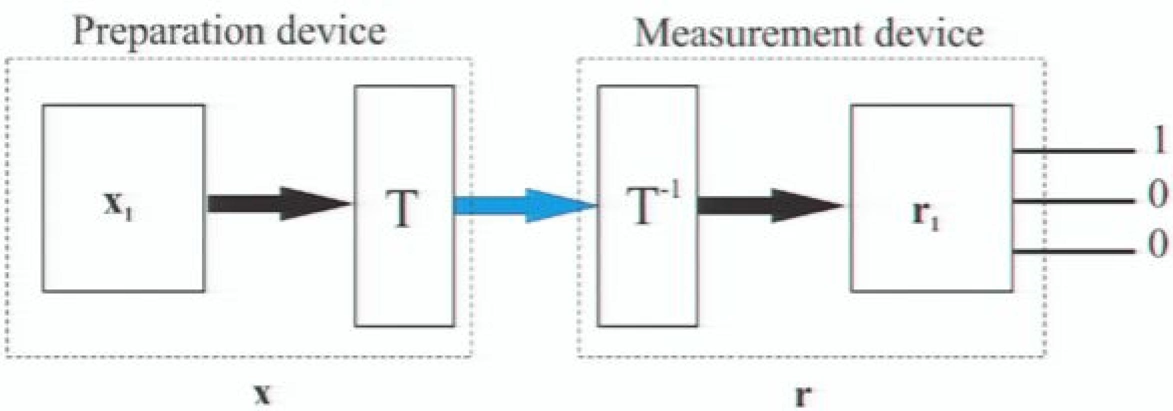


New coordinates $\qquad \mathbf{x} = T\mathbf{x}_1,$ $\qquad$ Old coordinates

$$\mathbf{x} = T\mathbf{x}_1,$$
$$\mathbf{r} = T^{-1T}\mathbf{r}_1$$

$$P_{\text{meas}} = \frac{1}{L}(1 + (L-1)\mathbf{r}^T\mathbf{x})$$

The set of measurement vectors



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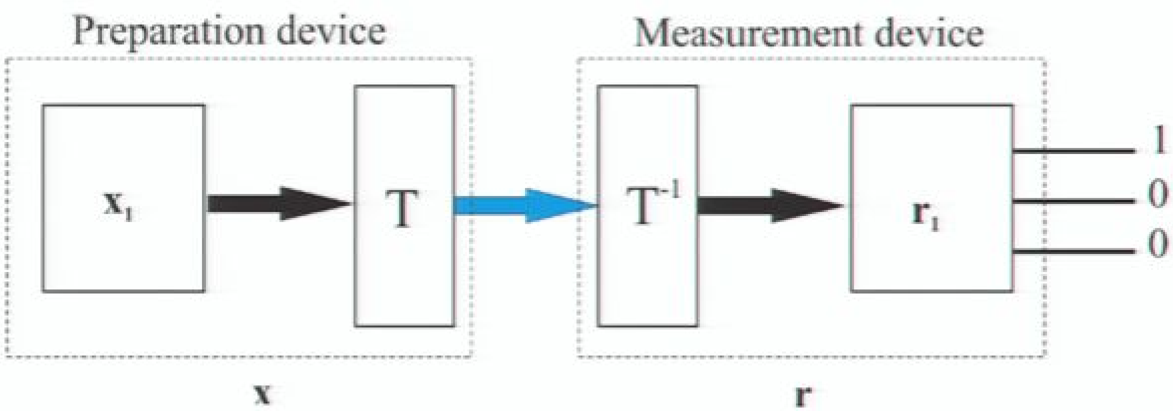
$$P_{\text{meas}} = \frac{1}{L}(1 + (L-1)\mathbf{r}^T\mathbf{x})$$

Representation picture: $\mathbf{y} = S\mathbf{x} \quad \mathbf{m} = S^{-1T}\mathbf{r}$

$$P_{\text{meas}} = \frac{1}{L}(1 + (L-1)\mathbf{m}^T\mathbf{y})$$

The set of measurement vectors

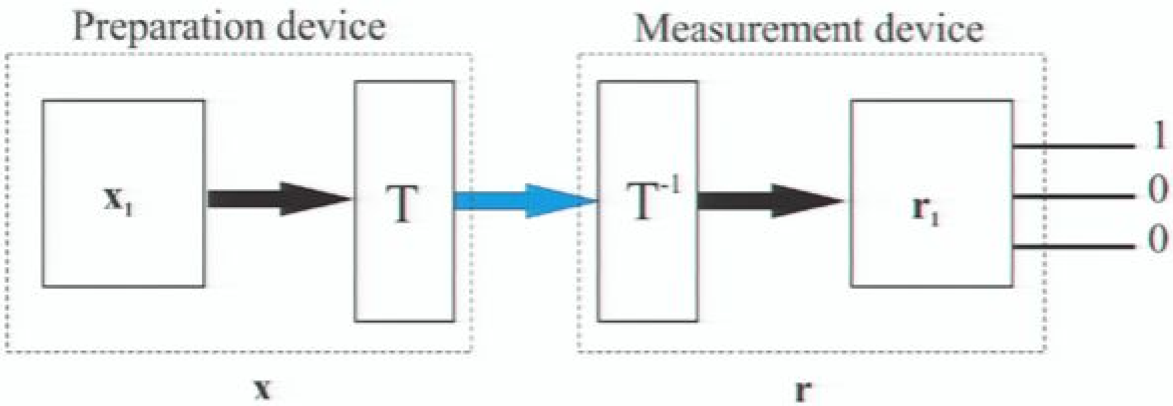


New coordinates $\begin{aligned} \mathbf{x} &= T\mathbf{x}_1, \\ \mathbf{r} &= T^{-1T}\mathbf{r}_1 \end{aligned}$ Old coordinates $P_{\text{meas}} = \frac{1}{L}(1 + (L-1)\mathbf{r}^T\mathbf{x})$

Representation picture: $\mathbf{y} = S\mathbf{x} \quad \mathbf{m} = S^{-1T}\mathbf{r}$

New coordinates $\begin{aligned} \mathbf{y} &= O\mathbf{y}_1, \\ \mathbf{m} &= O\mathbf{m}_1 \end{aligned}$ Old coordinates $P_{\text{meas}} = \frac{1}{L}(1 + (L-1)\mathbf{m}^T\mathbf{y})$

The set of measurement vectors



New coordinates / Old coordinates

$$\mathbf{x} = T\mathbf{x}_1, \qquad \mathbf{r} = T^{-1T}\mathbf{r}_1$$

$$P_{\text{meas}} = \frac{1}{L}(1 + (L-1)\mathbf{r}^T\mathbf{x})$$

Representation picture: $\mathbf{y} = S\mathbf{x} \quad \mathbf{m} = S^{-1T}\mathbf{r}$

New coordinates / Old coordinates

$$\mathbf{y} = O\mathbf{y}_1, \qquad \mathbf{m} = O\mathbf{m}_1$$

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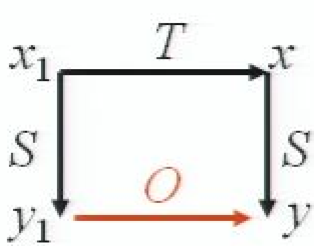
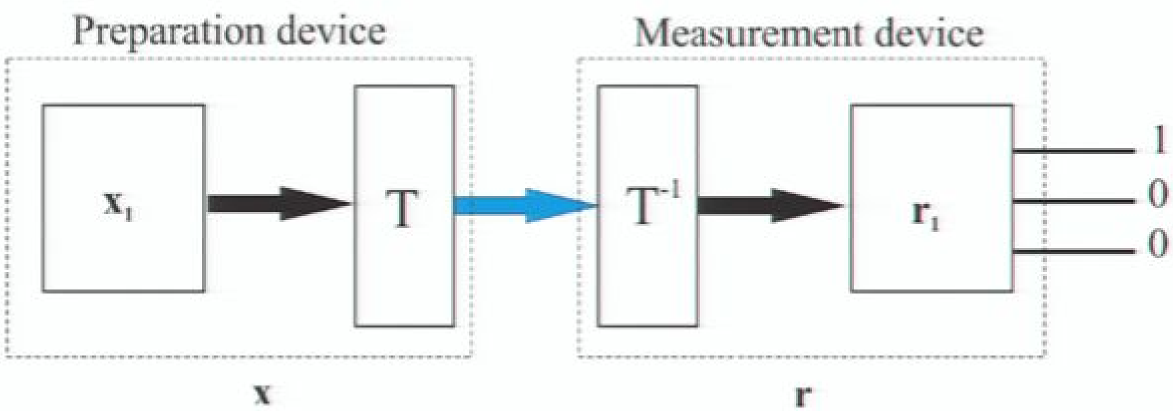


$STS^{-1} = O$

The set of measurement vectors



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$$\mathbf{x} = T\mathbf{x}_1, \quad \mathbf{r} = T^{-1T}\mathbf{r}_1$$

Old coordinates

$$P_{\rm meas} = \frac{1}{L}(1 + (L-1)\mathbf{r}^T\mathbf{x})$$

Representation picture: $\mathbf{y} = S\mathbf{x} \quad \mathbf{m} = S^{-1T}\mathbf{r}$

