Title: The quantum adiabatic theorem and eigenpath traversal

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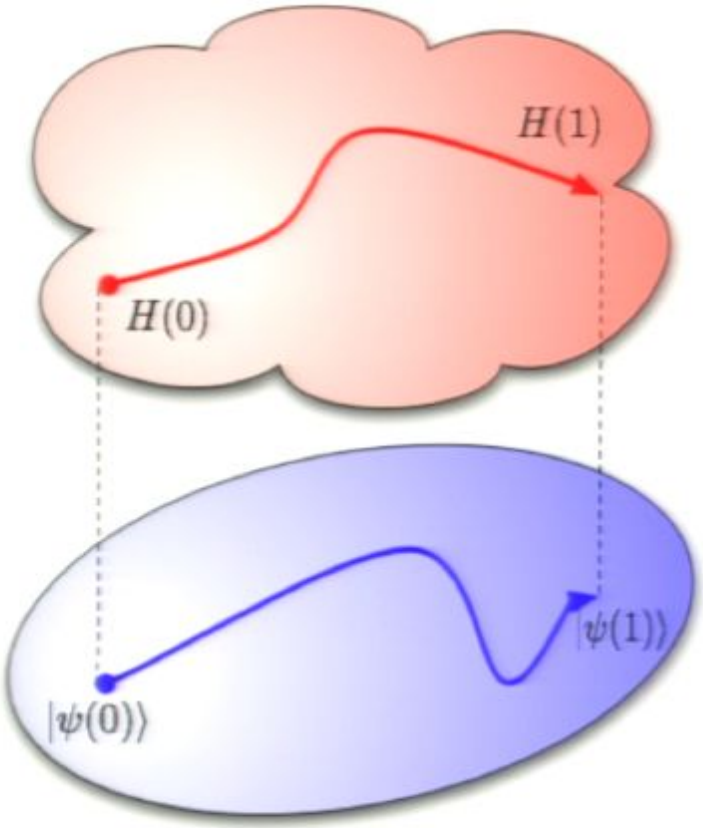
Abstract: We review situations under which standard quantum adiabatic conditions fail. We reformulate the problem of adiabatic evolution as the problem of Hamiltonian eigenpath traversal, and give cost bounds in terms of the length of the eigenpath and the minimum energy gap of the Hamiltonians. We introduce a randomized evolution method that can be used to traverse the eigenpath and show that a standard adiabatic condition is recovered. We then describe more efficient methods for the same task and show that their implementation complexity is close to optimal.

Eigenpath traversal

We are given

1. Initial (non-degenerate) eigenstate $|\psi(0)\rangle$
2. Hamiltonian path $\{H(s)\}$ (and $\{H(s) \otimes |1\rangle\langle 1|\}$), with $s \in [0, 1]$.

Starting with $|\psi(0)\rangle$, the path $\{H(s)\}$ defines a path $\{|\psi(s)\rangle\}$ with minimum gap Δ. The state $|\psi(1)\rangle$ is the objective.



Find $s(t)$, $t \in [0, \tau]$, and, maybe, $H_A(t)$ such that

$$i\partial_t|\phi(t)\rangle = (H(s(t)) + H_A(t))|\phi(t)\rangle \ ,$$

with $|\phi(0)\rangle = |\psi(0)\rangle \otimes |0\rangle$ and $|\phi(\tau)\rangle \approx |\psi(1)\rangle \otimes |0\rangle$.

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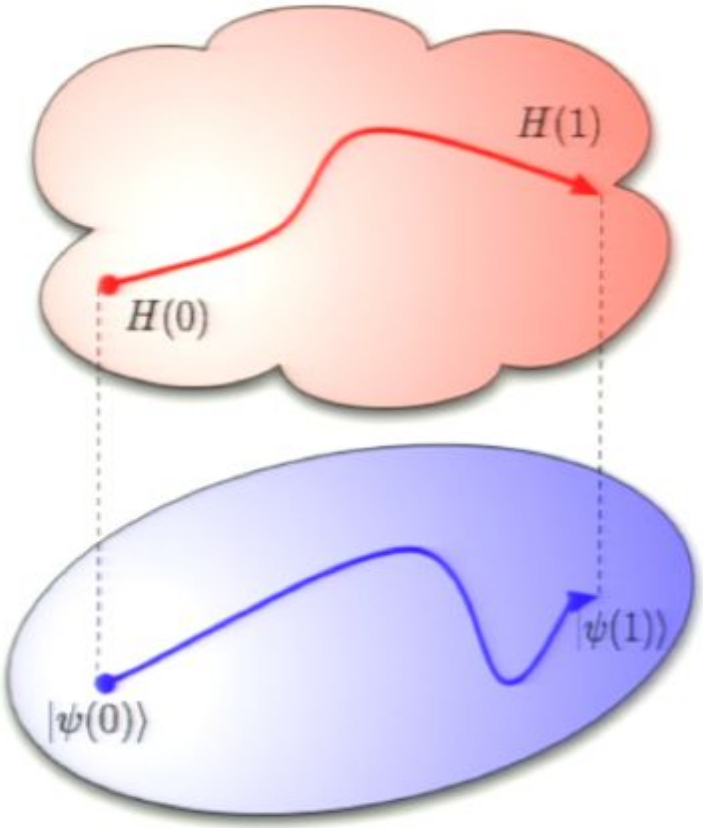

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Adiabatic conditions

Adiabatic condition: "folk" version

Evolve with $H(t/\tau)$, $t \in [0, \tau]$,

$$\frac{\|\partial_s H\|}{\Delta^2} \ll \tau \quad \text{or} \quad \frac{\|\partial_s H\|^2}{\Delta^3} \ll \tau \,,$$

where Δ is the gap. Born and Fock, 1928. Bohm. Messiah.

