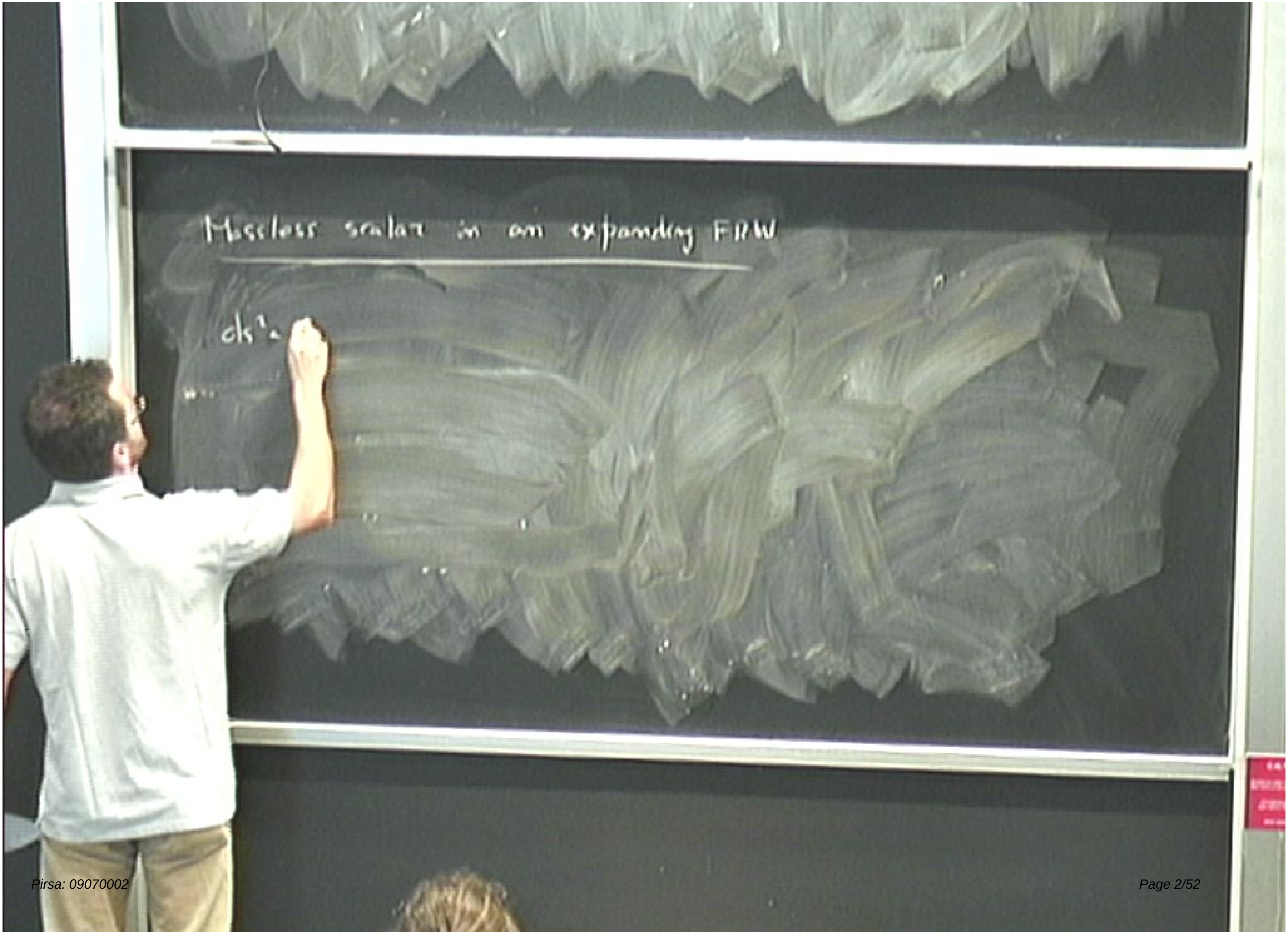
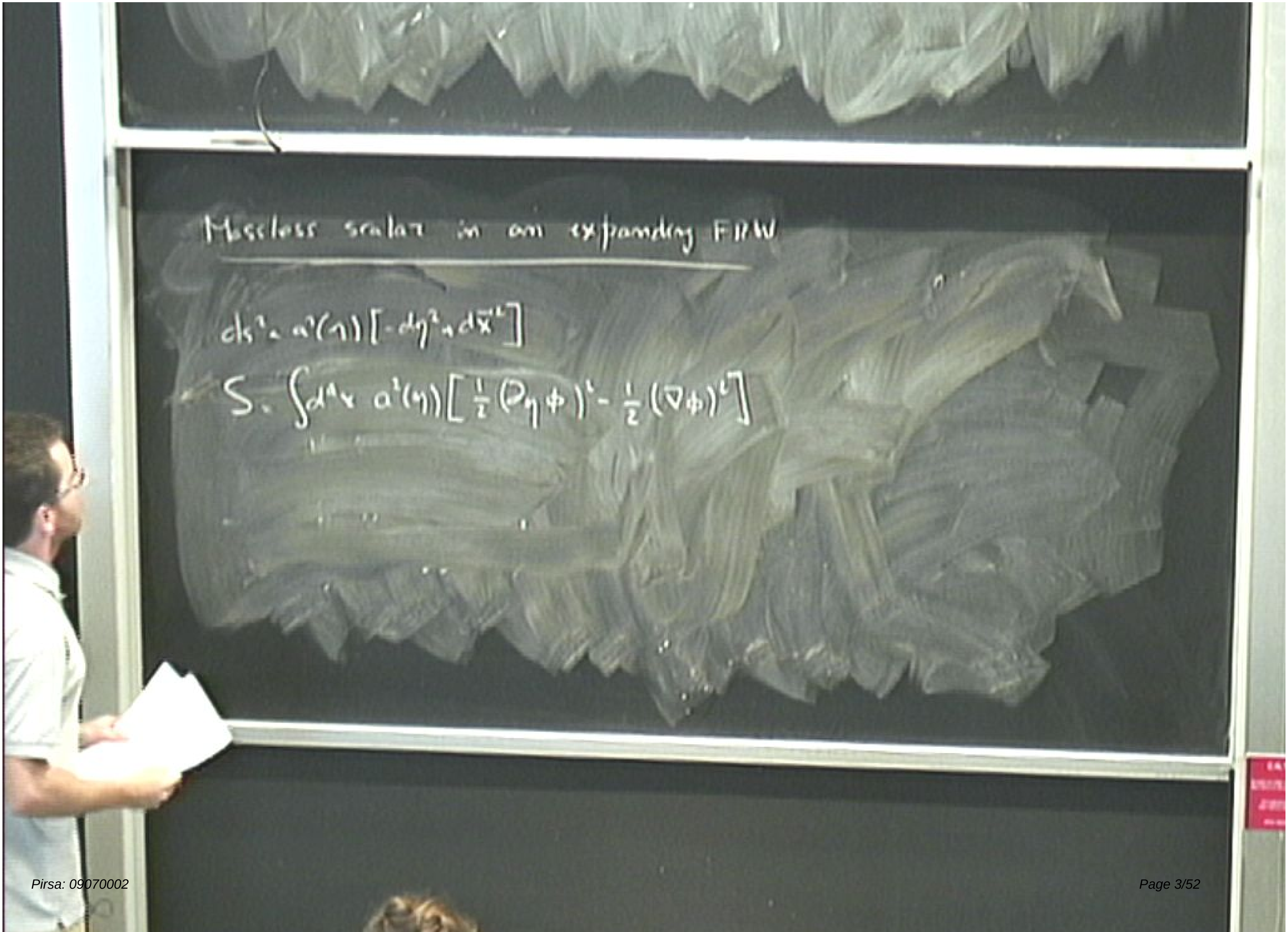
Title: Inflationary Cosmology - Lecture 4

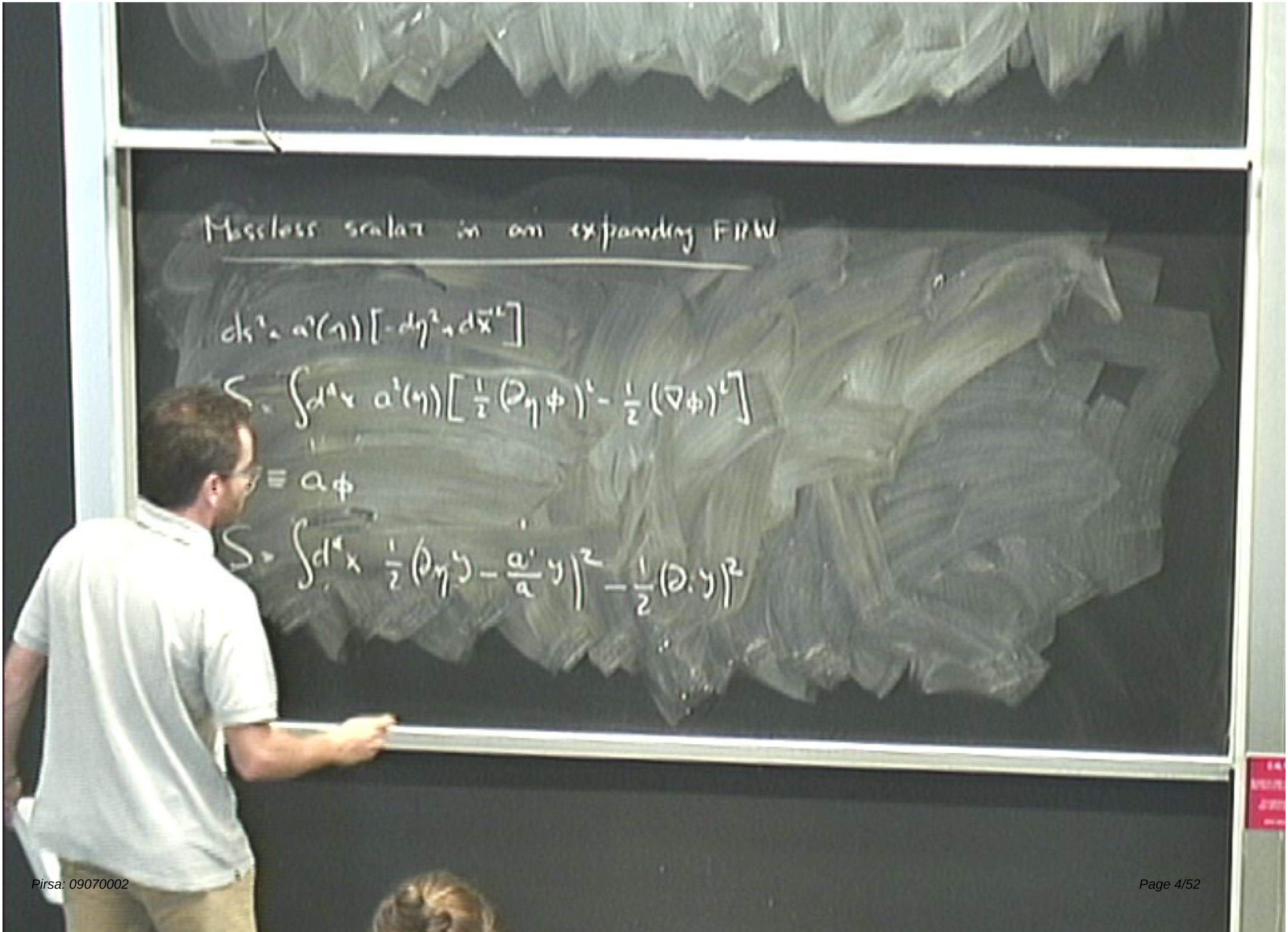
Date: Jul 01, 2009 10:00 AM

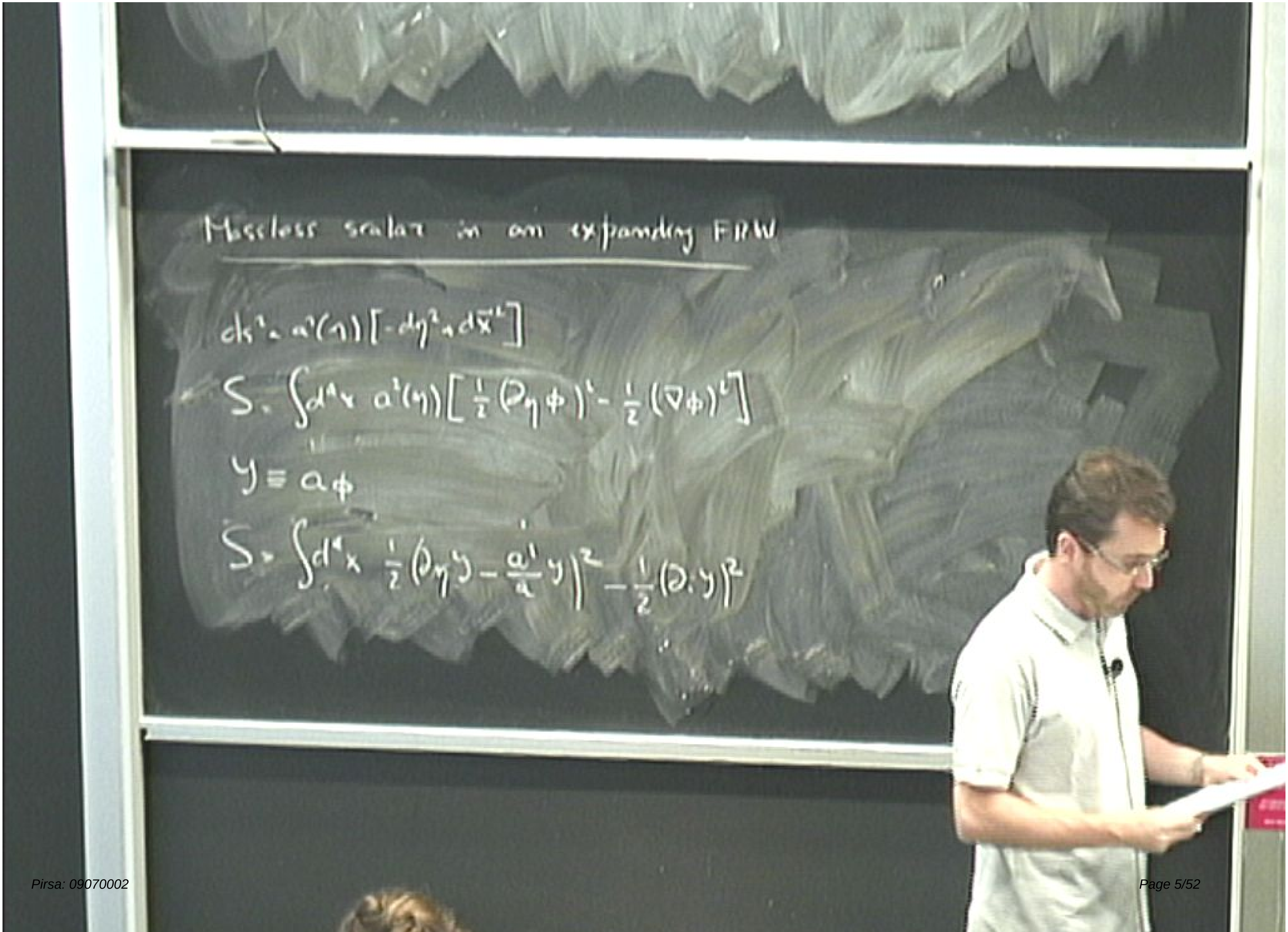
URL: http://pirsa.org/09070002

Abstract:


Massless scalar in an expanding FRW
ds² =


Massless scalar in an expanding FRW
$ds^2 = a^2(\eta)[-d\eta^2 + d\vec{x}^2]$
$S = \int d^4x \, a^2(\eta)\left[\frac{1}{2}(\partial_\eta \phi)^2 - \frac{1}{2}(\nabla\phi)^2\right]$


Massless scalar in an expanding FRW
$ds^2 = a^2(\eta)\left[-d\eta^2 + d\vec{x}^2\right]$
$S = \int d^4x \; a^2(\eta)\left[\frac{1}{2}(\partial_\eta \phi)^2 - \frac{1}{2}(\nabla\phi)^2\right]$
$y \equiv a\phi$
$S = \int d^4x \; \frac{1}{2}\left(\partial_\eta y - \frac{a'}{a} y\right)^2 - \frac{1}{2}(\partial_i y)^2$


Massless scalar in an expanding FRW
$ds^2 = a^2(\eta)[-d\eta^2 + d\vec{x}^2]$
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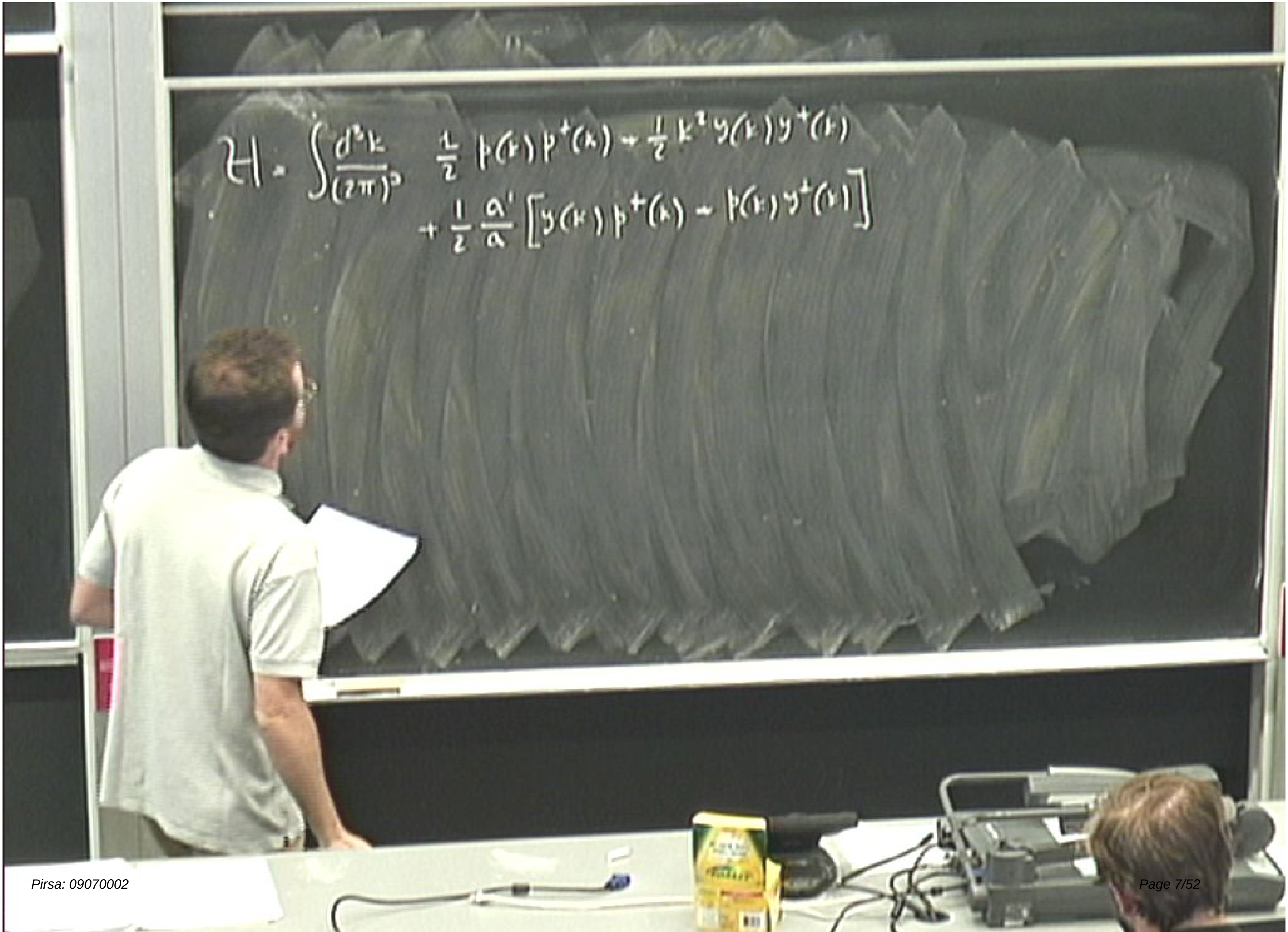
Massless scalar in an expanding FRW

$$ds^2 = a^2(\eta)\left[-d\eta^2 + d\vec{x}^2\right]$$

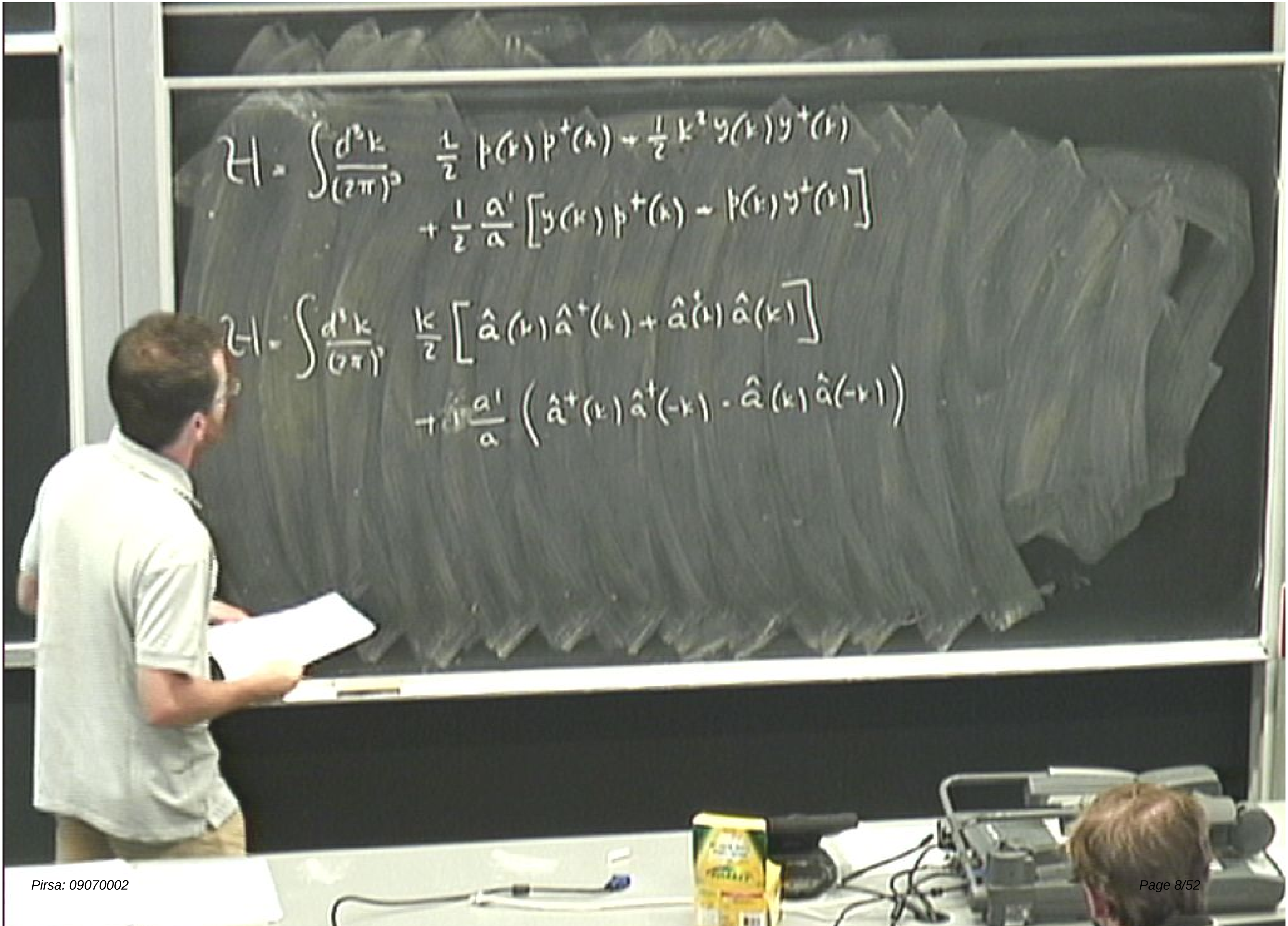
$$S = \int d^4x \; a^2(\eta)\left[\tfrac{1}{2}(\partial_\eta \phi)^2 - \tfrac{1}{2}(\nabla\phi)^2\right]$$

$$y \equiv a\phi$$

$$S = \int d^4x \; \tfrac{1}{2}\left(\partial_\eta y - \frac{a'}{a} y\right)^2 - \tfrac{1}{2}(\partial_i y)^2$$