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Old coordinates

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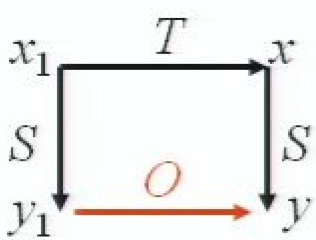
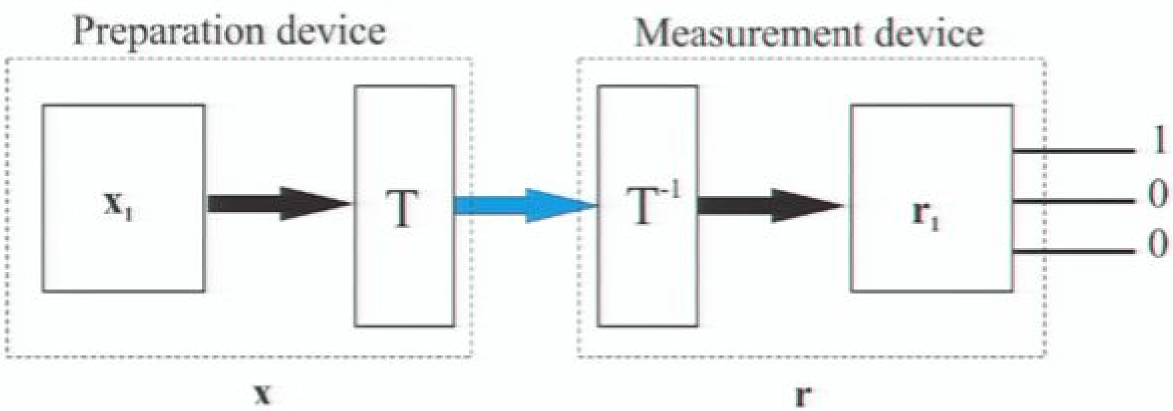


$STS^{-1}=O$

The set of measurement vectors



$$\mathbf{x} = T\mathbf{x}_1, \qquad \mathbf{r} = T^{-1T}\mathbf{r}_1$$

New coordinates — Old coordinates

$$P_{\text{meas}} = \frac{1}{L}(1 + (L-1)\mathbf{r}^T\mathbf{x})$$

Representation picture: $\mathbf{y} = S\mathbf{x} \quad \mathbf{m} = S^{-1T}\mathbf{r}$

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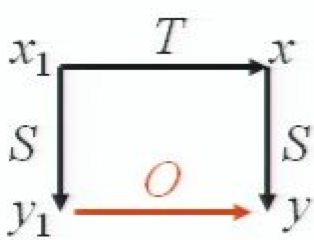
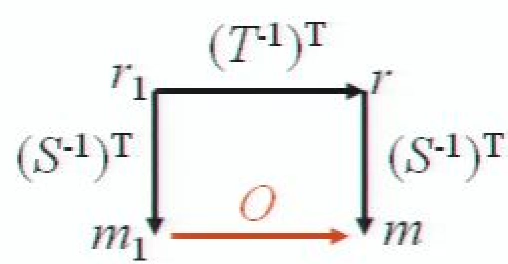
New coordinates — Old coordinates

$$P_{\text{meas}} = \frac{1}{L}(1 + (L-1)\mathbf{m}^T\mathbf{y})$$

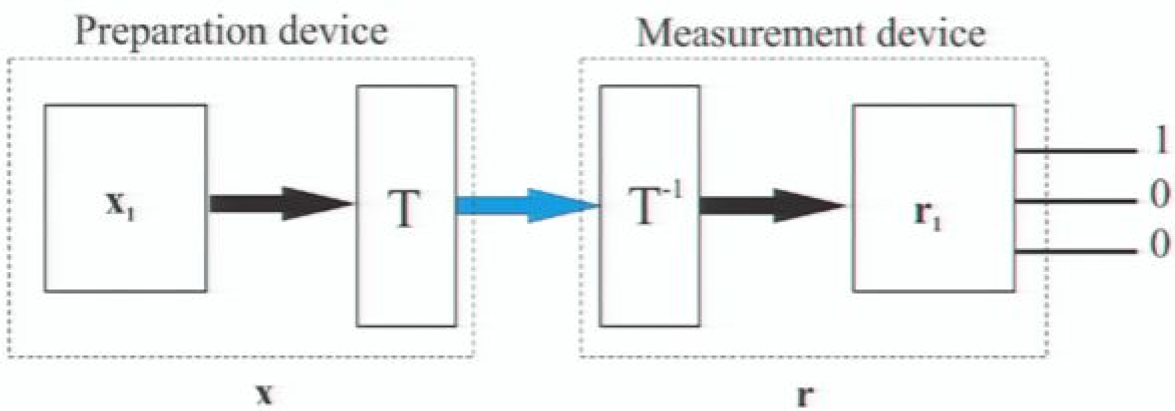


$STS^{-1}=O$



$S^T(T^{-1})^T(S^{-1})^T$
$=((O)^{-1})^T=(O)$

The set of measurement vectors



New coordinates → $\mathbf{x} = T\mathbf{x}_1, \quad \mathbf{r} = T^{-1T}\mathbf{r}_1$ ← Old coordinates

$$\mathbf{x} = T\mathbf{x}_1,$$
$$\mathbf{r} = T^{-1T}\mathbf{r}_1$$

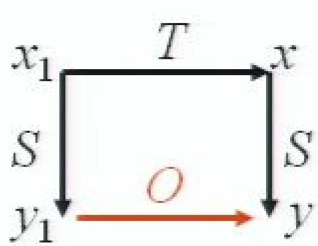
$$P_{\text{meas}} = \frac{1}{L}(1 + (L-1)\mathbf{r}^T\mathbf{x})$$

Representation picture: $\mathbf{y} = S\mathbf{x} \quad \mathbf{m} = S^{-1T}\mathbf{r}$

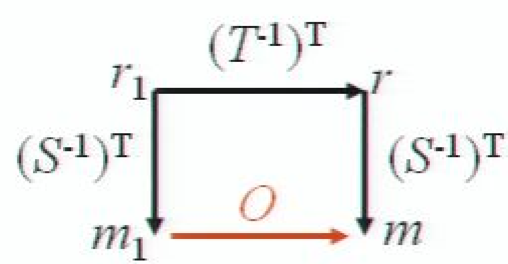
New coordinates → $\mathbf{y} = O\mathbf{y}_1, \quad \mathbf{m} = O\mathbf{m}_1$ ← Old coordinates

$$\mathbf{y} = O\mathbf{y}_1,$$
$$\mathbf{m} = O\mathbf{m}_1$$

$$P_{\text{meas}} = \frac{1}{L}(1 + (L-1)\mathbf{m}^T\mathbf{y})$$



$STS^{-1} = O$



$S^T(T^{-1})^T(S^{-1})^T = (O^{-1})^T = O$

The set of measurement vectors

$$\mathbf{x} = T\mathbf{x}_1, \qquad \mathbf{r} = T^{-1T}\mathbf{r}_1$$

Representation picture:

$$\mathbf{y} = S\mathbf{x} \quad \mathbf{m} = S^{-1T}\mathbf{r}$$

$$\mathbf{y} = O\mathbf{y}_1, \qquad \mathbf{m} = O\mathbf{m}_1$$

The set of measurement vectors

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Representation picture:

$$\mathbf{y} = S\mathbf{x} \quad \mathbf{m} = S^{-1T}\mathbf{r}$$

$$\mathbf{y} = O\mathbf{y}_1, \quad \mathbf{m} = O\mathbf{m}_1$$

$$||\mathbf{y}|| = c_1, \quad ||\mathbf{m}|| = c_2,$$

→ $\mathbf{y}_1/c_1$ and $\mathbf{m}_1/c_2$ are of the unit norm

The set of measurement vectors

$$\mathbf{x} = T\mathbf{x}_1, \qquad \mathbf{r} = T^{-1T}\mathbf{r}_1$$

Representation picture:

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$$\mathbf{y} = O\mathbf{y}_1, \qquad \mathbf{m} = O\mathbf{m}_1$$

$$||\mathbf{y}|| = c_1, \qquad ||\mathbf{m}|| = c_2,$$

→ $\mathbf{y}_1/c_1$ and $\mathbf{m}_1/c_2$ are of the unit norm → $\mathbf{m}_1 = \frac{c_2}{c_1}K\mathbf{y}_1$, $K^TK = 1\!\!1$ → $\mathbf{m} = \frac{c_2}{c_1}OKO^T\mathbf{y}$

The set of measurement vectors

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→

$$||\mathbf{y}|| = c_1$$

$$||OKO^T\mathbf{y}|| = c_1$$

for all $\mathbf{y}$

The set of measurement vectors

$$\mathbf{x} = T\mathbf{x}_1, \qquad \mathbf{r} = T^{-1T}\mathbf{r}_1$$

Representation picture:

$$\mathbf{y} = S\mathbf{x} \quad \mathbf{m} = S^{-1T}\mathbf{r}$$

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$$||\mathbf{y}|| = c_1, \quad ||\mathbf{m}|| = c_2, \longrightarrow$$

$\mathbf{y}_1/c_1$ and $\mathbf{m}_1/c_2$ are of the unit norm $\longrightarrow$ $\mathbf{m}_1 = \frac{c_2}{c_1} K \mathbf{y}_1$ $\longrightarrow$ $\mathbf{m} = \frac{c_2}{c_1} OKO^T\mathbf{y}$

$$K^T K = \mathbb{1}$$

$$\longrightarrow \; ||\mathbf{y}|| = c_1, \quad ||OKO^T\mathbf{y}|| = c_1 \; \text{for all } \mathbf{y} \longrightarrow K = \mathbb{1}$$

$$\mathbf{m} = \frac{c_2}{c_1}\mathbf{y}$$

$$\boxed{\mathbf{r} = \frac{c_2}{c_1} S^T S \mathbf{x} = D\mathbf{x}}$$

Hardy (2001, when the set T forms a Lie group)

The set of measurement vectors

$$\mathbf{x} = T\mathbf{x}_1, \qquad \mathbf{r} = T^{-1T}\mathbf{r}_1$$

Representation picture:

$$\mathbf{y} = S\mathbf{x} \quad \mathbf{m} = S^{-1T}\mathbf{r}$$

$$\mathbf{y} = O\mathbf{y}_1, \qquad \mathbf{m} = O\mathbf{m}_1$$

$$||\mathbf{y}|| = c_1, \quad ||\mathbf{m}|| = c_2, \longrightarrow \mathbf{y}_1/c_1 \text{ and } \mathbf{m}_1/c_2 \text{ are of the unit norm} \longrightarrow \mathbf{m}_1 = \frac{c_2}{c_1}K\mathbf{y}_1 \longrightarrow \mathbf{m} = \frac{c_2}{c_1}OKO^T\mathbf{y}$$

$$K^TK = \mathbb{1}$$

$$\longrightarrow \begin{array}{rcl} ||\mathbf{y}|| &=& c_1 \\ ||OKO^T\mathbf{y}|| &=& c_1 \end{array} \longrightarrow K = \mathbb{1} \qquad \mathbf{m} = \frac{c_2}{c_1}\mathbf{y} \qquad \boxed{\mathbf{r} = \frac{c_2}{c_1}S^TS\mathbf{x} = D\mathbf{x}}$$

for all $\mathbf{y}$

$$\mathbf{r} = D\mathbf{x} \qquad D = \frac{c_2}{c_1}S^TS$$

is positive symmetric invertible matrix

Hardy (2001, when the set T forms a Lie group)