Adiabatic condition: rigorous version

Evolve with $H(t/\tau)$, $t \in [0, \tau]$, with

$$\frac{\|\partial_s H\|^2}{\Delta^3} + \frac{\|\partial_s^2 H\|}{\Delta^2} \ll \tau \,.$$

Note that $s \in [0, 1]$. Linear interpolation gives, in general, the $1/\Delta^3$ folk version. Jansen, Ruskai and Seiler, 2007

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The folk version is wrong: not sufficient

Rabi oscillations

Resonant Hamiltonian path

$$H(s) = -\frac{\Delta}{2}\left(\cos(\theta)\sigma_z + \sin(\theta)\left(\cos(\nu s)\sigma_x - \sin(\nu s)\sigma_y\right)\right),$$

with $\nu = \Delta\cos(\theta)$. In the interaction picture

$$H_I(t) = -\frac{\Delta\sin(\theta)}{2}\sigma_x\,.$$

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The folk version of the adiabatic condition is not sufficient

$$\frac{\|\partial_s H\|}{\Delta^2} \ll \tau \quad \Rightarrow \quad \frac{\sin(2\theta)}{4} \ll \tau\,.$$

Schwinger, 1937. Schiff, 1949. Marzlin and Sanders, 2004. Tong et. al, 2005.

Powers of the folk version give $\theta \ll (2/\pi)^k$: second derivative is needed.

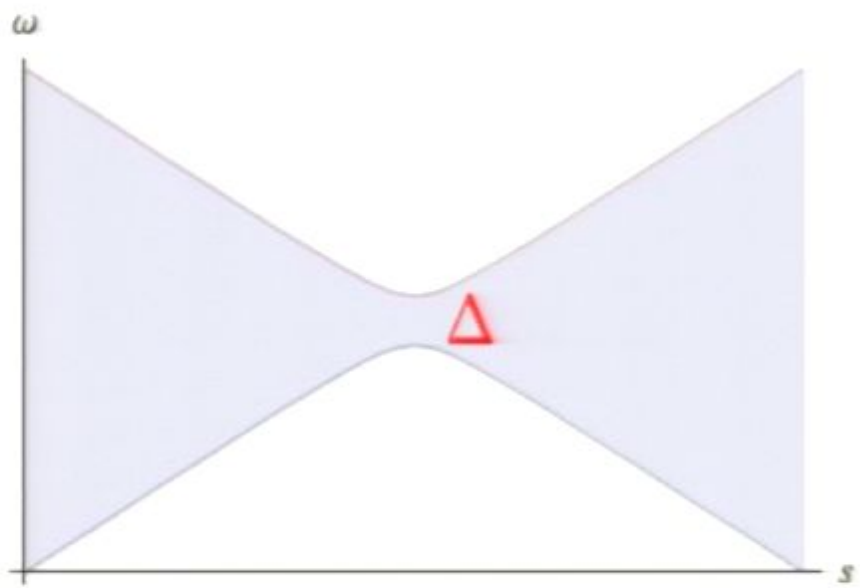
Standard adiabatic conditions are not necessary

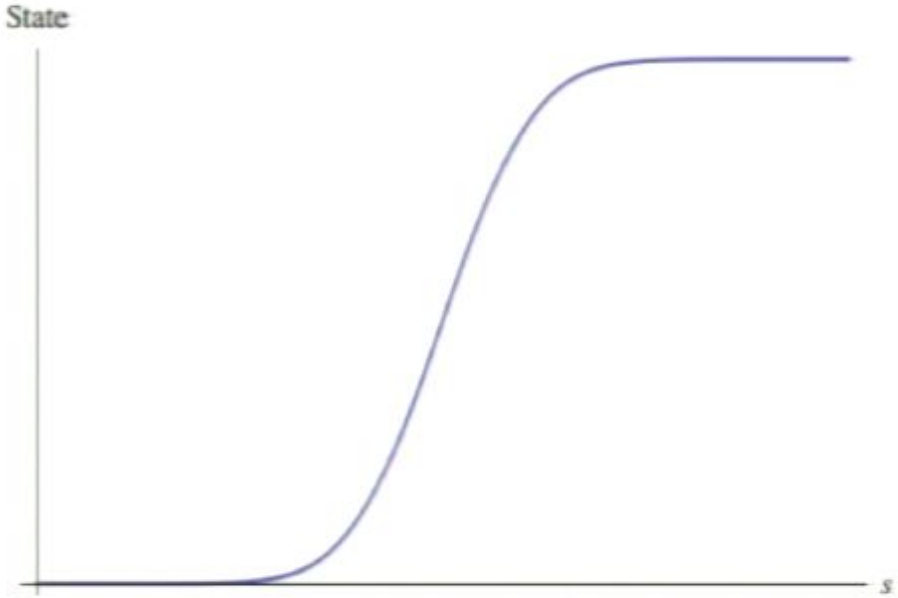
Grover

The gap in Grover is $1/\sqrt{N}$, so $\|\partial_s H\|/\Delta^2 = N$: no quantum speed-up!

Solution: go with variable speed. No linear interpolation!

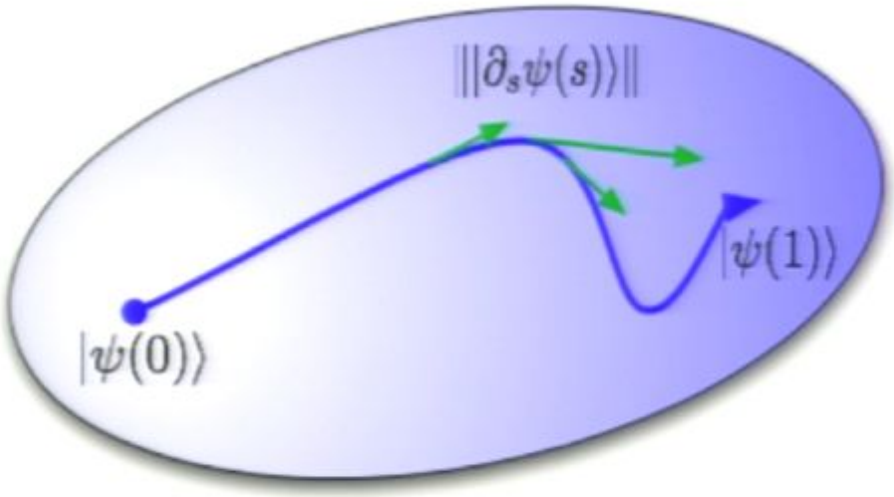
Roland and Cerf, 2002





Path length

The correct adiabatic condition



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The path length is

$$L = \int_0^1 \| |\partial_s \psi(s)\rangle \| ds \,,$$

if the phases are chosen such that $\langle \partial_s \psi(s) | \psi(s) \rangle = 0$.