$\mathcal{H} = \int \frac{d^3k}{(2\pi)^3} \; \frac{1}{2} p(k) p^+(k) + \frac{1}{2} k^2 y(k) y^+(k)$
$+ \frac{1}{2} \frac{a'}{a} \left[y(k) p^+(k) + p(k) y^+(k) \right]$


$\mathcal{H} = \int \frac{d^3k}{(2\pi)^3} \; \frac{1}{2} p(k) p^+(k) + \frac{1}{2} k^2 y(k) y^+(k)$
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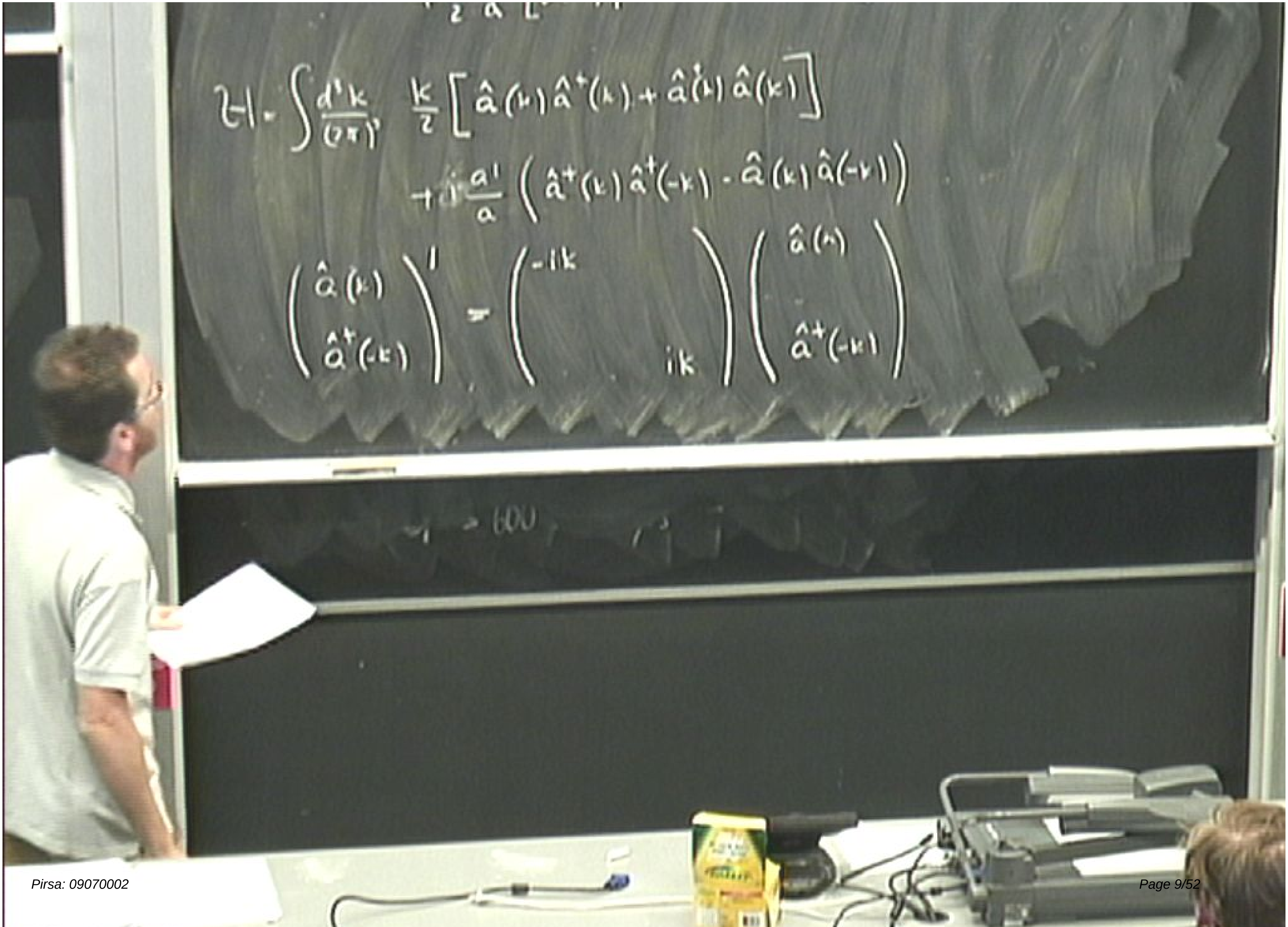
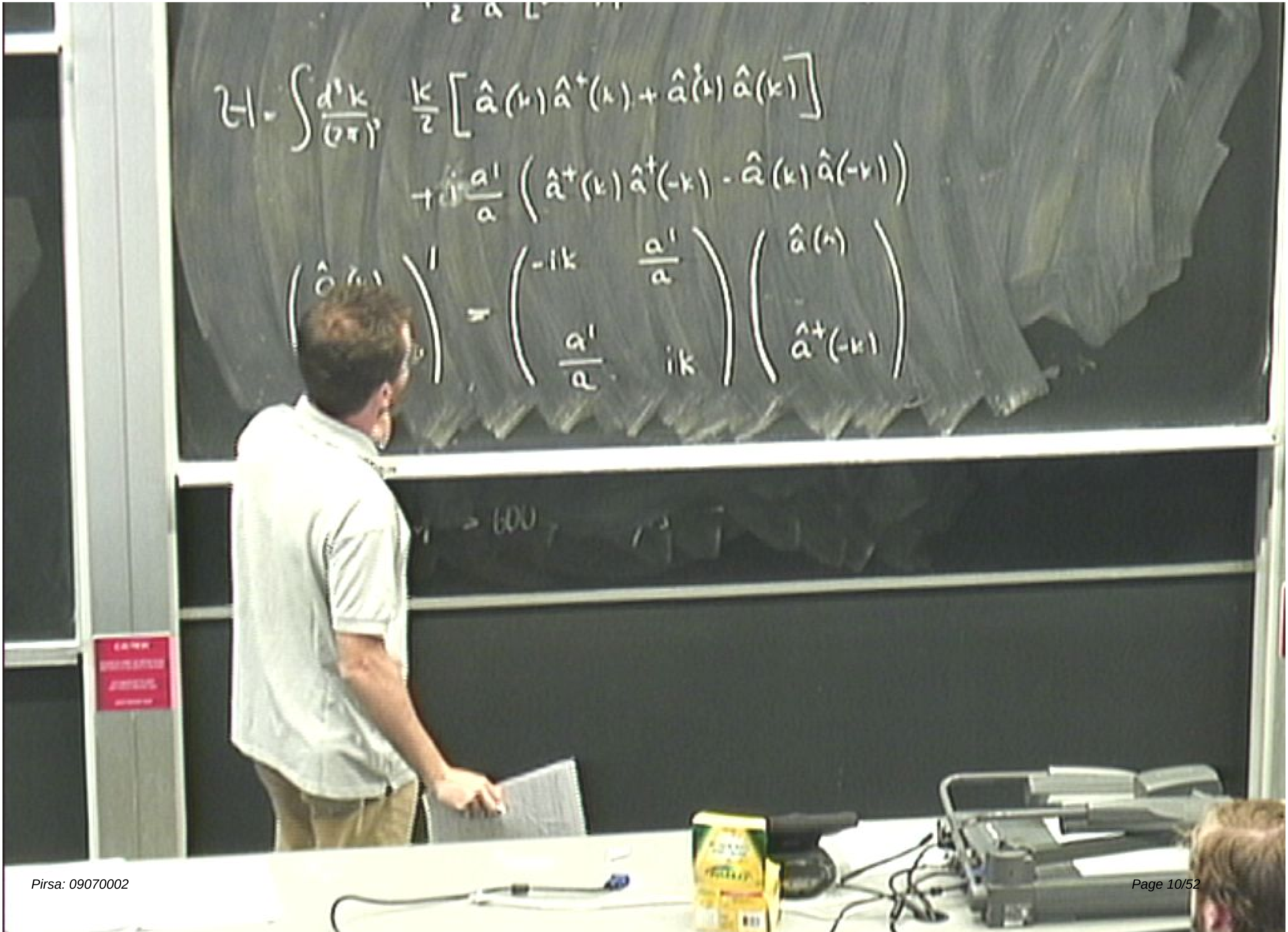
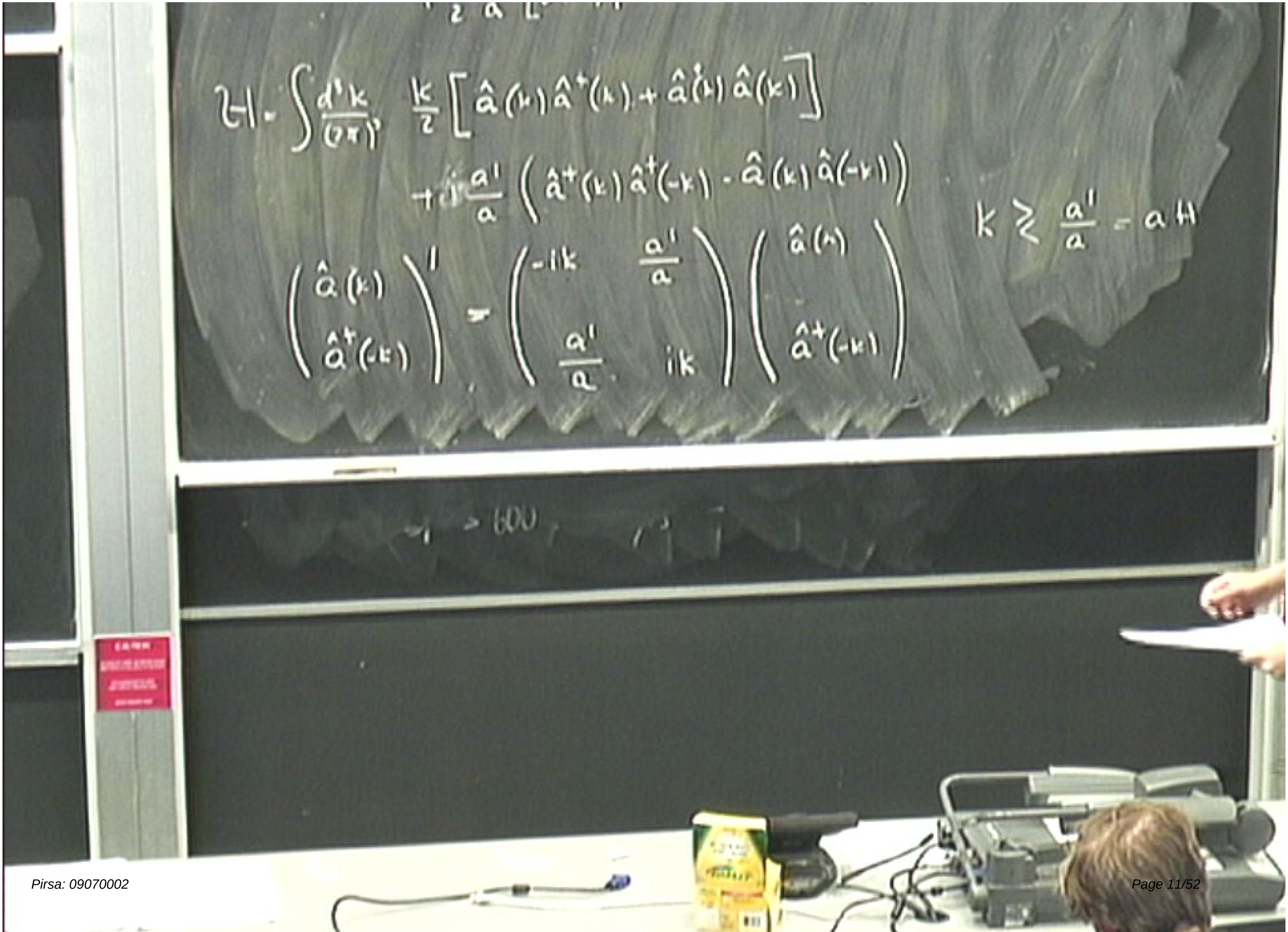
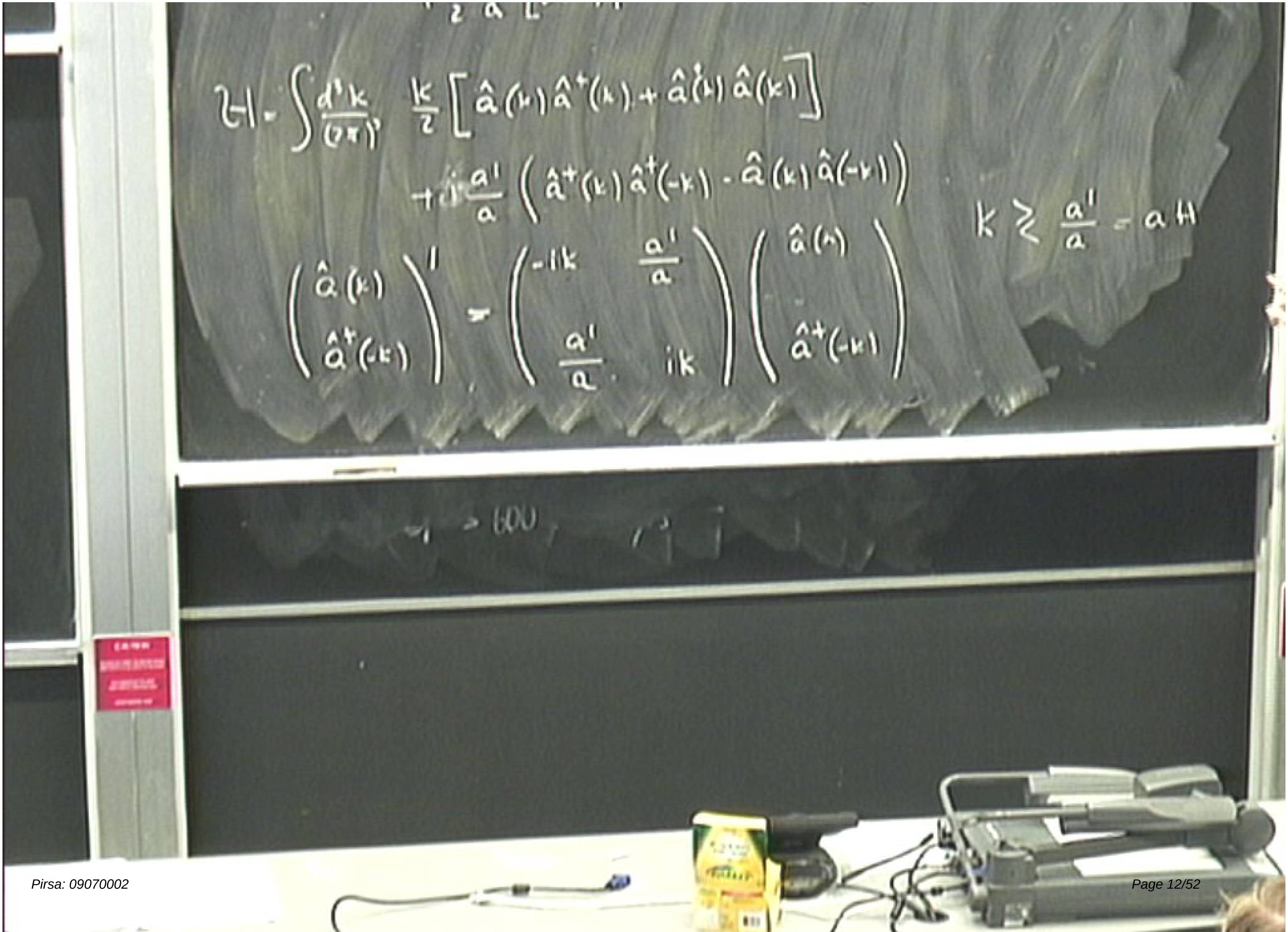
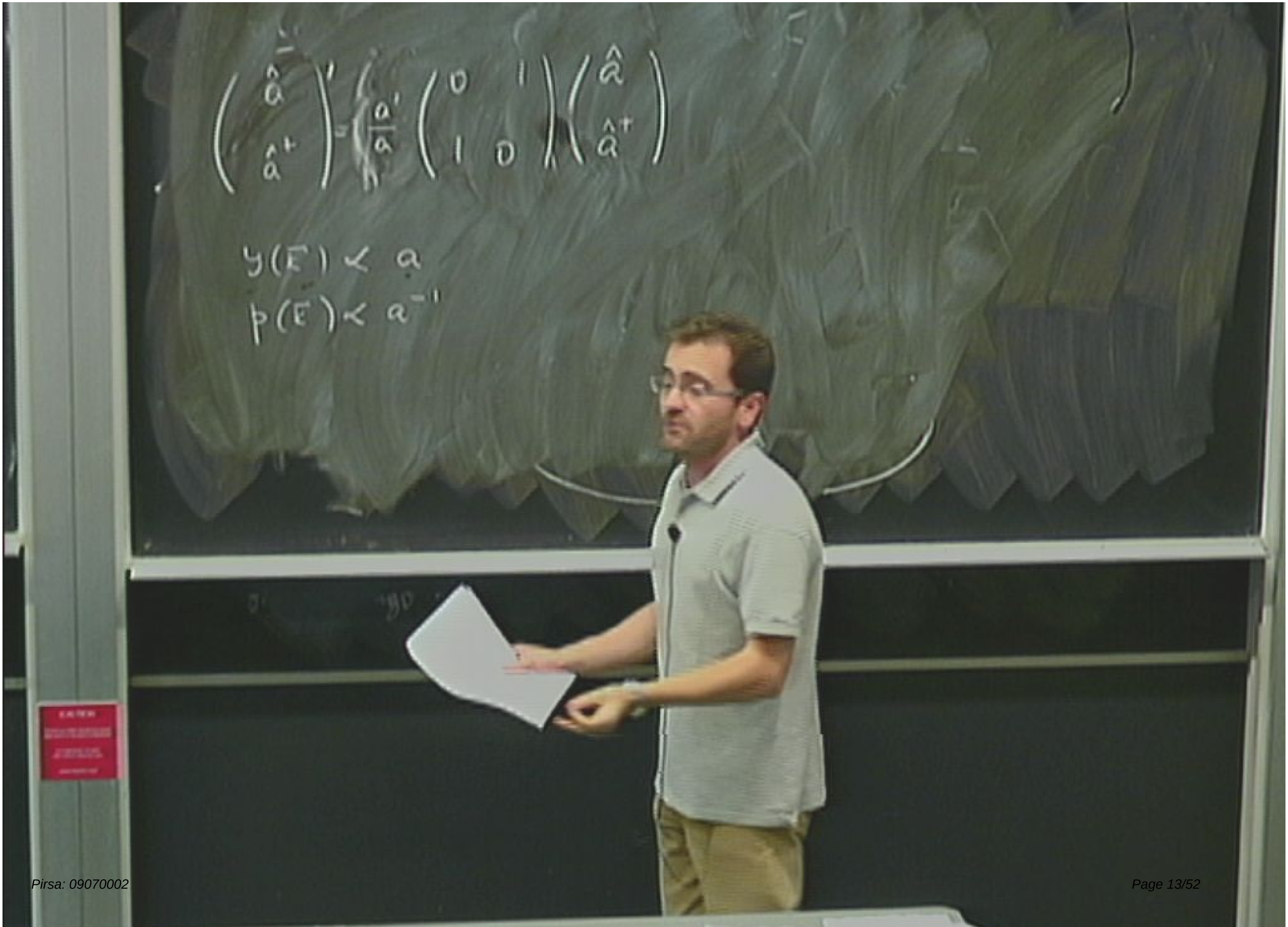
$\mathcal{H} = \int \frac{d^3k}{(2\pi)^3} \; \frac{k}{2} \left[\hat{a}(k) \hat{a}^+(k) + \hat{a}^+(k) \hat{a}(k) \right]$
$+ i \frac{a'}{a} \left(\hat{a}^+(k) \hat{a}^+(-k) - \hat{a}(k) \hat{a}(-k) \right)$