The set of pure states

$\mathbf{r} = D\mathbf{x}$ $\quad D = \frac{c_2}{c_1} S^T S$ is positive symmetric matrix

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$$P(\mathbf{x}_1, \mathbf{x}_2) = \frac{1}{L}(1 + (L-1)\mathbf{x}_1^T D \mathbf{x}_2)$$

$$P(\mathbf{x}, \mathbf{x}) = 1 \longrightarrow \mathbf{x}^T D \mathbf{x} = 1 \quad \Leftrightarrow \quad ||\sqrt{\frac{c_2}{c_1}} S\mathbf{x}|| = 1$$

x are points on a d-dimensional ellipsoid

The set of pure states

$$\mathbf{r} = D\mathbf{x}$$ $D = \frac{c_2}{c_1} S^T S$ is positive symmetric matrix

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Two-Level System

The set of pure states

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x are points on a d-dimensional ellipsoid

Two-Level System $\frac{1}{2}(1 + \mathbf{x}^T D \mathbf{y}) + \frac{1}{2}(1 + \mathbf{x}^{\perp T} D \mathbf{y}) = 1$

Two basis states: $\mathbf{x}^{\perp} = -\mathbf{x}.$

The set of pure states

$$\mathbf{r} = D\mathbf{x} \qquad D = \frac{c_2}{c_1} S^T S$$ is positive symmetric matrix

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$$\rightarrow \quad \mathcal{S}^{d-1} = \{\mathbf{x} \mid ||\mathbf{x}|| = 1\} \quad P(\mathbf{x}_1, \mathbf{x}_2) = \frac{1}{2}(1 + \mathbf{x}_1^T \mathbf{x}_2)$$

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Do physical states form the entire d-dimensional sphere?

Maximal set consistent with $P(\mathbf{x}_1, \mathbf{x}_2) = \frac{1}{2}(1 + \mathbf{x}_1^T \mathbf{x}_2) \geq 0$

Axiom 1

Any state (pure or mixed) of 2-level system can be prepared by mixing at most 2 orthogonal states.

Maximal set consistent with $P(\mathbf{x}_1, \mathbf{x}_2) = \frac{1}{2}(1 + \mathbf{x}_1^T \mathbf{x}_2) \geq 0$

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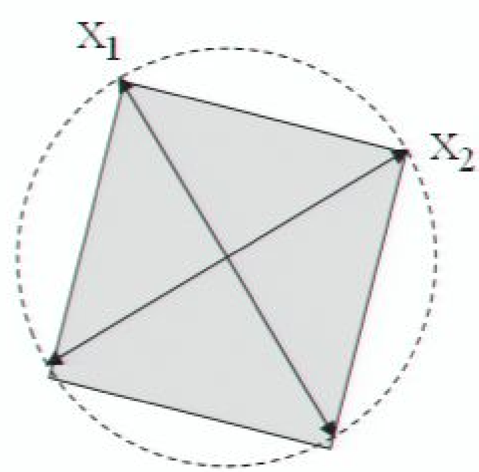

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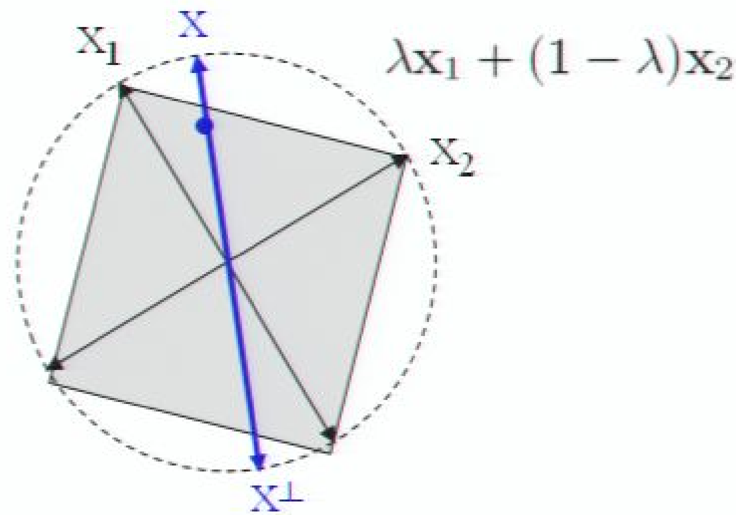
Decomposition of state: $\lambda' \mathbf{x} + (1-\lambda')\mathbf{x}^{\perp}$



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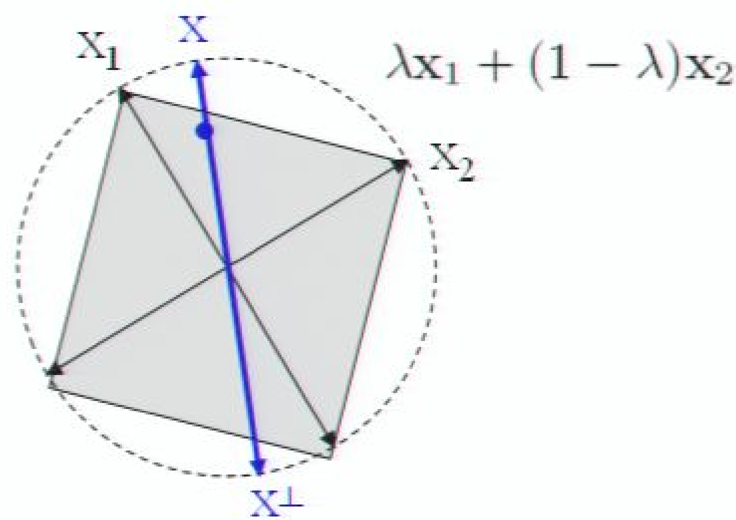

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Axiom 1

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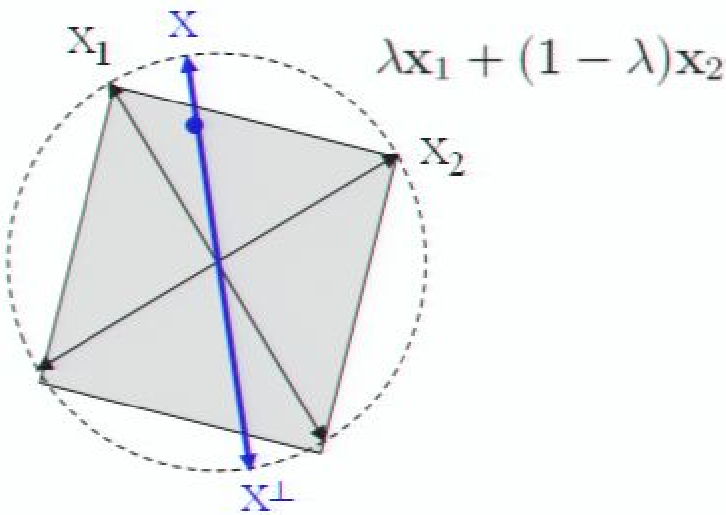
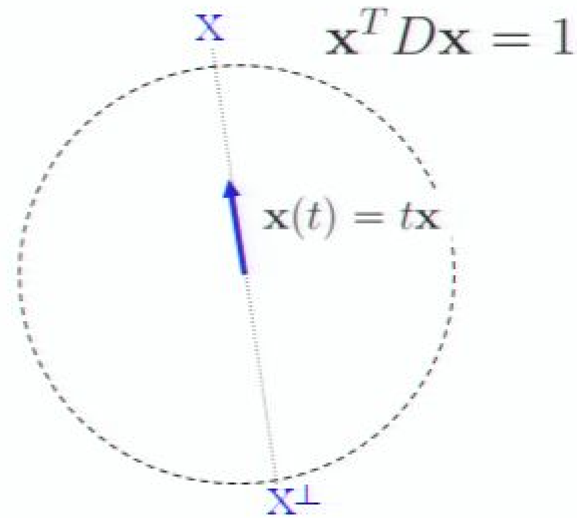
Existence of minimal entropy decomposition $H(p_x, 1-p_x) \leq H(p_{x1}, 1-p_{x1})$

Decomposition of state: $\lambda' \mathbf{x} + (1-\lambda')\mathbf{x}^{\perp}$



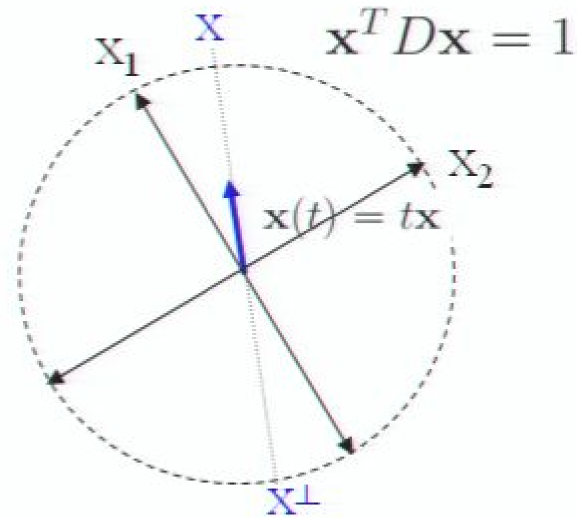
Maximal set consistent with $P(\mathbf{x}_1, \mathbf{x}_2) = \frac{1}{2}(1 + \mathbf{x}_1^T \mathbf{x}_2) \geq 0$