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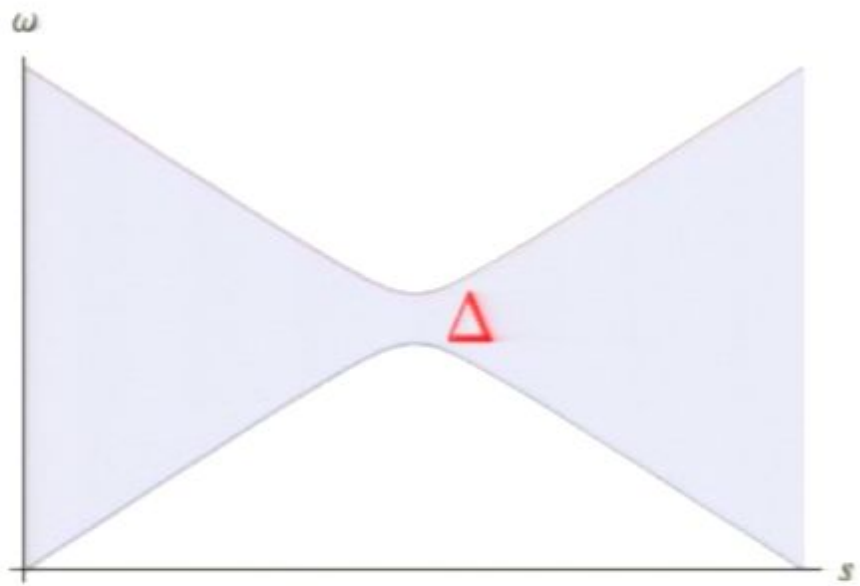

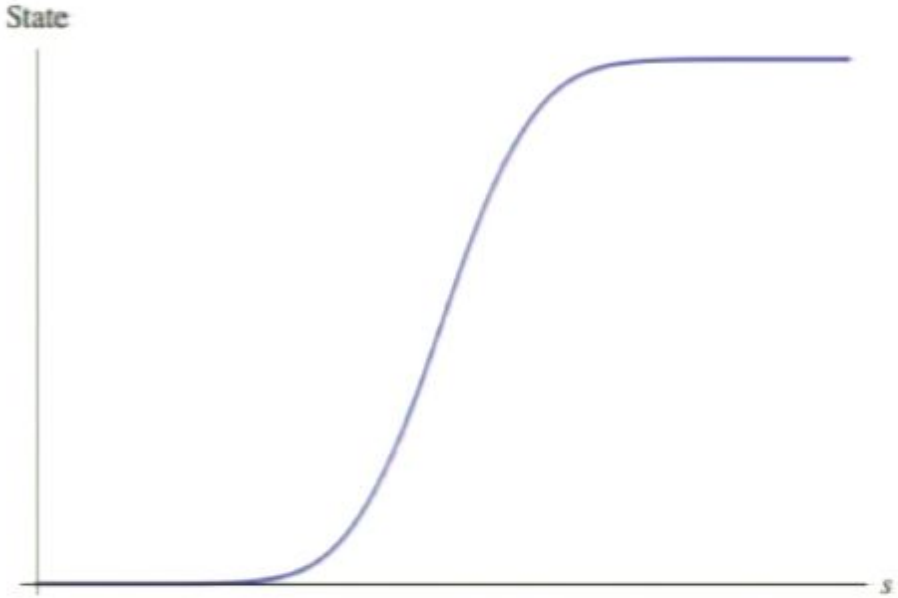

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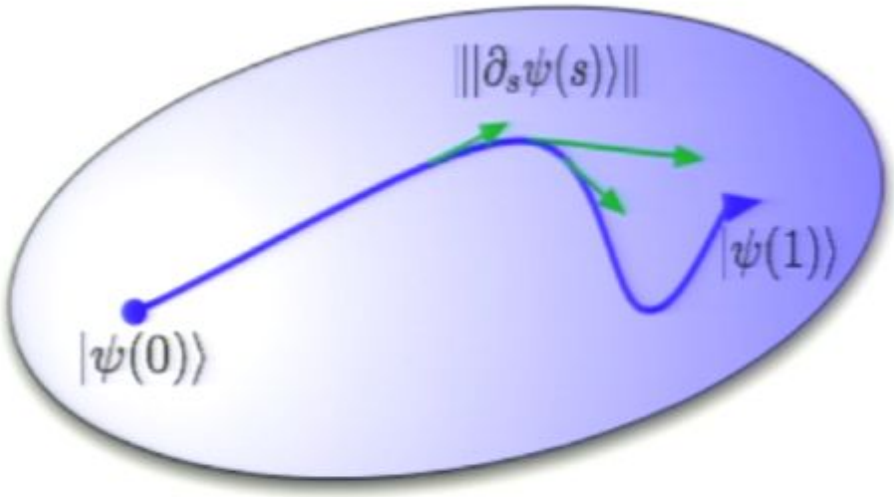

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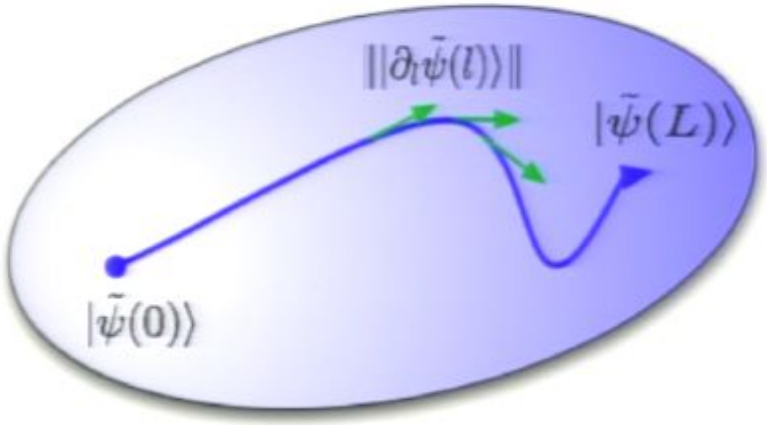
Uniform parametrization