$\mathcal{H} = \int \frac{d^3k}{(2\pi)^3} \frac{k}{2}\left[\hat{a}(k)\hat{a}^+(k) + \hat{a}^+(k)\hat{a}(k)\right]$
$+ i\frac{a'}{a}\left(\hat{a}^+(k)\hat{a}^+(-k) - \hat{a}(k)\hat{a}(-k)\right)$
$\begin{pmatrix} \hat{a}(k) \\ \hat{a}^+(-k) \end{pmatrix}' = \begin{pmatrix} -ik & \\ & ik \end{pmatrix} \begin{pmatrix} \hat{a}(k) \\ \hat{a}^+(-k) \end{pmatrix}$


$\mathcal{H} = \int \frac{d^3k}{(2\pi)^3} \frac{k}{2}\left[\hat{a}(k)\hat{a}^+(k) + \hat{a}^+(k)\hat{a}(k)\right]$
$+ i\frac{a'}{a}\left(\hat{a}^+(k)\hat{a}^+(-k) - \hat{a}(k)\hat{a}(-k)\right)$
$= \begin{pmatrix} -ik & \frac{a'}{a} \\ \frac{a'}{a} & ik \end{pmatrix}\begin{pmatrix} \hat{a}(k) \\ \hat{a}^+(-k) \end{pmatrix}$


$\mathcal{H} = \int \frac{d^3k}{(2\pi)^3} \frac{k}{2}\left[\hat{a}(k)\hat{a}^+(k) + \hat{a}^+(k)\hat{a}(k)\right]$
$+ i\frac{a'}{a}\left(\hat{a}^+(k)\hat{a}^+(-k) - \hat{a}(k)\hat{a}(-k)\right)$
$k \gtrsim \frac{a'}{a} = aH$
$\begin{pmatrix} \hat{a}(k) \\ \hat{a}^+(-k) \end{pmatrix}' = \begin{pmatrix} -ik & \frac{a'}{a} \\ \frac{a'}{a} & ik \end{pmatrix} \begin{pmatrix} \hat{a}(k) \\ \hat{a}^+(-k) \end{pmatrix}$

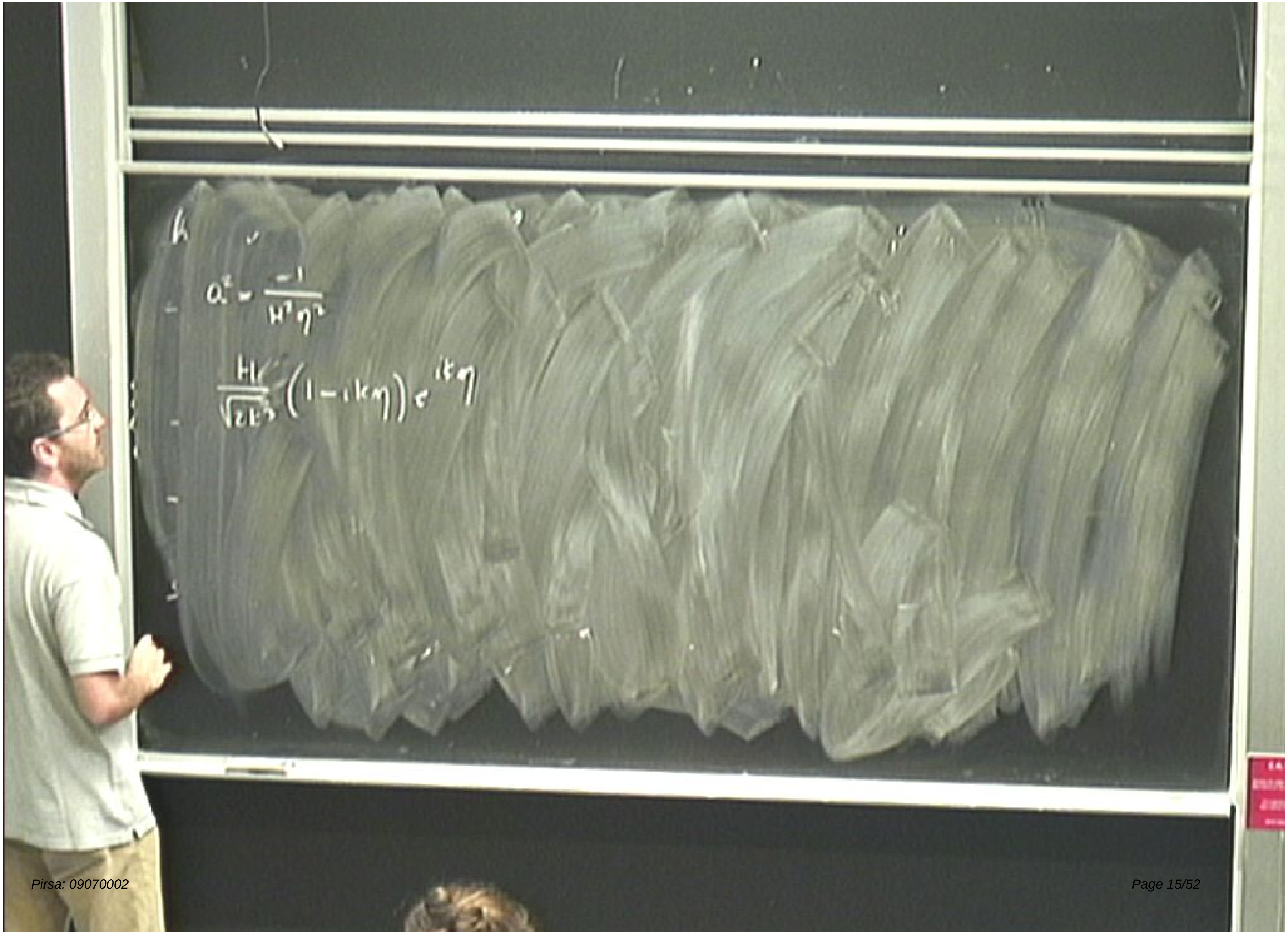

$\mathcal{H} = \int \frac{d^3k}{(2\pi)^3} \frac{k}{2}\left[\hat{a}(k)\hat{a}^+(k) + \hat{a}^+(k)\hat{a}(k)\right] + i\frac{a'}{a}\left(\hat{a}^+(k)\hat{a}^+(-k) - \hat{a}(k)\hat{a}(-k)\right)$
$k \gtrsim \frac{a'}{a} = aH$
$\begin{pmatrix}\hat{a}(k)\\ \hat{a}^+(-k)\end{pmatrix}' = \begin{pmatrix}-ik & \frac{a'}{a}\\ \frac{a'}{a} & ik\end{pmatrix}\begin{pmatrix}\hat{a}(k)\\ \hat{a}^+(-k)\end{pmatrix}$

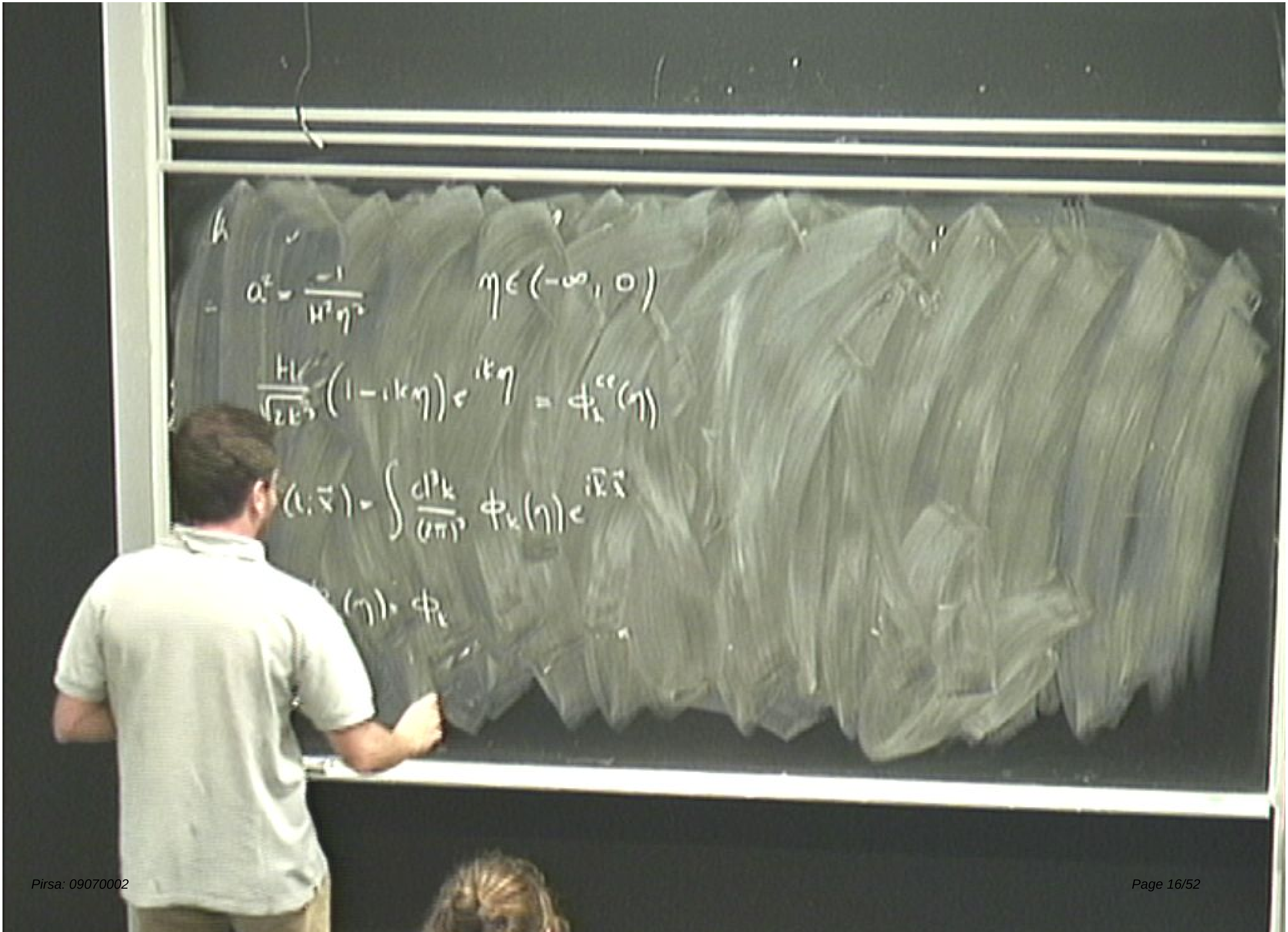

y(E) ∝ a
p(E) ∝ a⁻¹

$$\begin{pmatrix} \hat{a} \\ \hat{a}^{+} \end{pmatrix}' = \frac{a'}{a} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \hat{a} \\ \hat{a}^{+} \end{pmatrix}$$

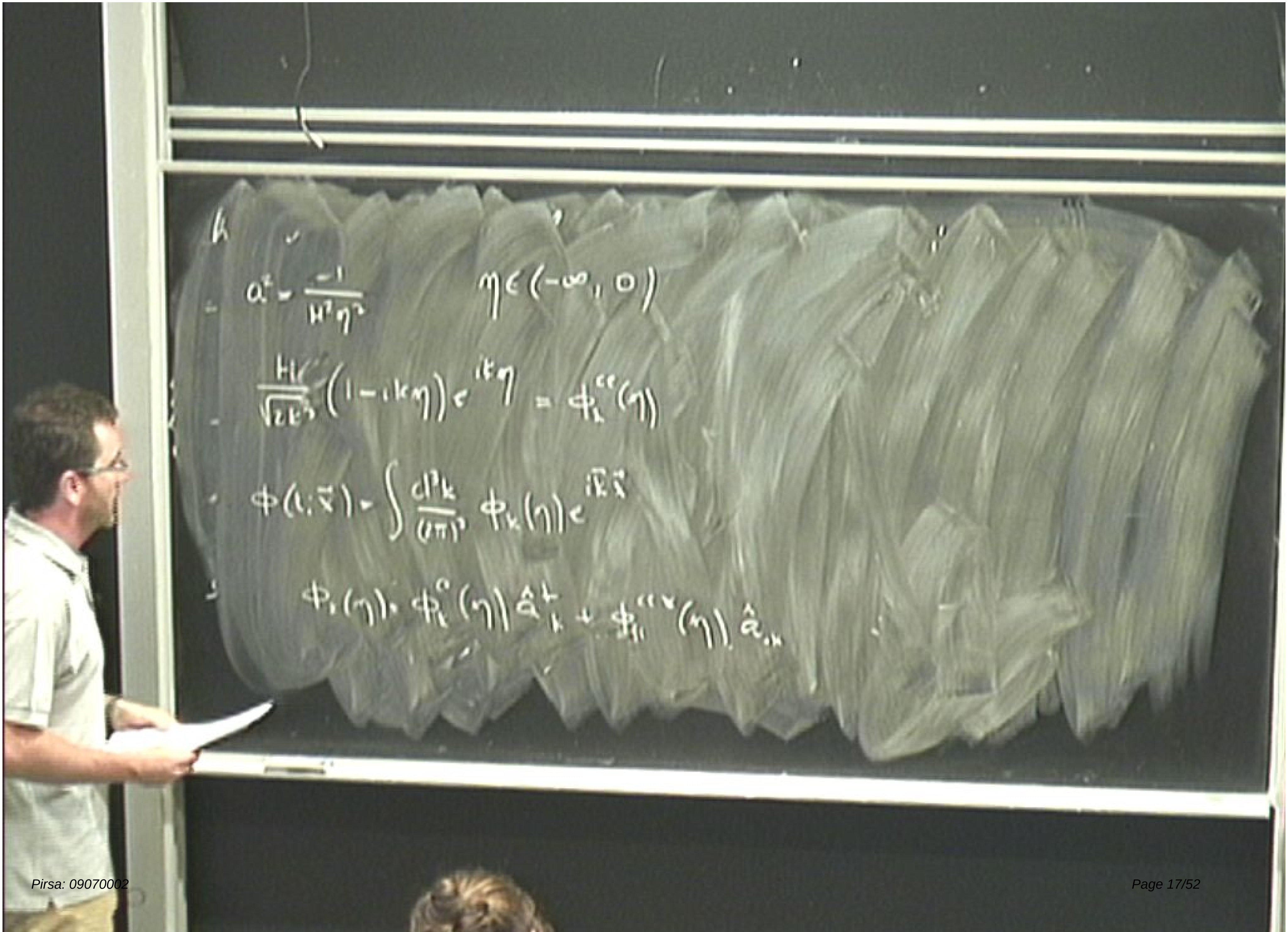
$$y(E) \propto a \quad \Rightarrow \quad \phi = \text{const}$$

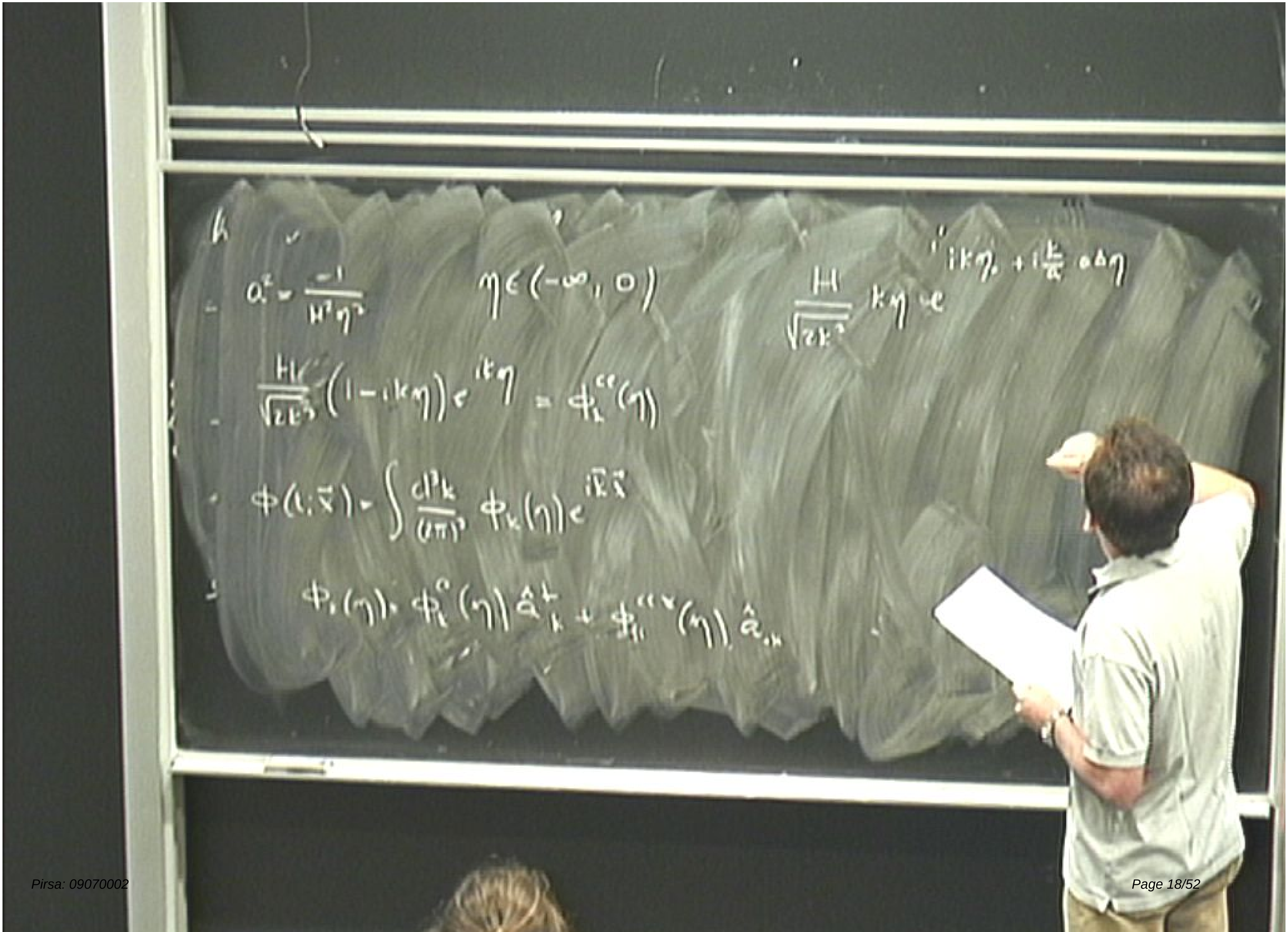
$$p(E) \propto a^{-1}$$


$a^2 = \frac{-1}{H^2\eta^2}$
$\frac{H}{\sqrt{2k^3}}(1 - ik\eta)e^{ik\eta}$



$a^2 = \frac{-1}{H^2\eta^2}$
$\eta \in (-\infty, 0)$
$\frac{H}{\sqrt{2k^3}}(1 - ik\eta)e^{ik\eta} = \phi_k^{cl}(\eta)$
$(t, \vec{x}) = \int \frac{d^3k}{(2\pi)^3} \phi_k(\eta) e^{i\vec{k}\cdot\vec{x}}$
$(\eta) = \phi_k$


$a^2 = \frac{-1}{H^2\eta^2}$ $\eta \in (-\infty, 0)$
$\frac{H}{\sqrt{2k^3}}(1 - ik\eta)e^{ik\eta} = \phi_k^{cl}(\eta)$
$\phi(t, \vec{x}) = \int \frac{d^3k}{(2\pi)^3}\phi_k(\eta)e^{i\vec{k}\cdot\vec{x}}$
$\phi_k(\eta) = \phi_k^{cl}(\eta)\hat{a}^\dagger_k + \phi_k^{cl*}(\eta)\hat{a}_{-k}$


$a^2 = \frac{-1}{H^2\eta^2}$
$\eta \in (-\infty, 0)$
$\frac{H}{\sqrt{2k^3}} k\eta\, e^{ik\eta_0 + i\frac{k}{a}\Delta\eta}$
$\frac{H}{\sqrt{2k^3}}(1 - ik\eta)e^{ik\eta} = \phi_k^{cl}(\eta)$
$\phi(t,\vec{x}) = \int \frac{d^3k}{(2\pi)^3}\phi_k(\eta)e^{i\vec{k}\cdot\vec{x}}$
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$$a^2 = \frac{-1}{H^2\eta^2} \qquad \eta \in (-\infty, 0)$$

$$\frac{H}{\sqrt{2k^3}}\, k\eta\, e^{ik\eta_0 + i\frac{k}{a}\Delta\eta}$$

$$k_{ph}$$

$$\frac{H}{\sqrt{2k^3}}(1 - ik\eta)\, e^{ik\eta} = \phi_k^{cl}(\eta)$$

$$\phi(t, \vec{x}) = \int \frac{d^3k}{(2\pi)^3}\, \phi_k(\eta)\, e^{i\vec{k}\cdot\vec{x}}$$

$$\phi_k(\eta) = \phi_k^{cl}(\eta)\, \hat{a}_k^{\dagger} + \phi_k^{cl*}(\eta)\, \hat{a}_{-k}$$