Physical States = Entire Sphere



Physical States = Entire Sphere

$\mathbf{x}(t) = t\mathbf{x} \longrightarrow \mathbf{x}(t) = t\sum_{i=1}^{d} c_i \mathbf{x}_i$



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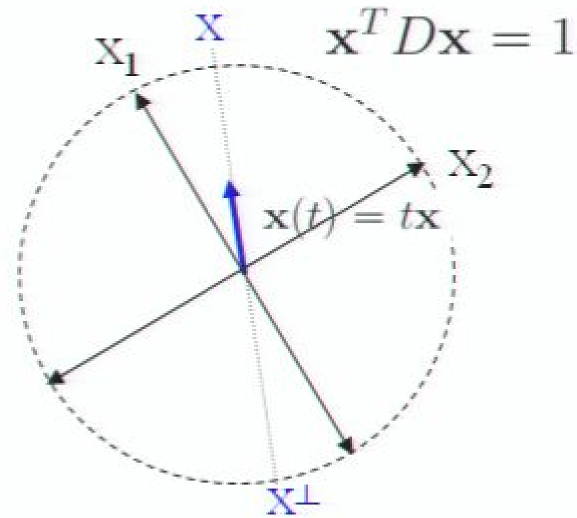
$\mathbf{x}(t) = \sum_{i=1}^{d} \lambda_i(t)\mathbf{x}_i + \lambda_i^{\perp}(t)\mathbf{x}_i^{\perp}$

For small t, positive numbers

$\lambda_i(t) = \frac{1}{2}(\frac{1}{d} + tc_i) \quad \lambda_i^{\perp}(t) = \frac{1}{2}(\frac{1}{d} - tc_i)$



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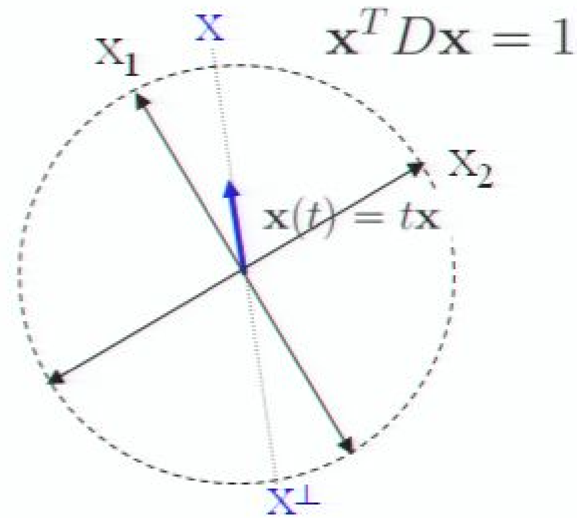

Axiom1 $\longrightarrow \mathbf{x}(t) = t\mathbf{x} = \alpha\mathbf{x}_0 + (1-\alpha)(-\mathbf{x}_0)$

$\alpha = \frac{1+t}{2}$ and $\mathbf{x} = \mathbf{x}_0$

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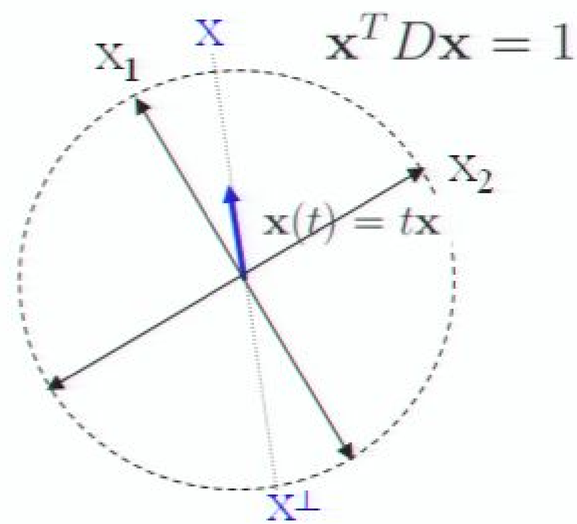

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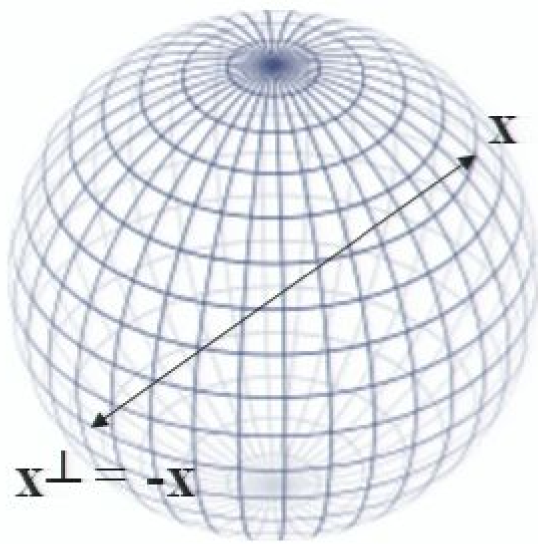
x(t) is physical state for all t – entire sphere

Physics of generalized bits

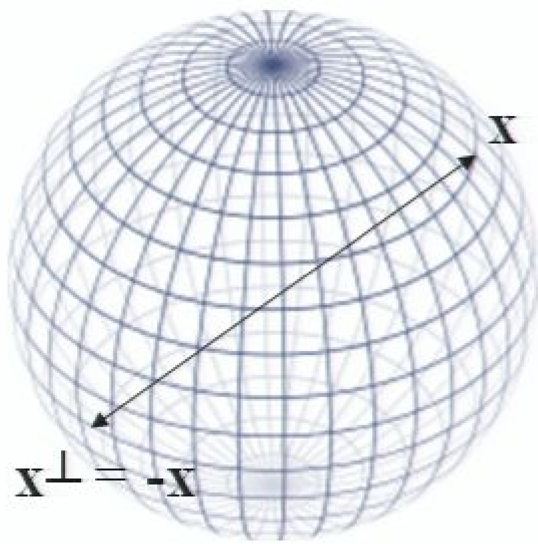


States, Measurement vectors
= S^{d-1}

$\mathbf{p} = (p_1, \ldots, p_d)$

$\mathbf{x} = (2p_1 - 1, \ldots, 2p_d - 1)$

Physics of generalized bits



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Transformations

preserving purity $||R\mathbf{x}|| = 1$

orthogonal matrix $R^T R = \mathbb{1}$

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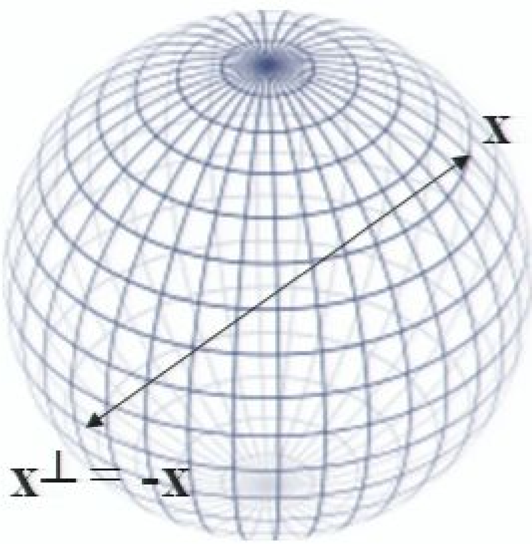

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1. How many complementary observables (MUBs) does g-bit have?
2. What are computational capabilities of g-bits?
3. Can two g-bits be entangled?

1 Bit Propositions

Boolean functions of a binary argument $x \in \{0,1\} \rightarrow y = f(x) \in \{0,1\}$

x	f_0	f_1	f_2	f_3
0	0	0	1	1
1	0	1	0	1

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b=0	b=1	
$f_0\, f_1$	$f_2\, f_3$	$f(0)$=b
$f_0\, f_2$	$f_1\, f_3$	$f(1)$=b
$f_0\, f_3$	$f_1\, f_2$	$f(1)$=$f(0)$+b

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$f_0\,f_3$	$f_1\,f_2$	$f(1)=f(0)$+b

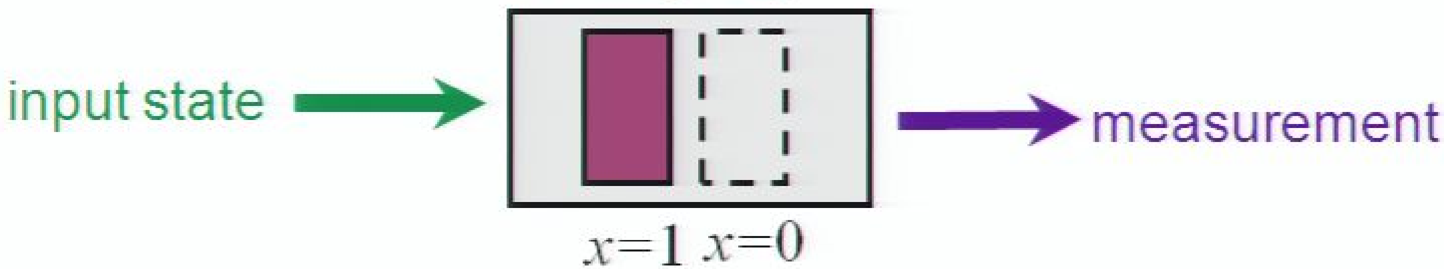
Given **1 bit of information resources** only 1 out of **3 complementary questions** can be answered.