If we know $\| |\partial_s \psi(s)\rangle \|$, we can use the *uniform parametrization*

$$|\tilde{\psi}(l)\rangle = |\psi(s(l))\rangle$$

with $l \in [0, L]$ such that (approximately)

$$\| |\partial_l \tilde{\psi}(l)\rangle \| \approx 1 \, .$$



Worst case parametrization

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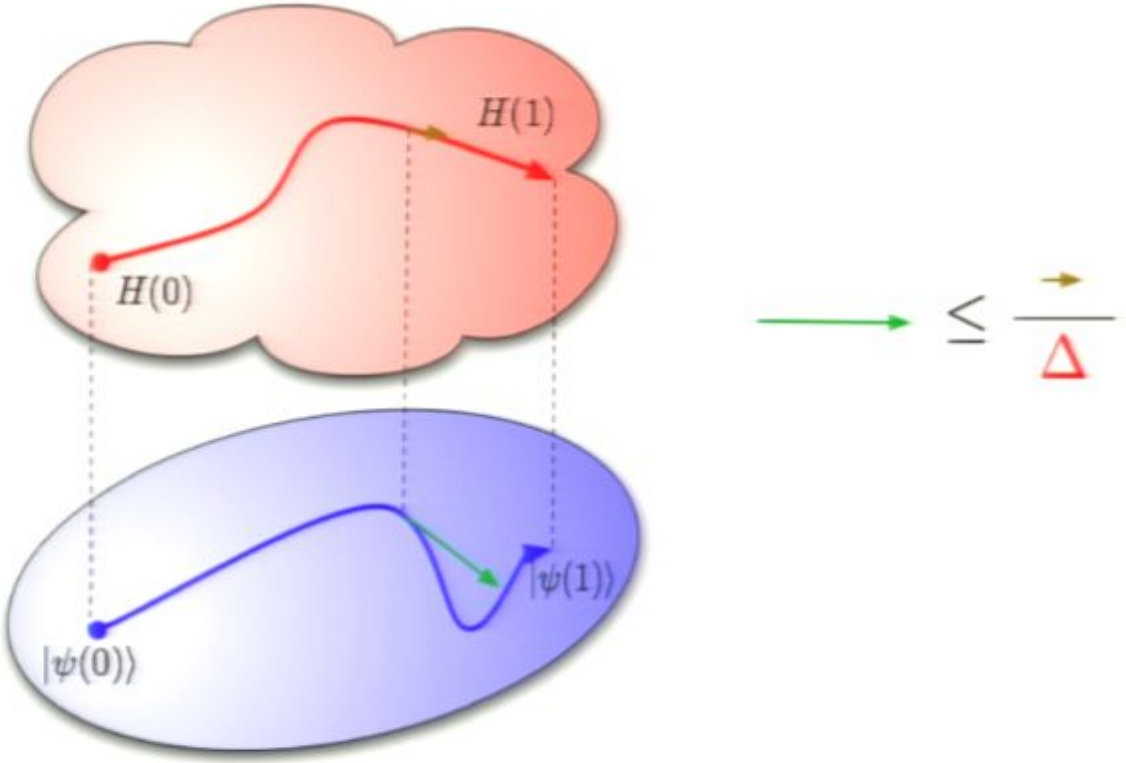
From $\|\|\partial_s\psi(s)\rangle\| \leq \|\partial_s H\|/\Delta$ choose and

$$s(l) = \frac{\Delta}{\|\partial_s H\|} l \quad \text{and} \quad L \ll \frac{\|\partial_s H\|}{\Delta}.$$



Summary of results

1. Randomizing adiabatic paths we obtain an expected cost (almost) like

$$\frac{L^2}{\Delta} \quad \text{or} \quad \frac{\|\partial_s H(s)\|^2}{\Delta^3}.$$

2. Using log copies of the initial state, we obtain a cost (almost) like

$$\frac{L}{\Delta} \quad \text{or} \quad \frac{\|\partial_s H(s)\|}{\Delta^2}.$$

3. If the energies are known, we obtain a cost (almost) like L/Δ. (There are other possible conditions).
4. We show that L/Δ is optimal.
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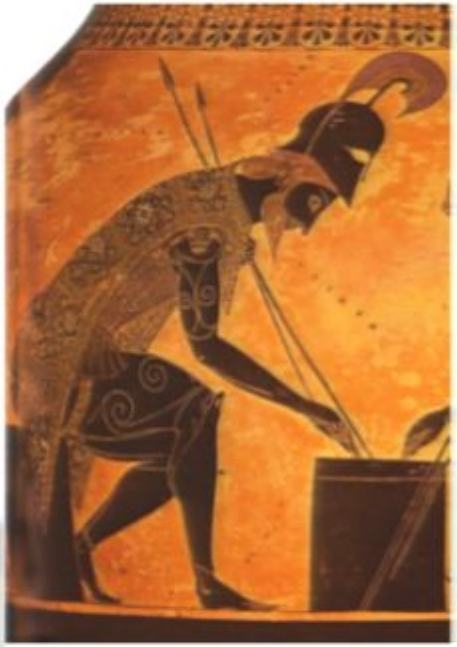
Path traversal with the Zeno effect

For a δ displacement, to first order

$$|\tilde{\psi}(l+\delta)\rangle \approx |\tilde{\psi}(l)\rangle + \delta|\tilde{\psi}(l)_{\perp}\rangle \ .$$



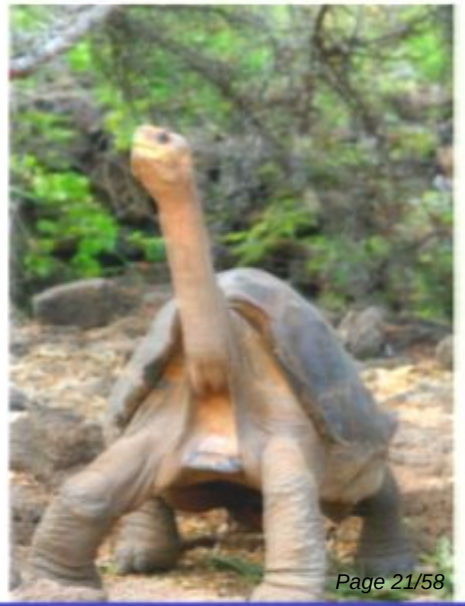
Overlap bound

$$|\langle\tilde{\psi}(l)|\tilde{\psi}(l+\delta)\rangle|^2 \geq 1-\delta^2 \ .$$

We project on $|\tilde{\psi}(l+\delta)\rangle$ at intervals δ, so as $\delta \to 0$ we can traverse the path with small total error.