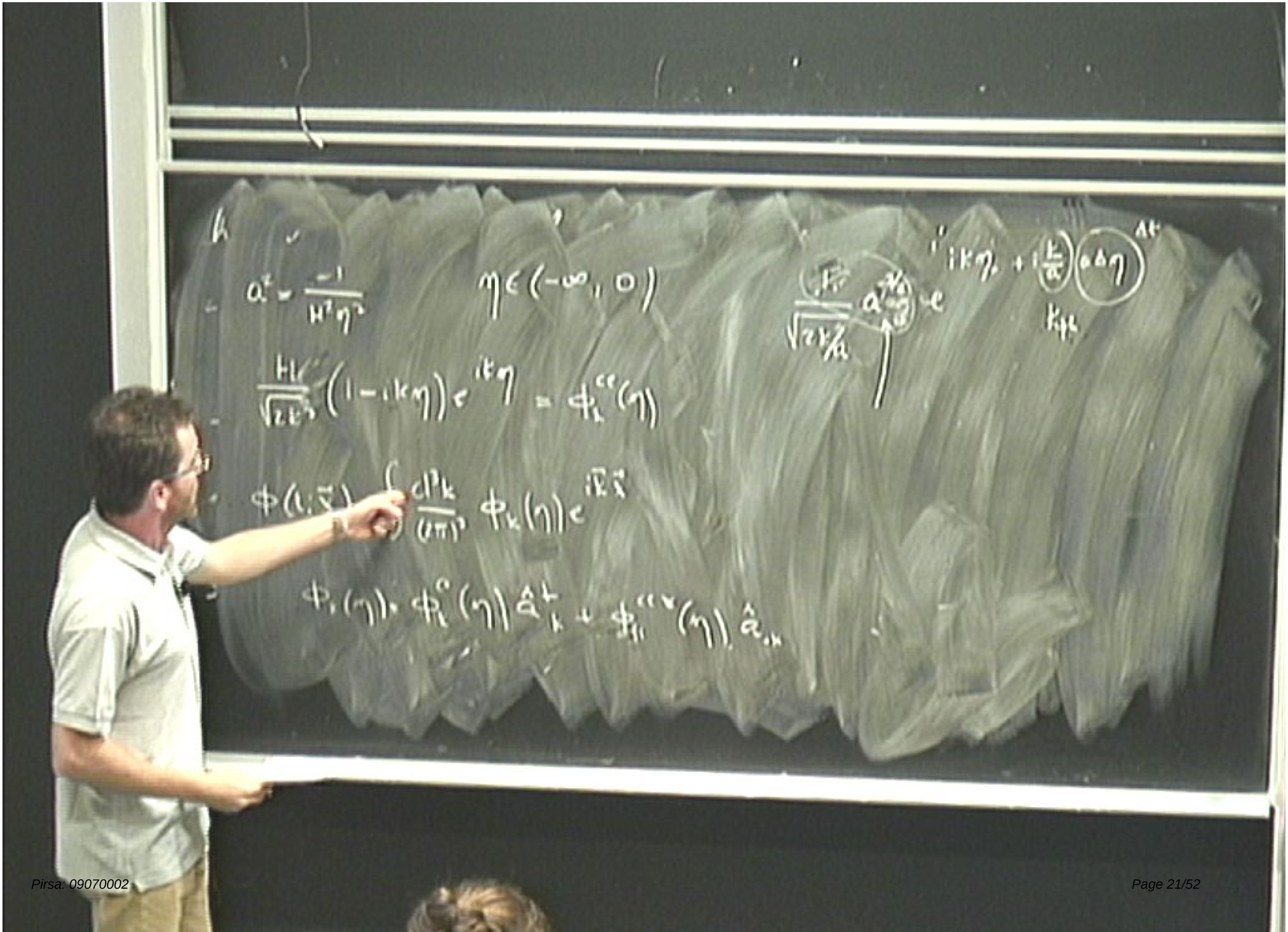
$$a^2 = \frac{-1}{H^2\eta^2} \qquad \eta \in (-\infty, 0) \qquad \frac{H}{\sqrt{2k^3}}\, k\eta\, e^{ik\eta_0 + i\frac{k}{a}\Delta\eta}$$

k_{ph} $\quad \Delta t$

$$\frac{H}{\sqrt{2k^3}}(1 - ik\eta)e^{ik\eta} = \phi_k^{cl}(\eta)$$

$$\phi(t, \vec{x}) = \int \frac{d^3k}{(2\pi)^3}\, \phi_k(\eta)\, e^{i\vec{k}\cdot\vec{x}}$$

$$\phi_k(\eta) = \phi_k^{cl}(\eta)\,\hat{a}^\dagger_k + \phi_k^{cl*}(\eta)\,\hat{a}_{-k}$$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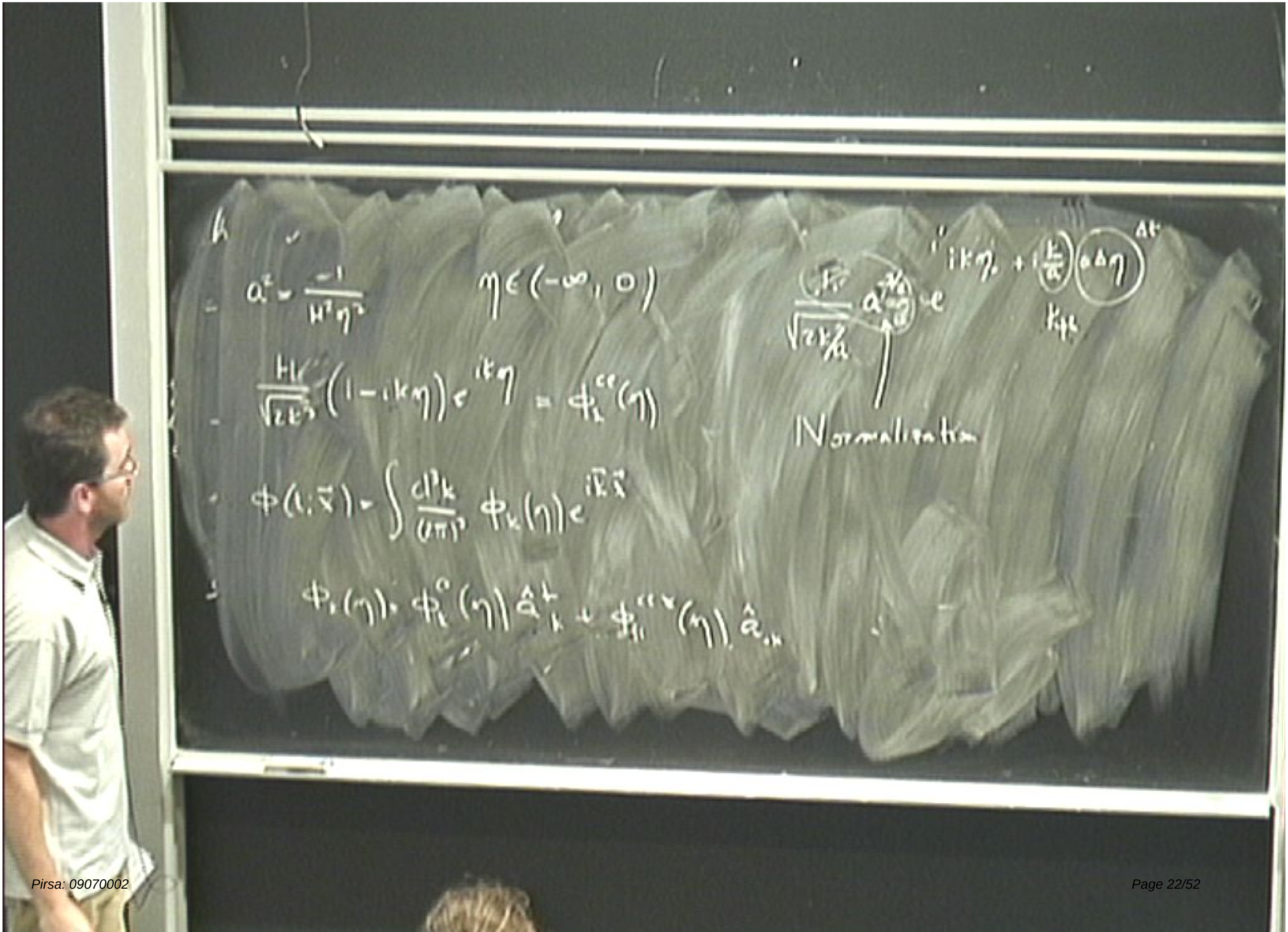
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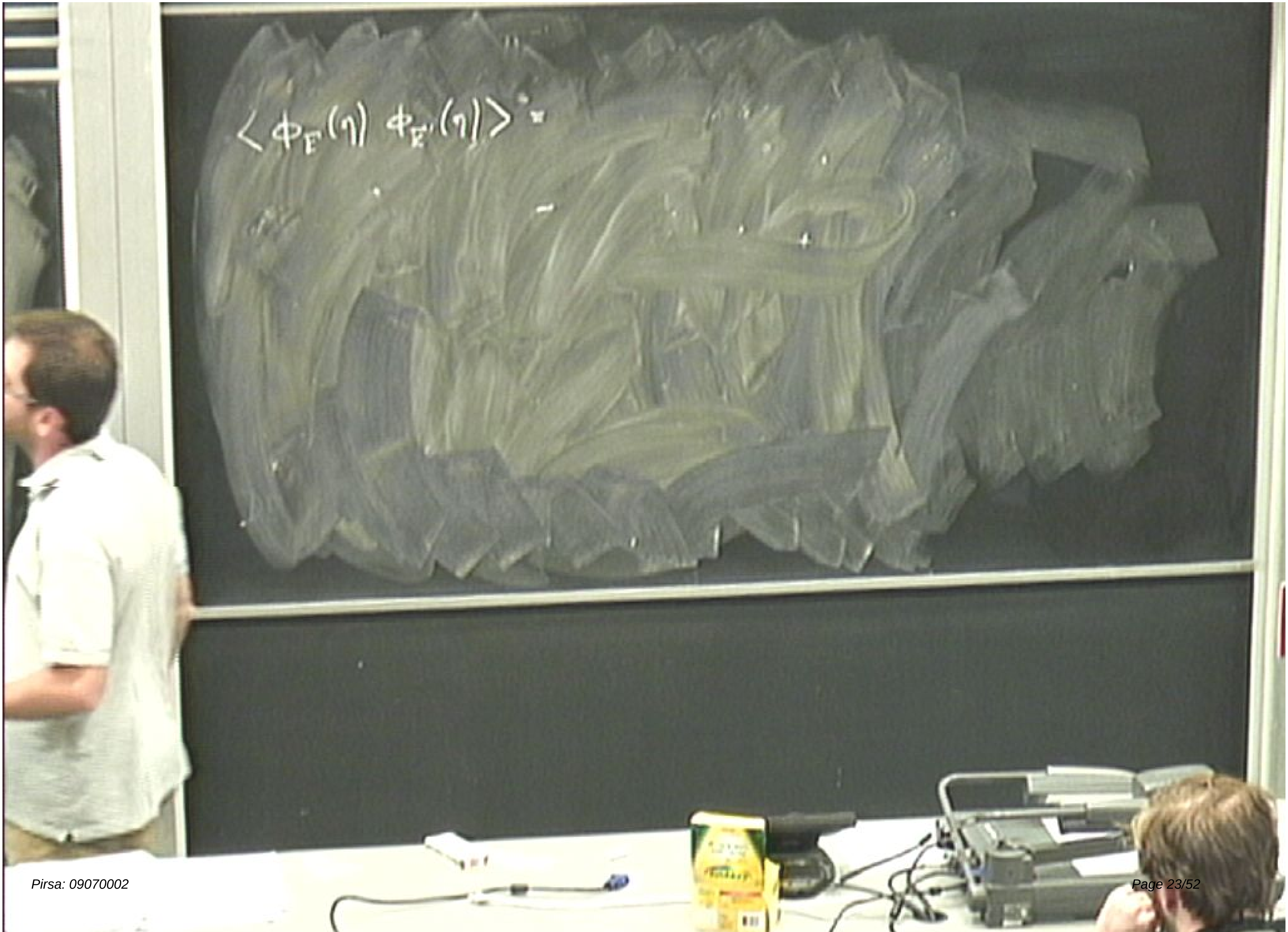
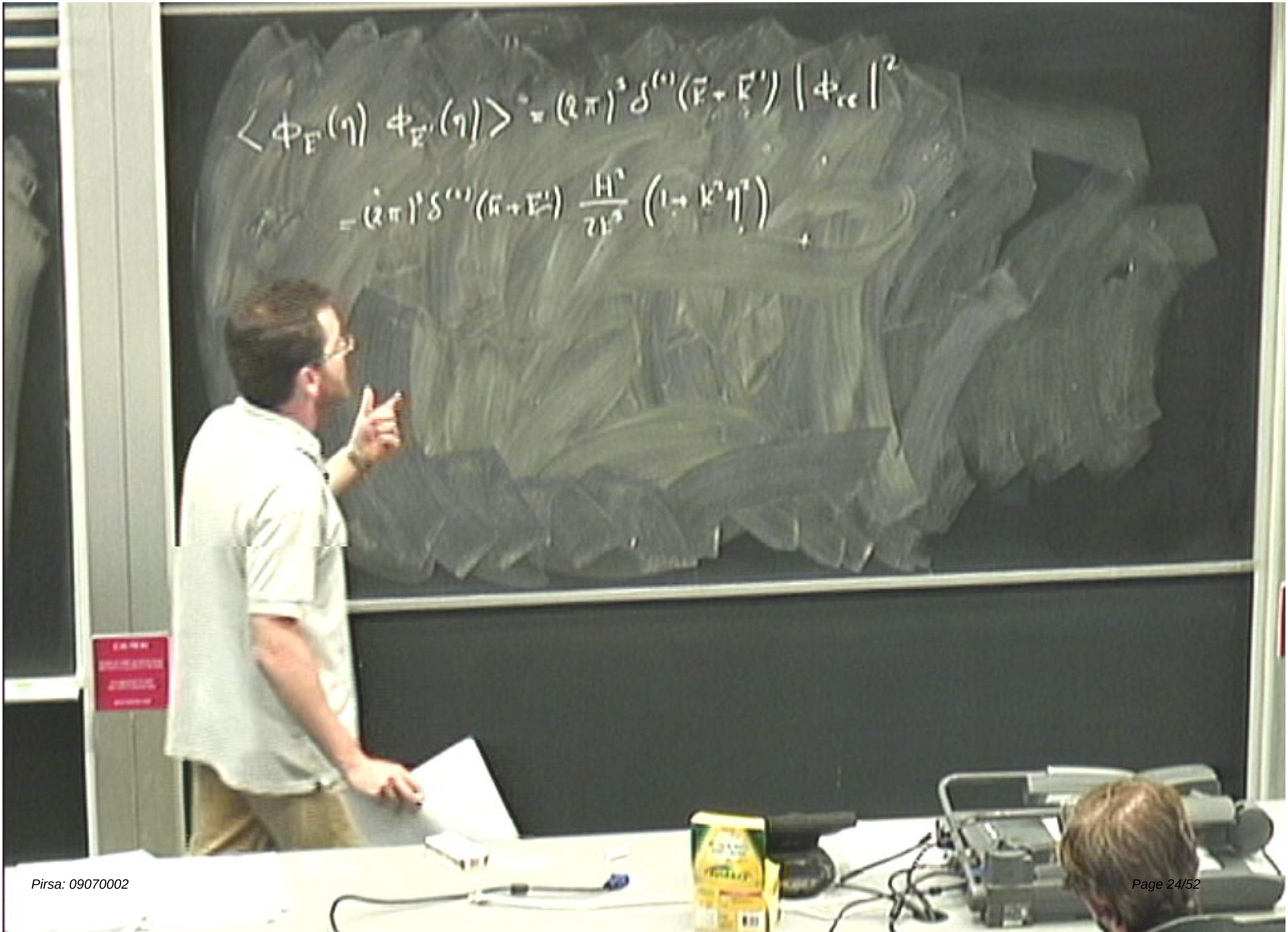
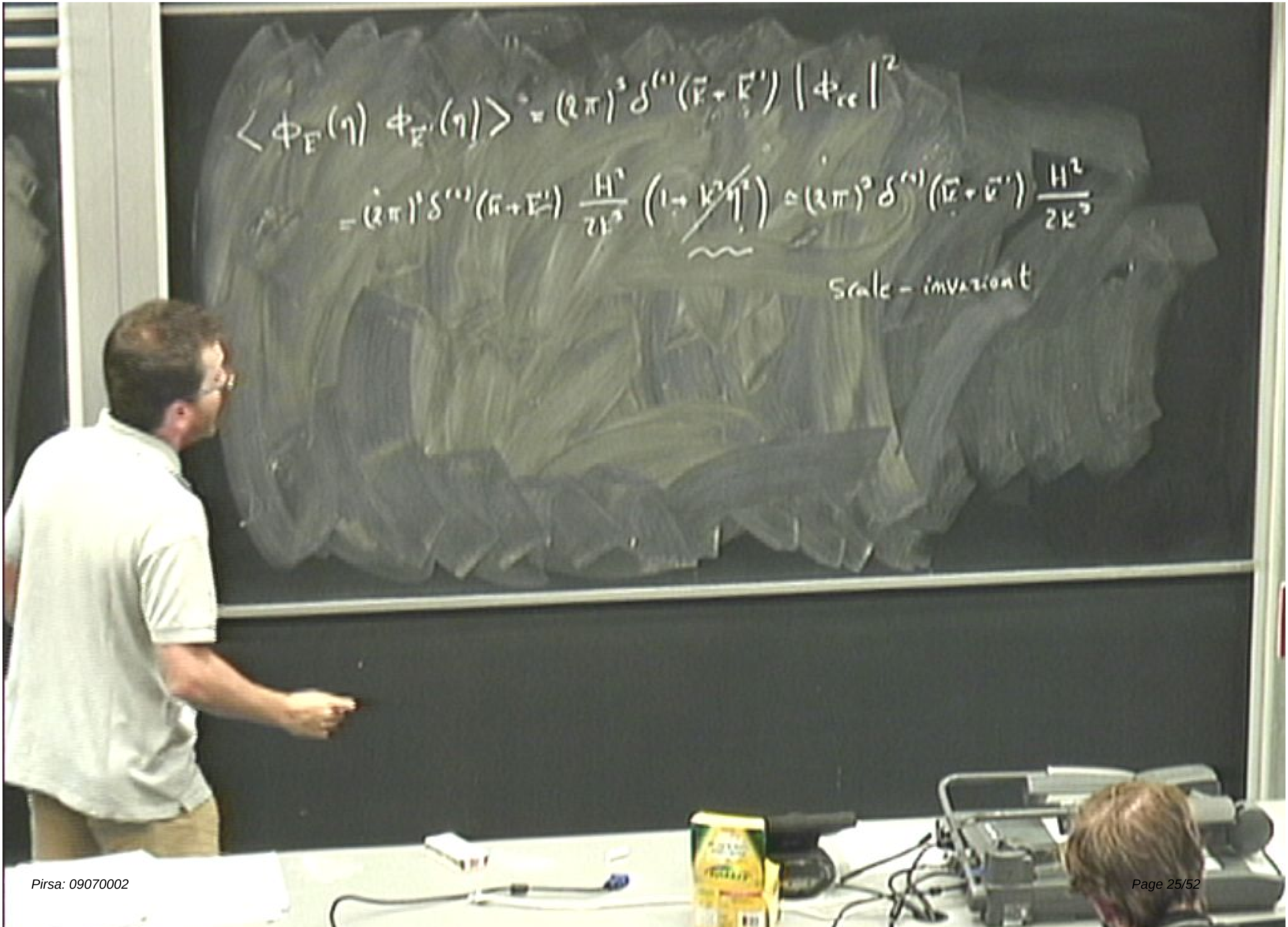
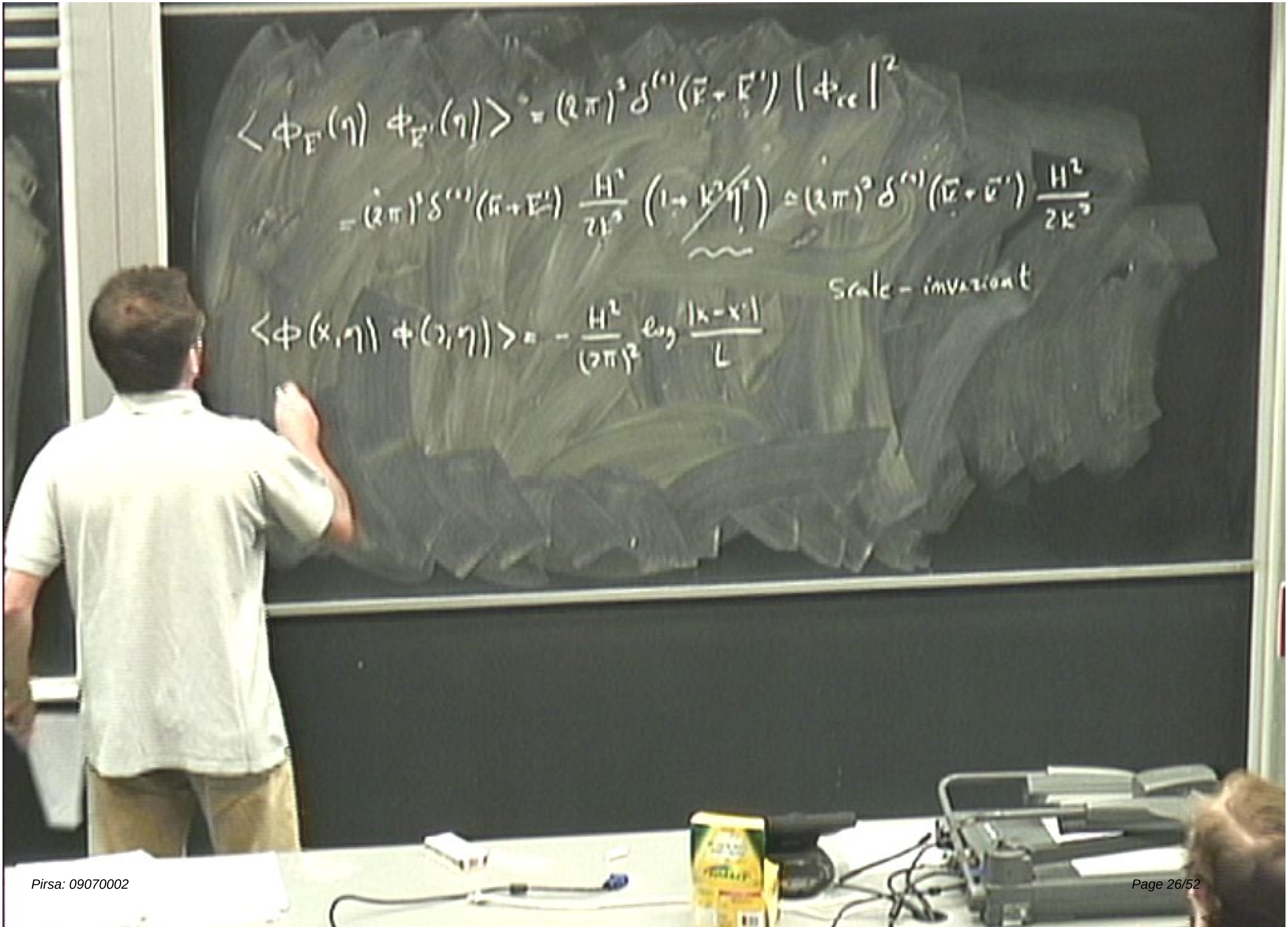
$\phi_k(\eta) = \phi_k^{cl}(\eta)\hat{a}^\dagger_k + \phi_k^{cl*}(\eta)\hat{a}_{-k}$
Normalization