1 qubit can only reveal 1 bit

The black box encodes the Boolean functions

$f(x)=0$ $f(0)=1$ $f(1)=1$

$U = \sigma_x^{f(0)} \sigma_z^{f(1)}$



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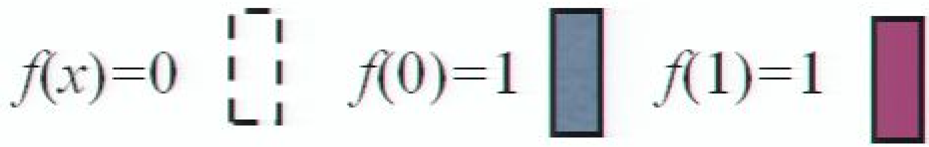

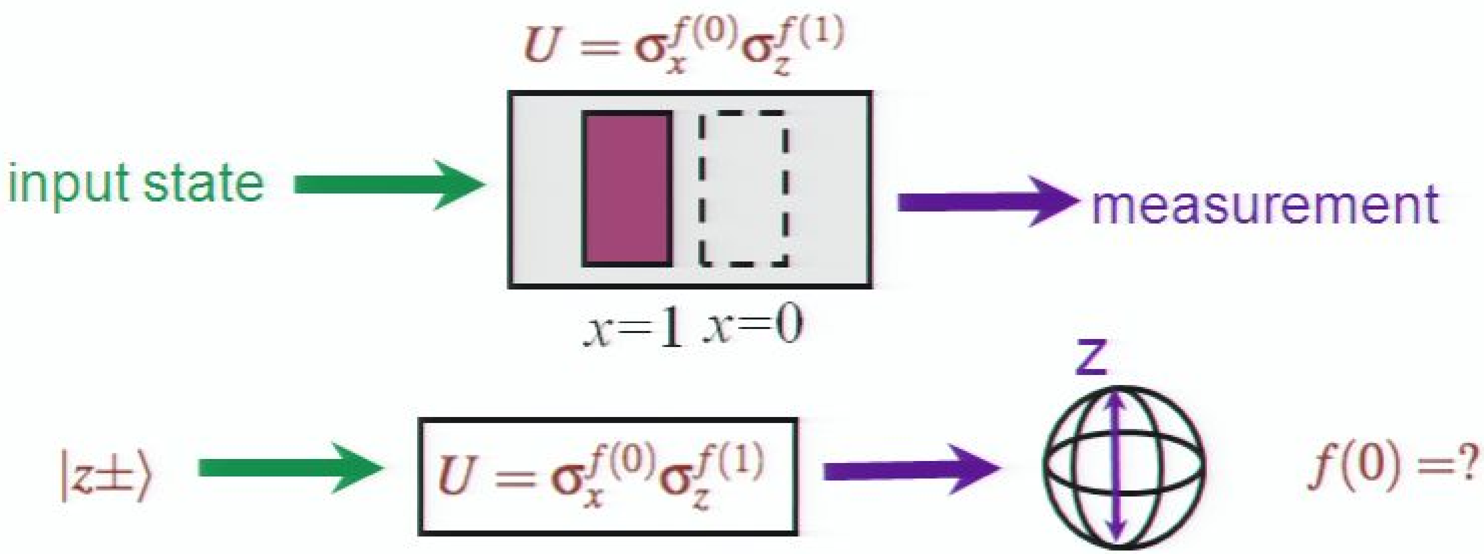

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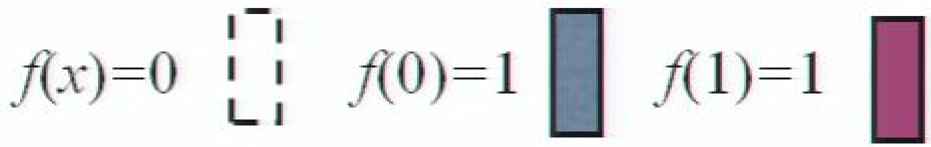

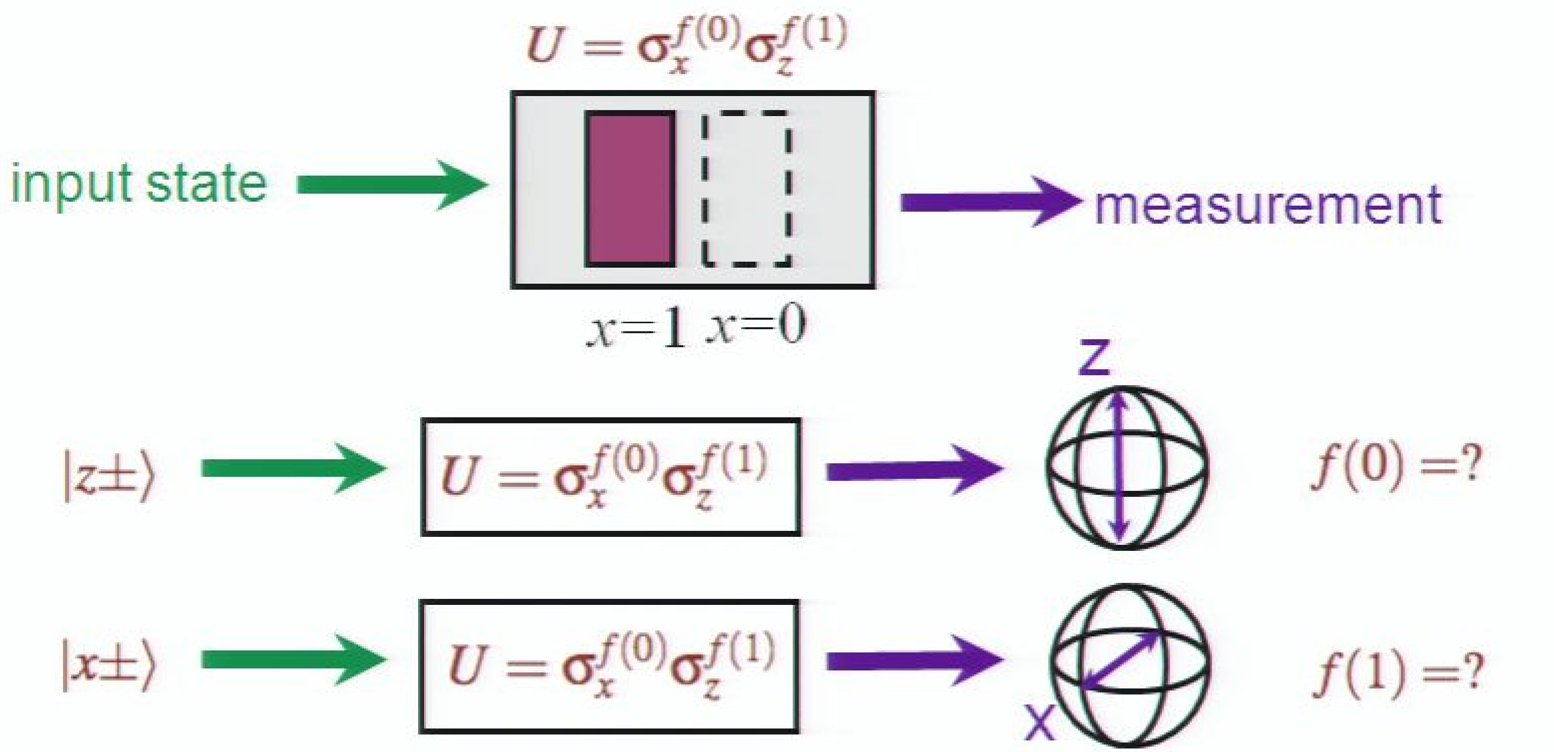

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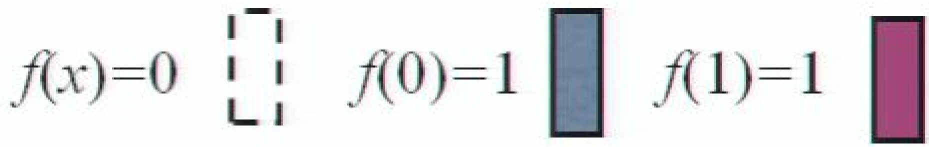