Childs, et al. 2002. Aharonov, and Ta-Shma, 2003. Somma,

Boixo, Barnum, Knill, 2008.



Zeno effect, formally

Zeno effect

Define the projective-measurement operations onto $|\tilde{\psi}(l)\rangle$ as

$$M_l(\rho) = P_l \rho P_l + \mathcal{E}((\mathbb{1} - P_l)\rho(\mathbb{1} - P_l)) \,,$$

with $P_l = |\tilde{\psi}(l)\rangle\langle\tilde{\psi}(l)|$ and $\mathcal{E}$ arbitrary quantum operations. Then the state $|\tilde{\psi}(L)\rangle$ can be prepared from $|\tilde{\psi}(0)\rangle$ with fidelity $p > 0$ by $q = \lceil L^2/(1-p) \rceil$ intermediate projective-measurement operations.

Proof

$$\mathrm{tr}\left[P_{l_q}(M_{l_q} \circ \cdots \circ M_{l_1}(\rho))\right] \geq |P_{l_q} \cdots P_{l_1} \tilde{\psi}(0)|^2$$
$$= \Pi_{j=1}^{q} |\tilde{\psi}(l_j)\tilde{\psi}(l_{j-1})|^2$$
$$\geq (1 - L^2)^q \geq 1 - L^2 q \geq 1 - (1-p) = p \,.$$

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Projective-measurements = phase decoherence

- Projective-measurement for the Zeno effect

$$M_l(\rho) = P_l \rho P_l + \mathcal{E}((\mathbb{1} - P_l)\rho(\mathbb{1} - P_l))$$

is phase decoherence.

- Random evolution with the instantaneous Hamiltonian $\tilde{H}(l)$ is also phase decoherence.
- Small errors: check the characteristic function of the random distribution, $\Phi(\Delta) \approx 0$.

$$\int e^{-iHt} \tilde{\psi}(l) \cdots \tilde{\psi}_j(l) e^{-iHt} d\mu(t) = \int e^{i\omega_j t} d\mu(t) |\tilde{\psi}(l) \cdots \tilde{\psi}_j(l)|$$
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Optimal dephasing

Average cost is $\mathcal{O}(1/\Delta)$.

$$\text{cost}(T) = \langle |T| \rangle \geq \frac{1 - |\Phi(\omega)|}{|\omega|}$$

$$\Rightarrow \quad \langle |T| \rangle \geq \frac{1-\epsilon}{\Delta} \quad \text{for } |\Phi(\omega)| = \epsilon \,.$$

This is just phase cancellation.

$$1 - |\Phi(\omega)| \leq |1 - \Phi(\omega)| \leq \int |1 - e^{j\omega t}| d\mu(t)$$

$$\leq \int |\omega t| d\mu(t) = \langle |T| \rangle |\omega| \,,$$

Optimal dephasing

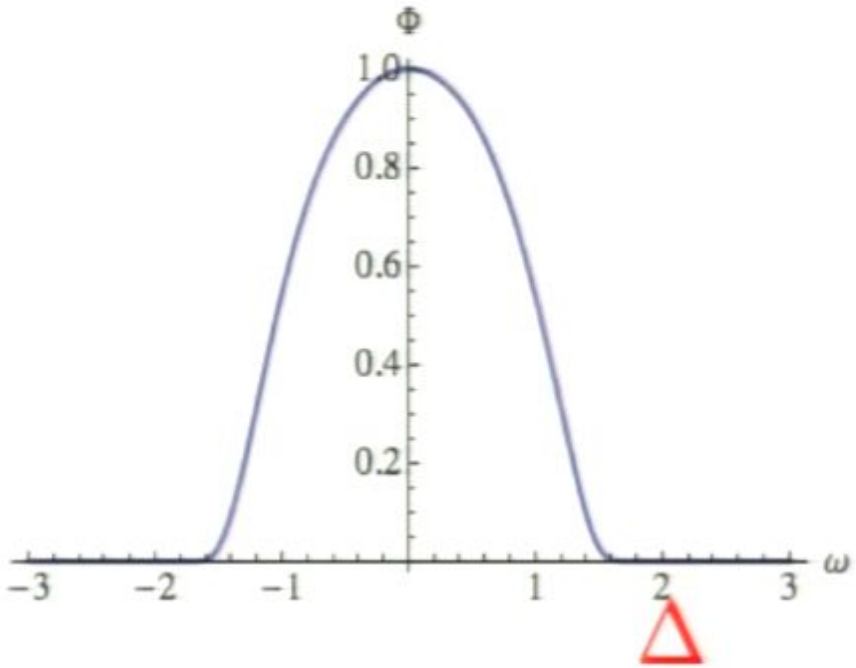
Zero error

If we can evolve with $-H$, then we can achieve:

- The optimal cost $\langle |T| \rangle = \Theta(1/\Delta)$.
- Zero error $\Phi(\omega) = 0$ for $|\omega| \geq \Delta$.
- And bounded moments of all order.

Φ is a rescaled Schwartz function:

- Of compact support.
- Smooth.

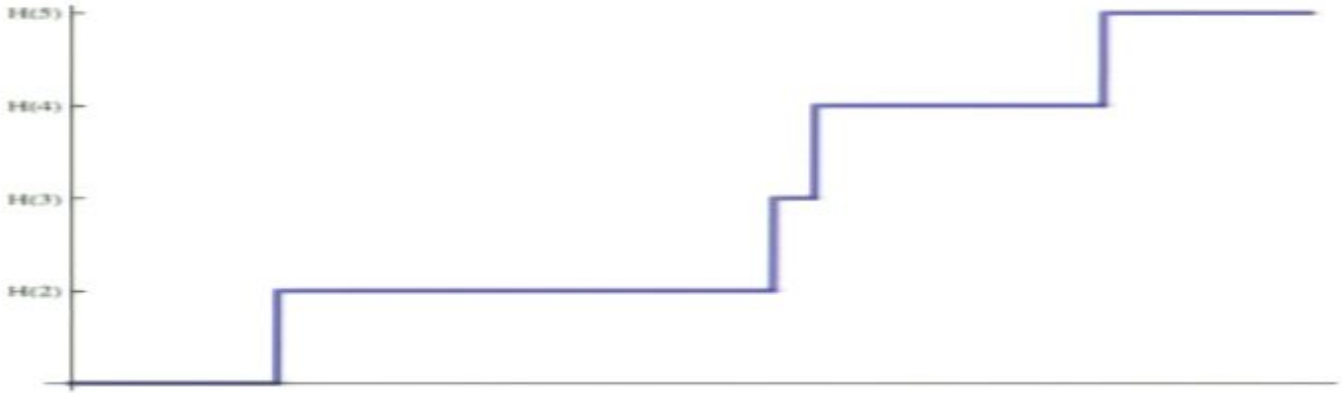


There is an unavoidable logarithmic overhead if $-H$ is not available.