$\langle \phi_E(\eta)\, \phi_{E'}(\eta) \rangle$


$\langle \phi_{\vec{k}}(\eta)\, \phi_{\vec{k}'}(\eta) \rangle = (2\pi)^3 \delta^{(3)}(\vec{k}+\vec{k}') \, |\phi_{cl}|^2$
$= (2\pi)^3 \delta^{(3)}(\vec{k}+\vec{k}') \, \frac{H^2}{2k^3} \left(1 + k^2\eta^2\right)$


$\langle \Phi_{\vec{k}}(\eta)\, \Phi_{\vec{k}'}(\eta) \rangle = (2\pi)^3 \delta^{(3)}(\vec{k}+\vec{k}')\, |\Phi_{kc}|^2$
$= (2\pi)^3 \delta^{(3)}(\vec{k}+\vec{k}')\, \frac{H^2}{2k^3}\left(1 + \cancel{k^2\eta^2}\right) \simeq (2\pi)^3 \delta^{(3)}(\vec{k}+\vec{k}')\, \frac{H^2}{2k^3}$
Scale-invariant

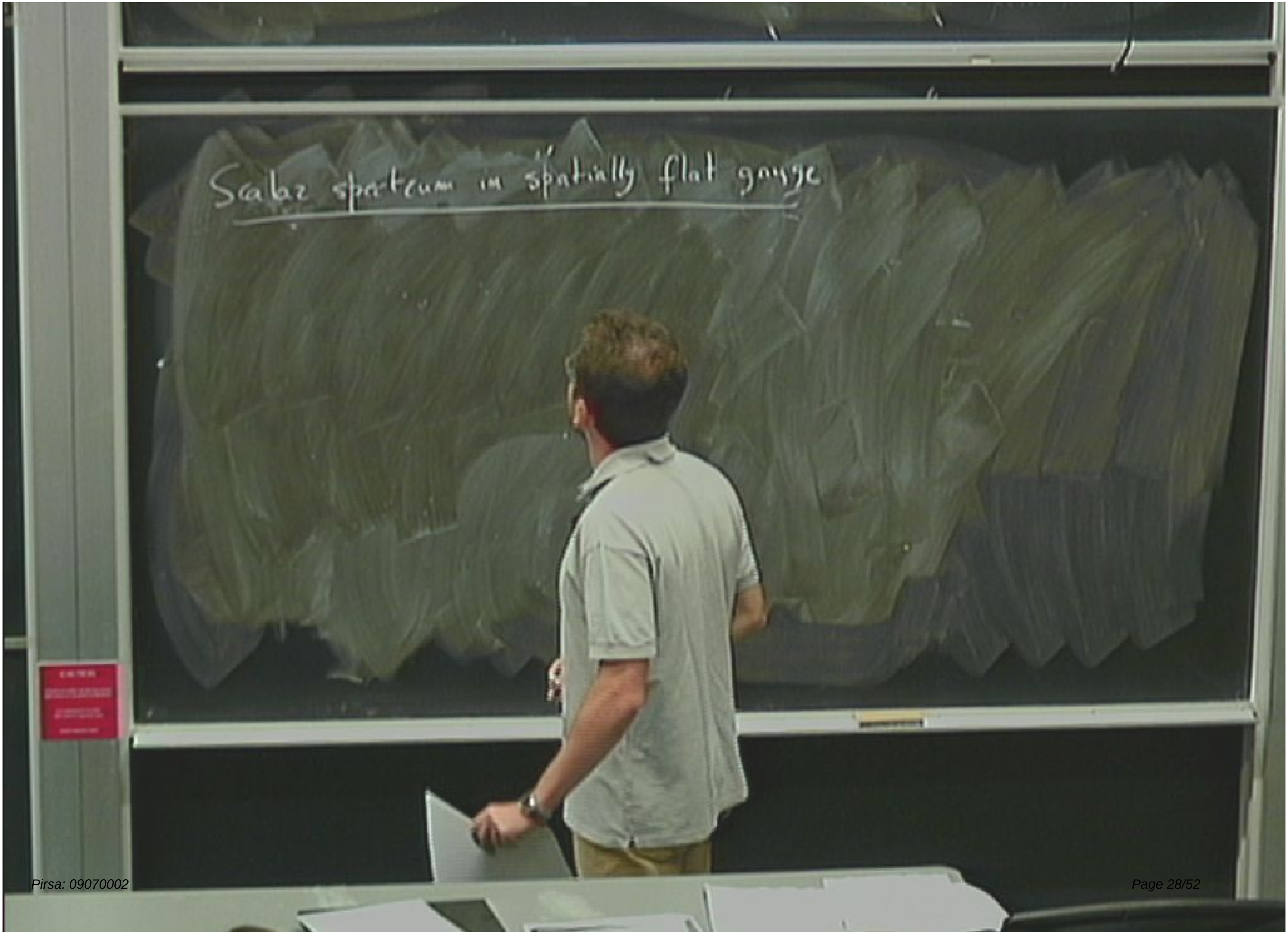

$\langle \phi_{\vec{k}}(\eta)\, \phi_{\vec{k}'}(\eta) \rangle = (2\pi)^3 \delta^{(3)}(\vec{k} + \vec{k}') \, |\phi_{kc}|^2$
$= (2\pi)^3 \delta^{(3)}(\vec{k} + \vec{k}') \frac{H^2}{2k^3} \left(1 + \cancel{k^2\eta^2}\right) \simeq (2\pi)^3 \delta^{(3)}(\vec{k} + \vec{k}') \frac{H^2}{2k^3}$
scale - invariant
$\langle \phi(x,\eta)\, \phi(x',\eta) \rangle = - \frac{H^2}{(2\pi)^2} \log \frac{|x - x'|}{L}$

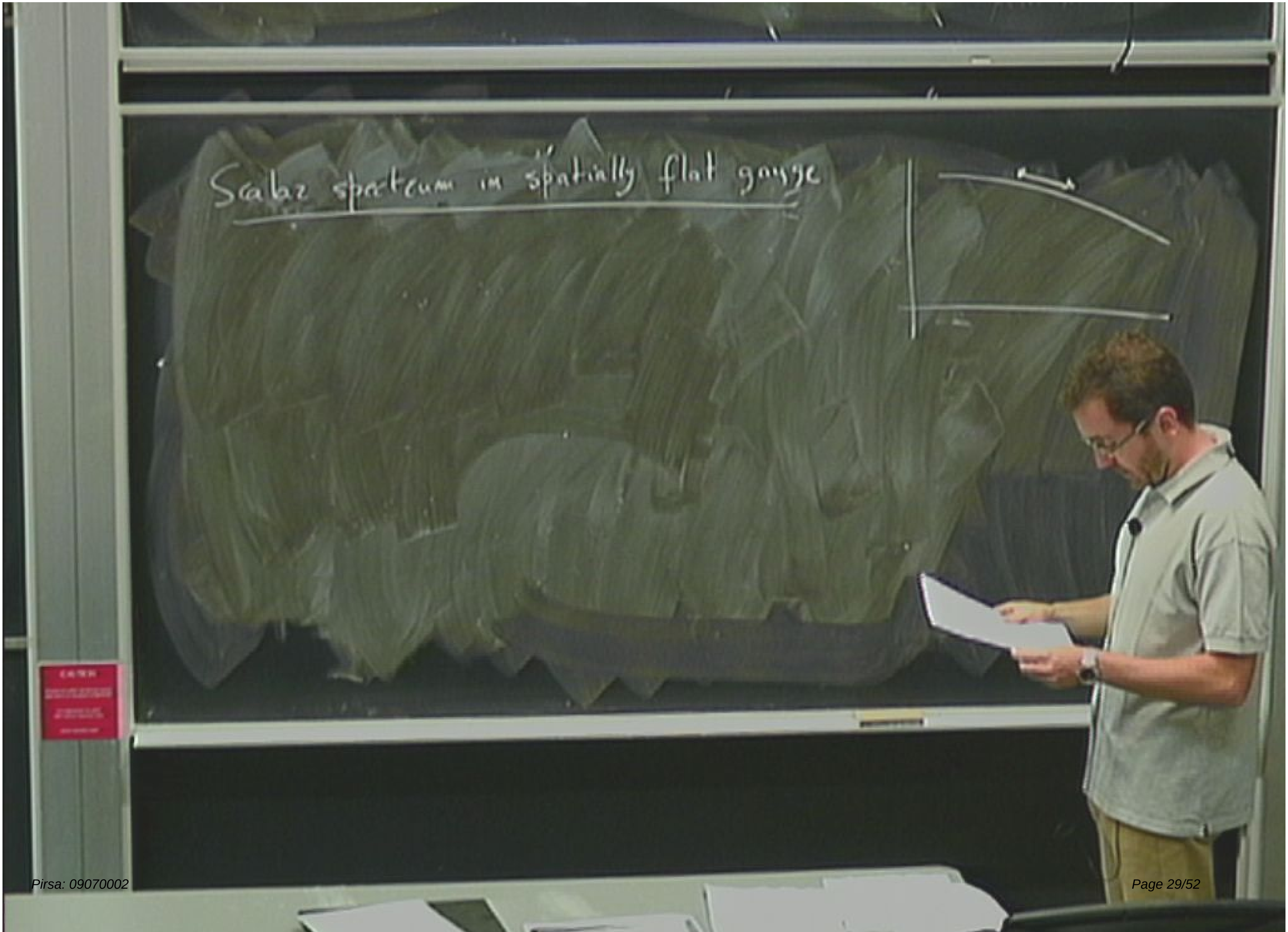
$$\langle \phi_{\vec{k}}(\eta)\, \phi_{\vec{k}'}(\eta) \rangle = (2\pi)^3 \delta^{(3)}(\vec{k} + \vec{k}') \left| \phi_{kc} \right|^2$$

$$= (2\pi)^3 \delta^{(3)}(\vec{k} + \vec{k}') \frac{H^2}{2k^3} \left(1 + \cancel{k^2\eta^2}\right) \simeq (2\pi)^3 \delta^{(3)}(\vec{k} + \vec{k}') \frac{H^2}{2k^3}$$

Scale - invariant

$$\boxed{\langle \phi(x,\eta)\, \phi(y,\eta) \rangle = -\frac{H^2}{(2\pi)^2} \log \frac{|x - x'|}{L}}$$


Scalar spectrum in spatially flat gauge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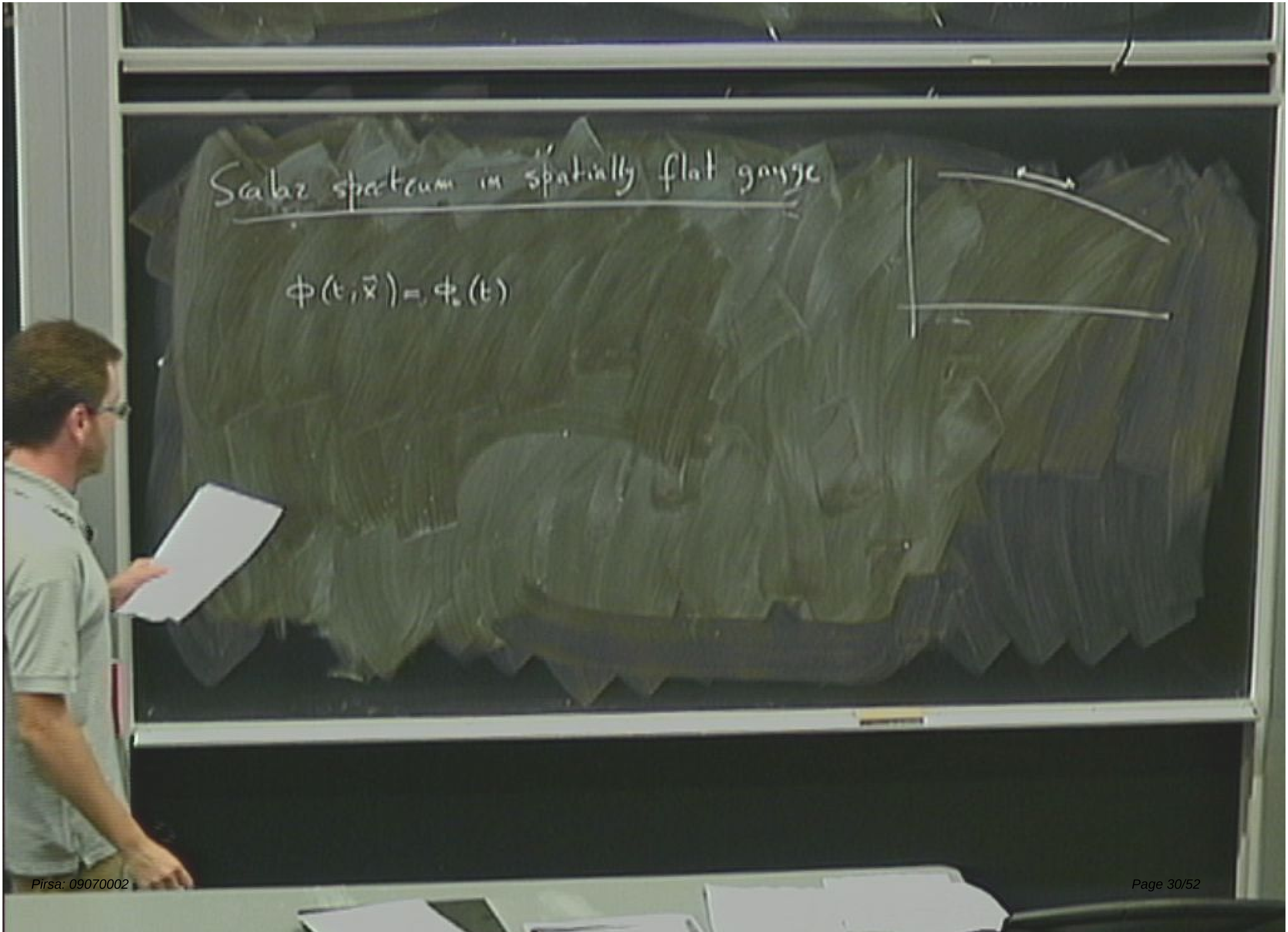
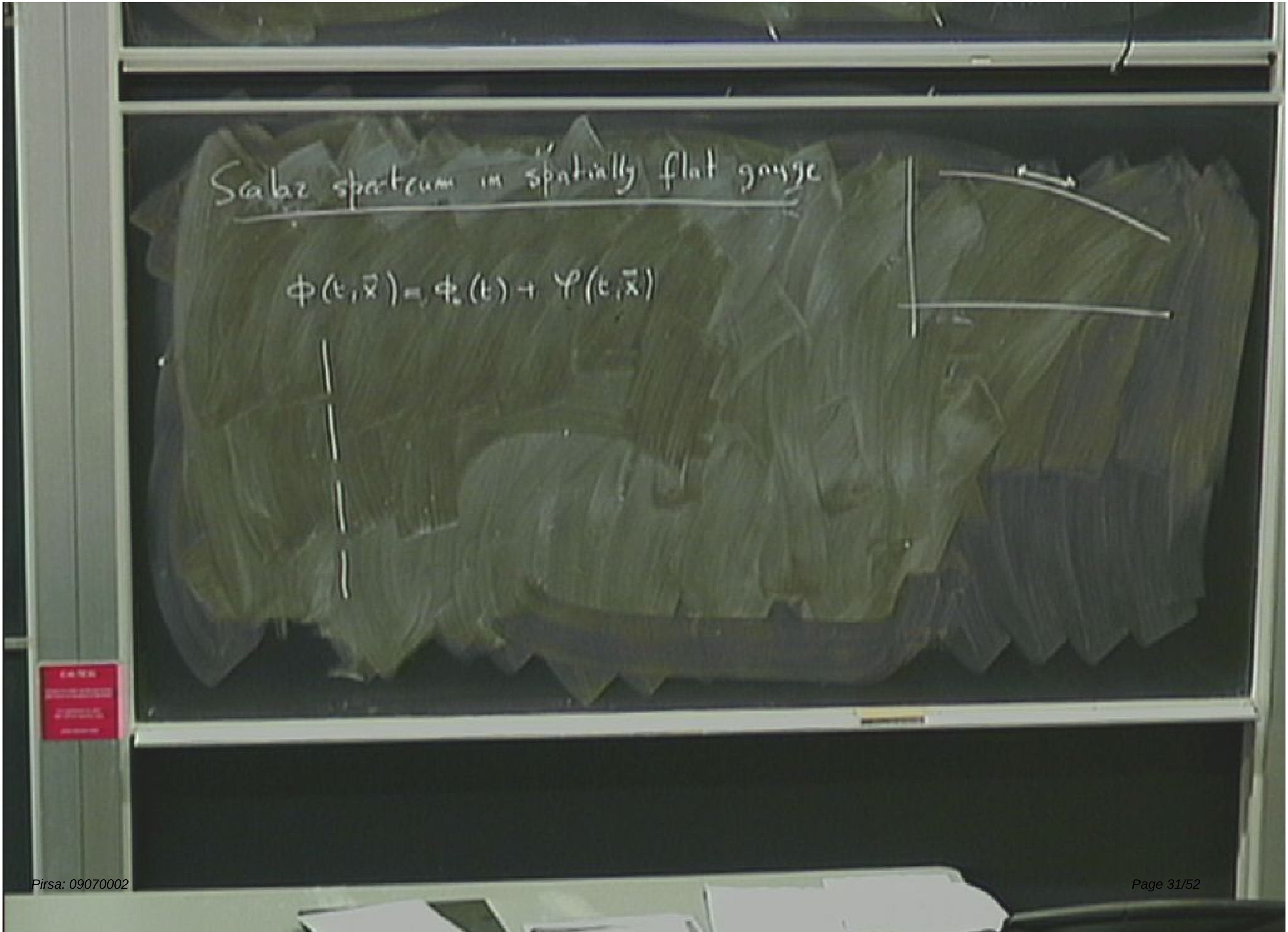
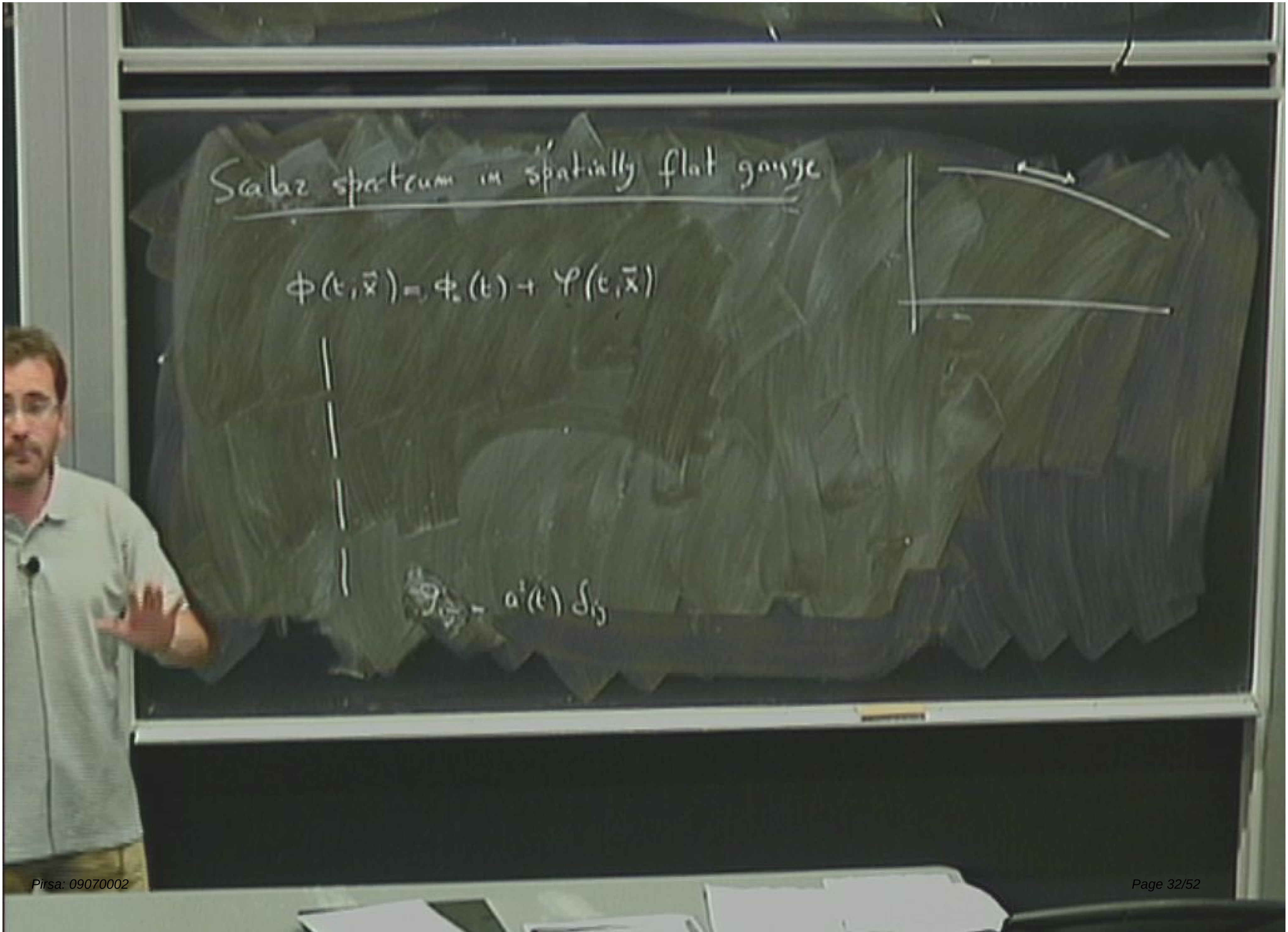
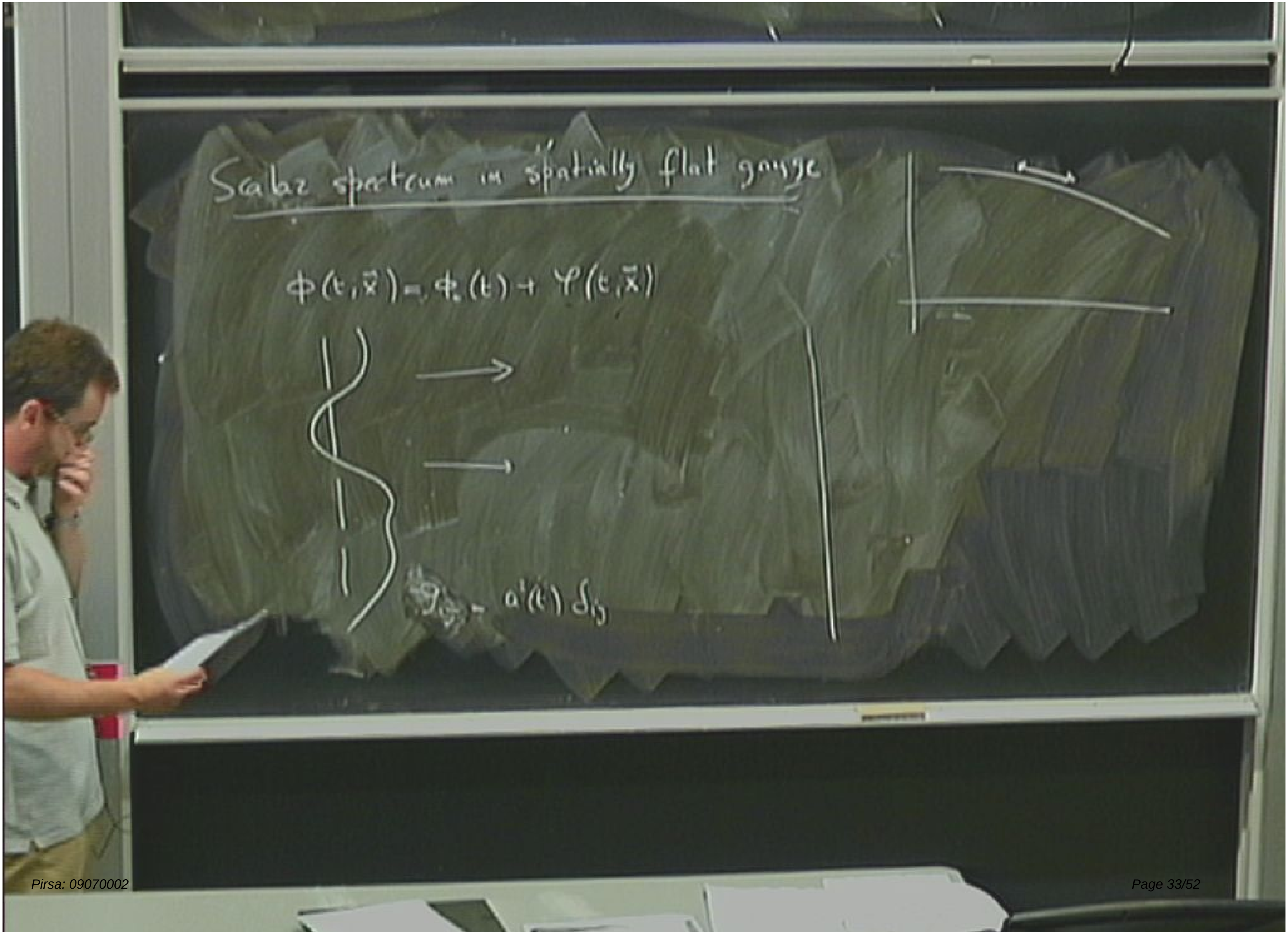