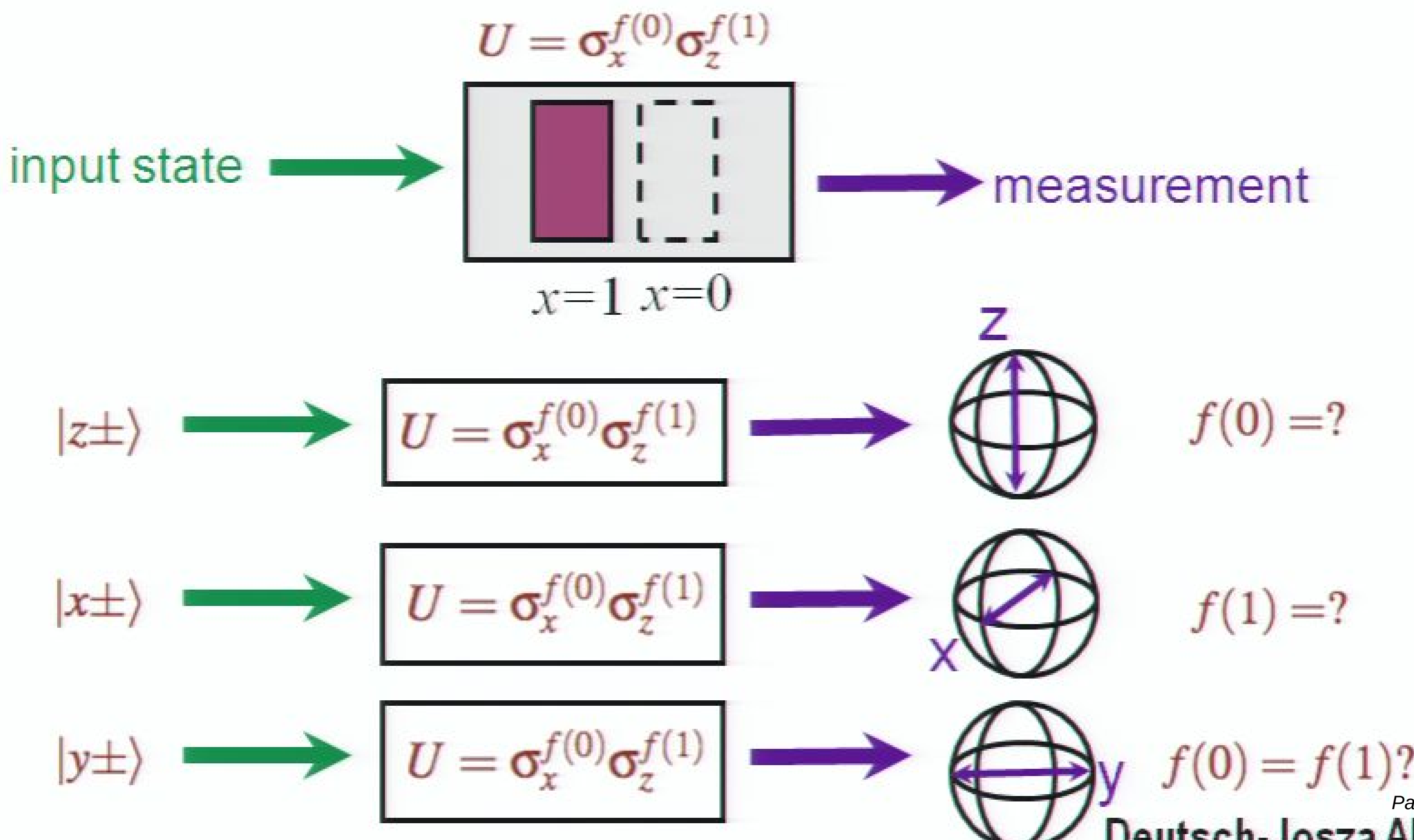

Deutsch-Josza Algorithm

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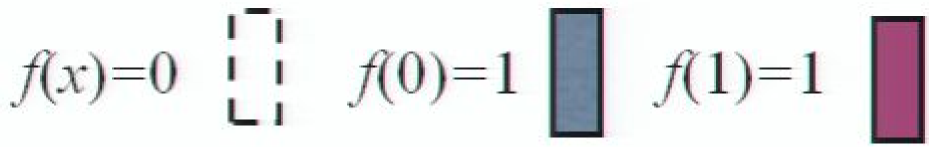
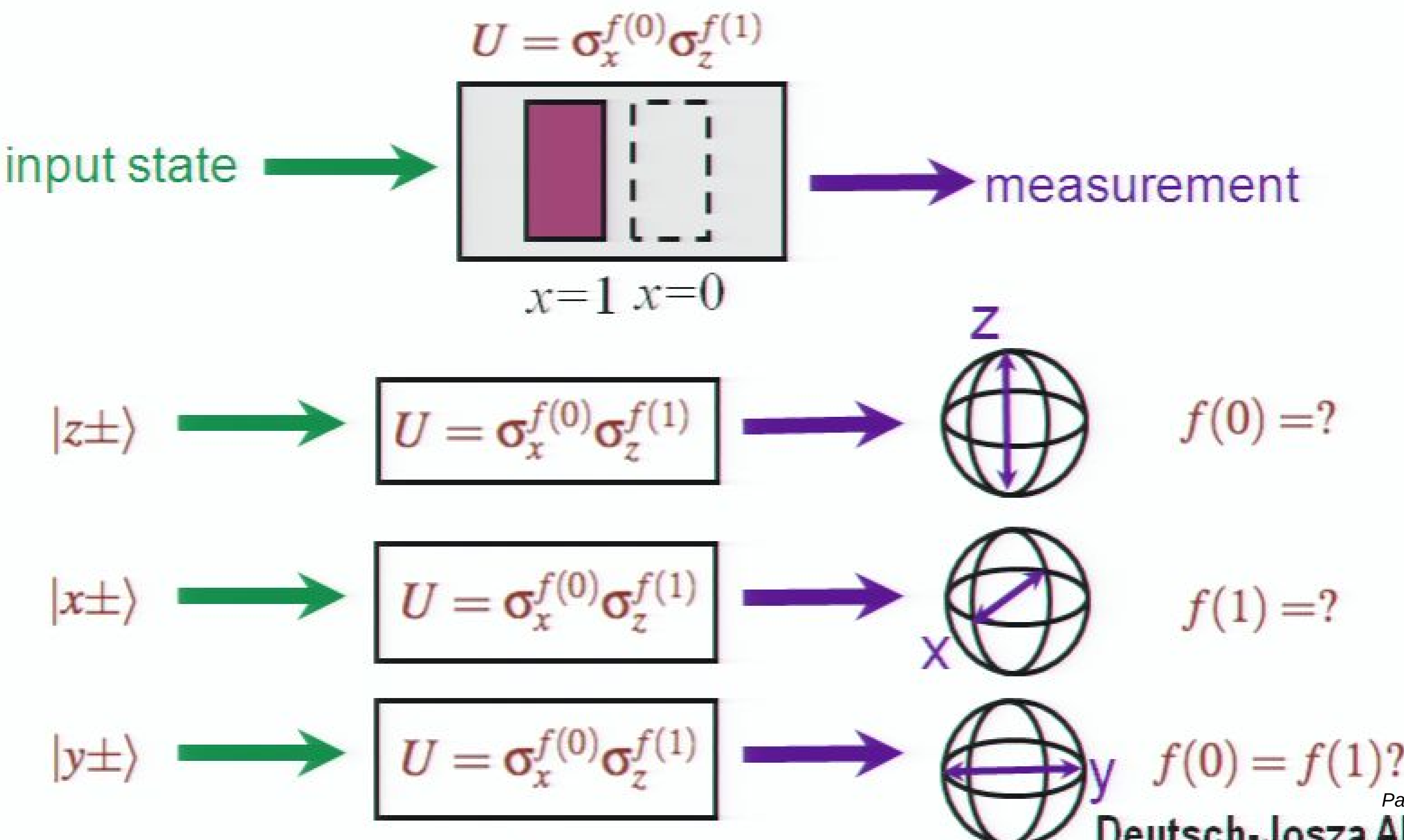
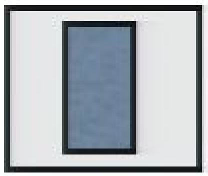




Deutsch-Josza Algorithm

analogous to Speekkens toy model

How many complementarity questions?



Classical bit, **1** question: Is there an object in the black box?

T. Paterek, B. Dakic, Č. Brukner, arXiv:0804.1423

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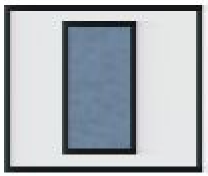

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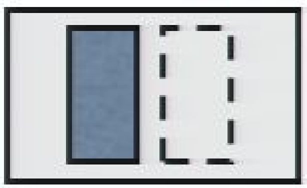

Qubit, **3** complementary questions

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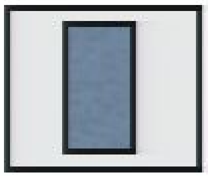

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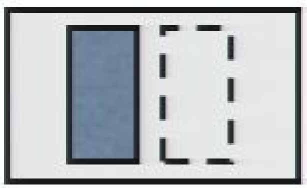

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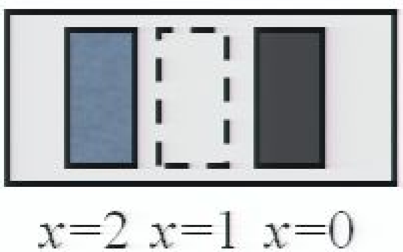

$x \in \{0,1,2\} \rightarrow y = f(x) \in \{0,1\}$

7 complementary questions

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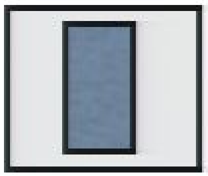

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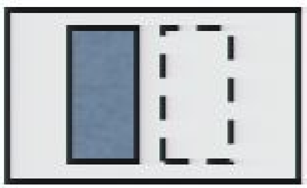

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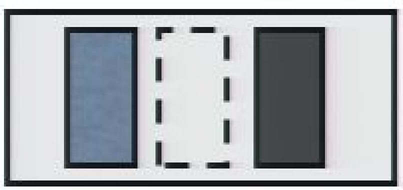

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$f(0)=?$ $f(1)=?$ $f(2)=?$

$f(0)+f(1)=?$

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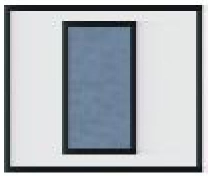

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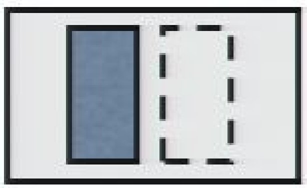

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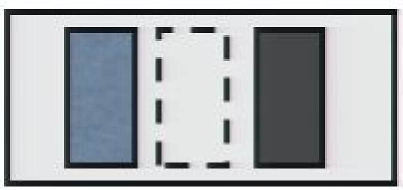

$x=2$ $x=1$ $x=0$

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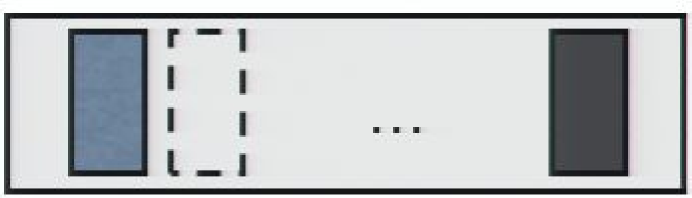
...