Randomized adiabatic evolution

Use random time distributions T with cost $\langle |T| \rangle \approx 1/\Delta$.

1. Initialize to $|\tilde{\psi}(0)\rangle$.
2. For each $j \in [1, \ldots, 2L^2/\epsilon]$ evolve with the instantaneous Hamiltonian for a random time T.



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Boixo, Knill, Somma, 2009

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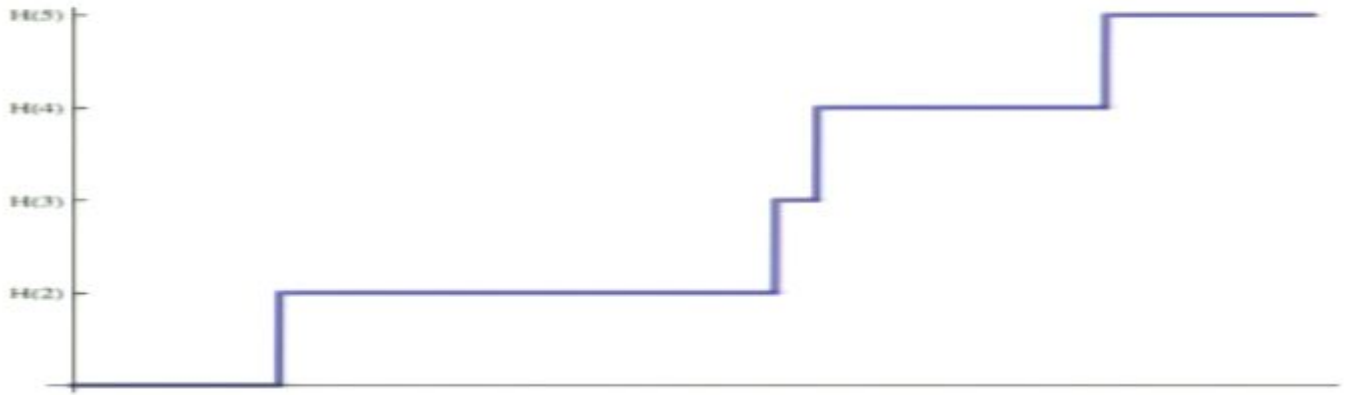

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Faster eigenpath traversal

Robust projections

1. The error in each step of the randomized evolution comes from the Zeno effect: the cost is linear in the error.
2. This gives the scaling of the randomized eigenpath traversal:

$$\text{step } \delta = \frac{1}{\frac{1}{\Delta}} \quad \Rightarrow \quad \text{cost} = \frac{L}{\delta}\frac{1}{\Delta} = \frac{L}{\Delta^2}.$$

3. Similar cost to the rigorous adiabatic theorem, but without a second derivative. Can we do better?
4. The scaling can be improved by using robust projections at each step: the cost can be made logarithmic in the error.

Abeyesinghe and Wocjan 2006

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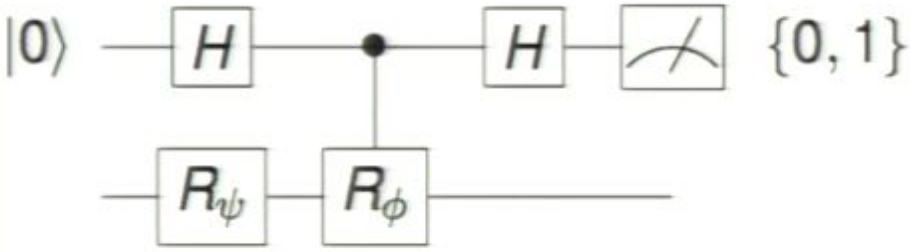
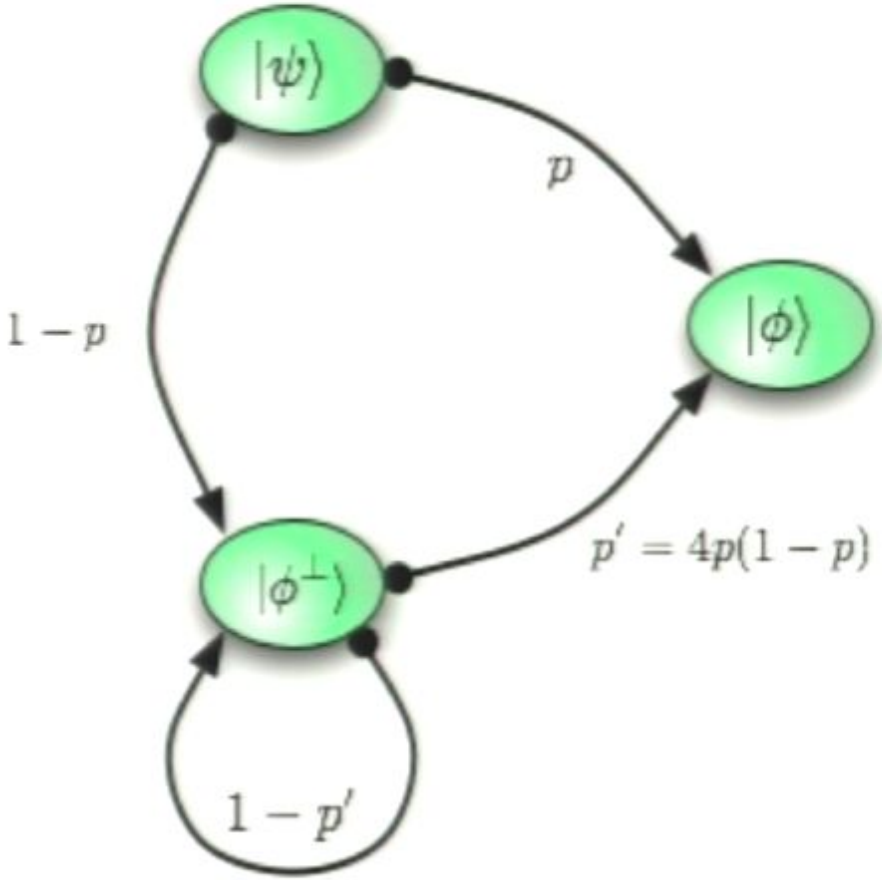
Reflect and project