Scalar spectrum in spatially flat gauge
φ(t, x⃗) = φ₀(t)


Scalar spectrum in spatially flat gauge
$\phi(t,\vec{x}) = \phi_0(t) + \varphi(t,\vec{x})$


Scalar spectrum in spatially flat gauge
$\phi(t,\vec{x}) = \phi_0(t) + \varphi(t,\vec{x})$
$g_{ij} = a^2(t)\,\delta_{ij}$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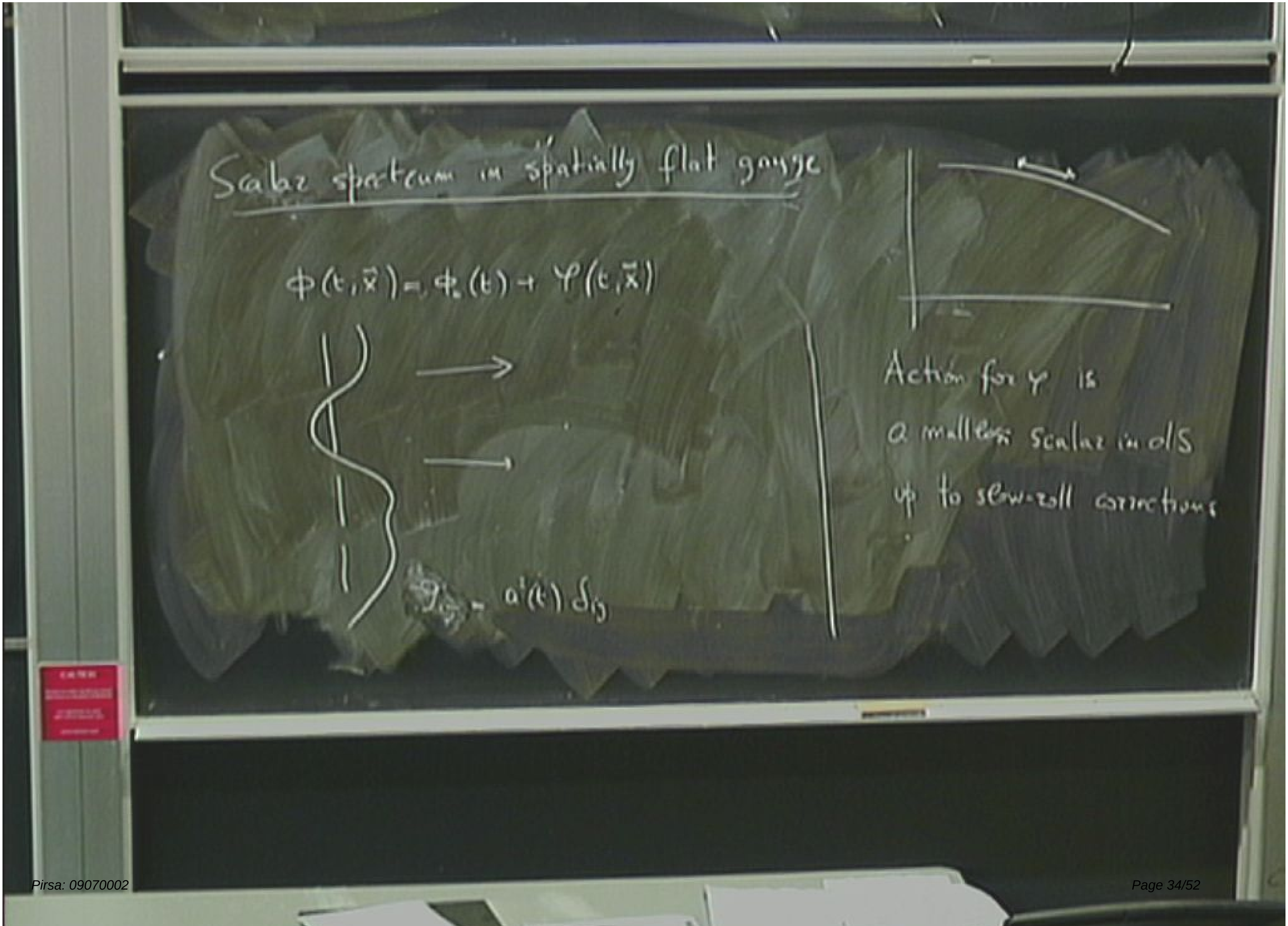
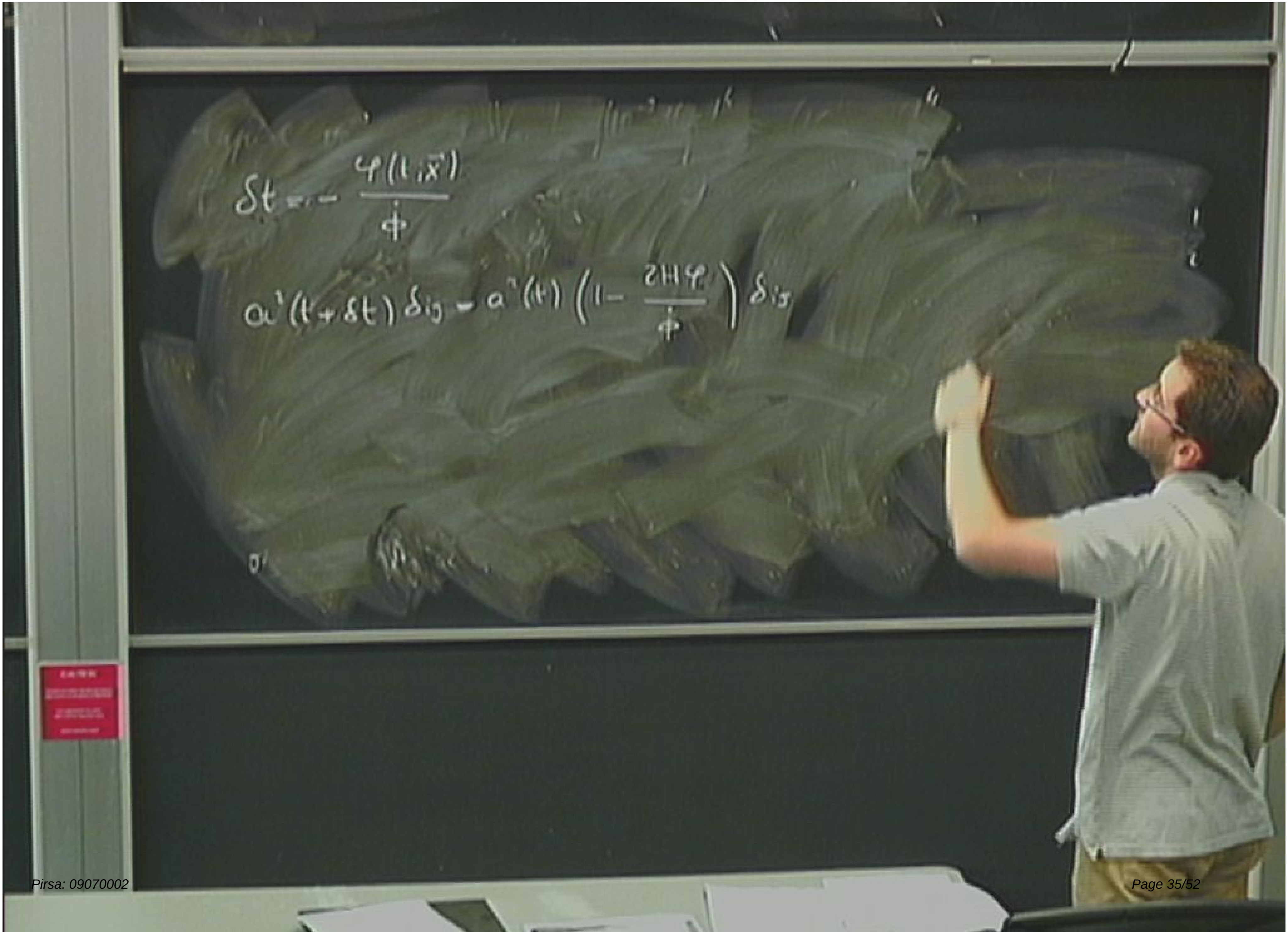
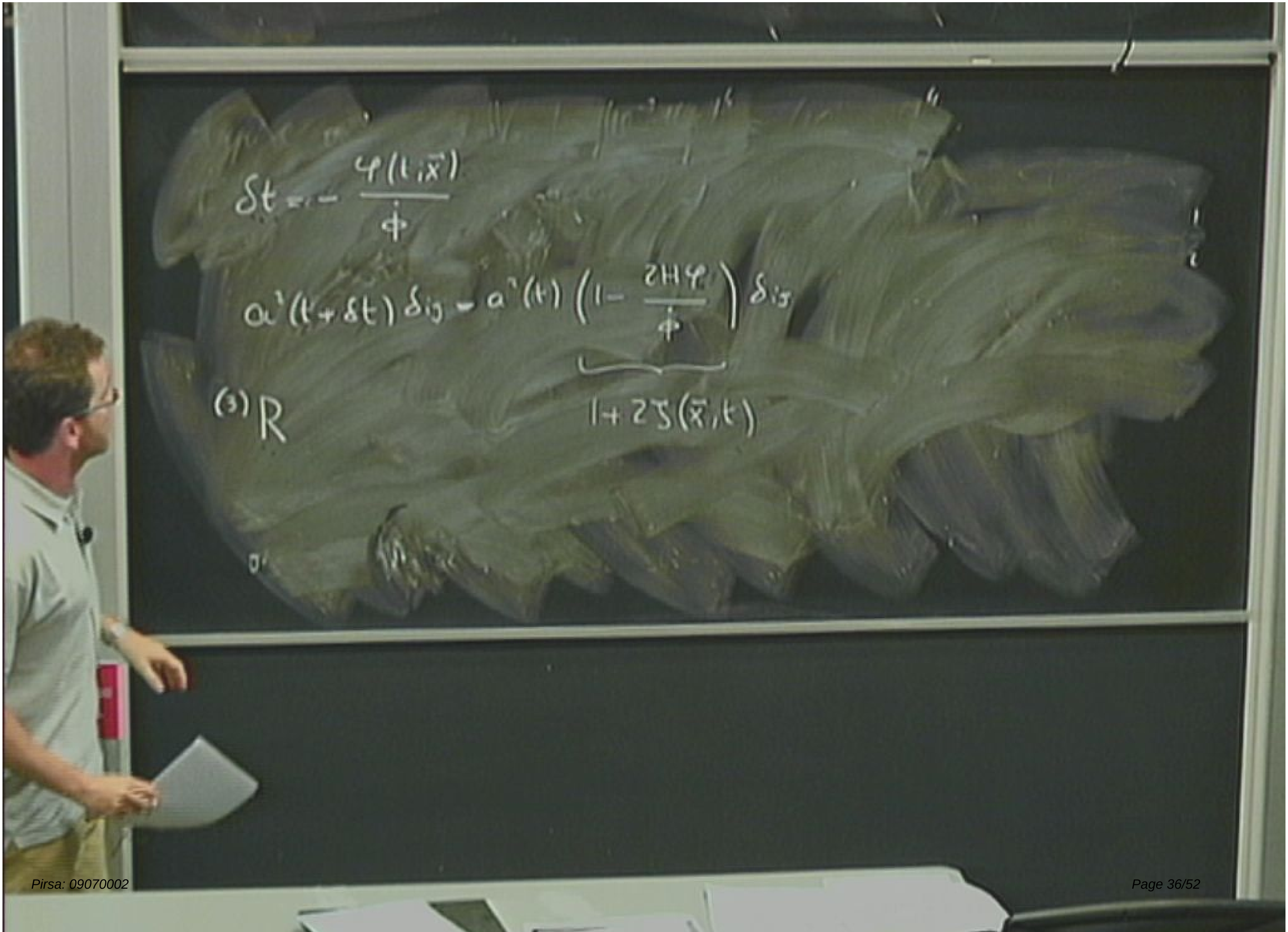

Scalar spectrum in spatially flat gauge
$\phi(t,\vec{x}) = \phi_0(t) + \varphi(t,\vec{x})$
$g_{ij} = a^2(t)\,\delta_{ij}$
Action for φ is
a massless scalar in dS
up to slow-roll corrections

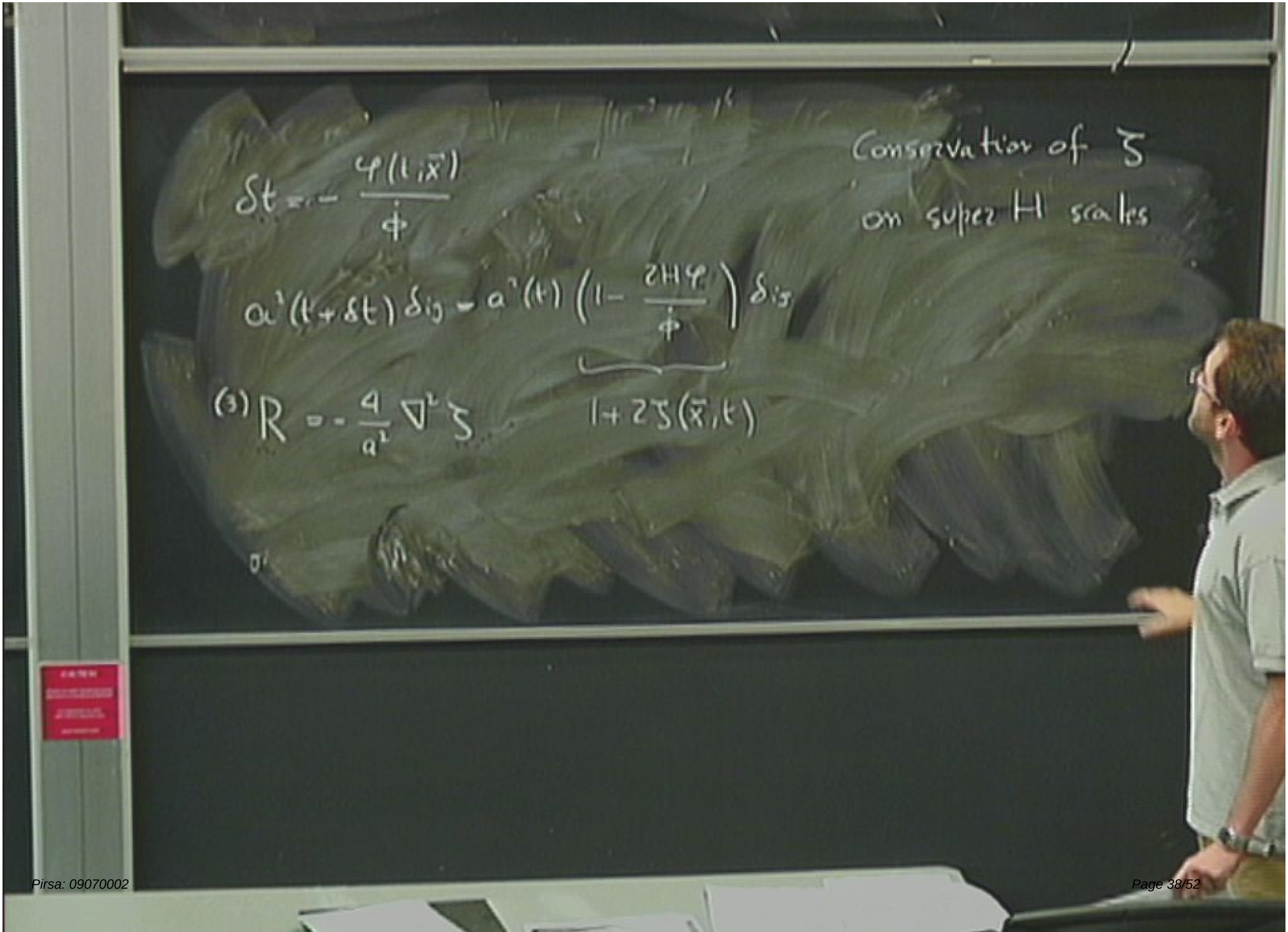

$\delta t = -\frac{\varphi(t,\vec{x})}{\dot{\phi}}$
$a^2(t+\delta t)\,\delta_{ij} = a^2(t)\left(1-\frac{2H\varphi}{\dot{\phi}}\right)\delta_{ij}$