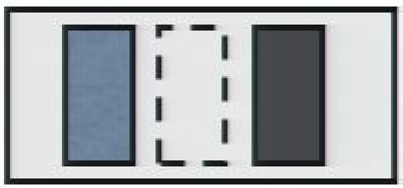
$x \in \{0,1,...,s-1\} \rightarrow y = f(x) \in \{0,1\}$

$$\binom{s}{1} + \binom{s}{2} + ... + \binom{s}{s} = 2^s - 1$$

complementary questions

T. Paterek, B. Dakic, Č. Brukner, arXiv:0804.1423

Case d=2^3-1=7



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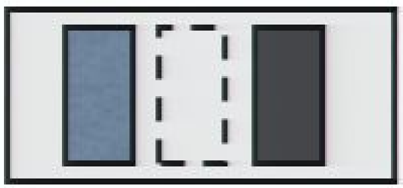

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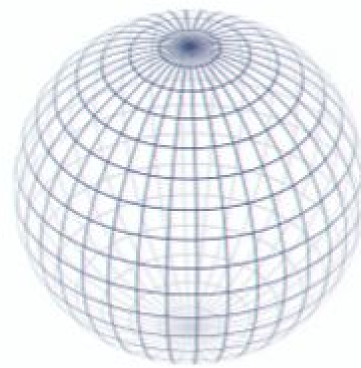
7 complementary questions

7-dimensional "Bloch" sphere,

states = vectors

Physical operations = rotations

Probability rule: $P(\vec{m}|\vec{n}) = \frac{1}{2}(1 + \vec{m} \cdot \vec{n})$



Discrete or Continuous Set States

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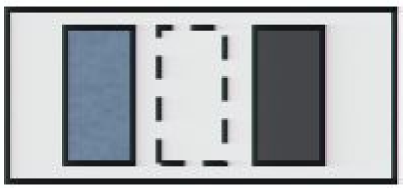

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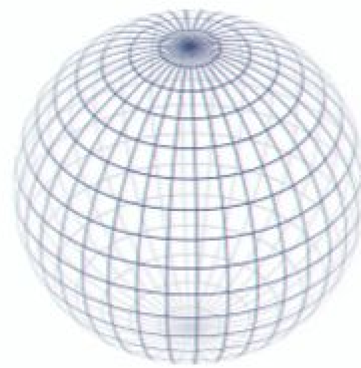


Discrete or Continuous Set States

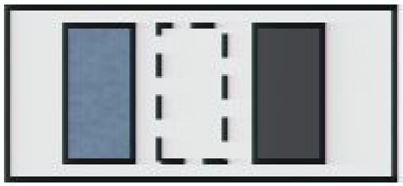
Black Box:

$R_0^{f(0)} R_1^{f(1)} R_2^{f(2)}$

$$R_0 \rightarrow \mathrm{diag}[-1, 1, 1, -1, -1, 1, -1]$$
$$R_1 \rightarrow \mathrm{diag}[1, -1, 1, -1, 1, -1, -1]$$
$$R_2 \rightarrow \mathrm{diag}[1, 1, -1, 1, -1, -1, -1]$$

"Clifford Operations" SO(7), det R=1

Case d=2^3-1=7



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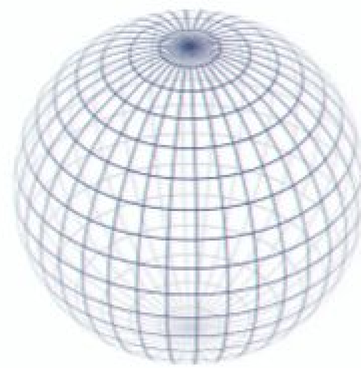

Discrete or Continuous Set States

Black Box:

$R_0^{f(0)} R_1^{f(1)} R_2^{f(2)}$

$$\begin{aligned} R_0 &\rightarrow \mathrm{diag}[-1,1,1,-1,-1,1,-1] \\ R_1 &\rightarrow \mathrm{diag}[1,-1,1,-1,1,-1,-1] \\ R_2 &\rightarrow \mathrm{diag}[1,1,-1,1,-1,-1,-1] \end{aligned}$$

"Clifford Operations" SO(7), det R=1

Note: Scaling d=2^r-1 the same as from the parameter counting argument for composite systems **-> coming next**!

Axiom (2) on Composite Systems:

The state of a composite system is completely determined by a set of probabilities for local measurements and the correlations between these measurements.

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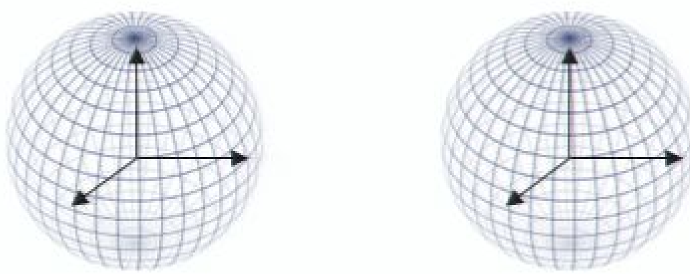

$$\psi = (\mathrm{x}, \mathrm{y}, T)$$

Local vectors Correlations

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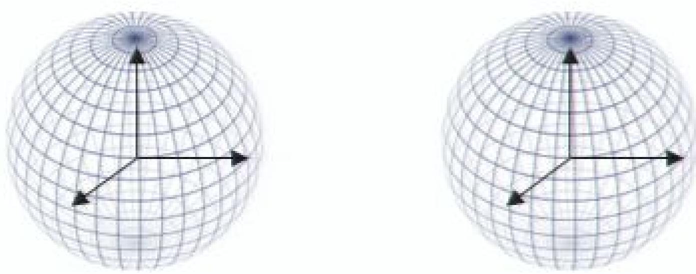

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Local vectors — Correlations

$$d(L_1)+d(L_2)+d(L_1)d(L_2) = d(L_1L_2) \longrightarrow d=L^r-1 \quad r=1,2,\ldots$$

Solution

Axiom (2) on Composite Systems:

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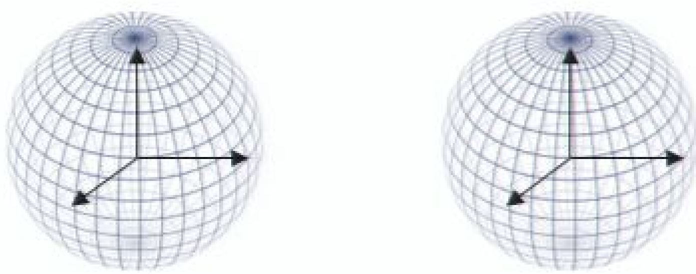

$$\psi = (\mathrm{x}, \mathrm{y}, T)$$

Local vectors (x, y) — **Correlations** (T)

$$d(L_1) + d(L_2) + d(L_1)d(L_2) = d(L_1 L_2) \longrightarrow d = L^r - 1 \quad r = 1, 2, \ldots$$

Solution

Real Quantum Mechanics: **2 + 2 + 2x2 < 9**

Quaternionic QM: **5 + 5 + 5x5 > 27**

Complex QM: **3 + 3 + 3x3 = 15**

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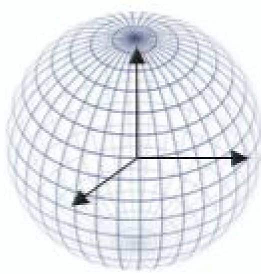

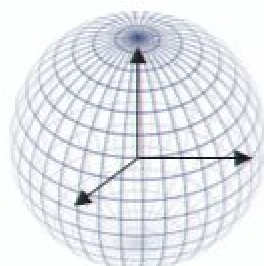

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Wootters (1986), Hardy (2001), Barrett (2005)

State of Composite System

$$\psi = (\mathrm{x}, \mathrm{y}, T)$$

Local vectors → x, y

Correlations → T

State of Composite System

$$\psi = (\mathbf{x}, \mathbf{y}, T)$$

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Product states: $\psi_p = (\mathbf{x}, \mathbf{y}, T = \mathbf{x}^T \mathbf{y})$

State of Composite System

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Product states: $\psi_p = (\mathbf{x}, \mathbf{y}, T = \mathbf{x}^T\mathbf{y})$

Probability rule: $P_{12}(\psi_1, \psi_2) = P_1(\mathbf{x}_1, \mathbf{x}_2)P_2(\mathbf{y}_1, \mathbf{y}_2)$ ← Product State

$$\begin{aligned} P_{12}(\psi_1, \psi_2) &= \frac{1}{4}(1 + \mathbf{x}_1^T\mathbf{x}_2 + \mathbf{y}_1^T\mathbf{y}_2 + \mathbf{x}_1^T\mathbf{x}_2\mathbf{y}_1^T\mathbf{y}_2) \\ &= \frac{1}{4}(1 + \mathbf{x}_1^T\mathbf{x}_2 + \mathbf{y}_1^T\mathbf{y}_2 + \mathrm{Tr}(T_1^T T_2)). \end{aligned}$$

← General State

State of Composite System

$$\psi = (\mathbf{x}, \mathbf{y}, T)$$

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Continuosly Connected with Idenitity

Local transformations: $(R_1, R_2)\psi = (R_1 \mathbf{x}, R_2 \mathbf{y}, R_1 T R_2^T)$ $\quad SO(d)$

$$\mathrm{diag}[t_1, \ldots, t_d] = R_1 T R_2^T$$ Singular value decomposition

An “Entanglement Witness“

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$$P(\psi, \phi_p) \geq 0 \longrightarrow 1 - ||\mathbf{x}||^2 - ||\mathbf{y}||^2 + \mathrm{Tr}(T^T T_0) = -1 + \mathrm{Tr}(T^T T_0) \geq 0$$

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d cannot be even

(in a theory with entanglement)

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Fliping **all** coordinates of a local (Bloch) vector x: $E\mathrm{x} = -\mathrm{x}$

Regular local transformation if d even as **det E = 1**

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Repeat the same for y: $E\mathbf{y} = \mathbf{-y}$

No entanglement, only product states!

Axiom (3) on Subspaces:

Upon any two linearly independent state ψ_1, ψ_2 *one can construct a two-dimensional subspace that is isomorphic to d-1 sphere* $\mathcal{S}^{d-1}$.