Assume $|\langle\psi|\phi\rangle|^2 = p > 0$. A reflection R_ϕ is $\mathbb{1} - 2|\phi\rangle\langle\phi|$.



$$\mathcal{E}_1 = \frac{\mathbb{1} - R_\phi}{2} R_\psi = P_\phi R_\psi$$

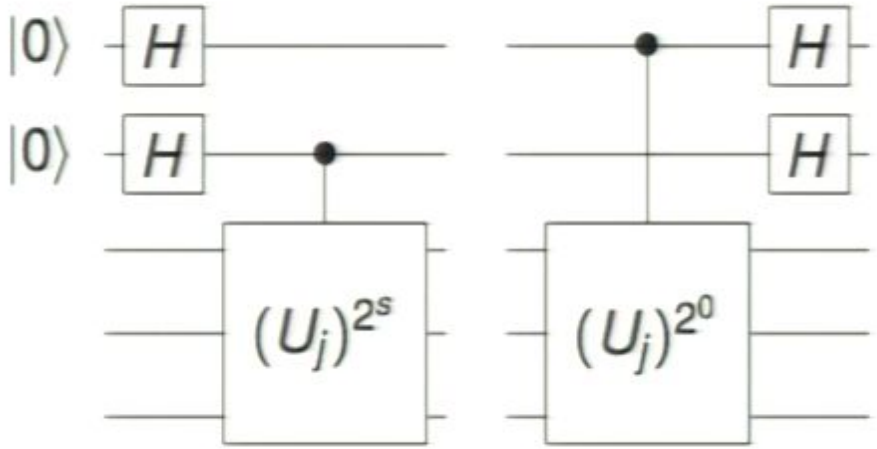
$$\mathcal{E}_0 = \frac{\mathbb{1} + R_\phi}{2} R_\psi = (\mathbb{1} - P_\phi) R_\psi$$



Tulsi, Grover, Patel, 2006. Boixo, Somma, Knill, in preparation.

Approximated reflections

To reflect a non-degenerate eigenstate $U_j|j\rangle = |j\rangle$: do a (high precision) simplified phase estimation, and reflect the ancillae.



Cost per step

Repeat $\log(1/\epsilon)$ phase estimations with precision $1/\Delta$. Reflect (coherently) the ancillae. The cost is

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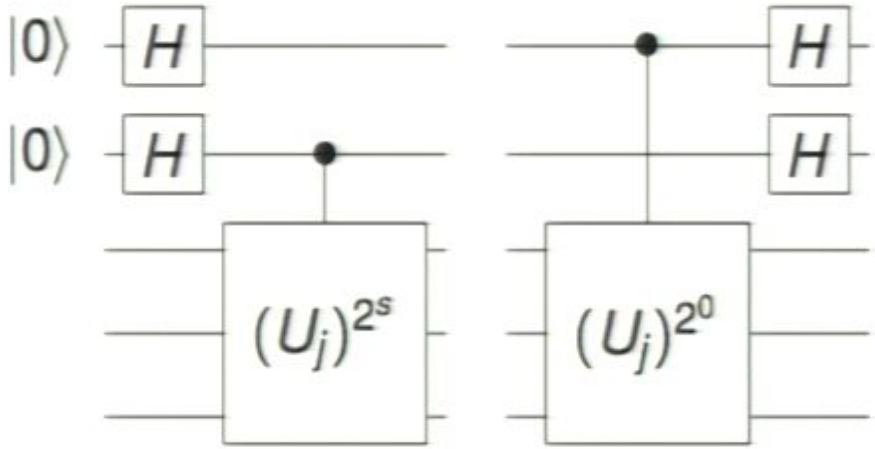

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Error control

We use quantum operations with many approximated reflections. We need to control the errors. But we have classical-quantum states

$$\mu = \sum_{\sigma} p_{\sigma} \mu_{\sigma}^{s} \otimes |\sigma\rangle\langle\sigma| \ ,$$

and operations

$$\mathcal{E}(\mu_{\sigma}^{s} \otimes |\sigma\rangle\langle\sigma|) = \sum_{\sigma'} A_{\sigma\sigma'} \mu_{\sigma}^{s} A_{\sigma\sigma'}^{\dagger} \otimes |\sigma'\rangle\langle\sigma'| \ .$$

Lemma

If $\|A_{\sigma\sigma'} - \hat{A}_{\sigma\sigma'}\| \leq \delta$ and e D is maximum number of possible σ' that can be reached from a given σ then

$$\|(\mathcal{E} - \hat{\mathcal{E}})(\mu)\|_{\mathrm{tr}} \leq 2D\delta \ ,$$

Eigenpath traversal with known energies

Boixo, Somma, Knill, in preparation

1. Break the path into steps so $|\langle j|j+1\rangle| \geq p$. This gives a scaling with L or $\|\partial_s H(s)\|/\Delta$.
   - At each step, reflect and project. This is done a expected constant number of times!
   - Approximate operations with phase estimation (cost is $1/\Delta$ plus logarithmic overhead)
   - Error control: introduce a cutoff for the total number of operations (use high concentration from martingale theory), and Lemma.

Cost with known energies

$$\frac{L}{\Delta} \quad \text{or} \quad \frac{\|\partial_s H(s)\|}{\Delta^2}$$

and logarithmic corrections.

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General case: unknown energies

Measuring energies coherently

1. Start with log copies of $|0\rangle$.
2. Do high precision phase estimation with all the copies and $H(\delta)$.
3. Do a coherent majority vote in the ancillae.
4. Undo the phase estimation: if the overlap is good, the undo works.