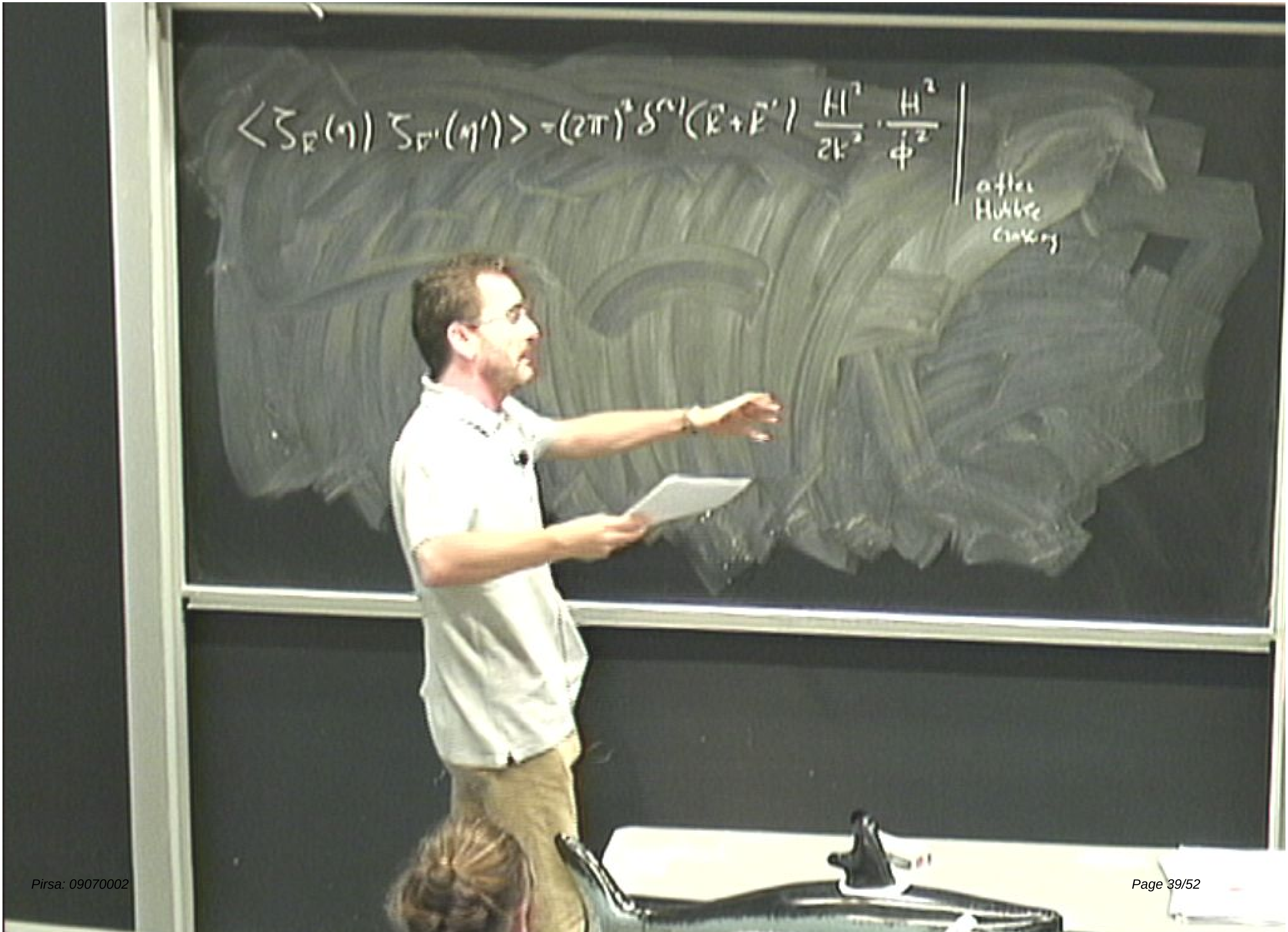

$\delta t = -\frac{\varphi(t,\vec{x})}{\dot{\phi}}$
$a^2(t+\delta t)\,\delta_{ij} = a^2(t)\underbrace{\left(1-\frac{2H\varphi}{\dot{\phi}}\right)}_{1+2\zeta(\vec{x},t)}\delta_{ij}$
${}^{(3)}R$

$$\delta t = -\frac{\varphi(t,\vec{x})}{\dot{\phi}}$$

$$a^2(t+\delta t)\,\delta_{ij} = a^2(t)\underbrace{\left(1-\frac{2H\varphi}{\dot{\phi}}\right)}_{1+2\zeta(\vec{x},t)}\delta_{ij}$$

$$^{(3)}R = -\frac{4}{a^2}\nabla^2\zeta$$

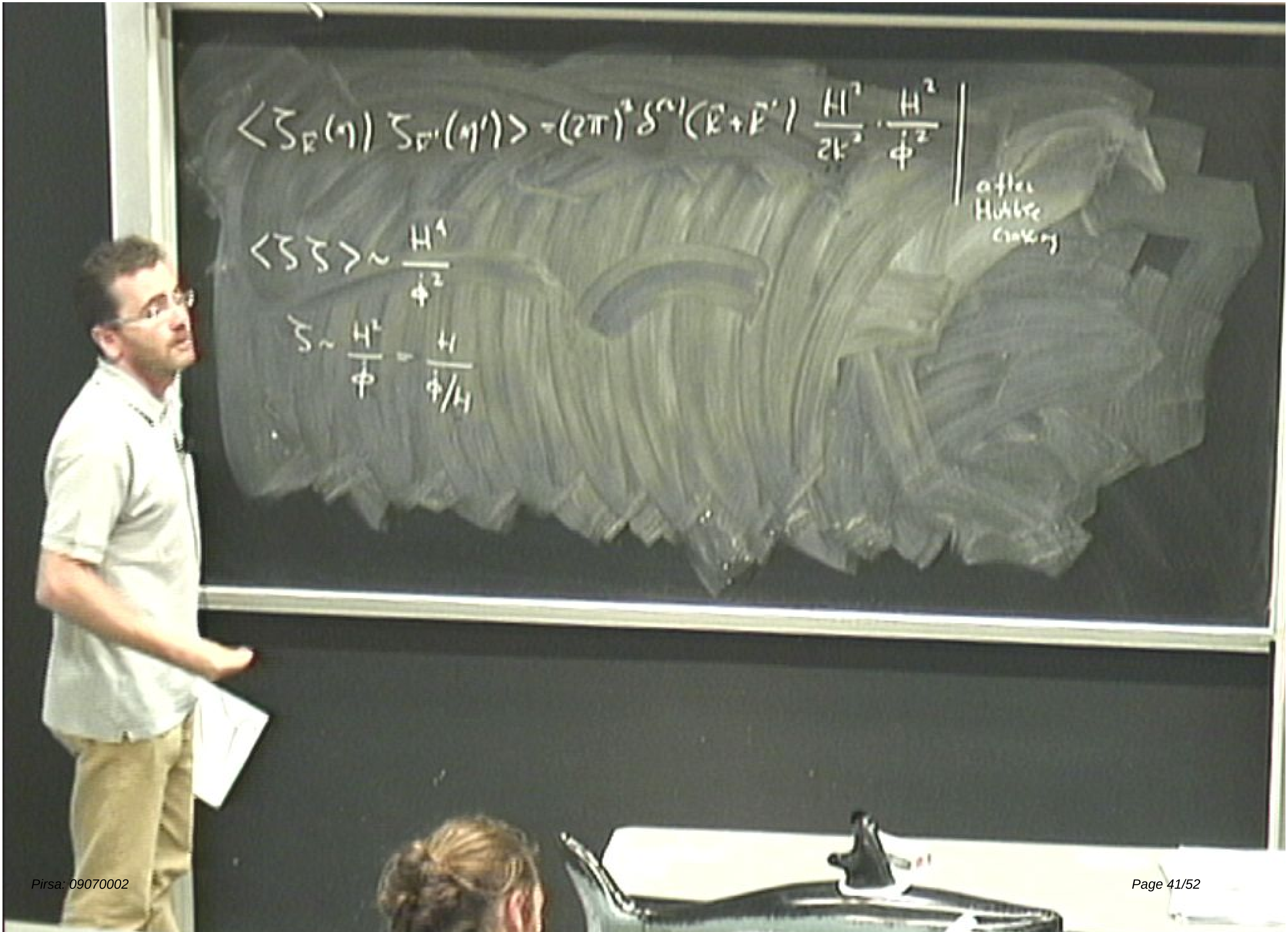

$\delta t = -\frac{\varphi(t,\vec{x})}{\dot{\phi}}$
$a^2(t+\delta t)\,\delta_{ij} = a^2(t)\underbrace{\left(1-\frac{2H\varphi}{\dot{\phi}}\right)}_{1+2\zeta(\vec{x},t)}\delta_{ij}$
${}^{(3)}R = -\frac{4}{a^2}\nabla^2\zeta$
Conservation of ζ
on super H scales


$\langle \zeta_{\vec k}(\eta)\, \zeta_{\vec k'}(\eta') \rangle = (2\pi)^3 \delta^{(3)}(\vec k + \vec k')\, \frac{H^2}{2k^3} \cdot \frac{H^2}{\dot\phi^2}$
after Hubble crossing

$$\langle \zeta_{\vec{k}}(\eta)\, \zeta_{\vec{k}'}(\eta') \rangle = (2\pi)^3 \delta^{(3)}(\vec{k} + \vec{k}')\, \frac{H^2}{2k^3} \cdot \frac{H^2}{\dot{\phi}^2} \Big|_{\text{after Hubble crossing}}$$

$$\langle \zeta \zeta \rangle \sim \frac{H^4}{\dot{\phi}^2}$$

$$\zeta \sim \frac{H^2}{\dot{\phi}}$$

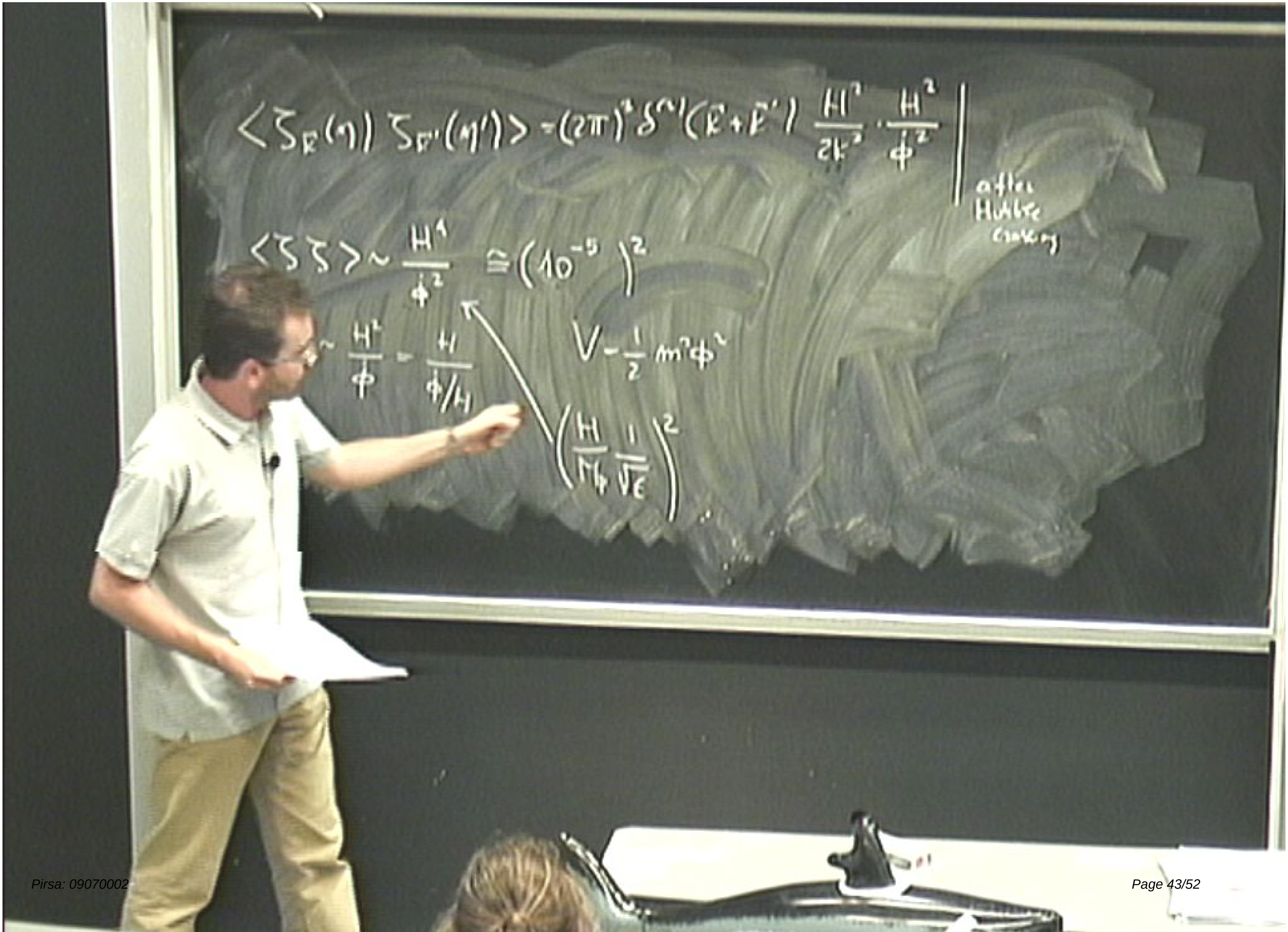

$\langle \zeta_{\vec k}(\eta)\, \zeta_{\vec k'}(\eta') \rangle = (2\pi)^3 \delta^{(3)}(\vec k + \vec k')\, \frac{H^2}{2k^3} \cdot \frac{H^2}{\dot\phi^2}$
after Hubble crossing
$\langle \zeta \zeta \rangle \sim \frac{H^4}{\dot\phi^2}$
$\zeta \sim \frac{H^2}{\dot\phi} = \frac{H}{\dot\phi / H}$

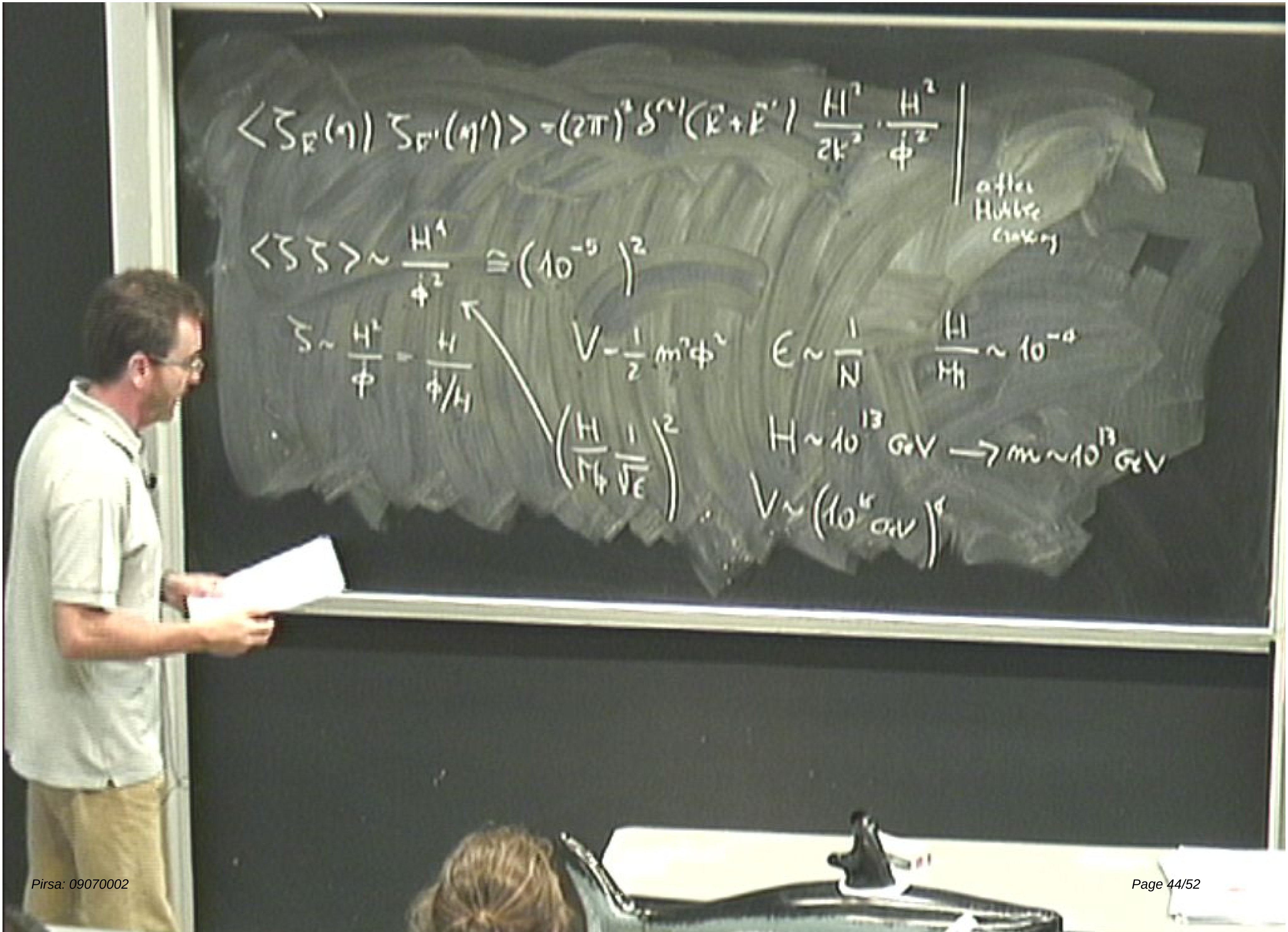
$$\langle \zeta_{\vec k}(\eta)\,\zeta_{\vec k'}(\eta')\rangle = (2\pi)^3\delta^{(3)}(\vec k+\vec k')\,\frac{H^2}{2k^3}\cdot\frac{H^2}{\dot\phi^2}$$