Hardy (2001)

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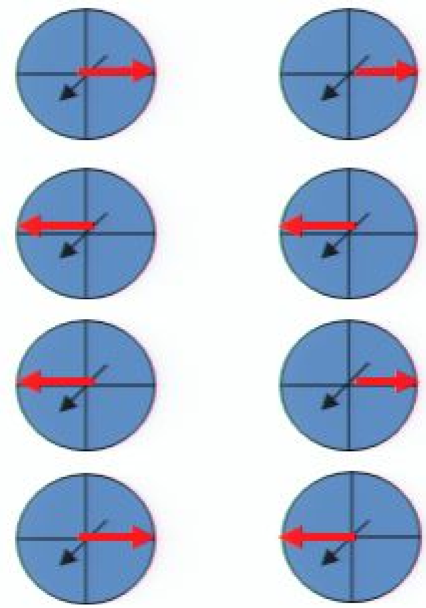

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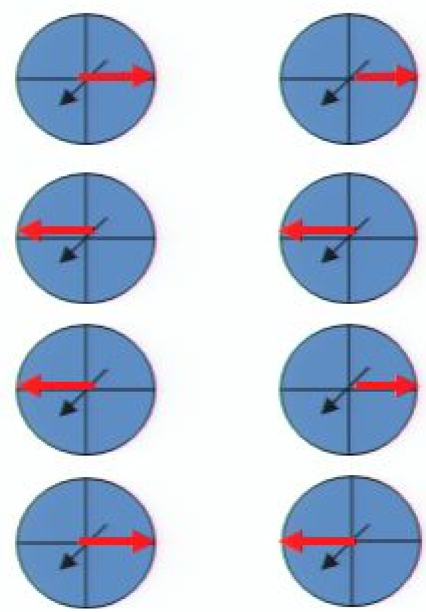

State Ψ belongs to 12 subspace iff:

$$P_{12}(\psi, \psi_1) + P_{12}(\psi, \psi_2) = 1$$

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The only product state belonging to S_{12} are Ψ_1 and Ψ_2.

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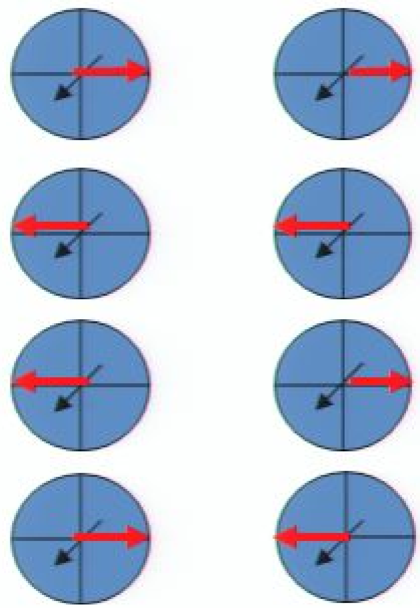

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Let $\psi_p = (\mathbf{x}, \mathbf{y}, \mathbf{x}\mathbf{y}^T) \in S_{12}$. We have

$$
\begin{aligned}
1 &= P_{12}(\psi_p, \psi_1) + P_{12}(\psi_p, \psi_2) \\
&= \frac{1}{4}(1 + \mathbf{x}\mathbf{e}_1 + \mathbf{y}\mathbf{e}_1 + (\mathbf{x}\mathbf{e}_1)(\mathbf{y}\mathbf{e}_1)) + \frac{1}{4}(1 - \mathbf{x}\mathbf{e}_1 - \mathbf{y}\mathbf{e}_1 + (\mathbf{x}\mathbf{e}_1)(\mathbf{y}\mathbf{e}_1)) \\
&= \frac{1}{2}(1 + (\mathbf{x}\mathbf{e}_1)(\mathbf{y}\mathbf{e}_1)) \\
&\Rightarrow \mathbf{x}\mathbf{e}_1 = \mathbf{y}\mathbf{e}_1 = 1 \ \vee \ \mathbf{x}\mathbf{e}_1 = \mathbf{y}\mathbf{e}_1 = -1 \\
&\Leftrightarrow \mathbf{x} = \mathbf{y} = \mathbf{e}_1 \ \vee \ \mathbf{x} = \mathbf{y} = -\mathbf{e}_1.
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Let $\psi_p = (\mathbf{x}, \mathbf{y}, \mathbf{x}\mathbf{y}^T) \in S_{12}$. We have

$$\begin{aligned}
1 &= P_{12}(\psi_p, \psi_1) + P_{12}(\psi_p, \psi_2) \\
&= \frac{1}{4}(1 + \mathbf{xe}_1 + \mathbf{ye}_1 + (\mathbf{xe}_1)(\mathbf{ye}_1)) + \frac{1}{4}(1 - \mathbf{xe}_1 - \mathbf{ye}_1 + (\mathbf{xe}_1)(\mathbf{ye}_1)) \\
&= \frac{1}{2}(1 + (\mathbf{xe}_1)(\mathbf{ye}_1)) \\
\Rightarrow\ & \mathbf{xe}_1 = \mathbf{ye}_1 = 1 \ \vee\ \mathbf{xe}_1 = \mathbf{ye}_1 = -1 \\
\Leftrightarrow\ & \mathbf{x} = \mathbf{y} = \mathbf{e}_1 \ \vee\ \mathbf{x} = \mathbf{y} = -\mathbf{e}_1.
\end{aligned}$$

→ All other pure states belonging to S_{12} (if exist) are entangled!

Lemma 2:

$\Pi e_j \quad -e_j$

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If the state $\psi \in S_{12}$ then, $\psi' = (R, 1\!\!1)\psi \in S_{34}$ and
$\psi'' = (1\!\!1, R)\psi \in S_{34}$

$$\begin{aligned}
\psi_1 &= (\mathrm{e}_1, \mathrm{e}_1, T_0 = \mathrm{e}_1\mathrm{e}_1^T) \\
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1 &= P_{12}(\psi, \psi_1) + P_{12}(\psi, \psi_2) \\
&= P_{12}((R, 1\!\!1)\psi, (R, 1\!\!1)\psi_1) + P_{12}((R, 1\!\!1)\psi, (R, 1\!\!1)\psi_2) \\
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d must be 3

(in a theory with entanglement)

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$\boxed{\psi \in S_{12}}$

1. Fliping **the first** and the **i-th** coordinate of a local vector x: $\psi_i = (R_i, \mathbb{1})\psi$

Regular local transformation as **det E = 1**

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$$\begin{aligned} 1 &= P_{12}(\psi, \psi_1) + P_{12}(\psi, \psi_2) \\ &= P_{12}((R, \mathbb{1})\psi, (R, \mathbb{1})\psi_1) + P_{12}((R, \mathbb{1})\psi, (R, \mathbb{1})\psi_2) \\ &= P_{12}(\psi', \psi_3) + P_{12}(\psi', \psi_4). \end{aligned}$$

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$\psi_{jkl} = (R_{jkl}, \mathbb{1})\psi$

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d = 3

After transformation 1 the state is in subspace 34:

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& \quad -||\mathbf{T}_1^{(x)}||^2 + ||\mathbf{T}_2^{(x)}||^2 + \cdots - ||\mathbf{T}_i^{(x)}||^2 + \cdots + ||\mathbf{T}_d^{(x)}||^2 \\
&= 1 - 2x_1^2 - 2x_i^2 - 2||\mathbf{T}_1^{(x)}||^2 - 2||\mathbf{T}_i^{(x)}||^2 + ||\mathbf{x}||^2 + ||\mathbf{y}||^2 + ||T||^2 \\
&= 2(2 - x_1^2 - x_i^2 - ||\mathbf{T}_1^{(x)}||^2 - ||\mathbf{T}_i^{(x)}||^2).
\end{aligned}$$

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After transformation 2 the state is in subspace 34:

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No entanglement,
only product states!

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Normalization $||x||^2 + ||y||^2 + ||\mathrm{T}||^2 + ||\eta||^2 = 3$

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→ Maximal violation of Bell's inequality possible!
Bell (CHSH) inequality violated **iff** $(T_{xx})^2 + (T_{zz})^2 \geq 1$ (=2 for max)

Horodecki et al. (1995)

Hardy's 5 Axioms

Axiom 1. (Probabilities) *Relative frequencies (measured by taking the proportion of times a particular outcome is observed) tend to the same value (which we call the probability) for any case where a given measurement is performed on a ensemble of n systems prepared by some given preparation in the limit as n becomes infinite.*

Axiom 2. (Simplicity) *K is determined by a function of N (i.e. $K = K(N)$) where $N = 1, 2, \ldots$ and where, for each given N, K takes the minimum value consistent with the axioms.*

Axiom 3. (Subspaces) *A system whose state is constrained to belong to an M dimensional subspace (i.e. have support on only M of a set of N possible distinguishable states) behaves like a system of dimension M.*

Axiom 4. (Composite system) *A composite system consisting of subsystems A and B satisfies $N = N_A N_B$ and $K = K_A K_B$.*

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with an information-theoretical flavour

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→ **Classical or Quantum Theory.**

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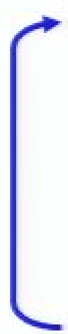
Classical Limit



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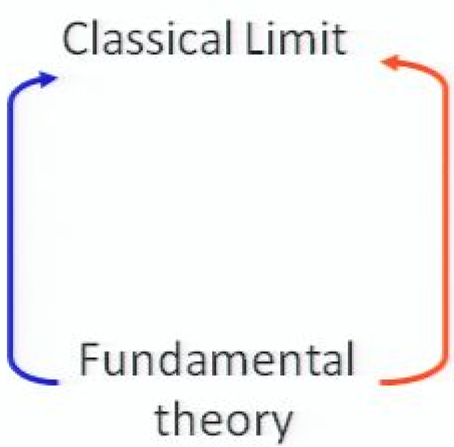

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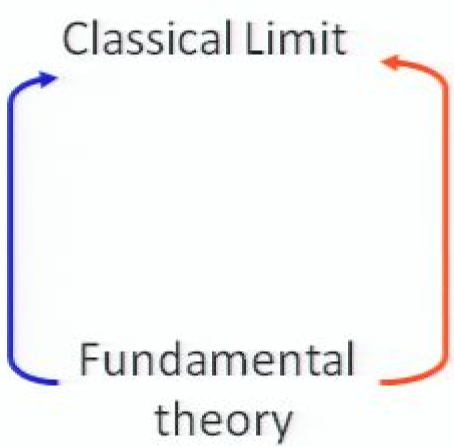

J. Kofler, C. B.,
Classical world arising out of quantum physics under the restriction of coarse-grained measurements
Phys. Rev. Lett. 99, 180403 (2007)

C. B. arXiv:0905.3363,
In the Kreisgang between classical and quantum physics (UniMolti modi della filosofia 2008/2)

Thank you for your attention!