Cost with unknown energies

$$\frac{L}{\Delta} \quad \text{or} \quad \frac{|\partial_s H(s)|}{\Delta^2}$$

and logarithmic corrections.

Dynamical speed

How do we fix the problem with the rate of change? This is a problem not only for Grover, but general quantum phase transitions. Consider the $1D$ Ising model

$$\sqrt{n} \approx L \leq \frac{\|\partial_s H(s)\|}{\Delta} \approx \frac{n}{1/n} = n^2$$

- Implement non-destructive preparation circuits. If a "jump" fails, we remain in the initial state.
- After a successful jump, try to jump twice further. After a failed jump, try twice closer.
- Non-destructive jumps can be implemented if the energy is known. Also if we know that there is no big overlap with other excited states.

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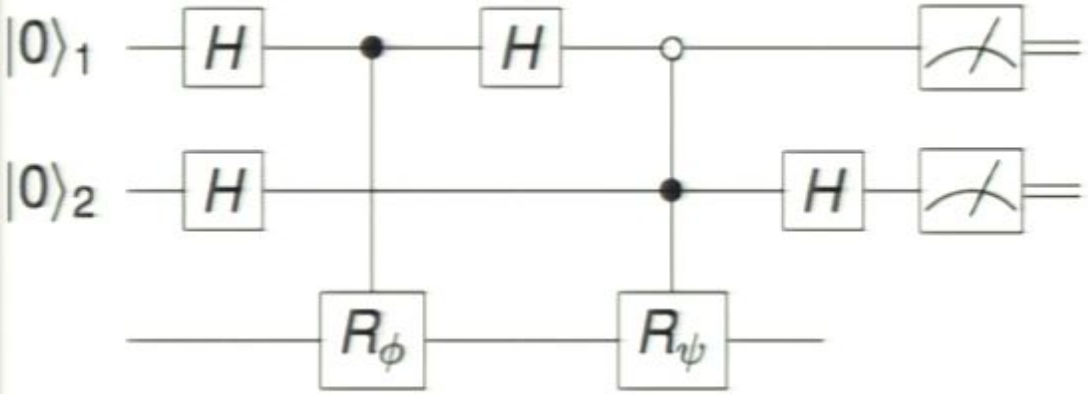
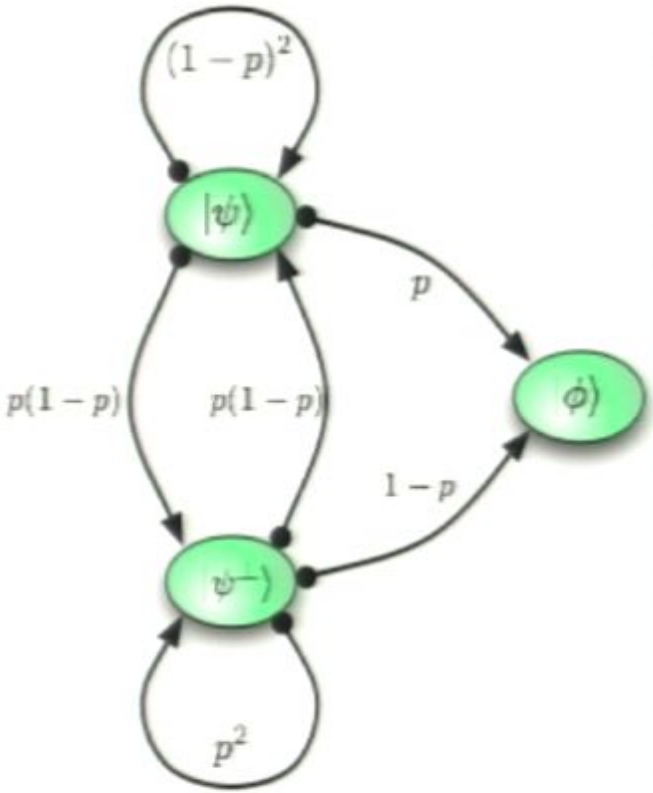
Non-destructive jump





Optimal cost: degenerate case

- This is proved by constant-gap piece-wise adiabatic evolutions. Zanardi and Rasetti, 1999. Kitaev, 2003. Bacon and Flammia, 2009. Oreshkov, 2009.
- For any quantum circuit $U = U_L \cdots U_1$ define the sequence of Hamiltonians

$$H_l = \mathbb{1} \otimes P_l - \mathbb{1} \otimes P_{l-1} - U_l \otimes |l\rangle\langle l-1| - U_l^\dagger \otimes |l-1\rangle\langle l| \quad (1)$$

- The adiabatic path

$$H_{l-1,l}(s) = (1-s)H_l - sH_l \quad (2)$$

effectively applies U_{l-1}, and has bounded gap.
- A concatenation of the paths $H_{l-1,l}$ gives an eigenpath of length $O(L)$ and gap $O(1)$, which applies the quantum circuit.
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Optimal cost: non-degenerate case

- For a "secret" word $x = x_1, \ldots x_n$ define $|x(j)^+\rangle = |x_1, \ldots x_j, +, \ldots, +\rangle$. The Hamiltonians are

$$H_{x,j} = \Delta |x(j)^+, c_j\rangle\langle x(j)^+, c_j| \ ,$$

where $|c_j\rangle = (|j\rangle + |j-1\rangle)/\sqrt{2}$ are "clock" states.

- Concatenation of $H_{x,j}$ is a path of length n and gap Δ.
- $H_{x,j}$ is a fractional oracle checking the first j digits of x.
- One full oracle can be implemented with 2 calls to the ordered search oracle, which checks if x is less than a

$$O_x\, a\rangle = (-1)^{x<a}\, a\rangle \ .$$

- This already (almost) implies the bound L/Δ using discretization of continuous oracles. Cleve et al. 2009.
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Conclusions and open questions

- The fundamental cost of (algorithmic) eigenpath traversal (in the oracle setting) is

$$\frac{L}{\Delta}$$

- Phase decoherence in the instantaneous eigenbasis is sufficient for path traversal (AQC). This randomized adiabatic path scales as L^2/Δ or $\|\partial_s H(s)\|^2/\Delta^3$. Randomization can be faster and more general.
- Often the path length is $L \ll 1/\Delta$. That's the case for continuous Grover and similar QPTs paths.
- Can we obtain the cost L/Δ with a monotone (adiabatic) path?

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