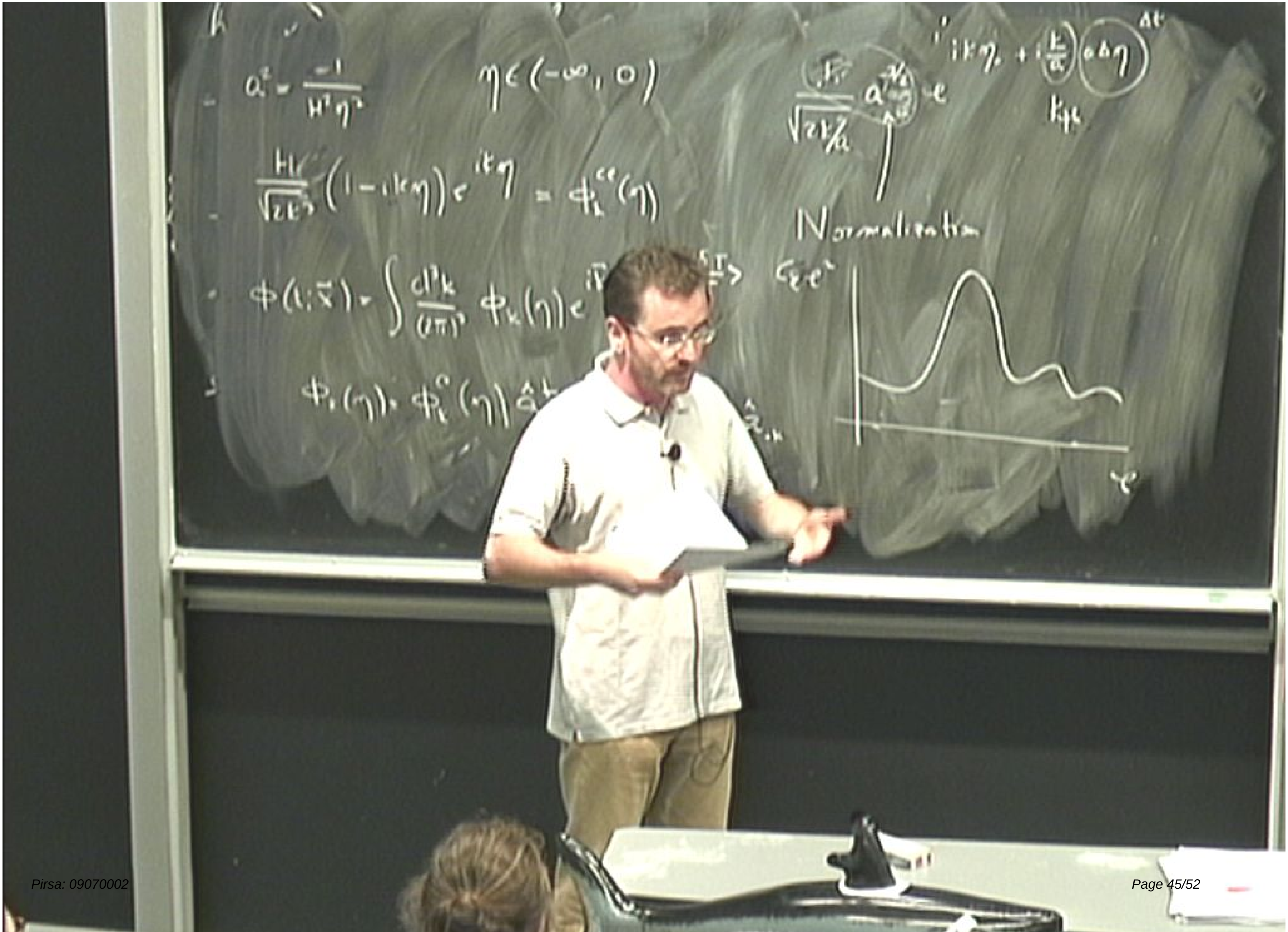
after Hubble crossing

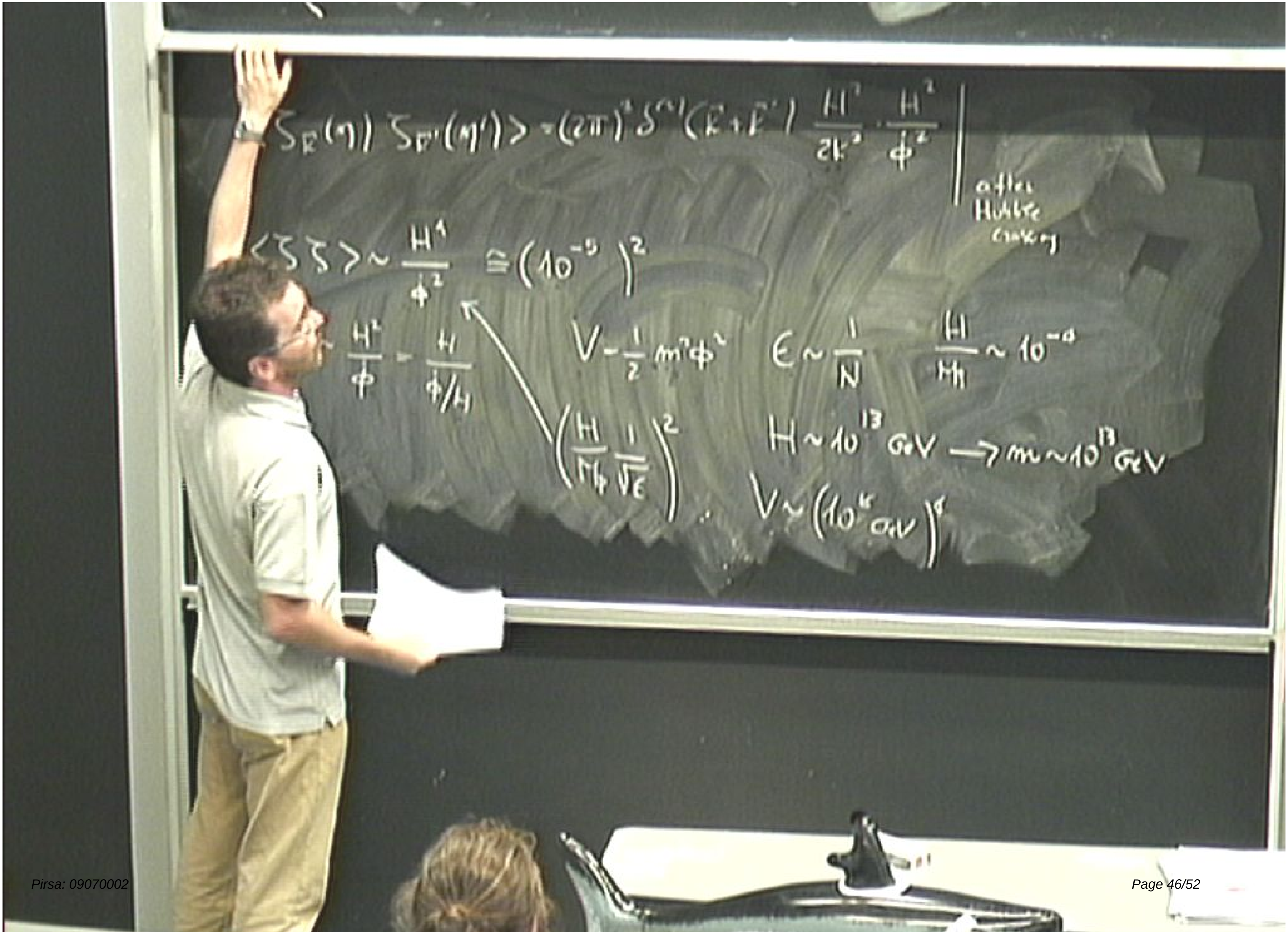
$$\langle\zeta\zeta\rangle \sim \frac{H^4}{\dot\phi^2} \cong (10^{-5})^2$$

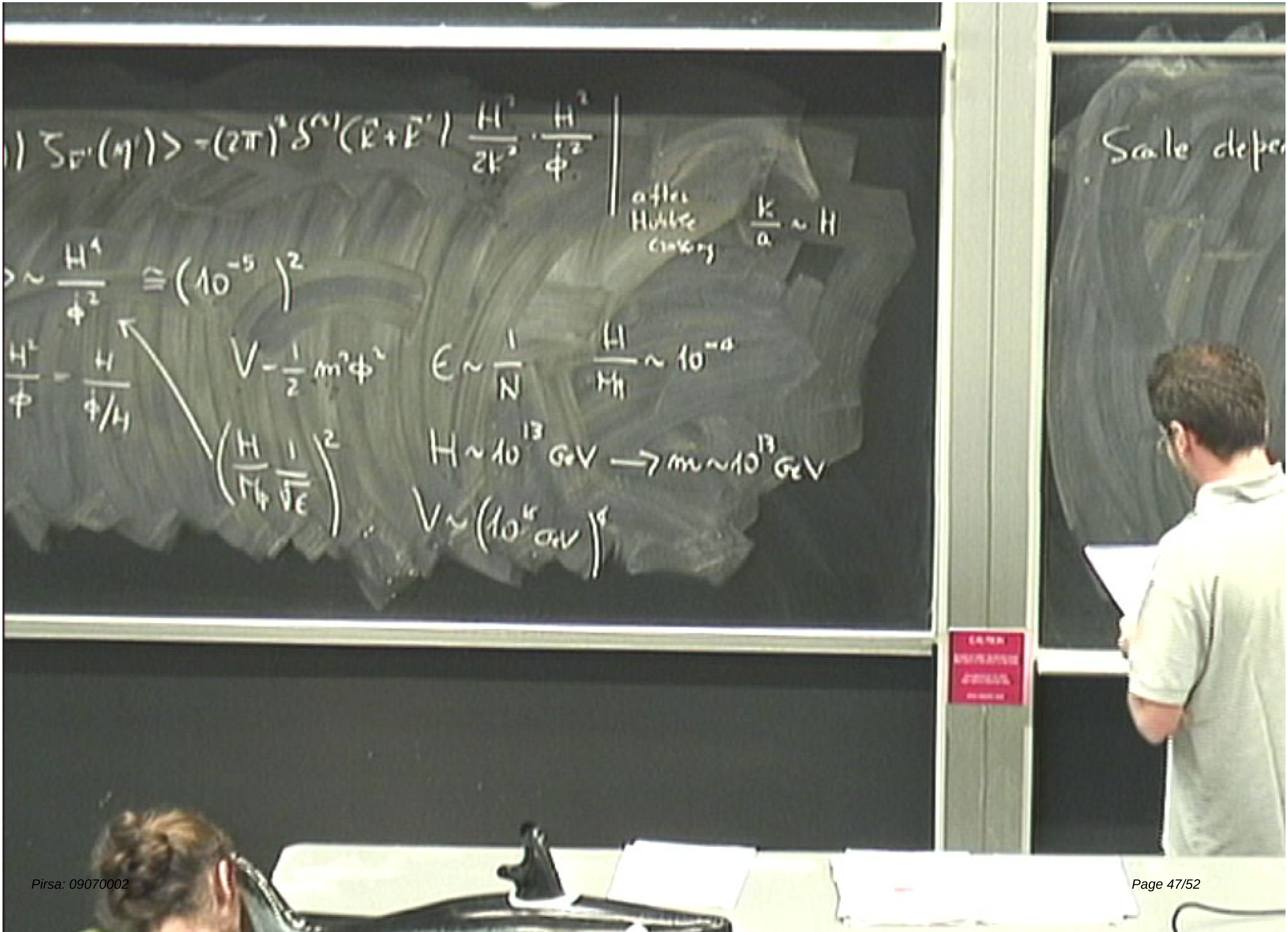
$$\zeta\sim\frac{H^2}{\dot\phi} = \frac{H}{\dot\phi/H}$$

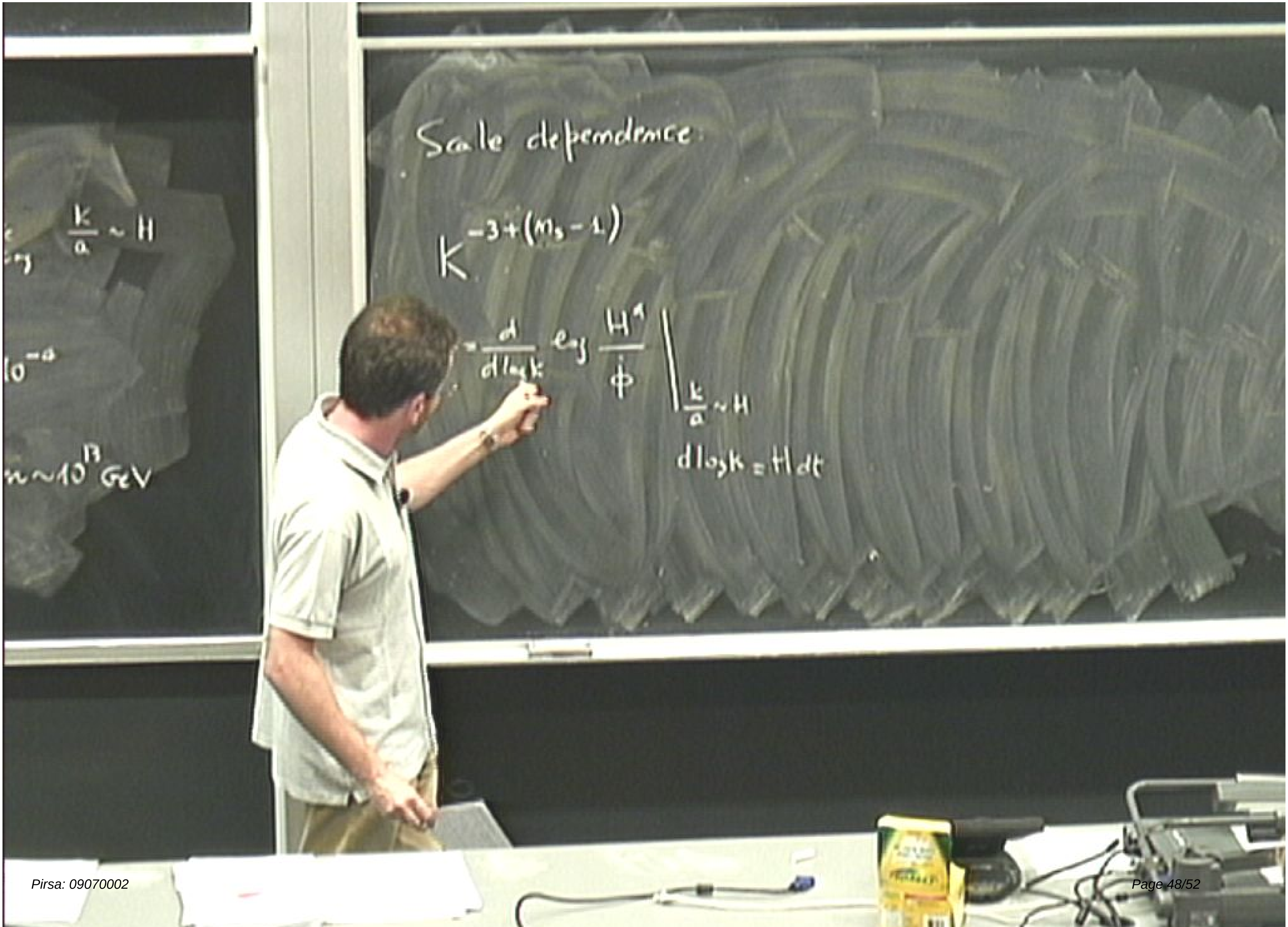

after Hubble crossing

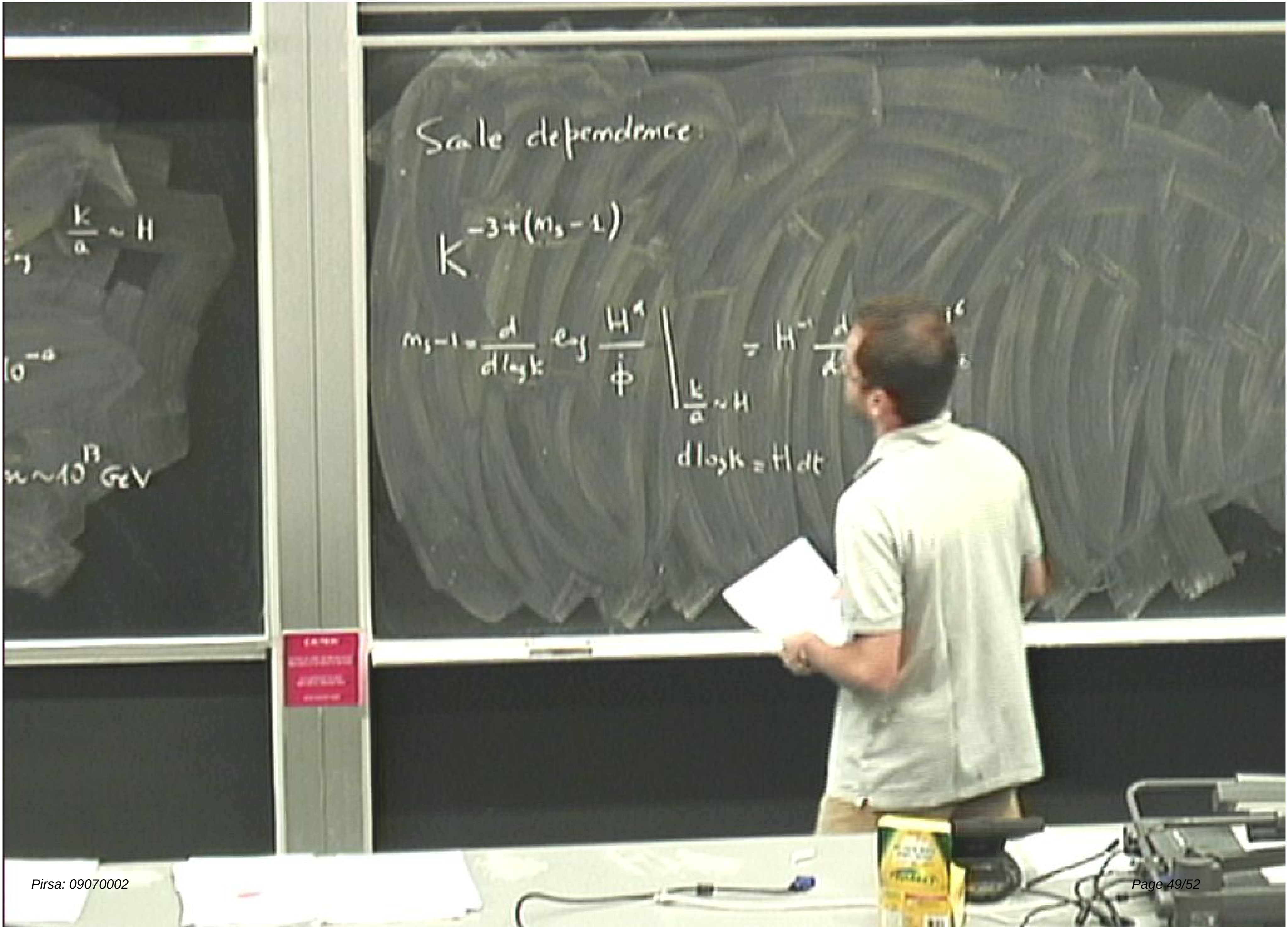

$\langle \zeta_k(\eta)\, \zeta_{k'}(\eta') \rangle = (2\pi)^3 \delta^{(3)}(\vec{k} + \vec{k}') \frac{H^2}{2k^3} \cdot \frac{H^2}{\dot\phi^2}$
after Hubble crossing
$\langle \zeta \zeta \rangle \sim \frac{H^4}{\dot\phi^2} \cong (10^{-5})^2$
$\zeta \sim \frac{H^2}{\dot\phi} = \frac{H}{\dot\phi/H}$
$\left(\frac{H}{M_{Pl}} \frac{1}{\sqrt{\epsilon}}\right)^2$
$V = \frac{1}{2} m^2 \phi^2$
$\epsilon \sim \frac{1}{N}$
$\frac{H}{M_{Pl}} \sim 10^{-4}$
$H \sim 10^{13}\ \mathrm{GeV} \to m \sim 10^{13}\ \mathrm{GeV}$
$V \sim (10^{16}\ \mathrm{GeV})^4$

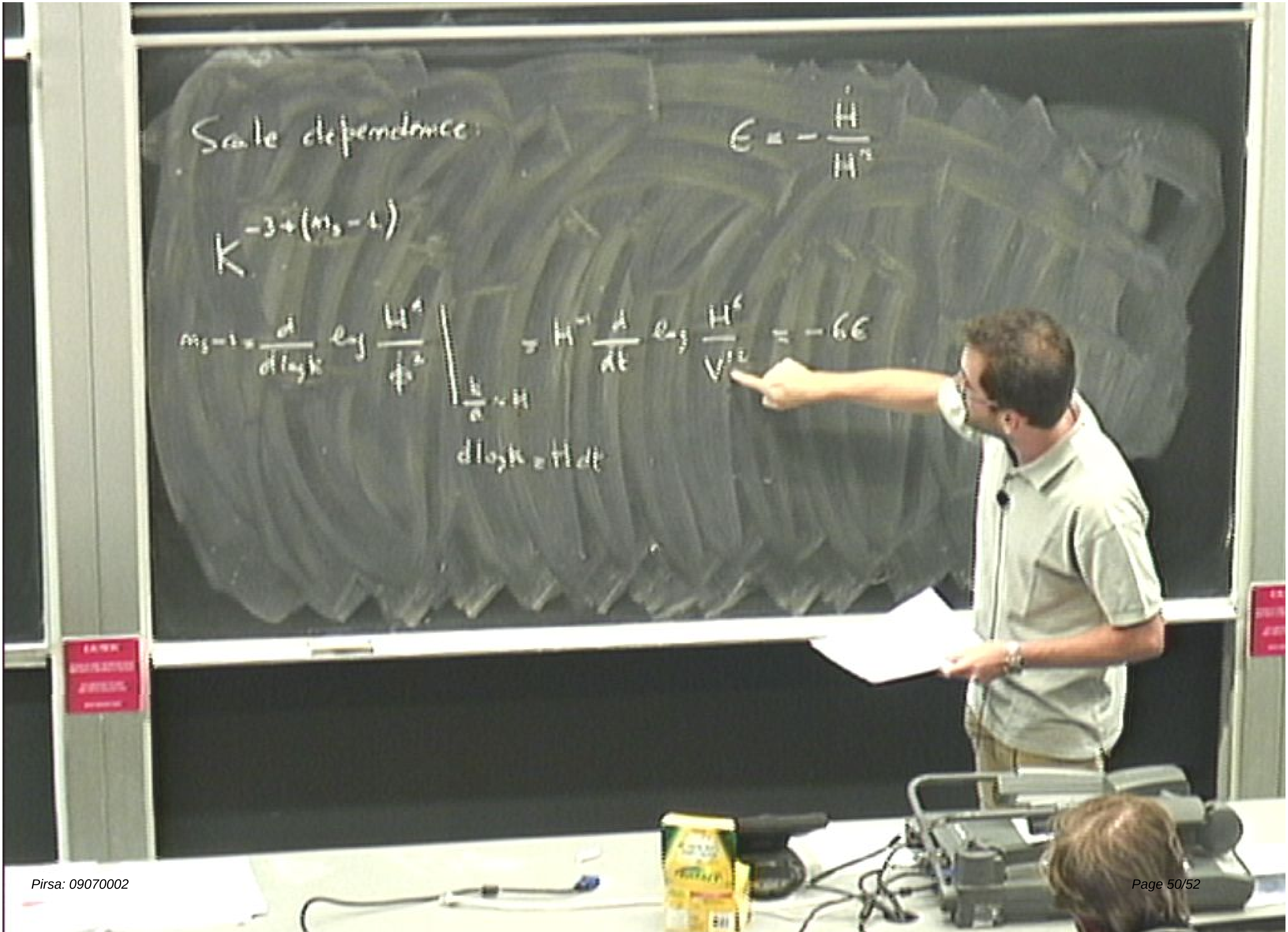



after
Hubble
crossing
H ~ 10^13 GeV → m ~ 10^13 GeV


after Hubble crossing
k/a ~ H
V = 1/2 m²φ²
ε ~ 1/N
H ~ 10^13 GeV → m ~ 10^13 GeV
Scale depe


Scale dependence:
k/a ~ H
~10^13 GeV
K^{-3+(n_s-1)}
d/dlog k log H^4/φ̇
k/a ~ H
dlog k = H dt


Scale dependence:
k^{-3+(n_s-1)}
n_s - 1 = d/d log k log H^4/φ̇ |_{k/a ~ H} = H^{-1} d/d
d log k = H dt
k/a ~ H
~10^13 GeV


Scale dependence:
ε = − Ḣ / H²
k^(−3+(n_s−1))
n_s − 1 = d / d log k log H⁴/φ̇² | k/a = H = H⁻¹ d/dt log H⁶/V'² = −6ε
d log k = H dt

Scale dependence:

$$\epsilon = -\frac{\dot{H}}{H^2} \qquad V'' \qquad \eta$$

$$k^{-3+(n_s-1)}$$

$n_s - 1 < 0$ red spectrum

$$n_s - 1 = \frac{d}{d\log k} \log \frac{H^4}{\dot{\phi}^2}\Bigg|_{\frac{k}{a}\sim H} = H^{-1}\frac{d}{dt}\log\frac{H^6}{V'^2} = -6\epsilon + 2\eta$$

$n_s - 1 > 0$ blue spectrum

$$d\log k = H\,dt$$

$$k^{-3+(n_s-1)}$$

$$n_s - 1 = \frac{d}{d\log k} \log \frac{H^4}{\dot{\phi}^2}\bigg|_{\frac{k}{a}\sim H} = H^{-1}\frac{d}{dt}\log\frac{H^6}{V'^2} = -6\epsilon + 2\eta$$

$$d\log k = H\,dt$$

$n_s - 1 < 0$ red spectrum

$n_s - 1 > 0$ blue spectrum

$n_s \simeq 0.96$